
© 2011 ICMA. 
Reprinted with permission.
ASPECTS OF SECOND-ORDER FEEDBACK AM SYNTHESIS

Victor Lazzarini†, Jari Kleimola‡‡, Joseph Timoney‡ and Vesa Välimäki‡‡
†Sound and Digital Music Technology Group
National University of Ireland, Maynooth
Ireland
victor.lazzarini@nuim.ie,
jtimoney@cs.nuim.ie
‡‡Dept. of Signal Processing and Acoustics
Aalto University School of Electrical
Engineering, Espoo, Finland
jari.kleimola@aalto.fi,
vesa.valimaki@tkk.fi

ABSTRACT
The technique of Feedback Amplitude Modulation (FBAM) is introduced in its second-order form. The basic aspects of the novel algorithm are discussed and its interpretation as a special case of a recursive periodic linear time-varying filter is explored. Extensions to the basic method are introduced, first with reference to existing variants of the first-order case, which have been previously studied. Second-order specific variations, such as pole-angle modulated resonator and frequency modulation, are discussed in some detail. The spectra produced with the proposed second-order methods are generally richer than those obtained using first-order methods, which is very desirable. Applications of the second-order FBAM methods complement the paper.

1. INTRODUCTION
Digital Feedback Amplitude Modulation (FBAM) synthesis is a class of synthesis techniques based on a self-modulating oscillator [1], [2]. It consists of a straightforward arrangement in which a delayed output is added to the oscillator amplitude. The simplest case, using a sinusoidal oscillator and a unit-sample delay, is defined by the following equation:

\[ y(n) = \cos(\omega_0 n) \left[ 1 + y(n-1) \right], \]  

(1)

where \( \omega_0 = 2\pi f_0 / f_s \), \( f_0 \) is the fundamental frequency, and \( f_s \) is the sampling rate.

This basic equation can then be used as the germ for a number of variations, which include, for instance, the addition of extra terms, waveshaping, heterodyning and longer feedback periods. FBAM can also be described a special case of a first-order recursive periodically linear time-variant (PLTV) filter [3], if Eq. (1) is recast as

\[ y(n) = x(n) + a(n)y(n-1), \]  

(2)

with \( x(n) = a(n) = \cos(\omega_0 n) \). Regarding FBAM as a PLTV filter proves to be very useful for the understanding of the system, as well as for developing variants to the basic technique, as extensively discussed in [2].

In this paper, we will extend the FBAM method from its first-order formulation (FBAM-1) to the second-order form (FBAM-2) and its variants. We will first examine the basic attributes of a straight extension of Eq. (1) into second-order and the definition of a basic FBAM-2 algorithm. This will be followed by a study of some of its derivative forms and applications.

2. SECOND-ORDER FBAM
A simple FBAM-2 expression can be defined as

\[ y(n) = \cos(\omega_0 n) \left[ 1 + y(n-1) + y(n-2) \right], \]  

(3)

where the amplitude of an oscillator is modulated by both its one-sample delay and its two-sample delay.

As with the first-order FBAM, this feedback expression can be expanded into an infinite sum of products given by

\[
y(n) = \cos(\omega_0 n) \cos(\omega_0 (n-1)) + \cos(\omega_0 (n-1)) \cos(\omega_0 (n-2)) + \ldots
\]

\[
+ \cos(\omega_0 (n-2)) \cos(\omega_0 (n-3)) + \ldots + \cos(\omega_0 (n-3)) \cos(\omega_0 (n-4)) + \ldots + \cos(\omega_0 (n-4)) \cos(\omega_0 (n-5)) + \ldots \]

(4)

which defines a pulse-like waveform made up of harmonics of the fundamental \( f_0 \). When compared to the first-order feedback oscillator,

\[ y(n) = \cos(\omega_0 n) \left[ 1 + y(n-1) \right] \]

(5)

we can see that a number of extra terms exist in the expansion. These will give rise to a narrower pulse and a richer spectrum with a wider bandwidth (see Fig. 1).

As with the original FBAM-1, it is useful to regard FBAM-2 as a second-order PLTV filter. In this case, Eq. (3) becomes

\[ y(n) = x(n) + a_1(n)y(n-1) + a_2(n)y(n-2), \]  

(6)
with \( x(n) = a_1(n) = a_2(n) = \cos(\omega_0 n) \). Of course, when developing the algorithm fully as a PLTV there will be no need to force the two coefficients \( a_1(n) \) and \( a_2(n) \) to be the same periodic signal or the filter input to be a sinusoid.

Figure 1. Comparison of FBAM-1 (dots) and FBAM-2 (continuous line) waveforms and spectra. \( f_0 = 500 \) Hz.

Finally, to complete the basic FBAM-2 algorithm, it is useful to include scaling parameters for the two feedback terms,

\[
y(n) = \cos(\omega_0 n) \left[ 1 + \beta_1 y(n-1) + \beta_2 y(n-2) \right],
\]

following the form seen in [1] for the first-order case, where it is called the ‘theme’ on which subsequent ‘variations’ are based. The flowchart of this basic FBAM-2 algorithm is shown in Fig. 2.

Figure 2. Flowchart of the basic FBAM-2 algorithm.

### 3. VARIANTS

#### 3.1. Second-order versions of FBAM-1 variations

The FBAM-1 coefficient-modulated allpass filter variation can be transformed into a second-order configuration by connecting two allpass stages into a cascade, and furnishing the latter stage with two feedback terms, as in

\[
\begin{align*}
    w(n) &= x(n-1) - a(n) \left[ x(n) - w(n-1) \right] \\
    y(n) &= w(n-1) - a(n) \left[ w(n) - \beta_1 y(n-1) - \beta_2 y(n-2) \right]
\end{align*}
\]

with \( x(n) = a(n) = \cos(\omega_0 n) \). The effect of the added allpass stage (\( \beta_2 = 0 \)) is depicted in Fig. 3, which shows a modest increase in bandwidth when compared to the original FBAM-1 form. However, increasing \( \beta_2 \) towards unity will gradually widen the bandwidth, until the spectrum reaches the shape shown in Fig. 1 (\( \beta_2 = 1 \)).

Other second-order FBAM-1 variants present similar characteristics. For example, the waveshaping variation, defined as

\[
y(n) = \cos(\omega_0 n) \left[ 1 + f[\beta_1 y(n-1) + \beta_2 y(n-2)] \right]
\]

and shown in Fig. 4 using a cosine waveshaper, behaves accordingly.

Figure 3. Allpass variation, comparison of FBAM-1 (dots) and FBAM-2 (continuous line). \( f_0 = 500 \) Hz, \( \beta_1 = 1, \beta_2 = 0 \).

Figure 4. Waveshaping variation – using \( f = \cos(\cdot) \) waveshaper, comparison of FBAM-1 (dots) and FBAM-2 (continuous line). \( f_0 = 500 \) Hz, \( \beta_1 = 1, \beta_2 = 0 \).

#### 3.2. Pole-angle modulated resonator

As discussed above, extensions of the basic FBAM-2 algorithm can be created by considering it as a second-order PLTV system. An interesting synthesis case is to consider an LTI resonator filter structure [4], defined by

\[
y(n) = x(n) + 2R \cos(\theta) y(n-1) - R^2 y(n-2),
\]

where the two complex-conjugate filter poles have radius \( R \) and angle \( \pm \theta \). In the PLTV case, our synthesis equation can be written as
\[ y(n) = x(n) + 2R \cos(\pi[3a(n) + \alpha])y(n-1) - R^2 y(n-2), \]

where \( x(n) = a(n) = \cos(\omega_0 n) \) and we are able to implement the filter pole-angle modulation.

Now we have three parameters to play with, the filter radius \( R \), the modulation amount \( \beta \), and the angle offset \( \alpha \). Various waveform shapes and spectra can be obtained with different values for these parameters, within their stability range. Fig. 5 shows the synthesis of a quasi-bandlimited square wave, generated by setting \( R = 0.5, \beta = 1 \) and \( \alpha = 0 \). Higher values of \( R \) will produce more harmonics, but with aliasing becoming more prominent.

\[ R = \frac{c}{f_i/2}, \quad \text{(12)} \]

This combination of parameters is very unstable and some \( c:m \) ratios are impossible (\( m \) defined as in [5] to be the modulator frequency). In particular, the cases \( c \leq m \) are problematic. Some ratios of small numbers are also unstable: 3:2, 2:1. The FM spectrum will be present for the duration of the envelope of the resonator impulse response, which is a decaying exponential defined by \( R^\alpha \). This allow us to generate an inharmonic attack based on a certain \( c:m \) ratio, which leads into an harmonic tone defined by the pole angle modulation synthesis after a certain amount of time.

Of course, since the pole-angle modulated resonator is PLTV, we can use distinct signals for its input \( x(n) \) and modulator \( a(n) \). An interesting case arises when we have a sinusoidal modulator and an arbitrary monophonic pitched input. In this case, we will be able to add components to the signal, creating a distorted output which is reminiscent of adaptive FM (AdFM) [6] and Adaptive Phase Distortion synthesis [7]. An example is shown on Fig. 6, where a C4 flute tone is used as an input to a pole-angle modulated resonator.

By setting the modulator frequency in relation to the input fundamental, it is possible to create harmonic or inharmonic spectra, depending on the modulator to input \( f_i \) ratio. This follows the similar principles of \( c:m \) ratios in FM (and AdFM) synthesis.

![Figure 5. Pole-angle modulation synthesis, with \( R = 0.5, \beta = 1 \) and \( \alpha = 0 \).](image)

Figure 6. Pole-angle modulated resonator with a C4 flute tone as input: (a) original steady-state spectrum and (b) spectrum of the pole-angle modulated filter output.

### 3.3. Frequency-modulated filters

Following these principles, it is possible to modulate the filter frequency and bandwidth directly, by converting the modulating signal into the filter pole parameters \( R \) and \( \theta \). In this case, in order to obtain a more stable behaviour from the filter, we keep a fixed \( Q \) ratio, which ultimately means that both \( R \) and \( \theta \) are modulated. To do that, we use Eq. (10) and the following identities:

\[ \theta = \frac{2\pi f_c(n)}{f_i} \quad \text{and} \quad R = \exp\left[ -\frac{\pi}{Q} \frac{f_c(n)}{f_i} \right], \quad \text{(13)} \]

with \( Q = f_C: B \), where \( B \) is the –3 dB bandwidth in Hz and \( f_c(n) \) is the time-varying centre frequency in Hz. The time-varying centre frequency can then be generated by sinusoidal modulation as

\[ f_c(n) = f_{i0} + A \cos\left[ \frac{2\pi f_m n}{f_i} \right], \quad \text{(14)} \]

where \( f_m \) is the modulation frequency in Hz. Care needs to be taken with \( Q \) and the frequency deviation \( A \) to keep the filter stable and reduce aliasing. The latter can be set to the product \( h_{i0} \), where \( I \) is a modulation index, as in classic FM synthesis. This set-up is much more stable than the basic pole-angle modulation FM and allows for yet another range of synthesis and processing effects.
4. APPLICATIONS

The Chamberlin state variable filter is a widely used second-order topology that enables decoupled control over the center frequency $f_c$ and Q factor, $Q_c = 1/Q$, of the filter [8], [9]. Applying the frequency-modulation variant of 3.3, we keep $Q_c$ fixed within the range 0...2, and modulate the center frequency of the filter using

$$f_c(n) = 2 \sin \left( \frac{\pi f_{\text{fo}}}{f_s} \right) + A \cos \left( \frac{2\pi f_{\text{fm}}}{f_s} n \right),$$  \hspace{1cm} (15)$$

The lowpass output of the Chamberlin filter can then be written in PLTV form as

$$y(n) = b_1(n)x(n-1) + \beta_1 a_1(n)y(n-1) - \beta_2 a_2(n)y(n-2),$$  \hspace{1cm} (16)$$

with $b_1(n) = f_0(n)^2$, $a_1(n) = 2 - f_0(n)Q_c - f_0(n)^2$, and $a_2(n) = 1 - f_0(n)Q_c$. Setting $\beta_2$ close or equal to 1, Eq. (16) produces a formant whose bandwidth and magnitude can be controlled using $Q_c$ and $A$. Parameter $\beta_1$ controls the center frequency of the resonance peak, as depicted in Fig. 7. The waveform plot shows also an initial transient, which damps rapidly with low $Q_c$ values, but stays more pronounced when $Q_c$ is increased. This is useful in inharmonic attack segment generation.

As a further extension of this principle, we proposed some new variants based on standard second-order filters, in particular looking at ways of modulating resonator parameters. This leads to novel possibilities, based on pole-angle and center-frequency modulation of second-order filters. The principles of second-order PLTV can be useful in the construction of interesting adaptive effects.

A remaining issue, currently under investigation, regards the filter stability, which is more complex here than in the first-order cases. Although beyond the scope of this initial study, this forms an important research question that will be tackled in subsequent work.

5. CONCLUSIONS

In this paper, we have studied some basic aspects of the second-order FBAM. We have presented it as a novel and natural extension of the first-order version of the synthesis method. It was demonstrated that the spectra of second-order FBAM variants are in general wider and richer than their first-order counterparts. This is a definite improvement on the original method, as it allows for a more complex output without any further modifications to the process. Following the methodology of previous studies for first-order cases, we have looked at the technique as a form of PLTV filtering with a sinusoidal input and modulator.

As a further extension of this principle, we proposed some new variants based on standard second-order filters, in particular looking at ways of modulating resonator parameters. This leads to novel possibilities, based on pole-angle and center-frequency modulation of second-order filters. The principles of second-order PLTV can be useful in the construction of interesting adaptive effects.

A remaining issue, currently under investigation, regards the filter stability, which is more complex here than in the first-order cases. Although beyond the scope of this initial study, this forms an important research question that will be tackled in subsequent work.

6. ACKNOWLEDGMENTS

This work has been supported by the Academy of Finland (project no. 122815).

7. REFERENCES


