
© 2008 Elsevier B.V.
Reprinted with permission.
Surface modes of negative-parameter interfaces and the importance of rounding sharp corners

Henrik Wallén*, Henrik Kettunen, Ari Sihvola

Department of Radio Science and Engineering, Helsinki University of Technology,
PO Box 3000, FI-02015 TKK, Finland

Received 9 January 2008; received in revised form 3 June 2008; accepted 17 July 2008
Available online 6 August 2008

Abstract

Surface (plasmon) resonances or electrostatic surface modes are possible on an interface between two materials with permittivity of opposite signs, and interfaces with permeability of opposite signs can support similar magnetostatic surface modes. This paper contains a review of the possible surface modes in several canonical geometries, and also numerical results demonstrating the importance of these surface modes for metamaterial modeling. Particular emphasis is placed on the unphysically singular surface modes that an ideal sharp wedge appears to support, and on how losses and rounded corners influence these modes.

© 2008 Elsevier B.V. All rights reserved.
PACS: 41.20.Cv; 42.25.Gy; 73.20.Mf

Keywords: Surface plasmons; Negative material parameters; Edge singularity

1. Introduction

Simultaneously negative permittivity and permeability give rise to negative index of refraction and allow backward waves to propagate in the medium. Equally interesting phenomena are the surface modes that can be excited on interfaces between media with material parameters of opposite signs. Consider, for instance, the ideal perfect lens [1], which has permittivity and permeability $\epsilon_r = \mu_r = -1$ relative to the environment. The focusing properties of this lens can be understood using ray-optics and negative refraction, but even more important is the amplification of evanescent waves that enables sub-wavelength resolution. This amplification is due to the excitation of surface modes in the sub-wavelength scale.

Looking at a length scale small enough compared with the wavelength, we can make a quasistatic approximation and consider electrostatics and magnetostatics separately. Negative-permittivity interfaces can support electrostatic surface modes or surface (plasmon) resonances [2–5], while negative-permeability interfaces support similar magnetostatic modes. In this paper, we only consider the electrostatic case with negative permittivity, since the magnetostatic case follows from simple duality.

The bulk effective permittivity of a metamaterial – or a natural plasmonic material – is not necessarily valid for a thin layer or sharp corner, but the very important question is outside the scope of this paper. Here, we a priori assume a model with a homogeneous object with bulk $\epsilon_r$ in free space, without any...
Metamaterials have been extensively studied in recent years, as is evident from the review articles [6–8], but it seems that the importance of the shape-dependent surface modes has been given little consideration in the metamaterial field. For instance, it is known that sharp negative-permittivity wedges can support unphysically singular surface modes [9,10], but several recent papers considering metamaterial wedges [11–14] pay little or no attention to the possible surface modes. Another recent study [15] shows that the numerical solution is non-convergent near a sharp wedge with negative permittivity and permeability, since unphysical resonant-like modes are excited near the edge, but the authors suggest that sufficiently large losses should be added to remove the problem.

In Section 2, we present a review of the possible surface modes in several important geometries, discussing the main features of the different modes and their dependence on losses. The purpose of this review is to give a better qualitative understanding of the kinds of surface modes that can occur also at more complicated metamaterial interfaces. We place particular emphasis on the importance of the shape-dependent surface modes.

In Section 3, we present some numerical results for a square negative-permittivity cylinder, which provide us some more quantitative insights on how losses and the rounding of sharp corners affect the numerical field solution.

This paper is an extended version of the conference presentation [16], which in turn was an outgrowth of research on the polarizability of a hemisphere [17–19].

### 2. Electrostatic surface modes

Dielectric objects with negative (effective) permittivity can support resonant-like electric fields – without external excitation – on a length scale much smaller than the wavelength. To get a better understanding of these surface modes, surface plasmons or electrostatic resonances, it is sufficient to consider the situation using classical electrodynamics. To justify this (quasi)static assumption, we assume that the whole object is much smaller than the wavelength. To get a better understanding of these surface modes, surface plasmons or electrostatic resonances, it is sufficient to consider the situation using classical electrodynamics. To justify the (quasi)static assumption, the electric field \( \mathbf{E} \) can be expressed in terms of the electrostatic potential as \( \mathbf{E} = -\nabla \phi \), and the task is to find a potential function satisfying the Laplace equation and the usual boundary conditions. To be more precise, assume that a dielectric object with relative permittivity \( \epsilon_r \) occupies the volume \( V \) bounded by the surface \( S \) in free space. The potential inside and outside of the object is denoted by \( \phi_i \) and \( \phi_o \), respectively, and \( \partial/\partial n \) is the normal derivative on the boundary \( S \). The potential should then satisfy

\[
\nabla^2 \phi_i = 0 \quad \text{in } V, \tag{1a}
\]
\[
\nabla^2 \phi_o = 0 \quad \text{outside } V, \tag{1b}
\]

and the boundary conditions

\[
\phi_i = \phi_o \quad \text{and} \quad \epsilon_r \frac{\partial \phi_i}{\partial n} = \frac{\partial \phi_o}{\partial n} \quad \text{on } S, \tag{2}
\]
\[
\phi_o(r) \rightarrow 0 \quad \text{when } r \rightarrow \infty. \tag{3}
\]

For an infinite object, the infinity-condition (3) must be slightly modified, so that both \( \phi_i \) and \( \phi_o \) stay bounded on the surface \( S \) and vanish when the distance from the surface grows to infinity.

Many surface-mode solutions from (1)–(3) have been published, see for instance [4,5,20] and the numerous references therein, and it should in principle be possible to construct explicit solutions in all the 13 coordinate systems [21] where the Laplace equation is separable. In the following subsections, we look at a few of the simpler ones, which together give good qualitative insights into the different kinds of surface modes that can occur in geometrically more complicated structures. These solutions can be constructed using the usual separation-of-variables technique in different coordinate systems, in some cases choosing the separation constants in a slightly unusual way to find solutions that oscillate along the interface \( S \) and decay away from it. The boundary conditions (2) then give some restrictions on the permittivity \( \epsilon_r \), for which the surface modes can occur.

#### 2.1. Half-space

The planar surface of a half-space \( z < 0 \) can support surface modes only when \( \epsilon_r = -1 \). The potential can be written using Cartesian coordinates in the form

\[
\phi(x, y, z) = \cos(k_x x) \cos(k_y y) e^{-s|z|}, \tag{4}
\]

where \( s > 0 \) and \( k_x^2 + k_y^2 = s^2 \); or using cylindrical coordinates as

\[
\phi(\rho, \varphi, z) = J_n(s\rho) \cos(n\varphi) e^{-s|z|}, \tag{5}
\]
where $s > 0$ and $n$ is an integer. The previous expressions with sin instead of cos, and any linear combination of such expressions with different values for the parameters $s$ and $(k_x, k_y)$ or $n$, are also valid surface-mode solutions.

2.2. Sphere

A sphere can support surface modes if

$$\epsilon_r = -\frac{n+1}{n}, \quad n = 1, 2, \ldots, \quad (6)$$

and if the sphere has radius $a$ and is placed at the origin, the potential can be expressed using spherical coordinates in the form

$$\phi_i(r, \theta, \varphi) = \left(\frac{r}{a}\right)^n P_n^m(\cos \theta) \cos(m\varphi), \quad (7a)$$

$$\phi_o(r, \theta, \varphi) = \left(\frac{r}{a}\right)^{n+1} P_n^m(\cos \theta) \cos(m\varphi), \quad (7b)$$

where $P_n^m$ is the associated Legendre function of degree $n$ and order $m = 0, 1, \ldots, n$. Also the solution $n = 0$ would be possible, but that corresponds to a charged perfectly conducting sphere. The lowest order mode is the dipolar resonance for $n = 1$ or $\epsilon_r = -2$, which is also called the Fröhlich mode [22]. Finally, when $n$ grows, the surface modes become more confined to the surface and in the limit $n \to \infty$ we get $\epsilon_r \to -1$ which seems very reasonable compared with the planar case.

2.3. Ellipsoid

The surface modes of spheroids look fairly complicated [4], and the ones of a general ellipsoid are certainly not simpler. Fortunately, the permittivity condition for the lowest order modes of a general ellipsoid can be written in the very simple form [22]

$$\epsilon_r = 1 - \frac{1}{L_i}, \quad i = 1, 2, 3, \quad (8)$$

where the shape factors $L_i$, which are determined by the semiaxis ratios, satisfy

$$0 < L_1 \leq L_2 \leq L_3 < 1, \quad L_1 + L_2 + L_3 = 1. \quad (9)$$

Clearly, we can get any $\epsilon_r < 0$ by varying the semiaxes of the ellipsoid. Some interesting special cases are disks, spheres and needles. For a disk $L_1 = L_2 \to 0$ and $L_3 \to 1$, which gives $\epsilon_r \to -\infty$ and $\epsilon_r \to 0$; the sphere has $L_1 = L_2 = L_3 = 1/3$, which gives $\epsilon_r = -2$ as above; and for a needle $L_1 \to 0$ and $L_2 = L_3 \to 1/2$, which gives $\epsilon_r \to -\infty$ and $\epsilon_r \to -1$. As another example, an ellipsoid with semiaxes $a, 2a$ and $3a$ gives $\epsilon_r = -0.734, \epsilon_r = -2.743$ and $\epsilon_r = -5.398$.

2.4. Cylinder

A cylinder $\rho < a$ supports two different kinds of surface modes. One set of modes is the 2D-version of the spherical surface modes

$$\phi_i(\rho, \varphi, z) = \left(\frac{\rho}{a}\right)^n \cos(n\varphi), \quad (10a)$$

$$\phi_o(\rho, \varphi, z) = \left(\frac{\rho}{a}\right)^{n+1} \cos(n\varphi), \quad (10b)$$

where $n = 1, 2, \ldots$ and these modes are only supported when $\epsilon_r = -1$. Another set of modes oscillates along the cylinder [5]

$$\phi_i(\rho, \varphi, z) = A I_n(s\rho) \cos(n\varphi) \cos(sz), \quad (11a)$$

$$\phi_o(\rho, \varphi, z) = B K_n(s\rho) \cos(n\varphi) \cos(sz), \quad (11b)$$

where $I_n$ and $K_n$ are the modified Bessel functions of order $n = 0, 1, 2, \ldots$, the constants satisfy $BK_n(s\rho) = AI_n(s\rho)$ and $s > 0$ is an arbitrary parameter. These surface modes are supported when

$$\epsilon_r = \frac{K_n'(sa)I_n(sa)}{K_n(sa)I'_n(sa)}, \quad (12)$$

which is also plotted in Fig. 1. The limit $\epsilon_r \to -\infty$, when $s \to 0$ and $n = 0$, corresponds to a charged ideally conducting cylinder, while the other limits $s \to 0$ and $n > 0$ approach the 2D-case (10), as one would expect. Finally, when $s \to \infty$, the modes approach the planar case, with $\epsilon_r \to -1$.

2.5. Sharp wedge

An ideal wedge can support surface modes that oscillate with constant amplitude and infinite spatial fre-
quency at the sharp edge [9]. In the following, we assume that the wedge occupies the region $-\alpha < \varphi < \alpha$ in cylindrical coordinates, and use the angle $\alpha < \varphi < 2\pi - \alpha$ for expressing the potential $\phi_0$ outside the wedge.

The wedge supports even modes for

$$\epsilon_e = \frac{\tanh[\nu(\pi - \alpha)]}{\tanh(\nu \alpha)},$$

(13)

and odd modes for

$$\epsilon_o = \frac{\tanh(\nu \alpha)}{\tanh[\nu(\pi - \alpha)]},$$

(14)

where $\nu > 0$ is an arbitrary parameter. The permittivity $\epsilon_e(\nu)$ is also plotted in Fig. 2 for three different wedge angles. Note that (13) and (14) approach $\epsilon_e = -1$ from opposite sides when $\nu \to \infty$, but surface modes for $\epsilon_e = -1$ and finite $\nu$ are possible only when $\alpha = \pi/2$, that is, when the wedge becomes a half-space. However, when $\epsilon_e = -1$, the flat sides of a wedge can locally support similar surface modes as a half-space.

The even modes can be expressed in the form

$$\phi_e(\rho, \varphi, z) = AK_{j\nu}(s\rho) \cosh(\nu \varphi) \cos(sz),$$

(15a)

$$\phi_0(\rho, \varphi, z) = BK_{j\nu}(s\rho) \cosh[\nu(\pi - \varphi)] \cos(sz),$$

(15b)

where $s > 0$ is an arbitrary parameter, the constants satisfy $B \cosh[\nu(\pi - \alpha)] = A \cosh(\nu \alpha)$, and $K_{j\nu}$ is the modified Bessel function of second kind and imaginary order. The even modes that are independent of $z$ can be expressed in the form

$$\phi_e(\rho, \varphi) = A \cos(\nu \ln \rho) \cosh(\nu \varphi),$$

(16a)

$$\phi_0(\rho, \varphi) = B \cos(\nu \ln \rho) \cosh[\nu(\pi - \varphi)].$$

(16b)

The modified Bessel function $K_{j\nu}$ decreases exponentially with large argument, and with small argument $sp \ll 1$ we get the approximation

$$K_{j\nu}(s\rho) \approx C \cos(\nu \ln(s\rho) + \psi),$$

(17)

where

$$C = |\Gamma(j\nu)|, \quad \psi = \arg \left\{ \frac{\Gamma(-j\nu)}{2^{j\nu}} \right\},$$

(18)

using the first term of the series expansion of $\Gamma_\nu$ and a few other formulas from [23]. Therefore, the solution (16) can also be considered a special case of (15) when $s \to 0$.

The odd modes can be expressed using sinh instead of cosh in (15) and (16).

The electric field at the tip of a sharp wedge can be singular. For an ordinary dielectric wedge, with positive permittivity $\epsilon_r > 0$, the potential behaves as $\rho^s$, with $s > 1/2$, when $\rho \to 0$ [24]. Then, the electric field behaves as $\rho^{s-1}$, and the energy density is proportional to $\rho^{2(s-1)}$ which is at most weakly singular, that is, the integrated field energy is always finite in the neighborhood of an ordinary positive-permittivity wedge.

However, the surface mode solutions (15) and (16) of a sharp negative-permittivity wedge are more singular. The potential behaves as $\cos(\nu \ln \rho)$, which is finite but oscillating with infinitely short spatial period when $\rho \to 0$. The electric field behaves like $\rho^{-1}$ (times a wildly oscillating function) and the energy density is hyper-singular as $\rho^{-2}$. That is, the integrated field energy is infinite in the neighborhood of the wedge, which is clearly unphysical. This result follows – mathematically – from the assumption that the corner is infinitely sharp and, therefore, the most obvious remedy is to round the sharp corner [10].

2.6. Losses

In any practical situation, we always have at least small losses, which we can model by including an imaginary part $\epsilon_r > 0$ in the relative permittivity $\epsilon_r = \epsilon_r' - j\epsilon_r''$, if we assume a quasistatic approximation based on time-harmonic fields with time dependence $e^{j\omega t}$.

Taking losses into account, the surface modes should not be possible without external excitation. For the sphere and half-space, it is obvious that the surface modes are suppressed if $\epsilon_r'' \neq 0$, but for the cylinder and wedge the solutions need more careful consideration.

A small sphere in a uniform external field would give an infinite dipolar response when $\epsilon_r = -2$, due to the excited surface mode. Including losses, the response is finite but it can still be very strong for metals such as silver, gold and aluminum when $\epsilon_r \approx -2$ [22].
The surface modes at a planar interface seem to be most sensitive to losses. Even a small imaginary part of the permittivity \( \varepsilon_r \ll \varepsilon_r' \) can have a dramatic impact on the amplitude of the excited surface modes. For instance, one problem with the perfect lens is that small losses can severely limit the performance of the lens [25–27].

For the cylinder, (12) appears to allow surface modes for complex \( \varepsilon_r \) if \( s \) is complex. However, in that case the potential (11) would be exponentially increasing in either the +z or −z direction, which is clearly not a valid surface-mode solution for an infinite cylinder. Therefore, the surface modes of the cylinder are also suppressed if we include losses.

The wedge is more problematic. Instead of (16), we can express the \( z \)-independent solution as
\[
\phi_i(\rho, \varphi) = A \rho^{-j\nu} \cosh(\nu \varphi),
\]
(19a)
\[
\phi_o(\rho, \varphi) = B \rho^{-j\nu} \cosh[\nu(\pi - \varphi)],
\]
(19b)
which is a valid surface-mode solution for complex \( \nu = \nu' - j\nu'' \) with \( \nu', \nu'' > 0 \). Then, (13) gives a complex \( \varepsilon_r = \varepsilon_r' - j\varepsilon_r'' \), and we can conclude that a lossy but infinitely sharp wedge appears to support surface modes! However, in reality, this is an anomaly or consequence of the idealized mathematical model – not a real physical phenomenon.

The obvious conclusion is that a negative permittivity material (with or without losses) should never be modeled using sharp corners.

2.7. Comparison of the surface modes

All the above presented surface modes share some common features, but there are also qualitative differences. In all cases, the relative permittivity \( \varepsilon_r \) must be negative, and the surface modes are oscillatory along the interface and decreasing with the distance from the interface.

In some cases the possible permittivity values form a discrete spectrum, and in other cases a continuous spectrum. In general, a bounded sufficiently smooth object should have a discrete spectrum [28], while an infinite object can have a continuous spectrum [20].

The half-space or planar interface is somewhat special as the spectrum is degenerated into one value, \( \varepsilon_r = -1 \), but instead the resonant modes are less restricted than in the other cases. The sphere and ellipsoid have discrete spectra, while the infinitely long cylinder and wedge have continuous spectra. However, for a fixed oscillation along the \( z \)-axis, that is, for a fixed parameter \( s \) in \( \cos(sz) \), the cylinder has a discrete spectrum (12) but the continuous spectrum of the sharp wedge (13) and (14) is independent of the parameter \( s \). Finally, for a rounded wedge, the spectrum is continuous but dependent on \( s \), as shown in [10]. That is, for fixed \( s \) we get a discrete spectrum of \( \varepsilon_r \), in the same manner as for the cylinder. Moreover, the \( \varepsilon_r \) spectrum for a rounded wedge is located within a smaller interval than for the sharp wedge.
3. Numerical experiments

3.1. Electrostatic effective permittivity

As a first numerical example, we consider the electrostatic effective permittivity of a mixture with square cylinders with negative permittivity $\epsilon_r$. The cylinders occupy a volume fraction of 1/8 in a square lattice with period $2a$, with the unit cell shown in Fig. 3. We solve the electrostatic potential $\phi(x, y)$ using Comsol Multiphysics (version 3.4) in 2D, using the initial mesh shown in Fig. 4 and also using one adaptive mesh refinement step. Using a plate-capacitor model, the effective relative permittivity becomes

$$\epsilon_{\text{eff}} = \frac{1}{V_0} \int_a \frac{\partial \phi}{\partial x}(x, y) \, dl,$$

(20)

where $V_0$ is the applied voltage and the integral is over the left boundary in Fig. 4.

The first result for $\epsilon_{\text{eff}}$ as a function of $\epsilon_r$ is shown in Fig. 5a. The computation seems to predict many infinities in the range $-3.2 < \epsilon_r < -0.38$, but the result is not converging in any sense using a finer discretization. For a sharp 90° wedge in free space, we should not get a unique solution for $-3 < \epsilon_r < -1/3$, due to the surface modes, and this is also confirmed by the computational result, although the permittivity range is slightly shifted compared with the single wedge case. This small shift seems reasonable since the surface modes at each corner are affected by the other corners of the same cylinder, and also by the other cylinders in the periodic mixture.

Adding losses as $\epsilon''_{\text{eff}} > 0$ changes the results, as shown in Fig. 5b, but the effective permittivity $\epsilon_{\text{eff}}$ is still not converging for all $\epsilon_r'$. As predicted in Section 2.6, it appears that the sharp edges can support unphysical sur-
face modes also for lossy materials. Near $\epsilon_r = -1$, the solution converges, since the surface modes at the planar surfaces are damped by the losses, and the edges do not support surface modes when the permittivity is exactly $\epsilon_r = -1$.

Using a small rounding of the corners of the square cylinder, the solution changes dramatically, as shown in Fig. 5c. The rounding is circular, with radius $R = 0.02a$, which changes the volume fraction marginally, from 1/8 to 0.1249. The effective permittivity still has singularities, but the solution is numerically stable, except near $\epsilon_r = -1$. Furthermore, the singularities occur for a discrete spectrum of $\epsilon_r$ values, with an accumulation point at $\epsilon_r = -1$, in agreement with the theoretical considerations above in Section 2.7.

Adding small losses to the rounded-corner case makes the solution stable for all $\epsilon_r$, and as shown in Fig. 5d the singularities in $\epsilon_{\text{eff}}$ are damped but their locations are stable.

Small roundings and small losses are sufficient to ensure that the solution is stable, but the actual solution clearly depends on the rounding. Fig. 6 shows the effective permittivity results using $R = 0.004a$, $R = 0.02a$ and $R = 0.1a$. The permittivity range where the singularities occur gets narrower with larger roundings, which is consistent with the analytical results in [10] for a different type of rounding.

### 3.2. Plane-wave scattering

As a second numerical example, we consider plane-wave scattering from a square cylinder with permittivity $\epsilon_r = -2 - 0.02j$ and side length $a = \lambda/2$, where $\lambda$ is the free-space wavelength. The cylinder is directed along the $z$-axis and the geometry of the computational setup in the $xy$-plane is shown in Fig. 7.

We solve the scattering problem in 2D using Comsol Multiphysics, using an incident plane wave and TM or TE polarization. In the TMz case, the incident plane-wave is $E_0 \mathbf{u}_z e^{-j k z}$, and the scattered electric field has only a $z$-component, which is used as the unknown. Similarly, in the TEz case, the magnetic field has only a $z$-component, while the electric field has $x$- and $y$-components.

The computational domain is terminated using a cylindrical perfectly matched layer (PML) with thickness $0.4\lambda$, which is further terminated using a scattering boundary condition. The initial mesh has 4822 second-order elements, with maximum element size $0.1\lambda$ and a similar mesh refinement at the sharp corners as in Fig. 4. In addition to the sharp corner case, we also solve the same problem using rounded corners, using a circular rounding with radius $0.02\lambda$. In the rounded corner case, the initial mesh has 4832 second-order elements, and the mesh refinement near the corners is again similar to the sharp corner case.

In the TEz case, surface modes are excited at the sharp corners, but not in the TMz case. In the TMz case, the electric field is tangential to the surface of the cylinder, and in particular, the field is orthogonal to the possible surface modes. Also without surface modes, the TMz case should be smoother since the fields are continuous across the boundary of the cylinder, while the electric field is discontinuous in the TEz case.

Instead of looking primarily at the fields near the cylinder, where it is obvious that the surface modes

---

Fig. 6. Real part of the effective permittivity $\epsilon_{\text{eff}}$ as a function of $\epsilon_r$, using different roundings $R/a = 0.004, 0.02, 0.1$ and small losses $-\epsilon_r/\epsilon_\ell = 0.01$.

Fig. 7. Geometry ($xy$-plane) for the plane-wave scattering test. The wave vector $k$ shows the direction of the incident plane wave, $\varphi$ is the scattering angle, and the near-to-far-field transformation is computed along the dashed circle with radius $0.9\lambda$.
should have a large impact, we are interested in the scattered far field or scattering cross section. To compute this, we solve the scattered (near) field, and thereafter, as a post-processing step, compute the scattered far-field pattern using the Stratton–Chu formula, integrating the near field along the circle shown in Fig. 7.

The surface modes of a planar surface or a wedge are very localized to the interface. The solutions (4) and (16) decrease exponentially with the distance from the interface. Therefore, the unphysically singular surface modes of the sharp edges should not influence the scattered far fields. The numerical results in Fig. 8 a and b, however, suggest that the numerical instability of the solution can also be visible in the far-field solution.

Fig. 8 a shows the scattering cross section as a function of the scattering angle for both the TMz and TEz cases. Actually, six results are plotted in the figure: four coinciding results for the TMz case and two for the TEz case. In the TMz case, the scattering cross section is almost exactly the same using sharp corners and rounded corners, but in the TEz case the result is only shown for the rounded corner case. In all these three cases, adding one adaptive mesh-refinement step does not significantly change the result.

In the TEz case with sharp corners, the near-field results are not converging with smaller discretization, and also the far-field results seems to be very unstable. Fig. 8 b shows the scattering cross section for the TEz case with sharp corner, using the initial mesh and two adaptive mesh-refinement steps. The results are somewhat similar to the rounded corner case, but it seems impossible to get accurate and stable results using sharp corners. In this case the losses are small, but even significantly increased losses do not make the solution stable. Using only losses to suppress the unphysically singular surface modes, it appears that $\epsilon'_r$ should be equal in magnitude as $\epsilon''_r$, or preferably larger.

4. Conclusions

Negative permittivity materials – artificial metamaterials as well as natural plasmonic materials – support localized electrostatic surface modes or surface (plasmon) resonances, on a length scale much smaller than the wavelength. The character of these surface modes depends highly on both the permittivity, which in all cases must be negative, and on the shape of the object or interface. In Section 2, a review was given about the main features of surface modes for basic canonical geometries. The number of electrostatic modes is infinite for a given particle. However, the character of the mode spectrum is very interesting as it can be either discrete or continuous, depending on the regularity of the surface and on whether the particle is finite or infinite in one direction.

A special emphasis was given to the wedge and its surface modes. For ordinary positive-permittivity objects, the electric field in the vicinity of an ideal sharp wedge becomes singular, but the model is still physically reasonable since the field energy is finite in the vicinity of the edge. However, the field solution becomes even more singular at the tip of a sharp negative-permittivity wedge, for a range of $\epsilon_r < 0$. The electric field energy would be infinite in the vicinity of the edge, and the sur-
face modes seem mathematically possible even if losses are included. This is, however, clearly an anomaly of the mathematical model and not a real physical phenomenon.

The characteristics of the modes were also illustrated with a numerical study where the effective quasistatic permittivity of a mixture with square cylinders was computed using the finite element method. The theoretical mode range in $\varepsilon_r$ for the wedge appeared clearly in the results and the details within the continuous spectrum did not stabilize when the computational grid was made denser. This situation clearly differs from “ordinary” plasmonic modes like the dipolar resonance at $\varepsilon_r = -2$ for a sphere, which can be captured numerically with very good accuracy.

There are two aspects in which the wedge problem can be made realistic: losses in the material response and a curvature in the corner. These were also analyzed numerically. A very important conclusion was that the wedge modes cannot be captured even if the medium is allowed to be lossy. The response of the particle did not converge with increasing computational effort. However, when the wedge corner was rounded, the solution converged clearly and the surface modes were stable. This happened even with lossless medium and a very small (but finite) radius of curvature.

Finally, the plane-wave scattering from a square cylinder with negative permittivity and losses was studied. The width of the square was one half wavelength. The numerical results confirmed the quasistatic finding that a rounding of the corners provides stable results. This time the response was the far-field scattering pattern. For the rounded corner geometry, the result did not converge mean-

Both the analytical and numerical results show that sharp corners can lead to unphysical solutions when modeling negative-parameter materials. The overall conclusion is that corners must be rounded if the real part of $\varepsilon_r$ (or $\mu_r$) is in the range where electrostatic (or magnetostatic) surface modes can occur. Adding losses is not sufficient.

References