Spatially dispersive metasurfaces

Viktar Asadchy
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Aalto University
School of Electrical Engineering
Department of Electronics and Nanoengineering
Abstract

Natural materials and substances possess a rich variety of electromagnetic properties over the entire electromagnetic spectrum. Despite this diversity, nature does not equip us with a full set of possible material tools for controlling electromagnetic waves and realizing all physically possible effects. Only the use of artificial, engineered substances can give us full control over electromagnetic properties of materials. For instance, while spatial dispersion effects are weak in natural materials, they can be strongly pronounced in artificially engineered composite materials, metamaterials. Metamaterials consist of inclusions whose dimensions are small but comparable with the operating wavelength, which enables existence of strong spatial dispersion effects in them such as bi-anisotropy, magnet-less magnetism, and gyrotropy.

This dissertation is devoted to the young and scantly explored field of spatially dispersive metasurfaces. Metasurfaces represent a two-dimensional arrangement of sub-wavelength inclusions engineered to manipulate in a prescribed fashion incident electromagnetic radiation. The first half of the dissertation contains a theoretical review of the research field essential for understanding of the obtained results outlined in the second half.

Presentation of novel results can be broken down into three parts. The first part describes a semi-analytical technique for polarizability extraction of an arbitrary electrically small bi-anisotropic scatterer. Subsequently, the technique was exploited for the design of a novel scatterer with extremely pronounced spatial dispersion of the first order.

The second part outlines the key ideas behind two designed spatially dispersive metasurfaces: A resonant gradient reflector and an absorber transparent outside the resonance band. It is demonstrated that such shadow-free operation of the metasurfaces requires spatial dispersion effects.

The last part presents the exact synthesis of gradient metasurfaces for ideal wavefront control in reflection and transmission regimes. The fundamental importance of spatial dispersion in such metasurfaces is demonstrated. As a proof of concept, an optical metasurface for perfect anomalous reflection is designed and measured.

Keywords spatial dispersion, metamaterials, metasurfaces, bi-anisotropy, gratings

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Preface

Our reality is like a game. We are all forced to follow the rules of this game. We do not know why we are pulled by gravitation to the Earth, but we know that it is one of the game rules. These rules are the only universal rules; they remain constant and do not depend on societies and civilizations. The name of this game is Physics.

The present work was carried out at the Department of Electronics and Nanoengineering of Aalto University, School of Electrical Engineering.

I would like to express my warmest gratitude to my supervisor, Prof. Sergei Tretyakov. Without your guidance and persistent help this dissertation would not have been possible. I very appreciate your numerous scientific advices and discussions with you from which I have learnt a lot. There is still so much I could learn from you. It was a great honour and privilege for me to work in your group and contribute to its high-quality research. I like the friendly and international spirit of the group as well as very important seminars. You make the crucial contribution to both of them.

I would like to thank my second supervisor at Francisk Skorina Gomel State University (Belarus), Prof. Igor Semchenko. I am very grateful to you for bringing me to the field of metamaterials and proposing to me several important research directions. You are always ready to support me in both academic and practical matters.

There are several other people who made crucial impact on my academic path. First of all, it is my teacher of physics at high school, Dr. Alexei Pavlov. His outstanding teaching skills and deep understanding of physics have determined my future. I thank Dr. Valerii Kapshai and other excellent professors and lecturers at Francisk Skorina Gomel State University who provided high-level physics education during my undergraduate studies at university. I am very grateful to Prof. Sergei Khakhomov for his
help with multiple practical and academic issues as well as for his good
and encouraging humour. I am also indebted to Prof. Martin Wegener and
Mr. Andreas Wickberg for providing me the opportunity to learn different
nanofabrication and optical measurement techniques during my research
visit in Karlsruhe Institute of Technology. The last but not least, I would
like to thank Prof. Constantin Simovski and Prof. Igor Nefedov for multi-
ple important scientific discussions during my doctoral studies.

I would like to express my thanks to all my collaborators and friends: Dr.
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mad Albooyeh, Dr. Ana Díaz Rubio, Dr. Mohammad Sajjad Mirmoosa,
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Mirza, Mr. Dimitrios Tzarouchis, Mr. Mikhail Omelyanovich, Mr. Sergei
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great and supportive to me these people have been throughout my life.

I would like to thank the preliminary examiners, Prof. Mario Silveirinha
and Dr. Dimitrios Sounas, for thorough examination of this thesis.

During this thesis work I have received financial support from the Aalto
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I address my warmest gratitude to my dear parents Natallia and Sergei,
aunt Elena, brother Ilia, and cousin Ekaterina for their unbounded kind-
ness and love to me, persistent support and presence in my life. Family
is of the highest importance to me. My special thanks goes to my grand-
mother Lyudmila who was an example of highest morality for me.

Finally, I wish to express my sincere thanks to my sweetheart wife
Huyen. Our meeting has divided my life into two parts: Incomplete “be-
fore” and very happy “after”.

Espoo, August 16, 2017,

Viktar Asadchy
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List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.


**Author’s Contribution**

Publication I: “Determining polarizability tensors for an arbitrary small electromagnetic scatterer”

The author formulated the idea based on his Diploma thesis prepared under the supervision of Prof. I.V. Semchenko. The analytical derivations were done by the author and verified by Dr. Y. Ra’di and Mr. I.A. Faniayeu. The author together with Mr. I.A. Faniayeu performed the simulations. The main body of the paper was written by the author. All the co-authors contributed to editing the paper. The work was conducted under the supervision of Prof. S.A. Tretyakov.

Publication II: “Purely bianisotropic scatterers”

The initial idea for the paper was proposed by Prof. S.A. Tretyakov. Results in Sections II and Appendix A were obtained by Dr. M. Albooyeh and Mr. S. Mirmoosa. Dr. M. Albooyeh proposed the scatterer in Section IIIA. The author proposed the purely bianisotropic scatterer in Section IIIB, performed the corresponding simulations, and prepared Section IV. The simulation results in Section IIIA were performed by the author, M. Albooyeh, R. Alaee, S.M. Hashemi, and M. Yazdi. The main body of the paper was written by Dr. M. Albooyeh and the author. All authors contributed to editing the text. The work was carried under the supervision of Prof. S.A. Tretyakov and Prof. C.R. Simovski.
Author's Contribution

Publication III: “Functional metamirrors using bianisotropic elements”

Prof. S.A. Tretyakov proposed the idea of the paper and supervised the theoretical analysis and numerical simulations. The author performed theoretical analysis, numerical simulations, device fabrication, near-field measurements, and wrote the manuscript. Dr. J. Vehmas and Prof. S.A. Tretyakov supervised the measurements. All authors contributed to editing the manuscript.

Publication IV: “Optical metamirror: all-dielectric frequency-selective mirror with fully controllable reflection phase”

Dr. M. Albooyeh proposed the idea of the paper and derived analytical formulas in Section 2. The author designed the metasurface, performed numerical simulations and interpretation of the results. All authors contributed to writing the manuscript. The work was conducted under the supervision of Prof. S.A. Tretyakov.

Publication V: “Broadband reflectionless metasheets: frequency-selective transmission and perfect absorption”

The initial idea for the paper was proposed by the author and subsequently developed by Prof. S.A. Tretyakov. The theoretical analysis and metasurface design were mainly performed by the author. The simulation results were obtained by the author and Mr. I.A. Faniayeu. The experimental realization and characterization were performed by Mr. I.A. Faniayeu at Francisk Skorina Gomel State University (Belarus) under the supervision of Profs. I.V. Semchenko and S.A. Khakhomov. Equivalent circuit shown in Fig. 7 was obtained by Dr. Y. Ra’di. The manuscript was mainly written by the author. All authors contributed to editing the manuscript. The work was conducted under the supervision of Prof. S.A. Tretyakov.
Publication VI: “Perfect control of reflection and refraction using spatially dispersive metasurfaces”

The idea for the paper was proposed by Prof. S.A. Tretyakov. The author, M. Albooyeh, S.N. Tcvetkova and Prof. S.A. Tretyakov equally contributed to the theoretical analysis in the paper. Dr. Y. Ra’di contributed to Section IIE.1, and Dr. A. Díaz-Rubio to Section IIE.2. The manuscript was mainly written by Prof. S.A. Tretyakov, the author, and Dr. M. Albooyeh. All authors contributed to editing the manuscript.

Publication VII: “Eliminating scattering loss in anomalously reflecting optical metasurfaces”

The initial idea for the paper was proposed by the author. The author performed theoretical analysis, design of the metasurface, numerical simulations, and optical measurements. Mr. A. Wickberg fabricated the metasurface. Dr. A. Díaz-Rubio derived formulas for the theoretical bound of conventional metasurface gratings. The manuscript draft was written by the author. All authors as well as Prof. S.A. Tretyakov contributed to editing the manuscript. The work was conducted under the supervision of Prof. M. Wegener.
List of Abbreviations

EMNZ  epsilon-and-mu-near-zero
TE    transverse electric
PEC   perfect electric conductor
FSS   frequency selective surface
HIS   high-impedance surface
List of Symbols

\(a\) linear size of an inclusion [m]
\(a_0\) complex coefficient
\(\mathbf{a}\) basis vectors of a coordinate system
\(a_{ij}\) susceptibility tensor of the second rank [F \cdot m\(^{-1}\)]
\(a_{ijk}\) susceptibility tensor of the third rank [F]
\(a_{ijkl}\) susceptibility tensor of the forth rank [F \cdot m]
\(a_w\) periodicity of an array of wires [m]
\(\mathbf{B}\) magnetic flux density [T]
\(\mathbf{B}_{in}\) magnetic flux density inside volume \(V\) [T]
\(\mathbf{B}_{out}\) magnetic flux density outside volume \(V\) [T]
\(c\) speed of light [m \cdot s\(^{-1}\)]
\(\mathbf{D}\) electric displacement field [C \cdot m\(^{-2}\)]
\(\mathbf{D}\) lattice periodicity [m]
\(\mathbf{D}_x\) lattice periodicity along the \(x\)-axis [m]
\(d\) distance between adjacent inclusions in a periodical array [m]
\(d_s\) slab thickness [m]
\(\mathbf{E}\) electric field [V \cdot m\(^{-1}\)]
\(\mathbf{E}_{av}\) electric field averaged over the two sides of a metasurface [V \cdot m\(^{-1}\)]
\(\mathbf{E}_{b}\) electric field of a wave scattered in the backward direction [V \cdot m\(^{-1}\)]
\(\mathbf{E}_{ext}\) external electric field [V \cdot m\(^{-1}\)]
\(\mathbf{E}_f\) electric field of a wave scattered in the forward direction [V \cdot m\(^{-1}\)]
\(\mathbf{E}_i\) electric field of the incident wave [V \cdot m\(^{-1}\)]
\(\mathbf{E}_{in}\) electric field inside volume \(V\) [V \cdot m\(^{-1}\)]
\(\mathbf{E}_{out}\) electric field outside volume \(V\) [V \cdot m\(^{-1}\)]
\(\mathbf{E}_r\) electric field of the reflected wave [V \cdot m\(^{-1}\)]
\(\mathbf{E}^\pm_t\) reflected electric field for incidence along the \(\pm n\) directions [V \cdot m\(^{-1}\)]
\(\mathbf{E}_{sc}\) scattered electric field [V \cdot m\(^{-1}\)]
\(\mathbf{E}_t\) electric field of the transmitted wave [V \cdot m\(^{-1}\)]
List of Symbols

\( E^\pm \) transmitted electric field for incidence along the ±n directions \([V \cdot m^{-1}]\)

\( E_{\text{tan}} \) total tangential electric field at the metasurface \([V \cdot m^{-1}]\)

\( F \) force \([N]\)

\( H \) magnetic field \([A \cdot m^{-1}]\)

\( H_0 \) external magnetic bias field \([A \cdot m^{-1}]\)

\( H_{\text{av}} \) magnetic field averaged over the two sides of a metasurface \([A \cdot m^{-1}]\)

\( H_i \) magnetic field of the incident wave \([A \cdot m^{-1}]\)

\( H_r \) magnetic field of the reflected wave \([A \cdot m^{-1}]\)

\( H_{\text{sc}} \) scattered magnetic field \([A \cdot m^{-1}]\)

\( H_{\text{tan}} \) total tangential magnetic field at the metasurface \([A \cdot m^{-1}]\)

\( I \) electric current \([A]\)

\( \overline{I} \) unit matrix (dyadic)

\( \overline{I}_t \) transverse unit dyadic

\( \overline{J} \) antisymmetric matrix

\( J_e \) averaged electric surface current density \([A \cdot m^{-1}]\)

\( J_{eV} \) electric current density inside volume \( V \) \([A \cdot m^{-2}]\)

\( J_{\text{ext}} \) averaged external (bound) electric current density \([A \cdot m^{-2}]\)

\( J_{\text{ind}} \) averaged induced (due to polarization) electric current density \([A \cdot m^{-2}]\)

\( J_m \) averaged magnetic surface current density \([V \cdot m^{-1}]\)

\( J_{mV} \) magnetic current density inside volume \( V \) \([V \cdot m^{-2}]\)

\( \overline{J}_t \) vector-product dyadic

\( j \) imaginary unit

\( \overline{K} \) transformation matrix

\( K_a \) vector dual to the antisymmetric part of the chirality tensor

\( K_\omega \) omega coupling coefficient

\( k \) wave vector \([m^{-1}]\)

\( k_0 \) wavenumber in vacuum \([m^{-1}]\)

\( k_b \) wave vector of a wave scattered in the backward direction \([m^{-1}]\)

\( k_f \) wave vector of a wave scattered in the forward direction \([m^{-1}]\)

\( k_i \) wavenumber of the incident wave \([m^{-1}]\)

\( k_p \) plasma wavenumber \([m^{-1}]\)

\( k_r \) wavenumber of the reflected wave \([m^{-1}]\)

\( k_t \) wavenumber of the transmitted wave \([m^{-1}]\)

\( M \) magnetization \([A \cdot m^{-1}]\)

\( \overline{M} \) matrix with zero trace

\( M_S \) magnetic polarization surface density \([A]\)

\( m \) magnetic dipole moment in notations used in the dissertation \([V \cdot s \cdot m]\)

\( m \) arbitrary integer number
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{\text{conv}}$</td>
<td>magnetic dipole moment in conventional notations [$A \cdot m^2$]</td>
</tr>
<tr>
<td>$N$</td>
<td>volume concentration of inclusions in material [$m^{-3}$]</td>
</tr>
<tr>
<td>$\overrightarrow{N}$</td>
<td>symmetric matrix</td>
</tr>
<tr>
<td>$n$</td>
<td>unit vector defining the normal to a surface</td>
</tr>
<tr>
<td>$n$</td>
<td>refractive index</td>
</tr>
<tr>
<td>$n_{\text{sc}}$</td>
<td>unit vector in the direction of observation of the scattered field</td>
</tr>
<tr>
<td>$P$</td>
<td>electric polarization density [$A \cdot s \cdot m^{-2}$]</td>
</tr>
<tr>
<td>$P_S$</td>
<td>electric polarization surface density [$A \cdot s \cdot m^{-1}$]</td>
</tr>
<tr>
<td>$p$</td>
<td>electric dipole moment [$A \cdot s \cdot m$]</td>
</tr>
<tr>
<td>$p^\pm$</td>
<td>electric dipole moment induced under excitation by incidence along the $\pm z$ directions [$A \cdot s \cdot m$]</td>
</tr>
<tr>
<td>$Q$</td>
<td>arbitrary differentiable vector quantity [$A \cdot s \cdot m^{-1}$]</td>
</tr>
<tr>
<td>$q$</td>
<td>electric charge [$C$]</td>
</tr>
<tr>
<td>$R_{\pm z}$</td>
<td>reflection coefficient for waves propagating in the $\pm z$-directions</td>
</tr>
<tr>
<td>$r$</td>
<td>position vector [$m$]</td>
</tr>
<tr>
<td>$r_0$</td>
<td>position vector of the central point [$m$]</td>
</tr>
<tr>
<td>$r_1$</td>
<td>local reflection coefficient</td>
</tr>
<tr>
<td>$r_{\text{sc}}$</td>
<td>distance between the scatterer and the observation point [$m$]</td>
</tr>
<tr>
<td>$r_w$</td>
<td>wire radius [$m$]</td>
</tr>
<tr>
<td>$S$</td>
<td>Poynting vector [$W \cdot m^{-2}$]</td>
</tr>
<tr>
<td>$S$</td>
<td>surface area [$m^2$]</td>
</tr>
<tr>
<td>$T$</td>
<td>transpose operator</td>
</tr>
<tr>
<td>$t$</td>
<td>time [$s$]</td>
</tr>
<tr>
<td>$V$</td>
<td>volume [$m^3$]</td>
</tr>
<tr>
<td>$V_a$</td>
<td>vector dual to the antisymmetric part of the non-reciprocity coupling tensor</td>
</tr>
<tr>
<td>$V_a$</td>
<td>artificial moving coupling coefficient</td>
</tr>
<tr>
<td>$v$</td>
<td>velocity [$m \cdot s^{-1}$]</td>
</tr>
<tr>
<td>$v_{\text{gr}}$</td>
<td>group velocity [$m \cdot s^{-1}$]</td>
</tr>
<tr>
<td>$v_{\text{ph}}$</td>
<td>phase velocity [$m \cdot s^{-1}$]</td>
</tr>
<tr>
<td>$X_{ij}$</td>
<td>imaginary part of $Z$-parameters [$\Omega$]</td>
</tr>
<tr>
<td>$Z_i$</td>
<td>impedance of the incident wave [$\Omega$]</td>
</tr>
<tr>
<td>$Z_{ij}$</td>
<td>$Z$-parameters [$\Omega$]</td>
</tr>
<tr>
<td>$Z_r$</td>
<td>impedance of the reflected wave [$\Omega$]</td>
</tr>
<tr>
<td>$Z_s$</td>
<td>surface impedance [$\Omega$]</td>
</tr>
<tr>
<td>$\hat{\alpha}_{\text{ee}}$</td>
<td>tensor of electric polarizability [$s \cdot m^2 \cdot \Omega^{-1}$]</td>
</tr>
<tr>
<td>$\overrightarrow{\hat{\alpha}_{\text{ee}}}$</td>
<td>collective electric polarizability dyadic [$s \cdot m^2 \cdot \Omega^{-1}$]</td>
</tr>
<tr>
<td>$\hat{\alpha}_{\text{ee}}$</td>
<td>co-component of collective electric polarizability dyadic [$s \cdot m^2 \cdot \Omega^{-1}$]</td>
</tr>
</tbody>
</table>
\( \hat{\alpha}_{ee}^{cr} \) cross-component of collective electric polarizability
dyadic \([s \cdot m^2 \cdot \Omega^{-1}]\)

\( \bar{\alpha}_{ee}^V \) tensor of electric polarizability with dimensions of volume \([m^3]\)

\( \bar{\alpha}_{em} \) tensor of electromagnetic polarizability \([s \cdot m^2]\)

\( \bar{\alpha}_{em} \) collective electromagnetic polarizability dyadic \([s \cdot m^2]\)

\( \hat{\alpha}_{em}^{co} \) co-component of collective electromagnetic polarizability dyadic \([s \cdot m^2]\)

\( \hat{\alpha}_{em}^{cr} \) cross-component of collective electromagnetic polarizability
dyadic \([s \cdot m^2]\)

\( \bar{\alpha}_{em}^V \) tensor of magnetoelectric polarizability with dimensions of volume \([m^3]\)

\( \bar{\alpha}_{me} \) tensor of magnetoelectric polarizability \([s \cdot m^2]\)

\( \hat{\alpha}_{me}^{co} \) co-component of collective magnetoelectric polarizability dyadic \([s \cdot m^2]\)

\( \hat{\alpha}_{me}^{cr} \) cross-component of collective magnetoelectric polarizability
dyadic \([s \cdot m^2]\)

\( \bar{\alpha}_{mm} \) tensor of magnetic polarizability \([s \cdot m^2 \cdot \Omega]\)

\( \bar{\alpha}_{mm} \) collective magnetic polarizability dyadic \([s \cdot m^2 \cdot \Omega]\)

\( \hat{\alpha}_{mm}^{co} \) co-component of collective magnetic polarizability dyadic \([s \cdot m^2 \cdot \Omega]\)

\( \hat{\alpha}_{mm}^{cr} \) cross-component of collective magnetic polarizability dyadic \([s \cdot m^2 \cdot \Omega]\)

\( \bar{\alpha}_{mm}^V \) tensor of magnetic polarizability with dimensions of volume \([m^3]\)

\( \beta \) scalar parameter modelling electric quadrupole polarization \([F \cdot m]\)

\( \beta_{+z} \) propagation constant of a wave carrying power towards
the \(+z\) direction \([m^{-1}]\)

\( \beta_{-z} \) propagation constant of a wave carrying power towards
the \(-z\) direction \([m^{-1}]\)

\( \gamma \) scalar parameter modelling artificial magnetic polarization \([F \cdot m]\)

\( \Delta_m \) auxiliary dyadic \([m^2]\)

\( \Delta_p \) auxiliary dyadic \([m^2]\)

\( \delta \) Dirac delta function

\( \delta_{ij} \) Kronecker delta

\( \epsilon \) permittivity \([F \cdot m^{-1}]\)

\( \bar{\epsilon} \) permittivity tensor \([F \cdot m^{-1}]\)

\( \epsilon_0 \) vacuum permittivity \([F \cdot m^{-1}]\)

\( \bar{\epsilon}_g \) permittivity tensor of a general linear bi-anisotropic medium \([F \cdot m^{-1}]\)

\( \epsilon_h \) permittivity of host medium \([F \cdot m^{-1}]\)

\( \bar{\epsilon}_n \) permittivity tensor of a non-reciprocal medium \([F \cdot m^{-1}]\)

\( \bar{\epsilon}_{or} \) permittivity tensor of material in the original coordinate system \([F \cdot m^{-1}]\)

\( \epsilon_s \) permittivity of spatially dispersive medium \([F \cdot m^{-1}]\)

\( \bar{\epsilon}_s \) permittivity tensor of spatially dispersive medium \([F \cdot m^{-1}]\)

\( \epsilon_t \) transverse permittivity of uniaxial bi-anisotropic medium \([F \cdot m^{-1}]\)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\epsilon}_{tr}$</td>
<td>permittivity tensor of material in the transformed coordinate system [$F \cdot m^{-1}$]</td>
</tr>
<tr>
<td>$\epsilon_{zz}$</td>
<td>permittivity along the z-direction [$F \cdot m^{-1}$]</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>efficiency</td>
</tr>
<tr>
<td>$\bar{\zeta}$</td>
<td>general bi-anisotropy tensor modeling magnetoelastic coupling</td>
</tr>
<tr>
<td>$\eta$</td>
<td>wave impedance [$\Omega$]</td>
</tr>
<tr>
<td>$\eta_0$</td>
<td>wave impedance in vacuum [$\Omega$]</td>
</tr>
<tr>
<td>$\eta_{\pm z}$</td>
<td>wave impedance for waves propagating in the $\pm z$-direction [$\Omega$]</td>
</tr>
<tr>
<td>$\theta_i$</td>
<td>incidence angle [rad]</td>
</tr>
<tr>
<td>$\theta_r$</td>
<td>reflection angle [rad]</td>
</tr>
<tr>
<td>$\theta_t$</td>
<td>transmission (refraction) angle [rad]</td>
</tr>
<tr>
<td>$\bar{\nu}$</td>
<td>general bi-anisotropy tensor modeling electromagnetic coupling</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>chirality parameter</td>
</tr>
<tr>
<td>$\bar{\kappa}$</td>
<td>chirality tensor</td>
</tr>
<tr>
<td>$\bar{\kappa}_s$</td>
<td>symmetric part of the chirality tensor</td>
</tr>
<tr>
<td>$\kappa_t$</td>
<td>symmetric part of uniaxial reciprocal bi-anisotropic dyadic</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>wavelength [m]</td>
</tr>
<tr>
<td>$\lambda_0$</td>
<td>wavelength in vacuum [m]</td>
</tr>
<tr>
<td>$\mu$</td>
<td>permeability [$N \cdot A^{-2}$]</td>
</tr>
<tr>
<td>$\bar{\mu}$</td>
<td>permeability tensor [$N \cdot A^{-2}$]</td>
</tr>
<tr>
<td>$\mu_0$</td>
<td>vacuum permeability [$N \cdot A^{-2}$]</td>
</tr>
<tr>
<td>$\bar{\mu}_g$</td>
<td>permeability tensor of a general linear bi-anisotropic medium [$N \cdot A^{-2}$]</td>
</tr>
<tr>
<td>$\bar{\mu}_n$</td>
<td>permeability tensor of a non-reciprocal medium [$N \cdot A^{-2}$]</td>
</tr>
<tr>
<td>$\bar{\mu}_{or}$</td>
<td>permeability tensor of material in the original coordinate system [$N \cdot A^{-2}$]</td>
</tr>
<tr>
<td>$\mu_t$</td>
<td>transverse permeability of a uniaxial bi-anisotropic medium [$N \cdot A^{-2}$]</td>
</tr>
<tr>
<td>$\bar{\mu}_{tr}$</td>
<td>permeability tensor of material in the transformed coordinate system [$N \cdot A^{-2}$]</td>
</tr>
<tr>
<td>$\xi$</td>
<td>reciprocal bi-anisotropy parameter [$\Omega^{-1}$]</td>
</tr>
<tr>
<td>$\bar{\xi}$</td>
<td>reciprocal bi-anisotropy tensor [$\Omega^{-1}$]</td>
</tr>
<tr>
<td>$\rho_{ext}$</td>
<td>averaged external (bound) electric charge density [$C \cdot m^{-3}$]</td>
</tr>
<tr>
<td>$\rho_{ind}$</td>
<td>averaged induced (due to polarization) electric charge density [$C \cdot m^{-3}$]</td>
</tr>
<tr>
<td>$\tau$</td>
<td>time [s]</td>
</tr>
<tr>
<td>$\Phi_r$</td>
<td>local phase of the reflected wave</td>
</tr>
<tr>
<td>$\Phi_t$</td>
<td>local phase of the transmitted wave</td>
</tr>
<tr>
<td>$\bar{\phi}_{ee}$</td>
<td>electric surface susceptibility tensor [m]</td>
</tr>
<tr>
<td>$\bar{\phi}_{em}$</td>
<td>electromagnetic surface susceptibility tensor [m]</td>
</tr>
<tr>
<td>$\bar{\phi}_{me}$</td>
<td>magnetoelectric surface susceptibility tensor [m]</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$\varphi_{mm}$</td>
<td>magnetic surface susceptibility tensor [m]</td>
</tr>
<tr>
<td>$\chi$</td>
<td>electric susceptibility</td>
</tr>
<tr>
<td>$\chi$</td>
<td>non-reciprocal bi-anisotropic tensor</td>
</tr>
<tr>
<td>$\chi_s$</td>
<td>symmetric part of the non-reciprocity coupling tensor</td>
</tr>
<tr>
<td>$\chi_t$</td>
<td>symmetric part of uniaxial non-reciprocal bi-anisotropic dyadic</td>
</tr>
<tr>
<td>$\psi$</td>
<td>arbitrary material tensor such as permittivity or chirality</td>
</tr>
<tr>
<td>$\omega$</td>
<td>angular frequency [rad s$^{-1}$]</td>
</tr>
<tr>
<td>$\omega_a$</td>
<td>angular frequency of full absorption regime [rad s$^{-1}$]</td>
</tr>
<tr>
<td>$\omega_e$</td>
<td>angular frequency of electric resonance [rad s$^{-1}$]</td>
</tr>
<tr>
<td>$\omega_m$</td>
<td>angular frequency of magnetic resonance [rad s$^{-1}$]</td>
</tr>
<tr>
<td>$*$</td>
<td>complex conjugate operator</td>
</tr>
<tr>
<td>$d$</td>
<td>differential of a function</td>
</tr>
<tr>
<td>$\frac{\partial}{\partial t}$</td>
<td>time derivative</td>
</tr>
<tr>
<td>$\dagger$</td>
<td>Hermitian conjugate operator</td>
</tr>
<tr>
<td>$\det{}$</td>
<td>determinant of a matrix</td>
</tr>
<tr>
<td>$\Im{}$</td>
<td>imaginary part operator</td>
</tr>
<tr>
<td>$\nabla$</td>
<td>nabla operator</td>
</tr>
<tr>
<td>$\nabla_j$</td>
<td>spatial derivative with respect to coordinate $r_j$</td>
</tr>
<tr>
<td>$\Re{}$</td>
<td>real part operator</td>
</tr>
<tr>
<td>$\text{tr}{}$</td>
<td>trace of a matrix</td>
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1. Introduction

Understanding of interaction of electromagnetic waves with matter is an important subject of physics. On the one hand, it carries out a cognitive function by explaining why the existing materials behave in the way they do. For example, why we see different colors and whether color is an intrinsic attribute of an object. On the other hand, wave-matter interaction performs an important applied function, devising recipes for engineering materials with desired electromagnetic properties: Holograms, lenses, surfaces reflecting more light than conventional mirrors, etc.

Interaction of materials with electromagnetic waves could be described, in a simple way, by wave scattering from atoms and molecules of these materials. Due to the inertial properties of electric charges, scattering response is different at different frequencies. Effect of retarded response to electromagnetic excitation is called frequency dispersion. The strongest scattering occurs when the frequency of external waves corresponds to the characteristic frequency of the material at which its constituents resonate. Characteristic times of electron transitions in atoms and molecules of most natural materials are comparable with the inverse frequency of visible light. Therefore, these materials have resonant interactions with light and we can see different colors. Since wave scattering from materials depends not only on their properties but also on the properties of the incident light, colors that we see are not intrinsic attributes of materials and vary under different illuminations. A good example is alexandrite stone which exhibits different colors for different illuminations. In the microwave region, resonant wave-matter interaction effects are predominantly due to orientational polarizations of polar molecules.

Although there is a variety of known natural materials with various electromagnetic properties, their response is still very bounded by nature, which does not allow us to uncover all potential opportunities for applica-
A naive solution for this problem could be found if we would be able to accurately engineer properties of atoms and molecules of bulk materials, i.e., their size, spatial distribution in the lattice, content as well as the electron cloud distribution and shape. Although this solution is not realistic, it gives us an idea of macro-engineering of matter constituents. Indeed, one could engineer macroscopic “atoms” whose sizes are big enough to be easily fabricated and adjusted, and at the same time, small enough compared to the wavelength of incident radiation. In such scenario, a three-dimensional lattice (with the periodicity smaller than the wavelength) of these “macro-atoms” can be homogenized and described as an ordinary material with microscopic constituents. The “macro-atoms” can be made of accessible materials with arbitrary shape and content, e.g. dielectric spheres of a specific radius. The lattice can be embedded in a host medium, e.g. dielectric. By adjusting the attributes of the “macro-atoms”, one could tailor their resonance frequency to increase interaction with incident radiation. This idea of engineering composites with sub-wavelength macroscopic inclusions underlies the concept of metamaterials. Metamaterials, actively studied from years 2000–2001 [1, 2], have yielded numerous applications and expanded our understanding of electromagnetism [3, 4]. The constituents of metamaterials obtained a new term “meta-atoms”.

The concept of metamaterials is strongly associated with the concept of spatial dispersion. When the size of scatterers or the distances between them become comparable to the wavelength, the composite constructed out of them possesses non-local polarization response and generally cannot be described in terms of the permittivity and permeability quantities solely. A demonstrative example of spatial dispersion effect is the response of a loop metal wire illuminated by a plane wave. If the wavelength of the incident wave is much greater than the loop radius, the electric field around the symmetric loop can be considered uniform and, therefore, it does not induce circulating currents in the wire (no magnetic polarization occurs). However, if the incident wavelength is equal to the double loop diameter, the electric fields at the opposite sides of the loop oscillate out of phase generating circulating currents in the wire, and the loop is strongly magnetically polarized (see Fig. 1.1). Thus, when spatial dispersion effects in material cannot be neglected, its response depends on the amplitude and orientation of the wave vector of incident waves (see Chapter 3 for more details). Spatial dispersion has played the crucial
Introduction

role in the field of metamaterials and enabled new exciting effects such as artificial magnetism with dielectrics and metals, gyrotropy, optical activity, and bi-anisotropy. They, in turn, have opened a way for realization of negative-index materials [1, 2], invisibility cloak [5], bi-anisotropic nihility [6, 7], and giant optical activity [8].

Very recently, two-dimensional counterparts of metamaterials, so-called metasurfaces, have been studied intensively. They represent two-dimensional arrangements of inclusions supported by a host layer. Although spatial dispersion in the form of artificial magnetism was widely exploited in known metasurfaces, a very limited number of works explored the potential of bi-anisotropy (first-order weak spatial dispersion, see Section 3.2) in metasurface design. This topic is the main subject of the present dissertation. The exposition of the content goes in the following order. Chapters 2 and 3 introduce to the reader the classical theory of frequency (temporal) and spatial dispersions [9–11]. Classification of bi-anisotropy (as a form of spatial dispersion) and constraints on material parameters are discussed. Chapter 3 is concluded with a review of recent studies on materials with extremely pronounced bi-anisotropy (so-called bi-anisotropic nihility). Chapter 4 presents a short overview of known metamaterials in which spatial dispersion plays the crucial role.

The last two chapters review the novel results obtained in this dissertation. Chapter 5 compares conventional techniques for polarizability extraction of single meta-atoms (an essential procedure for metasurface design) and outlines an alternative technique based on the results of Publication I. The novel technique, in contrast to previous approaches, is applicable for a general linear bi-anisotropic dipolar scatterer. Moreover, in Chapter 5 and Publication II electromagnetic constraints on proper-
ties of a general bi-anisotropic scatterer are examined. It is shown that while a lossless passive scatterer with pure bi-anisotropic response (direct electric and magnetic polarization effects are completely suppressed) cannot exist, it is not forbidden by nature to engineer a scatterer whose bi-anisotropic effects are dominant. The design of such a scatterer with extreme spatial dispersion of the first order is briefly outlined. Chapter 6 reviews the key ideas behind two synthesized bi-anisotropic metasurfaces: Frequency-selective gradient reflector (Publications III and IV) and broadband reflectionless absorber (Publication V). Spatial dispersion in both metasurfaces is essential to achieve the shadow-free operation, enabling cascades of several metasurface devices. Chapter 6 also briefly describes an exact synthesis of gradient metasurfaces for perfect wavefront control in reflection and transmission regimes. The results of Publication VI demonstrate fundamental importance of spatial dispersion in such metasurfaces. As a proof of concept, an optical metasurface for perfect anomalous reflection is designed and measured in Publication VII. Conclusions summarize the results obtained in the dissertation.
Part I

Research field
2. Frequency dispersion

Inertia is a fundamental attribute of any physical object expressed as its resistance to any changes of speed or direction of its motion. Inertial mass is the coefficient which is proportional to the strength of this resistance. An object with greater inertial mass requires greater force to change its speed than an object with smaller mass. Can an inertialess object exist? This would imply that its inertial mass consisting of the rest mass (invariant) and relativistic mass (increasing with velocity) is zero. Moreover, applying an external push to such an object, its speed would exceed the speed of light during a short period of time, which contradicts to the absolute limit on energy propagation velocity dictated by the theory of relativity.

Inertial properties of electrical charges in atoms and molecules greatly affect electromagnetic properties of all materials. Indeed, due to inertia, polarization (charge displacement) in materials occurs always with some delay with respect to an applied external electric field. Likewise, when the applied field vanishes, the induced polarizations for some time continue to oscillate. Thus, the coupling between the electric displacement field $D$ in a material and the external field $E$ is not instantaneous.

Consider a demonstrative example of impulse impact of an external electric field on a medium. The external field $E$ occurs at time $t = 0$, acts on the medium during time $dt$, and then instantly disappears. Here, it is assumed that the electric field is spatially uniform (the wavelength is much greater than the size of the medium constituents and the lattice periodicity), and the medium is linear and non-magnetic. The polarization increase of the medium $dP$ is proportional to $Edt$ and at time moment $t$ can be written as

$$dP(t) = \epsilon_0 \chi(t) E(0) dt,$$  

where function $\chi(t)$ is called susceptibility and depends on the medium
properties and time \( t \) passed from the beginning of the impulse until the observation moment, \( \epsilon_0 \) is the vacuum permittivity. This function should turn into zero at \( t = 0 \), due to inertial properties of electric charges and at \( t = \infty \), due to inevitable dissipation loss in all materials.

If the external field is applied during a long time period, it is convenient to split its impact into small time periods \( dt \), which is equivalent to a consequent of the impulse impacts considered above. Analogously to (2.1), the polarization increase at time \( t \) due to a previous impulse \( E(t')dt' \) can be expressed as \( dP(t) = \epsilon_0 \chi(t - t')E(t')dt' \). The total polarization \( P(t) \) at moment \( t \) is the superposition (sum) of the elementary polarization portions \( dP \) until this moment:

\[
P(t) = \epsilon_0 \int_{-\infty}^{t} \chi(t - t')E(t')dt'.
\] (2.2)

Here the integration starts from \( t' = -\infty \) to take into account the total impact from the external field that could generally start at any time in the past \( t' \). Importantly, the upper limit of integration is the present time moment \( t \) when the polarization is measured. In other words, the polarization induced in the material by moment \( t \) is determined only by earlier moments \( t' < t \) and does not depend on the actions from the future. Thus, electromagnetic polarization of materials follows such fundamental empirical law as causality principle [12]. Replacing \( t' \) with \( t - \tau \), Eq. (2.2) reads

\[
P(t) = \epsilon_0 \int_{0}^{\infty} \chi(\tau)E(t - \tau)d\tau,
\] (2.3)

and the electric displacement in the material becomes

\[
D(t) = \epsilon_0 E(t) + P(t) = \epsilon_0 \int_{0}^{\infty} [\delta(\tau) + \chi(\tau)]E(t - \tau)d\tau
\]
\[
= \int_{0}^{\infty} \epsilon(\tau)E(t - \tau)d\tau,
\] (2.4)

where \( \epsilon(\tau) = \epsilon_0[\delta(\tau) + \chi(\tau)] \) and \( \delta(\tau) \) is the Dirac delta function. Fourier transform of both sides of the equation

\[
\int_{-\infty}^{\infty} D(t)e^{-j\omega t}dt = \int_{0}^{\infty} \epsilon(\tau)d\tau \int_{-\infty}^{\infty} E(t - \tau)e^{-j\omega t}dt
\] (2.5)

gives a simple formula:

\[
D(\omega) = \epsilon(\omega)E(\omega),
\] (2.6)
where $\omega$ is the angular frequency and

$$
\epsilon(\omega) = \int_0^\infty \epsilon(\tau)e^{-j\omega\tau}d\tau.
$$

(2.7)

Thus, permittivities of all materials are functions of the frequency, i.e. they have frequency dispersion. It is completely determined by the susceptibility function of the material. Depending on the internal structure of the material, the permittivity can be described by various canonical models, such as the Debye and Lorentz models.

Similar derivations could be written also for magnetic response as well as bi-anisotropic coupling (described below), since all these effects obey causality. Frequency dispersion is a fundamental property of all materials, resulting from the inertial properties of their constituents. A hypothetical material without frequency dispersion should consist of electric charges of zero mass, however, any electrically charged particle has so-called non-zero electromagnetic mass (in addition to the gravitational mass) [13].

Due to limited and discrete variation of natural substances, in practice, one cannot always find proper materials with desired permittivity and permeability at specified frequency. For example, there are no materials with strong magnetic response in the optical range. Therefore, very often engineering spatial dispersion in artificial materials is the only solution for extending our limits for wave-matter interaction.
3. Spatial dispersion

3.1 Macroscopic Maxwell’s equations

Entire diversity of classical electromagnetic phenomena is described by Maxwell’s equations. Wave interaction in material media implies an enormously large number of charged particles and, therefore, its characterization requires macroscopic field equations. The macroscopic version of Maxwell’s equations is derived from the microscopic equations, assuming that all quantities are averaged over an electrically small volume. In the basic form the macroscopic Maxwell equations read

\[
\nabla \times E = -\frac{1}{\mu_0} \frac{\partial B}{\partial t}, \quad \frac{1}{\epsilon_0} \nabla \times B = \epsilon_0 \frac{\partial E}{\partial t} + J_{\text{ind}} + J_{\text{ext}},
\]

\[
\epsilon_0 \nabla \cdot E = \rho_{\text{ind}} + \rho_{\text{ext}}, \quad \nabla \cdot B = 0,
\]

(3.1)

where \(\rho_{\text{ind}}\) and \(J_{\text{ind}}\) are the induced due to polarization averaged electric charge and current densities, while \(\rho_{\text{ext}}\) and \(J_{\text{ext}}\) are the corresponding quantities describing external charges and currents (those which are not affected by the fields). Here, the equations are written in terms of directly measurable quantities: The electric field \(E\) and magnetic displacement vector \(B\). The conventional separation of electric charges to induced and external parts allows one to elegantly include all microscopic polarization effects in the material into macroscopic functions of electric \(P\) and magnetic \(M\) polarization densities.

Let us consider induced and external charges inside a material separately. In the absence of external charges (electrically neutral material), its total electric charge is zero, meaning

\[
\int_V \rho_{\text{ind}} dV = 0,
\]

(3.2)

where \(V\) is the material volume. Such an integral expression holds for materials with arbitrary shape, which is possible if the averaged induced
Spatial dispersion

charge density \( \rho_{\text{ind}} \) can be expressed as the divergence of a vector:

\[
\rho_{\text{ind}} = -\nabla \cdot \mathbf{P}, \tag{3.3}
\]

where \( \mathbf{P} = 0 \) outside the material. This vector is generally designated as \( \mathbf{P} \) and called the polarization vector of the material. Indeed, the electric charge density in the form (3.3) integrated over a volume \( V \) enclosing the material (and not touching it) always satisfies (3.2). The physical meaning of vector \( \mathbf{P} \) is the volume density of dipole moments induced in the material excited by external fields [14].

The conservation of induced electric charge implies that

\[
\frac{\partial \rho_{\text{ind}}}{\partial t} + \nabla \cdot \mathbf{J}_{\text{ind}} = \nabla \cdot \left( \mathbf{J}_{\text{ind}} - \frac{\partial \mathbf{P}}{\partial t} \right) = 0. \tag{3.4}
\]

The last equation is satisfied if the expression in the brackets equals to curl of some function:

\[
\mathbf{J}_{\text{ind}} - \frac{\partial \mathbf{P}}{\partial t} = \nabla \times \mathbf{M}. \tag{3.5}
\]

This function has the meaning of the magnetic polarization density in the material. Using (3.3) and (3.5), macroscopic equations (3.1) can be written as

\[
\begin{align*}
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\
\nabla \times \left( \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) &= \frac{\partial}{\partial t} \left( \epsilon_0 \mathbf{E} + \mathbf{P} \right) + \mathbf{J}_\text{ext}, \\
\n\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) &= \rho_{\text{ext}}, \\
\n\nabla \cdot \mathbf{B} &= 0.
\end{align*} \tag{3.6}
\]

Finally, using conventional notations

\[
\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}), \tag{3.7}
\]

the macroscopic Maxwell equations in a medium are given by

\[
\begin{align*}
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\
\nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_\text{ext}, \\
\n\nabla \cdot \mathbf{D} &= \rho_{\text{ext}}, \\
\n\nabla \cdot \mathbf{B} &= 0. \tag{3.8}
\end{align*}
\]

As it was explained in [15], splitting of induced current to the electric and magnetic parts dictated by (3.5) is not unique. Indeed, redefining the polarization densities as \( \mathbf{P}' = \mathbf{P} + \nabla \times \mathbf{Q} \) and \( \mathbf{M}' = \mathbf{M} - \frac{\partial \mathbf{Q}}{\partial t} \), they still exactly satisfy (3.3) and (3.5). Here, \( \mathbf{Q} \) is an arbitrary differentiable vector quantity. According to (3.6), the Maxwell equations are invariant with respect to the following transformation of macroscopic field quantities \( \mathbf{D} \) and \( \mathbf{H} \):

\[
\begin{align*}
\mathbf{D}' &= \mathbf{D} + \nabla \times \mathbf{Q}, \\
\mathbf{H}' &= \mathbf{H} + \frac{\partial \mathbf{Q}}{\partial t}.
\end{align*} \tag{3.9}
\]

Thus, the macroscopic field quantities \( \mathbf{D} \) and \( \mathbf{H} \) in (3.7) are not uniquely defined. However, this conclusion does not imply that Maxwell’s equations
applied for a particular medium do not have a unique solution. Indeed, to find the field solution inside a medium, one should also consider electromagnetic boundary conditions at the medium interfaces. These conditions depend on the choice of relations (3.7). Therefore, if one redefines vectors \( \mathbf{D} \) and \( \mathbf{H} \) according to (3.9), the boundary conditions will be also changed, but the solution for fields \( \mathbf{E} \) and \( \mathbf{B} \) will remain the same for each point of the medium.

### 3.2 First-order spatial dispersion and bi-anisotropy

Next, let us consider reciprocal materials, i.e. materials whose properties do not change under time-inversion transformation. Examples of such materials are dielectrics and metals. Non-reciprocal materials such as ferromagnetics and magnetized plasma will be considered in Section 3.7.

Spatial dispersion effects occur when the wavelength of electromagnetic radiation in a material is comparable with the size of its constituents or distances between them. In this scenario, to find the induced dipole moment in the constituent, it is not enough to know the local electric field at one point. Information about the electric field in the entire volume occupied by the constituent (inclusion) is required. Equivalently, it is enough to know the electric field vector and all its spatial derivatives at one point of the inclusion (e.g., its geometrical center at \( r_0 \)). The electric field in other points with coordinates \( r \) can be expressed via a Taylor series at the central point:

\[
E(r + r_0) = E(r_0) + \nabla_j E(r_0) r_j + \frac{1}{2} \nabla_j \nabla_k E(r_0) r_j r_k + \ldots \tag{3.10}
\]

where \( \nabla_j \) defines spatial derivative with respect to \( r_j \). The repeating indices imply summation according to the conventional tensor notations. If the local field around the inclusion can be approximated as that of a plane wave, it can be shown that the second, third and subsequent terms in (3.10) are of the order of \( (a/\lambda)E(r_0) \), \( (a/\lambda)^2 E(r_0) \), \ldots, respectively (\( a \) is the size of the inclusion and \( \lambda \) is the wavelength in the material).

The coupling between material polarization \( \mathbf{P} \) and external electric field \( \mathbf{E} \), as well as between \( \mathbf{D} \) and \( \mathbf{E} \), is in general non-local. Under monochromatic excitation, the electric displacement field \( D_i \) and magnetic field \( H_i \) in a spatially dispersive anisotropic medium are given by [9–11, 16]

\[
D_i = (\varepsilon_0 \delta_{ij} + a_{ij}) E_j + a_{ijk}(\nabla_k E_j) + a_{ijkl}(\nabla_l \nabla_k E_j) + \ldots, \quad H_i = \frac{1}{\mu_0} B_i, \tag{3.11}
\]
where $\delta_{ij}$ is the Kronecker delta, $a_{ij}$, $a_{ijk}$, and $a_{ijkl}$ are the susceptibility tensors.

It is easy to see that in the case when the material inclusions are electrically very small ($a \ll \lambda$), all terms with spatial derivatives in (3.10) can be neglected and equations (3.11) degenerate to the well-known constitutive relations in usual locally described dielectric:

$$D_i = (\epsilon_0 \delta_{ij} + a_{ij}) E_j, \quad H_i = \frac{1}{\mu_0} B_i.$$  \hspace{1cm} (3.12)

Here $a_{ij}$ represents susceptibility tensor which is commonly denoted as $\chi_{ij}$ (note that in the following $\phi_{ij}$ notations are used for the susceptibility tensor).

In the approximation of the first-order spatial dispersion (i.e. when the first two terms in (3.10) cannot be neglected) relations (3.11) can be simplified using the invariance of Maxwell's equations under transformation (3.9) to the so-called Post relations [10, 17, 18]:

$$D = \bar{\varepsilon} \cdot E - j \bar{\xi} \cdot B, \quad H = \frac{1}{\mu_0} B - j \bar{\xi}^T \cdot E.$$  \hspace{1cm} (3.13)

where $\bar{\varepsilon}$ and $\bar{\xi}$ are the tensor of permittivity and reciprocal bi-anisotropic tensor, respectively. Here time-harmonic field excitation in the form $e^{j\omega t}$ is assumed. Spatially dispersive materials which are modelled by constitutive relations (3.13) are called reciprocal bi-anisotropic or chiral materials. At the first glance it appears from these relations that the polarization effects in bi-anisotropic media are local: The field vectors are all connected through the expressions at the same point in space. Moreover, the relations could be interpreted in a way that electric polarization in a bi-anisotropic material is induced by both external electric and magnetic fields (“bi-” or double polarization). In fact, electric polarization is not caused by magnetic field $B$ directly, but instead by circulating electric field $E$ as it is seen from (3.11). Thus, the polarization effects in reciprocal bi-anisotropic materials are non-local, and such materials exhibit first-order spatial dispersion.

A classical example of a bi-anisotropic medium is an artificial composite comprising a three-dimensional arrangement of metal helices inside supporting dielectric material (see Fig. 3.1). Let us assume that the composite sample is illuminated by an incident wave whose wavelength is comparable to the size of each helix. In this case, the incident electric field moves free electrons along the helical inclusion, inducing an effective magnetic moment (circulating current) in the helix. Multiple induced magnetic moments in the inclusions correspond to macroscopic magnetic polarization.
of the composite. On the other hand, the incident electric field also generates an electric dipole moment in the inclusion. This electric dipole moment is caused by two different effects: Ordinary polarization in external electric field (local effect) and additional polarization due to the finite size of the helix and non-uniform circulating electric field in form of $\nabla \times \mathbf{E}$ (non-local effect). As it will be shown in Section 3.6, there are other important media having bi-anisotropic properties.

### 3.3 Second-order spatial dispersion and artificial magnetism

Another important notion is second-order spatial dispersion. Although it is a weaker effect compared to the first-order dispersion, it can become predominant for properly engineered inclusions. For example, a double split-ring resonator [19, 20] exhibits strongly pronounced effect of second-order spatial dispersion. Keeping only the first two terms with derivatives in (3.10) and applying an appropriate transformation (3.9), the following constitutive relations can be obtained in this case [10, 11]:

$$D = \epsilon \mathbf{E} - j \xi \mathbf{B} + \beta \nabla \nabla \cdot \mathbf{E}, \quad \mathbf{H} = \frac{1 - \omega^2 \mu_0 \gamma}{\mu_0} \mathbf{B} - j \xi \mathbf{E},$$  \hspace{1cm} (3.14)

where $\epsilon$, $\xi$, $\beta$, and $\gamma$ are scalar parameters defining the strength of corresponding polarization effect. These relations were derived for an isotropic medium, however, similar results can be obtained for the general case of anisotropic medium. Comparing these relations with (3.13), it is seen that the second-order dispersion yields two additional terms $\beta \nabla \nabla \cdot \mathbf{E}$ and $\frac{1 - \omega^2 \mu_0 \gamma}{\mu_0} \mathbf{B}$ which describe, respectively, polarization in the form of electric quadrupoles and magnetic dipole moments. Thus, effects of electric quadrupole polarization as well as artificial magnetism in reciprocal materials are manifestations of second-order spatial dispersion. It should be
Spatial dispersion

noted that although artificial magnetism is similar to natural magnetism of ferromagnetic materials, they are different phenomena: Non-local reciprocal effect and local non-reciprocal effect.

In the vast majority of studies on second-order spatial dispersion, materials with negligible quadrupole polarization properties are considered for simplicity since in this case the boundary conditions are the same as those for an ordinary anisotropic medium. For isotropic materials without quadrupole polarization properties, the constitutive relations are usually written (after solving the linear system of equations (3.14) with respect to \( D \) and \( B \)) in the following form:

\[
\begin{align*}
D &= \epsilon_s E - j\sqrt{\epsilon_0\mu_0\kappa} H, \\
B &= \mu H + j\sqrt{\epsilon_0\mu_0\kappa} E,
\end{align*}
\]

where \( \epsilon_s = \epsilon + \mu \xi^2 \) (subscript “s” corresponds to spatial dispersion), \( \mu = \mu_0/(1 - \omega^2\mu_0\gamma) \), and \( \kappa = \xi\mu/\sqrt{\epsilon_0\mu_0} \) is the chirality parameter. This form of constitutive relations takes into account both bi-anisotropy and artificial magnetism. The relations model isotropic chiral materials as well as reciprocal magnetic materials. The latter ones are widely studied in the literature [1, 2, 19, 20] due to the absence of natural materials with magnetic properties (\( \mu \neq \mu_0 \)) in the optical range.

Example of materials with pronounced dispersion of second order is a composite formed by double split-ring resonators which are shown in Fig. 3.2. To separately consider different polarization effects of a single inclusion, let us excite it by a standing plane wave. In the first scenario, the antinode of the electric field \( E_{\text{ext}} \) is at the inclusion center [as shown in Fig. 3.2(a)], and the electric currents induced in both rings \( J_1 \) and \( J_2 \) are circulating in the opposite directions. This current distribution corresponds to electric polarization of the inclusion (induced electric dipole

![Figure 3.2](image)

**Figure 3.2.** A double split-ring resonator in an external electric field \( E_{\text{ext}} \) of a standing wave. The resonator is situated (a) at the antinode and (b) at the node of the electric field.
moment \( p \), while the magnetic polarization (first-order dispersion effect) is suppressed. In the second scenario, the inclusion is positioned at the antinode of the magnetic field of a standing wave. As it was discussed above, it is not the magnetic field that moves the electrons, but spatially varying electric field. Figure 3.2(b) depicts the electric field distribution at the inclusion location in this case. Under such illumination, the induced currents in the rings \( J_1 \) and \( J_2 \) circulate in the same direction, resulting in a strong magnetic moment \( m \) (second-order dispersion effect). At the same time, this current distribution provides a suppressed electric dipole (first-order dispersion effect). Thus, a composite of double-split ring resonators can exhibit strong magnetic (diamagnetic \( \mu < \mu_0 \) or paramagnetic \( \mu > \mu_0 \) depending on the frequency) and moderate electric properties, while its bi-anisotropic properties are suppressed. The effect of artificial magnetism due to spatial dispersion should be distinguished from natural ferromagnetism which implies that a material has non-zero magnetization even in the absence of an external magnetic field.

Quadrupole polarization effects can be important in media whose inclusions are strongly excited by a non-uniform harmonic field, while weakly excited by a uniform field (e.g., [21] and Publication II). More discussions on this effect are given in Section 5.2.1.

3.4 Spatial dispersion of higher orders

Spatial dispersion of the first and second orders (without the quadrupole contribution) is commonly referred to as a *weak* dispersion effect. Here the name “weak effect” implies that the constitutive relations for materials with such spatial dispersion do not include field derivatives in explicit form and therefore “appear” to be local (see the discussion in Section 3.2). In contrast, “strong spatial dispersion” term stands for the case when higher-order derivatives in series (3.11) cannot be neglected and, therefore, they appear in the constitutive relations. Moreover, in this case, the boundary conditions at the medium interface become complicated and include field derivatives [10].

In the case of a medium with general dispersion properties, the constitutive relations can be written in a simple form under the assumption that the field structure in the medium is as that of a monochromatic plane wave \( \mathbf{E} = E_0 e^{j(\omega t - k \cdot \mathbf{r})} \). In this case, differentiation with respect to \( \nabla_m = \partial / \partial r_m \) reduces to multiplication by \( -jk_m \). Therefore,
Spatial dispersion relations (3.11) can be rewritten as [9]

$$\mathbf{D} = \bar{\varepsilon}(\mathbf{k}) \cdot \mathbf{E}, \quad \mathbf{B} = \mu_0 \mathbf{H},$$

(3.16)

where $\bar{\varepsilon}(\mathbf{k})$ is the tensor given by

$$\varepsilon_{ij}(\mathbf{k}) = \varepsilon_0 \delta_{ij} + a_{ij} - ja_{ijk}k_k - a_{ijkl}k_kk_l + \ldots$$

(3.17)

Thus, under plane-wave excitation, the constitutive relations can be written in a formally local representation, while the permittivity tensor depends on the wave vector $\mathbf{k}$ of the plane wave (naturally, it depends also on the frequency $\omega$ due to frequency dispersion).

Since the main subject of the present dissertation is spatial dispersion of the first order (bi-anisotropy) in metasurfaces, it is important to discuss classification of different kinds of bi-anisotropic effects and material inclusions which implement them.

### 3.5 Constituents of bi-anisotropic materials

Let us consider an anisotropic material with weak spatial dispersion (first- and second-order dispersion terms are present) described by constitutive relations in the following form (they can be derived similarly to relations (3.15)):

$$\mathbf{D} = \bar{\varepsilon}_s \cdot \mathbf{E} - j\sqrt{\varepsilon_0 \mu_0} \bar{\kappa} \cdot \mathbf{H}, \quad \mathbf{B} = \bar{\mu} \cdot \mathbf{H} + j\sqrt{\varepsilon_0 \mu_0} \bar{\kappa}^T \cdot \mathbf{E}.$$  

(3.18)

In the approximation of sparse inclusions concentration, the effective permittivity, permeability and chirality parameters can be written as [22]

$$\bar{\varepsilon}_s = \varepsilon_0 \bar{\kappa} + N\varepsilon_0 \bar{\alpha}^V_{ee}, \quad \bar{\mu} = \mu_0 \bar{\kappa} + N\mu_0 \bar{\alpha}^V_{mm}, \quad \bar{\kappa} = N \bar{\alpha}^V_{em},$$

(3.19)

where $N$ is the volume concentration of the material inclusions, $\bar{\alpha}^V_{ee}$, $\bar{\alpha}^V_{mm}$, and $\bar{\alpha}^V_{em}$ are the electric, magnetic, and magnetoelectric polarizability tensors of the inclusions with dimensions of volume, and $\bar{\kappa} = \delta_{ij}$ is the unit tensor in three-dimensional space. Taking into account (3.7), the constitutive relations can be transformed to the following microscopic form:

$$\mathbf{p} = \varepsilon_0 \bar{\alpha}^V_{ee} \cdot \mathbf{E} - j\sqrt{\varepsilon_0 \mu_0} \bar{\alpha}^V_{em} \cdot \mathbf{H}, \quad \mu_0 \mathbf{m}_{conv} = \mu_0 \bar{\alpha}^V_{mm} \cdot \mathbf{H} + j\sqrt{\varepsilon_0 \mu_0} (\bar{\alpha}^V_{em})^T \cdot \mathbf{E},$$

(3.20)

where $\mathbf{p} = \mathbf{P}/N$ is the electric dipole moment of each inclusion and $\mathbf{m}_{conv} = \mathbf{M}/N$ is the corresponding magnetic dipole moment written in conventional notations ($\mathbf{m}_{conv} = I \mathbf{S}$, where $I$ is the current of a loop with area
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$S$ creating the magnetic moment. These relations express that in a spatially dispersive material each inclusion can be polarized electrically and magnetically by both electric and magnetic fields. Indeed, the actual mechanism of polarization is spatial dispersion, however, the description of polarization in terms of local fields significantly simplifies theoretical analysis. Sometimes, it is convenient to redefine the polarizability tensors and to present the microscopic equations (3.20) in the following form free of constant coefficients (note that these notations for polarizabilities will be used in the following text of the dissertation):

\[
\begin{align*}
\mathbf{p} &= \mathbf{\alpha}_{ee} \cdot \mathbf{E} + \mathbf{\alpha}_{em} \cdot \mathbf{H}, \\
\mathbf{m} &= \mathbf{\alpha}_{mm} \cdot \mathbf{H} + \mathbf{\alpha}_{em}^T \cdot \mathbf{E},
\end{align*}
\]

(3.21)

where $\mathbf{m} = \mu_0 \mathbf{m}_{\text{conv}}$.

In natural non-magnetic materials, polarization response is predominantly determined by electric polarizability $\alpha_{ee}$ of atoms and molecules. Due to electrically small size of atoms ($a \ll \lambda$), the magnetoelectric $\alpha_{em}$ and magnetic $\alpha_{mm}$ polarizabilities are negligible, as weak spatial dispersion effects of $a/\lambda$ and $(a/\lambda)^2$ orders. Natural magnetic materials exhibit additionally strong magnetic polarization response (not due to spatial dispersion), however, it occurs only at low frequencies. Strong general polarization response can be achieved in artificial composites (metamaterials) whose resonant inclusions are comparable with the wavelength. The shape and internal structure of the inclusions can be engineered to enhance specific polarization effects, and in this case the magnitudes of the inclusion polarizabilities are not limited by $(a/\lambda)^m$ order (where $m = 1, 2$).

A typical inclusion with strong electric polarizability $\alpha_{ee}$ is a resonant straight metal wire of about $\lambda/2$ length. A double split-ring resonator (see Fig. 3.2) exhibits large magnetic polarizability $\alpha_{mm}$. Naturally, an inclusion with magnetoelectric polarizability $\alpha_{em}$ should in a sense combine characteristics of these two elements. As is seen from (3.20), there are two basic scenarios of magnetoelectric coupling depending on the mutual orientation of the field and the dipole moment which is induced by this field. The first scenario, where the moment and the field vectors are collinear, can be realized with a canonical metallic three-dimensional helix [23] shown in Fig. 3.3(a). Under excitation by vertically oriented electric field, the current induced in the wire forms a loop corresponding to a magnetic moment along the external electric field. The direction of the magnetic moment as well as the sign of the magnetoelectric polarizability depends on the helicity state of the helical inclusion. In the second
Figure 3.3. Topologies of various reciprocal bi-anisotropic inclusions. External electric field $E_{\text{ext}}$ induces magnetic moments $m$ in the inclusions. (a) A right-handed chiral canonical helix. (b) An omega inclusion with shape of the $\Omega$ letter. (c) A twisted omega inclusion. (d) A uniaxial omega inclusion. (e) A chiral inclusion with the shape of a true helix.

scenario, the induced moment and the field vectors are orthogonal. This can be realized by orienting the loop of the helix into another plane, as it is shown in Fig. 3.3(b). This planar geometry, often referred to as the omega shape [24] (after Greek letter $\Omega$), provides magnetoelectric polarization orthogonal to the exciting field. The sign of the magnetoelectric polarizability can be reversed by twisting the loop of the inclusion, as it is shown in Fig. 3.3(c). Uniaxial (isotropic in plane) omega response can be achieved by combining two orthogonal omega inclusions together [see Fig. 3.3(d)].

It should be noted that the canonical helix does not provide pure collinear magnetoelectric polarization response. Indeed, if it is excited by an external magnetic field along the $z$-axis [see Fig. 3.3(a)], the electric current induced in the wire forms two electric dipole moments (a large moment along the $z$-axis and a small moment along the $y$-axis). Thus, the canonical helix possesses two different polarization effects. In contrast, nearly pure collinear magnetoelectric polarization can be achieved in a true helix with a large number of turns (in practice, two turns are enough), shown in Fig. 3.3(e).
3.6 Classification of bi-anisotropic materials

It is convenient to classify macroscopic materials with bi-anisotropic properties, based on the form of the chirality tensor $\bar{\kappa}$ in relations (3.18). An arbitrary tensor can be always presented as [10,25]

$$\bar{\kappa} = \kappa I + \bar{M},$$

(3.22)

where $\kappa = \text{tr}\{\bar{\kappa}\}/3$ is a pseudoscalar (see more details in Section 3.8) complex chirality parameter, and $\bar{M}$ is a matrix with zero trace (sum of the diagonal elements). It can be proven [10] that parameter $\kappa$ is non-zero only if the material has a mirror-asymmetric structure, or in other words, if the material and its mirror image cannot be superposed onto one another (e.g., human hand). It should be noted that the aforementioned result is related to the entire material structure and not to single inclusions. Materials with non-zero parameter $\kappa$ are called chiral (derived from the Greek $\chi$ειρ, “hand”) bi-anisotropic materials. Thus, not all materials formed by chiral inclusions are chiral (e.g., Publication V), and vice versa, not all chiral materials consist of chiral inclusions (e.g., [26]). Materials with non-zero isotropic electromagnetic chirality parameter $\kappa I$ can be conceptually constructed by an array of uniaxial multi-turn helices of the same helicity state arranged with equal density along the basis unit vectors in a lattice (alternatively, the helices can be randomly distributed as is shown in Fig. 3.1).

Matrix $\bar{M}$ with zero trace can be always decomposed into a sum of symmetric and antisymmetric parts:

$$\bar{M} = \bar{N} + \bar{J},$$

(3.23)

where $\bar{N} = (\bar{M} + \bar{M}^T)/2$ is a symmetric matrix and $\bar{M} = (\bar{M} - \bar{M}^T)/2$ is an antisymmetric matrix. Matrix $\bar{N}$ can be always diagonalized by transforming the coordinate system. In other words, one can find a coordinate system with basis vectors $a_i$ in which matrix $\bar{N}$ has non-zero components only at the main diagonal. Thus, non-zero symmetric matrix $\bar{N}$ can be modelled by uniaxial chiral inclusions oriented along the basis vectors $a_i$ (in the general lossy case, three complex vectors $a_i$ can be decomposed to nine real basis vectors). The trace of the diagonalized matrix, likewise that of original matrix $\bar{N}$, should be equal to zero. The aforementioned property implies that the chiral inclusions oriented along the basis vectors $a_i$ must be of different handedness so that in total chirality in the material is compensated. Interestingly, a material with $\kappa = 0$ and $\bar{N} \neq 0$
exhibits such chiral effect as electromagnetic activity for specific directions of propagation (when the propagation direction is perpendicular to either vector \( a_i \)), despite that the material is not chiral. Due to this behaviour such materials are called pseudochiral [10]. In fact, the inclusions of a pseudochiral bi-anisotropic material should not necessary have a mirror-asymmetric geometry (since \( \kappa = 0 \)) and can have various two-dimensional topologies. Some examples of pseudochiral materials include composites with specifically arranged omega inclusions [25] and metasurfaces composed of “planar chiral” geometries [27].

An arbitrary asymmetric tensor

\[
\mathbf{J} = \begin{bmatrix}
0 & -b_3 & b_2 \\
b_3 & 0 & -b_1 \\
-b_2 & b_1 & 0
\end{bmatrix}
\]  

(3.24)

in three-dimensional space can be characterized in some coordinate system by a dual vector \( \mathbf{b} = [b_1; b_2; b_3]^T \) (see also discussion in Section 3.8). In general, the vector can be complex and therefore decomposed into two unit vectors multiplied by real and imaginary constants as \( \mathbf{b} = K_1 \mathbf{b}' + j K_2 \mathbf{b}'' \). These two vectors define two axes in space around which the material has rotational symmetry (the material possesses the same response illuminated along the axis by a wave with arbitrary polarization). Moreover, illuminated along the axes, the material exhibits bi-anisotropic response of omega type (the induced moment is orthogonal to external field). Therefore, to model a material with asymmetric chirality tensor \( \mathbf{\kappa} = \mathbf{J} \), one should construct a three-dimensional composite from uniaxial omega inclusions oriented along vectors \( \mathbf{b}' \) and \( \mathbf{b}'' \). An example of a uniaxial omega inclusion is shown in Fig. 3.3(d). Such bi-anisotropic materials are called omega media [10]. It is important to note that in contrast to chiral bi-anisotropy effects, omega effects cannot be observed in isotropic composites. This is explained by the fact that omega media always have asymmetric properties with respect to the propagation direction. Thus, if one constructs a composite of uniaxial omega inclusions oriented along the three coordinate vectors, the microscopic omega effects would be effectively compensated at the macro-level, and the composite would behave as if it were made of electrically and magnetically polarizable inclusions only (\( \pi_{ee} \neq 0, \pi_{mm} \neq 0, \text{ and } \pi_{em} = 0 \)). This scenario was implemented in two-dimensional metasurfaces in [28].

Although there are three basic classes of reciprocal bi-anisotropic materials (chiral, pseudochiral, and omega), in practice, designed bi-anisotropic
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materials often have properties which combine those of different classes. It is convenient to decompose electromagnetic response of an arbitrary material to the three basic classes [10, 25]. An analogous idea was recently proposed for individual meta-atoms [21, 29]. It states that an arbitrary linear weakly-dispersive meta-atom can be decomposed to a few basic “modules”. This concept named “materiatronics” provides a universal route for understanding and possibly designing materials with general electromagnetic properties.

To clarify the presented classification of bi-anisotropic effects, let us consider an example material with arbitrary bi-anisotropic properties whose chirality tensor \( \kappa \) (dimensionless) in some coordinate system is given by

\[
\kappa = \begin{bmatrix}
2 & 1 & j \\
1 & 2 & -1 - j \\
-j & 1 + j & 2
\end{bmatrix}
\] (3.25)

Complex chirality tensor refers to a material with dissipation loss or gain. Following decompositions (3.22) and (3.23), the given tensor can be represented as the following sum:

\[
\kappa = 2I + \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & j \\
0 & 0 & -1 - j \\
-j & 1 + j & 0
\end{bmatrix}
\] (3.26)

The first term \( 2I \) corresponds to isotropic chiral magnetoelectric coupling and can be implemented using a composite whose unit cell comprises three uniaxial helices oriented along the basis vectors as is shown in Fig. 3.4(a). The helices should be right-handed, corresponding to positive chirality parameter \( \kappa = 2 \). The size of the unit cell (periodicity of the composite) should be adjusted appropriately to ensure the given strength of the chirality coupling.

The zero-trace symmetric part of the chirality tensor in (3.26) can be diagonalized to the following tensor

\[
\tilde{D} = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] (3.27)

in a new coordinate system whose basis vectors \( \mathbf{a}_i \) are expressed in terms of the original basis vectors as \( \mathbf{a}_1 = [-1; 1; 0]^T \), \( \mathbf{a}_2 = [0; 0; 1]^T \), and \( \mathbf{a}_3 = [1; 1; 0]^T \). Thus, the symmetric part of the tensor in (3.26) can be modelled by a unit cell with one left-handed uniaxial helix oriented along the \( \mathbf{a}_1 \)-axis (component \( D_{11} = -1 \)) and one right-handed uniaxial helix oriented
Figure 3.4. Conceptual realization of an arbitrary reciprocal bi-anisotropic material with the use of basic elements. The grey frame box depicts the unit cell. Unit cells modelling (a) isotropic chiral response, (b) pseudochiral response (the right- and left-handed helices are shown in dark and light blue, respectively), and (c) uniaxial omega response (the lossless and lossy inclusions are shown in dark and light green, respectively). (d) The conceptual unit cell of the bi-anisotropic material with magnetoelectric coupling described by Eq. (3.26).

along the $a_3$-axis (component $D_{33} = 1$). This unit cell is illustrated in Fig. 3.4(b). Obviously, as discussed above, this is not a unique conceptual physical realization. It should be noted that the unit cell is in overall achiral. This can be easily verified by taking a mirror image (in either plane) of the cell and comparing it with the original cell.

The antisymmetric part in representation (3.26) is described by a dual vector written in the original coordinate system as $b = [1 + j; j; 0]^T$. The complex vector can be decomposed to a linear combination of real unit vectors $b = K_1 b' + j K_2 b'' = [1; 0; 0]^T + j \sqrt{2} \cdot [\frac{\sqrt{2}}{2}; \frac{\sqrt{2}}{2}; 0]^T$. Thus, the antisymmetric tensor in (3.26) can be modelled by a unit cell with two uniaxial omega inclusions as shown in Fig. 3.4(c). One of the inclusions should be lossless (real $K_1$) and oriented along vector $b'$, while another one should be lossy (positive imaginary $j K_2$) and oriented along $b''$. Since $K_1 \neq K_2$, the two inclusions should possess different strength of magne-
Thus, the bi-anisotropic material with the chirality tensor $\mathbf{\kappa}$ given by (3.26) can be constructed as a composite with a unit cell combining all previously determined inclusions. The final unit cell is shown in Fig. 3.4(d). As is seen from the figure, the electromagnetic response of the material can be easily determined for different illumination directions. For example, the maximum absorption in the material appears when electromagnetic waves propagate along the bisection of the angle between the $+x$ and $+y$ axes, so that the waves strongly interact with the virtual lossy omega inclusions. For the same propagation direction, one can achieve the most pronounced optical activity effect from the material. In this scenario, the incident wave does not excite the left-handed helix, but excites three right-handed helices (along the $x$ and $y$ axes as well as along the bisection angle between $+y$ and $-x$ directions). Furthermore, the highest suppression of omega response occurs for waves propagating along the $z$-axis, when the omega inclusions are weakly excited.

In summary, classification of bi-anisotropic effects is very important not only from the theoretical point of view but also as an intelligent tool for designing general reciprocal materials with specified electromagnetic properties.

### 3.7 Time inversion and non-reciprocity in electrodynamics

Symmetry properties of material relations under time inversion and spatial inversion are of crucial importance for material science. Knowledge of the symmetry properties of a given material allows us to judge of its internal structure and properties of its constituents. This knowledge is especially critical for designing metamaterials with required electromagnetic response. Based on this response, one can extract general information on material properties of meta-atoms (either it should be made of dielectrics, or metals, or magnetics, etc.) and their geometries (mirror symmetry properties).

Let us first consider time inversion in electromagnetics (hereafter, time inversion implies inversion of the time flow direction). Maxwell’s equations are symmetric with respect to time direction inversion $dt \rightarrow -dt$ ($dt$ is positive if time goes from the past to the future), i.e., all electromagnetic processes taking place in a closed lossless system (without any external
sources or fields) are reversible in time. This statement holds for both microscopic and macroscopic Maxwell’s equations under the assumption of linear medium (note it is not applicable to “active” media since they imply the existence of external to the system sources which break the time symmetry). It is commonly assumed that when the time direction is reversed $dt \to -dt$, the electric charge does not change (an even quantity under time inversion), therefore, the electric current changes sign (an odd quantity under time inversion) since it is the time derivative of the charge. Equivalent statements hold also for the corresponding density quantities $\rho$ and $J$. In fact, there is an alternative agreement that the electric charge changes sign under time inversion, which is equally possible [10] but will not be considered here. Since the Lorentz force acting on electric charge $q$ (in general moving with the linear speed $v$)

$$F = qE + qv \times B$$

(3.28)

is even under time inversion (acceleration is an even parameter), the electric field $E$ is invariant and the magnetic induction $B$ changes sign. Now the invariance of macroscopic Maxwell’s equations under time reversal is obvious from (3.8). From Gauss’s law it follows that the electric induction $D$ is an even parameter under time reversal, while from Ampere’s law, the magnetic field $H$ is a time-odd parameter.

If an electromagnetic system under consideration is not closed so that there are some external perturbations acting on it and they are odd with respect to time inversion, time-reversal symmetry of Maxwell’s equations in the system becomes broken. Such perturbations can be of non-electromagnetic as well as electromagnetic nature (in this case they should be external to the system). A good example is an external static magnetic field bias. Materials such as metals and ferromagnetics placed in this field would exhibit different response for different directions of time flow in the system. Here it is assumed that the external magnetic field is invariant to the time flow since it is external to the considered system. Materials that possess different electromagnetic response under time reversal are called non-reciprocal.

Let us consider a classical optical example which illustrates the difference between reciprocal and non-reciprocal materials. It is well known that chiral materials, such as sugar solutions, exhibit different refractive indices for electromagnetic waves with left- and right-handed polarizations. The reason for this effect is that in chiral materials right-handed and left-handed inclusions are in different proportions. Let us consider
the case when a chiral material slab is illuminated by a linearly polarized wave as shown in Fig. 3.5(a). The incident wave can be represented as a sum of two waves with opposite circular polarizations and equal phases. These waves of different handedness travel through the material with different speeds, which results in accumulation of the phase difference between them. At the second interface, the superposition of the two circularly polarized waves with different phases is a linearly polarized wave with a tilted polarization axis. Thus, chiral materials rotate polarization plane of waves propagating through them. This effect is called optical activity or isotropic gyrotropy effect. Since chiral materials are reciprocal (no external perturbations are required for its operation), the incident wave, which has passed the material, under time reversal would return back along the same way and its field vectors would form the same traces in space [see Fig. 3.5(b)].

Similar effect of polarization plane rotation is observed for waves travelling through magneto-optical (gyrotropic) materials placed in external bias field $H_0$ (a time-odd parameter). Polarization rotation in such materials occurs due to anisotropic permeability tensor and is called the Faraday effect [shown in Fig. 3.5(c)]. It can be observed, for example, for light incident on a slab of magneto-optical material. In contrast to the previously considered reciprocal scenario, the incident wave under time reversal ex-
Spatial dispersion experiences different response from the magnetized magneto-optical material and the traces of the field vectors do not coincide [see Fig. 3.5(d)]. This effect is non-reciprocal and is the basis of very important electromagnetic devices such as isolators and circulators.

It appears that non-reciprocal materials break time-reversal symmetry, and Maxwell's equations are not reversible with respect to time. This happens because the system under consideration is not closed. If one adds also the source of the magnetic bias field into the system, it becomes closed and time-inversion symmetry will hold. Indeed, any source of static magnetic field must include some circulating direct currents which maintain this field: It can be an electromagnet with a solenoid coil or natural magnets consisting of ordered microscopic current loops (magnetic dipole moments) formed by rotating electrons. Thus, under time reversal in this electrodynamic system, the currents which form the static magnetic field will change their directions and the magnetization field will be reversed. In this case, the wave propagation inside the ferromagnetic material becomes reversible in time.

Other examples of non-reciprocal materials are magnetized plasma and magnetized graphene. Non-reciprocal response can be achieved also by other means: Materials moving with some speed, magnetless active materials mimicking electron spin precession of natural magnets [30, 31], and non-linear materials [32].

3.8 Spatial inversion and non-reciprocal bi-anisotropic materials

Analogously to anisotropic materials, non-reciprocal effects can be observed also in bi-anisotropic materials. As it was discussed in Section 3.7, the necessary condition for the existence of non-reciprocal effects in a material is that its response depends on an external parameter which has time-odd symmetry. However, in order to achieve non-reciprocal bi-anisotropic response of a specific type, one should also consider spatial symmetry properties of the inclusions.

Let us consider an arbitrary vector \( \mathbf{a} = a_x \mathbf{x} + a_y \mathbf{y} + a_z \mathbf{z} \) in \( xyz \) coordinate system (\( x, y, \) and \( z \) are the basis unit vectors). Under spatial inversion, i.e. when one of the coordinate axes changes sign, the coordinates of the vector change in a specific way. For example, if the \( y \) axis is reversed, the vector in the new system will have form \( \mathbf{a}' = a_x \mathbf{x} - a_y \mathbf{y} + a_z \mathbf{z} \). This transformation is equivalent to mirror inversion of the vector in the initial
Figure 3.6. Transformation of (a) a true vector and (b) a pseudovector under spatial inversion (y-axis is reversed). Vector projections on the xy plane are shown as shadows in grey.

coordinate system with respect to the \(xz\)-plane [see Fig. 3.6(a)]. Thus, the transformation of the vector’s projections is given by

\[
a_x = a'_x, \quad a_y = -a'_y, \quad a_z = a'_z.
\]  

(3.29)

Vectors which obey this transformation rule are called true vectors or even vectors with respect to spatial inversion of the coordinates. Physical vectors such as linear speed, force, and differential operator \(\nabla\) are true vectors.

Next, let us consider a vector which is a result of the vector product of two true vectors, i.e., \(\mathbf{a} = \mathbf{b} \times \mathbf{c}\). In the initial coordinate system, this vector has the following form:

\[
\mathbf{a} = (b_y c_z - b_z c_y)\mathbf{x} + (b_z c_x - b_x c_z)\mathbf{y} + (b_x c_y - b_y c_x)\mathbf{z}.
\]  

(3.30)

Under spatial inversion of the coordinates \(y\)-axis is reversed), both true vectors \(\mathbf{b}\) and \(\mathbf{c}\) are transformed according to (3.29), while their vector product \(\mathbf{a}' = \mathbf{b}' \times \mathbf{c}'\) has the following form:

\[
\mathbf{a}' = -(b_y c_z - b_z c_y)\mathbf{x} + (b_z c_x - b_x c_z)\mathbf{y} - (b_x c_y - b_y c_x)\mathbf{z}.
\]  

(3.31)

Hence, the transformation of vector \(\mathbf{a}\) in this case is described by

\[
a_x = -a'_x, \quad a_y = a'_y, \quad a_z = -a'_z.
\]  

(3.32)

Figure 3.6(b) depicts the spatial inversion of such a vector. This transformation is different from that given by (3.29). Thus, vectors which are formed as cross products of two true vectors are transformed under spatial reversal in a different way than true vectors. They are called pseudovectors or odd vectors with respect to spatial inversion. As is seen in
Fig. 3.6(b), under spatial inversion a pseudovector transforms into its mirror image with an additional sign flip. It should be noted that the difference in the properties of true vectors and pseudovectors occurs only under spatial inversion and does not appear under rotational coordinate transformations. Similarly, it can be shown that a cross product of a true vector and a pseudovector results in a true vector. An example of pseudovectors in electrodynamics is the magnetic induction vector $\mathbf{B}$. Indeed, according to (3.28), the force which is acting on a moving electric charge by the field $\mathbf{B}$ can be a true vector only if $\mathbf{B}$ is a pseudovector. This conclusion is based on the assumption that the charge $q$ is a true scalar, i.e., an even parameter with respect to spatial inversion. This assumption, despite being usually made, is not necessary true. An example of a pseudoscalar is the complex chirality parameter $\kappa$ discussed in Section 3.6. Under spatial reversal of chiral isotropic material formed, for example, by helical inclusions, the sign of the chirality parameter of the material changes since the handedness of the helices changes. Similarly to vectors and scalars, pseudotensors transform under spatial inversion as true tensors with an additional sign flip.

Using the expression for the Lorentz force (3.28), one can deduce that the electric field $\mathbf{E}$ is a true vector. From the macroscopic Maxwell equations it follows that $\mathbf{H}$ is a pseudovector and $\mathbf{D}$ is a true vector.

Taking into account aforementioned symmetry properties of the electromagnetic field quantities under time and space reversal, let us re-examine the constitutive relations for a reciprocal bi-anisotropic material (3.18). Reciprocity of the material response requires that the tensors of permittivity $\varepsilon_s$, permeability $\mu$, and chirality $\kappa$ do not depend on any external time-odd parameter. Let us represent the chirality tensor in a dyadic form:

\[
\kappa = \frac{(\kappa + \kappa^T)}{2} + \frac{(\kappa - \kappa^T)}{2} = \kappa_s + \mathbf{K}_a \times \mathbf{I},
\]

where the first term $\kappa_s$ and the second term $\mathbf{K}_a \times \mathbf{I}$ represent, respectively, symmetric and antisymmetric dyadics, and $\mathbf{K}_a$ is a vector dual to the antisymmetric dyadic (see also Section 3.6). The constitutive relations for a general reciprocal media are written in the following form:

\[
\mathbf{D} = \varepsilon_s \cdot \mathbf{E} - j \sqrt{\varepsilon_0 \mu_0} (\kappa_s + \mathbf{K}_a \times \mathbf{I}) \cdot \mathbf{H}, \quad \mathbf{B} = \mu \cdot \mathbf{H} + j \sqrt{\varepsilon_0 \mu_0} (\kappa_s - \mathbf{K}_a \times \mathbf{I}) \cdot \mathbf{E}.
\]

(3.34)

From both equations it follows that $\kappa_s$ must be a pseudodyadic (or pseudotensor), while $\mathbf{K}_a$ a true vector because both sides of each equation must be vectors of the same spatial symmetry. For instance, let us consider the
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first expression. In this case, dyadic \((\overline{\epsilon}_a + K_a \times \overline{I})\) is a pseudodyadic, and its scalar product with pseudovector \(\mathbf{H}\) results in a true vector, which is of the same symmetry as vector \(\mathbf{D}\) at the left side of the expression. Thus, from the spatial inversion symmetry of the constitutive relations, it follows that the symmetric part of the chirality tensor is non-zero only in a composite with mirror-asymmetric internal structure. On the other hand, to engineer a reciprocal material with an anti-symmetric bi-anisotropic tensor, one should use inclusions with mirror-symmetric geometries, for example, omega inclusions. These conclusions are in agreement with the results of Section 3.6.

Constitutive relations for a general non-reciprocal bi-anisotropic medium (assuming no spatial dispersion of the first order) are written as [10]

\[
\begin{align*}
\mathbf{D} &= \overline{\epsilon}_n \mathbf{E} + \sqrt{\epsilon_0 \mu_0} (\overline{\chi}_s + V_a \times \overline{I}) \cdot \mathbf{H}, \\
\mathbf{B} &= \overline{\mu}_n \mathbf{H} + \sqrt{\epsilon_0 \mu_0} (\overline{\chi}_s - V_a \times \overline{I}) \cdot \mathbf{E},
\end{align*}
\]

(3.35)

where \(\overline{\epsilon}_n\) and \(\overline{\mu}_n\) are the permittivity and permeability dyadics (subscript “\(n\)” corresponds to non-reciprocity), \(\overline{\chi}_s\) and \(V_a \times \overline{I}\) are the symmetric and anti-symmetric parts of the general non-reciprocity coupling dyadic. Based on the time-reversal properties of these constitutive relations, it is seen that both parts of the non-reciprocity dyadic must be time-odd or, in other words, the material response must depend on an external time-odd parameter (e.g., magnetic bias field). Space-reversal symmetry of relations (3.35) implies that the symmetric dyadic \(\overline{\chi}_s\) is a pseudodyadic, while \(V_a\) is a true vector. Thus, the spatial-symmetry properties of corresponding dyadics in non-reciprocal bi-anisotropic materials are equivalent to those in reciprocal materials. However, in the non-reciprocal case, material properties also depend on an external time-odd parameter (usually magnetic field) which is a pseudovector and additionally flips sign of the dyadic under spatial inversion. Therefore, to construct a non-reciprocal material with a symmetric bi-anisotropic dyadic \(\overline{\chi}_s\), its internal structure should be designed to be mirror-symmetric. Such materials or media were named after Bernard Tellegen who suggested the idea of an electromagnetic gyrator as a building block of these media [33]. Non-reciprocal bi-anisotropic materials with asymmetric dyadic \(V_a \times \overline{I}\), in contrast, must have structure which cannot be superposed onto its mirror image. This class of media was named an artificial “moving” medium. Although it is in fact at rest, its electromagnetic response is equivalent to that of an ordinary material which is truly moving with some speed \(v = V_a\) (a true vector) [10, 34].
Figure 3.7 demonstrates topologies of non-reciprocal bi-anisotropic inclusions that can be used as building blocks in composite materials of the mentioned two types. Both inclusions consist of a ferrite sphere (here, ferrite is chosen since it is non-conductive in contrast to other magnetic alloys) magnetized by external magnetic field $H_0$ and located in the proximity of metal wires of specific shape. The electric field of an incident wave excites the electric current along the wires which, in turn, excites alternating magnetic field around the wires. This magnetic field induces a magnetic moment in the ferrite sphere. Likewise, the incident magnetic field excites an electric dipole moment in the wires through magnetization of the sphere. The induced moment in the inclusion shown in Fig. 3.7(a) is co-directed with the electric field which caused it. This inclusion was firstly proposed in [35] and experimentally tested in [36]. It should be noted that in addition to the non-reciprocal Tellegen response, the inclusion exhibits also reciprocal bi-anisotropic properties of a uniaxial omega inclusion. The inclusion shown in Fig. 3.7(b), sometimes named as an artificial moving element [35], in addition to the non-reciprocal bi-anisotropic response, exhibits reciprocal chiral effects due to its mirror-asymmetric shape. The analytical polarizabilities of the non-reciprocal inclusions shown in Fig. 3.7 were reported in [37]. More discussion on polarizabilities of non-reciprocal inclusions can be found in Section 5.1.1.

Summarizing the presented results, let us write the constitutive relations for a general bi-anisotropic material with possible reciprocal and non-reciprocal magnetoelectric properties [10]:

$$D = \varepsilon_g \cdot E + \sqrt{\varepsilon_0 \mu_0} \left( \bar{\chi} - j \bar{\kappa} \right) \cdot H,$$

$$B = \mu_g \cdot H + \sqrt{\varepsilon_0 \mu_0} \left( \bar{\chi} + j \bar{\kappa} \right) T \cdot E,$$

(3.36)
where \( \kappa = \kappa_s + \mathbf{K}_a \times \mathbf{I} \) and \( \chi = \chi_s + \mathbf{V}_a \times \mathbf{I} \) are the chirality and non-reciprocity tensors, respectively, \( \varepsilon_g \) and \( \mu_g \) are the permittivity and permeability dyadics (tensors) of the material with general bi-anisotropic response (subscript “\( g \)” corresponds to a general linear medium). Note that the form of the magnetoelectric dyadic in these constitutive relations is not unique and was chosen for convenience based on the constraints on the material parameters (see more details in Section 3.9).

It is important to mention that non-reciprocal bi-anisotropic coupling is not an effect of spatial dispersion. In contrast to reciprocal spatially dispersive inclusions where magnetoelectric and magnetic response can occur only due to their finite sizes, non-reciprocal inclusions exhibit these responses even in locally uniform external fields (when the particle size is negligibly small compared to the wavelength). For example, in the considered ferrite-based inclusions, uniform electric field excites electric current in the wires which in turn induce magnetic moments in the ferrite sphere. Thus, bi-anisotropic properties of a medium can be caused by spatial dispersion effects or non-reciprocal magnetoelectric coupling.

### 3.9 Constraints on material parameters

There are three different constraints imposed on the material parameters of an arbitrary linear medium. The first two constraints follow from the energy conservation principle and spatial symmetry of the medium structure. In a certain sense, these restrictions are not universal since they are applicable only to passive media [38] or media with specific crystallographic symmetry. The third constraint is universal since it is based only on the linearity of the medium and time-reversal symmetry of Maxwell’s equations.

The energy conservation principle applied for lossless bi-anisotropic media with constitutive relations (3.36) dictates the following restrictions on the material parameters [10]:

\[
\varepsilon_g = \varepsilon_g^\dagger, \quad \mu_g = \mu_g^\dagger, \quad \kappa = \kappa^\dagger, \quad \chi = \chi^\dagger,
\]

(3.37)

where symbols * and † denote complex conjugate and Hermitian conjugate (complex conjugate and transposed matrix) operations. Therefore, in a lossless material, both reciprocal \( \kappa \) and non-reciprocal \( \chi \) bi-anisotropic parameters are purely real tensors (for this reason the imaginary unit was placed in front of the reciprocal tensor in (3.36)). The same conclusion is
Spatial dispersion applied to the symmetric parts of the permittivity \( \bar{\epsilon}_g \) and permeability \( \bar{\mu}_g \) tensors, while the corresponding anti-symmetric parts are purely imaginary.

Material parameters of a medium with specific crystallographic structural symmetry obey additional restrictions expressed as

\[
\bar{\psi} = \bar{K} \cdot \bar{\psi} \cdot \bar{K}^T ,
\]

(3.38)

if they are described by a true tensor and

\[
\bar{\psi} = \det{\{\bar{K}\}} \bar{K} \cdot \bar{\psi} \cdot \bar{K}^T ,
\]

(3.39)

if they are described by a pseudotensor. Here \( \bar{\psi} \) is a material parameter such as the permittivity or chirality tensor, \( \bar{K} \) is a transformation matrix that defines spatial transformation under which the medium does not change its response, and \( \det{\{\bar{K}\}} \) is the determinant of the transformation matrix. For example, a transformation matrix for a composite formed by helical inclusions with orientation of each inclusion shown in Fig. 3.3(a) is diagonal with matrix elements \( K_{11} = 1, K_{22} = -1, \) and \( K_{33} = -1 \) (in the coordinate system specified in the figure; indices 1, 2, and 3 correspond to axes \( x, y, \) and \( z \)). In this case, using (3.38) and (3.39), one can find that material parameters described by true tensors (\( \bar{\epsilon}_g \) and \( \bar{\mu}_g \)) must have the following zero components due to spatial symmetry of the composite: \( \psi_{12} = \psi_{13} = \psi_{21} = \psi_{31} = 0 \) (here \( \bar{\psi} \) denotes both \( \bar{\epsilon}_g \) and \( \bar{\mu}_g \)). The chirality pseudotensor \( \kappa \), in contrast, has other zero components \( \kappa_{11} = \kappa_{22} = \kappa_{23} = \kappa_{32} = \kappa_{33} = 0 \). Importantly, symmetry restrictions (3.38) and (3.39) can be applied likewise to microscopic parameters such as the polarizabilities of inclusions.

The third constraint on material parameters of media results from the time-reversal symmetry of Maxwell’s equations. It is usually expressed in form of the Lorentz reciprocity theorem. The Lorentz theorem is formulated for a pair of sources with current densities \( J_A \) and \( J_B \) which create fields \( E_A \) and \( E_B \) [38,39] (see illustration in Fig. 3.8). The theorem states that in reciprocal media, the reaction of field \( E_A \) on a source with current density \( J_B \) should be the same as that of field \( E_B \) on a source with \( J_A \). In other words, interactions between any pair of electromagnetic sources are reciprocal. This formulation, in fact, does not imply strictly time direction reversibility \( dt \rightarrow -dt \) (similarly to video rewind) of all electromagnetic processes in the medium. Instead, it just emulates time reversibility by interchanging the locations of the sources acting on one another \( (dt \) is always positive).
The reaction of field $E_A$ on a source with current density $J_B$ is defined by the volume integral [40]:

$$
\langle A, B \rangle = \int_{V_B} E_A \cdot J_B dV,
$$

(3.40)

where the volume $V_B$ contains the source $B$, and $dV$ is the volume element. Thus, the Lorentz reciprocity theorem can be written as

$$
\langle A, B \rangle - \langle B, A \rangle = \int_{V_{AB}} E_A \cdot J_B dV - \int_{V_{AB}} E_B \cdot J_A dV = 0,
$$

(3.41)

where $V_{AB}$ is the volume containing $A$ and $B$ sources. The theorem can be generalized also to a general bi-anisotropic medium with possible non-reciprocal effects [41], assuming that the field $E_B$ generated by the source $B$ is calculated in the time-reversed version of the medium (to emulate time reversibility). As it was discussed in Section 3.7, in the time reversed medium, all external non-reciprocal parameters (let us denote them as the magnetization vector $H_0$) switch signs and the material parameters can be written as $\overline{\psi}(-H_0)$.

Let us write the constitutive relations for a general bi-anisotropic medium in the following form:

$$
D = \varepsilon \cdot E + \psi \cdot H, \quad B = \mu \cdot H + \zeta \cdot E.
$$

(3.42)

Here, in contrast to (3.36), no assumptions on the magnetoelectric dyadic are made. Using the standard manipulations with Maxwell’s equations [38, 39, 41], for a bi-anisotropic medium described by (3.42), the requirement

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.8.png}
\caption{Electromagnetic interaction between a pair of sources with the current densities $J_A$ and $J_B$.}
\end{figure}
(3.41) can be written in the following form:

\[
\int_{V_{AB}} (E_A \cdot J_B - E_B \cdot J_A) dV = \int_{S_{AB}} (E_A \times H_B - E_B \times H_A) \cdot dS \\
+ j\omega \int_{V_{AB}} \left\{ E_B \cdot [\bar{\xi}_g^T (-H_0) - \bar{\xi}_g (H_0)] \cdot E_A - E_B \cdot [\bar{\psi}^T (-H_0) + \bar{\psi} (H_0)] \cdot H_A \\
+ H_B \cdot [\bar{\psi}^T (-H_0) + \bar{\psi} (H_0)] \cdot E_A - H_B \cdot [\bar{\psi}_g^T (-H_0) - \bar{\psi}_g (H_0)] \cdot H_A \right\} dV = 0,
\]

(3.43)

where \(S_{AB}\) is the closed surface of volume \(V_{AB}\), vector \(dS\) points outwards from volume \(V_{AB}\). By choosing enough large volume \(V_{AB}\), the surface integral in (3.43) becomes zero. Indeed, assuming that the medium has non-zero dissipation loss (a purely lossless medium is an abstraction), the power density quantity \(E_A \times H_B\) decays faster than the square of the distance to surface \(S_{AB}\) and, therefore, at enough large distance its surface integral becomes negligibly small. Relation (3.43) must be satisfied for an arbitrarily chosen volume \(V_{AB}\), which is possible only if all the expressions in the square brackets are identically zero. The obtained requirements are the generalized Onsager-Casimir relations for material parameters [41–44]:

\[
\bar{\xi}_g (H_0) = \bar{\xi}_g^T (-H_0), \quad \bar{\psi}_g (H_0) = \bar{\psi}_g^T (-H_0), \quad \bar{\psi} (H_0) = -\bar{\psi}^T (-H_0). \tag{3.44}
\]

Here, construction of type \(\bar{\psi}^T (-H_0)\) denotes a transposed tensor of material parameter \(\bar{\psi}\) for the same medium when all the external non-reciprocal parameters switched signs. If the medium is reciprocal \((H_0 = 0)\), then equations (3.44) simplify to \(\bar{\xi}_g = \bar{\xi}_g^T, \bar{\psi}_g = \bar{\psi}_g^T, \) and \(\bar{\psi} = -\bar{\psi}^T\). In this case, the constitutive relations become equivalent to (3.18) under the replacement \(\bar{\psi} = -j\nu_0 \mu_0 \bar{\mu}\).

Let us obtain the constitutive relations in the form (3.36). One can decompose the magnetoelectric tensors into \(\bar{\psi}(H_0) = C_1 \bar{\kappa}_1 (H_0) + C_2 \bar{\kappa}_2 (H_0)\) and \(\bar{\xi}(H_0) = C_3 \bar{\kappa}_2 (H_0) + C_4 \bar{\kappa}_2 (H_0)\), where \(\bar{\kappa}_{1,2}\) and \(\bar{\kappa}_{1,2}\) denote non-reciprocal and reciprocal parts, respectively, and \(C_{1-4}\) are some complex scalars. The last expression in (3.44) is written as

\[
C_{1} \bar{\kappa}_1 (H_0) + C_2 \bar{\kappa}_1 (H_0) = C_3 \bar{\kappa}_2 (H_0) - C_4 \bar{\kappa}_2 (H_0), \tag{3.45}
\]

where the time-reversal properties of reciprocal \(\bar{\kappa}_2 (-H_0) = \bar{\kappa}_2 (H_0)\) and non-reciprocal \(\bar{\kappa}_2 (-H_0) = -\bar{\kappa}_2 (H_0)\) media were used. Equation (3.45) can be satisfied only if the non-reciprocal/reciprocal terms in both sides of the equation are equal (otherwise, under reversal of \(H_0\), the equation does not hold). Thus, the following limitations exist: \(C_1 = C_3, C_2 = -C_4, \bar{\kappa}_1 = \)
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\[ \chi_{T_2} = \chi, \quad \kappa_{T_2} = \kappa. \] By choosing \( C_1 = \sqrt{\varepsilon_0 \mu_0} \) and \( C_2 = -j \sqrt{\varepsilon_0 \mu_0} \), the constitutive relations (3.42) take the same form as those in (3.36).

The Onsager-Casimir relations can be also written for the microscopic polarizabilities [10, 45]:

\[ \begin{align*}
\bar{\alpha}_{ee}(H_0) &= \bar{\alpha}_{T_2 ee}(-H_0), \\
\bar{\alpha}_{mm}(H_0) &= \bar{\alpha}_{T_2 mm}(-H_0), \\
\bar{\alpha}_{me}(H_0) &= -\bar{\alpha}_{T_2 em}(-H_0).
\end{align*} \]

(3.46)

The constraints on material parameters play an important role for studying existing materials as well as for designing novel composite materials.

3.10 Wave propagation in bi-anisotropic materials and extreme bi-anisotropy

Bi-anisotropy is an important physical concept which extends our understanding of macroscopic electromagnetics of different media. This concept has led to a variety of important applications unattainable with anisotropic materials. To determine possible applications of a bi-anisotropic material of a specific class, one should find its eigenwaves, i.e., harmonic plane waves with specific characteristics (such as polarization, direction, and propagation constant) that can propagate in the material. Based on the knowledge of eigenwaves, wave propagation inside a material sample as well as its reflection and transmission properties can be determined.

Although first known bi-isotropic material dates back to 1811 when François Arago observed rotation of the polarization plane of linearly polarized light in quartz, wave propagation in bi-anisotropic media was extensively studied only in the 1990-s [10, 46–48]. In the early 2000-s, Akhlesh Lakhtakia proposed a concept of a “nihility” material whose both permittivity and permeability are equal to zero at some frequency [49]. Subsequently, a similar concept was formulated for bi-anisotropic media of different classes [6, 7, 50, 51]. In this scenario, while permittivity and permeability of such media are zero, the magnetoelectric coupling coefficients are non-zero. Thus, bi-anisotropic effects in these media become most pronounced (extreme) and completely determine the electromagnetic response of the media. This Section briefly summarizes results on wave propagation in bi-anisotropic materials of two reciprocal (chiral and omega) and two non-reciprocal (Tellegen and artificial moving) classes. The corresponding cases of extreme bi-anisotropic response are also reported.

Let us consider a general bi-anisotropic medium [see Eqs. (3.36)] with
uniaxial symmetry, i.e., when there is only one preferred direction (z-axis) in the medium structure (medium properties do not change under rotation around this axis). Axial plane-wave propagations in the form $e^{j(\omega t - \beta z)}$ and $e^{j(\omega t + \beta z)}$ are assumed, where $\beta_{+z}$ and $\beta_{-z}$ are the propagation constants of the waves carrying power towards the $+z$ and $-z$ directions, respectively. This case is, probably, the most interesting and practical since the material response is independent on the orientation of electric and magnetic fields of propagating waves. The constitutive relations for a general uniaxial bi-anisotropic medium have the following form:

$$\begin{align*}
\mathbf{D} &= \varepsilon_g \cdot \mathbf{E} + \sqrt{\varepsilon_0 \mu_0} \left[ (\chi_t - j\kappa_t) \mathbf{I}_t + (V_a - jK_a) \mathbf{J}_t \right] \cdot \mathbf{H}, \\
\mathbf{B} &= \mu_g \cdot \mathbf{H} + \sqrt{\varepsilon_0 \mu_0} \left[ (\chi_t + j\kappa_t) \mathbf{I}_t + (V_a - jK_a) \mathbf{J}_t \right] \cdot \mathbf{E},
\end{align*}$$

(3.47)

where $\varepsilon_g = \varepsilon_0 \varepsilon_t I_t + \varepsilon_0 z z$, $\mu_g = \mu_0 \mu_t I_t + \mu_0 z z$, $\mathbf{I}_t = xx + yy$ is the transverse unit dyadic, $\mathbf{J}_t = z \times \mathbf{I}_t = yx - xy$ is the vector-product operator, and subscript $t$ denotes transverse components. These relations can be obtained from (3.36), assuming that $\chi = \chi_t I_t + V_a J_t$ and $\kappa = \kappa_t I_t + K_a J_t$, where $\chi_t$, $V_a$, $\kappa_t$, and $K_a$ measure the strength of the corresponding magnetoelectric effects in the transverse plane (with respect to the axial direction).

If the medium is lossless, all the material parameters in (3.47) are real quantities.

It is well known, that the eigenwaves of a uniaxial chiral medium ($\chi_t = V_a = K_a = 0$) are circularly polarized waves that can propagate symmetrically in both directions along the $z$ axis ($\beta_{+z} = \beta_{-z}$) [47]. The propagation constants of the right and left circularly polarized eigenwaves equal $\beta_{+z1} = k_0(\sqrt{\varepsilon_t \mu_t} + \kappa_t)$ and $\beta_{+z2} = k_0(\sqrt{\varepsilon_t \mu_t} - \kappa_t)$, respectively, where $k_0$ is the free-space wavenumber [see Fig. 3.9(a)]. The different propagation constants of the two circularly polarized eigenwaves yield known optical rotation (optical activity) and circular dichroism effects for waves transmitted through a chiral layer (see also Section 3.7). A particularly interesting scenario is when $\sqrt{\varepsilon_t \mu_t} = \kappa_t$ and $\beta_{+z2}$ becomes equal zero. Such a scenario resembles recently introduced epsilon-and-mu-near-zero (EMNZ) media with exotic properties (see a review in [52]), while in the present case, all the material parameters are non-zero. The impedance of a chiral medium does not depend on the chirality parameter $\kappa_t$ and is defined as $\eta = \eta_0 \sqrt{\mu_t / \varepsilon_t}$, where $\eta_0$ is the free-space wave impedance. Hence, the reflection coefficient from a chiral slab also does not depend on $\kappa_t$. This property was exploited for designing various reflection-less metasurfaces where $\varepsilon_t = \mu_t$ regime can be achieved using chiral inclusions with balanced electric and magnetic properties (see Publication V
Figure 3.9. Wave propagation in uniaxial (a) chiral, (b) chiral nihility, (d) Tellegen, (e) artificial moving, and (f) artificial nihility moving media. (c) Reflection of electromagnetic waves from different sides of an omega slab. Symbols $v_{ph}$ and $v_{gr}$ denote the phase and group velocity of waves, respectively.

and (53, 54)). Interestingly, in the case of chiral nihility when $\epsilon_t = \mu_t = 0$ and $\kappa_t \neq 0$, the propagation constants of the two eigenwaves become opposite in sign: $\beta_{+z1} = -\beta_{+z2} = k_0 \kappa_t$. One of these waves is a backward wave since its phase velocity $v_{ph2}$ and group velocity $v_{gr2}$ are oppositely directed [see Fig. 3.9(b)]. The highest possible amplitude of the negative refractive index achieved in a chiral nihility material makes it optimal for realizing media with negative refractive index [6, 55–57].

The eigenwaves of a uniaxial omega medium ($\chi_t = \kappa_t = V_a = 0$) have an arbitrary polarization and propagate along the $+z$ and $-z$ directions.
with the same propagation constant $\beta_{+z} = \beta_{-z} = k_0 \sqrt{\varepsilon_t \mu_t - K_a^2}$ [48]. If coupling parameter $K_a$ is enough large, no propagating waves are supported by the medium (if the medium is lossless). Interestingly, the wave impedance of an omega medium is different for different propagation directions $\eta_{\pm z} = \eta_0 (\sqrt{\varepsilon_t \mu_t - K_a^2 \mp j K_a}) / \varepsilon_t$. Hence, reflected waves from opposite sides of an omega slab have different phases, as shown in Fig. 3.9(c) (the amplitudes are the same due to reciprocity). Such property is unique for reciprocal media and plays an important role for applications where asymmetry in reflection or scattering is required (see Publications III, IV, and VI, and [58, 59]). It should be noted that zero reflection coefficient from an omega slab can be achieved only if $K_a = \mp j(\mu_t - \varepsilon_t)/2$ [48], which is not possible for the case of lossless slabs ($\mu_t$, $\varepsilon_t$, and $K_a$ are real). A similar conclusion is true for wire omega metasurfaces (slabs with single-inclusion thicknesses) [53]. In a lossless omega nihility material ($\varepsilon_t = \mu_t = 0$ and $K_a \neq 0$), no waves can propagate due to the imaginary propagation constant $\beta_{+z} = \beta_{-z} = -j k_0 K_a$ (note that in the presence of loss wave propagation is allowed). The asymmetry of the wave impedance goes into extreme: $\eta_{+z} = j \infty$ and $\eta_{-z} = 0$ [50, 51, 60]. Therefore, the reflection coefficient from different sides of an omega nihility slab is equal to that of a perfect magnetic ($R_{+z} = +1$) or perfect electric conductor ($R_{-z} = -1$).

In the case of non-reciprocal Tellegen medium ($\kappa_t = K_a = V_a = 0$), the eigenwaves have arbitrary polarization and propagate in both directions along the symmetry axis with the same propagation constant $\beta_{+z} = \beta_{-z} = k_0 \sqrt{\varepsilon_t \mu_t - \chi_t^2}$ [7, 51, 60, 61]. Distinguishing feature of a uniaxial Tellegen medium is non-orthogonality of the electric and magnetic fields of waves propagating along its axis. The angle between the electric and magnetic fields depends on the Tellegen parameter $\chi_t$ [60, 61]. In particular, one can tune the Tellegen parameter in such a way that the electric and magnetic fields of the propagating wave oscillate in two different planes crossed at an angle of $45^\circ$ [see illustration in Fig. 3.9(d)]. An alternative scenario is possible: The electric field has elliptical polarization, while the magnetic field is linearly polarized. A uniaxial Tellegen slab exhibits polarization plane rotation in reflection, which is a non-reciprocal effect [46]. Tellegen slabs with active elements can be used as isolators [62]. In the extreme case of lossless Tellegen nihility, no waves can travel in the medium along its axis since the propagation constant turns purely imaginary $\beta_{+z} = \beta_{-z} = -j k_0 \chi_t$. The wave impedance of the medium tends to
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infinity in this case.

In an artificial moving medium \((\chi_t = \kappa_t = K_a = 0)\), waves of arbitrary polarization can propagate. The propagation constants of an artificial moving medium are \(\beta_{+z} = k_0(\sqrt{\epsilon_t \mu_t} - V_a)\) and \(\beta_{-z} = k_0(\sqrt{\epsilon_t \mu_t} + V_a)\) for waves travelling inside the medium along the \(+z\) and \(-z\) directions, respectively [7, 35, 51, 60]. The difference between the propagation constants for waves travelling in opposite directions is a non-reciprocal effect. Figure 3.9(e) illustrates wave propagation in such a medium when \(V_a < \sqrt{\epsilon_t \mu_t}\). The impedance of an artificial moving medium does not depend on the coupling parameter \(V_a\) and is equal to \(\eta = \eta_0 \sqrt{\mu_t / \epsilon_t}\). The case of artificial moving nihility arouses particular interest. In this case, along the \(+z\) direction only backward waves of arbitrary polarization can propagate \(\beta_{+z} = -k_0 V_a\) if \(V_a > 0\), while in the opposite direction the eigenwaves are ordinary forward waves \(\beta_{-z} = k_0 V_a\) if \(V_a > 0\) as shown in Fig. 3.9(f). Such behaviour resembles chiral nihility, however, a non-reciprocal moving nihility medium operates with any polarization of propagating radiation and possesses asymmetric operation from different directions. Thus, an artificial moving nihility medium has a unique feature: For the illumination coming from the opposite directions the composite behaves either as an ordinary medium or as a Veselago medium [63]. In paper [62] it was shown that a single-layer composite of artificial moving inclusions can be tuned in such a way that from one side it is transparent for incoming radiation (vacuum behaviour), while from the opposite side it fully transmits incident waves, changing their phase (Veselago medium behaviour).

To summarize, studies and realizations of bi-anisotropic materials open up new and unique possibilities for creation of exotic light-matter interaction effects. Most of analytical studies were completed in the 1990-s during so-called “bi-anisotropy” boom. However, as it can be seen by analyzing modern scientific publications, only recently bi-anisotropic effects became widely exploited in various practical applications in the field of complex artificial materials and metamaterials (see e.g., Publication VI and [64–66]).
4. Metamaterials and role of spatial dispersion

The field of metamaterials has significantly expanded over the last two decades and includes such directions as negative-index materials, subwavelength focusing, invisibility cloak, wire media, metasurfaces, etc. The present section provides a brief overview of these research directions with the emphasis on the role of spatial dispersion for realization of the corresponding phenomena and functionalities.

4.1 Negative refraction

Let us consider a classical problem of reflection and transmission of plane waves at an infinite interface between two media with refractive indices \( n_1 \) and \( n_2 \), as shown in Fig. 4.1(a). Assuming TE (transverse electric) polarization, the electric field of the incident, reflected, and transmitted waves at the interface \( (x = 0) \) are given by

\[
E_i = e^{j(\omega t - k_i \sin \theta_i z)}, \quad E_r = e^{j(\omega t - k_r \sin \theta_r z)}, \quad \text{and} \quad E_t = e^{j(\omega t - k_t \sin \theta_t z)},
\]

respectively. Since the tangential fields in the two media at the interface must be continuous \( (E_i + E_r = E_t) \), the arguments in the exponential functions must be equal, mean-

![Figure 4.1](image)

**Figure 4.1.** Two possible scenarios of wave refraction at an interface between two media. (a) Ordinary refraction. (b) Negative refraction.
ing that the tangential wavevectors of the three plane waves are equal:
\[ k_i \sin \theta_i z = k_r \sin \theta_r z = k_t \sin \theta_t z. \] These conditions result in well-known laws of reflection \( \theta_i = \theta_r \) (since \( k_i = k_r \)) and refraction \( k_i \sin \theta_i = k_t \sin \theta_t \).

Usually it is silently assumed that no other plane waves can propagate in this case. However, as it was, probably, for the first time noticed by Leonid Mandelshtam [67], there can exist one more propagating plane wave in the second medium with the refractive index \( n_2 \). Indeed, mathematically condition \( k_i \sin \theta_i = k_t \sin \theta_t \) does not change if angle \( \theta_t \) is replaced by angle \( \pi - \theta_t \). Physically, this solution corresponds to a plane wave whose wavevector \( k_t \) is directed at the angle \( \pi - \theta_t \) (or equivalently at \( -\theta_t \)) from the +x-direction [see Fig. 4.1(b); the usual wave refracted at \( +\theta_t \) is not shown for clarity]. Such wave refraction appears unusual since wave oscillations propagate from a source-free region towards the source. However, this scenario is possible, because the direction of wavevector \( k_t \) defines only the phase velocity direction \( v_{ph} = k_t \omega / k_t^2 \) and not the energy propagation direction. The latter one is defined by the group velocity \( v_{gr} = \nabla_k \omega (k_t) \) (here \( \nabla_k \) is the gradient of the angular frequency \( \omega \) as a function of the wave vector \( k_t \)) and is directed away from the interface.

Such unusual refraction at a negative angle \( -\theta_t \) is called negative refraction. This phenomenon occurs if the phase and group velocities of waves propagating in the second medium are oppositely directed [68].

A possibility of existence of media with oppositely directed phase and group velocities of wave propagation was theoretically predicted in [69, 70]. It was shown that a material with specifically chosen non-unit permittivity and permeability (described by the Lorentz-Drude model) can exhibit anomalous dispersion \( d\omega(k)/dk < 0 \) and, hence, negative group velocity in a certain frequency range. In this range, both permittivity \( \epsilon(\omega) \) and permeability \( \mu(\omega) \) are negative. Thus, already in 1940-s and 1950-s, negative refraction phenomenon was known and well understood, even though it was not experimentally confirmed. After several years, in 1968, Victor Veselago published his famous theoretical review paper about substances with both negative permittivity and permeability in which he envisaged such exotic phenomena as negative refraction and planar-slab lensing [63]. The media with such double-negative properties were subsequently named backward or left-handed. These names are related to the physical properties of the media. As it can be derived from the Maxwell equations for plane waves in form \( e^{j\omega t - jkr} \)

\[ k \times E = \omega \mu H, \quad k \times H = -\omega \epsilon E, \] (4.1)
if simultaneously $\Re\{\varepsilon\} < 0$ and $\Re\{\mu\} < 0$, electromagnetic waves can propagate in such medium and the Poynting vector $\mathbf{S}$ is oppositely directed with respect to the wavevector:

$$\mathbf{S} = \frac{1}{2} \Re(\mathbf{E} \times \mathbf{H}^*) = \frac{|E|^2}{2\omega\Re\{\mu\}} \mathbf{k} = \frac{|H|^2}{2\omega\Re\{\varepsilon\}} \mathbf{k}. \quad (4.2)$$

In this case, vectors $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{k}$ form a left-handed basis of vectors. An alternative way to achieve backward wave propagation using chiral materials was proposed in the early 80-s [55].

Nevertheless, for several decades the exciting research direction about negative refraction slowly developed due to the absence of natural materials with required electric and magnetic properties. Natural substances with negative permittivity (ionosphere at radio frequencies and metals at optical frequencies where the real part of the permittivity dominates over the imaginary part) were known. However, to achieve negative-refraction response, one should ensure that both permittivity and permeability become negative in a single material in the same frequency range. Fulfillment of the latter condition became possible with the developments of artificial composite materials with engineered effective permittivity [71, 72] and permeability [19, 20]. Such materials behave as electric plasma and an artificial magnetic medium (with the Lorentz-type response) in the desired frequency range, respectively. By combining the electric plasma and artificial magnetic materials in a single composite material and tuning its response, one can realize negative refraction phenomenon. This idea was proposed in [1] and experimentally verified in [2]. Note that this idea closely resembles the one proposed back in the 1950-s [69, 70]. Figure 4.2 illustrates the double-negative material formed by straight metallic wires (representing a wire medium with negative permittivity) and double split-ring resonators (representing an artificial magnetic medium with negative permeability).

It is important to note that the negative permeability regime in this case is a manifestation of spatial dispersion of the second order, as it was discussed in Section 3.3. Naturally, negative refractive index can be achieved also in media with spatial dispersion of the first order, namely, chiral nilhility media [6, 57] as it was discussed in Section 3.10. Realization of negative-index media without spatial dispersion effects is possible but requires the use of ferromagnetic materials which are available only at relatively low frequencies.
4.2 Sub-wavelength focusing and transformation optics

In the early 2000-s, it was realized that artificial composite materials with independently tunable electric and magnetic responses can find multiple important applications. In particular, a planar slab of thickness $d_s$ made of a negative-index material with the refractive index $n = \varepsilon/\varepsilon_0 = \mu/\mu_0 = -1$ and firstly proposed in [63] can be used as a perfect lens with sub-wavelength resolution (breaking the established diffraction limit) [73]. Conceptually, such a lens in free space focuses all propagating waves from a point source in phase into one point at a distance $2d_s$ from the source behind the slab [63] [see Fig. 4.3(a)]. The focusing occurs for any location of the source. Reflections at the slab interfaces are zero since its wave impedance is equal to that of free space $\eta_0$ for all propagating modes. At the same time, all evanescent waves radiated from the source decay in free space but grow in the negative-index slab in the same way [73] as shown in Fig. 4.3(b). Thus, both propagating and evanescent fields at the

![Figure 4.2. Illustration of the first double-negative metamaterial.](image)

![Figure 4.3. Concept of a perfect lens.](image)
image are exactly the same (in the amplitude and phase) as those of the original source. In this case the imaging is perfect and not limited by the diffraction limit. It should be noted that in a realistic scenario when the negative-index slab of a finite size possesses some dissipation loss and its material parameters are not precisely tuned, the effect of sub-wavelength focusing can be significantly compromised [29, 74].

Another exciting applications of artificial materials with designed permittivity and permeability are various devices engineered based on transformation optics. The basic idea of transformation optics is the invariance of the Maxwell equations with respect to coordinate transformations (a particular case, spatial inversion transformation, was discussed in Section 3.8). Figure 4.4 illustrates this idea. Let us consider a part of space in the original coordinate system \(xyz\) filled with a material with permittivity \(\varepsilon_{\text{or}}(x, y, z)\) and permeability \(\mu_{\text{or}}(x, y, z)\) (see the left side of the figure). One can transform the coordinate system \(xyz\) into a new arbitrary system \(x'y'z'\), for example, as it is shown on the right side of Fig. 4.4 (the shaded region in the middle denotes the absence of space in the electromagnetic sense). According to the invariance of the Maxwell equations, all the electromagnetic fields inside the space in the transformed coordinate system are the same as those in the original system, i.e., \(E'(x', y', z') = E(x, y, z) \neq E'(x, y, z)\) (here the electric field \(E\) is considered as an example). On the other hand, the material parameters such as permittivity and permeability change under the transformation according to a known law (i.e., \(\varepsilon_{\text{tr}}(x, y, z) \neq \varepsilon_{\text{or}}(x, y, z)\)). The main appealing feature of coordinate transformation is that both parts of space shown in Fig. 4.4 behave equivalently under external illumination. In other words, excited by external sources, both parts of space will scatter the same electromag-

![Figure 4.4. Concept of transformation optics. The shaded region denotes the absence of space in the electromagnetic sense.](image-url)
netic fields (note that inside the two parts of space the fields are very different). This conclusion has several important implications. First, electromagnetic objects with complex (anisotropic and inhomogeneous) material properties may be substituted in some cases by objects of different shape with different material parameters which are easier to implement in practice. Secondly, one can exploit the “empty” region/regions (shown in Fig. 4.4) appeared under coordinate transformation (these regions can be also at the surface of the object) by placing inside some arbitrary objects scattering from which should be eliminated. Alternatively, the same approach is applicable for cancelling scattering from arbitrary inhomogeneities: The inhomogeneities should be covered by an object with specific material properties which under inverse coordinate transformation should represent an object with the properties of free space. These ideas were formulated, probably, for the first time in 1961 in [75].

Interestingly, the general principle of coordinate transformations in electromagnetics was understood even earlier by Albert Einstein. He correctly calculated ray bending of light from stars passing near the Sun on the way to the Earth. The light bending occurs due to the fact that massive objects like the Sun curve the space-time around them (in this curved space light propagates by a straight line and the Maxwell equations are satisfied). Thus, observing star light from the Earth during solar eclipse, it falsely appears that the Sun is surrounded by a transparent shell made of an inhomogeneous material (in fact, this effect cannot be noticed with the naked eye).

In 2006, when it was already demonstrated that metamaterials can exhibit almost arbitrarily electromagnetic response, John Pendry and Ulf Leonhardt independently proposed the idea of electromagnetic invisibility cloak [76, 77]. Apparently, they were not aware of the earlier work about the similar idea [75]. The idea of the invisibility cloak is to place an object to be hidden inside the “empty” regions discussed above. As it was shown in these works, the invisibility cloak material must have inhomogeneous non-unity permeability. Hence, realization of such magnetic response requires spatially dispersive materials (ferromagnetic materials are available only at low frequencies). Experimental demonstrations of an invisibility cloak based on the second-order and first-order spatial dispersion effects were reported [5, 78]. Subsequently, based on the transformation optics, various exotic devices have been proposed (see a review in [79]).
4.3 Wire media

Spatial dispersion often plays the crucial role in artificial wire media, despite the fact that such media usually do not exhibit magnetic response. A simple wire medium represents a two-dimensional periodical array of infinitely long parallel metal wires embedded in a host medium with permittivity $\varepsilon_h$ (see illustration in Fig. 4.5). The axes of the wires are along the z-direction, the wire radius is $r_w$, and the periodicity in the $yz$-plane is $a_w$. The wire media were first studied in [71, 72] as artificial electric plasmas with tunable plasma frequency whose effective permittivity can be smaller than unity (important for antenna lenses applications) or even negative. The latter case did not attract much interest (no propagating waves are allowed in such plasma) until a double-negative metamaterial incorporating a wire medium was proposed in year 2000 [1] (wave propagation is allowed in such material). In earlier works [71, 72], it was assumed that for arbitrary propagation direction under the assumption $r_w < a_w \ll \lambda$, the wire medium can be modelled as a uniaxial dielectric with the local permittivity dyadic $\varepsilon = \varepsilon_{zz} zz + \varepsilon_h (xx + yy)$, where the axial component of permittivity is the function of the frequency only:

$$\varepsilon_{zz}(\omega) = \varepsilon_h \left(1 - \frac{k_p^2}{\varepsilon_h \omega^2/c^2}\right). \quad (4.3)$$

Here $c$ is the speed of light and $k_p$ is the plasma wavenumber. For wire media with very thin PEC (perfect electric conductor) wires with $r_w < 0.1 a_w$, the wavenumber is given by [80]:

$$k_p^2 = \frac{2\pi/a_w^2}{\ln \left(\frac{a_w}{2\pi r_w}\right) + 0.5275}. \quad (4.4)$$

In year 2003, it was demonstrated that this local model of permittivity is applicable only for the case of transverse wave propagation, i.e., when

---

**Figure 4.5.** Illustration of a wire medium.
\( k_z = 0 \) (\( k_z \) is the \( z \)-component of the wavevector) [81]. In the general case of wave propagation, \textit{even in the very large wavelength limit} \( \lambda \gg a_w \), the axial permittivity \( \epsilon_{zz} \) cannot be described locally and strongly depends on the wavevector \( k \):

\[
\epsilon_{zz}(\omega, k_z) = \epsilon_h \left( 1 - \frac{k_p^2}{\epsilon_h \omega^2 / c^2 - k_z^2} \right). \tag{4.5}
\]

For the axial propagation \( k = k_z \), the axial component of permittivity tends to infinity, physically meaning that the axially propagating waves are transverse (\( E_z = H_z = 0 \)). Thus, the wire medium is, probably, the only class of metamaterials, and materials in general, which exhibits strong spatial dispersion even in the very large wavelength limit (as discussed in Section 3.2, spatial dispersion in metamaterials with dipole inclusions is negligible when the wavelength is much larger than the characteristic period). Strong spatial dispersion in an electrically dense wire medium appears due to the electrically long length of the wires in the axial direction.

As it was shown in [82], the effect of strong spatial dispersion does not always occur in wire media and can be neglected under special conditions. In particular, for wire media operating in the near infrared and visible frequency regions, spatial dispersion effect is nearly completely suppressed due to the high kinetic inductance of metal nanowires. For some applications of wire media spatial dispersion can be disadvantageous, e.g., frequency filtering of wave beams. On the other hand, spatial dispersion in microwave wire media leads to new exciting applications such as canalization of wave beams and far-field super-resolution imaging (see a review paper [83]).

### 4.4 Bragg scattering and electromagnetic crystals

In the previous sections, materials whose electromagnetic response can be described by some effective parameters were considered. Homogenization of material properties not always can be performed. Materials with electrically large periodicity (typically larger than half of the wavelength) represent a separate big class of media with unique properties. A brief description of such materials is given in the present section.

It is known that a natural crystal (highly ordered microscopic structure forming a three-dimensional lattice) illuminated by electromagnetic radiation with the wavelength comparable to atomic spacings diffracts some
Metamaterials and role of spatial dispersion

Figure 4.6. Bragg scattering from periodical structures. (a) Scattering from a natural crystal. (b) Scattering from an electromagnetic crystal.

part of the radiation [84] [see Fig. 4.6(a)]. Under the Bragg condition, i.e., when the radiation wavelength is \( \lambda = 2D \cos \theta_i / m \) (\( D \) is the lattice periodicity, \( m \) is a positive integer number, and \( \theta_i \) is the incidence angle), the scattered waves from each plane of atoms in the lattice interfere constructively, yielding maximized specular reflection. Thus, depending on the wavelength of incident radiation and the incident angle, electromagnetic waves can penetrate the lattice or be totally reflected. The latter case corresponds to the so-called electromagnetic stop-band. Lattices with different scattering properties (shape and size of atoms) can have different stop-band characteristics.

Since the lattice periodicity of most natural crystals is of the order of 0.1 nm, the Bragg scattering occurs from them only for radiation of very low wavelengths, such as X-rays. In nature, Bragg scattering can be observed also for optical radiation of the ultraviolet (opal crystal) and visible (moth eye) spectra. It is important to note that electromagnetic properties of crystals cannot be characterized by usual means such as effective material parameters \( \epsilon \) and \( \mu \) at frequencies when \( D \geq \lambda / 2 \). Indeed, scattering response (for any excitation) of a medium described by given effective material properties must be uniquely determined by them. However, due to diffraction phenomena, material parameters of crystals can be introduced only for particular excitation, and for any other excitation the parameters will be modified. Thus, natural crystals exhibit strong spatial dispersion in the specific frequency range.

At this point, it is convenient to classify all electromagnetic volumetric materials with respect to significance of spatial dispersion effects they exhibit. Figure 4.7 schematically displays this classification for different ratios between the lattice periodicity \( D \) and the operating wavelength \( \lambda \).
When $D/\lambda$ is on the order of 0.01 and smaller, the material can be considered as homogeneous and described by local permittivity $\bar{\epsilon}(\omega)$. Permeability $\bar{\mu}(\omega)$ and the bi-anisotropic parameter $\bar{\chi}(\omega)$ are essential only for non-reciprocal media. Spatial dispersion effects (chirality and omega couplings) are weak, non-resonant effects observable only in large compared to the wavelength samples. The examples of the homogeneous materials are the majority of natural crystals at optical and lower frequencies and amorphous natural materials (assuming that $D$ in this case is the averaged periodicity). Materials with $0.01 < D/\lambda < 0.5$ can be described either by formally local parameters such as permittivity $\bar{\epsilon}(\omega)$, permeability $\bar{\mu}(\omega)$, reciprocal $\bar{\kappa}(\omega)$, and non-reciprocal $\bar{\chi}(\omega)$ bi-anisotropic tensors (bi-anisotropic metamaterials), or by strongly non-local permittivity $\bar{\epsilon}(k,\omega)$ (wire media). Periodic materials with $D/\lambda > 0.5$ are not described by material parameters (e.g., natural crystals at X-radiation or artificial electromagnetic crystals at lower frequencies discussed below).

Is it possible, by analogy with metamaterials, to extend the idea of Bragg scattering to the optical frequencies? For simplicity, let us consider a one-dimensional scenario (along the $z$-axis). Let us virtually construct a lattice whose each period includes two layers of dielectric materials with different refractive indices [see Fig. 4.6(b)]. Such a period represents a discontinuity from which incident radiation would be scattered (analogously to planes of atoms in natural crystals). The physics of scattering from this crystal is the same as that of natural crystals. By adjusting the thickness of the dielectric layers and their refractive indices, one can tune the operating frequency of this crystal. Such an artificial one-dimensional crystal is also called a dielectric mirror. It is of great importance for various applications in modern optics and nanophotonics.
frequency, the mirror fully reflects incident light with record high reflection efficiency unattainable with metallic mirrors (typically $98 - 99.9\%$). Likewise, artificial crystals can also have a three-dimensional structure.

Probably, first theoretical works predicting exciting electromagnetic properties of artificial crystals were published in the 1970-s \cite{85, 86}. In year 1987, two experimental papers reported two important effects in three-dimensional artificial crystals: an unprecedented spatial concentration of the electromagnetic field and inhibited spontaneous emission of quantum emitters placed in a crystal \cite{87, 88}. Artificial crystals operating at optical and lower frequency spectra were called photonic and electromagnetic crystals, respectively. At present time, photonic crystals is a relatively well-studied field resulted in a myriad of applications for nanophotonics such as thin-film optics (gratings, lenses, coatings, and mirrors), optical fibers, near-field confinement and guiding of light, and negative refraction (see a review \cite{89}).

### 4.5 Metasurfaces

Unique properties of metamaterials have lead to numerous applications in applied physics, filling the gaps of electromagnetic response of natural materials, alloys, and chemical compounds. Nevertheless, manufacturing of volumetric metamaterials is often a serious challenge even with the present-day technologies, especially when it comes to three-dimensional optical metamaterials. Since most metamaterial applications imply wave propagation through electrically thick volumes, there are significant dissipation losses that can severely distort the device operation. On the other hand, small thickness of metamaterial-based devices can be critical, e.g., for nanophotonics and microwave applications. Finally, designing a volumetric metamaterial, one must often engineer proper electromagnetic response (e.g., polarizability tensors of the inclusions) in all three dimensions.

Fortunately, it is not always necessary to design a bulk metamaterial and analogous response can be achieved with an electrically thin counterpart representing a single-layer metamaterial, the so-called metasurface. This fact can be proven based on the generalized Huygens’ principle \cite{90, 91}. Let us consider a volume $V$ filled with arbitrary sources of electromagnetic radiation, electric and virtual magnetic charges $q_i$ and currents $J_i$ [see the left side of Fig. 4.8(a)]. These sources create electric
Figure 4.8. (a) Illustration of the Huygens’ principle applied to scattering from volumetric electromagnetic sources. (b) Application of the Huygens’ principle to the concept of metamaterials. Electromagnetic response from any volumetric material (or metamaterial) in principal can be always reproduced with a two-dimensional layer of electric and magnetic currents of arbitrary shape. (c) Diffraction of electromagnetic waves on periodic planar structures.

According to the Huygens’ principle, this system of scatterers can be always replaced by an arbitrarily thin layer of specific electric $\mathbf{J}_{eV}$ and magnetic $\mathbf{J}_{mV}$ currents enclosing the volume $V$. The thickness of the layer can be electrically small but non-zero since magnetic currents can be generated only via loops of electric currents with finite thickness. Importantly, the equivalent currents $\mathbf{J}_{eV}$ and magnetic $\mathbf{J}_{mV}$ (surface currents if the layer thickness is electrically negligible) scatter electromagnetic fields only outwards of volume $V$ and these fields are the same as those created by the original system of sources, i.e., $\mathbf{E}$ and $\mathbf{B}$ [see the right side Fig. 4.8(a)]. Such system of currents, which scatter only into one side are called Huygens’ surfaces or Huygens’ sources. This concept possesses important implication to the metamaterials paradigm. Let us consider an arbitrary
volumetric metamaterial sample [shown in the left side of Fig. 4.8(b)] excited by an arbitrary external wave with the wavevector $k_i$. The external wave induces some charges $q_i$ and currents $J_i$ in the inclusions of the sample which re-radiate secondary fields $E_{\text{out}}$ and $B_{\text{out}}$ into space outside volume $V$ which encloses the sample. According to the previously described principle, one can replace the bulky metamaterial sample with the induced polarization charges and currents by equivalent surface currents $J_{eV}$ and $J_{mV}$ which would scatter the same fields $E_{\text{out}}$ and $B_{\text{out}}$ outside volume $V$. Next, knowing these equivalent currents, one can in principle determine appropriate topologies of meta-atoms (placed at the surface of volume $V$) which under excitation by the external wave with $k_i$ would generate the same currents $J_{eV}$ and $J_{mV}$ [see the right side of Fig. 4.8(b)]. Such an arrangement of meta-atoms, namely metasurface, would possess identical electromagnetic response to that of the original metamaterial sample for the given excitation $k_i$. For other excitations, the metasurface generally would not imitate the bulky sample. Thus, one can conclude that in the cases when specific electromagnetic response is required for particular excitation, volumetric metamaterials can be replaced by electrically thin and in general curved metasurfaces. During the last decade, planar metasurface devices have successfully substituted their bulky counterparts for various applications: negative refraction [92], scattering cancellation cloaking [93], reciprocal [94] and non-reciprocal [62] optical activity, absorption [54], multi-channel surfaces [95], general reflection and transmission control ([58, 96–99] and Publications III, IV, V, VI, and VII), etc.

It is important to discuss the place of metasurfaces in the framework of general periodic artificial electrically-thin structures. All planar periodic structures are characterized by the spatial period $D$ (in the surface plane) which determines if they exhibit diffraction effects. Let us consider a two-dimensional planar periodic structure illuminated by a plane wave at an angle $\theta_i$ as shown in Fig. 4.8(c). The structure is located between two media with refractive indices $n_1$ and $n_2$. Calculating the optical length difference between two rays (which pass through the structure at two points separated by $D$) accumulated during their propagation, one can find the condition of diffraction maximum in the first medium at an angle $\theta_r$ (or, alternatively, in the second medium at an angle $\theta_t$). These classical diffraction conditions for reflection and transmission scenarios
read:
\[ n_1 D (\sin \theta_i - \sin \theta_r) = m_r \lambda_0, \quad D (n_1 \sin \theta_i - n_2 \sin \theta_t) = m_t \lambda_0, \quad (4.6) \]

where \( \lambda_0 \) is the wavelength in vacuum, \( m_r \) and \( m_t \) are some integer numbers defining the diffraction orders. As it is seen from these conditions, diffraction occurs only when the period \( D \) is comparable with or larger than the wavelengths in both media. Assuming for simplicity that \( n_1 = n_2 = 1 \), relations (4.6) can be written as
\[ \frac{D}{\lambda_0} = \frac{m_{r,t}}{\sin \theta_i - \sin \theta_{r,t}}. \quad (4.7) \]

From this relation, two well-known implications follow. First, for normally incident waves (\( \theta_i = 0^\circ \)) diffraction occurs only if the periodicity of the structure is \( D \geq \lambda_0 \) \((m_{r,t}/\sin \theta_{r,t} \geq 1 \text{ for any } m_{r,t} \neq 0)\). Second, for other incident angles (\( \theta_i \neq 0^\circ \)) diffraction may appear if \( D \geq \lambda_0/2 \). These two important conditions are often used, for example, in the antenna theory where the periodicity is chosen based on them to avoid diffraction modes (lobes).

It is convenient to classify different periodical structures into several groups based on the distance between adjacent inclusions/elements, \( d \), and type of electromagnetic response (resonant or non-resonant). For clarity, only uniform structures \((D = d)\) are considered. Figure 4.9 illustrates the classification of different types of periodical planar structures with typical topologies of each type.

The first group of periodical structures is formed by dense two-dimensional arrays of electrically long conducting wires. Such structures usually are designed for achieving full reflection regime and are used as lightweight reflectors and screens in various microwave applications (e.g., meshes in microwave ovens). To ensure high reflection level, the periodicity of the wire arrays usually is made small compared to the wavelength in free space \( d \ll \lambda/2 \). An incident wave with electric field oscillating along the wires induces the electric current in them which radiate symmetrically (forward and backward) waves with the phase opposite to that of the incident wave. Therefore, in the forward direction the scattered and incident waves interfere destructively, resulting in low transmission. Probably, the first study on dense arrays of wires appeared in 1898 [100]. Being a two-dimensional analogue of wire medium, dense arrays of wires also exhibit effects of strong spatial dispersion [101]. Dense arrays of wires are non-resonant structures and, due to sub-wavelength distance between the wires, can be homogenized in the plane.
The second group of periodical planar structures is commonly referred to as frequency-selective surfaces or FSS [102]. These structures are usually formed by periodically arranged patches (on a dielectric substrate) or slots (in a metal sheet) of various shapes [102] (see illustration in Fig. 4.9). In contrast to inductive arrays of wires which usually operate at non-resonant frequencies, FSSs are resonant structures (resonance of single inclusions) due to the presence of both inductive and capacitive properties (because of the gaps in the direction of the electric field between adjacent elements). Hence, FSSs can be designed to resonate at frequencies where the element size is comparable to the half-wavelength. Due to the intermediate sizes of the elements ($\lambda/2 \leq d < \lambda$), FSSs do not have diffraction lobes (for the specified incidence angle) and cannot be homogenized to a continuous impedance sheet. FSSs are widely used as microwave reflection/transmission filters and low-profile antennas. Non-uniform non-periodic FSSs elements called reflectarrays and transmitarrays are used mainly as antenna arrays (see more discussions in Section 6.3).

In the end of the twentieth century, two important periodical structures were proposed based on the so-called Jerusalem crosses [103, 104]. They comprise the third group in the present classification. These structures with sub-wavelength inclusions (can be homogenized) resonate due to the capacitive coupling between adjacent elements (so-called distributed resonance). According to classification in some literature [102], these structures can be referred as FSSs with sub-wavelength inclusions. So-called

![Figure 4.9. Classification of different types of periodical planar structures with typical topologies of each type. The red and green regions denote non-resonant and resonant types of structures, respectively. The FSSs with the periodicity $\lambda/2 < D < \lambda$ are inhomogenizable for oblique excitations.](image-url)
mushroom-type high-impedance surfaces (HIS) introduced in [105] can be attributed to the same type of resonant periodic structures. Typically, sizes of their elements are very small compared to the wavelength \(d \sim \lambda/10\) due to additional inductance caused by a dielectric slab covered with a ground plane. Mushroom-type surfaces possess a stop-band for propagating surface waves, and therefore, can be considered as a kind of a two-dimensional electromagnetic crystal.

Diffraction gratings [106] (see illustration in Fig. 4.9) representing the forth group in the present classification have the longest history: Probably the first grating was made in 1785 using strung hairs between two finely threaded screws [107]. In contrast to the previous groups of periodic structures, gratings are intentionally designed to reflect/transmit incident waves mostly into a diffraction mode, therefore, their periodicity is always comparable to or larger than the wavelength \(d > \lambda/2\). Gratings which diffract incident energy towards only one direction are called blazed gratings (e.g., [108]). Usually gratings are designed for the optical range and represent a two-dimensional version of photonic crystals (see discussion in Section 4.4).

Metasurfaces are homogenizable \((\lambda/30 < d < \lambda/2)\) single-layer composites of inclusions which can be referred to both first (non-resonant) and third (resonant) groups in the present classification. Recently, metasurface concept has been extended to non-uniform structures, yielding the so-called metasurface-inspired diffraction gratings [92]. The distinctive feature of such gratings is the sub-wavelength distance between the elements over one period \((d < \lambda/2, \text{while } D > \lambda/2)\). In this case, the grating can be described by the surface averaged impedance.

Metasurfaces can exhibit spatial dispersion effects of the first (bi-anisotropy) and second (artificial magnetism) orders. Thus, on the one hand, the recently proposed concept of metasurfaces resembles and combines previously known planar structures (wire arrays, FSS with sub-wavelength elements, reflect- and transmitarrays, mushroom-type surfaces), while, on the other hand, it generalizes the theory of periodical structures and enables novel previously unattainable devices: wavefront transformers [98], non-reciprocal sheets [62], shadow-free mirrors and absorbers (Publications III, IV, and V), cascaded antennas [53], 100%-efficient ([96,109,110], Publication VII) and multi-channel [95] gratings, etc.
4.6 Homogenization models of metasurfaces

There exist different homogenization models of metasurfaces [111–113]. As an example, let us consider a model based on collective polarizability tensors (dyadics) [94]. The advantage of this model is that, by fixing the required scattering response from a metasurface, one can find all the polarizability components of individual meta-atoms which ensure this response. Interactions between meta-atoms are taken into account. Knowledge of the required polarizabilities of a meta-atom gives us good insight about its possible structure and the composition of the metasurface. Precise tuning of the meta-atom dimensions can be performed using various numerical techniques (e.g., Publication I). The homogenization model works best for uniform arrays (see another homogenization model applied to non-uniform arrays in Publication VI).

Let us assume that a planar periodic uniform metasurface of infinite size (the orientation of the metasurface plane is defined by the normal unit vector \( \mathbf{n} \)) is illuminated by a plane wave with the wavevector \( \mathbf{k}_i \) from one of its sides, as shown in Fig. 4.10 (metasurface is in a homogeneous medium with the impedance \( \eta \)). As it was discussed earlier, the scattered fields from a metasurface are completely determined by equivalent surface averaged electric \( \mathbf{J}_e \) and magnetic \( \mathbf{J}_m \) current densities (note that they are not volumetric currents) induced in the metasurface. Infinitely large sheets of electric and magnetic surface currents scatter two plane waves in the backward and forward directions with respect to the incidence. The corresponding electric fields of these two plane waves are given by [94]

\[
\mathbf{E}_b = -\frac{\eta}{2} \mathbf{J}_e + \frac{1}{2} \mathbf{n} \times \mathbf{J}_m, \quad \mathbf{E}_f = -\frac{\eta}{2} \mathbf{J}_e - \frac{1}{2} \mathbf{n} \times \mathbf{J}_m, \tag{4.8}
\]

where the top and bottom signs denote the cases when \( k_i \uparrow \downarrow \mathbf{n} \) and \( k_i \uparrow \uparrow \mathbf{n} \), respectively. The averaged currents can be expressed through the induced...
Metamaterials and role of spatial dispersion

electric \( p \) and magnetic \( m \) dipole moments of a single inclusion as \( J_e = j \omega p / S \) and \( J_m = j \omega m / S \), where \( S \) is the unit-cell area.

Taking into account that the reflected wave is the back-scattered wave and the transmitted wave is the sum of the incident and forward-scattered waves, the corresponding electric fields in terms of the dipole moments read

\[
E_r = -\frac{j \omega}{2S} (\eta p \mp n \times m), \quad E_t = E_i - \frac{j \omega}{2S} (\eta p \pm n \times m),
\]

(4.9)

where \( E_r \), \( E_t \), and \( E_i \) are the electric fields of the reflected, transmitted, and incident waves, respectively. The induced dipole moments in the metasurface inclusions can be expressed [similarly to (3.21)] in terms of the so-called collective polarizability dyadics and incident fields [94]:

\[
p = \hat{\alpha}_{ee} \cdot E_i + \hat{\alpha}_{em} \cdot H_i, \quad m = \hat{\alpha}_{mm} \cdot H_i + \hat{\alpha}_{me} \cdot E_i.
\]

(4.10)

Here, the hats mark the collective polarizability dyadics which take into account the interactions between the inclusions and can be expressed in terms of the individual polarizability dyadics. To simplify the analysis, let us assume that the metasurface possesses uniaxial symmetry along the normal vector \( n \). In this case, the polarizability dyadics can be decomposed onto symmetric (“co” terms) and antisymmetric (“cross” terms) parts:

\[
\hat{\alpha}_{ee} = \hat{\alpha}_{ee}^{co} t + \hat{\alpha}_{ee}^{cr} t, \quad \hat{\alpha}_{mm} = \hat{\alpha}_{mm}^{co} t + \hat{\alpha}_{mm}^{cr} t
\]

(4.11)

\[
\hat{\alpha}_{em} = \hat{\alpha}_{em}^{co} t + \hat{\alpha}_{em}^{cr} t, \quad \hat{\alpha}_{me} = \hat{\alpha}_{me}^{co} t + \hat{\alpha}_{me}^{cr} t.
\]

By substituting (4.10) and (4.11) in (4.9), one can obtain the expressions for the reflected and transmitted fields in terms of the collective polarizabilities:

\[
E_r = -\frac{j \omega}{2S} \left\{ \eta \hat{\alpha}_{ee}^{co} \mp \hat{\alpha}_{em}^{cr} + \hat{\alpha}_{me}^{cr} - \frac{1}{\eta} \hat{\alpha}_{mm}^{co} \right\} \times t \cdot E_i, \quad \text{(4.12)}
\]

\[
E_t = \left\{ \left[ 1 - \frac{j \omega}{2S} \left( \eta \hat{\alpha}_{ee}^{co} \mp \hat{\alpha}_{em}^{cr} + \hat{\alpha}_{me}^{cr} + \frac{1}{\eta} \hat{\alpha}_{mm}^{co} \right) \right] \times t \cdot E_i, \quad \text{(4.13)}
\]

These expressions obtained in [94] provide a straightforward way for the analysis of metasurfaces formed by general bi-anisotropic inclusions (reciprocal and non-reciprocal). One important implication of these equations is that the zero-reflection regime is not achievable with metasurfaces possessing solely electric or magnetic properties. To eliminate re-
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flection, the metasurface should have either balanced electric and magnetic response (so-called Huygens’ condition \( \eta \hat{\alpha}_{ee} = \hat{\alpha}_{mm}/\eta \)) or proper biaxial anisotropic response [so that all the expressions in the square brackets of (4.12) become zero]. Other important implications of expressions (4.12) and (4.13) are summarized in Table 4.1. They became subjects of a series of publications [53, 54, 58, 62, 94].

<table>
<thead>
<tr>
<th>Electromagnetic response</th>
<th>Possible realizations</th>
<th>Required collective polarizabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polarization rotation in reflection (( E_r = a_0 \mathbf{J}_t \cdot E_i ), where ( a_0 ) is a complex coefficient and ( 0 &lt;</td>
<td>a_0</td>
<td>\leq 1 ))</td>
</tr>
<tr>
<td>Polarization rotation in transmission (( E_t = a_0 \mathbf{J}_t \cdot E_i ))</td>
<td>1) Metasurface with reciprocal uniaxial chiral inclusions [94] (optical activity) 2) Non-reciprocal materials such as magnetized plasmas and ferrites (the Faraday effect)</td>
<td>( \hat{\alpha}<em>{ee} = -\hat{\alpha}</em>{mm} \neq 0 ) ( \eta \hat{\alpha}<em>{ee} = \hat{\alpha}</em>{mm}/\eta \neq 0 )</td>
</tr>
<tr>
<td>Reflection asymmetry for illuminations from opposite sides (( E_r^+ \neq E_r^- ))</td>
<td>Metasurface with reciprocal uniaxial omega inclusions [58]</td>
<td>( \hat{\alpha}<em>{ee} = \hat{\alpha}</em>{mm} \neq 0 )</td>
</tr>
<tr>
<td>Transmission asymmetry for illuminations from opposite sides (( E_t^+ \neq E_t^- ))</td>
<td>Metasurface with non-reciprocal uniaxial artificial moving inclusions [62]</td>
<td>( \hat{\alpha}<em>{ee} = -\hat{\alpha}</em>{mm} \neq 0 )</td>
</tr>
</tbody>
</table>

Table 4.1. Typical response of bi-anisotropic uniaxial metasurfaces for plane-wave illumination.

Besides the presented above homogenization model of metasurfaces there are two other widely known models advantageous for some particular cases. One of them, based on the equivalent impedance matrix, is especially powerful for description of non-uniform metasurfaces as well as metasurfaces containing a ground plane. This model is presented in detail in Section 6.3. Another homogenization model is based on the macroscopic surface susceptibility tensors. It relates the surface averaged polarization densities in the metasurface plane (\( P_S = \mathbf{p}/S \) and \( M_S = \mathbf{m}/(\mu_0 S) \)) to electric \( \mathbf{E}_{av} \) and magnetic \( \mathbf{H}_{av} \) fields averaged over the two sides of the
Metamaterials and role of spatial dispersion:

\[ P_S = \varepsilon_0 \bar{\phi}_{ee} \cdot E_{av} + \sqrt{\mu_0 \varepsilon_0} \bar{\phi}_{em} \cdot H_{av}, \]
\[ M_S = \bar{\phi}_{mm} \cdot H_{av} + \sqrt{\mu_0} \bar{\phi}_{me} \cdot E_{av}, \]

(4.14)

where \( \bar{\phi} \) denotes macroscopic surface-averaged susceptibilities. After some derivations, the susceptibilities can be expressed in terms of the collective polarizabilities [defined by (4.10)]:

\[ \bar{\phi}_{ee} = \frac{1}{\varepsilon_0} (\Delta_p)^{-1} \left[ \bar{\alpha}_{ee} + \frac{j \omega \bar{\alpha}_{em}}{2 \eta_0} \cdot (\bar{S} \bar{I}_t - \bar{\alpha}_{mm} \frac{j \omega}{2 \eta_0} \cdot \bar{\alpha}_{me}) \right], \]
\[ \bar{\phi}_{em} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} (\Delta_p)^{-1} \left[ \bar{\alpha}_{em} + \frac{j \omega \bar{\alpha}_{em}}{2 \eta_0} \cdot (\bar{S} \bar{I}_t - \bar{\alpha}_{mm} \frac{j \omega}{2 \eta_0} \cdot \bar{\alpha}_{me}) \right], \]
\[ \bar{\phi}_{me} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} (\Delta_m)^{-1} \left[ \bar{\alpha}_{me} + \frac{j \omega \bar{\alpha}_{me}}{2 \eta_0} \cdot (\bar{S} \bar{I}_t - \bar{\alpha}_{ee} \frac{j \omega}{2 \eta_0} \cdot \bar{\alpha}_{me}) \right], \]
\[ \bar{\phi}_{mm} = \frac{1}{\mu_0} (\Delta_m)^{-1} \left[ \bar{\alpha}_{mm} + \frac{j \omega \bar{\alpha}_{me}}{2 \eta_0} \cdot (\bar{S} \bar{I}_t - \bar{\alpha}_{ee} \frac{j \omega}{2 \eta_0} \cdot \bar{\alpha}_{me}) \right], \]

(4.15)

where

\[ \Delta_p = S \bar{I}_t - \bar{\alpha}_{ee} \frac{j \omega}{2} \eta_0 + \frac{\omega^2 \bar{\alpha}_{em}}{4} \cdot (S \bar{I}_t - \bar{\alpha}_{mm} \frac{j \omega}{2 \eta_0} \cdot \bar{\alpha}_{me}), \]
\[ \Delta_m = S \bar{I}_t - \bar{\alpha}_{mm} \frac{j \omega}{2} \eta_0 + \frac{\omega^2 \bar{\alpha}_{me}}{4} \cdot (S \bar{I}_t - \bar{\alpha}_{ee} \frac{j \omega}{2 \eta_0} \cdot \bar{\alpha}_{me}). \]

(4.16)

The main advantage of the homogenization model based on the macroscopic surface susceptibility tensors is that, knowing the required tensors for some specific electromagnetic response, one can immediately see if the metasurface inclusions must be lossy (\( \Im\{\bar{\phi}_{ij}\} < 0 \)) or possess some gain (\( \Im\{\bar{\phi}_{ij}\} > 0 \)).

Although bi-anisotropy phenomena have been known and studied for a long time, there is just a few studies on bi-anisotropic metasurfaces (spatial dispersion effects of the first order). The main objective of the present dissertation is to explore what exotic and unique functionalities are available using metasurfaces with general bi-anisotropic response. The synthesis of novel spatially dispersive metasurfaces (bi-anisotropic as well as strongly non-local) is also presented.
Part II

Results
5. Single bi-anisotropic inclusions

Electromagnetic properties of a metamaterial are determined by properties and spatial arrangement of its individual constituents. This idea underlies the very concept of metamaterials. Therefore, most studies on electromagnetic artificial materials and devices require engineering individual inclusions with prescribed properties (e.g., polarizabilities).

The relations between the macroscopic properties of a composite material and the microscopic properties of its constituents were discussed in Section 3.5. Expressions (3.19) describe these relations for materials with dilute concentrations of inclusions. If the inclusions concentration is not low so that the interaction between the inclusions should be taken into account, the macroscopic and microscopic quantities can be related using one of the approximate mixing rules. Probably, the simplest, while effective, rule is the so-called Maxwell Garnett approach [114, 115]. Extension of this approach to general bi-anisotropic materials can be found, e.g., in [116, 117].

When the required polarizabilities of individual inclusions are known, one can estimate what inclusions topology and material can be used to ensure the required microscopic response. In some simple cases, it is possible to roughly determine the dimensions and material characteristics of the inclusions based on known theoretical models. However, in most cases of bi-anisotropic inclusions, the correspondence between their polarizabilities and internal structure can be set up only with the use of numerical and semi-analytical techniques.

5.1 Polarizabilities extraction techniques

It is convenient to classify all approaches for polarizability retrieval of a single scatterer into two groups: Fully analytical and semi-analytical ap-
Single bi-anisotropic inclusions

Probably, the most universal approximate technique belonging to the first group is one which models a scatterer as an effective circuit consisting of reactive and resistive lumped elements [118, 119]. Naturally, if the scatterer is anisotropic, the effective circuit schematic depends on the incidence direction and polarization. Although this technique is usually applicable only for scatterers with sub-wavelength dimensions, it adequately models spatial dispersion effects of the first and second orders. Another fully analytical approach applicable for dielectric spherical scatterers with arbitrary sizes and refractive indices relies on the spherical harmonic expansion theory formulated by Mie in 1908 [120]. Subsequently, this theory was extended to scatterers of arbitrary shapes [121].

Recently, the extended Mie theory has been used by many researchers to study the multipolar behaviour of scatterers of different shapes, e.g. [122, 123]. These techniques are semi-analytical because they imply the knowledge of scattered fields from the scatterer under specific illumination (these fields are commonly determined using full-wave simulations). Writing the scattered fields in terms of vector spherical harmonics, multipolar moments of an arbitrary scatterer can be calculated. However, this approach implies computationally heavy integrations of scattered fields over the sphere surrounding the scatterer that complicates the implementation of the method. Furthermore, it appears problematic to use such methods for extracting polarizabilities from experimentally measured response of the inclusion.

Alternatively, polarizability retrieval can be performed if the induced currents and charges in the scatterer are known (e.g., based on simple theoretical models or full-wave simulations) under specific illumination. Such semi-analytical approach is rigorous and applicable beyond the long-wavelength approximation [124, 125]. The disadvantage of this method is expressed in the fact that it requires relatively complicated integration over the charge volume of the scatterer and for different scatterers different volumetric integrals should be calculated.

A separate class of methods constitutes polarizability extraction techniques for a scatterer located inside a uniform periodic array or a waveguide [59, 126–128]. Such techniques retrieve the polarizability tensors based on the known coupling coefficients in periodic arrays or waveguides and numerically calculated (or measured) scattering coefficients. It should be noted that the relations between the scattering coefficients from

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1Publications on this method appeared after Publication I.
the array and polarizabilities of the individual scatterers are not trivial and in most cases are just approximate.

5.1.1 Author’s contribution

Since by definition metamaterials are composed of scatterers with sub-wavelength dimensions, the electric and magnetic dipolar moments are the most significant and important moments in the Mie expansion for such scatterers. This assumption allows one to dramatically simplify the scattering-based polarizability retrieval and propose a straightforward semi-analytical method for extracting polarizability tensors of an arbitrary small scatterer from its response in the far zone. This technique for the polarizability extraction is based on the assumption that in the far zone the contributions to the scattered fields from electric and magnetic dipoles dominate over those from higher-order multipole moments of the scatterer. The validity of this assumption can be tested by numerical evaluation of the scattering pattern or, experimentally, by repeating the measurements with several orientations of the scatterer. It should be noted that for the needs of designing metasurfaces and metamaterials the knowledge of higher-order polarizabilities is usually not required (the plane-wave reflection and transmission coefficients from regular metasurfaces as well as the effective parameters of metamaterials are defined by dipolar responses of their inclusions).

In contrast to the approach based on spherical harmonic expansion [122, 123] which requires knowledge of scattered fields at all points on a sphere surrounding the scatterer, the far-field approach allows one to determine all 36 polarizability components by probing scattered fields only in 6 specific directions using 6 specific wave illuminations. Alternatively, to extract one specific polarizability component of the scatterer, the method implies determination of the scattered fields only in two special directions for two wave illuminations. This feature significantly simplifies the realization of the method in full-wave simulations. Furthermore, the discrete and minimal number of directions in which the scattered fields must be probed allows one to utilize the method also experimentally.

The basic idea of the polarizability retrieval technique based on the scattered far-field is the following. To determine one specific polarizability component of an arbitrary bi-anisotropic scatterer (let us consider $\alpha_{11}^{ee}$, where numerical indices 1, 2, and 3 represent the projections on the $x$, $y$, and $z$ axes, respectively), it should be illuminated by two incident
plane waves (equivalent to a single standing wave), e.g., as illustrated in Fig. 5.1. According to (3.21), component $\alpha_{11}^{ee}$ can be expressed in terms of the induced dipole moments in the scatterer:

$$\alpha_{11}^{ee} = \frac{1}{2E_i} (p_1^+ + p_1^-),$$  \hfill (5.1)

where $p_1^+$ and $p_1^-$ stand for the induced electric dipole moments (the $x$ components) under excitation by the plane waves propagating along the $+z$ and $-z$ directions, respectively.

Under the assumption of electrically small size of the scatterer, the induced dipole moments can be expressed in terms of the scattered fields in the far zone [14]:

$$E_{sc} = \frac{k^2}{4\pi\epsilon_0 r_{sc}} e^{-jkr_{sc}} \left[ (n_{sc} \times p) \times n_{sc} - \frac{1}{c\mu_0} n_{sc} \times m \right],$$  \hfill (5.2)

where $n_{sc}$ is the unit vector in the direction of observation of the scattered field and $r_{sc}$ is the distance between the scatterer and the observation point. In the considered example, the $x$ component of the electric dipole moment can be found,

$$p_1^\pm = \frac{2\pi\epsilon_0 r_{sc} e^{jk r_{sc}}}{k^2} ( +z E_{sc1}^\pm - z E_{sc1}^\pm ),$$  \hfill (5.3)

where notation $+z E_{sc1}^-$ denotes the $x$ component of the scattered field in the $+z$ direction if the scatterer is illuminated by the incident wave propagating towards the $-z$ direction. The scattered fields can be calculated using full-wave simulations or measured. By combining (5.1) and (5.3), one can calculate the polarizability component $\alpha_{11}^{ee}$ of the scatterer. Analogously, other polarizability components can be retrieved.

The presented polarizability retrieval method applicable for arbitrary, electrically small bi-anisotropic scatterers is the subject of Publication I.
For more detail on the method as well as its implementation for polarizability retrieval of two bi-anisotropic scatterers (reciprocal and non-reciprocal), the reader is referred to the full text of the publication.

5.2 Purely bi-anisotropic scatterers

As it was discussed in Section 3.5, spatial dispersion effects of the first (bi-anisotropy) and the second (artificial magnetism) orders in a medium can be enhanced through elaborate engineering of the inclusions. For example, arrays of chiral inclusions with balanced electric and magnetic properties exhibit equally strong bi-anisotropic properties compared to the electric and magnetic responses (see more detail in Section 6.2.1). Furthermore, by precise engineering, one can synthesize a metamaterial whose bi-anisotropic properties (indirect polarization response or coupling) are more pronounced than electric and magnetic ones [6] (direct polarization response); see also Section 3.10. Although these conclusions are correct for composite materials, they are not necessary applicable to the individual constituents. Indeed, as is seen from expressions (3.19) for dilute inclusion concentrations, maximized indirect polarization coupling of a medium ($\vec{\kappa} \neq 0$, $\vec{\epsilon} = \vec{\mu} = 0$) does not imply that its inclusions exhibit maximal indirect coupling. On the contrary, as it follows from the expressions, both direct and indirect coupling effects in the individual inclusions exist in this case and can be of the same order ($\vec{\alpha}_{ee} \neq 0$, $\vec{\alpha}_{mm} \neq 0$, $\vec{\alpha}_{em} \neq 0$). Thus, the scenario of individual scatterers with maximized indirect polarization coupling is not trivial and deserves a separate study.

Some fundamental restrictions on the attainable values of the polarizabilities of individual scatterers are well known. The time-symmetry restrictions are dictated by the Onsager-Casimir principle [see (3.46)]. For all passive scatterers the imaginary part of the six-dyadic $\bar{\alpha} - \bar{\alpha}^\dagger$ is positive-definite (see e.g., [10]), where the six-dyadic $\bar{\alpha}$ is formed by the four polarizabilities as

$$\bar{\alpha} = \begin{bmatrix} \bar{\alpha}_{ee} & \bar{\alpha}_{em} \\ \bar{\alpha}_{me} & \bar{\alpha}_{mm} \end{bmatrix}.$$  \hspace{1cm} (5.4)

The causality requirement leads to Kramers-Kronig relations for the polarizabilities and to sum rules [10].

Classical constraints on the polarizabilities of an arbitrary scatterer do not impose an explicit limitation on the relative strength of the indirect
polization coupling compared to the direct ones. Assuming that an incident wave excites polarization in a scatterer which is described by four main polarizability components $\alpha_{ee}$, $\alpha_{mm}$, $\alpha_{em}$, and $\alpha_{me}$ (approximately correct for a number of classical scatterers such as uniaxial helices, splitting resonators, and omega inclusions), the tensor polarizability quantities can be replaced by scalar ones. Since the components $\alpha_{em}$ and $\alpha_{me}$ are related through (3.46), it is convenient to describe the strength of direct and indirect polarization couplings by the products $\alpha_{ee}\alpha_{mm}$ and $\alpha_{em}\alpha_{me}$, respectively. It should be noted that the passivity limitation on the imaginary part of the six-dyadic $\bar{\alpha} - \bar{\alpha}^\dagger$ for the case of scalar polarizabilities leads to the condition $\Im \{\alpha_{em}\alpha_{me}\} < \Im \{\alpha_{ee}\alpha_{mm}\}$ [46]. Nevertheless, the real parts, or the absolute values, of these products are not restricted by the passivity.

Based on known results [46, 118, 129], it is expected that the indirect coupling effects cannot be stronger than the direct ones. For example, in [129], the following constraint was reported:

$$|\alpha_{em}\alpha_{me}| \leq |\alpha_{ee}\alpha_{mm}|.$$ (5.5)

However, this limitation exists only under the assumption that the scatterer is modeled as a single-resonant RLC circuit. In such a scatterer, both induced electric and magnetic dipole moments are formed by the same current distribution and have nearly the same frequency dispersion close to the fundamental resonance.

### 5.2.1 Author’s contribution

As it was demonstrated in Publication II, if a scatterer possesses multi-resonance behaviour (the electric and magnetic dipole moments resonate at different frequencies and are formed by different current distributions), inequality (5.5) generally does not hold. In order to maximize the indirect coupling and suppress the direct polarization effects in the scatterer, it was proposed to push the idea of multi-mode scatterers to the limit by designing a nanodimer whose two constituents have electric polarizabilities of the opposite signs. An example of such nanodimer is shown in Fig. 5.2. It consists of two closely spaced, deeply sub-wavelength dielectric spheres with the relative permittivity $\epsilon_1$ and $\epsilon_2$ and equal radii.

Despite the fact that dimer scatterers have been intensively studied in the past (see e.g., [130]), their potentials for enhancing bi-anisotropic coupling as compared to the direct ones appear not to be realized. Commonly,
previous works were devoted to designing dimers that have no excited electric dipole moment under illumination of an incident plane wave (e.g., to realize scattering cancellation cloaking:

\[
p = \bar{\alpha}_{ee} \cdot E_i + \bar{\alpha}_{em} \cdot H_i \approx 0.
\] (5.6)

However, this condition is drastically different from the goal to minimize the direct coupling. Indeed, Eq. (5.6) implies that the indirect coupling should be of the same order as the direct ones.

A notion of “purely bi-anisotropic scatterer” implies a unique scenario when the magnetic polarization of the scatterer is generated predominantly by external electric field and vice versa:

\[
p \approx \bar{\alpha}_{em} \cdot H_i, \quad m \approx \bar{\alpha}_{me} \cdot E_i.
\] (5.7)

Properly choosing the materials and the dimensions of the nanodimer spheres, one can ensure that the electric polarizabilities of the spheres have equal amplitudes and opposite signs, resulting in nearly zero total electric polarization coupling of the nanodimer (\(\alpha_{ee} \approx 0\)). On the other hand, this configuration of the opposite electric dipoles forms an electric quadrupole moment and a magnetic dipole moment [which, according to (5.7), corresponds to the non-zero indirect coupling coefficient \(\alpha_{me}\)].

Importantly, under illumination by an external magnetic field, predominantly electric dipole moment is induced in the nanodimer, according to (5.7). The external magnetic field due to the Faraday law creates circulation of the electric field around the center of the dimer. Due to the opposite electric polarizabilities of the nanodimer spheres, this circulating external electric field excites non-circulating electric dipoles directed along the same direction. Therefore, the total magnetic response at excitation by magnetic fields is almost completely suppressed, and the scatterer radiates as a pure electric dipole (electric quadrupole moment is also very small).

Using polarizability retrieval approach discussed in Section 5.1.1, the polarizability of the designed nanodimer were determined, revealing that in a very wide frequency range the magneto-electric coupling coefficient
Single bi-anisotropic inclusions

(\(\alpha_{me}\)) is at least one order of magnitude stronger than both electric and magnetic ones (\(\alpha_{ee}\) and \(\alpha_{mm}\)), so that the limitation (5.5) is largely exceeded (more than 27 times). The unique property of pure bi-anisotropic response implies several exciting consequences. For example, such a nanodimer has an equivalent response (along its axis) to that of Kerker’s magnetoelectric sphere with \(\epsilon = \mu\) [131]. In this regime, the backscattering from the dimer is zero for any polarization of the incident wave. Moreover, the forward scattered wave has always the same polarization as that of the incident wave. Another interesting property of purely bi-anisotropic scatterers is predominantly lateral scattering for specific illuminations. In this case the scatterer radiates very weakly in the forward and backward directions, while scattering along the lateral directions is relatively strong. This effect can find important applications in nanophotonics for engineering optical forces.

It should be noted that the designed nanodimer can be used also as an inclusion in various metasurfaces. Positioned in a two-dimensional or three-dimensional periodical array, the nanodimer will no longer exhibit purely bi-anisotropic response due to the interaction between the array elements. However, the effective (collective) direct polarizabilities \(\bar{\alpha}_{ee}\) and \(\bar{\alpha}_{mm}\) are still smaller than the indirect ones \(\bar{\alpha}_{em}\) and \(\bar{\alpha}_{me}\) (the relations between the effective and individual polarizabilities for arbitrary dipolar particles were derived in [94]). Such a combination of the effective polarizabilities is unique and especially necessary for metasurface designs which require \(|\bar{\alpha}_{ee}\bar{\alpha}_{mm}| \neq |\bar{\alpha}_{em}\bar{\alpha}_{me}|\). In particular, a metasurface formed of properly tuned nanodimers can asymmetrically reflect incident radiation from opposite sides with +1 and −1 reflection coefficients, mimicking an electric wall and a magnetic wall in a single-layer device. In contrast, a metasurface made of usual wire omega inclusions reflects incident radiation with ±j reflection coefficients from the opposite sides.

The results outlined in this section as well as additional results on purely bi-anisotropic dimers with dimensions comparable to the wavelength can be found in Publication II.
6. Spatially dispersive metasurfaces and their applications

As it was discussed above, spatial dispersion effects play an important role in artificial composite materials. Using intelligently engineered metallic or dielectric inclusions, one can achieve magnetic properties without any magnetic materials. Moreover, it is possible to attain various bi-anisotropic effects which are several orders of magnitudes stronger than those observed in natural materials. Although the first designed metasurfaces incorporated only electrically polarized elements [100], it was subsequently realized that extending their functionalities requires the use of magnetically polarized inclusions. For example, to create low-reflective metasurfaces for manipulation of transmitted waves, one should ensure proper magnetic polarization properties of their inclusions [132, 133] (see also Section 4.6). As is seen from (4.12) and (4.13) as well as from Table 4.1, bi-anisotropic effects significantly expand the design freedom and functionalities available with metasurfaces. As it will be shown in Section 6.3.1, some important applications require bi-anisotropy of a specific kind. The present section is devoted to various novel and unique applications of spatially dispersive metasurfaces. Obviously, the described examples do not embrace all of the possible applications of such metasurfaces, and there is much more to be discovered in this field.

6.1 Shadow-free gradient reflectors

Reflectors are natural or artificial structures which fully reflect incident electromagnetic radiation of specific frequency towards some direction. Conventional mirrors, known since the dawn of civilization, obey the simple law of reflection: The reflection angle is equal to the incidence angle. If the mirror surface can be engineered to enable general control over the phase of the reflected wave (gradient surface), it is possible to change
the direction of the reflected energy flow at will [92]. Developments in the field of antennas enabled creation of reflectarrays [134], layers with any desired phase of reflection at microwaves. Conceptually, reflectarrays are conventional mirrors, modified by some additional phase-shifting elements positioned close to fully reflecting surfaces.

Most known reflectors, from microwave parabolic dish antennas to optical mirrors and gratings, incorporate a metal back surface (a ground plane). Such surface is an essential element which ensures zero transmission and high level of reflection. On the other hand, the presence of a metal ground plane forbids transmission at all practically important frequencies (casting a “shadow” from the device) and limits application possibilities. One can envision several unique devices enabled with shadow-free reflectors. At microwaves, being practically transparent for infrared and visible radiation, such reflectors have a clear potential for breakthroughs in the design of antennas for various applications, in particular for satellites and for radio astronomy. For example, a shadow-free reflector can work as a large parabolic reflector for radio waves, while being deposited on solar-cell panels of a satellite, not disturbing the panel operation [see Fig. 6.1(a)]. Such compact deployment would save room on the spacecraft. In radio astronomy as well as in satellite technologies, it

Figure 6.1. Applications of shadow-free reflectors. (a) Deposition of the reflector antenna onto a solar panel of a satellite. The reflector is transparent for visible light and does not disturb operation of the solar panel. (b) Cascading different shadow-free reflectors in a stack. Each reflector performs a particular functionality at specific frequency and practically does not interact with other reflectors.
will become possible to realize multi-frequency or multi-beam antennas using a parallel stack of shadow-free reflectors, each tuned to emulate a parabolic dish antenna at different frequency and, if needed, with a different focal point [see Fig. 6.1(b)].

Clearly, such shadow-free gradient reflectors possess electromagnetic properties which are not available in natural materials, and potential realizations require the use of metamaterials or metasurfaces. The last few years have witnessed remarkable progress in the development of gradient metasurfaces capable of general control over the reflected and transmitted wavefronts. However, while metasurfaces tailoring wavefronts in transmission [92, 132, 133] usually consist of sub-wavelength inclusions and are transparent (to some extent) outside of the operating frequency band, most metasurfaces manipulating reflection are metal-backed and create reflections over the entire frequency range, see e.g., [135–137].

6.1.1 Author's contribution

Design of a shadow-free gradient reflector must lack a ground plane. In this case, the full-reflection regime can be achieved with a two-dimensional array of sub-wavelength resonant meta-atoms. Due to sub-wavelength distances between the adjacent inclusions, their induced dipole moments (a discrete array of such dipole moments) can effectively model a sheet of continuous electric and magnetic currents. Through adjustments of the inclusions geometries it is possible to change the distribution of the effective current and tune the desired reflection properties of the metasurface. To control the wavefront of reflection, the inclusions in the array should be engineered in such a way that they re-radiate waves in the backward direction with different phases \( E_b = e^{j\Phi_b(x)}E_i \) (varying along the \( x \) coordinate), while in the forward direction they scatter waves with the same phase, opposite to that of the incident plane wave \( E_f = -E_i \) (to destructively interfere with the incident wave, yielding zero transmission behind the metasurface). This feature of controllable asymmetric scattering dramatically distinguishes shadow-free reflectors from other reflectors utilizing a ground plane [135–137]. Figure 6.2 illustrates the asymmetric scattering property by the example of a gradient metasurface with linear phase variations over the \( x \) coordinate. In this example, the metasurface reflects normally incident plane waves at an angle of 45°.

Let us explore all possible scenarios for realization of a shadow-free reflector using general reciprocal bi-anisotropic metasurfaces. Assuming
that the metasurface is locally uniform (each unit element of the metasurface is designed under the assumption that it is located in a uniform periodical array of identical elements), one can write scattered fields from the metasurface at some arbitrary point. According to (4.8) and (4.10), the electric fields of the backward $E_b$ and forward $E_f$ (local fields) scattered plane waves from the metasurface illuminated by an incident plane wave are given by

$$E_b = -\frac{j\omega}{2S} \left( \eta \hat{\alpha}_{ee} + 2\hat{\alpha}_{em} - \frac{1}{\eta} \hat{\alpha}_{mm} \right) E_i,$$

$$E_f = -\frac{j\omega}{2S} \left( \eta \hat{\alpha}_{ee} + \frac{1}{\eta} \hat{\alpha}_{mm} \right) E_i. \quad (6.1)$$

Here the metasurface is polarized only under illumination by waves of one specific polarization. As is seen from the equations, asymmetric scattering properties in the backward and forward directions cannot be accomplished if the metasurface inclusions possess solely electric or magnetic response. It should be noted that, in principle, the asymmetric scattering can be achieved by exploiting an array of anisotropic inclusions with zero $\hat{\alpha}_{em}$ and non-zero $\hat{\alpha}_{ee}$ and $\hat{\alpha}_{mm}$ polarizabilities such as simple electric and magnetic dipoles. However, the design of such an array becomes very challenging since the inclusions of these two types must operate in the metasurface at non-resonant frequencies [58] and be adjusted individually and very precisely, taking into account their mutual interactions. One can overcome these difficulties by exploiting inclusions with strong bi-anisotropic coupling of the omega type $\hat{\alpha}_{em} \neq 0$. Such inclusions at the resonance create the same forward-scattered fields ($E_f = \text{const}$), while different electromagnetic coupling strength $\hat{\alpha}_{em}$ ensures the desired dif-

---

**Figure 6.2.** Field distribution of the incident wave and waves scattered from a gradient metasurface. The metasurface reflects normally incident radiation at an angle of $45^\circ$. The forward scattered waves from each inclusion have identical phases and opposite to the phase of the incident wave, which yields zero transmission. In the backward direction the inclusions radiate with discrete phase shifts from 0 to $5\pi/3$, exhibiting anomalous reflection.
6.2 “Invisible” absorbers

Total absorption of electromagnetic radiation requires elimination of all wave propagation channels: reflection, transmission, and scattering. It is known that incident electromagnetic energy can be nearly fully absorbed in thin layers, but only in a narrow frequency band [138, 139]. The maximal absorption bandwidth of any passive layer obeys a fundamental limitation (as follows from the causality principle) and is proportional to the layer thickness [140]. On the other hand, apparently it has not been noticed before that there is no such fundamental limitation on the frequency range in which the reflection from a thin resonant absorbing layer can be made negligible. Conceptually, it is possible to realize a thin layer which fully absorbs the incident power in a narrow frequency band and allows the wave to freely pass through at other frequencies, thus, producing no reflections at all (within the band where the layer remains electrically thin and the inclusions forming the absorber remain electrically small). The existence of such a structure does not contradict known fundamental limitations. Obviously, exploitation of the opportunity to design a resonant absorber, which is transparent outside of the absorption band, could open up a number of novel possibilities in applications, for example, in ultra-thin filters for wave trapping, selective multi-frequency bolometers and sensors. Such an all-frequencies-matched resonant absorber would be “invisible” from the illuminated sides, still acting as a band-stop filter in transmission.

In fact, most of the known designs of thin absorbers contain a continuous metal ground plane (a mirror) behind the absorbing layer (e.g., [138,139]). The mirror obviously produces nearly full reflection outside of the absorption band. Although this feature is crucial for some applications, it forbids designing resonant absorbers which are transparent outside of the absorption band. The use of a ground plane can be avoided in absorbers based on arrays of subwavelength Huygens’ elements (Huygens’ metasurfaces, see Section 4.5) that possess the appropriate level of dissipative
loss. Such Huygens’ elements scatter secondary waves only in the forward direction (without reflection) which destructively interfere with the incident wave, yielding zero transmission. Some topologies of Huygens’ inclusions were introduced in [131, 141, 142]. Subsequently, Huygens’ inclusions of different topologies have been used as structural elements in sheets to control transmission wavefronts [132, 133, 143]. Recently, there have been proposed several topologies of absorbers based on cut wire arrays separated by a dielectric layer (see e.g., [144, 145]). However, in all these structures the Huygens’ balance between the electric and magnetic responses (which is necessary for cancellation of the reflected waves) holds only inside a narrow-frequency band for which the dimensions have been optimized. Outside of this band, reflections appear due to prevailing excitation of either electric or magnetic modes. Figure 6.3(a) illustrates this scenario by plotting electric and magnetic polarizability of individual inclusions of typical absorbers versus frequency. Here the polarizabilities are modeled using the conventional Lorentz dispersion model, which, near the resonance, qualitatively describes electric and magnetic dipolar responses of the inclusions of an arbitrary shape. The physical reason for inevitable reflections appearing outside the resonance is that different resonant modes exhibit different frequency dispersions. Thus, within this scenario it is impossible to realize a resonant absorber which is reflectionless over a wide frequency range.

**Figure 6.3.** (a) Illustration of the absorption regime in a metasurface with unit cells containing electrically and magnetically polarizable inclusions resonating at the same frequency $\omega_e = \omega_m$ and having different frequency dispersions. Red and blue lines depict, respectively, normalized electric and magnetic polarizabilities. Solid and dashed lines show the real and imaginary parts of the polarizabilities, respectively. Frequency where full absorption regime occurs is $\omega_a$. More detail about the calculation parameters for the Lorentz dispersion model are given in Publication V. (b) Individual normalized polarizabilities of the designed double-turn helical inclusions. Frequency dispersions of all the polarizabilities are nearly the same in a wide frequency range.
6.2.1 Author's contribution

To ensure the Huygens’ balance between the electric and magnetic dipole moments induced in the metasurface over a wide frequency range (to cancel reflection), one can exploit inclusions designed in such a manner that their both electric and magnetic polarizations are created by excitation of the same resonant mode. The induced current distribution of this resonant mode should be such that both electric and magnetic moments are excited and can be tuned to the desired balance. These properties are inherent for bi-anisotropic elements such as chiral or omega wire inclusions. Therefore, the presence of spatial dispersion effects in the metasurface are important for realizing “invisible” absorbers. Examining expressions (4.12) and (4.13), however, manifests that the requirements of total symmetric (from both sides) absorption in a uniaxial metasurface $E_r = E_i = 0$ can be satisfied only if all bi-anisotropic effects (reciprocal and non-reciprocal) are suppressed. In order to surmount this obstacle, one can use bi-anisotropic inclusions on the level of the unit cell but arranging the inclusions in the array so that the bi-anisotropy is compensated on the level of the entire array. Such scenario can be achieved in an array of alternating bi-anisotropic inclusions of two sorts which differ only by the sign of the electromagnetic coupling parameter (a racemic array). Therefore, combination of them yields overall bi-anisotropy compensation. Figure 6.4 depicts an example of the metasurface which consists of bi-anisotropic resonant inclusions (smooth chiral helices) with the opposite handedness and balanced electric and magnetic responses. The balanced properties can be ensured simply by proper choosing the helix dimensions. The loss factors of the electric and magnetic polarizabilities of

![Figure 6.4](image)

Figure 6.4. Arrangement of the double-turn helical inclusions in the array. Blue and red colors denote right- and left-handed helices, respectively.
the helices are identical because both polarizabilities depend on the sum of the radiation resistances of a small electric dipole and a small magnetic dipole excited in the helix [118]. Thus, the electric and magnetic polarizabilities of the inclusions have nearly identical dispersions and the proposed array of the inclusions acts as a Huygens’ surface in a very wide frequency range (the upper bound is the frequency where the higher-order resonance occurs in the helices). The individual polarizabilities of the designed helices retrieved using the technique reported in Publication I are plotted in Fig. 6.3(b). The non-zero electromagnetic polarizability $\alpha_{em}$ of a single helix is compensated in the array of the left- and right-handed helices. The full-absorption regime is accomplished in the metasurface due to a proper level of dissipative loss in the helical inclusions (helices are made of a nichrome alloy).

More detail on theoretical and experimental characterization of “invisible” absorbing metasurfaces can be found in Publication V.

### 6.3 Exact synthesis of metasurfaces for wavefront control

Control of light by tailored light-matter interaction is a key aspect of optics. Refractive optics provides a broad range of functionalities, utilizing phase accumulation due to light propagation through bulky optical components such as conventional lenses and waveplates. For many applications, however, it is preferable to have flat optical components which operate based on diffraction. Engineering high-efficiency components is a non-trivial task and requires appropriate analytical and numerical techniques. For the simplest reflection scenario, for which an incident plane wave should be diffracted (steered) into a desired reflection direction, high performance can be achieved with blazed gratings [106]. Typically, they consist of grooves patterned on a metal surface and provide up to 80%–99.6% of reflection efficiency into the $n = -1$ diffraction order (Littrow configuration), which corresponds to reflection in the direction of incidence (retroreflection) [108]. However, high-efficiency diffraction has not been obtained under conditions where the incident and anomalously reflected beams include an angle approaching 90°.

Metasurfaces can impart arbitrary phase profiles on an incident beam, enabling a number of devices (e.g., holograms, complex lenses, beam splitters, etc.). Planar metasurfaces can be fabricated along the lines of standard microelectronics. This is an advantage compared to conventional...
grooved gratings, which require more demanding fabrication processes [146]. However, it was very recently reported [96, 110, 147] that known metasurface designs can efficiently operate only for moderate angular separations between the incident and reflected beams (not exceeding approximately the angle of $45^\circ$). For larger separations, inevitable parasitic reflections into undesired directions appear, reducing efficiency. This drawback is related to the approach used for the metasurface design which approximates the metasurface as locally uniform. Gradient reflecting metasurfaces considered in Section 6.1.1 were also designed based on this approach.

Likewise, known structures for wavefront manipulation in transmission regime, such as blazed gratings [148] and metasurfaces [132, 133], suffer from parasitic reflections, especially for high separation angles between the incident and refracted beams. Thus, there is a strong need in finding a methodology of designing gradient metasurfaces capable of ideal control of reflected and transmitted wavefronts. The main subject of the present section is an exact synthesis technique of such metasurfaces (off-resonance operation of metasurfaces is not considered). It is assumed here that a metasurface is illuminated by a given plane wave and it is designed in such a way that all the energy is reflected (or refracted) as a plane wave at a given angle without energy loss.

Let us consider in detail the conventional approach for the design of wave-reflecting metasurface gratings (gradient reflectors). It requires that the local reflection coefficient $r_l(x)$ (the ratio between the tangential components of the reflected and incident electric fields at a specific point on the metasurface, defined by the coordinate $x$) equals unity at each point and its phase changes linearly versus $x$ [92, 135]

$$r_l(x) = 1 \cdot e^{jk_0(\sin \theta_i - \sin \theta_r)x} = e^{j\Phi_r(x)},$$

(6.2)

where $\theta_i$ and $\theta_r$ are the angles of incidence and reflection with respect to the surface normal, and $\Phi_r$ is the phase of the local reflection coefficient. Here, TE polarization of the incident wave is assumed, as illustrated in Fig. 6.5(a). Next, in the traditional design procedure, the locally uniform surface (on the sub-wavelength scale) approximation is used. This means that each unit element of the metasurface is designed under the assumption that it is located in a uniform periodical array of identical elements. Such a uniform array generates only specular reflection. Therefore, the
total tangential fields $E_{\text{tan}}$ and $H_{\text{tan}}$ at the uniform array are given by

$$
E_{\text{tan}} = E_i e^{-jk_0 \sin \theta_i x} + E_r e^{-jk_0 \sin \theta_r x},
$$
$$
H_{\text{tan}} = \frac{1}{\eta_0} (E_i \cos \theta_i e^{-jk_0 \sin \theta_i x} - E_r \cos \theta_r e^{-jk_0 \sin \theta_r x}).
$$

(6.3)

After designing each unit element under this assumption, the final non-uniform metasurface is constructed. The surface impedance $Z_s$ of such a metasurface, determined as $E_{\text{tan}} = Z_s n \times H_{\text{tan}}$ ($n$ is the normal vector to the metasurface plane pointing towards the source), reads

$$
Z_s = j \frac{\eta_0}{\cos \theta_i} \cot[\Phi_i(x)/2].
$$

(6.4)

Here, it was taken into account that the local reflection coefficient is unity at each point, i.e. $E_r = E_i$. This impedance is imaginary for all coordinates $x$ and, therefore, can be realized using passive lossless structures. By calculating the local reflection coefficient from this impedance as $r_1 = (Z_s - \eta_0)/(Z_s + \eta_0)$, one can fulfil expression (6.2). The amplitude of the reflection coefficient is unity everywhere, while the phase changes by $2\pi$ over each period $D_x = \lambda/|\sin \theta_i - \sin \theta_r|$. Although such a scenario provides the required phase variations, it does not take into account the impedance matching of the incident and reflected waves, which results in parasitic reflections. Without proper understanding of the physical properties of metasurfaces for wavefront manipulation of reflected and transmitted radiation it is not possible to create 100%-efficient structures with desired properties.

### 6.3.1 Author's contribution

**Reflection scenario**

The level of the parasitic reflections occurring due to the mismatch between the incident and reflected waves can be calculated. According to (6.3), the metasurface is designed assuming that the impedances of both
incident $Z_i$ and reflected $Z_r$ waves are equal to $\eta_0 / \cos \theta_i$. However, for perfect anomalous reflection (at an angle $\theta_r$), the total fields at the metasurface should obey

$$E_{\tan} = E_i e^{-j k_0 \sin \theta_i x} + E_r e^{-j k_0 \sin \theta_r x},$$

$$H_{\tan} = 1/\eta_0 (E_i \cos \theta_i e^{-j k_0 \sin \theta_i x} - E_r \cos \theta_r e^{-j k_0 \sin \theta_r x}),$$

and the impedances of the corresponding waves are $Z_i = \eta_0 / \cos \theta_i$ and $Z_r = \eta_0 / \cos \theta_r$. It is clear that the conventional design procedure of gradient metasurfaces fails to provide high efficiency: The metasurfaces are not properly designed to compensate the mismatch between the impedance of the input (incident plane wave) and output (reflected plane wave). This mismatch inevitably results in parasitic reflections (specular reflection and reflection into other diffraction modes). In this case, the amount of power delivered into the desired $\theta_r$ direction normalized to the incident power (the efficiency of the metasurface) can be easily calculated as

$$\zeta = 1 - \left( \frac{Z_r - Z_i}{Z_r + Z_i} \right)^2 = \frac{4 \cos \theta_i \cos \theta_r}{(\cos \theta_i + \cos \theta_r)^2}.$$  

(6.6)

This expression for the efficiency is applicable only for lossless metasurface gratings. Figure 6.6 shows the ultimate maximum efficiency of conventional lossless metasurface gratings dictated by (6.6) when the incidence angle is fixed to $\theta_i = 0^\circ$. It should be noted that the metasurface gratings proposed in Publications III and IV were designed based on the conventional approach and their efficiencies are also limited by (6.6).

To overcome the theoretical limit (6.6) on the efficiency of anomalous reflection, one should consider the appropriate boundary condition at the metasurface (6.5). In this case the required surface impedance $Z = E_{\tan}/H_{\tan}$ can be written as

$$Z_s = \frac{\eta_0}{\sqrt{\cos \theta_i \cos \theta_r}} \frac{\sqrt{\cos \theta_i} + \sqrt{\cos \theta_r} e^{j \Phi_r(x)}}{\sqrt{\cos \theta_i} - \sqrt{\cos \theta_r} e^{j \Phi_r(x)}}.$$  

(6.7)

Figure 6.6. Theoretical bound on the efficiency of conventional metasurface reflective gratings when the incidence angle is $\theta_i = 0^\circ$. 

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The same expression was independently obtained in [147]. Interestingly, this impedance is a complex number meaning that in some regions of the metasurface the $z$-component of the total Poynting vector (normal to the surface) must be positive, and in others it should be negative. The same fundamental property was found in blazed gratings [149]. It should be stressed that this behaviour of the Poynting vector does not imply that the metasurface cannot be passive and lossless. On the contrary, a properly tuned metasurface with strongly non-local response (strong spatial dispersion) can emulate such an impedance surface: The power which passes through the input surface in the “lossy” regions is not absorbed but it is reradiated from the “active” regions. This conclusion was firstly reported in Publication VI. Similar conclusion was independently suggested by the authors of [110] who proposed theoretical realization of the gradient reflector in form of a bi-anisotropic metasurface supporting two evanescent waves on the side opposite to the illuminated one. The interference of these two evanescent waves allows one to achieve the required oscillations of the Poynting vector without utilizing active elements. Another solution for realizing anomalous reflection with unitary efficiency was proposed based on bi-anisotropic metasurfaces with one individual scatterer per unit cell [150].

Probably, the first practical realization of the gradient reflector diffracting incident waves at an arbitrary angle with 100%-efficiency was reported in [96]. The required complex surface impedance dictated by (6.7) was implemented at microwaves using the concept of leaky-wave antenna arrays exhibiting strong spatial dispersion. The designed leaky-wave metasurface represents an array of metal strips with different lengths separated from a ground plane by a dielectric spacer. An incident wave illuminating the metasurface is coupled into a surface wave propagating along the interface which in turn, due to the proper phase tuning, is coupled into a reflected plane wave radiated from the metasurface at the designed angle. Naturally, such a metasurface cannot be characterized with local parameters and possesses strong spatial dispersion. Publication VII reports on design and implementation of the leaky-wave metasurface grating in the near-infrared frequency range. In principle, perfect anomalous reflection can be obtained also in metasurface gratings without a ground plane such as ones presented in Publications III and IV. However, in this case proper and not trivial engineering of evanescent fields at the metasurface plane is necessary.
For practical realization of metasurface gratings the operational bandwidth is an important issue. It should be noted that the bandwidth is determined not only by the resonant properties of the metasurface inclusions but also by the grating phase gradient. Bandwidth of gratings designed based on non-linear gradient with impedance (6.7) is smaller than that of conventional gratings with impedance (6.4) [96]. Moreover, it is expected that gratings designed for higher deflection angles possess narrower bandwidths due to stronger non-local response.

**Refraction scenario**

Analogously to the reflection scenario, metasurfaces for wavefront manipulation in transmission designed under the assumption of locally uniform surface [92, 132, 133] also suffer from parasitic energy channelling into undesired propagating modes. The efficiency limit in this case is similar to (6.6) where $\theta_r$ should be replaced with $\theta_t$ [151].

Let us consider a metasurface diffracting an incident plane wave in medium 1 (with the wavenumber $k_1$ and the electric field vector $E_i$) into a single refracted (transmitted) plane wave in medium 2 (with the wavenumber $k_2$ and the electric field vector $E_t$). The geometry of the problem is shown in Fig. 6.5(b). The tangential fields at the illuminated side $E_{\text{tan}1}$ and $H_{\text{tan}1}$ and the opposite side $E_{\text{tan}2}$ and $H_{\text{tan}2}$ (at $z = 0$) read

\[
E_{\text{tan}1} = E_i e^{-jk_1 \sin \theta_1 x}, \quad n \times H_{\text{tan}1} = E_i \frac{1}{\eta_1} \cos \theta_i e^{-jk_1 \sin \theta_1 x},
\]

\[
E_{\text{tan}2} = E_t e^{-jk_2 \sin \theta_2 x}, \quad n \times H_{\text{tan}2} = E_t \frac{1}{\eta_2} \cos \theta_t e^{-jk_2 \sin \theta_2 x},
\]

(6.8)

where $\eta_1$ and $\eta_2$ are the wave impedances of media 1 and 2, respectively.

The energy conservation imposes the requirement of continuity of the normal component of the total Poynting vector at each point of the metasurface

\[
\frac{1}{2} \text{Re}(E_{\text{tan}1} \times H_{\text{tan}1}^*) = \frac{1}{2} \text{Re}(E_{\text{tan}2} \times H_{\text{tan}2}^*),
\]

(6.9)

which implies the following relation for the electric field amplitudes in the two media:

\[
E_t = E_i \sqrt{\frac{\cos \theta_1}{\cos \theta_2}} \sqrt{\frac{\eta_2}{\eta_1}}
\]

(6.10)

To understand the required physical properties the metasurface, let us model it using an equivalent $T$-circuit shown in Fig. 6.7. The tangential fields at both interfaces are related to the $Z$-parameters of the circuit as follows:

\[
E_{\text{tan}1} = Z_{11} n \times H_{\text{tan}1} + Z_{12} (-n \times H_{\text{tan}1}),
\]

\[
E_{\text{tan}2} = Z_{21} n \times H_{\text{tan}1} + Z_{22} (-n \times H_{\text{tan}1}).
\]

(6.11)
Looking for a solution where all the $Z$-parameters are purely imaginary $Z_{ij} = jX_{ij}$ (a lossless metasurface) and substituting the field values from (6.8) and (6.10) into (6.11), one can obtain the following equations for the $X$-parameters:

$$X_{11} = \frac{\eta_1}{\cos \theta_i} \cot \Phi_t(x), \quad X_{22} = \frac{\eta_2}{\cos \theta_t} \cot \Phi_t(x),$$

$$X_{12} = X_{21} = \frac{1}{\sqrt{\eta_1 \eta_2}} \frac{\sin \Phi_t(x)}{\cos \theta_i \cos \theta_t},$$

where $\Phi_t(x) = k_1 \sin \theta_i x - k_2 \sin \theta_t x$. Formulas (6.12) are in complete agreement with the results of [109], obtained using the generalized scattering parameters approach.

Knowing the $Z$-parameters of a metasurface, one can determine suitable topologies of constitutive elements (the unit-cell structures) which will realize the desired functionality of 100%-efficient refraction. The metasurface modeled by (6.12) is reciprocal since $X_{12} = X_{21}$. The required physical properties of such metasurfaces can be understood from the corresponding equivalent $T$-circuit shown in Fig. 6.7. The circuit is asymmetric, because $X_{11} \neq X_{22}$ (which is equivalent to $Z_{11} \neq Z_{22}$). This structure of the $Z$-matrix can be realized only with a bi-anisotropic metasurface of the omega type (weak spatial dispersion), see a discussion in [152, 153]. Possible appropriate topologies include arrays of $\Omega$-shaped inclusions, arrays of split rings, double arrays of patches (patches on the opposite sides of the substrate must be different to ensure proper magnetoelectric coupling), etc. Probably the first experimental realizations of ideal refracting metasurfaces were reported very recently [154, 155]. In both works, the metasurface comprised inclusions with the omega bi-anisotropic coupling.

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**Figure 6.7.** Equivalent $T$-circuit of a reciprocal metasurface for the considered case of TE polarization.
7. Conclusions

The present dissertation is devoted to an important field of applied electromagnetics: Spatially dispersive metasurfaces. This subject merges such areas of physics as metamaterials, bi-anisotropic media, antenna arrays, and diffraction gratings. Due to the multidisciplinary nature of the subject, phenomena and devices resulting from it may carry significant implications for applied electromagnetics from microwaves to optical frequencies (moreover, for all wave processes, i.e., mechanical, acoustic, quantum, etc.). Frequency-selective metasurfaces proposed in Publications III and V have a clear potential for various applications in telecommunications and space industry. It is expected that such metasurfaces can be designed for fabrication with standard printed-circuit-board technology suitable for mass production. The metasurfaces can be cascaded one behind the other, replacing entire complexes of different conventional antennas.

Findings reported in Publications VI and VII are only the first essential steps towards realization of surfaces for general wavefront manipulations. Although the presented synthesis approach was applied to a plane-wave-to-plane-wave diffraction scenario, it can be extended to scenarios of complex wave transformations: focusing lenses, beam splitters, filters, holograms, etc. Due to the maximal possible efficiency (100% in the absence of dissipation loss), such novel structures will outperform the conventional counterparts. Interestingly, the presence of spatial dispersion effects in these structures is a necessary condition.

In conclusion, it is important to note that the subject of spatially dispersive metasurfaces is not limited to the studies conducted within the present dissertation. There remains much to be done, which is always the case in this fascinating enigmatic world, world of Physics.


——————————— Publications
The present dissertation is devoted to an important field of applied electromagnetics: Spatially dispersive metasurfaces. Metasurfaces represent a two-dimensional arrangement of sub-wavelength inclusions engineered to manipulate incident electromagnetic radiation in a prescribed fashion. The subject of spatially dispersive metasurfaces merges such areas of physics and engineering as metamaterials, biaxial anisotropic media, antenna arrays, and diffraction gratings. Due to the multidisciplinary nature of the subject, phenomena and devices resulting from it may carry significant implications for applied electromagnetics from microwaves to optical frequencies. The dissertation presents design and measurement data of several spatially dispersive metasurfaces: Shadow-free gradient reflector and absorber as well as an optical anomalously reflecting grating.