A comparison of complete vs. simplified viscous terms in boundary layer flow and lid-driven cavity flow.

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ABSTRACT

Viscous terms are derived in curved co-ordinates using finite volume formulation. An approximation in common use called Thin Shear Layer approximation is derived. Turbulent flow over a flat plate with a $k - \epsilon$ -model as well as a laminar lid-driven flow are computed and the results are compared between the full viscous terms and the TSL approximation.

MAIN RESULT

An implementation of full viscous terms in FINFLO solver.

KEY WORDS

Thin Shear Layer, viscous flow, boundary layer flow, lid-driven cavity flow

APPROVED BY

Timo Siikonen December 19,1997
1 Introduction

At high Reynolds numbers wall shear layers, wakes or free shear layers will be of limited size, and if the viscous region remains limited then the dominating influence of the shear layers comes from the gradients transverse to the main flow direction. If we consider an arbitrary curvilinear system of co-ordinates with $\xi^1$ and $\xi^2$ along the surface and $\xi^3$ directed towards the normal of the surface, then the thin shear layer (TSL) approximation of the Navier-Stokes equations consists of neglecting all $\xi^1$ and $\xi^2$ -derivatives occurring in the turbulent and viscous shear stress terms. Also, at high Reynolds numbers the mesh is made dense in the direction normal to the shear layer, and therefore the neglected terms are computed with a lower accuracy [1]. In the following applications derivatives are computed in a simplified manner in all curved co-ordinate directions which is referred to as TSL approximation.

2 Finite Volume Formulation

2.1 Derivation of the viscous terms

Transport equation in conservation form can be written as

$$\frac{d}{dt} \int_V U \, dV + \int_S \vec{F}(U) \cdot d\vec{S} = \int_V Q \, dV,$$

which in discretized form is written as

$$\frac{d}{dt} (V_i U_i) + \sum_{k=1}^{faces} \vec{F}(U) S_k = V_i Q_i.$$

The flux at the surface is written as

$$\vec{F} = (F - F_v) \vec{n}_x + (G - G_v) \vec{n}_y + (H - H_v) \vec{n}_z,$$

where the components of the unit normal vector are written as $\vec{n}_x = \frac{\vec{s}_x}{|\vec{s}_x|}$, $\vec{n}_y = \frac{\vec{s}_y}{|\vec{s}_y|}$ and $\vec{n}_z = \frac{\vec{s}_z}{|\vec{s}_z|}$. Each surface of the cell has a unit normal $\vec{n}$ (see Fig. 1). Viscous fluxes $F_v$, $G_v$, and $H_v$ are

$$F_v = \begin{bmatrix} 0, \tau_{xx}, \tau_{xy}, \tau_{xz}, u\tau_{xx} + v\tau_{xy} + w\tau_{xz} - q_x, \mu \phi \frac{\partial \phi}{\partial x} \end{bmatrix}^T,$$

$$G_v = \begin{bmatrix} 0, \tau_{yx}, \tau_{yy}, \tau_{yz}, u\tau_{yx} + v\tau_{yy} + w\tau_{yz} - q_y, \mu \phi \frac{\partial \phi}{\partial y} \end{bmatrix}^T,$$

$$H_v = \begin{bmatrix} 0, \tau_{zx}, \tau_{zy}, \tau_{zz}, u\tau_{zx} + v\tau_{zy} + w\tau_{zz} - q_z, \mu \phi \frac{\partial \phi}{\partial z} \end{bmatrix}^T,$$
where the last term is viscous part of generic scalar equation, including turbulent kinetic energy and dissipation of kinetic energy. The diffusion coefficient is approximated as $\kappa \phi = \mu + \frac{\mu_T}{\sigma_\phi}$, where $\mu_T$ is the turbulent viscosity and $\sigma_\phi$ is appropriate Schmidt’s number. Heat conduction flux is

$$q = -(k + k_T) \nabla T = - \left( \mu \frac{c_p}{P_T} + \mu_T \frac{c_p}{P_T} \right) \nabla T,$$

where $k$, $k_T$ are laminar and turbulent thermal conductivities respectively, $P_T, P_T$ are laminar and turbulent Prandl numbers respectively and $c_p$ is specific heat at constant pressure. A stress tensor for newtonian viscous fluid is written as

$$\tau_{ij} = \mu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{2}{3} (\nabla \cdot \vec{V}) \delta_{ij} \right) - \rho u_i u_j$$

where for the Reynolds stresses Boussinesq’s approximation is used

$$-\frac{\rho u_j u_i}{\rho} \delta_{ij} = \mu_T \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{2}{3} (\nabla \cdot \vec{V}) \delta_{ij} \right)$$

The stress tensor is symmetric $\tau_{ij} = \tau_{ji}$. Viscous fluxes are obtained from Eqs. (2) and (4)

$$\hat{F}_v^1 = 0$$

$$\hat{F}_v^2 = \int_S (\tau_{xx} n_x + \tau_{yx} n_y + \tau_{zx} n_z) dS$$

$$\hat{F}_v^3 = \int_S (\tau_{xy} n_x + \tau_{yy} n_y + \tau_{zy} n_z) dS$$

$$\hat{F}_v^4 = \int_S (\tau_{xz} n_x + \tau_{yz} n_y + \tau_{zz} n_z) dS$$

$$\hat{F}_v^5 = \int_S [(u \tau_{xx} + v \tau_{xy} + w \tau_{xz}) n_x + (u \tau_{xy} + v \tau_{yy} + w \tau_{yz}) n_y]$$
divided into incompressible and compressible parts

\[ F = \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) n_x + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) n_y + \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) n_z \]

These equations are valid on every surface as corresponding components of the surface areas \( S_i \) and normals \( n_i \) are used. A detail concerning a computation of incompressible flow is mentioned here. The stress tensor of Eq. (6) can be divided into incompressible and compressible parts

\[ \tau_{ij} = (\mu + \mu_T) \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} (\nabla \cdot \nabla) \delta_{ij} \right) = \sigma_{ij} - (\mu + \mu_T) \frac{2}{3} (\nabla \cdot \nabla) \delta_{ij}. \]
\begin{align*}
+ \quad w \left( \sigma_{xx} n_x + \sigma_{xy} n_y + \left[ \sigma_{xz} - (\mu + \mu_T) \frac{2}{3}(\nabla \cdot \hat{V}) \right] n_z \right) \\
- \quad q_x n_x - q_y n_y - q_z n_z \ dS, \\
\end{align*}
(11)

For an incompressible flow $\nabla \cdot \hat{V} = 0$ mathematically. In conservative equations the mass flux is in balance when $\int_{S} \rho \hat{V} \ dS = 0$. In a pseudo-compressibility method density $\rho$ is not necessarily constant. Even with a constant density, the mass flux balance does not indicate that $\nabla \cdot \hat{V} = 0$, since the fluxes are numerically calculated from different formulae from $\nabla \cdot \hat{V} = 0$. Hence compressible viscous effects may be present to some extent. For truly incompressible i.e. constant density flows this effect is of the same order as the truncation order.

In order to compute the cartesian derivatives a somewhat less known version of the generalized divergence theorem of Gauss is applied (see e.g. [2]),

\[
\int_{V} \nabla \phi \ dV = \int_{S} \phi \ d\hat{S}
\]
(12)

Integration is performed over the volume $V_{ijk}$ on the left side,

\[
\int_{V_{ijk}} \nabla \phi \ dV = V_{ijk} \left( \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right)
\]
(13)

and over the corresponding faces on the right side,

\[
\int_{S} \phi \ d\hat{S} = \left( S_{x,i+\frac{1}{2},j,k} \phi_{i+\frac{1}{2},j,k} - S_{x,i-\frac{1}{2},j,k} \phi_{i-\frac{1}{2},j,k} \right) \hat{i} + \left( S_{x,i,j+\frac{1}{2},k} \phi_{i,j+\frac{1}{2},k} - S_{x,i,j-\frac{1}{2},k} \phi_{i,j-\frac{1}{2},k} \right) \hat{j} + \left( S_{x,i,j,k+\frac{1}{2}} \phi_{i,j,k+\frac{1}{2}} - S_{x,i,j,k-\frac{1}{2}} \phi_{i,j,k-\frac{1}{2}} \right) \hat{k}
\]
(14)

Hence the derivative of $\phi$ in $x$ -direction can be solved

\[
V_{ijk} \left( \frac{\partial \phi}{\partial x} \right)_{ij} = S_{x,i+\frac{1}{2},j,k} \phi_{i+\frac{1}{2},j,k} - S_{x,i-\frac{1}{2},j,k} \phi_{i-\frac{1}{2},j,k} + S_{x,i,j+\frac{1}{2},k} \phi_{i,j+\frac{1}{2},k} - S_{x,i,j,k+\frac{1}{2}} \phi_{i,j,k+\frac{1}{2}}
\]
(15)

If the variables are defined in the middle of volumes, then the values at surfaces must be interpolated as averages. These integrations can also be performed over a different volume e.g. $V_{i+1/2,j,k}$ which is equivalent to increasing all $i$ -indeces by $\frac{1}{2}$ in Eqs. (14) and (15).
2.2 Thin Shear Layer Approximation

Computing derivatives accurately is computationally heavy and the required memory size is increased, which of course also depends on the structure of the code. At high Reynolds numbers only the terms in the direction of curved co-ordinates are preserved. This is obtained by dropping the cross derivatives from equation (15) [3]. The computational grid should be as orthogonal and dense as possible in the directions against the surfaces, where the dominating viscous effects are present. Also, in the following, the integration is performed over the volume \( V_{i+1/2,j,k} \),

\[
V_{i+1/2,j,k} \frac{\partial \phi}{\partial x}_{i+1/2,j,k} = S_{x,i+1,j,k} \phi_{i+1,j,k} - S_{x,i,j,k} \phi_{i,j,k}.
\]

(16)

In the thin-layer approximation the surface components are replaced by the interpolated surface component between them. This is usually readily available and the volume \( V_{i+1/2,j,k} \) can be approximated as an average \( \frac{1}{2}(V_i + V_{i+1}) \). The Eq. (16) can be further modified as

\[
\left( \frac{\partial \phi}{\partial x} \right)_{i+1/2,j,k} = \frac{S_{x,i+1/2,j,k}}{V_{i+1/2,j,k}} (\phi_{i+1,j,k} - \phi_{i,j,k})
\]

\[
= \frac{S_{i+1/2,j,k}}{V_{i+1/2,j,k}} n_{x,i+1/2,j,k} (\phi_{i+1,j,k} - \phi_{i,j,k})
\]

\[
= \frac{n_{x,i+1/2,j,k} (\phi_{i+1,j,k} - \phi_{i,j,k})}{d_{i+1/2,j,k}}.
\]

(17)

As the approximate derivatives of Eq. (17) are inserted into the fluxes given by Eqs.(9)

\[
F_{v,i+1/2}^{\alpha} = \left\{ \begin{array}{l}
\left[ \frac{2}{3} \mu \left[ \frac{2}{d_{i+1/2}} (u_{i+1} - u_i) - \frac{n_{y,i+1/2}}{d_{i+1/2}} (v_{i+1} - v_i) \right]
\right. \\
- \frac{n_{z,i+1/2}}{d_{i+1/2}} (w_{i+1} - w_i) \left. \right] n_{x,i+1/2}
\end{array} \right.
\]

\[
+ \mu \left[ \frac{n_{y,i+1/2}}{d_{i+1/2}} (u_{i+1} - u_i) + \frac{n_{x,i+1/2}}{d_{i+1/2}} (v_{i+1} - v_i) \right] n_{y,i+1/2}
\]

\[
+ \mu \left[ \frac{n_{x,i+1/2}}{d_{i+1/2}} (u_{i+1} - u_i) + \frac{n_{z,i+1/2}}{d_{i+1/2}} (w_{i+1} - w_i) \right] n_{z,i+1/2}
\]

\[
= \left[ \frac{A}{3} n_{x,i+1/2}^2 + n_{y,i+1/2}^2 + n_{z,i+1/2}^2 \right] \frac{\mu}{d_{i+1/2}} (u_{i+1} - u_i)
\]

\[
+ \frac{n_{x,i+1/2} n_{y,i+1/2}}{3d_{i+1/2}} \mu (v_{i+1} - v_i)
\]
The TSL approximation for the energy and the generic scalar equation becomes

\[
\frac{n_{x,i+\frac{1}{2}}n_{y,i+\frac{1}{2}}}{3d_{i+\frac{1}{2}}} \mu (w_{i+1} - w_i) \quad S_{i+\frac{1}{2}}
\] (18)

Other momentum flux components are derived in a similar manner,

\[
\hat{F}^{3}_{v,i+\frac{1}{2}} = \left[ \frac{n_{x,i+\frac{1}{2}}n_{y,i+\frac{1}{2}}}{3d_{i+\frac{1}{2}}} \mu (u_{i+1} - u_i) + \left( \frac{4}{3} n_{x,i+\frac{1}{2}} + n_{x,i+\frac{1}{2}} + n_{z,i+\frac{1}{2}} \right) \frac{\mu}{d_{i+\frac{1}{2}}} (v_{i+1} - v_i) \right] S_{i+\frac{1}{2}}
\]

\[
\hat{F}^{4}_{v,i+\frac{1}{2}} = \left[ \frac{n_{x,i+\frac{1}{2}}n_{y,i+\frac{1}{2}}}{3d_{i+\frac{1}{2}}} \mu (u_{i+1} - u_i) + \left( \frac{4}{3} n_{x,i+\frac{1}{2}} + n_{x,i+\frac{1}{2}} + n_{z,i+\frac{1}{2}} \right) \frac{\mu}{d_{i+\frac{1}{2}}} (w_{i+1} - w_i) \right] S_{i+\frac{1}{2}}.
\] (19)

The TSL approximation for the energy and the generic scalar equation becomes

\[
\hat{F}^{5}_{v,i+\frac{1}{2}} = u_{i+\frac{1}{2}} \hat{F}^{2}_{v,i+\frac{1}{2}} + v_{i+\frac{1}{2}} \hat{F}^{3}_{v,i+\frac{1}{2}} + w_{i+\frac{1}{2}} \hat{F}^{4}_{v,i+\frac{1}{2}}
\]

\[
- \left( k + k_T \right) \left( \frac{\partial T}{\partial x} n_{x,i+\frac{1}{2}} + \frac{\partial T}{\partial y} n_{y,i+\frac{1}{2}} + \frac{\partial T}{\partial z} n_{z,i+\frac{1}{2}} \right) S_{i+\frac{1}{2}}
\]

\[
= u_{i+\frac{1}{2}} \hat{F}^{2}_{v,i+\frac{1}{2}} + v_{i+\frac{1}{2}} \hat{F}^{3}_{v,i+\frac{1}{2}} + w_{i+\frac{1}{2}} \hat{F}^{4}_{v,i+\frac{1}{2}}
\]

\[
- \frac{c_p \mu}{Pr d_{i+\frac{1}{2}}} \left( 1 + \frac{Pr \mu_T}{Pr T} \right) (T_{i+1} - T_i) S_{i+\frac{1}{2}}
\]

\[
\hat{F}^{6}_{v,i+\frac{1}{2}} = \mu \phi \left( \frac{\partial \phi}{\partial x} n_{x,i+\frac{1}{2}} + \frac{\partial \phi}{\partial y} n_{y,i+\frac{1}{2}} + \frac{\partial \phi}{\partial z} n_{z,i+\frac{1}{2}} \right) S_{i+\frac{1}{2}}
\]

\[
= \frac{\mu \phi}{d_{i+\frac{1}{2}}} (\phi_{i+1} - \phi_i) S_{i+\frac{1}{2}}
\] (20)

The momentum fluxes can also be written in a compact form

\[
\hat{F}^{2}_{v,i+\frac{1}{2}} = \frac{S_{i+\frac{1}{2}} \mu_{i+\frac{1}{2}}}{d_{i+\frac{1}{2}}} \left[ (u_{i+1} - u_i) + \frac{n_{x,i+\frac{1}{2}}}{3} (\overline{m}_{i+1} - \overline{m}_i) \right]
\]

\[
\hat{F}^{3}_{v,i+\frac{1}{2}} = \frac{S_{i+\frac{1}{2}} \mu_{i+\frac{1}{2}}}{d_{i+\frac{1}{2}}} \left[ (v_{i+1} - v_i) + \frac{n_{y,i+\frac{1}{2}}}{3} (\overline{m}_{i+1} - \overline{m}_i) \right]
\]

\[
\hat{F}^{4}_{v,i+\frac{1}{2}} = \frac{S_{i+\frac{1}{2}} \mu_{i+\frac{1}{2}}}{d_{i+\frac{1}{2}}} \left[ (w_{i+1} - w_i) + \frac{n_{z,i+\frac{1}{2}}}{3} (\overline{m}_{i+1} - \overline{m}_i) \right],
\] (21)

where \(\overline{m}_i\) and \(\overline{m}_{i+1}\) are the scaled contravariant velocities in the direction of the surface \(S_{i+\frac{1}{2}}\)

\[
\overline{m}_i = n_{x,i+\frac{1}{2}} u_i + n_{y,i+\frac{1}{2}} v_i + n_{z,i+\frac{1}{2}} w_i
\] (22)
In the implicit phase the equations are linearized in a simplified manner. The effect of the viscous terms $T$ is added on the Jacobian matrices $A = \partial F/\partial U$, $B = \partial G/\partial U$ and $C = \partial H/\partial U$ as

$$
A^\pm = R_A(\Lambda_A^\pm + TI) R_A^{-1},
$$
$$
B^\pm = R_B(\Lambda_B^\pm + TI) R_B^{-1},
$$
$$
C^\pm = R_C(\Lambda_C^\pm + TI) R_C^{-1}.
$$

Here $R$ is the transformation matrix from conservative variables $U$ to characteristic variables $W$, $\delta W = R^{-1}\delta U$. $\Lambda^\pm$ are the diagonal matrices containing the positive and the negative eigenvalues of the Jacobians. The diagonal weight factor $T$ is computed as

$$
T = \frac{2(\mu + \mu_T)}{\rho d},
$$

where $d$ is the length of the cell in the direction of the computational sweep.

### 3 Test cases

#### 3.1 Flat Plate

The case is a flow over flat plate with a high free-stream turbulent intensity. Inlet velocity was 9.4 m/s and the pressure gradient was zero. Measurements have been made down to $x = 1.495$ m that corresponds to $Re_x \approx 940000$. Upstream turbulence intensity is $Tu = 6.0\%$. Dissipation is set so that decay of free stream is in balance with production. The calculation is started 16 cm before the plate. The length of the flat plate is 1.6 m. The height is 30 cm and the height of the first row of cells was $2.5 \times 10^{-5}$ [4] that is equal to $y^+ \approx 0.7$ at most of the domain (at the leading edge $y^+ = 2.1$). Grid is clustered near the wall though the nearest three rows are kept constant. The ratio between neighbouring cells is $\Delta y_{n+1}/\Delta y_n = 1.125$. The grid size is $96 \times 64$ and the grid is seen in figure 2.
3.2 Lid-driven cavity flow

The second test case is an isothermal cavity flow problem. The grid size for the problem is $80 \times 80$ and the grid is seen in figure 3. The top wall is moving with velocity $0.006108 \, \text{m/s}$ and the resulting Reynolds number is 400.

4 Results

4.1 Flow over the Flat Plate

The second-order upwind discretization with Roe’s splitting was used for inviscid fluxes. The case was scaled so that the free stream velocity $U_e$ was increased from $9.4\, \text{m/s}$ to $94\, \text{m/s}$ to ensure convergence. Computation with full fluxes was done with four multigrid levels whereas the TSL computation converged only with three levels maximum. These cases were iterated excessive 30000 iterations at the finest grid level, which was due to poor convergence of some properties in the TSL computation shown in Fig. (4). The computation of the full viscous fluxes took about 5% more CPU-time/cycle and the memory need was $21.09 \, \text{MB}$ for the full terms and $19.08 \, \text{MB}$ for the TSL approximation ($19.47 \, \text{MB}$ with the same four levels). The results were nearly identical from both calculations. Skin friction $c_f = \tau_w / \left( \frac{1}{2} \rho U_e^2 \right)$, where $\tau_w$ is a shear stress at the wall, is presented in Fig. 5. The velocity profiles are compared in Fig. 6 in terms of $u^+$, which is a dimensionless velocity defined as $u^+ = u / u_r$, where $u_r = \sqrt{\tau_w / \rho}$ is a friction velocity. Dimensionless distance
Fig. 4: Convergence histories of drag coefficients, total mass and turbulent kinetic energy with full viscous terms (a) and with TSL approximation (b).

Fig. 5: Skin friction coefficient.
from the wall is approximated as $y^+ = \frac{\rho u_y u_r}{\mu}$. Turbulence level is defined as $T_u = \sqrt{\frac{\mu}{\nu}}$, where $k$ is the turbulent kinetic energy. The turbulence level profiles are presented in Fig. 7.

In this case a pseudo-compressibility method was used with the second-order upwind discretization. 10 000 iterations were computed. The $u/v$-velocity distributions were plotted at $y/x$ -positions $0.0625L$, $0.25L$, $0.5L$, $0.75L$ and $0.9375L$ respectively, where $L$ stands for the length and the height of the cube. The plots are represented in Fig. 8. If $u$-velocity results are compared with the reference solution [5] at $y = 0.5L$, a suspicion arised that numerical dissipation has spoilt the solution. The computation with the full viscous terms was continued with 5000 iterations with the third-order upwind biased fluxes and these results are closer to the reference solution. In this case the full viscous terms give results closer to the reference solution than the TSL approximation.

5 Discussion

In the boundary layer flow there was practically no difference between the results computed with the TSL approximation and the full viscous terms. In this case the dominating viscous forces might be those in the direction of the plate surface and hence in direction of the grid line. In the cavity flow there was a small difference between the two cases and the full viscous terms produced slightly better results. This might result from the nature of the case, at small Reynolds numbers the TSL approximation is not valid. If the last 'contravariant' terms are neglected in equations (21), then correct terms are obtained for any orthogonal grid with incompressible flow. It was first thought that this 'compressible' term might not vanish as well as $\nabla \cdot V$ -term in the full viscous terms, but both terms were of the same order of magnitude. Also, the pseudo-compressibility method was suspected to cause numerical dissipation in the solution.
Fig. 6: Dimensionless velocity profiles at various downstream positions.
Fig. 7: Turbulence intensities at various downstream positions.
Fig. 8: Scaled u/v -velocity profiles at different x/y -positions.
References


