Nonlocal nonlinear potential theory and fractional integral operators

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Abstract

This thesis develops Potential Theory for nonlinear fractional Laplace type equations. These equations are nonlocal integro-differential equations defined as singular integrals. We study weak solutions and weak supersolutions of the equations, demonstrating that they behave as in the case of standard elliptic partial differential equations. We also define a related notion of superharmonic functions via a comparison with weak solutions. The superharmonic functions are used to give a nonlocal version of Perron’s method for solving Dirichlet problems with general boundary data.

To obtain all the required properties of superharmonic functions, we use a related obstacle problem as a tool. For this, several regularity results for the solution to the obstacle problem are proved. In addition, we study a notion of viscosity solutions to the considered equations. The results reveal that the classes of viscosity supersolutions and superharmonic functions are the same, and for bounded solutions, they coincide with the class of weak supersolutions.

The thesis also studies the regularity of maximal functions by extending the regularity results of a fractional maximal operator to its local counterpart. Finally, we consider finitely randomized dyadic systems on metric measure spaces and apply them to functions of bounded mean oscillation.

Keywords fractional Laplace equation, nonlocal operator, potential theory, superharmonic function, Perron’s method, obstacle problem

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Tiivistelmä


Superharmonisten funktioiden ominaisuuksien johtamisessa käytetään työkaluna vastaavan yhtälön liittyvää esteongelmaa. Tätä silmällä pitäen esteongelman ratkaisulle todistetaan useita säänöllisyystuloksia. Lisäksi tutkitaan viskositeettiratkaisujen käsitettä tarkasteltaville yhtälöille. Osittautuu, että viskositeettisuperratkaisut muodostavat täsmälleen saman luokan kuin superharmoniset funktiot, ja rajoitetuun ratkaisun tapauksessa tämä luokka yhtyy heikkojen superratkaisujen luokkaan.

Työssä tutkitaan myös maksimaalfunktioita säänöllisyystä yleistämällä frakcionaalisen maksimaaliperiaattorin säänöllisyystuloksia lokaaliin tilanteeseen. Lisäksi tarkastellaan äärellisesti satunnaisetettuja dyadisia systeemejä metrisissä mitta-avaruuksissa ja sovelletaan niitä rajoitettun keskiväärähtelyn funktioihin.

Avainsanat  frakcionaalinen Laplacen yhtälö, epälokaali operaattori, potentiaaliteoria, superharmoninen funktio, Perronin menetelmä, esteongelma

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Preface

I started my studies on partial differential equations under the supervision of Prof. Juha Kinnunen by writing my Bachelor's thesis in the summer 2010. Working with him continued later with my Master's thesis, and further, the project leading to this thesis started in the spring 2013. I am deeply grateful for having had the privilege to benefit from Juha's advice and guidance during these years. I also want to thank him for giving me several opportunities to participate international conferences and workshops.

Secondly, I wish to express my deep gratitude to another of my advisors, Dr. Tuomo Kuusi, for leading me into the world of nonlocal equations. He has always had wonderful ideas how to overcome mathematical difficulties arising in our research. I also want to thank my other co-authors Toni Heikkinen, Erik Lindgren, Giampiero Palatucci and Heli Tuominen for pleasant and fruitful collaboration. I want to express my gratitude to Prof. Moritz Kassmann and Prof. Daniel Spector for carrying out the preliminary examination of the thesis and to Prof. Lorenzo Brasco for agreeing to be my opponent. For financial support, I am very grateful to the Magnus Ehrnrooth Foundation and the Department of Mathematics and Systems Analysis of Aalto University.

I give special thanks to current and former members of the "Coffee room gang". It has always been fun to have lunch together – and some after-work beer as well. Finally, I want to thank my family and all my other friends. Especially, I am grateful to my girlfriend Saana for all the beautiful moments she has given me during the preparation of this thesis.

Espoo, October 10, 2016,

Janne Korvenpää
6. Summaries of the articles III–V

6.1 Properties of weak supersolutions
6.2 Solution to the obstacle problem
6.3 Regularity of the solution to the obstacle problem
6.4 Perron’s method
6.5 Equivalence of solutions
6.6 Further discussion

References

Publications
List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.


Author’s Contribution

Publication I: “Regularity of the local fractional maximal function”

The author has been responsible for Section 3 that contains the main results in the Euclidean setting.

Publication II: “Finitely randomized dyadic systems and BMO on metric measure spaces”

The author has performed a substantial part of the research.

Publication III: “The obstacle problem for nonlinear integro-differential operators”

The author has performed a substantial part of the research.

Publication IV: “Fractional superharmonic functions and the Perron method for nonlinear integro-differential equations”

The author has performed a substantial part of the research.

Publication V: “Equivalence of solutions to fractional $p$-Laplace type equations”

The author has performed a substantial part of the research.
1. Introduction

In this thesis, a central role is played by fractional Laplace type equations in $\mathbb{R}^n$. The fractional Laplace equation of order $s \in (0, 1)$ is

$$(-\Delta)^s u = 0,$$

that is, the usual Laplacian operator $-\Delta = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$ is raised to power $s$. The motivation for the fractional Laplacian originally comes from Fourier Analysis where the Fourier transform $\mathcal{F}$ changes derivation to multiplication enabling the definition of fractional derivatives on the Fourier side. Thus, one can define the fractional Laplacian of differentiability order $s$, of a smooth enough function $u$, as

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F} u),$$

where $\xi$ is the variable on the Fourier side. Substituting $s = 1$ provides the usual Laplacian $-\Delta u$. Alternatively, the fractional Laplacian can be written as an integral

$$(-\Delta)^s u(x) = C_{n,s} \int \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy$$

with a constant $C_{n,s}$ depending on $n$ and $s$. This makes (1.1) a nonlocal integro-differential equation.

Fractional Laplacian operators have been intensively studied in recent years. They form a nonlocal counterpart of the classical theory of elliptic operators in partial differential equations with many similarities. The methods are mostly based on the linearity of the operator $(-\Delta)^s$, whereas its nonlocal character is explicitly visible in the theory. Fractional Laplace equations have applications particularly in physics, applied mathematics, and mathematical finance, in phenomena with the presence of long-range interactions. As typical examples, we mention: the analysis of anomalous diffusion ([6]), quasi-geostrophic flow models ([51]), materials science in...
This thesis considers nonlinear versions of (1.1) where $(-\Delta)^s$ is replaced by a nonlinear nonlocal operator. In such a nonlinear setting, the existing theory is much more limited since one has to deal with the nonlinear growth behavior of the equation, in addition to the typical issues given by its nonlocal feature. In particular, effective linear methods based on Fourier Analysis or Hilbert spaces cannot be applied. Instead, the equation itself must be tackled together with different notions of solutions to the equation, as is usually carried out in the case of standard nonlinear partial differential equations.

Our aim is to develop Potential Theory for nonlocal nonlinear equations. Nonlinear Potential Theory is based on comparison principles stating that if two solutions are in order on the boundary of an open set, they must also be in the same order in the set. The comparison principle is not a linear phenomenon; therefore, it allows the development of Potential Theory without having a linear solution space. In particular, there is a comprehensive Potential Theory for the $p$-Laplace equation

$$\text{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad 1 < p < \infty, \quad (1.4)$$

which is a nonlinear counterpart of the Laplace equation, and for other equations of that type. It is worth noting that by using a concept of superharmonic functions based on the comparison principle, one can define so-called Perron solutions of Dirichlet boundary value problems that apply for very general boundary data. We intend to apply similar techniques in the nonlocal framework.

The thesis also contains a section concentrating on Harmonic Analysis. First, regularity of maximal functions is considered in Sobolev spaces. More precisely, we study a local fractional maximal operator that combines the smoothing feature of fractional maximal operators with restrictions given by the locality. We aim to extend the regularity results of the global fractional maximal function to its local counterpart. Secondly, randomized dyadic systems are considered in metric measure spaces. The randomized dyadic system is a structure of families of sets resembling the Euclidean dyadic cubes, where the construction is randomized in a suitable way. We intend to study finitely randomized dyadic systems and their connection to functions of bounded mean oscillation (BMO).

The thesis is organized as follows. In Chapter 2 below, we introduce different maximal operators and regularity results known for them. Chap-
ter 3 discusses BMO functions and randomized dyadic systems on metric measure spaces. In Chapter 4, we then summarize our results on the Harmonic Analysis section. Chapter 5 is devoted to introducing nonlocal integro-differential equations, different notions of solutions to the equations, and the most relevant known results. Chapter 6 finally summarizes our results in nonlocal nonlinear Potential Theory. The last part of the thesis contains the five original articles.
2. Maximal operators

2.1 Maximal functions

Maximal operators are standard tools in Harmonic Analysis, Partial Differential Equations, and Potential Theory. The most typical maximal function is the (centered) Hardy–Littlewood maximal function in $\mathbb{R}^n$, defined as

$$M_u(x) := \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u(y)| \, dy, \quad x \in \mathbb{R}^n,$$

(2.1)

for a locally integrable function $u$. Here

$$\int_{B_r(x)} f \, dy := \frac{1}{|B_r(x)|} \int_{B_r(x)} f \, dy$$

denotes the integral average of $f$ over $B_r(x)$, the ball of radius $r$ centered at $x$. The celebrated theorem of Hardy, Littlewood, and Wiener asserts that the maximal operator is bounded in $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$, more precisely

$$\|M_u\|_{L^p(\mathbb{R}^n)} \leq C\|u\|_{L^p(\mathbb{R}^n)}$$

(2.2)

for some constant $C$ depending only on $n$ and $p$. The proof can be found in many earlier works, such as in [56].

In Potential Theory, fractional versions of maximal operators are usually more useful than the Hardy–Littlewood maximal operators. The fractional maximal function in $\mathbb{R}^n$ is defined as

$$M_\alpha u(x) := \sup_{r>0} r^\alpha \int_{B_r(x)} |u(y)| \, dy, \quad x \in \mathbb{R}^n,$$

(2.3)

where the parameter $\alpha$ satisfies $0 \leq \alpha \leq n$. Compared to the Hardy–Littlewood maximal function, there is an extra coefficient $r^\alpha$ in front of the integral average, implying that bigger balls are preferred by the supremum. When $\alpha = 0$, the extra coefficient vanishes and one obtains the
Maximal operators

Hardy–Littlewood maximal function. In the case $0 < \alpha < n$, there is a close connection between the fractional maximal function and the Riesz potential

$$I_\alpha u(x) := \int_{\mathbb{R}^n} \frac{|u(y)|}{|x - y|^{n-\alpha}} \, dy, \quad x \in \mathbb{R}^n.$$  \hfill (2.4)

Using the connection to the Riesz potential, one can prove the following boundedness result for $M_\alpha$: if $1 < p < \infty$ and $0 \leq \alpha < \frac{n}{p}$, then

$$\|M_\alpha u\|_{L^p(\mathbb{R}^n)} \leq C \|u\|_{L^p(\mathbb{R}^n)}$$ \hfill (2.5)

for some constant $C$ depending only on $n$, $p$, and $\alpha$. Here $p^* := \frac{np}{n - \alpha p}$ is the fractional critical exponent, also appearing in fractional Sobolev inequalities. More details can be found in [56].

When the function $u$ is defined only in an open subset $\Omega \subset \mathbb{R}^n$, it is natural to consider a local version of the maximal function. The local Hardy–Littlewood maximal function $M_\Omega u$ is defined as in (2.1), but the supremum is taken only over all radii satisfying $0 < r < \text{dist}(x, \mathbb{R}^n \setminus \Omega)$. That is, only balls inside $\Omega$ are taken into account. Similarly, one defines the local counterpart of the fractional maximal operator, $M_{\alpha, \Omega}$. For the local Hardy–Littlewood maximal operator, boundedness in $L^p(\Omega)$ for $p > 1$ directly follows from (2.2):

$$\|M_\Omega u\|_{L^p(\Omega)} \leq \|M(u\chi_\Omega)\|_{L^p(\mathbb{R}^n)} \leq C \|u\chi_\Omega\|_{L^p(\mathbb{R}^n)} = C \|u\|_{L^p(\Omega)},$$ \hfill (2.6)

where $\chi_\Omega$ denotes the characteristic function of $\Omega$. In the same way, from (2.5) we obtain the boundedness result

$$\|M_{\alpha, \Omega} u\|_{L^{p^*}(\Omega)} \leq C \|u\|_{L^p(\Omega)}$$ \hfill (2.7)

for $1 < p < \infty$ and $0 \leq \alpha < \frac{n}{p}$. The definition of $M_{\alpha, \Omega} u$ suggests that its values tend to be small close to the boundary $\partial \Omega$. Indeed, at such points, the feasible radii are very small due to the restriction $r < \text{dist}(x, \mathbb{R}^n \setminus \Omega)$; thus, the coefficient $r^\alpha$ prevents the quantity $M_{\alpha, \Omega} u(x)$ from increasing excessively.

When balls and Lebesgue measures are replaced with spheres and surface measures, respectively, in the definition of maximal operators above, we obtain spherical maximal operators. Similar notation and terminology as above will be used in the spherical setting. The local spherical fractional maximal function of $u$ is defined as

$$S_{\alpha, \Omega} u(x) := \sup_{r} r^\alpha \int_{\partial B_r(x)} |u(y)| \, d\mathcal{H}(y), \quad x \in \Omega,$$ \hfill (2.8)
where $\mathcal{H}$ denotes the $n-1$-dimensional Hausdorff measure and the supremum is taken over all radii $r$ for which $0 < r < \text{dist}(x, \mathbb{R}^n \setminus \Omega)$. When $\Omega = \mathbb{R}^n$, $\alpha = 0$, or $\Omega = \mathbb{R}^n$ and $\alpha = 0$, we have the spherical fractional maximal operator $S_\alpha$, the local spherical maximal operator $S_{\Omega}$, and the spherical maximal operator $S$, respectively. Even if the spherical maximal function may be $+\infty$ at some points, it is finite almost everywhere for an $L^p$-function $u$. There is a similar boundedness result for $S_\alpha$ as (2.5): if $n \geq 2$, $p > \frac{n}{n-1}$, and $0 \leq \alpha \leq \min\left\{\frac{n-1}{p}, n - \frac{2n}{(n-1)p}\right\}$, then

$$
\|S_\alpha u\|_{L^p(\mathbb{R}^n)} \leq C\|u\|_{L^p(\mathbb{R}^n)}
$$

(2.9)

for some constant $C$ depending only on $n$, $p$, and $\alpha$. It is notable that (2.9) does not hold without the additional restrictions for the parameters compared to (2.5). Proofs towards (2.9) can be found, for example, in [57, 7, 53, 52] for different values of parameters. Again, we obtain the corresponding boundedness result for $S_{\alpha, \Omega}$ from (2.9), as done in (2.6).

### 2.2 Regularity of maximal functions

Since maximal operators preserve $L^p$-spaces, it is natural to ask whether they might also preserve any regularity. In [41], Juha Kinnunen asked and answered the question for the Hardy–Littlewood maximal operator in Sobolev spaces $W^{1,p}(\mathbb{R}^n)$. Sobolev spaces are defined by

$$
W^{1,p}(\Omega) := \{ u \in L^p(\Omega) : |Du| \in L^p(\Omega) \}
$$

(2.10)

for $1 \leq p \leq \infty$ and $\Omega \subset \mathbb{R}^n$, that is, the space of $L^p$-functions whose weak (or distributional) gradients are also in $L^p$. He proved that for $u \in W^{1,p}(\mathbb{R}^n)$ there is a pointwise estimate

$$
|\nabla M u(x)| \leq M|Du|(x) \quad \text{for a.e. } x \in \mathbb{R}^n,
$$

(2.11)

and by combining it with the Hardy–Littlewood–Wiener theorem (2.2), obtained that $M$ is bounded from $W^{1,p}(\mathbb{R}^n)$ to $W^{1,p}(\mathbb{R}^n)$ whenever $p > 1$. The estimate (2.11) demonstrates that the weak gradient of the maximal function is controlled by the maximal function of the weak gradient.

In [42], Kinnunen and Lindqvist extended the result of Kinnunen to subsets of $\mathbb{R}^n$. For a general open $\Omega \subset \mathbb{R}^n$, they proved a pointwise estimate

$$
|\nabla M_\Omega u(x)| \leq 2M_\Omega|Du|(x) \quad \text{for a.e. } x \in \Omega
$$

(2.12)
for the local maximal function of \( u \in W^{1,p}(\Omega) \), and by combining it with (2.6), obtained that \( \mathcal{M}_\Omega \) is bounded from \( W^{1,p}(\Omega) \) to \( W^{1,p}(\Omega) \) whenever \( p > 1 \). It is notable that the pointwise estimate essentially remains the same for general \( \Omega \); The only difference between (2.11) and (2.12) is the coefficient 2 in the latter one. Kinnunen and Lindqvist also proved that the local Hardy–Littlewood maximal function preserves the boundary values of a nonnegative function in Sobolev’s sense. More precisely, \( \mathcal{M}_\Omega u - |u| \) belongs to \( W^{1,p}_0(\Omega) \) for \( u \in W^{1,p}(\Omega) \). Here \( W^{1,p}_0(\Omega) \) denotes the closure of smooth compactly supported functions, \( C_\infty^0(\Omega) \), with respect to the norm

\[
\|u\|_{W^{1,p}(\Omega)} := \left( \int_\Omega |u|^p \, dx + \int_\Omega |Du|^p \, dx \right)^{\frac{1}{p}}.
\]

(2.13)

The articles [44, 50] are also noteworthy as they extend the regularity results of Kinnunen and Lindqvist to fractional Sobolev spaces.

In [43], Kinnunen and Saksman extended the results of [41] to the fractional maximal operator. They proved a similar pointwise estimate as (2.11), namely

\[
|DM_\alpha u(x)| \leq \mathcal{M}_\alpha |Du|(x) \quad \text{for a.e. } x \in \mathbb{R}^n
\]

(2.14)

for \( u \in W^{1,p}(\mathbb{R}^n) \), and combining it with the boundedness result (2.5), obtained that \( \mathcal{M}_\alpha \) is bounded from \( W^{1,p}(\mathbb{R}^n) \) to \( W^{1,p}(\mathbb{R}^n) \) whenever \( 1 < p < \infty \) and \( 0 \leq \alpha < \frac{n}{p} \). This means that \( \mathcal{M}_\alpha \) locally improves the Sobolev regularity since \( p^* > p \) for a positive \( \alpha \). Kinnunen and Saksman also proved that the fractional maximal function is locally a Sobolev function, even if the function \( u \) itself is just an \( L^p \)-function. They obtained a pointwise estimate

\[
|DM_\alpha u(x)| \leq C_M \alpha \mathcal{M}_{\alpha-1} u(x) \quad \text{for a.e. } x \in \mathbb{R}^n
\]

(2.15)

for \( u \in L^p(\mathbb{R}^n) \) with a constant \( C \) depending only on \( n \) and \( \alpha \), and by combining it with (2.5), that \( |DM_\alpha u| \in L^q(\mathbb{R}^n) \) for \( q : = \frac{np}{n-(\alpha-1)p} \) whenever \( 1 < p < \infty \) and \( 1 \leq \alpha < 1 + \frac{n}{p} \). In particular, since \( q < p^* \), it holds \( \mathcal{M}_\alpha u \in W^{1,q}_{loc}(\mathbb{R}^n) \) if the parameters satisfy \( 1 \leq \alpha < \frac{n}{p} \).

Our goal in Publication I is to extend the results of Kinnunen and Saksman to subsets of \( \mathbb{R}^n \). An especial question of interest is to discover the counterparts of the pointwise estimates (2.14) and (2.15) for the local fractional maximal operator \( \mathcal{M}_{\alpha,\Omega} \). Since the corresponding estimate for the Hardy–Littlewood maximal function remains essentially the same when moving from \( \mathbb{R}^n \) to a general subset, one might also expect the same event in the fractional setting. However, this will not be the case. An additional term will appear in both, the counterpart of (2.14) and of (2.15) for \( \mathcal{M}_{\alpha,\Omega} \).
Simple examples show that, in general, these additional terms cannot be omitted. Consequently, obtaining regularity results as good as in $\mathbb{R}^n$ cannot be guaranteed for general subsets.

All the regularity results above are stated for centered maximal functions in which averages are taken over balls. Furthermore, uncentered maximal functions or maximal functions over cubes could be considered. Simple examples suggest that uncentered maximal functions should be at least as regular as their centered versions. For instance, the uncentered Hardy–Littlewood maximal function of the characteristic function of a ball is continuous, but the centered maximal function is not. In [43], it is noted that in the global case the maximal operator over cubes behaves similarly as the maximal operator over balls. In the local case, however, there are examples showing that the smoothing properties of the maximal operator over cubes are much worse.
Maximal operators
3. Geometric structures

3.1 Metric measure spaces and dyadic cubes

A triple \((X, d, \mu)\) consisting of a set \(X\), a metric \(d\), and a Borel regular doubling measure \(\mu\) is called a metric measure space with doubling measure. The doubling condition means that there exists a constant \(C_\mu > 1\) such that

\[
0 < \mu(B_{2r}(x)) \leq C_\mu \mu(B_r(x)) < \infty
\]  

(3.1)

for all metric balls \(B_r(x) := \{ y \in X : d(x, y) < r \}\), \(x \in X, 0 < r < \infty\).

It is quite standard to add some more assumptions to those mentioned above, such as completeness, local compactness, a measure lower bound condition, and validity of a Poincaré inequality. As general references on the subject, we mention [30] and [5].

The existence of a doubling measure is sufficient to construct a family of sets in metric spaces that has many similar properties to the standard dyadic cubes

\[
\left\{ 2^{-k}([0,1)^n + j) : k \in \mathbb{Z}, j \in \mathbb{Z}^n \right\}
\]

in Euclidean spaces \(\mathbb{R}^n\). Such sets are called dyadic cubes in metric measure spaces. They form a natural tree structure of generations such that two dyadic cubes are either disjoint or one is contained in the other, and each generation covers the whole space \(X\). The generation determines the size of the dyadic cube. A dyadic cube \(Q\) of generation \(k \in \mathbb{Z}\) satisfies

\[
B_{C_0 \delta^k}(z) \subset Q \subset B_{C_0 \delta^{k-1}}(z),
\]

(3.2)

for some \(z \in Q\) and for universal constants \(c_0 > 0, C_0 > 0,\) and \(\delta \in (0,1)\) depending only on the doubling constant \(C_\mu\). That is, dyadic cubes are almost like metric balls with radii exponentially proportional to their generations.
The family of all dyadic cubes will be denoted by $D$ and the family of dyadic cubes of generation $k$ by $D_k$. If $Q \subset Q^*$ for some $Q \in D_{k+1}$ and $Q^* \in D_k$, then $Q^*$ is called the parent of $Q$ and $Q$ is called a child of $Q^*$. Every dyadic cube has exactly one parent, whereas the number of children a dyadic cube can have is bounded by a universal constant. In particular, the dyadic doubling property

$$\mu(Q^*) \leq C^* \mu(Q) \quad (3.3)$$

holds for a universal constant $C^* > 1$. Dyadic cubes were first constructed in metric measure spaces by Michael Christ in [18] and the construction was later improved, for example, by Hytönen and Kairema in [32].

### 3.2 Functions of bounded mean oscillation

A locally integrable function $f$ has bounded mean oscillation in $(X,d,\mu)$, denoted by $f \in \text{BMO}$, if

$$\|f\|_{\text{BMO}} := \sup_B \int_B |f - f_B| \, d\mu < \infty, \quad (3.4)$$

where the supremum is taken over all metric balls $B \subset X$. Here

$$f_B := \int_B f \, d\mu := \frac{1}{\mu(B)} \int_B f \, d\mu$$

denote the integral average. Similarly, $f$ has bounded dyadic mean oscillation, denoted by $f \in \text{BMO}_D$, if (3.4) holds with balls $B$ replaced with dyadic cubes in $D$. It follows from (3.2) that $\text{BMO} \subset \text{BMO}_D$, but the converse is not true, which can already be seen in the Euclidean setting. Local versions of BMO are defined by taking the supremum in (3.4) only over balls or dyadic cubes contained in a measurable subset $A \subset X$. The local BMO spaces are denoted by $f \in \text{BMO}(A)$ and $f \in \text{BMO}_D(A)$, respectively. In particular, it is often convenient to consider $\text{BMO}_D(Q_0)$ for some initial dyadic cube $Q_0$.

In $\mathbb{R}^n$, the connection between BMO and dyadic BMO was widely studied first in [27], in which Garnett and Jones gave new proofs for certain theorems concerning BMO functions. Their idea was first to prove the easier dyadic version of each theorem and then to obtain the general version by averaging over the dyadic results over translations in $\mathbb{R}^n$. A crucial part of their proofs is that the translation average of a suitable family of dyadic BMO functions belongs to BMO. In [58], Sergei Treil offered a different way to obtain BMO from dyadic BMO by showing that the BMO
norm is comparable to the expectation of dyadic BMO norms over suitable randomized dyadic systems. Since this approach does not need the translation structure of $\mathbb{R}^n$, it can be extended to metric measure spaces.

### 3.3 Randomized dyadic systems

In metric spaces, randomized dyadic systems have been constructed, for instance, by Hytönen, Martikainen, and Kairema in [33, 32], and the construction has been simplified by Hytönen and Tapiola in [34]. In the randomized dyadic systems, there is a probability space $(\Omega, \mathbb{P})$ defining a system of dyadic cubes $\mathcal{D}(\omega)$ for each $\omega \in \Omega$ with the basic properties of dyadic systems, also satisfying the following property. There exist constants $C > 0$ and $\eta > 0$ such that

$$\mathbb{P}\left( \left\{ \omega \in \Omega : x \in \bigcup_{Q \in \mathcal{D}_k(\omega)} \partial_\varepsilon Q \right\} \right) \leq C \left( \frac{\varepsilon}{\delta^k} \right)^\eta \quad (3.5)$$

for every $x \in X$, $k \in \mathbb{Z}$, and $\varepsilon > 0$, where $\delta$ is as in (3.2) and

$$\partial_\varepsilon Q := \left\{ x \in Q : d(x, X \setminus Q) < \varepsilon \right\} \cup \left\{ x \in X \setminus Q : d(x, Q) < \varepsilon \right\}$$

is a boundary region around $\partial Q$. The condition (3.5) means that the probability of a point ending up near the boundary of a random dyadic cube is small. In the construction of randomized dyadic systems, all scales are typically randomized independently. In particular, [34] uses a sample space of the form

$$\Omega = \left\{ 0, 1, \ldots, \left\lfloor \frac{1}{\delta} \right\rfloor \right\}^{\mathbb{Z}} \quad (3.6)$$

and independent uniform probability distributions in all generations $k \in \mathbb{Z}$. The construction is based on randomizing radii of certain balls determining the dyadic cubes.

In [17], Chen, Li, and Ward studied the connection between BMO and dyadic BMO in metric measure spaces using randomized dyadic systems satisfying (3.5). They showed that if there is a family $\{ f^\omega \}_{\omega \in \Omega}$ of functions with uniformly bounded dyadic BMO norms in $\omega$ such that the mapping $\omega \mapsto f^\omega$ is measurable, then the expectation $f = \mathbb{E}[f^\omega]$ over $\omega \in \Omega$ belongs to BMO and satisfies

$$\|f\|_{BMO} \leq C \sup_{\omega \in \Omega} \|f^\omega\|_{BMO_{D(\omega)}} \quad (3.7)$$

for some universal constant $C$. However, applying their result is problematic since the desired measurability is difficult to check. Even in such a
simple case as (3.6), $\Omega$ is uncountable and its connection to the related probability measure $P$ does not allow trivial ways to conclude the measurability.

Our goal in Publication II is to consider finitely randomized dyadic systems in order to overcome the measurability questions of [17]. We define a finite version of the probability space $(\Omega, P)$ in (3.6) for every $m \in \mathbb{N}$ as $(\Omega_m, P_m)$, where the sample space is given by

$$\Omega_m = \prod_{k < -m} \{0\} \times \prod_{-m \leq k \leq m} \left\{0, 1, \ldots, \left\lfloor \frac{1}{\delta} \right\rfloor \right\} \times \prod_{k > m} \{0\} \quad (3.8)$$

and $P_m$ has a uniform distribution in $\Omega_m$. That is, the construction of dyadic systems is randomized only in a finite number of generations $k$. Consequently, every subset of $\Omega_m$ is automatically measurable with respect to $P_m$. In addition, all the basic properties of dyadic systems remain to hold since $\Omega_m \subset \Omega$. By slightly modifying the construction in [34], we obtain the following weaker version of (3.5) for $(\Omega_m, P_m)$. There exist constants $C > 0$ and $\eta > 0$ such that

$$P_m \left( \left\{ \omega \in \Omega_m : x \in \bigcup_{Q \in D_k(\omega)} \partial \varepsilon Q \right\} \right) \leq C \left( \frac{\varepsilon}{\delta^k} \right)^{\eta} \quad (3.9)$$

for every $m \in \mathbb{N}, x \in X, k \in \{-m, \ldots, m\}$, and $\varepsilon \geq \delta^m$. When $m \to \infty$, the estimate (3.9) essentially becomes (3.5).
4. Summaries of the articles I–II

4.1 Regularity of the local fractional maximal function

In Publication I, we consider regularity of the local fractional maximal function extending the results in [43] to a general open subset $\Omega \subset \mathbb{R}^n$. Let us first focus on the case where we only assume $u \in L^p(\Omega)$. We are able to prove a pointwise estimate

$$|DM_{\alpha,\Omega} u(x)| \leq C(M_{\alpha-1,\Omega} u(x) + S_{\alpha-1,\Omega} u(x)) \quad \text{for a.e. } x \in \Omega \quad (4.1)$$

with some constant $C$ depending only on $n$ and $\alpha$. This is the counterpart of (2.15) containing an additional term with the local spherical fractional maximal function. Due to the presence of $S_{\alpha-1,\Omega}$, the ranges of parameters are restricted: $n \geq 2$, $p > \frac{n}{n-\alpha}$, and $1 \leq \alpha < \min \left\{ \frac{n-1}{p}, n - \frac{2n}{(n-1)p} \right\} + 1$. The estimate (4.1) is proved similarly as the estimate (2.12) in [42]. It is first shown for the fractional averages

$$u_t^\alpha(x) := (t\delta(x))^\alpha \int_{B_t(\delta(x))} u(y) \, dy, \quad \delta(x) := \text{dist}(x, \mathbb{R}^n \setminus \Omega), \; t \in (0, 1),$$

relative to the distance from the boundary, firstly in the case of smooth functions explicitly and then for general $L^p$-functions by approximation. The spherical maximal function appears when using Gauss’ theorem. By replacing the parameter $t$ with an enumeration of the rationals in $(0, 1)$ and applying a weak compactness argument, we have a convergence for the weak gradients of a sequence of fractional averages. The sequence of the weak gradients converges to $|DM_{\alpha,\Omega} u|$. Combining the pointwise estimate (4.1) with the boundedness result (2.7) and with the local counterpart of (2.9), implies that $|DM_{\alpha,\Omega} u| \in L^q(\Omega)$ with $q = \frac{np}{n-(\alpha-1)p}$. In particular, $M_{\alpha,\Omega} u \in W^{1,q}_{\text{loc}}(\Omega)$ which is the same result as in $\mathbb{R}^n$, although holding in a smaller range of parameters.
Moreover, by applying a Hardy-type result determining when a Sobolev function has zero boundary values in Sobolev’s sense, we actually obtain that \( M_{\alpha, \Omega} u \in W^{1,q}_0(\Omega) \) if \( \Omega \) has a finite measure. This is what one expects according to the definition of the local fractional maximal function.

Let us then concentrate on the case where \( u \) itself belongs to a Sobolev space \( W^{1,p}(\Omega) \). In the same way as (4.1), we obtain a pointwise estimate

\[
|D M_{\alpha, \Omega} u(x)| \leq 2 M_{\alpha, \Omega} |Du|(x) + \alpha M_{\alpha-1, \Omega} u(x) \quad \text{for a.e. } x \in \Omega \tag{4.2}
\]

whenever \( n \geq 2, 1 < p < n \), and \( 1 \leq \alpha < \frac{n}{p} \). The only essential difference in the proof is that one of the Green’s formulas is used instead of Gauss’ theorem. Again, there is an additional term in the estimate compared to its counterpart (2.14) for \( D M_{\alpha} u \). Due to the additional term \( M_{\alpha-1, \Omega} u \), one cannot obtain a mapping result including \( W^{1,p^*}_\text{loc} \) in a general subset \( \Omega \), unlike in \( \mathbb{R}^n \). Instead, we have \( M_{\alpha, \Omega} u \in W^{1,q}_\text{loc}(\Omega) \) as in the case of \( L^p \)-functions. However, the parameter range is wider since no spherical maximal function is needed. On the other hand, if \( \Omega \) supports the Sobolev inequality

\[
\|u\|_{L^{p^*}(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}, \quad p^* := \frac{np}{n-p}, \tag{4.3}
\]

then it follows that \( M_{\alpha, \Omega} u \in W^{1,p^*}(\Omega) \) with \( p^* = \frac{np}{n-\alpha_0 p} \), that is, the regularity is the same as in \( \mathbb{R}^n \). This is the case, for instance, if \( \Omega \) is bounded with a \( C^1 \)-boundary.

We also extend our regularity results to metric measure spaces with a doubling measure satisfying a measure lower bound and supporting a Poincaré inequality. We use a discrete version of the maximal operator, because the standard maximal operators do not have the required regularity properties without any additional assumptions on the metric and measure. Discrete maximal operators have been studied earlier, for example, in [1, 29].

4.2 Finitely randomized dyadic systems on BMO

In Publication II, we consider finitely randomized dyadic systems on BMO extending the results in [17] to the finitely randomized case. The proof for the connection between BMO and dyadic BMO goes similarly as the proof of (3.7) in [17], by using a telescoping sum argument to decompose the expectation into small and large scales and considering them separately. However, due to the weaker probability condition (3.9), we get a BMO estimate only for balls with radius of magnitude between \( \delta^m \) and \( \delta^{-m} \).
We also have to assume certain uniform boundedness in $\omega$ in addition to uniformly bounded BMO norms. We get roughly the following result when taking into account the assumptions mentioned above. If there is a family $\{f^\omega\}_{\omega \in \Omega_m}$ of functions with uniformly bounded dyadic BMO norms in $\omega$, then the expectation $f_m = E_m[f^\omega]$ over $\omega \in \Omega_m$ satisfies

$$\int_B |f_m - (f_m)_B| \, d\mu \leq C \sup_{\omega \in \Omega_m} \|f^\omega\|_{\text{BMO}_D(\omega)} + C_m$$

(4.4)

for some universal constant $C$ and for a ball $B \subset X$. When $m \to \infty$, the quantity $C_m$ tends to 0 and all the restrictions of $B$ vanish, leading to (3.7).

As an application of our result, we prove a theorem of Uchiyama [60], on a construction of certain BMO functions, in metric measure spaces. It is one of the results considered by Garnett and Jones in [27], where they use translated dyadic structures in $\mathbb{R}^n$ to derive results from their dyadic counterparts. The dyadic version of Uchiyama’s theorem follows in the metric setting similarly as in [27], since all the arguments are based on dyadic structures, for instance, dyadic Calderón–Zygmund type decompositions. After having the dyadic theorem, it is relatively simple to derive the non-dyadic version from the dyadic one by applying (4.4).
Summaries of the articles I–II
5. Integro-differential equations

5.1 Fractional Laplace equations

We start this chapter by more precisely considering fractional Laplace equations. We adopt the definition of the fractional Laplacian of order $s \in (0, 1)$ as the singular integral

$$\mathcal{L}^s u(x) := \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy,$$

(5.1)

Here the symbol P.V. means "in the principal value sense" and is defined in the latter equation when the limit exists. Principal values are needed in (5.1) because the integrand is typically not integrable in the usual sense, but cancellations make the limit exist for nice enough functions, such as $C^2$-functions. Similarly as the derivative of a function may not exist at every point, the principal value does not have to exist everywhere. The operator $\mathcal{L}^s$ is nonlocal in the sense that $\mathcal{L}^s u(x)$ depends on the values of $u$ arbitrarily far from $x$, not only in a small neighborhood. Due to the nonlocality, $u$ is always assumed to be defined in the whole $\mathbb{R}^n$.

The singular integral in (5.1) appears, for instance, as a limit of a long jump random walk. That is, the fractional Laplacian is similarly related to certain Lévy processes as the usual Laplacian is related to the Brownian motion. To see this, one considers a discrete random walk of a particle in a grid $h\mathbb{Z}^n$, $h > 0$, with probability $\mathcal{P}(k) \propto |k|^{-n-2s}$ to move along the vector $hk$, $k \in \mathbb{Z}^n$, in a time step $h^{2s}$. Defining $u(x,t)$ to be the probability
that the particle lies at \( x \in h\mathbb{Z}^n \) at time \( t \in h^{2s}\mathbb{Z} \), one has
\[
\frac{u(x, t + h^{2s}) - u(x, t)}{h^{2s}} = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \mathcal{P}(k) \frac{u(x + hk, t) - u(x, t)}{h^{2s}} \sim \sum_{k \in \mathbb{Z}^n \setminus \{0\}} h^n \frac{u(x + hk, t) - u(x, t)}{|hk|^{n+2s}},
\]
and letting \( h \to 0 \) implies
\[
\partial_t u(x, t) \propto \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x + z, t) - u(x, t)}{|z|^{n+2s}} \, dz.
\]
The right-hand side above coincides with the opposite of (5.1) after a change of variables \( y = x + z \) and dropping the time dependence. For an elementary reference on the fractional Laplacian, we mention [61], see also [13].

The definition of the fractional Laplacian generalizes to nonlinear settings. The fractional \( p \)-Laplacian with differentiability order \( s \in (0, 1) \) and summability growth \( p > 1 \) is defined as
\[
(-\Delta)^s_p u(x) := \text{P.V.} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+sp}} \, dy. \tag{5.2}
\]
When \( p = 2 \), we obtain the fractional Laplacian. When \( p \neq 2 \), the operator \((-\Delta)^s_p\) is not only nonlocal, but also nonlinear. The fractional \( p \)-Laplacian can be seen as a nonlocal counterpart of the \( p \)-Laplacian\( \Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) \). For instance, solutions of related Dirichlet problems for the fractional \( p \)-Laplace equation \((-\Delta)^s_p u = 0\) converge, up to a multiplicative constant, to solutions of the \( p \)-Laplace equation \( \Delta_p u = 0 \) as \( s \to 1 \), as seen in [35]. For related convergence results as \( s \to 1 \), we mention [8, 11]. There are also different approaches in nonlinearizing the fractional Laplacian, such as in [4, 54, 16].

The most typical framework to study fractional Laplace equations consists of Dirichlet boundary value problems in a domain \( \Omega \subset \mathbb{R}^n \), of the form
\[
\begin{cases}
(-\Delta)^s_p u = 0 & \text{in } \Omega, \\
u = g & \text{on } \mathbb{R}^n \setminus \Omega.
\end{cases} \tag{5.3}
\]
The boundary values are set in the whole complement of the considered domain, not just on the boundary \( \partial \Omega \), since, by the nonlocality, the values on \( \partial \Omega \) are insufficient to uniquely determine the interior behavior. In the fractional Laplacian setting, there are solution formulas that are somewhat similar to those for the usual Laplacian, see, for example, [12]. In particular, when the domain is a ball \( B_r(0) \subset \mathbb{R}^n \), there is a Poisson
formula for the solution of the Dirichlet problem (5.3) with $p = 2$, read as

$$u(x) = c_{n,s} \int_{\mathbb{R}^n \setminus B_r(0)} g(y) \left( \frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^s |x - y|^{-n} \, dy, \quad x \in B_r(0),$$

(5.4)

where $c_{n,s}$ is a constant depending on $n$ and $s$. For a general $p$, it is challenging to create nontrivial solutions of equations such as (5.3). However, all affine functions solve $(-\Delta)^s p u = 0$ because of symmetric cancellations.

It is usual to consider more general nonlocal integro-differential equations of the form

$$Lu(x) := \text{P.V.} \int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}(u(x) - u(y)) K(x, y) \, dy = 0,$$

(5.5)

where $K: \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty]$ is a uniformly elliptic and symmetric kernel of order $(s, p)$, that is,

$$\Lambda^{-1} \leq K(x, y)|x - y|^{n+sp} \leq \Lambda \quad \text{for all } x, y \in \mathbb{R}^n, x \neq y,$$

(5.6)

$$K(x, y) = K(y, x) \quad \text{for all } x, y \in \mathbb{R}^n,$$

(5.7)

with $\Lambda \geq 1$, $s \in (0, 1)$, and $p > 1$. When dealing with solutions in a viscosity approach, we also assume translation invariance and continuity, that is,

$$K(x, y) = K(x + z, y + z) \quad \text{for all } x, y, z \in \mathbb{R}^n, x \neq y,$$

(5.8)

$$x \mapsto K(x, y) \text{ is continuous in } \mathbb{R}^n \setminus \{y\} \text{ for all } y \in \mathbb{R}^n.$$

(5.9)

In the literature, when considering viscosity solutions, the model kernel $|x - y|^{-n - sp}$ is typically generalized in a slightly different way. However, the generalizations coincide when (5.6)–(5.8) hold. In addition to the fractional $p$-Laplace equation, (5.5) includes fractional Laplace equations with measurable coefficients as special cases. The reader may always keep in mind the model case $K(x, y) = |x - y|^{-n - sp}$.

### 5.2 Fractional Sobolev spaces

Fractional Sobolev spaces are needed when defining weak solutions for equations as (5.5). Let $s \in (0, 1)$, $p \geq 1$, and let $\Omega \subset \mathbb{R}^n$ be an open set. The fractional Sobolev spaces are defined as

$$W^{s,p}(\Omega) := \left\{ f \in L^p(\Omega) : \frac{|f(x) - f(y)|}{|x - y|^{n+sp}} \in L^p(\Omega \times \Omega) \right\}.$$

(5.10)

They are intermediary Banach spaces between $L^p(\Omega)$ and $W^{1,p}(\Omega)$ in the sense that

$$W^{1,p}(\Omega) \subset W^{s,p}(\Omega) \subset L^p(\Omega)$$

(5.11)
when $\Omega$ is regular enough. In addition, $W^{s,p}(\Omega) \subset W^{\sigma,p}(\Omega)$ when $0 < \sigma \leq s < 1$. The fractional Sobolev space $W^{s,p}(\Omega)$ is endowed with the natural norm
\[
\|f\|_{W^{s,p}(\Omega)} := \|f\|_{L^p(\Omega)} + [f]_{W^{s,p}(\Omega)}
\]
\[
= \left( \int_{\Omega} |f|^p \, dx \right)^{\frac{1}{p}} + \left( \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{np + sp}} \, dx \, dy \right)^{\frac{1}{p}},
\]
where $[f]_{W^{s,p}(\Omega)}$ is called the Gagliardo seminorm of $f$. Notation $f \in W^{s,p}_{loc}(\Omega)$ means that $f \in W^{s,p}(D)$ for every compactly contained open set $D \Subset \Omega$. Fractional Sobolev functions are assumed to be extended to 0 outside $\Omega$, and by $W^{s,p}_0(\Omega)$ we denote the closure of $C^\infty_0(\Omega)$ in $W^{s,p}(\mathbb{R}^n)$. There are also two alternative notions of fractional Sobolev spaces with zero boundary values in the literature: the closure of $C^\infty_0(\Omega)$ in $W^{s,p}(\Omega)$ and those functions in $W^{s,p}(\mathbb{R}^n)$ that vanish outside $\Omega$. The latter class is equal to $W^{s,p}_0(\Omega)$ when $\Omega$ is a bounded open Lipschitz set, and they all coincide if in addition $sp \neq 1$.

Fractional Sobolev spaces have various fractional Poincaré and fractional Sobolev inequalities. In fractional Poincaré inequalities, $L^p$-norms are controlled by Gagliardo seminorms. Two useful ones are the following: If $f \in W^{s,p}(B_r)$ for a ball $B_r \subset \mathbb{R}^n$, then
\[
\left( \int_{B_r} |f - f_{B_r}|^p \, dx \right)^{\frac{1}{p}} \leq cr^s[f]_{W^{s,p}(B_r)},
\]
and if $f \in W^{s,p}_0(\Omega)$ and $\Omega \Subset \Omega'$, then
\[
\|f\|_{L^p(\Omega')} \leq C[f]_{W^{s,p}(\Omega')},
\]
where $c$ depends only on $n$, $s$, and $p$, and $C$ is independent of $f$. Fractional Sobolev inequalities, in turn, have $L^{p^*}$-norms on the left-hand side with the fractional Sobolev exponent $p^* = \frac{np}{n - sp}$. In particular, there are compact embeddings of the form
\[
W^{s,p}(B_r) \hookrightarrow L^q(B_r) \quad \text{for all } q \in [1, p^*)
\]
when $sp \leq n$. In the complementary case $sp > n$, fractional Sobolev functions are Hölder continuous satisfying compact embeddings
\[
W^{s,p}(B_r) \hookrightarrow C^{0,\alpha}(\overline{B_r}) \quad \text{for all } \alpha \in (0, s - \frac{n}{p}).
\]

The Hölder continuity condition for $\alpha \in (0, 1]$ means that
\[
\|f\|_{C^{0,\alpha}(\Omega)} := \sup_{x,y \in \Omega, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.
\]
We refer to [25] for more central properties of fractional Sobolev spaces, see also [2, 22, 10, 11].

Tail spaces, which are closely related to fractional Sobolev spaces, are needed for controlling nonlocal contributions. They are defined as

\[ L^{p-1}_{sp}(\mathbb{R}^n) := \left\{ f \in L^{p-1}_{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|f(x)|^{p-1}}{(1 + |x|)^{n+sp}} \, dx < \infty \right\} \quad (5.18) \]

for \( s \in (0, 1) \) and \( p > 1 \). The space \( L^{p-1}_{sp}(\mathbb{R}^n) \) consists of functions that do not grow unduly fast at a distance. In particular, bounded functions and the functions in \( W^{s,p}(\mathbb{R}^n) \) belong to the tail space. The nonlocal contributions typically appear in the form of

\[ \text{Tail}(f; z, r) := \left( r^{sp} \int_{\mathbb{R}^n \setminus B_r(z)} |f(x)|^{p-1} |x - z|^{-n-sp} \, dx \right)^{\frac{1}{p-1}}, \quad (5.19) \]

which is called the nonlocal tail of the function \( f \) in ball of radius \( r > 0 \) centered in \( z \in \mathbb{R}^n \). The coefficient \( r^{sp} \) and the power \( \frac{1}{p-1} \) render the tail homogeneous in such a way that the tail of a constant is just the constant itself multiplied by a universal constant. If \( f \in L^{p-1}_{sp}(\mathbb{R}^n) \), then \( \text{Tail}(f; z, r) < \infty \) for all \( z \in \mathbb{R}^n \) and \( r > 0 \).

5.3 Weak solutions

As in the local setting, a formula for weak solutions can be derived from \( \mathcal{L}u = 0 \) by multiplying by a test function \( \eta \in C_0^\infty(\Omega) \) and integrating over \( \mathbb{R}^n \). To make the formula symmetric, instead of integrating by parts in the standard manner, one has to integrate by parts fractionally. This is achieved by adding another term with the roles of \( x \) and \( y \) interchanged.

This leads to the following weaker concept of solutions. A function \( u \in W^{s,p}_{loc}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n) \) is a fractional weak solution of (5.5) in \( \Omega \) if

\[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2} (u(x) - u(y)) (\eta(x) - \eta(y)) K(x, y) \, dx \, dy = 0 \quad (5.20) \]

for every \( \eta \in C_0^\infty(\Omega) \). The class \( W^{s,p}_{loc}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n) \) above is essentially the largest function class in which the left-hand side of (5.20) is always finite. By approximation, any \( \eta \in W^{s,p}_0(D) \) with \( D \subseteq \Omega \) works as a test function in (5.20). Furthermore, if \( u \) is known to be in \( W^{s,p}(\Omega') \) for some \( \Omega' \supseteq \Omega \), any \( \eta \in W^{s,p}_0(\Omega) \) suffices. The formula (5.20) also follows when minimizing the following fractional energy functional

\[ f \mapsto \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - f(y)|^p K(x, y) \, dx \, dy, \quad (5.21) \]
which coincides with \([f]_n^{\alpha} \in W^{\alpha,p}(\mathbb{R}^n)\) in the model case \(K(x, y) = |x - y|^{-n-\alpha}\).

Often it is more flexible to work with supersolutions and subsolutions instead of solutions. A function \(u \in W^{\alpha,p}_{\text{loc}}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)\) is a fractional weak supersolution of (5.5) in \(\Omega\) if
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}(u(x) - u(y))(\eta(x) - \eta(y))K(x, y) \, dx \, dy \geq 0
\]
for every nonnegative \(\eta \in C_0^\infty(\Omega)\). Similarly, a function \(u\) is a fractional weak subsolution of (5.5) in \(\Omega\) if \(-u\) is a fractional weak supersolution, or equivalently
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}(u(x) - u(y))(\eta(x) - \eta(y))K(x, y) \, dx \, dy \leq 0
\]
for every nonnegative \(\eta \in C_0^\infty(\Omega)\). It is not difficult to see that \(u\) is a fractional weak solution, if it is both a fractional weak supersolution and a subsolution. In short, the functions above will be called weak solutions, weak supersolutions, and weak subsolutions, respectively.

In [24] and [23], Di Castro, Kuusi, and Palatucci have proved essential regularity results for weak solutions defined in (5.20). In all their estimates, the nonlocal tail (5.19) appears on the right-hand side. In the linear setting, similar results have been proved by Moritz Kassmann in [38].

It is usual that only either the positive part \(u_+ := \max\{u, 0\}\) or the negative part \(u_- := \max\{-u, 0\}\) of \(u\) is needed in the tail. The first key result is a Caccioppoli-type estimate in \(B_r(x_0) \subset \Omega\) for controlling the fractional energy
\[
\int_{B_r(x_0)} \int_{B_r(x_0)} |w(x)\phi(x) - w(y)\phi(y)|^p K(x, y) \, dx \, dy
\]
of the truncations \(w = (u - k)_+\) of a weak subsolution \(u\), for any \(\phi \in C_0^\infty(B_r(x_0))\) and \(k \in \mathbb{R}\). From the Caccioppoli estimate, it follows that weak subsolutions are locally bounded from above and satisfy the following boundedness estimate
\[
\sup_{B_r(x_0)} u \leq c \left( \int_{B_{2r}(x_0)} \frac{u^p}{r^p} \, dx \right)^{\frac{1}{p}} + \text{Tail}(u_+; x_0, r)
\]
for any \(B_{2r}(x_0) \subset \Omega\).

Furthermore, a suitable De Giorgi–Nash–Moser iteration leads to local Hölder continuity and certain nonlocal Harnack inequalities for weak solutions. The oscillation \(\text{osc}_B u := \sup_B u - \inf_B u\) of a weak solution \(u\) satisfies the Hölder continuity estimate
\[
\text{osc}_{B_r(x_0)} u \leq c \left( \frac{r^\alpha}{r} \right)^{\frac{\alpha}{p}} \left[ \left( \int_{B_{2r}(x_0)} |u|^p \, dx \right)^{\frac{1}{p}} + \text{Tail}(u; x_0, r) \right]
\]
for any \(B_{2r}(x_0) \subset \Omega\).
for any $B_{2r}(x_0) \subset \Omega$ and $\rho \in (0,r]$, and for some Hölder continuity exponent $\alpha$ depending on $n$, $p$, and $s$. If the weak solution is nonnegative in $B_{2r}(x_0) \subset \Omega$, it satisfies a nonlocal Harnack inequality of the form

$$\sup_{B_r(x_0)} u \leq c \inf_{B_{2r}(x_0)} u + c \text{Tail}(u_-, x_0, 2r).$$

(5.27)

For weak supersolutions, there is a weaker Harnack inequality

$$\left( \int_{B_r(x_0)} u^t \, dx \right)^{\frac{1}{t}} \leq c \inf_{B_{2r}(x_0)} u + c \text{Tail}(u_-, x_0, 2r)$$

(5.28)

for any $t \in (0,\bar{t})$, for a suitable $\bar{t}$. Additional related estimates can be found in [46] which considers equations as (5.5) with measure data on the right-hand side. There are also higher regularity results [9] by Brasco and Lindgren in the case $p \geq 2$.

### 5.4 $(s, p)$-harmonic functions

In Potential Theory, the concept of harmonic functions is more useful than weak solutions. In nonlinear context, the notion of superharmonic functions appeared first in [28]. In [49], Peter Lindqvist defines $p$-superharmonic functions for the $p$-Laplace equation as lower semicontinuous functions that are not identically infinity and obey a comparison principle with continuous solutions to the $p$-Laplace equation. The $p$-superharmonic functions form a nice class of functions which plays a central role in nonlinear Potential Theory, allowing one to prove various nonlinear counterparts of the results in the classical linear Potential Theory [26]. In particular, they are used in Perron’s method to construct solutions for general boundary data. As a general reference on nonlinear Potential Theory, we mention [31].

We define $(s, p)$-superharmonic functions in the nonlocal framework for the equation (5.5). Our approach is similar to the definition of $p$-superharmonic functions with two essential differences needed due to the nonlocal character. Firstly, $(s, p)$-superharmonic functions are defined in the whole $\mathbb{R}^n$, thus requiring that they are controlled far away. More precisely, the negative part of the function has to belong to the tail space. Secondly, in the comparison principle, the functions have to be in order in the whole complement, not just on the boundary. The reason for this is that the boundary behavior is not, in the nonlocal setting, sufficient to determine the events occurring inside the domain, as already mentioned in connection with Dirichlet problems.
**Definition 5.4.1.** We say that a function $u: \mathbb{R}^n \to [-\infty, \infty]$ is an $(s,p)$-superharmonic function in $\Omega$ if it satisfies the following four assumptions.

(i) $u < +\infty$ almost everywhere in $\mathbb{R}^n$, and $u > -\infty$ everywhere in $\Omega$.

(ii) $u$ is lower semicontinuous in $\Omega$.

(iii) $u$ satisfies the comparison in $\Omega$ against solutions, that is, if $D \subset \subset \Omega$ is an open set and $v \in C(\overline{D})$ is a weak solution of (5.5) in $D$ such that $u \geq v$ on $\partial D$ and almost everywhere on $\mathbb{R}^n \setminus D$, then $u \geq v$ in $D$.

(iv) $u_-$ belongs to $L_p^{s-1}(\mathbb{R}^n)$.

Similarly, a function $u$ is an $(s,p)$-subharmonic function in $\Omega$ if $-u$ is $(s,p)$-superharmonic. Finally, if it is both $(s,p)$-superharmonic and $(s,p)$-subharmonic in $\Omega$, we call it $(s,p)$-harmonic in $\Omega$.

It is notable that $(s,p)$-superharmonic functions are defined pointwise in $\Omega$, but only up to a measure zero in the complement. From the definition, it is clear that the most obvious properties of $p$-superharmonic functions are also satisfied by $(s,p)$-superharmonic functions. In particular, if $u$ and $v$ are $(s,p)$-superharmonic, so are the pointwise minimum $\min\{u,v\}$ and affine transforms $au + b$ for $a > 0$ and $b \in \mathbb{R}$. Moreover, if $u$ is $(s,p)$-superharmonic in $\Omega$, it is $(s,p)$-superharmonic in any open subset of $\Omega$.

### 5.5 Viscosity solutions

The third concept of solutions we are going to consider are viscosity solutions. The notion of viscosity solutions was introduced by Crandall and Lions in [21], and is based on the pointwise evaluation of the considered partial differential equation with a test function. A lower semicontinuous function is said to be a viscosity supersolution of a given equation if every $C^2$-function touching the function from below, is a pointwise supersolution to the equation in the touching point. In [37], Juutinen, Lindqvist, and Manfredi proved that for the $p$-Laplace equation viscosity supersolutions and $p$-superharmonic functions coincide. Later, Julin and Juutinen provided a simpler proof in [36].

In [47], Erik Lindgren defines viscosity solutions for fractional $p$-Laplace type equations based on the approach of Caffarelli and Silvestre in the pure fractional Laplacian case [14]. Compared to its local analogy, an essential feature in the definition is that the test function is chosen to be the solution candidate itself in the complement of a neighborhood of the touching point. Such a test function is needed because of the nonlocal character of the operator $\mathcal{L}$ in (5.5). We adopt a slightly more general
Integro-differential equations

Definition, which is valid for every $p > 1$ and does not assume the function to be bounded. Here the definition of viscosity supersolutions is stated only in the case $p > \frac{2}{2-s}$. When $1 < p \leq \frac{2}{2-s}$, the condition (iii) below needs certain additional assumptions on the test function $\phi$ if $\nabla \phi(x_0) = 0$, see Publication V.

**Definition 5.5.1.** We say that a function $u: \mathbb{R}^n \to [-\infty, \infty]$ is an $(s, p)$-viscosity supersolution of (5.5) in $\Omega$ if it satisfies the following four assumptions.

(i) $u < +\infty$ almost everywhere in $\mathbb{R}^n$, and $u > -\infty$ everywhere in $\Omega$.

(ii) $u$ is lower semicontinuous in $\Omega$.

(iii) If $\phi \in C^2(B_r(x_0))$ for some $B_r(x_0) \subset \Omega$ such that $\phi(x_0) = u(x_0)$ and $\phi \leq u$ in $B_r(x_0)$, then $L_\phi r(x_0) \geq 0$, where

$$
\phi_r = \begin{cases}
\phi & \text{in } B_r(x_0), \\
u & \text{on } \mathbb{R}^n \setminus B_r(x_0).
\end{cases}
$$

(iv) $u_-$ belongs to $L^{p-1}_{\text{loc}}(\mathbb{R}^n)$.

Similarly, a function $u$ is an $(s, p)$-viscosity subsolution in $\Omega$ if $-u$ is an $(s, p)$-viscosity supersolution. Finally, if it is both an $(s, p)$-viscosity supersolution and a subsolution in $\Omega$, we call it an $(s, p)$-viscosity solution in $\Omega$.

Conditions (i), (ii), and (iv) above are exactly the same as for $(s, p)$-superharmonic functions. Here (i) and (iv) are needed to exclude infinities in overly large sets in the case of unbounded functions.

### 5.6 Obstacle problems

Obstacle problems form a central tool in Potential Theory. They are closely related to partial differential equations and can be formulated in several ways. Roughly speaking, when solving an obstacle problem, one is seeking the minimal weak supersolution above a given obstacle function in an open set. In the fractional Laplacian setting, obstacle problems have been studied, for example, by Caffarelli, Silvestre, and Salsa [15, 55]. In particular, their regularity estimates include results stating that the solution to the obstacle problem inherits the regularity of the obstacle.

When considering obstacle problems related to (5.5), we take an approach based on variational inequalities, such as in [40] in the classical local setting. To formulate our definition, we first introduce some notation.
Let $\Omega \Subset \Omega'$ be bounded open subsets of $\mathbb{R}^n$. Here $\Omega$ is the set where the obstacle problem is solved, whereas $\Omega'$ can be seen as a reference set determining the energy class for the solution candidates. Let $h: \mathbb{R}^n \to [-\infty, \infty)$ be an extended real-valued function, which is considered to be the obstacle, and let $g \in W^{s,p}(\Omega') \cap L^{p-1}_{sp}(\mathbb{R}^n)$ be the boundary values. The set of feasible functions is defined as

$$K_{g,h}(\Omega, \Omega') := \left\{ u \in W^{s,p}(\Omega') : u \geq h \text{ in } \Omega, u = g \text{ on } \mathbb{R}^n \setminus \Omega \right\},$$

that is, we are looking for functions with the given boundary values that are above the given obstacle function. If $h \equiv -\infty$, we are just seeking solutions to a Dirichlet boundary value problem.

We define a functional $A_u$ given by

$$A_u(v) := A_1(u) + A_2(u)$$

for every $u \in K_{g,h}(\Omega, \Omega')$ and $v \in W^{s,p}(\Omega')$, where

$$A_1(u) := \int_{\Omega'} \int_{\Omega'} |u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y)) K(x, y) \, dx \, dy,$$

$$A_2(u) := 2 \int_{\mathbb{R}^n \setminus \Omega'} \int_{\Omega} |u(x) - g(y)|^{p-2}(u(x) - g(y))(v(x) - v(y)) K(x, y) \, dx \, dy.$$

The motivation for the functional above is that when also assuming $v \in W^{s,p}_0(\Omega)$, we have

$$A_u(v) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y)) K(x, y) \, dx \, dy,$$

that is, $A_u(v)$ coincides with the double integral in the weak formulation (5.20). Moreover, the functional $A_u$ belongs to the dual of $W^{s,p}(\Omega')$ since for every $u \in K_{g,h}(\Omega, \Omega')$ and $v \in W^{s,p}(\Omega')$ it holds by Hölder’s inequality

$$|A_u(v)| \leq C \left( \|u\|_{W^{s,p}(\Omega')}^{p-1} + \text{Tail}(g; z, r)^{p-1} \right) \|v\|_{W^{s,p}(\Omega')},$$

where $z \in \Omega$, $r = \text{dist}(\Omega, \partial \Omega') > 0$, and $C$ is independent of $u, v,$ and $g$. Here we need the fact that the functions are fractional Sobolev functions in the bigger reference set $\Omega'$.

Now, we provide the definition of solutions to the obstacle problem in the general nonlocal framework considered here.

**Definition 5.6.1.** We say that $u \in K_{g,h}(\Omega, \Omega')$ is a solution to the obstacle problem in $K_{g,h}(\Omega, \Omega')$ if

$$A_u(v - u) \geq 0$$

whenever $v \in K_{g,h}(\Omega, \Omega')$. 

36
When considering boundary regularity of the obstacle problem, the only additional regularity assumption we will need for \( \Omega \) is the following measure density condition for its complement: there are \( r_0 > 0 \) and \( \delta_\Omega \in (0,1) \) such that

\[
\inf_{0 < r < r_0} \frac{|(\mathbb{R}^n \setminus \Omega) \cap B_r(x_0)|}{|B_r(x_0)|} \geq \delta_\Omega
\]  

(5.32)

for every \( x_0 \in \partial \Omega \). If (5.32) holds for a bounded open set \( \Omega \), we call \( \Omega \) a regular set. To obtain any boundary continuity, we will also need to have the boundary data \( g \) itself in the class \( \mathcal{K}_{g,h}(\Omega, \Omega') \). The boundary continuity breaks down, for instance, in such a simple case as \( s p < 1 \), \( \Omega = B_1(0), \Omega' = B_2(0), g \equiv 0, \) and \( h \equiv 1 \). Indeed, \( \chi_{B_1(0)} \in W^{s,p}(B_2(0)) \) when \( s p < 1 \), and a straightforward calculation shows that \( \chi_{B_1(0)} \) solves the obstacle problem (5.31). Anyway, the condition \( g \in \mathcal{K}_{g,h}(\Omega, \Omega') \) is not very restrictive since the most useful obstacles will be \( h \equiv -\infty \) and \( h = g \).

### 5.7 Objectives

We aim at developing nonlocal nonlinear Potential Theory by extending the results of [31] to the nonlocal setting, for equations of the form (5.5) in a bounded open set \( \Omega \). First, this will include proving several properties for fractional weak supersolutions that are known in the local theory, continuing from those proved in [24] and [23]. Then we will need to consider central properties of \((s,p)\)-superharmonic functions to become convinced that they really form a natural function class playing the role of the usual \(p\)-superharmonic functions. For this, we will have to study the nonlocal obstacle problem and show that it has all the essential properties of obstacle problems related to partial differential equations. Applying the \((s,p)\)-superharmonic functions, our main goal will be a nonlocal version of Perron’s method for solving Dirichlet problems with general boundary data. In addition to this, we will focus on the nonlocal counterpart of [37], that is, the equivalence between the notions of \((s,p)\)-superharmonic functions and \((s,p)\)-viscosity supersolutions.
6. Summaries of the articles III–V

6.1 Properties of weak supersolutions

In Publication IV, we first consider fractional weak supersolutions defined in (5.22). We show that they have all the same main properties as weak supersolutions in the standard elliptic theory, such as in the case of the $p$-Laplacian. Even though our results are very expected, the proofs are more complicated than in the local framework since, due to the double integral in (5.22), we also have to deal with nonlocal contributions. Indeed, the nonlocal contributions in $\mathbb{R}^n \setminus \Omega$ do not vanish as the double integral splits

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} = \int_{\Omega} \int_{\Omega} + \int_{\mathbb{R}^n \setminus \Omega} \int_{\mathbb{R}^n \setminus \Omega} + \int_{\mathbb{R}^n \setminus \Omega} \int_{\mathbb{R}^n \setminus \Omega} = \int_{\Omega} \int_{\Omega} + 2 \int_{\mathbb{R}^n \setminus \Omega} \int_{\mathbb{R}^n \setminus \Omega}$$

when the integrand is symmetric and vanishes when both of the variables are outside $\Omega$. The first term on the right-hand side, the local contribution, can usually be treated somewhat similarly to weak formulations in the local setting, by applying the properties of fractional Sobolev spaces instead of the usual Sobolev spaces. The latter term, the nonlocal contribution, typically corresponds to a nonlocal tail defined in (5.19) and can be controlled by means of tail spaces (5.18). Due to the singular kernel, we also need a positive distance between the two sets in the second integral to be able to separate the variables.

The first of the properties of weak supersolutions that we need is a natural nonlocal comparison principle between weak supersolutions and subsolutions. It reveals that if a weak supersolution $u$ is above a weak subsolution $v$ in the complement of an open subset of $\Omega$, the functions must also be in the same order in the interior of the set. The comparison principle follows from the definitions in a straightforward way when choosing $(u-v)_-$ as a test function, splitting the double integrals in (5.22) and (5.23)
into sets of the forms \( \{ u \geq v \} \) and \( \{ u < v \} \), and summing up. The second main property is that weak supersolutions can be chosen to be lower semi-continuous. The result is obtained by carefully interpolating between the local and nonlocal contributions in the boundedness estimate (5.25).

We also obtain various convergence results for sequences of weak supersolutions. The most important one is for a sequence with a locally uniform upper bound that is between two fixed functions in the tail space, indicating that if such a sequence converges pointwise, the limit function is also a weak supersolution. The result is largely based on recent estimates for fractional supersolutions. Iterating the boundedness estimate (5.25), we obtain a locally uniform lower bound for the sequence, and combining this with a Caccioppoli-type estimate yields a locally uniform bound for Gagliardo seminorms. This allows us, applying Hölder’s inequality, to conclude that the difference between the weak formulations of the limit function and a function in the sequence tends to zero.

### 6.2 Solution to the obstacle problem

In Publication III, we prove the existence of a solution to the obstacle problem (5.31) when the set of feasible functions \( \mathcal{K}_{g,h}(\Omega, \Omega') \) is non-empty. The proof is based on the standard theory of monotone operators. We only need to show that the operator \( A \) defined in (5.29) is monotone, coercive, and weakly continuous, and the existence follows, thanks to our functional analytic definition (5.31). A similar approach has been used, for instance, in [40] and [31]. It easily follows from the definition that the solution to the obstacle problem is unique. Indeed, if we assume that there are two solutions, we can test them against each other and conclude that they have to coincide almost everywhere.

When choosing \( v = u + \eta \) in (5.31) with a test function \( \eta \), we see that the solution to the obstacle problem is a weak supersolution. Similarly, choosing \( v = u - \varepsilon \eta \) with small enough \( \varepsilon > 0 \) implies that it is also a weak subsolution in the set where it is strictly above the obstacle. In particular, when the obstacle is identically \(-\infty\), the solution to the obstacle problem is a fractional weak solution. This corresponds to solving a Dirichlet boundary value problem. It is noteworthy that our results generalize the existence theory of Dirichlet problems where the boundary data is required to be in \( W^{s,p}(\mathbb{R}^n) \), since we only need to assume \( g \in W^{s,p}(\Omega') \cap L^{p-1}_{sp}(\mathbb{R}^n) \).
Publication III continues by considering the regularity of the solution to the obstacle problem. To obtain Hölder regularity for the solution, we take a similar approach as in [24]. Since fractional Sobolev functions are automatically Hölder continuous when \( p > \frac{n}{s} \) by (5.16), only the range \( 1 < p \leq \frac{n}{s} \) needs to be considered. The main estimates needed are analogous for interior regularity and boundary regularity. First, certain perturbations of the solution satisfy inequalities similar to the definition of weak subsolutions or supersolutions, and consequently, we obtain Caccioppoli-type estimates as in (5.24). Then we can prove local boundedness estimates, such as (5.25), for certain truncations of the solution, and by applying the boundedness estimates and suitable iteration techniques, we finally achieve Hölder continuity.

Let us discuss the interior regularity in more detail. The solution to the obstacle problem inherits the regularity of the obstacle inside the domain. In particular, if the obstacle function is locally Hölder continuous in \( \Omega \), so is the solution to the obstacle problem. The same also holds for the ordinary continuity. In the set where the solution to the obstacle problem is strictly above the obstacle, it is a weak solution, and the Hölder continuity follows as done in [24]. On the contact set where the solution \( u \) touches the obstacle \( h \), the Hölder continuity follows by iterating an oscillation decay estimate of the form

\[
\text{osc}_{B_r(x_0)} u + \text{Tail}(u - h(x_0); x_0, \sigma r) \\
\leq \frac{1}{2} \left( \text{osc}_{B_r(x_0)} u + \text{Tail}(u - h(x_0); x_0, r) \right) + c \text{ osc}_{B_r(x_0)} h
\]

(6.1)

for \( B_r(x_0) \subset \Omega \) with small enough \( \sigma \). The estimate (6.1) is derived by using the boundedness estimate and a weak Harnack estimate.

Let us then consider the boundary regularity for a regular set \( \Omega \). The solution to the obstacle problem inherits the regularity of the boundary values in a small enough ball \( B_r(x_0) \) for \( x_0 \in \partial \Omega \). Again, the result holds for both the local Hölder continuity and the ordinary continuity. The regularity on the boundary does not depend on the obstacle at all due to the assumption \( g \in K_{g,h}(\Omega, \Omega') \), that is, the boundary values are above the obstacle. One of the keys for proving the boundary regularity is the following
logarithmic estimate

\[ \int_{B_r(x_0)} \int_{B_r(x_0)} \left| \log \frac{w_+(x)}{w_-(y)} \right|^p K(x, y) \, dx \, dy \leq c r^{n-s_p} \left( 1 + \varepsilon^{1-p} \right) \text{Tail}((w_\pm)_-, x_0, 2r^{p-1}) \] (6.2)

satisfied by the truncations

\[ w_\pm := \sup_{B_{2r}(x_0)} (u - k_\pm) - (u - k_\pm) + \varepsilon \]

of the solution \( u \) for every \( \varepsilon > 0 \), for large enough \( k_+ \), and for small enough \( k_- \). The estimate (6.2) follows similarly as a logarithmic estimate proved in [24], when having chosen a suitable test function. The Hölder continuity follows by iterating an oscillation decay estimate behaving as (6.1). The oscillation decay estimate is obtained from the boundedness estimate and the logarithmic estimate (6.2). An additional summary of our results with possible further developments can be found in [45].

### 6.4 Perron’s method

Publication IV also considers \((s, p)\)-superharmonic functions defined in Definition 5.4.1. We are able to extend several essential properties of \(p\)-superharmonic functions to the nonlocal framework, including a connection to weak supersolutions, pointwise behavior, summability results, convergence results, and a natural comparison principle between \((s, p)\)-superharmonic and \((s, p)\)-subharmonic functions. Due to the nonlocality, the proofs are more complicated than in the local case, even though no double integrals are present in the notion of \((s, p)\)-superharmonic functions. The obstacle problem is repeatedly used as a central tool. In particular, it plays the role of solving Dirichlet problems to obtain continuous weak solutions with the right boundary values. To obtain continuous solutions to the obstacle problem, we often need to approximate obstacle functions and boundary data from below by continuous functions.

The flexibility of \((s, p)\)-superharmonic functions suffices to extend Perron’s method to the nonlocal framework determining Perron solutions. We define the upper Perron solution \( H_g \) as the pointwise infimum of the upper
class of functions $u$ satisfying

(i) $u$ is $(s, p)$-superharmonic in $\Omega$,

(ii) $u$ is bounded from below in $\Omega$,

(iii) $\liminf_{\Omega \ni y \to x} u(y) \geq \limsup_{\mathbb{R}^n \setminus \Omega \ni y \to x} g(y)$ for all $x \in \partial \Omega$,

(iv) $u = g$ in $\mathbb{R}^n \setminus \Omega$,

for a bounded open set $\Omega \subset \mathbb{R}^n$ and boundary data $g \in L^{p-1}_{\text{loc}}(\mathbb{R}^n)$. The lower Perron solution $H_g$ is defined symmetrically using $(s, p)$-subharmonic functions. It easily follows from the definitions that the Perron solutions are always in order $\overline{H}_g \geq \underline{H}_g$, and if there is a weak solution $h$, continuous up to the boundary, to the Dirichlet problem with continuous boundary data, then the Perron solutions coincide $\overline{H}_g = h = \underline{H}_g$. We are also able to show that the Perron solutions can be either $(s, p)$-harmonic in $\Omega$ or identically $\pm \infty$ in $\Omega$.

### 6.5 Equivalence of solutions

Publication V shows the equivalence between the notions of $(s, p)$-superharmonic functions and $(s, p)$-viscosity supersolutions under the assumptions (5.6)–(5.9). When considering viscosity solutions, one has to work with principal values appearing in (5.5). As in the case of weak solutions, integrals often have to be split into local parts and nonlocal parts, but now difficulties arise from the principal values instead of the double integrals of the weak formulation. The local part can be handled due to regularity of the function, whereas the nonlocal part is controlled in terms of tail spaces.

Carefully analyzing the principal values in (5.5), we obtain regularity properties for $L$, including that the principal value $L u(x)$ of an $(s, p)$-viscosity solution exists at any touching point with a smooth function. The most central result towards the desired equivalence is a comparison principle between $(s, p)$-viscosity supersolutions and subsolutions. According to the equivalence result, all the properties of $(s, p)$-harmonic functions also hold for $(s, p)$-viscosity solutions. In particular, the notions of weak solutions and viscosity solutions coincide, up to a measure zero, for bounded functions.
6.6 Further discussion

We have developed nonlocal nonlinear Potential Theory having been able to define a natural class of superharmonic functions. It has the same basic properties as in the local case and allows us to construct nonlocal Perron solutions. There are still many results in nonlinear Potential Theory that have not been completely extended to the nonlocal framework. One of them is the resolutivity of the Perron’s method, that is, when the upper and lower Perron solutions coincide. Recently, Lindgren and Lindqvist have provided a positive answer [48] in the case of the fractional $p$-Laplacian with continuous and bounded boundary data, but something more general could be expected.

Another interesting question is the nonlocal Wiener criterion, that is, providing a geometric condition for a boundary point characterizing when the solution for continuous boundary data is continuous at the boundary point. Such points are called regular boundary points. Solving the Wiener criterion would probably also require studying capacities and Wolff potentials in the fractional nonlocal framework. In the local nonlinear setting, the Wiener criterion was solved by Kilpeläinen and Malý in [39], see also [59].
References


