Multi Period Balance Sheet Scenario Optimisation

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Abstract

The objective of this study is to formulate and model a multi period scenario optimisation problem and to obtain recommendations for optimal balance sheet item portfolio. To do this the study gathers data from the public financial statements of the case company Aktia PLC, explains the relevant framework and model specifications, describes the actual optimisation model and finally presents the analysis of the results. The process combines many aspects of finance, decision making and optimisation. The portfolio optimisation problem is done over multiple periods with simulated future scenarios by formulating a tree representing the possible future outcomes.

The academic background combines aspects from information management and finance. The frameworks are selected from several different journals and fields in business studies and economics. The study explains the theoretical framework around the main topics of financial portfolio management, decision making and scenario optimisation. For the methodology the thesis aims to form qualitative research on the theories behind the model and the case company data in the form of literature review. The more quantitative research is done on the actual optimisation model and the analysis of these results. The case company was selected to allow for quantitative research and to show that the model works in real life.

The results from the model conclude with an optimal solution and a suggestion for changes made in one year for the balance sheet item portfolio. The results were heavily affected by the inputs and thus different solutions could be obtained easily. The base line model performed very well in comparison to other options. When comparing to the actual realisation for the most recently published data, the model showed a slight reduction in profits with the strong positive effect on balance sheet growth. In this solution the profits were 2 384 000€ lower in the model than those observed in year 2013 but the balance sheet total was 1 959 281 000€ larger.

Keywords Portfolio optimisation, multi period, scenario, optimisation, optimization, scenario tree, decision making, scenario optimisation, balance sheet optimisation, simulation
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1. INTRODUCTION

Ever since the financial crisis 2007-2008, bank decision making has been under the microscope (Shiller, 2008). Although optimal bank capital and balance sheet optimisation have been a popular topic in the 20th century (Modigliani & Miller, 1958), the developing investment options and financial innovations have brought more variables to the equation (Schwartz, 2008). Thus solving optimal bank balance sheet is not easy, with difficulties in uncertain future and forecasting asset returns.

From a wider perspective, having a stable economy, where bank risk taking is not excessive, is preferred by the entire economy. The problem is that when banks aim for better profitability and growth, problems with volatility tend to arise (Hnatkovska & Loayza, 2003). From the bank decision maker’s point of view, the aim is to maximise profits while avoiding bankruptcy. To counter this conflict of interests, bank regulation has been introduced to prevent banks from taking too much risk (Dothan & Williams, 1980).

In this setup the aim of this study is to find optimal bank balance sheet structure for the case company and analyse the results. The model used for this optimisation problem generates scenarios to simulate the future outcomes of the uncertain returns on assets. The model solves the optimisation problem through several time periods. The problem can be referred to as multi period balance sheet scenario optimisation.

1.1. Motivation

The reason for taking on such a challenging problem is that it offered the author a chance to combine knowledge from studies with experience from working in Santander Consumer Finance Oy. This thesis is a continuation of previous work on the author’s Bachelor’s thesis, where a simple one period optimisation model was formulated. The thesis combines studies in optimisation, finance and statistical tools.
Relevant master’s degree courses include advanced studies in optimisation, investment science, time series analysis, advanced statistical methods and simulation. These courses are offered in the information management program under information and service management master’s degree in Aalto University. Thus the thesis is a continuation of earlier work put together with more advanced studies and real life work experience.

1.2. Background of the research and target group

Previous studies on balance sheet optimisation are often portfolio optimisation and risk focused. For example Markowitz (Markowitz, 1959) portfolio theory and various mathematical models have been used to estimate and analyse optimal bank portfolios (Crane, 1971) and (Kalman & Hammer, 1967). Since the banking industry and assets have become more complex over time, more complex models have been introduced to estimate asset risks and profitability (Glantz & Kissell, 2013).

These previous studies have optimised bank balance sheet items or individual investments by maximising profits or profits with respect to risk. However, this thesis deviates from standard portfolio optimisation and uses random scenarios to solve the maximisation problem. Previous scenario optimisation research and research with large scale optimisation solver software include studies for example in the forest industry in Finland (Rämö, 2013) or South America (Kallio, et al., 2012) and real investment options (Hilli, et al., 2007). Scenario optimisation is often used in stochastic problems with uncertainty, including economics, and can be applied to portfolio optimisation problems (Mausser & Rosen, 2001).

The target group for this thesis are students and people interested in scenario optimisation. Another target group is financial and other organisations interested in balance sheet or product mix optimisation. The results of this study can be applied to budgeting problems, and with some modifications to product management. The thesis assumes some understanding on mathematical optimisation and finance.
1.3. Research problem and aims of the study

Although bank balance sheet optimisation is complex, it offers interesting study options. This thesis aims to solve a multi period scenario optimisation problem. To do this it presents a mathematical optimisation model that is built to produce realistic results that provide real life implications. These results are presented and analysed in chapter 6. In short the research goals can be stated as:

1. Obtaining, editing and analysing the data from the case company
2. Explaining relevant framework and theory
3. Selecting and formulating a mathematical optimisation model
4. Using the model to solve optimal solution for selected data and time horizon
5. Analysing the results

The expected results should be somewhat close to the real life values of the case company. Moreover, the results should yield some implications and provide useful information about the case company.

Portfolio scenario optimisation and the methods used are important as a study subject because they make complicated portfolio optimisation easy to solve with a model that can be easily modified. Adding more information and complexity in the model is fairly easy and provides more precise results. Thus the model can respond to user needs or changing problem settings. Moreover, the method of scenario optimisation deviates from the standard portfolio optimisation, making it a great tool to cross check initial optimal values obtained with other methods.

1.4. Limitations

The study is limited by the fact that the focus is on information management systems and not finance or traditional portfolio optimisation. Compared to traditional budgeting problems where balance sheet items are selected to yield optimal portfolio, the model selected for this thesis is fairly simple and uses scenarios. Although the selected method of scenario optimisation model
allows adding more information to the model, the model presented in this thesis does not include many properties. Some problems and limitations may occur due to the following factors:

- Balance sheet items are cut down to five: cash, loans, other assets, savings and debt
- Time horizon is one year with 4 quartiles, in reality decision making is more continuous
- Risk is taken into account in the form of exponential utility function and decreasing marginal returns
- Future returns are forecast with a simple model based on historical averages
- Emphasis on information management and decision making rather than finance
- The data analysis is done with available case company specific data
- Quarterly profits are maximised and expected to be paid out, rather than contributing to balance sheet growth

For better end results, the model could be improved by adding more financial information in the model for example in the form of better time series analysis or more precise inputs. Moreover, if the model was used in practise, the company could provide far better estimates of asset returns, forecast returns or possible future scenarios. For these improvements further studies are encouraged.

The results of this study can be used to get indication on what are the optimal values for the selected balance sheet items with the selected model and specifications. The results obtained this way represent the average yearly changes with the generated scenarios. In reality the final outcome and the realisation is one of the endpoints of the scenario tree, rather than the average.

1.5. Structure of the thesis

The structure of this thesis is as follows. The first two chapters are introductory, including necessary information about the case company and the collected data. Research problem, questions and main findings are also included in this chapter.

The third chapter goes through the necessary framework for this thesis and the fourth presents the model specifications. These two chapters form the literature review for this thesis. They include
various topics in stochastic optimisation, financial portfolio optimisation, modelling and calculus. The fifth chapter explains the mathematical model used to solve the optimisation problem. This chapter advances from introducing the scenario tree structure further on to variables and constraints.

The results chapter reports the output of the optimisation model with two different inputs. It then compares the outcomes in terms of profits and changes in the balance sheet total items. These results are compared with historical changes and the actual changes from the year 2013 financial statement. The sensitivity analysis, which uses different input variables to obtain alternative output and results, is reported at the end of this chapter.

The final chapter concludes that the model performs well and is a good option compared to the actual changes for the year 2013. The model suggests on average a higher total for the balance sheet items with some reductions in profitability. This leads to higher market share and profits in the long run, arguing that the model produces realistic and efficient results. The real life implications are thus significant and the model could be applied to the decision making process.
2. CASE COMPANY AKTIA

The case company Aktia Oyj is a fairly small Finnish bank with a market share of 3.8% on Finnish loans and 3.0% on savings (Finanssialan Keskusliitto, 2012). It was founded 1826 in Helsinki and merged later in 1991 with other local banks (Aktia, 2014).

The main reason why this case company was chosen over several other options is that it has relatively simple balance sheet. The most important drivers of Aktia’s business include giving out loans, managing different assets and holding on to savings. They do not specialise in complex investment options and have for example only derivatives that aim to counter economic fluctuations (Aktia, 2013).

However, there are some complications with the data related to the case company. For one, the data used for the purpose of this study only spans from 2006 to 2012. The historic data has only 7 data points from yearly data. For better precision one should aim to use more precise and current data. Furthermore, the company could have more company specific information about the future than the public data used in this thesis can predict.

One issue regarding the historic data is that it includes the economic crisis that hit especially the banking industry, preceding the euro crisis that started later in 2009 (Hall, 2012). Even if Finland did manage to maintain high credit rating, the economic recession greatly affected the interest rates in that period (Santis, 2012). Since the data is from this high risk period with low interest rates and high fluctuations, it might be seen as a disturbance when trying to forecast future data.

2.1. Case company data

The data used in this thesis was gathered from the public financial statements of the case company Aktia from the years 2006-2012 (Aktia, 2014). It was edited to fit the purpose of this research. The values of the balance sheet items were fairly easy and straightforward to obtain from the documents, but the corresponding profits needed some calculations and assumptions. Appendix 1 and 2 show the tables for the return calculations and Table 1 below explains the key aspects.
Calculations were made for the balance sheet items in order to obtain the desired 5 classes. The company used somewhat differing notations over the years and the income for the balance sheet items often consisted of several different income variables, so some assumptions needed to be made. Moreover, since the other assets didn’t really have reported income, the income was calculated with a formula:

\[
Other\ Assets\ income \nonumber \\
= All\ interest\ income - Cash\ income - Income\ from\ Loans \\
+ direct\ Other\ Asset\ income \\
+ direct\ Other\ Asset\ costs\ (negative)income
\]

Thus the income on other assets is a sum of other interest incomes and various other income costs and profits (Appendix 2). For the purpose of the study it was important to include this variable in the model, in order to avoid having loans and cash as only assets. The returns displayed below in Table 1 show the development of asset returns in the past six years.

There are significant changes in balance sheet item values and their returns. Some of these differences can be explained by the financial crisis and some variation by the random walk in the balance sheet item returns. These balance sheet item changes and returns are used to model the problem.
One important observation from this data is that the cost of borrowing is in the observation period always lower than the interest incomes on the assets. In order to be profitable, the bank needs to make money by getting higher interest for the investment than the interest rate of the debtors (Freixas & Rochet, 2008). Moreover, the bank should aim to keep the relation between the costs and income reasonable. Since the return on assets is higher than the cost of liabilities, on average it would seem profitable to increase the balance sheet items excessively.

If the bank would like increase savings or debt, the costs for borrowing would start to rise. This is due to the increasing cost of debt as debt increases by increased default risk or increased interest rates (Angbazo, 1997). Similarly, if the bank would like to invest in loans or other assets, after a while available investment options would start to yield smaller interest profits. In order to consider this, the model includes marginal effects on returns. Decreasing marginal effects mean that as the volume of a balance sheet item is increased, the return for that variable decreases.
For each increase in loans, cash or other assets, the rate of return decreases. Similarly, for each increase in debt and savings, the cost of additional increase goes up. Adding these marginal effects to the model improves the interpretation of the data and the model better predicts the real life situations. This idea of decreasing marginal effects is supported by the data presented.

Further analysis on the returns and marginal effects is done in the model specifications chapter, where the return variables are formulated and analysed. The historical prices presented in this chapter are used to formulate the model.
3. LITERATURE REVIEW AND FRAMEWORK

The field of bank and balance sheet optimisation has long been a topic of interest in the academic field, with studies from 1960s (Kalman & Hammer, 1967), (Koehn & Santomero, 1980), (Diamond & Rajan, 2000). Because of the financial crisis in 2007-2009, the topic of optimal bank capital has generated numerous new articles, journals and discussions. The theory of modern bank balance sheet optimisation dates back to Markowitz portfolio theory (Markowitz, 1952). Relevant to the topic of portfolio optimisation is the randomness and stochasticity of the future outcomes (Aoki, 1989).

In addition to portfolio optimisation theory, this thesis includes scenario optimisation (Dembo, 1991). As financial assets become more complex, it might be easier to simulate reality rather than solve absolute values. Both optimisation topics have been thoroughly covered in financial literature and this thesis aims to combine these two main topics. In addition to these two main topics, this chapter introduces topics on time series modelling, large scale optimisation software and other theoretical framework behind the mathematical model presented in chapter 3.

3.1. Stochasticity

Stochasticity refers to a situation where a system or a process is non-deterministic and must be analysed using probability theory (Aoki, 1989). In other words there is a random element or an unpredictable parameter in the system. The randomness must be controlled as this thesis does with the use of mathematical programming (Luhandjula, 2004). In this thesis the unknown parameters are simulated from the selected distribution, thus taking into account the stochasticity.

In this study the randomness comes in the form of unknown future returns. This can be seen as a result of the banking industry being prone to fluctuations. Crises and changing economic conditions greatly affect the industry and investment returns (Eickmeier, et al., 2006). In order to take this randomness into account, this study generates random scenarios of different economic conditions.
Historical data is used to predict possible future returns and scenarios are simulated in order to give possible future outcomes for returns. The different future scenarios are then used to solve the optimal solution. The stochasticity and uncertainty can in this way be controlled, modelled and taken into account. In this thesis the stochasticity is countered with scenario optimisation and simulation.

Another option to solve problems with uncertainty is by using solving a theoretical absolute value. For example, in portfolio optimisation expected profits are often maximised with respect to risk (Markowitz, 1952). In scenario optimisation future forecasts can be simulated using historic data and the complicated problem can be solved fairly easily. This way the actual optimisation model can be more complex and additional changes can be made easily.

3.1.1. Modelling financial time series

There are several options for modelling financial time series. As the main focus of this thesis is optimisation and not time series analysis, a simple model was selected. However, there are more sophisticated options available for analysing financial time series.

For example different autoregressive models (Akaike, 1969) assume that the future values of the dependent variable depend linearly on its previous values. More complex models include for example AutoRegressive Conditional Heteroskedasticity ARCH models (Engle, 1982) and others derived with the same principles. Moving average models are also widely used, and the model used in this thesis is somewhat related to the idea of average returns (Pindyck & Rubinfeld, 1998). These models often show a clear trend in the data.

When modelling and forecasting financial time series, it is important to choose an appropriate model. However, no matter how well the model performs on historical data, it might fail to predict changes or shocks to the economy (Grabel, 2003). Financial models often include some degree of random walk (Spitzer, 1964). Most often a model is formed to give an estimate for future outcomes and the random walk occurs around this estimate.
A non-complex solution for forecasting future values for prices or returns $R$ is by a simple model using only the mean $\mu$ and an error term $\epsilon$:

$$R = \mu + \epsilon$$

This simple model does not take directly into account the momentums or the movement of the time series data, but rather uses the mean to estimate future values. For better results, one might consider formulating a more complex model. Moreover, this model does not include any other variables but the mean. For example using a separate variable for general economic conditions might help in forecasting.

The graph below illustrates the observed historical data and provides simulated future forecasts that are generated randomly using the model selected for this thesis. This comparison is done to illustrate that the model produces reasonable returns by simulation. The model is presented later on in chapter 3.

![Historic quarterly returns vs Randomly generated returns from selected forecasting model](image)

**Figure 1: Historic quarterly returns and forecast made with the thesis model**

This figure captures the element of randomness and uncertainty in financial forecasting and provides an estimate for future values. The output of the model does not efficiently predict or forecast similar economic recessions as can be seen in the historic data 2006-2008, but rather relies more on the average values and random walk. Relying on the average values is the downside of the model used in this thesis.
However, for the purpose of this study, this model predicts future values with sufficient precision. Moreover, for scenario optimization it can be seen as a desirable quality to have the development of the returns close to the mean and to have generally less extreme outcomes. In other words the model generates numerous random scenarios. Some of these scenarios will have extreme values such as those observed in times of economic turbulence, but on average it provides forecasts close to the average values. When there is no time trend, the scenario tree branches do not often wander off to far extremes.

### 3.2. Scenario optimisation

Scenario optimisation is an important tool when dealing with uncertainty. In scenario optimisation, when faced with uncertainty, possible outcomes and their corresponding probabilities are estimated (Dembo, 1991). Two important topics discussed in this thesis under scenario optimisation are large scale optimisation problems and optimisation software.

The model used in this thesis estimates 5 random scenarios for each time period. The probabilities for each outcome are the same and thus the decision tree follows a random walk with the number of decision nodes increasing exponentially. Thus for the scenario optimisation setup the scenarios are random, given equal probabilities and increasing exponentially in numbers each time period.

#### 3.2.1. Large scale optimisation problems

Since scenario optimisation is used over multiple periods, the number of decision nodes is increasing exponentially. For a one period model with four quartiles and 5 scenarios on each period, the number of individual final outcomes is $5^4 = 625$. If 5 scenarios was used in a monthly data for modelling a yearly decision making process, the last stage alone would have $5^{12} = 244,140$ decision nodes. Since this is a model with multiple decision periods, decisions need to be made on every period.

There can be thousands of decisions to be made in a multi period scenario optimisation model. Moreover, the decisions made earlier often affect the decisions later on in the scenario tree. For each node, there are previous decisions, possible future scenarios and current scenario situation...
that affect the decision making process. Thus the problems become quickly very complex and complicated to solve with traditional optimisation models.

In order to solve large scale scenario optimisation problems, one needs to use software in order to avoid infinite solver calculation process.

3.2.2. **AMPL optimisation software**

AMPL provides optimisation software for large scale optimisation problems for linear and nonlinear problems (Fourer, et al., 2002). The program includes numerous different solvers that work in AMPL to find optimal solutions. These different solvers can be used to solve linear, nonlinear, quadratic, mixed integer and other types of optimisation problems (AMPL Optimization LLC, 2014). The distribution of different solver software and different solvers can be seen in the following graph (NEOS Server, 2013):

![Solver problem submissions by solver](image)

**Figure 2: Market share of solver software and solvers 2000-2014 (NEOS Server 2013)**

Since it is possible for anyone to create a solver using AMPL, there are plenty of options to choose from. However, there are a few software and solvers that dominate the market in terms of market share. In software AMPL and GAMS are most used, when in solvers MINLP and MINOS are most
preferred. For the purpose of this study, the solver MOSEK was selected, as it offers a free and efficient solver for nonlinear optimisation.

The model needs to be written in AMPL code language in order to solve the large scale optimisation problem. Different solver software and solvers have different requirements for how the code should be represented. The model is presented in the way it is inserted into AMPL solver software using MOSEK solve (Appendix 6, 7 and 8)

The problem consists of three separate files: the model (Appendix 6), the data (Appendix 7) and the run (Appendix 8) file. The run file is used to operate the other two files and print the output. For each of the files, the code is written on the left hand side and comments indicated with # are written on the right hand side. They have no operational function but are only included to explain the code. In addition to the AMPL software, in order to operate, one needs to obtain the MOSEK solver.

The run file orders the commands to operate the model and the data files. Moreover, it prints the output of the optimisation solution, presented in Appendix 5. The output includes returns from the data file, all the decision variables and expected profits, final stage outcome node expected profits and at the end the expected average yearly changes in each balance sheet item.

3.3. Portfolio optimisation

Portfolio optimisation refers to a set of assets which are chosen to find an optimal solution, most often by maximising profits with respect to risk. As mentioned before, this theory dates back to Markowitz theory explained in his book (Markowitz, 1959). As the financial markets have developed throughout the years, the portfolio optimisation problem settings and models have become more complex, offering for example solutions for large scale financial problems (Perold, 1984).

The focus in the field of bank balance sheet optimisation is often in portfolio management and investment decisions, where risk with respect to maximum profits is being optimised (Brodt, 1978). Previous portfolio optimisation research include studies with generally two main themes:
theoretical and mathematical emphasis often related to operations research (Oguzsoy & Guven, 1997) or more practical background with focus on analytics and financial studies (Selhausen, 1977). In addition to portfolio optimisation, scenario optimisation is used with simulation in economics (Pflug, 2001). However, studies using scenario optimization with simulation in multi period model for bank balance sheet portfolio optimisation were not discovered.

Scenario tree optimisation might not always be the best alternative. For example, the use of more advanced computer algorithms may be able to forecast with the given data more precisely, leading to better results (Beraldi, et al., 2013). The more advanced forecasting methods and tools may be superior in financial forecasting. On the other hand, often in finance the key to success is personal experience with the markets and the ability to forecast economic conditions and financial situations (Selhausen, 1977). To counter the problem of relying too much on data or expert knowledge, models that combine expert opinions with mathematical optimisation modelling have been produced (Lutgens & Schotman, 2010).

To answer the question of which optimisation method to use there is no single answer. Different options have different pros and cons. Simple models can usually be easily expanded where as more complex methods usually provide greater precision. To select one model over the others one needs to justify the reasons for selecting the model.

One of the most used portfolio optimisation models is the Markowitz model, that optimises portfolio returns with respect to risk (Markowitz, 1959). Figure 3 illustrates the Markowitz optimisation of profits with respect to risk. However, as Markowitz model is nonlinear, there have been several suggestions to a linear programming portfolio optimisation model. Two main linear programming models include the mean absolute deviation model (MAD) (Konno & Yamazaki, 1991) and the conditional value at risk model (CVaR) (Rockafellar & Uryasev, 2000).
As seen from the figure, balance sheet optimisation can be seen as a nonlinear trade-off between profits and risk. In the picture, the expected profits $E$ is on the horizontal axis and volatility $V$ is on the vertical axis. The circle represents the possible portfolios, with corresponding expected profits and volatility. For the decision maker, the best options lie on the frontier that has the highest expected profits or lowest volatility. The efficient frontier is marked on bottom right with a thicker line, where no point is dominated by lower volatility and higher profits. This curve represents feasible solutions for an optimisation problem. The decision maker must choose the optimal point according to his profits per volatility trade off.

The different optimisation models often combine at least one of the two: profits and risk. However, this thesis uses a nonlinear model which includes the cost of risk within the model in the parameter returns. Thus it aims to maximise the profits, where the risk is included in the model in the form

Figure 3: Markowitz portfolio optimisation and the efficient frontier (Markowitz, 1952)
of decreasing returns and exponential utility function. These topics are discussed further in this chapter.

3.4. Bank regulation and capital requirements

The idea of bank regulation aims to prevent banks from taking too much risk. It should be clear that with the financial crisis in 2007 the banks were taking too much risk and operating in a way that was not optimal for the whole economy or the banks in the long run (Miles, et al., 2013). As a result of the crisis it can be seen that the bank regulatory framework which was based on the Basel II recommendations was unsuccessful in preventing the crisis (Danielsson, et al., 2001).

When talking about optimal solution for banks' balance sheet it is important to see that the optimal level for bank capital is different from the banks' point of view when compared to the whole society's interests. The society would prefer a stable banking sector with no crises, but the banks need to take risk in order to stay competitive. The banks with better loan and service prices attract more customers and thus banks are encouraged to take more risk. To solve this problem of increasing risks, bank regulation aims to prevent banks from taking too much risk. Stricter bank regulation results in a more stable banking sector, but there are also costs related to it.

Although the mathematical model in this thesis does not include bank capital, it is important to understand how it affects decision making and how bank regulation works.

3.4.1. Bank equity and leverage

Bank decision making is about managing different balance sheet items or the bank portfolio in changing market situations. A decision maker chooses to change the different bank balance sheet items by taking into account the future prospects of the returns on these items. Looking at it this way the optimisation problem for the banks' decision maker is to maximise the expected utility of

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1 This master’s thesis is a direct continuation to my earlier work on bachelor's thesis and I wanted this chapter to be published with some modifications in this framework section.
the profits by changing certain decision variables on the balance sheet. (Fletcher, 1995) and (Sinkey, 2002)

Capital requirements and bank regulation work as constraints in the optimisation problem for the decision maker. For example Berger, Herring and Szegö discuss the role of capital in financial institutions in their article (Berger, et al., 1995). Bank capital most often refers to the bank capital stock which consists of shareholders' equity and disclosed reserves (Basel Committee on Banking Supervision, 2006). Another form of bank regulation is minimum requirements for cash, which can be directly and easily imposed on banks.

Capital requirements mean that banks have to hold a certain amount of equity with respect to their total assets. The capital ratio is defined as the share of equity in total assets \( \frac{\text{Equity}}{\text{Total Assets}} \). Setting a minimum capital requirement means a bank has to acquire equity and cannot expand unlimitedly by borrowing money and increasing debt.

Because this method of measuring capital ratio does not take into account the risks related to different assets, most of the capital requirement models use weights to valuate risk. To estimate the risk weighted assets, the banks categorise their assets and have each category multiplied with the corresponding risk weight. The Basel accords set the minimum requirement capital according to \( \frac{\text{Equity}}{\text{Risk weighted assets}} \) and they have different categories for equity – Tier 1 capital, Tier 2 capital etc. (Balin, 2008).

Leverage is closely related to the idea of capital requirements and is defined as \( \frac{\text{Total Assets}}{\text{Equity}} = \left( \frac{\text{Equity}}{\text{Total Assets}} \right)^{-1} \). Thus in a highly leveraged company the equity is small compared to the total assets. This means that imposing a capital requirement would mean setting a maximum value for leverage. To understand the effects of capital requirements one needs to understand how leverage affects the banks.

Whether high leverage is a good or a bad thing depends on the market situation (Chew, 1996). This means that in a good economic situation the return on equity increases with leverage since
the profits of a profitable company, with high leverage, are shared with a small amount of shareholders. Similarly a high leverage in poor economic situation causes the return on equity to decrease and even turn negative. The article by the Committee on the Global Financial System (Committee on the Global Financial System, 2009) discusses how leverage is closely linked with risk as high leverage increases risk but also the potential profits and amplifies the effects of the economic situation.

Limiting leverage is important for the stability of the banking sector, which has a huge impact on the welfare of the country (Miles, et al., 2013). When bank regulation and capital requirements are increased, the decision makers will have to increase the capital stock beyond the original optimal value for the bank (Koehn & Santomero, 1980). The total effect from increased bank regulation is a more stable banking sector, but decreased returns for the individual banks.

3.4.2. Bank regulation development

The reason why individual banks would not operate on the optimal bank capital level for the whole economy comes from the fact that they need to take risk in order to stay competitive on the markets (Boyd & De Nicoló, 2005). Bank regulation was introduced to solve this issue. By forcing banks with laws to have enough equity and cash in the bank the government can ensure a safe banking sector.

In general, this means a bank would be required to have enough cash and other liquid assets to cover for the customers’ need for cash. If the amount of cash and other liquid assets drops or remains low for a long period of time, the bank is likely to have liquidity problems and may experience a bank run (Diamond & Dybvig, 1983). The cash constraint introduced in the model specifications is an important influencer in the optimal solution of the model presented in this thesis.

In addition to the bank run, another disaster scenario is when the bank does not have enough capital or equity. When a bank runs out of equity to cover for the losses of falling asset prices it goes bankrupt. A bank that goes bankrupt has a significant negative effect on the whole economy. (Gruber & Warner, 1977). This is what happened in the financial crisis in 2007 when the losses
from risky assets grew large and the banks ran out of equity to cover for the losses. Equity is dropped out of the optimisation problem due to its complexity and the bad fit with the model specifications and other balance sheet items.

Comparing the two effects of better stability of the banking sector and poorer profitability explains the main issue of bank regulation. Banks need to take risks in order to stay competitive, but the risks related to highly leveraged banks can cause bankruptcy or bank runs that have major negative impacts to the whole society. One of the most used models for estimating the cost of capital was presented in an article by Modigliani and Miller (Modigliani & Miller, 1958).

In addition to the problem of optimal bank capital, there are significant challenges in the implementation of bank regulation. The problem with capital requirements is that as banks have an incentive to increase leverages beyond the values of regulative limits to maximize profits, they will find ways to increase risks and leverages in other ways in order to avoid the higher capital requirements (Tett, 2010). For example before the financial crisis, banks increased risks and leverages with new financial innovations as explained in the article discussing the causes of the financial crisis (Baily, et al., 2008). The increase in risks was mainly possible because of financial innovations that enabled banks to increase risks and leverages (Culp & Neves, 1998).

When the capital requirements fail to effectively stop banks from increasing leverages above the optimal solutions for the whole society, regulation is needed. There are several suggestions on how to improve the regulation on the banking sector to efficiently stop the banks from taking too much risk.

The two major proposals on how to improve the regulation include the Basel Accords and the Independent Banking Commission’s report - also known as Vickers report. They all aim to solve the problem with the banking sector by combining higher capital requirements with stricter regulation. The Basel I and Basel II agreements had been widely implemented already before the financial crisis (Drumond, 2009), but as they did not prevent the financial crisis they have received criticism (Balin, 2008). Moreover, even before the financial crisis, some of the weak points of Basel II were already pointed out by researchers (Danielsson, et al., 2001).
Basel II already included a capital requirement of 8% out of risk weighted assets. This meant that the assets were given weights according to their risk. As explained before the use of financial innovations allowed the banks to work around such capital requirements and thus the Basel II was insufficient to prevent the financial crisis alone. The Basel III agreement aimed to update the Basel accords in order to successfully prevent future crises.

Vickers report was released September 2011 and the implementation of the recommendations was started in the UK immediately. The recommendations were aimed for the UK’s government and were even more complex than those of Basel recommendations (Independent Commission on Banking, 2011).

Both Basel III and Independent Banking Commission’s recommendations included higher capital requirements combined with other regulatory changes to prevent the banks from taking too much risk and increasing leverages beyond the values optimal for the whole economy. The Basel I and II failed to prevent the financial crisis mainly because they lacked regulation and the banks were able to go around the capital requirements and increase risk beyond the point where they can still bear the losses.

The Basel III and Vickers recommendations aim to fix this problem and introduce a sufficient regulatory reform for the financial sector to prevent future crises spreading from the banking sector to the real economy. In addition to these capital requirements and stricter bank regulation, both Basel III and the Vickers report included liquidity requirements.

The following table shows the capital requirements for different capital ratios in the Basel III framework.

Table 2: Basel III requirements for bank capital

<table>
<thead>
<tr>
<th>Tier 1 Capital/Risk Weighted Assets</th>
<th>Basel III requirement</th>
<th>Additional countercyclical buffer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tier 1 + 2 Capital / RWA</td>
<td>8 %</td>
<td>+2,5%</td>
</tr>
<tr>
<td>Common equity / RWA</td>
<td>6 %</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4,5 %</td>
<td>+2,5%</td>
</tr>
</tbody>
</table>
As Aktia’s $\frac{\text{Tier 1 Capital}}{\text{Risk Weighted Assets}} = 11.8\%$, it clearly has a good financial standing and any capital constraints on equity would not be constraining (Aktia, 2013). Most Nordic banks have very high levels of capital. Moreover, since equity is not included in the model because it fits poorly the model setting, trying to force any equity constraints would be illogical. Thus, bank regulation should be included in the model in the form of cash constraints.
4. MODEL SPECIFICATIONS AND METHODS

This thesis formulates a mathematical optimisation model to solve a multi period balance sheet optimisation problem. Since the model is stochastic, scenario optimisation is used. The framework necessary for modelling is introduced in chapter 2 and the model is presented in the following chapter. In order to solve the problem, this thesis uses the AMPL optimisation software. In other words the methodology of this thesis is quite straightforward: the necessary framework is explained, then the model is presented and finally the obtained results are analysed.

The model selected for thesis uses several theories in order to solve the maximisation problem. The framework provided in the following subchapters is aimed to explain some concepts that are required to fully understand the model. In addition to explaining the theories, some justification for the selection of these theories is provided.

4.1.1. Lognormal distribution and lognormal returns

The lognormal distribution refers to a variable whose logarithm is normally distributed (Balakrishnan & Chen, 1999). Lognormal distribution $X$ values can be created from normal distribution $Y$. It is possible to express the lognormal distribution $Y$ as the logarithm of the normal distribution $X$ or the other way around:

$$Y = \log(X) \quad \text{and thus also} \quad X = \exp(Y)$$

The values of the lognormal distribution $X$ are always positive with a mean and variance that can be defined. It is important that the return distribution follows a lognormal distribution with strictly positive values and long tail representing unlikely crisis situations. Since the returns need to follow a lognormal distribution, the return vector $R$ for all assets $i$ can be generated using normal distribution, with the observed mean $\mu$ and the variances from the covariance matrix $V$.

$$\log(R) \sim N(\mu, V)$$
The lognormal distribution can be drawn as a graph. Here is a simple illustration of the lognormal distribution with a standard normal distribution of mean 0 and variance of 1.

![Lognormal distribution of X](image)

**Figure 4: Lognormal distribution with standard normal distribution N(0,1)**

The reason why lognormal distribution was selected for this study is that it produces returns that are always positive and they simulate returns realistically. This means that returns are generated with the same average and corresponding variances, with some extreme values in high returns or returns close to 0. Scenarios with very high returns are possible but not very likely, similarly scenarios with returns that are close to 0 are possible. This would be the case for example in a financial crisis, where assets returns might have strong fluctuations.

### 4.1.2. Exponential utility function

Utility theory is often used in economics and has a background on behavioural sciences. It aims to provide a measurement of utility for monetary gain (von Neumann & Morgenstern, 1953). Important theories include Von Neumann–Morgenstern utility theorem with four key axioms for utility theorem and Bernoulli's formulation, originally published in Latin 1738 and later translated in 1954 (Bernoulli, 1954).
In a stochastic system that includes uncertainty, using exponential utility function and maximising expected utility is often used in portfolio optimisation (Pratt, 1964). Using this method, each possible outcome is denoted with a utility according to the selected utility function. The expected utility is then calculated multiplying the utility of the outcome by the probability of that event. The total sum of different expected utilities is the expected utility of the decision.

Exponential utility function can be expressed in terms of portfolio profits \( p \) in a single time period and the Arrow-Pratt coefficient of absolute risk aversion \( \gamma \) (Arrow, 1965) (Pratt, 1964).

\[
\begin{align*}
u(p) &= -e^{-\gamma p} \\
A(p) &= -\frac{u''(p)}{u'(p)} = \gamma
\end{align*}
\]

The exponential utility function resembles a risk averse decision maker preference, where the utility function is exponentially concave (Rabin, 2000). In this function very small values are given more weight, having a large impact on the decision making where as high returns yield much less additional utility. The decreasing marginal utilities represent the risk averseness of the decision maker.

A concave exponential utility function resembles the decision maker’s relation to risk. For a larger risk factor \( \gamma \), the curve is sharper, resulting in more risk averse behaviour. Similarly, if \( \gamma \) is very small, the decision maker is more risk neutral. For a convex utility function the decision maker can be classified as a risk taker. To estimate the risk averseness, the Arrow-Pratt measure of absolute risk-aversion (ARA) can be calculated (Pratt, 1964) and (Arrow, 1965):

\[
\begin{align*}
u''(p) \quad u'(p)
\end{align*}
\]

Since the absolute risk-aversion coefficient is a constant \( \gamma \) and not a function of profits \( p \), it can be concluded that changes in profits \( P \) do not affect preferences of risk (Pratt & Zeckhauser, 1987). The fact that profits do not affect risk preferences can be argued to be unrealistic, but for the profits obtained in this thesis, the problem is quite insignificant.

Decision makers in the banking sector are often trying to minimise volatility and maximise profits according to portfolio theory (Prigent, 2007). To include this risk averseness in the optimisation
model, this thesis uses exponential utility function. However, exponential utility function has received some criticism for the reason that the absolute risk aversion is assumed constant (French & Buccola, 1978).

The gamma factor $\gamma$ is estimated so that the utilities from the profits form a reasonable concave curve. This can be challenging when the true gamma factor of the company is unknown. When the selected $-\gamma$ is multiplied by profits ($p$) and included inside an exponential function, the results from the utilities form a curve that resembles the risk tolerance of the bank (Pratt, 1964).

![Exponential utility function](image)

**Figure 5: Exponential utility function randomised example with $\gamma = 3.4E-05$**

In order to form a reasonable utility curve, the gamma factor $\gamma$ is selected so that the exponent $-\gamma p$ is close to negative one. The reason why the utility function is scaled to this number is that since the utility is changing exponentially, $e$ raised to the power of one result in reasonable changes in the utility for values below and above the profits $p$. If some other number was selected, the steepest part of the curve would move from the middle towards one of the ends.
The selected scaling factor proposes an estimation of the decision maker preferences, but for better results, one should select a utility curve that matches the company’s unknown true gamma factor. The value for the gamma factor can be estimated with the profits from historical data. The profits per quartile are one fourth of the average yearly returns. Using the average yearly returns (calculated from Appendix 1), the gamma factor is estimated to be:

\[
\gamma = \frac{1}{p} \quad \Rightarrow \quad \gamma = \frac{1}{\text{average yearly returns}} = \frac{4}{117345} \approx 0.000034
\]

Using this gamma yields a utility function that gives the portfolio values reasonable concave functions. Other gamma factors can be used and the risk tolerance should be estimated and selected by the decision maker. However, since there is no knowledge of Aktia’s true gamma factor or risk tolerance, this rough estimate is used for the purpose of this study.

4.1.3. **Cholesky factorisation**

To use the Cholesky decomposition one needs a positive definite covariance matrix from the data (Watkins, 2004). This matrix can be obtained by calculating the covariance between error terms. As mentioned before, the model uses lognormal returns, mean and error term to model future returns. The returns can be expressed in vector form \( \mathbf{R} = (R_i) \) for a single time period. The model for the returns can thus be written as:

\[
\log(\mathbf{R}) = \mu + \epsilon \quad \text{where} \quad \log(\mathbf{R}) \equiv \{\log(R_i)\}
\]

The observed error terms of this model form m-by-n matrix denoted the error term matrix \( \mathbf{E} \). In the data used, this matrix was 5-by-7 and included the data from 2006-2012 and 5 different balance sheet items. The error terms are calculated from the mean.

\[
\mathbf{E} = \begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \ldots \\
\varepsilon_{21} & \ldots & \ldots \\
\ldots & \ldots & \varepsilon_{mn}
\end{bmatrix} \quad \text{where} \quad \varepsilon_m = \{\varepsilon_{m1}, \varepsilon_{m2}, \ldots, \varepsilon_{mn}\}
\]
The error term covariance matrix $V$ can be calculated from the error term matrix $E$ by solving the covariance between the rows of the balance sheet item error terms $\epsilon_m$. In other terms the covariance is calculated between two balance sheet item rows at a time from the data. This error term covariance matrix is denoted $V$ and can be expressed as:

$$V = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & \ldots \\
\sigma_{21} & \sigma_{22} & \ldots \\
\vdots & \vdots & \ddots & \sigma_{m1} \\
\sigma_{1m} & \sigma_{2m} & \ldots & \sigma_{mm}
\end{bmatrix}$$

where

$$\sigma_{kl} = cov(\epsilon_k, \epsilon_l)$$

$$\sigma_{kl} = \sigma_{lk}$$

$$\sigma_k^2 = \sigma_{kk} = var(\epsilon_k)$$

To use the Cholesky decomposition, one needs to solve the Cholesky factor $C$ from the covariance matrix $V = C^T C$. Here the covariance matrix is decomposed into upper and lower triangular Cholesky matrixes. The lower matrix, named Cholesky factor $C$, is the matrix that need to be obtained. In the following calculations $c_{kk}$ denotes the elements in the Cholesky factor $C$, with arbitrary $c$ and $k$ denoting specific row and column element. The solution can be found by using matrix calculations that are explained further in the literature. (Watkins, 2004).

$$V = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} \\
\sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & \sigma_{35} \\
\sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \sigma_{45} \\
\sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma_{55}
\end{bmatrix} = CC^T$$

$$= \begin{bmatrix}
c_{11} & 0 & 0 & 0 & 0 \\
c_{21} & c_{22} & 0 & 0 & 0 \\
c_{31} & c_{32} & c_{33} & 0 & 0 \\
c_{41} & c_{42} & c_{43} & c_{44} & 0 \\
c_{51} & c_{52} & c_{53} & c_{54} & c_{55}
\end{bmatrix} \begin{bmatrix}
c_{11} & c_{21} & c_{31} & c_{41} & c_{51} \\
c_{22} & c_{32} & c_{42} & c_{52} & \ast \\
c_{33} & c_{43} & c_{53} & \ast & \ast \\
c_{44} & c_{54} & \ast & \ast & \ast \\
c_{55} & \ast & \ast & \ast & \ast
\end{bmatrix}$$

Now for the diagonal elements:

$$c_{kk} = \sqrt{\sigma_{kk} - \sum_{n=1}^{k-1} {\sigma_{kn}^2}}$$

for example

$$c_{33} = \sqrt{\sigma_{33} - (c_{31}^2 + c_{32}^2)}$$
For the elements below the diagonal line, where \( l > k \) in \( c_{lk} \) then:

\[
c_{lk} = \frac{1}{c_{kk}} \sqrt{\sigma_{lk} - \sum_{n=1}^{k-1} c_{ln} c_{kn}}
\]

for example \( c_{32} = \frac{1}{c_{22}} \sqrt{\sigma_{32} - c_{31} c_{21}} \)

Thus solution for the Cholesky factorisation \( C =
\begin{bmatrix}
c_{11} & 0 & 0 & 0 & 0 \\
c_{21} & c_{22} & 0 & 0 & 0 \\
c_{31} & c_{32} & c_{33} & 0 & 0 \\
c_{41} & c_{42} & c_{43} & c_{44} & 0 \\
c_{51} & c_{52} & c_{53} & c_{54} & c_{55}
\end{bmatrix}
\) can be obtained.

The Cholesky factor \( C \) can be used to solve linear equations. Moreover, multiplying the Cholesky factor \( C \) with a random uncorrelated vector \( U \) of standard normal variables, the output vector \( CU \) has the same covariance properties as the original covariance vector \( V \) (Rubinstein & Kroese, 2011). As mentioned before, this random vector \( R \) follows a multivariate lognormal distribution, since the logarithm of returns is used and the error terms are lognormally distributed.

The simulation of new returns can be performed as a function of \( C, U, \) the identity matrix \( I \) and the mean vector \( \mu \) of historical logarithmic returns:

\[
\log(R) = \mu + CU \quad \text{where} \quad U \sim (0, I)
\]

Here the Cholesky decomposition is used to correlate randomly generated lognormally distributed error terms of the returns. The error terms are used to simulate returns that follow a correlated lognormal distribution. The returns are thus similar in terms of mean values and correlations to those of the historical values.
5. MULTI PERIOD SCENARIO OPTIMISATION MODEL

To model a financial stochastic model, this thesis uses multi period scenario optimisation model. It aims to optimise case company Aktia’s balance sheet by creating scenarios based on historical data and finding the optimal solution over a time period of 4 quartiles.

The returns for scenarios are randomly created using error terms that are lognormally distributed. Lognormal distribution is used so that the returns would follow a realistic nonnegative and long tailed distribution. Return for each scenario is created separately and different scenarios do not follow any pre-set pattern. However, based on historical data, the model does use methods that affect the randomness of the returns described later in chapter 5.2. These methods are used in order to take into account several real life factors, such as the drift in he observed data and the decreasing marginal returns.

This chapter explains the formulation of the optimisation model. It starts with the structure of the scenario tree and the decision variables. It then moves on to the return simulations, which use the theories presented in the previous chapter to simulate future returns. The marginal effects and the drift factor is shown in here since they are a part of the return simulation. Next the objective function is explained and the constraints are listed. The chapter finishes with the AMPL files used to solve this problem.

5.1. Scenario tree structure

The time horizon for this problem is one year, including four quartiles. Different time periods are denoted with the STAGE variable and for example STAGE = 0 represents the start of the year and the first quartile. The final time period, STAGE = 4, does not include any decisions but is used to assess the outcomes of the final decision period, denoted STAGE = 3. Each of the quartiles represents one stage, with the first stage only including decisions and the last stage only including results. The stage can be represented as a set with STAGE = \{0, 1, 2, 3, 4\}

For each stage there are five possible future scenarios. Scenarios represent different outcomes in the returns for the balance sheet items. There are 5 balance sheet items: Loans (L), Other Assets
(O), Savings (S), Debt (D) and Cash (Cs). For each stage the decision maker must choose any changes to be made for any of the five balance sheet items. Nodes represent different scenarios and decisions that need to be made. Thus the scenario tree can be drawn (with the nodes after stage 1 overlapping):

![Decision tree structure with individual nodes overlapping after stage 1](image-url)

Figure 6: Decision tree structure with individual nodes overlapping after stage 1

Each node in each stage has an identification number. The first stage has one node 0 and the following stage has nodes 1, 2, 3, 4 and 5. Each decision node has 5 scenarios in the next stage. For example next nodes for node 1 are 7, 8, 9, 10 and 11 on stage 2.
The total number of nodes is growing exponentially. Looking from the current moment at stage 0 and forecasting future returns for four periods leads to hundreds of different possible outcomes. For each time period a decision should be made with these forecast future scenarios. The total number of nodes per stage can be expressed as:

**Table 3: Number of nodes in each stage**

<table>
<thead>
<tr>
<th></th>
<th>Nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stage 0</td>
<td>1</td>
</tr>
<tr>
<td>Stage 1</td>
<td>5</td>
</tr>
<tr>
<td>Stage 2</td>
<td>25</td>
</tr>
<tr>
<td>Stage 3</td>
<td>125</td>
</tr>
<tr>
<td>Stage 4</td>
<td>625</td>
</tr>
</tbody>
</table>

For example in stage 3 there are 125 separate decision nodes, with 5 outcomes each. The decision maker has to make a decision based on the possible future scenarios in each node. After the decision, there are 5 outcomes for each decision node in stage 3, resulting in a total of 625 final outcomes for the problem in stage 4. In this model, the decisions are made in stages 0, 1, 2 and 3 whereas the outcomes of each decision can be calculated for the next stages, namely 1, 2, 3 and 4.

Throughout the thesis, node[t] refers to the set of nodes in period t. Similarly pre[k] stands for the previous node of node k and next[k] refers to the set of nodes that are the succeeding scenarios of node k. For example the next nodes of node 0 are 1, 2, 3, 4 and 5, while the previous node of 4 is 0.

### 5.1.1. Decision variables and budget constraints

The decision variables, denoted $X_{ik}$, are chosen for each balance sheet item (i) for each decision node (k) in stages 0, 1, 2 and 3. Each decision variable represents the quarterly change for a balance sheet item i in thousand euros (1000 €). They are chosen in order to maximise the utility of the expected future profits as explained in the objective function. Each decision variable $X_{ik}$ is multiplied by the corresponding return affected by the decreasing marginal effects. In addition to these returns from the decision variables, the total profits include returns from the entire portfolio, denoted $Y_{ik}$ and explained later.
The decision variables $X_{ik}$ represent the changes for Loans ($L$), Other Assets ($O$), Savings ($S$) and Debt ($D$) for each stage. Since Cash ($Cs$) depends on the other assets, this constraint can be written as a function of the decision variables $X_{ik}$, where $i$ denotes different balance sheet items and $k$ different decision nodes:

$$X_{1k} = -X_{2k} - X_{3k} + X_{4k} + X_{5k}$$

In other words, cash can be increased by taking in savings or debt and decreased by issuing loans or buying other assets. These five assets are the decision variables to be chosen in each decision node to maximise expected future profits.

One important aspect of a multi-period portfolio optimisation problem is that for each decision, the increase in portfolio affects the returns for future periods. In this model, the decision variable $Y_{ik}$ represents the total portfolio value for balance sheet item $i$ in node $k$. The value of the portfolio is always the value in the previous period plus the selected optimal increase $X_{ik}$. Notice that since the objective is to maximise the utility of profits and these profits are expected to be paid out to investors, the only increases in the portfolio come from $X_{ik}$, rather than increases from the returns of the portfolio.

In the optimisation software AMPL, this is done by adding a separate decision variable $Y_{ik}$ for the total portfolio value, with $k$ denoting the node and $\text{pre}[k]$ denoting the previous node of $k$ and $i$ denoting balance sheet item. This constraint can be expressed as a function of the portfolio value variable $Y_{ik}$, where $\text{pre}[k]$ stands for the previous node preceding node $k$:

$$Y_{i0} = \text{portfolio value at 2012} \quad \text{and for } k > 0: \quad Y_{ik} = Y_{i\text{pre}[k]} + X_{i\text{pre}[k]}$$

Although this portfolio value variable $Y_{ik}$ is directly dependent on the decision variables $X_{ik}$, they need to be separately defined for optimisation purposes. This way the model takes into account the direct effects of increases or decreases in balance sheet items with the variable $X_{ik}$ and captures the effect of portfolio development with the variable $Y_{ik}$. 
5.1.2. **Cash constraint**

Constraints are used in the model in order to impose certain rules to the maximisation problem. Here the most important rules deal with the minimum amount of cash and how cash depends on increases in liabilities and assets. The budget constraint, discussed in under decision variables is also considered as a constraint and a decision variable. The cash constraint is another important part of the optimisation problem, but it is not considered as a decision variable.

The bank is required to have certain amount of its assets in cash or other liquid assets so that it can allow its customers to withdraw money or make other transactions. These requirements are often directly imposed on the banks by legislation (Fama, 1980). For the model the bank regulatory minimum cash requirement is included as a simple constraint for the Cash asset.

In this thesis, since the real cash per assets requirement is unknown, the cash per assets ratio is estimated from most recent historical data. In other words it is assumed that the future cash per assets ratio must be higher than the future cash per assets ratios. This constraint can be calculated from the year 2012 historical data (Appendix 1) and be expressed with cash (Cs), all assets (a) and total yearly changes in the balance sheet (Δ). The changes in cash (ΔCs) and total balance sheet assets (Δa) can be easily calculated from the data. In the AMPL model, the easiest way to solve the change in total balance sheet is from the liabilities, Savings and Debt.

\[
\frac{Cs}{a} \leq \frac{Cs + \Delta Cs}{a + \Delta a} \Rightarrow \Delta Cs \geq \frac{Cs}{a} * \Delta a
\]

Since the total amount of assets and cash are always positive in a balance sheet, it is possible to calculate the value of cash per assets and formulate the constraint. From the data of 2012 the value of cash per assets is approximately 5.23%. If the sum of the decision variables for all the assets is multiplied by the cash per asset value, then it should always be greater or equal to the decision variable for cash. Thus the constraint for cash per assets can be written as:

\[
\Delta Cs \geq \frac{Cs}{a} * \Delta a = 0.052 * \Delta a
\]
The change in cash assets should always be greater or equal to the change in the sum of all assets multiplied by a constant of 0.052. Since bank regulation gives requirements for liquid assets such as cash, these assets can and should be implemented in the model in the form of constraints. The decision maker and the bank has usually more precise information on the cash, but for this thesis a simple cash requirement is used. For more realistic results, more regulative rules on cash or any other asset can be implemented in the model in the form of constraint.

5.2. Simulating returns

The method for simulating future returns is selected to model the observed data. This thesis uses the covariance table of the error terms to simulate new error terms by using Cholesky factorisation (Rubinstein & Kroese, 2011). This leads to the error terms of the simulated returns to have same covariance properties as the original data. Moreover, returns are expressed as logarithms to simulate realistic, non-negative values. Thus the simulated returns have the same mean and correlation with the historical data.

The returns are treated as positive quarterly net returns for all balance sheet items, ranging from 1.5% to 0.005% (Appendix 8). For the liabilities, the return values are changed into negative for the final return values. This is due to the fact that the lognormal distribution used for the return simulation (Figure 4) creates positive returns with the specified mean and variance. The first three balance sheet items result in positive returns and the remaining two result in negative costs, since the final simulated returns always yield negative values.

The returns are net returns, suggesting that the total quarterly return for any balance sheet item is obtained by multiplying the volume of that balance sheet item multiplied by the return. Further in the model specifications we learn that the marginal effects affect the return for the specific quarterly change in the balance sheet. Thus the total return is affected by the simulated return, the amount of quarterly change and the marginal effect affecting the return of this increase in balance sheet item.
5.2.1. Simulating error terms and generating returns for assets

First the logarithms of the returns from historical data (RH$_{ij}$) are calculated. The notation used refers to different assets with the subscript $i$ and for different historical time periods with $j$. Next the error terms ($\varepsilon_{ij}$) of each observation from the mean ($\mu_i$) are calculated. Thus the returns can be expressed as a function of the mean and the error term:

$$\log RH_{ij} = \mu_i + \varepsilon_{ij} \quad \text{with} \quad \varepsilon_{ij} \sim N(0, \sigma_i^2)$$

With this simple model the error terms can be calculated. The logarithm of returns, mean of these logarithms and the calculated error term matrix $E$ is reported in Appendix 3. The error terms can be seen to be approximately normally distributed. The normal distribution is later used to generate new error terms and it was selected since it is the best distribution with the following historic distribution of error terms:

![Histogram](image)

**Figure 7: Evidence of normal error term frequency distribution**

Next, a covariance matrix $V$ is calculated with these error terms. As shown before, the covariance matrix $V$ required to solve the Cholesky factorisation can be obtained from the error term matrix $E$. Each covariance presented in the covariance matrix is the covariance between two balance sheet item error term row vectors.

$$V = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{21} & \cdots & \cdots & \sigma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1} & \cdots & \cdots & \sigma_{mm} \end{bmatrix}$$
Using this covariance matrix $V$ it is possible to calculate the Cholesky decomposition $V = CC^T$ (Watkins, 2004). The matrix $C$ has the property that when multiplied with a random vector of standard normal variables, the resulting vector has the same variances and covariance as the original covariance matrix $V$. This means that by calculating the Cholesky factor $C$, one can multiply it with vector $U$ to generate randomly simulated error terms that have the same covariance as the historic data. The vector $U$ variables are distributed normally, with $I$ denoting the identity matrix.

\[
U = \begin{bmatrix} U_1 \\ \vdots \\ U_n \end{bmatrix} \sim N(0, I) \quad \text{thus when simulating new values} \quad \log R_{ik} = \mu_i + (CU)_k
\]

By first calculating the covariance matrix $V$, then solving the Cholesky factor $C$ and multiplying it with $U$, one can simulate new error term values. For each decision node $k$, a new random vector $U$ should be created. Since the error terms were historically approximately normally distributed, variables following the normal distribution should be selected to generate the new values. The random vector $U$ is then normally distributed with a mean of 0 and variance equal to that of the identity matrix $I$, and the returns are lognormally distributed.

Simulating correlated returns is important since the data supports the idea of the individual balance sheet items having significant correlations. For example in good and bad economic situations the returns on assets tend to move together. The covariance matrix $V$ and the Cholesky factor $C$ are reported in Appendix 4. The following figure shows the importance of correlation, as the historic data seems to move together, having significant correlation.
This effect of the asset returns moving together can essentially be captured by using the Cholesky decomposition. The historic error terms displayed in Figure 8 support the idea of a time trend in the returns. This trend can be included in the model with the drift factor, explained in chapter 5.2.3.

Further deriving the formula for the new simulated returns yields the following formula, with $k$ denoting nodes and $i$ balance sheet items. The final outcome for returns follows a lognormal distribution.

$$R_{ik} = e^{\mu_i + (CU)_k}$$

By using the formula above, the Cholesky factor $C$ and a random vector $U$ with normally distributed values it is possible to simulate new returns that are lognormally distributed, have the same mean and same correlation as the original historical data. Here $R_{ik}$ represents the return $R$ for balance sheet item $i$ in node $k$.

The returns for all the scenarios are generated using random numbers. For scenario optimisation, it is possible to generate pre-selected scenarios, for example good, average and bad. It is also possible to select different scenarios directly. These two methods can easily be added to the model, but they require the decision maker to make the choices different scenarios. For the purpose of this
thesis, random scenarios are used. In order to create selected scenarios, more knowledge from the industry is required.

5.2.2. **Decreasing marginal effect on returns**

The idea of decreasing marginal effects comes from real life situations. In order to get more loan applications from customers or more depositors, the bank must offer better rates or invest in marketing. Similarly if the bank wants to increase its debt, give out more loans or buy more various assets, the quality of these options become worse as volume increases. These lead to increased costs, increased default risk or decreased returns. For all the balance sheet items, as volume is increased the returns are decreasing. (Knight, 1944).

The marginal effects make it impossible to exploit such situations where the return on loans is higher than cost of debt – as the historical data suggests – by increasing loan amounts infinitely. It offers better real life implications in the multi period model as it makes large scale changes less likely and more costly. Increasing any asset yields profits only to the point where it can be financed with the balance sheet liabilities. Increasing it further will decrease profits.

Since this thesis uses public data from Aktia PLC, the marginal effects need to be estimated from the available data (Aktia, 2014). The methods used in this thesis to estimate the marginal effects is limited to the small amount of data available. For better results, one should use company specific data obtained from the managers.

In the simulation all the returns and marginal effects are considered positive. The change is done after the returns are simulated for the liabilities, so that the returns are negative and can be considered as costs. However, in and throughout the simulation the marginal effects are positive in all observations. The returns with marginal effects \( r_{ik} \) for the decision variables \( X_{ik} \), which represent the selected quarterly changes in the balance sheet, can be expressed with the return without marginal effects \( R_{ik} \), where \( i \) denotes the balance sheet item, \( k \) denotes the node and \( \text{pre}[k] \) denotes the previous node.

\[
     r_{ik} = R_{ik} - X_{i\text{pre}[k]} * m_i
\]
The marginal effects \( m_i \) for each asset can be estimated using historic data. In reality, marginal effects can depend on many factors and can be difficult to measure. Here total historical balance sheet item is denoted as \( a_{ij} \) with the subscript \( i \) for each balance sheet item and \( j \) for different historical time periods. The change in a balance sheet item \( i \) for a specific year \( j \) can be expressed as \( \Delta a_{ij} \) and the historical returns \( RH_{ij} \). Moreover, while \( RH_{ij} \) is the year specific return, \( rh_{ij} \) represents the return with marginal effect \( m_i \). The mean for the returns of all the balance sheet items for a specific year is \( \mu_j \).

\[
rh_{ij} = RH_{ij} - \Delta a_{ij} * m_i
\]

Notice that since the return simulation uses positive numbers for both assets and liabilities until the final phase, all returns, mean returns and marginal effects are considered positive in this phase. Moreover it makes the marginal effect estimation simpler when one does not need to consider assets and liabilities separately. Since only yearly data is available, the yearly returns are transformed into quarterly returns by dividing the yearly returns by four.

To estimate the marginal effects, the model assumes that \( RH_{ij} \) represents the return without marginal effects and can be calculated from the data. In reality the return calculated by dividing total returns is usually slightly less because the marginal effects decrease the return for the newly increased balance sheet item \( \Delta a_{ij} \). However, since the marginal effects and \( \Delta a_{ij} \) is relatively small to the entire portfolio, this estimate is used.

\[
RH_{ij} = \text{historical returns} = \text{total income for asset } i \text{ in year } j / a_{ij}
\]

Next the impact of the marginal effect is estimated. To achieve this one should consider the following graphical presentation of the return. The marginal effect represents the decrease in return that depends on the change in the balance sheet item \( \Delta a_{ij} \) and change in the return:
Figure 9: the effect of marginal effect on balance sheet item return

The marginal effect can be obtained by estimating the relation between the changes in return and change in balance sheet item increase. The change in balance sheet item can be directly obtained from the data, but the change in the return is more difficult to estimate.

For the purpose of this study and considering the optimisation model used, it is estimated that the marginal effects tend to fix the balance sheet item returns towards the mean. This means that the decision maker would increase the balance sheet items to the point where the asset returns are close to the cost of liabilities. This rough estimate cannot be applied to all situations or claimed to be accurate, but on average with data from several years it yields results that can be used for the purpose of this study.

The marginal effect for one year can be written by using the change in the asset, historical return without marginal effects and the historical mean return. The change in balance sheet item return is here expressed as the deviation from the average return. Thus the marginal effect can be estimated as the change in the difference between the returns divided by the change in balance sheet item.

\[ m_{ij} = \frac{RH_{ij} - \mu_j}{\Delta a_{ij}} \]

When calculated for each year one can see that the marginal effects vary from year to year. For the purpose of this study one needs to obtain one single marginal effect for each balance sheet item. To get this estimate for the marginal effect, the average from all the years T in the historic dataset is calculated:
Thus the marginal effect $m_i$ for a specific balance sheet item can be estimated by calculating the average of all the marginal effects for a specific balance sheet items through all the historical time periods. The resulting marginal effects for each balance sheet item is reported in the following table:

| Table 4: Marginal effects for each year and average marginal effects $m_i$ |
|---|---|---|---|---|---|---|---|---|
|    | 2006  | 2007  | 2008  | 2009  | 2010  | 2011  | 2012  | Average |
| Cash | 8.72E-08 | 8.09E-08 | 9.40E-09 | 3.84E-09 | 7.64E-09 | 1.83E-09 | 5.43E-09 | 2.80E-08 |
| Loans | 1.33E-08 | 8.55E-09 | 1.35E-08 | 9.77E-09 | 7.93E-09 | 9.88E-09 | 2.26E-08 | 1.22E-08 |
| Other Assets | 7.14E-09 | 4.74E-09 | 2.00E-08 | 6.36E-09 | 8.14E-09 | 1.79E-08 | 5.74E-08 | 1.74E-08 |
| Savings | 6.55E-09 | 2.49E-08 | 8.13E-09 | 2.21E-08 | 1.22E-08 | 1.16E-08 | 6.12E-08 | 2.10E-08 |
| Debt | 9.87E-09 | 9.08E-09 | 2.65E-08 | 7.31E-09 | 7.62E-09 | 2.01E-08 | 5.68E-08 | 1.96E-08 |

5.2.3. **Drift factor**

The drift factor $\varphi$ improves the model by generating returns that are affected by the previous returns, instead of generating random returns. Thus for example shocks or extreme values tend to affect the economy longer than just one period. For a factor of 0 the error terms are generated with given covariance from a random lognormal distribution. However, for a factor of 0.5 the return is the average of the randomly simulated return and the previous return.

The data suggests there is some drift, where new returns are affected by the returns for the previous year. Adding this drift factor to the model modifies it to be very close to the Autoregressive model (Madsen, 2007). This is a simple model used to simulate simple drift, where the new return is a certain percentage of the old return and the rest is the newly simulated return. In this formulation $i$ stands for different assets, $k$ for different decision nodes and $\text{pre}[k]$ for the previous node of node $k$.

$$\log R_{ik} = (1 - \varphi)(\mu_i + \varepsilon_{ik}) + \varphi(\log R_{\text{pre}[k]})$$

Whereas the Autoregressive model would state:
\[
\log R_{ik} = \mu_i + \varphi \log R_{ipre[k]} + \varepsilon_{ik}
\]

To avoid increasing the returns over time, the mean and error term are multiplied by \((1 - \varphi)\) ensuring that the mean remains close to the historic mean \(\mu_i\). A reasonable drift factor of \(\varphi = 0.25\) is selected for the purpose of this study in order to get more realistic results. This means that 75% of the returns are random, and the remaining 25% is carried from previous returns.

Notice that when using the drift factor \(\varphi\), the previous returns affect the simulation of the new returns. Thus for the first simulated returns, the drift factor does not affect the simulation. For better results, another option would be to use the latest returns, obtained from the historical data. However, this does not have significant impact on the end results and thus the first period simulated returns do not include the drift factor.

### 5.3. Objective function

The objective function defines the maximisation problem, where the decision maker tries to maximise the utility of the portfolio profits \(P\) in the multi period problem with respect to set constraints. The utility of the profits are calculated in each time period and each node \(p_k\). Thus the utility of the entire optimisation problem is referred to as \(u(P)\) and the utility of profits in individual nodes \(k\) is referred to as \(u(p_k)\). Each utility of individual profits is multiplied by the probability of the node occurring, giving the expected utility of the entire system \(u(P)\).

The profits are maximised and they are not invested back into the system. In this model, it is expected that the obtained profits are paid out to the shareholders. The reason why profits are the focus of this thesis and not equity or portfolio growth is that the thesis is from the department of Information and Service Management. Thus the optimisation objective is closer to operations research and profit optimisation rather than financial portfolio optimisation. For further studies the model could take into account the growth in the balance sheet items, so that the balance sheet items are increasing with respect to the corresponding returns and maximising equity would be the objective.
The model maximises the expected utility of the portfolio on each stage. Since the profits are calculated in periods 1, 2, 3 and 4, with 5 scenarios the probabilities are always $5^t$ depending on the stage $t$. By using $ke[t]$ for the last node of stage $t$ and thus $kf[t] = ke[t-1] + 1$ for the first node on stage $t$, the total expected utilities can be written as:

$$u[P] = \sum_{t=1}^{T} \sum_{k=ke[t]}^{kf[t]} \left[ \frac{1}{5^t} \cdot u(p_k) \right]$$

The profits depend on the returns and the value of the balance sheet items. As discussed before in chapter 3.2 the returns are generated with the specified model for each asset $(i)$ and the marginal effects $(m_i)$ affect this simulation. The total profits in node $k$ can be written as a function of the decision variables $X_{ij}$, the returns with marginal effects $r_{ik}$, returns without marginal effects $R_{ik}$ and the current balance sheet item values in the portfolio $Y_{ik}$, where $pre[k]$ stands for the previous node:

$$p_k = \sum_{i=1}^{5} \left[ X_{i,pre[k]} \cdot r_{ik} + Y_{i,pre[k]} \cdot R_{ik} \right]$$

The notation here includes the increases in the asset in the previous year $X_{i,pre[k]}$, the marginal effects $m_i$, the portfolio value for each balance sheet item in the previous time period $Y_{i,pre[k]}$ and the simulated returns $R_{ik}$.

The notation should be interpreted such that the amount of selected change in the previous time period $X_{i,pre[k]}$ is multiplied by the return, which is obtained by reducing the total marginal effect from the simulated return. The total marginal effect is calculated by multiplying the marginal effect with the volume of change in the previous time period, $X_{i,pre[k]}$. In addition to these profits, the total portfolio profits in the previous time period $Y_{i,pre[k]}$ are calculated by multiplying the portfolio value with the simulated return.

This way it is possible to calculate the profits in each node. For the optimisation problem, the utility function or $u(p) = -e^{-\gamma \cdot p}$ presented in the next chapter is needed. One important aspect of the optimisation problem noticeable in the total profits function above, is its concave nature. Modifying the formula it is possible to see the negative quadratic form using vectors:
\[ p_k = \sum_{i=1}^{5} \left[ -X_{ipre[k]}^2 \times m_i + X_{ipre[k]} \times R_{ik} + Y_{ipre[k]} \times R_{ik} \right] \]

The objective function is a sum of concave functions. In other words increasing any balance sheet item will at some point start to decrease profits. Notice that even when the portfolio values include previous decision variables \( Y_{ipre[k]} \), it can be assumed concave and thus the profits are decreasing if any decision variable is increased excessively. The second derivative of the function above is always negative and the function is concave. Objective function concavity is important when determining optimal solution. For a concave objective function, the found maximum solution is known to be global. When compared to Markowitz portfolio theory, this model has one optimal solution instead of an efficient frontier. (Dantzig, 1955).

The complete objective function can be written by combining the exponential utility function, the utility of the portfolio in node \( k \) and the probabilities of each node through all the stages. In this formulation \( ke[t] \) is used to denote the last node of stage \( t \), \( kf[t] \) for the first node on stage \( t \) and the function utility function \( u(P) = -e^{-\gamma P} \) denoting the utility of all portfolio profits in all stages.

\[
u(P) = \sum_{t=1}^{\tau} \frac{1}{\gamma^t} \sum_{k=\text{ke}[t]}^{\text{kf}[t]} -e^{-\gamma \times (\sum_{i=1}^{5} \left[ X_{ipre[k]} \times R_{ik} + Y_{ipre[k]} \times R_{ik} \right])}
\]

The objective of the optimisation problem is to maximise the expected utility of the portfolio profits over the selected time period. This can be achieved by maximising \( u(P) \) presented above.

5.3.1. **Optimisation problem mathematical presentation**

The entire optimisation problem and the constraints can be expressed mathematically in the following formulation.

\[
\max \sum_{t=1}^{\tau} \frac{1}{\gamma^t} \sum_{k=\text{ke}[t]}^{\text{kf}[t]} -e^{-\gamma \times (\sum_{i=1}^{5} \left[ X_{ipre[k]} \times R_{ik} + Y_{ipre[k]} \times R_{ik} \right])}
\]

**Subject To:**
\[ X_{1k} + X_{2k} + X_{3k} = X_{4k} + X_{5k} \]

\[ X_{1k} \geq 0,052 \times (X_{4k} + X_{5k}) \]

\[ Y_{ik} = Y_{i\text{pre}[k]} + X_{i\text{pre}[k]} \quad \text{for } 0 < k \]

\[ Y_{i0} = a_{i2012} \]

\textit{where:}

\begin{itemize}
  \item \textit{k denotes different nodes}
  \item \textit{i denotes different balance sheet items}
  \item \textit{t stands for different time periods, quartiles}
  \item \textit{kf[t] stands for first node of time period t}
  \item \textit{ke[t] stands for last node of time period t}
  \item \textit{γ denotes gamma factor}
  \item \textit{r_{ik} denotes return on the change on balance sheet asset with marginal effects}
  \item \textit{R_{ik} denotes return on the balance sheet item in previous time period without marginal effects}
\end{itemize}

\section{6. MODEL RESULTS AND ANALYSIS}

The model explained in the previous chapter is used to simulate balance sheet item returns for multiple periods and for 5 scenarios branching off from each node. This means that the problem follows a decision tree with 4 stages of decision nodes. As discussed before, this method leads to having 156 decision nodes and 780 nodes with resulting profits.

This chapter compares the results from two models with different outputs. The results are analysed and compared in order to estimate model efficiency. To provide better comparison, these obtained
results are compared with historical data and most recent data from 2013. The profits and balance sheet totals are compared. Finally this chapter presents the sensitivity analysis, which aims to study the effect of changing the input variables on the final output.

6.1. Comparison of two models with different inputs

Here the model is solved with two different sets of inputs. Since these inputs are based on estimates and should in reality be chosen by the decision maker, the outputs offer alternative views. Two solutions are given in order to perform comparison and analysis on the differences.

The first solution presents a simple model with no drift factor $\phi = 0$ and the marginal effects are equal to the averages of the marginal effects estimated from the data, as explained in the previous section. The second model offers an alternative view with more precise inputs. It solves the optimal solution using a drift factor of $\phi = 0.25$ (which means 25% of the return on a balance sheet item is carried to the next period) together with the estimated marginal effects from year 2009.

6.1.1. Results with no drift factor $\phi = 0$ and average decreasing marginal effects calculated from historical data

To look at the results of this model with no drift factor and selected marginal effects, one needs to start from the balance sheet item return simulations. The marginal effects calculated from the data are presented earlier in chapter 5.2.2.

The results of the simulation of asset returns are shown in graph 10. The graph shows the distribution of the simulated prices and how they fluctuate in different scenarios in all nodes. The simulation produces some extreme values, which would resemble crises or boom periods in real life.
This graph can be used to analyse the nature of the optimisation problem. Giving out loans yields on average the best returns, but other assets have smaller volatility. Cash is the least profitable, with return smaller than the cost of debt or savings. However, bank regulation dictates that in order for bank to operate and meet the minimum liquidity requirements, it needs to hold certain percentage amount of its assets in cash.

On the liabilities side of the balance sheet it is possible to see that savings cost less on average, but have slightly higher volatility compared to the cost of debt. Debt on the other hand is linked to the bank’s leverage and might cause trouble if increased excessively (Baxter, 1967). Similarly increasing savings might have drawbacks as the bank is required to hold higher level of cash reserves and it might be costly to try to increase savings in a competitive market. The average returns and variances for each asset can be seen in the following table:
Table 5: Balance sheet item historical mean net quarterly returns and variance with corresponding simulated values

<table>
<thead>
<tr>
<th></th>
<th>Historical mean</th>
<th>Simulated mean</th>
<th>Historical variance</th>
<th>Simulated variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash</td>
<td>3.44E-03</td>
<td>3.48E-03</td>
<td>6.89E-06</td>
<td>1.10E-05</td>
</tr>
<tr>
<td>Loans</td>
<td>8.29E-03</td>
<td>8.28E-03</td>
<td>6.80E-06</td>
<td>6.20E-06</td>
</tr>
<tr>
<td>Other Assets</td>
<td>8.01E-03</td>
<td>8.02E-03</td>
<td>7.33E-07</td>
<td>5.58E-07</td>
</tr>
<tr>
<td>Savings</td>
<td>-4.82E-03</td>
<td>-4.78E-03</td>
<td>4.53E-06</td>
<td>3.73E-06</td>
</tr>
<tr>
<td>Debt</td>
<td>-5.89E-03</td>
<td>-5.87E-03</td>
<td>1.69E-06</td>
<td>1.44E-06</td>
</tr>
</tbody>
</table>

The optimisation software AMPL makes decisions with the returns simulated in the previous Figure 10. It takes into account all the future values of the successor scenarios and chooses balance sheet items $X_{ik}$, where $i$ stands for balance sheet items and $k$ for nodes. This optimal solution is listed in Appendix 5. The expected yearly increase can be calculated by multiplying the decision variable with the probability of occurrence and compared with the historical changes:

Table 6: Optimal solution expected yearly changes and historical yearly changes, 1000 €

<table>
<thead>
<tr>
<th></th>
<th>Expected total asset change in a year:</th>
<th>Historical average yearly changes 2006-2012:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash</td>
<td>56 021</td>
<td>43 421</td>
</tr>
<tr>
<td>Loans</td>
<td>326 719</td>
<td>583 867</td>
</tr>
<tr>
<td>Other Assets</td>
<td>306 327</td>
<td>327 957</td>
</tr>
<tr>
<td>Savings</td>
<td>493 561</td>
<td>340 068</td>
</tr>
<tr>
<td>Debt</td>
<td>195 505</td>
<td>517 561</td>
</tr>
</tbody>
</table>

From expected yearly changes in the balance sheet items (Table 6), it can be seen that the model output is somewhat close to the historical averages. However, the historical values seem to be putting more emphasis on giving out loans and debt while savings are increasing less.

This difference can be explained with a few different answers. It is highly likely that the cost for attracting more savings is higher than estimated in the model. This means that it is more costly to attract new savings customers than the model estimates. Thus the true value of the marginal effect on savings is higher and would result in an optimal solution with fewer savings. Since savings are decreasing due to the increased negative marginal effect, more debt needs to be accumulated in
order for the cash to remain positive. Suggesting that the bank aims not to increase debt may be more profitable, but not necessarily realistic because of the cash constraint.

The high historical increases in loans can be explained by strategic decisions: the bank tries to capture higher market share in loans even if they are less profitable for optimal solution. This kind of strategic decision is not taken into account in the mathematical model. Another reason could be that the profits for other assets were not clearly reported in the financial statements and there is some problem estimating the returns for other assets.

It can be seen from the balance sheet average returns that the positive return on issued loans is on average higher than the cost of debt (0.83% versus 0.59%). This is one reason why the bank might choose to increase loans and debt and it is taken into account in the model in the form of decreasing marginal effects (the \( m_i \) factor). In other words, there may be some estimation error in the marginal effects that contribute the solved values to have lower loans and debt than the historical values.

To understand the objective function and the solution to the problem one needs to look at the profits. As explained before, the model aims to maximise expected utility with exponential utility function. By selecting the decision variables \( X_{ik} \), the profits for each node can be calculated. The following graph presents the profits on each node for the simulated prices with marginal effects and solved balance sheet item changes \( X_{ik} \). The objective is to select decision variables to maximise utility.
Figure 11: Profits in each node using a solved optimal portfolio $X_{ik}$

The graph presents the profits from the decision variables $X_{ik}$ and illustrates the direct profits from the decisions made. Notice that here, the portfolio profits are excluded from this illustration. By maximising the expectation of the utilities from the profits on all the nodes, the system finds the optimal solution with maximal utility. As explained before, the expected utility of the whole system can be calculated by multiplying utility of the profits in each node with the probability of its occurrence. In the solution presented here, the solver gives the following output for the maximisation problem:

Table 7: MOSEK solver optimal solution for $\phi=0$ and average marginal effects

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Primal objective       :</td>
<td>-1.0108</td>
</tr>
<tr>
<td>Dual objective         :</td>
<td>-1.01021</td>
</tr>
</tbody>
</table>

This output does not really tell much about the actual problem, but it can be compared later with other models to see which performs better with the chosen utility function. For more information about the optimal solution, one should look at balance sheet item returns and average yearly changes presented before in graph 10 and table 6.
6.1.2. Results with drift factor of $\phi = 0.25$ and marginal effects from year 2009

In this simulation, the returns are simulated with a drift factor. This means that a return has 25% impact on the next return and thus also 6.25% impact on the return after the next return and so forth. There tends to exist less extreme values as seen from the graph, but the time trend is clearer. Consecutive returns tend to be close to one another and a shock persists longer, affecting returns on several years.

![Simulated balance sheet item net quarterly returns for all nodes](image)

**Figure 12: Simulated balance sheet item returns for all nodes with a drift factor of 25\%**

Because of the time trend the mean net quarterly returns and the variances are slightly affected. The distributions are clearly different to those without a drift factor. To fully understand the distributions one should look at the mean and variance of each of these simulated returns. These can be seen in the following table:
Table 8: Historical and simulated mean net quarterly returns and variances with drift

<table>
<thead>
<tr>
<th></th>
<th>Historical mean</th>
<th>Simulated mean</th>
<th>Historical variance</th>
<th>Simulated variance</th>
</tr>
</thead>
<tbody>
<tr>
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<td>3.06E-03</td>
<td>6.89E-06</td>
<td>4.37E-06</td>
</tr>
<tr>
<td>Loans</td>
<td>8.29E-03</td>
<td>8.18E-03</td>
<td>6.80E-06</td>
<td>3.48E-06</td>
</tr>
<tr>
<td>Other Assets</td>
<td>8.01E-03</td>
<td>8.00E-03</td>
<td>7.33E-07</td>
<td>3.39E-07</td>
</tr>
<tr>
<td>Savings</td>
<td>-4.82E-03</td>
<td>-4.65E-03</td>
<td>4.53E-06</td>
<td>2.00E-06</td>
</tr>
<tr>
<td>Debt</td>
<td>-5.89E-03</td>
<td>-5.83E-03</td>
<td>1.69E-06</td>
<td>8.53E-07</td>
</tr>
</tbody>
</table>

One clear difference in the simulated returns with these inputs is that with the drift factor of $\phi=0.25$ the variances are lower than those of the original data. This is due to the fact that part of the simulated returns is carried from previous time period return. Thus on average the returns are closer to the mean, leading to the decrease in volatility. Using the drift factor in return simulation affects the mean, variance and randomness. This might be seen to have a negative effect on the model results as they yield less extreme values and might lead into problems in forecasting.

By using the simulated returns, it is possible to solve the model. The expected yearly increase can be calculated and compared with the historical changes. The changes here are due to the selected inputs, namely drift factor and marginal effects.

Table 9: Optimal solution expected yearly changes and averages of historical changes (1000 €)

<table>
<thead>
<tr>
<th></th>
<th>Expected total asset change in a year:</th>
<th>Historical average yearly changes 2006-2012:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash</td>
<td>74 606</td>
<td>43 421</td>
</tr>
<tr>
<td>Loans</td>
<td>375 484</td>
<td>583 867</td>
</tr>
<tr>
<td>Other Assets</td>
<td>889 350</td>
<td>327 957</td>
</tr>
<tr>
<td>Savings</td>
<td>519 340</td>
<td>340 068</td>
</tr>
<tr>
<td>Debt</td>
<td>820 099</td>
<td>517 561</td>
</tr>
</tbody>
</table>

As seen here, the optimal solution puts a lot of emphasis on other assets, while increasing savings and debt. Moreover, the general tolerance towards risks seems to be higher as balance sheet items grow more on a yearly level than before. These outcomes can be explained by the selected inputs. Since the variances are lower with the drift factor, the volatility of the returns is smaller, leading to more risk taking. This can be seen in the following graph. There seem to be a few extreme values, and values are closer together.
Figure 13: Profits in each node using a solved optimal portfolio $X_{ik}$

The graph presents the profits from the chosen optimal balance sheet item changes $X_{ik}$ and the returns with marginal effects $m_i$. With the profits closer together and with less extreme values, the decision makers are now more willing to take risk. Including the drift factor in the model is a bit problematic because it decreases the volatility leading to more risk taking. However, this idea is supported by the observed data. Thus further research on the time trend is encouraged.

It is possible to solve the optimisation problem with these profits. The model maximises the expectation of the utilities on all the nodes and finds the optimal solution with maximal utility. The output from the solver with the final objective function value is expressed in the following table.

**Table 10: MOSEK solver optimal solution for $\phi=0.25$ and marginal effects from 2009**

<table>
<thead>
<tr>
<th>Primal objective :</th>
<th>-0.97178</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dual objective</td>
<td>-0.97114</td>
</tr>
</tbody>
</table>

This objective function value does not explain much of the problem alone, but can be compared with the solution from the first model. In this model the drift factor causes the variance in balance
sheet items to be lower, leading to more risk taking. This risk taking can be seen in the form of higher proposed balance sheet average changes. Because of these higher changes, the utility in this problem solution is higher. Thus model 2 gives out a solution with more changes and higher expected profits. This is due to the inputs used in the model. In order to understand the difference between the two models, more comparisons need to be made.

6.1.3. **Most recent data released by the case company after the model was formulated**

The data used in formulating the model was obtained from Aktia’s financial statements from the years 2006-2012 and the model used in this thesis was created with this data. During the final stages of the thesis work, new data was released from 2013. This new data provides an opportunity to compare the different optimal solutions and the actual realisation of Aktia’s balance sheet. By comparing these alternatives it is also possible to look at and analyse the reasons why the model fails or succeeds in predicting certain changes in the balance sheet.

**Table 11: Balance sheet items 2013 and percentage changes from 2012 (1000 €)**

<table>
<thead>
<tr>
<th>Balance sheet</th>
<th>2010</th>
<th>2011</th>
<th>2012</th>
<th>2013</th>
<th>Change</th>
<th>% change</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Assets</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cash (C_i)</td>
<td>273 364</td>
<td>475 042</td>
<td>587 613</td>
<td>414 328</td>
<td>-173 285</td>
<td>-29.49 %</td>
</tr>
<tr>
<td>Loans (L_i)</td>
<td>6 637 551</td>
<td>7 152 124</td>
<td>7 360 225</td>
<td>6 897 349</td>
<td>-462 876</td>
<td>-6.29 %</td>
</tr>
<tr>
<td>Other Assets (A_i)</td>
<td>4 108 238</td>
<td>3 428 897</td>
<td>3 292 352</td>
<td>3 622 129</td>
<td>329 777</td>
<td>10.02 %</td>
</tr>
<tr>
<td><strong>Sum</strong></td>
<td>11 019 153</td>
<td>11 056 063</td>
<td>11 240 190</td>
<td>10 933 806</td>
<td>-306 384</td>
<td>-2.73 %</td>
</tr>
<tr>
<td><strong>Liabilities</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Savings (Y_i)</td>
<td>4 356 327</td>
<td>4 757 179</td>
<td>4 689 040</td>
<td>4 892 982</td>
<td>203 942</td>
<td>4.35 %</td>
</tr>
<tr>
<td>Debts (D_i)</td>
<td>4 827 366</td>
<td>4 464 037</td>
<td>4 584 724</td>
<td>4 106 018</td>
<td>-478 706</td>
<td>-10.44 %</td>
</tr>
<tr>
<td>Equity (E_i)</td>
<td>497 290</td>
<td>523 756</td>
<td>657 409</td>
<td>641 709</td>
<td>-15 700</td>
<td>-2.39 %</td>
</tr>
<tr>
<td>Other Liabilities (O_i)</td>
<td>1 338 170</td>
<td>1 311 091</td>
<td>1 309 017</td>
<td>1 293 097</td>
<td>-15 920</td>
<td>-1.22 %</td>
</tr>
<tr>
<td><strong>Sum</strong></td>
<td>11 019 153</td>
<td>11 056 063</td>
<td>11 240 190</td>
<td>10 933 806</td>
<td>-306 384</td>
<td>-2.73 %</td>
</tr>
</tbody>
</table>

From the table above it is possible to see the actual realisations for the yearly changes in Aktia’s balance sheet items from year 2012 to year 2013. The percentage changes are also listed, to give an idea about the scale of the change. One important observation is that in general the trend in the balance sheet items and the totals is upwards up until the year 2013.
There is a 30% drop in cash assets, while only the other assets were increasing. On the liabilities side debt was cut down by 10% while more savings were accumulated. The balance sheet shrunk a little which contributes to some loss of market share. To understand why the balance sheet would be decreased by the decision maker, the profits of each balance sheet item need to be considered.

Table 12: Balance sheet item profits and returns 2013 and comparison to year 2012 (1000 €)

<table>
<thead>
<tr>
<th>Income</th>
<th>2010</th>
<th>2011</th>
<th>2012</th>
<th>2013</th>
<th>Change</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash (C_i)</td>
<td>2 485</td>
<td>3 290</td>
<td>1 237</td>
<td>775</td>
<td>-462</td>
<td>-37.35%</td>
</tr>
<tr>
<td>Loans (L_i)</td>
<td>152 164</td>
<td>186 132</td>
<td>173 496</td>
<td>126 325</td>
<td>-47 171</td>
<td>-27.19%</td>
</tr>
<tr>
<td>Other Assets (A_i)</td>
<td>134 868</td>
<td>109 172</td>
<td>107 504</td>
<td>126 458</td>
<td>18 954</td>
<td>17.63%</td>
</tr>
<tr>
<td>Sum</td>
<td>240 326</td>
<td>262 563</td>
<td>232 296</td>
<td>172 952</td>
<td>-59 344</td>
<td>-25.55%</td>
</tr>
<tr>
<td>Savings (Y_i)</td>
<td>-54 411</td>
<td>-63 252</td>
<td>-57 149</td>
<td>-45 970</td>
<td>11 179</td>
<td>-19.56%</td>
</tr>
<tr>
<td>Debts (D_i)</td>
<td>-81 692</td>
<td>-107 039</td>
<td>-105 647</td>
<td>-94 832</td>
<td>10 815</td>
<td>-10.24%</td>
</tr>
<tr>
<td>Equity (E_i)</td>
<td>-59 675</td>
<td>-37 187</td>
<td>-55 880</td>
<td>-51 978</td>
<td>3 901</td>
<td>-6.98%</td>
</tr>
<tr>
<td>Other Liabilities (O_i)</td>
<td>45 084</td>
<td>36 343</td>
<td>47 779</td>
<td>80 493</td>
<td>32 714</td>
<td>68.47%</td>
</tr>
<tr>
<td>Sum</td>
<td>-91 019</td>
<td>-133 948</td>
<td>-115 017</td>
<td>-60 309</td>
<td>54 708</td>
<td>-47.57%</td>
</tr>
</tbody>
</table>

From the table it is obvious that the returns and costs on almost all of the balance sheet items have gone down. As the returns go down, it is rational for the bank to decrease the total balance sheet. If the profits would go up, increasing would be a good solution. The returns show a significant increase in profitability of other assets, which can explain the increase in other assets. For the costs there have been some improvements in profitability. By comparison the costs have gone down more than the returns, which would argue for better profitability, combined with smaller balance sheet.

The positive changes in other assets and the negative changes in the rest of the balance sheet emphasises the importance of the market conditions. It would seem that in a bad market situation the returns go down, but since the volatility of the return on other assets is small, it becomes more desirable. Thus in a bad market situation it is rational to decrease balance sheet totals and put more emphasis on other assets. These other assets include for example buildings and other real investments.
It can be argued that some of these changes may be stochastic random walk, which can be taken into account in the simulation and some are not random, meaning that strategic planning or decision making is taking place. For random differences, one could say that for example the realisation of the balance sheet item returns were very negative. After observing and forecasting more negative future returns, the bank decision makers adjust and decrease balance sheet item values. The other explanation is that the changes were not based on a random factor, but rather a strategic decision.

The model cannot take into account strategic choices. This is because by default it maximises expected utility of the profits. However, if one would apply game theory to the process, it would no longer be possible or reasonable to solve the problem using multi period scenario optimisation (von Neumann & Morgenstern, 2007). For decision making purposes and actual budgeting issues, this model and the solutions could provide significant and important information. In the current situation however, it is very likely that the decisions are made with other methods, where strategic positioning is given more emphasis than pure maximisation of expected profits. This would argue that the problem setting follows more closely game theory than profit maximisation problem.

In the long run increasing and decreasing balance sheet yearly probably leads to high costs related to acquiring the assets and financing them with liabilities. Moreover, the benefit of steadily increasing the balance sheet and market share most likely leads to higher profits. In the model and assumptions presented in this thesis, there exists no such optimal solution in multi period optimisation where balance sheet items are increased and decreased in consecutive years.

6.1.4. **Comparison of different solutions and expected profits**

To compare all the results from the two models presented and the actual observed outcome, one needs to understand the big picture. For the models, the solutions depend heavily on the inputs selected. Thus one can obtain many results by using the model with different inputs. Even the same solution that can be observed in real life may be obtained by selecting the correct inputs.

When considering the observed outcome of 2013, it is the same as looking back and seeing only final node of the entire scenario tree of possible outcomes. Thus comparison of the 2013 outcome
and model optimal solutions can be done, but the differences can be big because one is looking backwards where the others are looking forward. The following table summarises the changes in balance sheet items through different time periods and model solutions.

**Table 13: Comparison of balance sheet between different solutions and observations (1000 €)**

<table>
<thead>
<tr>
<th></th>
<th>Historical averages</th>
<th>2012 changes</th>
<th>2013 changes</th>
<th>1st model</th>
<th>2nd model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash ($C_i$)</td>
<td>43 421</td>
<td>112 571</td>
<td>-173 285</td>
<td>56 021</td>
<td>74 606</td>
</tr>
<tr>
<td>Loans ($L_i$)</td>
<td>583 867</td>
<td>208 101</td>
<td>-462 876</td>
<td>326 719</td>
<td>375 484</td>
</tr>
<tr>
<td>Other Assets ($A_i$)</td>
<td>327 957</td>
<td>-136 545</td>
<td>329 777</td>
<td>306 327</td>
<td>889 350</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>955 246</td>
<td>184 127</td>
<td>-306 384</td>
<td>689 067</td>
<td>1 339 440</td>
</tr>
</tbody>
</table>

**Liabilities**

<table>
<thead>
<tr>
<th></th>
<th>Historical averages</th>
<th>2012 changes</th>
<th>2013 changes</th>
<th>1st model</th>
<th>2nd model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Savings ($Y_i$)</td>
<td>340 068</td>
<td>-68 139</td>
<td>203 942</td>
<td>493 561</td>
<td>519 340</td>
</tr>
<tr>
<td>Debts ($D_i$)</td>
<td>517 561</td>
<td>120 687</td>
<td>-478 706</td>
<td>195 505</td>
<td>820 099</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>857 628</td>
<td>52 548</td>
<td>-274 764</td>
<td>689 066</td>
<td>1 339 439</td>
</tr>
</tbody>
</table>

It can be seen from table above how the two model solutions are fairly close to the historical values, providing more similar optimal solutions. The selected model uses historical averages in the return simulation, which contributes to the solution values to follow the historical values to some degree. The model solutions are fairly close to the observed values in 2012 as well. However, for the year 2013 the model was unable to predict the radical change in the asset returns that caused the decrease in balance sheet items.

It is important to understand why the model did not predict the 2013 values. The 2013 balance sheet item values presented in the table represent the end values and the outcome. When making decisions with unknown outcomes, it is much harder to predict radical changes. Year 2013 data represents such radical changes where all returns decrease significantly. Even if some simulations did predict such extreme values, on average the returns were higher and the optimal solution has increasing balance sheet items. These changes can be taken into account in the model by decreasing the average returns.

The model did succeed in predicting high increases in other assets compared to the rest and the first model the increase in savings compared to debt. However, such situations where in year 2012
the savings are decreasing and debt is increasing when in year 2013 savings are increasing and
debt is decreasing are very hard to predict. For any model without a separate economic variable it
would be hard to predict changes that are exact opposites in consecutive years. It seems like there
is some strategic agenda, or the decision maker may have done some new analysis on future
economic conditions.

If for example the decision maker was expecting a growth period in 2012 and thus wanted to
increase debt, loans and cash, but after some change in 2013 changed the views of the future to a
more negative forecast. Thus the year 2012 and 2013 differ greatly in terms of balance sheet item
yearly changes. It is also harder to solve optimal bank balance sheet item yearly changes in such
situations.

For better precision that may possibly take into account the radical changes in 2013 the decision
maker may adjust the expected average returns and marginal effects in the model. Furthermore,
the gamma factor represents the decision makers risk tolerance and can be used to counter risky
scenarios whereas the cash constraint can be used to meet regulatory requirements on cash assets.
Another option would be to include an economic indicator that forecasts future economic situation.
Such indicator could be for example the skirt length indicator. (van Baardwijk & Franses, 2010)

6.2. Analysis

To analyse the performance of the different solutions, one must look at the profits of the different
options. For comparison, one should use the actual realisation of balance sheet item returns from
year 2013. These returns can be used to estimate what would have been the impact of selecting a
model and using the averages as yearly budget. It is not possible to determine the actual profits for
a specific model solution, since the data is not publicly available. The profits can be calculated
with the marginal effects, the returns from the yearly changes can be calculated with a simple
formula, where \( X_i \) represents the selected optimal solution and \( R_i \) the return on balance sheet item
i for year 2013:
It is important to understand that in scenario optimisation there are several time periods, and the yearly averages expressed as optimal solutions for models 1 and 2 are only averages. If in the scenario optimisation the decision maker would realize they are in a difficult node, namely in a bad economic situation, he would adjust the future possible scenarios and the optimal balance sheet items. Looking at the final changes for year 2013 or comparing profits to these values thus gives unfair disadvantage for any model that reports yearly averages. However, this way some comparison on profits can be made across different solutions or decision maker options.

Table 14: Profits from yearly changes in different solutions, with marginal effects (1000 €)

<table>
<thead>
<tr>
<th></th>
<th>Historical</th>
<th>2012 changes</th>
<th>2013 changes</th>
<th>1st model</th>
<th>2nd model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash (Ci)</td>
<td>28.37</td>
<td>-144.63</td>
<td>-1165.78</td>
<td>16.82</td>
<td>-16.46</td>
</tr>
<tr>
<td>Loans (Li)</td>
<td>6523.69</td>
<td>3281.66</td>
<td>-11098.30</td>
<td>4678.17</td>
<td>5152.45</td>
</tr>
<tr>
<td>Other Assets (Ai)</td>
<td>9579.67</td>
<td>-5091.33</td>
<td>9622.39</td>
<td>9063.06</td>
<td>17296.65</td>
</tr>
<tr>
<td>Savings (Yi)</td>
<td>-5618.68</td>
<td>542.87</td>
<td>-2787.75</td>
<td>-9742.48</td>
<td>-10531.92</td>
</tr>
<tr>
<td>Debts (Di)</td>
<td>-17203.87</td>
<td>-3072.86</td>
<td>6564.49</td>
<td>-5264.53</td>
<td>-32123.44</td>
</tr>
<tr>
<td>Total</td>
<td>-6690.81</td>
<td>-4484.29</td>
<td>1135.05</td>
<td>-1248.96</td>
<td>-20222.72</td>
</tr>
</tbody>
</table>

It can be seen from the table 14 how in the market condition of 2013 and with the corresponding returns, the bank decision making was efficient. Using the averages from model 1 or model 2 would have not been as efficient in terms of pure profit. Another significant aspect is the big difference between the two models. Model 1 outperforms the historical average values, 2012 values and model 2. Model 2 on the other hand performed worst of all the selected solutions in a bad economic situation.

The reasons behind these results lie in the economic situation. It would seem from looking at the returns that there is an economic downturn or a bad period in 2013. Model 2 has the biggest proposed total balance sheet increase, whereas year 2012 has the smallest. The model 2 solution takes most risk and 2012 solution takes least amount of risk in terms of total balance sheet changes.
This can be seen as the reason why model 2 performs worst and the 2012 solution offer better profitability with a more risk averse solution.

The most important observation is the good performance of model 1. The solution suggests increasing the total balance sheet, which can be seen beneficial in the long run as the company would lose less market share and customers. Even when it performs worse in the bad economic condition, it might be argued that it gives better setting for the future. Increasing and decreasing balance sheet totals every second year might yield better short term gains, but in the long run the multi period model aims for increasing the balance sheet totals, leading to bigger market share and profits.

6.2.1. **Sensitivity analysis**

A sensitivity analysis helps to understand the relationship between the input and output variables. It gives important information on how the output is affected by the input and thus the results can be better understood. Moreover, a sensitivity analysis improves the reliability of the model. (Saltelli, et al., 2000)

Because the model is not linear, a simplified analysis is performed. In this analysis, the inputs are changed to yield different results which are still in a reasonable range. The changes are done one input at a time and the amount of change is selected to show reasonable results. The base case used here is the model 1 setup without drift factor and the marginal effects. The marginal effects equal to the mean marginal effects from the data, as explained in the model specification section. The inputs that are changed in this sensitivity analysis are marginal effects $m_i$, cash requirements and gamma factor $\gamma$.

For the first analysis, the marginal effects are changed. If one considers the output of the 1st model and compares it with other results, it would seem that other assets are over emphasised, cost for acquiring more savings might be higher compared to the marginal effect of debt. In other words, acquiring more other assets might provide less profits as the investment options become fever, when on the other side it seems like it would be more costly to increase savings compared to debt. Using these ideas it is possible to give new values for the marginal effects.
Table 15: Sensitivity analysis with marginal effects increased for other assets and savings, decreased for debt

<table>
<thead>
<tr>
<th>Marginal effects sensitivity analysis</th>
<th>Original $m_i$</th>
<th>New $m_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash</td>
<td>2.80E-08</td>
<td>2.80E-08</td>
</tr>
<tr>
<td>Loans</td>
<td>1.22E-08</td>
<td>1.22E-08</td>
</tr>
<tr>
<td>Other Assets</td>
<td>1.74E-08</td>
<td>3.74E-08</td>
</tr>
<tr>
<td>Savings</td>
<td>2.10E-08</td>
<td>4.10E-08</td>
</tr>
<tr>
<td>Debt</td>
<td>1.96E-08</td>
<td>9.60E-09</td>
</tr>
</tbody>
</table>

Using these new marginal effects it is possible to solve the new optimal solution. Note that here the marginal effects for other assets and savings is increased and to control these increased costs, the marginal effect for debt are decreased. This means that the relation between the liabilities savings and debt is changed so that the cost of increasing savings is higher and the cost of increasing borrowing is lower. By not changing the marginal effects too much in one direction the total balance sheet item sum should be close to the original. In this situation with the changed marginal effects, the optimal solution with average yearly changes would be:

Table 16: Average yearly changes with modified marginal effects (1000 €)

<table>
<thead>
<tr>
<th></th>
<th>Original $m_i$</th>
<th>New $m_i$</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash</td>
<td>56 021</td>
<td>63 508</td>
<td>7 487</td>
</tr>
<tr>
<td>Loans</td>
<td>326 719</td>
<td>442 049</td>
<td>115 330</td>
</tr>
<tr>
<td>Other Assets</td>
<td>306 327</td>
<td>168 967</td>
<td>-137 360</td>
</tr>
<tr>
<td>Savings</td>
<td>493 561</td>
<td>246 864</td>
<td>-246 697</td>
</tr>
<tr>
<td>Debt</td>
<td>195 505</td>
<td>427 660</td>
<td>232 155</td>
</tr>
<tr>
<td>Total</td>
<td>1 378 133</td>
<td>1 349 048</td>
<td>-29 085</td>
</tr>
</tbody>
</table>

As can be seen from table 17, the marginal effects directly affect the relationship and size of the proposed changes on balance sheet items. This can be argued to be one of the reasons why there are significant changes in the balance sheet throughout the years. By changing and estimating the
marginal effects more precisely, one can obtain better solutions. If the relation between two balance sheet items should be changed, it is possible to increase and decrease the marginal effects which was done in this example where the marginal effect of savings was increased and marginal effect of debt was decreased.

For the cash requirement the cause and effect relationship should be quite straightforward. Increasing cash requirements would cause banks to hold more of its assets in cash, which would limit investing in something more profitable. On the other hand it brings more stability as the bank is less likely to meet liquidity problems due to changing economic conditions. Bank regulation, capital and cash requirements were discussed earlier in this study.

For the purposes of this sensitivity analysis, the cash requirement constraint is increased from 0.05228 to 0.15228. This is a significant 191% increase, but still the cash assets remain the smallest and least profitable unit. To further analyse the costs of cash constraints and bank regulation, one should run separate models and study the costs for the entire economy. The results of this sensitivity analysis can be seen below:

Table 17: Average yearly changes with model 1 and increased cash requirements (1000 €)

<table>
<thead>
<tr>
<th></th>
<th>Basic model</th>
<th>New cash requirements</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash</td>
<td>56 021</td>
<td>102 092</td>
<td>46 071</td>
</tr>
<tr>
<td>Loans</td>
<td>326 719</td>
<td>320 009</td>
<td>-6 710</td>
</tr>
<tr>
<td>Other Assets</td>
<td>306 327</td>
<td>129 223</td>
<td>-177 104</td>
</tr>
<tr>
<td>Savings</td>
<td>493 561</td>
<td>229 082</td>
<td>-264 479</td>
</tr>
<tr>
<td>Debt</td>
<td>195 505</td>
<td>322 241</td>
<td>126 736</td>
</tr>
<tr>
<td>Total</td>
<td>1 378 133</td>
<td>1 102 647</td>
<td>-275 486</td>
</tr>
</tbody>
</table>

It is possible to see a significant 20% drop in total sum of these balance sheet items when cash constraints are increased. This has a direct negative effect on profitability, but on the other hand the bank is less likely to meet problems with liquidity. The results could be used to further estimate
the costs of increasing cash constraints. However, as this is not the emphasis of this research, it will be merely pointed out, but further research could be done with this model.

The final input factor to consider is the gamma factor $\gamma$ determining the slope of the exponential utility function. For a higher gamma factor, the utility curve is steeper and it results in the decision maker to be more risk averse. Any deviation for lower profits would result in large negative utility. This would lead to the decision maker choosing safer and less volatile portfolios. These would include putting more emphasis on cash assets, other assets and savings. The results for increasing the gamma factor $\gamma$ from 3.41E-05 to 7.41E-05 can be seen below:

Table 18: The effect of changing the gamma factor on average yearly changes in balance sheet

<table>
<thead>
<tr>
<th></th>
<th>Basic model</th>
<th>Increased gamma</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash</td>
<td>56 021</td>
<td>128 170</td>
<td>72 149</td>
</tr>
<tr>
<td>Loans</td>
<td>326 719</td>
<td>189 014</td>
<td>-137 705</td>
</tr>
<tr>
<td>Other Assets</td>
<td>306 327</td>
<td>438 074</td>
<td>131 747</td>
</tr>
<tr>
<td>Savings</td>
<td>493 561</td>
<td>590 528</td>
<td>96 967</td>
</tr>
<tr>
<td>Debt</td>
<td>195 505</td>
<td>164 730</td>
<td>-30 775</td>
</tr>
<tr>
<td>Total</td>
<td>1 378 133</td>
<td>1 510 516</td>
<td>132 383</td>
</tr>
</tbody>
</table>

As gamma factor is increased in table 19, more emphasis is put on the safer assets and liabilities. Moreover, the table argues how changing the gamma factor has complex effects on the output. Even when the decision maker is more risk averse, the balance sheet totals are increased when gamma is increased. As the gamma factor is changed, the objective function values are no longer comparable, since the values are on a different scale. Thus the decision maker should aim to choose the correct or most appropriate gamma factor in an early phase of the optimisation process.

All in all the effects of changing the inputs are significant. Changing the marginal effects and cash constraint had a more direct and predictable results, whereas changing the gamma factor had more complex effect on the resulting average balance sheet changes. By changing these inputs, the decision maker can influence the output. By selecting the correct inputs, the output can be very close to the true optimal values in real life.
The benefit of scenario optimization is seen in these results and analysis. One can make adjustments to the model fairly easily and solve the optimal solution. The closer to the real values the inputs are, the closer the output is to a theoretical situation where maximum profits are obtained. Scenario simulation proves to be a fairly efficient and fast way of finding optimal solutions for portfolio optimization problems with the methods shown in this thesis.
7. CONCLUSIONS

The aim of this thesis was to find optimal bank balance sheet structure in a multi period scenario optimisation model. In order to reach these goals, this thesis analysed and reported the public case company data that was presented in chapter 2. The next chapters summarised the most important methods, frameworks and model specifications. The model itself was presented in chapter 5. Chapter 6 reported the results from the model and the analysis on these results, including the sensitivity analysis.

To determine the success of this study the research problems need to be revisited. The research goals were set as:

1. Obtaining, editing and analysing the data from the case company
2. Explaining relevant framework and theory
3. Selecting and formulating a mathematical optimisation model
4. Using the model to solve optimal solution for selected data and time horizon
5. Analysing the results

All of the above mentioned goals were reached within the limits of this study. For better results one should consider using the company’s own private data, improving the time series analysis on the data and improve the analysis on the results. These are excellent areas for further research and could each be made into a study of its own. For the purpose of this study, the goals were met successfully.

The data gathering and analysing step was quite straightforward. The data was successfully obtained from the financial statements and proved to be very useful in later parts of the study. Although the historical data was limited to only six years and only 5 balance sheet items were analysed in the end, the quality of the data was sufficient to produce accurate and significant results. Moreover it was possible to predict future returns and balance sheet changes efficiently even with the little amount of data available.
The important findings of this thesis are related to the relevant frameworks, the formulated model, obtained results and company specific analysis. The literature review revealed portfolio optimisation studies in both finance and operations management, but few combined the two or made significant connections. The third chapter summarised the main elements in stochastic optimisation, financial portfolio optimisation and scenario optimisation. Even in such challenging environment as bank portfolio optimisation, the model performed fairly well and the results obtained fell in line with the previous studies and observed outcomes.

Another important finding from the literature review was the role of bank regulation. The analysis suggested that Aktia had sufficient capital levels, but a cash constraint should be included in the model. The model itself was presented in the fifth chapter where it was successfully explained in simple terms. The main output of this chapter was the model in AMPL, presented at the end of the chapter.

In the results the importance of the inputs and the data was highlighted. For example bank regulation or changes in bank risk aversion coefficient would have direct effects on the output. The uncertainty and high importance of the input variables created challenges for the optimisation model, but the results were realistic because of the large number of simulated scenarios. Moreover, with scenario optimisation the inputs could easily be modified and changed to meet the needs of the decision maker or the market situation.

Combining portfolio optimisation with scenario optimisation in the model presented in this thesis proved to be very efficient when compared to other historical values. It can be argued that the 1st model performed very well when comparing to other alternative options. Here are the average yearly changes of the original model and profits with the later observed 2013 returns data:

**Table 19: Proposed or observed yearly changes and profits related to these changes (1000 €)**

<table>
<thead>
<tr>
<th>Balance sheet item yearly proposed or observed changes</th>
<th>Profits from yearly changes using returns from 2013 with marginal effects</th>
</tr>
</thead>
<tbody>
<tr>
<td>Historical 2012 changes</td>
<td>Historical 2012 changes</td>
</tr>
<tr>
<td>2013 changes</td>
<td>2013 changes</td>
</tr>
<tr>
<td>1st model</td>
<td>1st model</td>
</tr>
<tr>
<td>2nd model</td>
<td>2nd model</td>
</tr>
<tr>
<td>C</td>
<td>43 421</td>
</tr>
<tr>
<td>L</td>
<td>583 867</td>
</tr>
<tr>
<td>O</td>
<td>327 957</td>
</tr>
<tr>
<td>S</td>
<td>340 068</td>
</tr>
<tr>
<td>D</td>
<td>517 561</td>
</tr>
<tr>
<td>Totals</td>
<td>1 812 874</td>
</tr>
</tbody>
</table>
The 1\textsuperscript{st} model solution proposes a change where there is a significant increase in the balance sheet assets and relatively small negative effect on the total profits of the company related to these changes. On the other hand the changes are not free, as the return on these increased assets is negative. The balance sheet for the 1\textsuperscript{st} model was significantly larger with assets totalling 9.1\% larger than the observed 2013 data. Moreover, since the balance sheet is growing, future profits and prospects are improved. Thus one could even argue that the model solution presented in this thesis is strategically better than the one observed in the in the financial statement of 2013.

One reason for decreasing the balance sheet size might be the maximisation of short term profits in year 2013. Maximising short term profits might be more tempting for the decision maker and avoiding any kind of costs or losses related to these new changes is desirable. The model 1 solution proposed changes that would have cost the company 1.2 million whereas the 2013 changes resulted in 1.1 million profits.

This difference of approximately 2.4 million which occurred between the baseline case and the observed outcome of 2013 can be further analysed. In terms of absolute profits the 2013 was fairly efficient. However, on average over several years, the 9.1\% increase in the total balance sheet would bring more profits in the future. It can be argued that this increase in the balance sheet items is an investment. Furthermore, it can be concluded that the increased balance sheet would quickly pay itself back because of the improved future profit prospects.

As explained before, since the returns on assets were higher than the cost of liabilities, increasing the balance sheet steadily should yield higher profits in the long run. The difference between short term and long term profits is vital in decision making. It might be more beneficial for the decision maker to aim for short term profits.

The model presented in this thesis is a multi-period model and optimisation is done over several periods. It does not take into account strategic short term win maximisation but provides a more general and long term solution. Moreover, looking back after observing the outcome of 2013 is the same as looking back from the final node of the decision tree and comparing it to the average solution which includes the entire decision tree.
For the results the general conclusion is clear. The model outperformed many other alternatives and even when the profits were smaller compared to the observed results of 2013 the total balance sheet was bigger. The inputs had a significant effect on the output and the decision maker could possibly reach far better results when using the correct inputs together with more precise data. With more work on the model and better data the model would perform even better.

7.1. Implications

This thesis successfully builds a multi period scenario optimisation model that is efficient and provides realistic results. The model can be implemented directly or with easy modifications to both academic research and company decision making. The model used here can be improved by using more precise inputs as the sensitivity analysis shows, in order to gain more precise results for business purposes. The model can be modified to fit the needs of either different academic interests or different business optimisation problem.

In the academic field, similar techniques have already been applied to for example the forest industry (Kallio, et al., 2012). Other research that can benefit from this study are stochastic optimisation research that use large scale optimisation software to find optimal solutions. The main contributions from this thesis for other academic research are the scenario creation process, optimisation over multiple periods and the use of AMPL optimisation software.

In business life the model could be used in different fields to solve optimal production or budgeting problems. With some modifications, the methods and model specifications could be used for example in product portfolio optimisation or financial portfolio management. The main contribution to decision makers in the field of business is the use of scenario optimisation. With the methods presented here, scenario optimisation could be applied to numerous problems. The further the field deviates from a simple portfolio optimisation problem, the more customisation and modifications need to be made in the model.

The most important thing in scenario optimisation is that the model should generate realistic scenarios that represent possible events from real life. If the scenario creation process is successful, then finding the optimal solution is fairly straightforward. Thus creating random scenarios can be
very efficient when dealing with complex large scale problems. As this thesis shows, it is possible to apply scenario optimisation to a bank balance sheet portfolio optimisation problem.

7.2. Improvements in the model for future research

After the optimal solution was found there were findings as to how the model could have been improved or made better. Because of the time limitations as a master’s thesis, these improvements are merely pointed out for further research and they work as a way of reporting important learning points that were discovered during the research. The thesis has a clear focus on optimisation and simulation from information management point of view. To improve the financial impact one should consider the improvements discussed in this section.

The most significant improvement in the model would be including equity in the model, such that each balance sheet item would have the returns reinvested in that balance sheet item. This way the balance sheet items and the portfolio would grow according to the simulated returns. This method would better explain the frequent changes in the balance sheet in the historical data. Including equity in the model would mean that the maximisation problem would aim to maximise the expected utility of the increase in equity (or decrease).

Equity would be equal to the increase in balance sheet asset side minus the increase in the balance sheet liabilities side. By introducing equity in the model, it would be possible to further study the relationships between bank regulation, bank equity and decision making. Moreover, this way the model would be more realistic, where instead of always paying out the profits they would be reinvested in the company, and dividends would be paid out to the investors on a yearly basis.

Another important point for future research would be to improve the time series analysis of the model. In the thesis this was done with an elementary drift factor, but further research could implement a more complex model for return simulation based on time series analysis. One option for improving the time series analysis would be to include an economic indicator to the model, such as the hemline index discussed earlier in this paper (van Baardwijk & Franses, 2010). Related to the time series analysis, one point for further study would be to improve the data analysis phase, by either obtaining more specific data or by performing deeper analytics on the data.
Other improvement options include analysing the data for better inputs as discussed in the sensitivity analysis, analysing the impact of bank regulation on the entire economy as discussed under bank regulation development or the optimal risk for the economy with the model presented in this thesis. In addition to these topics, it would be great if studies in other fields related to large scale scenario optimisation would benefit from the methods and model presented in this thesis.
8. REFERENCES


## Appendix 1: Balance sheet items and returns over historic data period from 2006-2012 (1000€)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Assets</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cash (C&lt;sub&gt;i&lt;/sub&gt;)</td>
<td>307,907</td>
<td>235,273</td>
<td>506,311</td>
<td>340,960</td>
<td>273,364</td>
<td>475,042</td>
<td>587,613</td>
</tr>
<tr>
<td>Loans (L&lt;sub&gt;i&lt;/sub&gt;)</td>
<td>3,797,018</td>
<td>4,757,011</td>
<td>5,526,194</td>
<td>6,141,562</td>
<td>6,637,551</td>
<td>7,152,124</td>
<td>7,360,225</td>
</tr>
<tr>
<td>Other Assets (A&lt;sub&gt;i&lt;/sub&gt;)</td>
<td>1,385,455</td>
<td>2,960,529</td>
<td>3,507,568</td>
<td>4,073,317</td>
<td>4,108,238</td>
<td>3,428,897</td>
<td>3,292,352</td>
</tr>
<tr>
<td><strong>Sum</strong></td>
<td>5,490,380</td>
<td>7,952,813</td>
<td>9,540,073</td>
<td>10,555,839</td>
<td>11,019,153</td>
<td>11,056,063</td>
<td>11,240,190</td>
</tr>
<tr>
<td><strong>Liabilities</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Savings (Y&lt;sub&gt;i&lt;/sub&gt;)</td>
<td>3,340,385</td>
<td>3,729,991</td>
<td>5,015,277</td>
<td>4,753,586</td>
<td>4,356,327</td>
<td>4,757,179</td>
<td>4,689,040</td>
</tr>
<tr>
<td>Debts (D&lt;sub&gt;i&lt;/sub&gt;)</td>
<td>1,728,973</td>
<td>2,740,892</td>
<td>3,130,482</td>
<td>4,045,926</td>
<td>4,827,366</td>
<td>4,464,037</td>
<td>4,584,724</td>
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<tr>
<td><strong>Sum</strong></td>
<td>5,490,380</td>
<td>7,952,813</td>
<td>9,540,073</td>
<td>10,555,839</td>
<td>11,019,153</td>
<td>11,056,063</td>
<td>11,240,190</td>
</tr>
<tr>
<td><strong>Income</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cash (C&lt;sub&gt;i&lt;/sub&gt;)</td>
<td>5,291</td>
<td>8,065</td>
<td>9,320</td>
<td>3,126</td>
<td>2,485</td>
<td>3,290</td>
<td>1,237</td>
</tr>
<tr>
<td>Loans (L&lt;sub&gt;i&lt;/sub&gt;)</td>
<td>140,774</td>
<td>208,798</td>
<td>280,123</td>
<td>190,364</td>
<td>152,164</td>
<td>186,132</td>
<td>173,496</td>
</tr>
<tr>
<td>Other Assets (A&lt;sub&gt;i&lt;/sub&gt;)</td>
<td>41,380</td>
<td>88,421</td>
<td>139,865</td>
<td>123,030</td>
<td>134,868</td>
<td>109,172</td>
<td>107,504</td>
</tr>
<tr>
<td><strong>Sum</strong></td>
<td>181,306</td>
<td>272,404</td>
<td>386,129</td>
<td>285,576</td>
<td>240,326</td>
<td>262,563</td>
<td>232,296</td>
</tr>
<tr>
<td><strong>Savings (Y&lt;sub&gt;i&lt;/sub&gt;)</strong></td>
<td>-61,920</td>
<td>-107,152</td>
<td>-171,208</td>
<td>-78,997</td>
<td>-54,411</td>
<td>-63,252</td>
<td>-57,149</td>
</tr>
<tr>
<td>Debts (D&lt;sub&gt;i&lt;/sub&gt;)</td>
<td>-37,764</td>
<td>-73,128</td>
<td>-105,199</td>
<td>-82,170</td>
<td>-81,692</td>
<td>-107,039</td>
<td>-105,647</td>
</tr>
<tr>
<td><strong>Sum</strong></td>
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<td>-183,526</td>
<td>-285,176</td>
<td>-133,328</td>
<td>-91,019</td>
<td>-133,948</td>
<td>-115,017</td>
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<tr>
<td><strong>Returns %</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cash (C&lt;sub&gt;i&lt;/sub&gt;)</td>
<td>1.72%</td>
<td>3.43%</td>
<td>1.84%</td>
<td>0.92%</td>
<td>0.91%</td>
<td>0.69%</td>
<td>0.21%</td>
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<tr>
<td>Loans (L&lt;sub&gt;i&lt;/sub&gt;)</td>
<td>3.71%</td>
<td>4.39%</td>
<td>5.07%</td>
<td>3.10%</td>
<td>2.29%</td>
<td>2.60%</td>
<td>2.36%</td>
</tr>
<tr>
<td>Other Assets (A&lt;sub&gt;i&lt;/sub&gt;)</td>
<td>2.99%</td>
<td>2.99%</td>
<td>3.99%</td>
<td>3.02%</td>
<td>3.28%</td>
<td>3.18%</td>
<td>3.27%</td>
</tr>
<tr>
<td>Savings (Y&lt;sub&gt;i&lt;/sub&gt;)</td>
<td>-1.85%</td>
<td>-2.87%</td>
<td>-3.41%</td>
<td>-1.66%</td>
<td>-1.25%</td>
<td>-1.33%</td>
<td>-1.22%</td>
</tr>
<tr>
<td>Debts (D&lt;sub&gt;i&lt;/sub&gt;)</td>
<td>-2.18%</td>
<td>-2.67%</td>
<td>-3.36%</td>
<td>-2.03%</td>
<td>-1.69%</td>
<td>-2.40%</td>
<td>-2.30%</td>
</tr>
<tr>
<td><strong>Equity (E&lt;sub&gt;i&lt;/sub&gt;)</strong></td>
<td>-16.80%</td>
<td>-17.90%</td>
<td>-1.80%</td>
<td>-8.70%</td>
<td>-12.00%</td>
<td>-7.10%</td>
<td>-8.50%</td>
</tr>
</tbody>
</table>
### Appendix 2: Other Asset return calculations (1000€)

<table>
<thead>
<tr>
<th>Other Assets income and costs</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
<th>2011</th>
<th>2012</th>
</tr>
</thead>
<tbody>
<tr>
<td>Muut korkotuotot</td>
<td>35 241</td>
<td>55 541</td>
<td>96 868</td>
<td>92 086</td>
<td>85 677</td>
<td>73 141</td>
<td>57 563</td>
</tr>
<tr>
<td>Osinkotuotot</td>
<td>1 222</td>
<td>1 541</td>
<td>1 395</td>
<td>607</td>
<td>1 105</td>
<td>173</td>
<td>53</td>
</tr>
<tr>
<td>Palkkiotuotot netto</td>
<td>40 061</td>
<td>47 346</td>
<td>41 034</td>
<td>46 346</td>
<td>57 030</td>
<td>60 565</td>
<td>65 319</td>
</tr>
<tr>
<td>Henkilökustus netto</td>
<td>0</td>
<td>138 078</td>
<td>91 037</td>
<td>13 991</td>
<td>16 477</td>
<td>22 732</td>
<td>27 304</td>
</tr>
<tr>
<td>Rahoitusvarojen ja velkojen nettotuotot</td>
<td>736</td>
<td>2 257</td>
<td>-3 359</td>
<td>826</td>
<td>-5 585</td>
<td>-14 815</td>
<td>2 940</td>
</tr>
<tr>
<td>Kiinteistöt</td>
<td>4 880</td>
<td>480</td>
<td>6 010</td>
<td>406</td>
<td>518</td>
<td>266</td>
<td>338</td>
</tr>
<tr>
<td>Liiketoiminnan muut tuotot</td>
<td>5 433</td>
<td>3 067</td>
<td>14 958</td>
<td>3 565</td>
<td>7 916</td>
<td>4 361</td>
<td>4 682</td>
</tr>
<tr>
<td><strong>Sum of income</strong></td>
<td>87 573</td>
<td>248 310</td>
<td>247 761</td>
<td>157 827</td>
<td>163 138</td>
<td>146 423</td>
<td>158 199</td>
</tr>
<tr>
<td>Muut korkokulut</td>
<td>2 512</td>
<td>-3 246</td>
<td>-8 769</td>
<td>27 839</td>
<td>45 084</td>
<td>36 343</td>
<td>47 779</td>
</tr>
<tr>
<td>Vakuutuskorvauskulut ja vastuuvelan nettomuutos</td>
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<td>-75664</td>
<td>0</td>
<td>0</td>
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<td>Henkilöstö</td>
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<td>-60605</td>
<td>-79219</td>
<td>-82842</td>
<td>-73 203</td>
<td>-75 352</td>
</tr>
<tr>
<td>IT/muut kulut</td>
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<td>-35501</td>
<td>-38419</td>
<td>-44783</td>
<td>-50247</td>
<td>-26 380</td>
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**Other Assets income calculation:**

| All income from assets +   | 181 306   | 272 404   | 386 129   | 285 576   | 240 326   | 262 563   | 232 296   |
| Cash (Ci) income -         | 5 291     | 8 065     | 9 320     | 3 126     | 2 485     | 3 290     | 1 237     |
| Loans (Li) income -        | 140 774   | 208 798   | 280 123   | 190 364   | 152 164   | 186 132   | 173 496   |
| **Sum of Other Assets income +** | 87 573   | 248 310   | 247 761   | 157 827   | 163 138   | 146 423   | 158 199   |
| **Sum of Other Assets costs +** | -81434  | -215 430  | -204 582  | -126 883  | -113 947  | -110 392  | -108 258  |
| **Other Assets Income**    | 41 380    | 88 421    | 139 865   | 123 030   | 134 868   | 109 172   | 107 504   |
## Appendix 3: Historic quarterly returns, logarithms of returns, means and error term matrix \( \mathbf{E} \)

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## Appendix 4: Covariance matrix \( \mathbf{V} \) and Cholesky factor \( \mathbf{C} \)

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Appendix 5: Optimal solution $X_{ik}$, 1 out of 2 pages in 1000€

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Appendix 5: Optimal solution $X_{ik}$, 2 out of 2 pages

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Appendix 6: MOSEK solver optimisation model, gradu.mod file

```plaintext
# input variables
param gamma = 1.409E-05;
param factor = 0.1;
param cashasset = 0.05228;

# formulating the tree properties
set ASSET := 1..5;
param T := 4;
param S := 3;

# tree structure
set NODES := 0..((S^T+1)-1)/(S-1)-1;
set STAGE := 0..T;
set NODE[t in STAGE];
set next[k in NODES];
param ke [t in STAGE];

# returns, budget and covariance matrix
param cov [a in ASSET, b in ASSET];
param C [a in ASSET, b in ASSET];
param K [a in ASSET, k in NODES];

# variables for return calculations
param h := 7;
param orl[asset[asset]]; # initial asset returns
param meanreturn[asset]; # mean asset returns
param histp [a in ASSET, k in 1..H]; # historical prices
param error [a in ASSET, m in 1..M]; # error terms from historical data
param mareff [a in ASSET]; # decreasing marginal effects of each asset, based on estimate from data

# arbitrary parameters for error terms, simulated distribution values
param p[1..1];
param q[2];
param r[1..2];
param r[1];

# option randseed 0;
var X [a in ASSET, k in NODES; k := ke[T-1]);
var Y [a in ASSET, k in NODES];

# utility function
maximize expected utility:
sum[i in STAGE: 0 < i < T+1] (1/(S^T))*
[exp(-gamma) * sum[a in ASSET]
[x[a,pre[i]]*(cov[a,pre[i]]*mareff[a])
+ (a,pre[i])*K[i,(a,k)])];

subject to cashconstraint [k in NODES; k := ke[T-1]]: X[k] := X[0,k];

subject to cashassets [k in NODES; k := ke[T-1]]: Y[k] := origasset[a];

subject to initialbudget [a in ASSET]:
Y[a,0] = origasset[a];

subject to budget [a in ASSET, k in NODES: 0 < k := ke[T-1]]:
Y[a,k] = Y[a,pre[i]]*X[a,pre[i]];"
Appendix 7: MOSEK solver gradu.dat data file

```
# original balance sheet values in 2012
paraset :=
  1 38763 2 736025 3 3292352 4 4689040 5 4584724;
# historical quarterly returns show in appendix
paraset :=
  1 0.00426532 0.000461821 0.000570485 0.0002284219 0.0002264905 0.0001726947 0.0000525867
  2 0.00942373 0.001497030 0.001248505 0.000586053 0.001681734 0.000443631 0.001982182
  3 0.00684658 0.007384457 0.000952111 0.0007463922 0.0008103024 0.0007868195 0.000665568
  4 0.00460337 0.001194692 0.0008471908 0.0004328937 0.0003108006 0.000307182 0.000315318
  5 0.00316303 0.000640376 0.0008297412 0.0005039113 0.0004204086 0.0005941358 0.000571694;
paraset :=
  1 0.603668 2 1.213222 3 1.758838 4 2.059868 5 1.980508;

let ke[0] := 0;
for (t in 1..7) [let ke[t] := ke[t-1] + 0.0c;]
# calculating last nodes of each stage

let node[0] := 0;
for (t in 1..7) [let nod[t] := ke[t-1]+1..ke[t];]
# calculating nodes in stage t

let ff := 0;
for (k in nodes: k := ke[t-1]) [
  let next[k] := ff+f;ff:=ff+1c;
]
# calculating next nodes of node k

for (k in nodes: k := ke[t-1]) [
  let next[k] := ff=f;
]
# calculating previous nodes of node k

for (a in Asset) [
  let a := hist[a,k];
]
# calculating logarithms of price

for (k in Asset) [
  let histp[a,k] := log(a);
]
# calculating mean of the logarithmic returns

for (k in Asset) [
  let histr[a,k] := sum(k in Asset)[histp[a,k]/n];
]
# calculating error terms in model

for (a in Asset) [
  let err[a,k] := histp[a,k]-meann[a];
]
# calculating covariances on error terms

for (a in Asset) [
  let cov[a,b] := sum(k in Asset)[err[a,k]*err[b,k]/n];
]
# to be used as error term covariance matrix for cholesky composition

let l := 1;
for (l in Asset) [let l := l+1c;]
# at the end C is a lower triangular matrix and
# CC'T = cholesky factorization of the correlation matrix

let l1 := norma[0,1];
let l2 := norma[0,2];
let l3 := norma[0,3];
let l4 := norma[0,4];
let l5 := norma[0,5];
let l6 := norma[0,6];
let l7 := norma[0,7];
let l8 := norma[0,8];
let l9 := norma[0,9];

for (k in stages: k := 1)
# generating scenarios for the rest (with the drift factor):

let l1 := exp(1-factor)*(meanreturn[1]+factor*log(l1,pre[1]));
let l2 := exp(1-factor)*(meanreturn[2]+factor*log(l2,pre[2]));
let l4 := exp(1-factor)*(meanreturn[4]+factor*log(l4,pre[4]));
let l5 := exp(1-factor)*(meanreturn[5]+factor*log(l5,pre[5]));
let l6 := exp(1-factor)*(meanreturn[6]+factor*log(l6,pre[6]));
let l7 := exp(1-factor)*(meanreturn[7]+factor*log(l7,pre[7]));
let l8 := exp(1-factor)*(meanreturn[8]+factor*log(l8,pre[8]));
let l9 := exp(1-factor)*(meanreturn[9]+factor*log(l9,pre[9]));

]);
display n;
# used to print out the simulated returns in the output
```
Appendix 8: MOSEK solver gradu.run executive file

```
reset;
model gradu.md;
data  graduadata.dat;
option solver mosek;  # MOSEK #
param err1 := 0;  # MOSEK #
param err2 := 130;  # MOSEK #
param err3 := 99;  # MOSEK #
solve > out;

let solver := (if (solve_result_num-err1 or solve_result_num-err2 or solve_result_num-err3) then 0 else 1);

# The following commands merely print out the decision variables,
# profits in result nodes, and the total average changes in the balance sheet (reported as the results)
print the decision variables in different nodes and corresponding profits
for[t in 0..T]{
  for[k in node[t]]{
    if k = 0 then
      # for the first stage there are no profits
      printf"Node %1.0f Stage %1.0f\n Asset1 = %.0f\n Asset2 = %.0f\n Asset3 = %.0f\n Asset4 = %.0f\n Asset5 = %.0f\n",k,t,X[1,k],X[2,k],X[3,k],X[4,k],X[5,k])
    else if t = T then
      # for the last node there are no decisions to be made
      printf"Node %1.0f Stage %1.0f\n Asset1 = %.0f\n Asset2 = %.0f\n Asset3 = %.0f\n Asset4 = %.0f\n Asset5 = %.0f\n",t,X[1,k],X[2,k],X[3,k],X[4,k],X[5,k],
      (sum[a in Asset] (X[a,pre[k]]*k[a,k]-X[a,pre[k]]*mareff[a])+con[a,pre[k]]*R[a,k]));
    else
      printf"Node %1.0f Stage %1.0f\n Asset1 = %.0f\n Asset2 = %.0f\n Asset3 = %.0f\n Asset4 = %.0f\n Asset5 = %.0f\n",t,X[1,k],X[2,k],X[3,k],X[4,k],X[5,k],
      (sum[a in Asset] (X[a,pre[k]]*k[a,k]-X[a,pre[k]]*mareff[a])+con[a,pre[k]]*R[a,k]));
    }
};

# print total asset changes per asset
print "Mean total asset change in a year (if T=4):
for[a in Asset] {
  printf "Asset %.0f = %.0f\n",a,sum[t in STAGE] (1/(50*T))*sum[k in node[t]: k = ke[t-1]] X[a,k]);
```

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