Extremal statistics in the energetics of domain walls

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We study at \(T=0\) the minimum energy of a domain wall and its gap to the first excited state, concentrating on two-dimensional random-bond Ising magnets. The average gap scales as \(\Delta E \sim L^\theta/(N_s)\), where \(\theta\) is the energy fluctuation exponent, \(L\) is the length scale, and \(N_s\) is the number of energy valleys. The logarithmic scaling is due to extremal statistics, which is illustrated by mapping the problem into the Kardar-Parisi-Zhang roughening process. It follows that the susceptibility of domain walls also has a logarithmic dependence on the system size.

The energy landscapes of random systems are often assumed to be described at low temperatures by scaling exponents that follow from the behavior of the ground states. In renormalization group (RG) language this means that temperature is an irrelevant variable. In most quenched random systems, the energy landscape contains many low-lying metastable minima separated by high barriers. Examples can be found in the realm of random magnets, the most famous one being spin glasses [1]. The dynamical behavior at finite temperatures, as a result of a temperature change or the application of an external field, will naturally depend on the associated barriers and energy differences between the minima.

It is often assumed that energy differences or barriers between configurations (\(\delta E\)) relate to the length \(l\) involved by a scaling relation \(\delta E \sim l^\theta\), where \(\theta\) is an energy fluctuation exponent. It measures the dependence of the first nonanalytic correction to the ground state or free energy on the length scale. Here we show that, for extended manifolds, or Ising exponent. It measures the dependence of the first nonanalytic component that follows from the behavior of the ground states. In between configurations (minima. The dynamical behavior at finite temperatures, as a result of a temperature change or the application of an external field, will naturally depend on the associated barriers and energy differences between the minima.

Let us now analytically derive the scaling of the “extreme statistics” contributions to the lowest minimum \(E_0\), and the gap between two lowest minima, \(\Delta E = E_1 - E_0\). We consider the case of many independent valleys in the landscape \(N_s > 1\), which means that the DP’s can have an arbitrary starting or end point, and that \(L_s > 1\). For the “single valley” boundary condition case (one end of the manifold fixed), it is known numerically that near its mean the distribution is Gaussian [9]. Hence we draw the energies \(E\) from the distribution

\[
P(E) = k \exp \left[ -\frac{(E - \langle E \rangle)}{\Delta E} \right]^n.
\]
where $\langle E \rangle \sim L^\theta$ is the average energy of the manifold, $\Delta E \sim L^\theta$ measures its fluctuations, and $k$ normalizes the integral so $k \sim 1/L^\theta$. The exponent $\eta$ is not constant [9,3]. Near the peak, $\eta = 2$. In the low energy tail numerical simulations indicate that $\eta_\ll 1.6$, while in the high energy tail the best estimate is $\eta_\gg 2.4$ [9]. At this stage we allow $\eta$ to be variable, but note that it is the behavior near the mean and the low energy tail which is the most important in this calculation. In a system with $N_z \sim L_z/L^L$ independent local minima, the probability that the global minimum has energy $E$ is given by

$$L_{N_z}(E) = N_z P(E)\{1 - C_1(E)\}^{N_z - 1},$$

where $C_1(E) = \int P(e) d\varepsilon$ [10]. The gap $\Delta E_1$ follows similarly. Its distribution, $G_{N_z}(\Delta E_1, E)$ is given by

$$G_{N_z}(\Delta E_1, E) = N_z (N_z - 1) P(E)P(E + \Delta E_1) \times \{1 - C_1(E + \Delta E_1)\}^{N_z - 2}.$$  

$G_{N_z}(\Delta E_1, E)$ is the probability that the lowest energy manifold has an energy $E$, then the gap to the next lowest energy level is $\Delta E_1$. The average value of the global minimum is given by

$$\langle E_0 \rangle = \int_{-\infty}^{\infty} E L_{N_z}(E) dE,$$

which is not analytically integrable. The typical value of the lowest energy may be estimated using an extreme scaling estimate. It follows from the fact the term inside the $\{\}$ in Eq. (3) becomes unity if $C_1$ is small enough. This has proven useful in other contexts, for example breakdown of random networks, and here reads [11]

$$1/kN_z P(\langle E_0 \rangle) \approx 1$$

which yields

$$\langle E_0 \rangle \approx \langle E \rangle - \Delta E \ln(N_z)^{1/\eta},$$

where $\Delta E \sim L^\theta$.

To estimate the typical value of the gap, we use, similarly to Eq. (6),

$$1/k^2 N_z (N_z - 1) P(\langle E_0 \rangle)P(\langle E_0 \rangle + \langle \Delta E_1 \rangle) \approx 1,$$

which, with Eq. (7), and the fact that $|\langle \Delta E_1 \rangle| \leq |\langle E_0 \rangle|$, yields

$$\langle \Delta E_1 \rangle \approx \frac{\Delta E^\eta}{\eta \langle E \rangle - \langle E_0 \rangle} \eta^{1/(\eta - 1)} \frac{\Delta E}{\eta \langle \ln(N_z) \rangle^{1/(\eta - 1)/\eta}}.$$  

We thus find that, in addition to the usual sample to sample variations in the energy $\Delta E \sim L^\theta$, there is a slow reduction in the gap which scales as $\ln(N_z)^{1/(\eta - 1)/\eta}$, provided $N_z > 1$. Our case is closely related to the weakly broken replica symmetry [12] of DP’s; also see Ref. [13], where the relation between replica methods and extremal statistics is discussed.
are important; hence we expect where we have used Eq. 11. The line $-0.41 + 0.53[\ln(2.78L_z/L^z)]^{1/2}$ is a guide to the eye. We have subtracted the expected dependence of $\langle E \rangle$ from $\langle E_0 \rangle$ (see the text). In Figs. 2–4 we use RB disorder, with a $J_{ij,z} \in [0 \sim 1]$ uniform distribution and $J_{ij,x}=0.5$. The number of realizations ranges from $N=500$ for $L=300$ and $L_z=500$ to $N=2000$ for $L=200$ and $L_z=600$.

reduced, so that bonds in and above the window are neglected and the new ground state, its $E_1$, and the corresponding gap energy $\Delta E_1$ are found. We studied at least $N=500$ realizations of system sizes up to $L=300$ and $L_z=500$. Figure 2 starts the discussion of the numerical data by showing how the ground state energy $\langle E_0 \rangle$ behaves as a function of $L$ and $L_z$. The scaling result [Eq. (7)] shows that the correction to the energy follows a logarithmic dependence on $N_{L_z}$, which is confirmed in the figure. Note that the extraction of this correction from the data requires an educated guess of how $\langle E \rangle$, the single valley energy, behaves with $L$. We have used an ansatz $\langle E \rangle \sim aL + b$, with the values of $a$ and $b$ demonstrated in Fig. 2, so that the exponent value $\eta=2$ corresponds to a Gaussian distribution. Due to the nature of the procedure, it would probably be possible to obtain a reasonable fit for, e.g., $\eta=\eta_-$ as well.

For small sample sizes, $L_z < L^z$, the value of the energy $E_0$ is affected by confinement. Similarly, the gap is controlled by confinement effects in this limit. When $L_z$ is large there are many independent valleys and extreme statistics effects are important; hence we expect

$$\langle \Delta E_1(L,L_z) \rangle = \begin{cases} \tilde{f}(L_z), & L_z \ll L, \\ L^0/\ln(L_z/L^z)^{\eta-1}/\eta, & L_z \gg L \end{cases}$$

where we have used Eq. (9) and $N_z \sim L_z/L^z$. We attempt to collapse the data by using the reduced variables $\langle \Delta E_1(L,L_z) \rangle /L^0$ versus $L_z/L^z$ for various $L$ and $L_z$. As seen in Fig. 3 we find a nice agreement with the extreme scaling form, with the ratio $(\eta-1)/\eta=1/2$, i.e., by using a Gaussian distribution.

Next we consider the relation of the extremal statistics to the susceptibility of these manifolds. In the $D$-dimensional case the susceptibility is defined by

![FIG. 2. The scaling of the ground state energy $E_0$ as a function of scaled transverse system size $L_z/L^z$ for the system sizes $L=100$, 200, and 300. The line $-0.41 + 0.53[\ln(2.78L_z/L^z)]^{1/2}$ is a guide to the eye. We have subtracted the expected dependence of $\langle E \rangle$ from $\langle E_0 \rangle$ (see the text). In Figs. 2–4 we use RB disorder, with a $J_{ij,z} \in [0 \sim 1]$ uniform distribution and $J_{ij,x}=0.5$. The number of realizations ranges from $N=500$ for $L=300$ and $L_z=500$ to $N=2000$ for $L=200$ and $L_z=600$.](image1)

![FIG. 3. The scaling function $f(y)$ of the scaled disorder average of the energy difference $\langle \Delta E_1 \rangle /L^0$ as a function of scaled transverse system size $L_z/L^z$ for the system sizes $L=100$, 200, and 300, each with $\tilde{z}_0/L=\text{const.}$ $\theta=1/3$ and $\xi=2/3$. The line has a shape $f(y)=0.23(\ln y)^{-1/2}$. The configurations are the same as in Fig. 2.](image2)

where the change in the magnetization of the whole $d$ dimensional system is calculated in the limit of the vanishing external field from the positive side [16,17], and the brackets imply a disorder average. We recently showed that the general behavior follows from a level-crossing phenomenon, which involves an extra potential $V_{\parallel}(z)=hz$, dependent on the height of the interface, in Hamiltonian (1), and that $h$ is an applied external field to the manifold. In any particular configuration when $h$ is varied, the manifold position changes in macroscopic “jumps” [16], the first one occurring at $h_1$.

One may write the susceptibility [Eq. (11)] with the help of the probability distribution of the fields $h_1 P(h_1)$, in the form

$$\chi = \lim_{h \to 0+} \left\langle \frac{\Delta z}{\Delta h} \right\rangle = \left\langle \frac{\Delta z_1}{L_z} \right\rangle \lim_{h \to 0+} P(h_1),$$

because the magnetization of a system $m(h) \sim \tilde{z}(h)/L_z$, and since the distance in the jump between the minima $\langle \Delta z_1 \rangle \sim L_z$ [16], independently of the sample-dependent $h_1$. It is expected that a scaling form $P(h_1)=P(h_1,L_z)$ applies, and that $P$ remains finite in the limit $h_1 \to 0$. Next we compare the average susceptibility as a function of the number of valleys $N_z$ to the conjecture that, in the presence of the field, the average gap for the original and excited state follows an extremal statistics form similar to Eq. (9).

The simulations are done again using a fixed height window in which the original ground state without a field is found. After this the external field $h$ is slowly applied by increasing the coupling constant values $J_{ij}(z)=J_{ij,\text{random}} + h z$, where $J_{ij}$ is perpendicular to the $z$ direction, until the first jump is observed with the corresponding $h_1$ and $\Delta z_1$. In order to find the scaling relation for the first jump field $h_1$, we perform the ansatz $\langle \Delta E_1 \rangle = (h_1 L_z)^{\beta(D=1)}$, since the field contributes to a polymer energy proportional to $L^D(D=1)$, and...
The scaling function $f(y)$ of the scaled disorder-average of the jump field $(h_1)L^{1-\theta}L_z$ as a function of scaled transverse system size $L_z/L^5$ for the system sizes $L=100, 150, 200, 250,$ and $300$, each with $z_0/L_z=\text{const.}$. The line has a shape $f(y)=0.41\ln(y)^{1/2}$. Here the number of realizations ranges from $N=500$ for $L=300$ and $L_z=500$ to $N=2600$ for $L=200$ and $L_z=600$.

$L_z\\sim \langle \Delta z_1 \rangle$ is the difference in the field contributions $h_z$ to the energy at finite $h$ at different average valley heights $z_0$ and $z_1$. Hence

$$\langle h_1(L, L_z) \rangle L_z \sim L^\theta f\left(\frac{L_z}{L^5}\right), \quad (13)$$

where the scaling function $f(y)=\lfloor \ln(L/L^5) \rfloor^{y-1/\eta}$. Figure 4 shows the scaling function [Eq. (13)] with a collapse of $(h_1(L, L_z))L^{1-\theta}L_z$ versus $L_z/L^5$ for various $L$ and $L_z$ which is again in good agreement with the logarithmic extreme scaling correction. Generalizing to arbitrary dimensions, one has the behavior of $(h_1(L, L_z))\sim L^{\theta-D}L_z^{-\theta} \lfloor \ln(L/L^5) \rfloor^{\eta y-1/\eta}$. For the susceptibility [Eq. (12)], one obtains, using $(h_1)$ for the normalization factor at $P(h_1)=0$,