Tailoring Josephson Coupling through Superconductivity-Induced Nonequilibrium

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The distinctive quasiparticle distribution existing under nonequilibrium in a superconductor-insulator-normal metal-insulator-superconductor mesoscopic line is proposed as a novel tool to control the supercurrent intensity in a long Josephson weak link. We present a description of this system in the framework of the diffusive-limit quasiclassical Green-function theory and take into account the effects of inelastic scattering with arbitrary strength. Supercurrent enhancement and suppression, including a marked transition to a \( \pi \) junction, are striking features leading to a fully tunable structure.

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Nonequilibrium effects in mesoscopic superconducting circuits have been receiving rekindled attention during the last few years [1]. The art of controlling Josephson coupling in superconductor-normal metal-superconductor (SNS) weak links is at present in the spotlight: a recent breakthrough in mesoscopic superconductivity is indeed represented by the SNS transistor, where supercurrent suppression as well as its sign reversal (\( \pi \) transition) were demonstrated [2,3]. This was achieved by driving the quasiparticle distribution in the weak link far from equilibrium [4–6] through external voltage terminals, viz., normal reservoirs. Such behavior relies on the two-step shape of the quasiparticle nonequilibrium distribution, typical of diffusive mesoscopic wires and experimentally observed by Pothier and coworkers [7].

The purpose of this Letter is to demonstrate that it is possible to tailor the quasiparticle distribution through superconductivity-induced nonequilibrium in order to implement a unique class of superconducting transistors. This can be achieved when mesoscopic control lines are connected to superconducting reservoirs through tunnel barriers (\( I \)), realizing a superconductor-insulator-normal metal-insulator-superconductor (SINIS) channel. The peculiar quasiparticle distribution in the \( N \) region, originating from biasing the \( S \) terminals, allows one to access several regimes, from supercurrent enhancement with respect to equilibrium to a large amplitude of the \( \pi \) transition passing through a steep supercurrent suppression. These features are accompanied by a large current gain (up to some 10\(^5\) in the region of larger input impedance) and reduced dissipation. The ultimate operating frequencies available open the way to the exploitation of this scheme for the implementation of ultrafast current amplifiers.

The investigated mesoscopic structure (see Fig. 1) consists of a long diffusive weak link of length \( L_J \) much larger than the superconducting coherence length \( \xi_0 \) oriented along the \( x \) direction. This defines the SNS junction of cross section \( A_J \). The superconducting terminations belonging to the SNS junction, labeled \( S_J \) (3 and 4), are kept at zero potential. The SINIS control line is oriented along the \( y \) direction and consists of a normal wire, of length \( L_C \) and cross section \( A_C \), connected through identical tunnel junctions of resistance \( R_T \) to two superconducting reservoirs \( S_C \) (1 and 2), biased at opposite voltages \( \pm V_C/2 \). The superconducting gaps of \( S_J \) and \( S_C \) (\( \Delta_J \) and \( \Delta_C \)) are in general different.

The supercurrent \( I_J \) flowing across the SNS junction is described by [5,6]

\[
I_J(V_C) = \frac{\alpha A_J}{e L_J} \int_0^\infty dE \left[ f(-E; V_C) - f(E; V_C) \right] \text{Im}[j_E],
\]

and depends on the quasiparticle distribution function \( f(E) \). In Eq. (1), \( \alpha \) is the normal-state conductivity which determines the normal-state resistance of the junction according to \( R_N = L_J/\alpha A_J \). The distribution function \( f \) reduces to the equilibrium Fermi distribution when \( V_C = 0 \). The energy-dependent spectral supercurrent [8,9], \( \text{Im}[j_E] \), can be calculated by solving the Usadel equations [10]. Following the parametrization of the Green functions given in Ref. [8], these equations in the \( N \) region can be written

FIG. 1 (color). Scheme of the Josephson transistor. The supercurrent \( I_J \) (along the white dashed line) is tuned by applying a bias \( V_C \) across the SINIS symmetric line connected to the center of the weak link. All normal wires are assumed quasi-one-dimensional.
\[ j_E = -\sinh^2(\theta) \partial_x \chi, \quad \partial_x j_E = 0, \]
\[ \hbar D \delta^2 \theta + 2iE \sinh \theta - \frac{\hbar D}{2}(\partial_x \chi)^2 \sinh(2\theta) = 0, \] (2)

where \( D \) is the diffusion coefficient and \( E \) is the energy relative to the chemical potential in \( S_f \). \( \theta(x, E) \) and \( \chi(x, E) \) are in general complex functions. For perfectly transmissive contacts, the boundary conditions at the \( S_fN \) interfaces reduce to \( \theta = \arctanh(\Delta_j/E) \) and \( \chi = \pm \phi/2 \) in the reservoirs \( S_f \), where \( \phi \) is the phase difference between the superconductors.

As required by Eq. (1), we must determine the actual quasiparticle distribution in the \( N \) region of the SINIS structure. This is controlled by voltage \( (V_C) \) and temperature and by the amount of inelastic scattering in the control line. In the case of a short control wire with no inelastic interactions, the quasiparticle distribution, according to Ref. [11], is given by

\[
f(E, V_C) = \frac{\mathcal{N}_{1,2} \mathcal{F}_1 + \mathcal{N}_{1,2} \mathcal{F}_2}{\mathcal{N}_{1} + \mathcal{N}_{2}},
\] (3)

where \( \mathcal{N}_{1,2} = \mathcal{N}_{S_f}(E \pm eV_C/2) \) and \( \mathcal{F}_{1,2} = \mathcal{F}_0(E \pm eV_C/2) \). The former are the BCS densities of states in the reservoirs \( S_C \) (labeled 1 and 2 in Fig. 1). \( \mathcal{F}_0(E) \) is the Fermi function at lattice temperature \( T \) [12]. In this case Eqs. (1) and (3) yield the dimensionless transistor output characteristics shown in Fig. 2(a). The latter plots the supercurrent \( I_J \) vs control bias \( V_C \) at different temperatures for a long junction (i.e., \( \Delta_j > \Theta_1^c \), where \( \Theta_1^c = \hbar D/L_j^2 \) is the Thouless energy of the SNS junction, as this is the limit where the supercurrent spectrum varies strongly with energy). We assumed \( \phi = \pi/2, \) \( T_C^c/T_c^f = 0.2, \) where \( T_C^c(T) \) and \( T_c^f \) are the critical temperatures of the superconductors \( S_C \) and \( S_f \) such that \( \Delta_j/\Theta_1^c = 300 \).

At the lowest temperatures, increasing \( V_C \) leads to a large supercurrent enhancement with respect to equilibrium slightly below \( V_C = 2 \Delta_c(T)/e = V_C^{\ast}(T) \) (region I in Fig. 2). Further increase of bias leads to a \( \pi \) transition (region II) and finally to a decay for larger voltages [14]. This behavior is explained in Figs. 2(b)–2(d), where the spectral supercurrent (solid line) is plotted together with \( f(-E) - f(E) \) (dash-dotted line) for values of \( V_C \) and \( T \) corresponding to regions I, II, and III, respectively. Hatched areas represent the integral of their product, i.e., the supercurrent \( I_J \) of Eq. (1). In particular, region I corresponds to the cooling regime where hot quasiparticles are extracted from the normal metal \([11,15]\). The origin of the \( \pi \) transition in region II is illustrated by Fig. 2(c), where the negative contribution to the integral is shown. We remark that the intensity of the supercurrent inversion is very significant. It reaches about 60% of the maximum value of \( I_J \) at \( V_C \approx V_C^{\ast}(T) \) in the whole temperature range, nearly doubling the \( \pi \)-state value of the supercurrent as compared to the case of an all-normal control channel \([5,6]\). In the high-temperature regime \( T/T_C^c \approx 0.6 \), when the equilibrium critical current is vanishing, the supercurrent first undergoes a low-bias \( \pi \) transition (region III in Fig. 2), then enters regions I and II. This recovery of the supercurrent from vanishingly small values at equilibrium is again the consequence of the peculiar shape of \( f \) [see Fig. 2(d)]. Notably, the supercurrent enhancement around \( V_C^{\ast}(T) \) remains pronounced even at the highest temperatures, so that \( I_J \) attains values largely exceeding 50% of the junction maximum supercurrent. This demonstrates the full tunability of the supercurrent through nonequilibrium effects induced by the superconducting control lines. We remark that this is a unique feature stemming from the superconductivity-induced nonequilibrium population in the weak link.

The length \( L_C \) of the SINIS control line can be additionally varied to control the supercurrent by changing the effective strength of inelastic scattering in the \( N \) region. For \( R_T \gg R_C = L_C/\alpha A_C \), the distribution function \( f(E) \) in the \( N \) region is essentially \( y \) independent and we have

\[
\frac{1}{e^2 R_T \Omega_C \nu_F} \left\{ \mathcal{N}_1 [\mathcal{F}_1 - f(E)] + \mathcal{N}_2 [\mathcal{F}_2 - f(E)] \right\} + \kappa \int \text{d}q \text{d}e q^\alpha I(\omega, e, E) = 0.
\] (4)
Here $\nu_F$ is the normal-metal density of states at the Fermi energy, $\Omega_C$ is the volume of the $N$ region, and $I$ is the net collision rate at energy $E$. At low temperatures, the most relevant scattering mechanism is electron-electron scattering [16] and we can neglect the effect of electron-phonon scattering. Then [7,17],

$$I(\omega, e, E) = I^{in}(\omega, e, E) - I^{out}(\omega, e, E),$$  \hspace{1cm} (5)

and

$$I^{in}(\omega, e, E) = [1 - f(\omega)[1 - f(E)]f(\omega) f(E + \omega)],$$  \hspace{1cm} (6)

where $\mathcal{K}_{coll} = (R_T/R_C)(L^2_C / D) \sqrt{\Delta_c} = \sqrt{2(R_T/R_K) \times (\Delta_c/E_{Fth}^c)}$, $R_K = \hbar / 2 e^2$, and $E_{Fth}^c = \hbar D/L^2_C$. In the absence of electron-electron interaction ($\mathcal{K}_{coll} = 0$), Eq. (3) is recovered.

The influence of inelastic scattering on $I_f$ is shown in Fig. 3, which displays the critical current of a long junction at $T = 0.1 T^c_c$ for several values of $\mathcal{K}_{coll}$. Here $I_f$ is obtained by numerically solving Eq. (8). The effect of electron-electron interaction is to strongly suppress the $\pi$ state and to widen the peak around $V^*_c$. The $\pi$ transition vanishes for $\mathcal{K}_{coll} \approx 100$, but the $I_f$ enhancement due to quasiparticle cooling still persists in the limit of even larger inelastic scattering [20]. The disappearance of the $\pi$ state can be understood by looking at the right inset of Fig. 3, which clearly shows how $f$ (calculated at $eV^*_c = 2.5\Delta_c$) gradually relaxes from nonequilibrium towards a Fermi function upon increasing $\mathcal{K}_{coll}$. The left inset shows how $f$ (evaluated at $eV^*_c = 1.5\Delta_c$) sharpens, thus enhancing $I_f$, by increasing $\mathcal{K}_{coll}$. This effect follows from the fact that inelastic interactions redistribute the occupation of quasiparticle levels in the $N$ region, thus increasing the occupation at higher energy. As a consequence, higher-energy excitations are more effectively removed by tunneling, even for biases well below and not only around $V^*_c$ (as in the case of $\mathcal{K}_{coll} = 0$). At the same time, supercurrent recovery at high temperature is gradually weakened upon enhancing $\mathcal{K}_{coll}$. Notably, these calculations show that a rather large amount of inelastic scattering is necessary to weaken and completely suppress the $\pi$ state. For example, using Al/Al$_2$O$_3$/Cu as materials composing the SINIS line, $\mathcal{K}_{coll} = 1$ corresponds to use a fairly long control line with $L_c \approx 2.3 \mu m$ [21].

![FIG. 3. Supercurrent vs $V_C$ for various $\mathcal{K}_{coll}$ with $T = 0.1 T^c_c$ and $T^c_c = 0.2 T^c_f$. Insets show the distribution function at $eV^*_c = 1.5\Delta_c$ (left) and $eV^*_c = 2.5\Delta_c$ (right) calculated for the same $\mathcal{K}_{coll}$ values.](image)

Changing the ratio $T^c_c/T^c_f$ shifts the $I_f$ response along the $V_C$ axis, the shape of the characteristics being

$$I^{out}(\omega, e, E) = [1 - f(\omega - \omega)][1 - f(E + \omega)]f(\omega)f(E).$$  \hspace{1cm} (7)

Electron-electron interaction is either due to direct Coulomb scattering [18,19] or mediated by magnetic impurities [16]. Below, we concentrate on the former but the latter would yield a similar qualitative behavior. From the calculation of the screened Coulomb interaction in the diffusive channel, it follows [18] that $\alpha = -3/2$ for a quasi-one-dimensional wire and $\kappa = (\pi \sqrt{D/2h^{3/2}p_F A_c^{-1}})$ [19]. We note that $\Delta_c$ is the most relevant energy scale to describe the distribution function for different voltages $V_C$. It is thus useful to replace $\omega \rightarrow \omega / \Delta_c$ and $e \rightarrow e / \Delta_c$ in order to obtain a dimensionless equation. Multiplying Eq. (4) by $e^2 R_T \Omega_C \nu_F$, we obtain

$${\mathcal{N}}_1[1 - f(E)] - \mathcal{N}_2[1 - f(E) - f^2] = \mathcal{K}_{coll} \int d\omega e\omega^{-3/2} I(\omega, e, E),$$  \hspace{1cm} (8)

where $\mathcal{K}_{coll} = (R_T/R_C)(L^2_C / D) \sqrt{\Delta_c} = \sqrt{2(R_T/R_K) \times (\Delta_c/E_{Fth}^c)}$. $R_K = \hbar / 2 e^2$, and $E_{Fth}^c = \hbar D/L^2_C$. In the absence of electron-electron interaction ($\mathcal{K}_{coll} = 0$), Eq. (3) is recovered.

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![FIG. 4. (a) Power dissipated in the SINIS line vs $V_C$ calculated for various ratios $T^c_c/T^c_f$ and $T = 0.01 T^c_f$. (b) Noise power $S$ vs $V_C$ calculated for the same parameters as in (a). (c) Differential current gain $G_I$ vs $V_C$ for $T^c_c/T^c_f = 0.2$. The inset shows $G_I$ in the high-bias region. In all these calculations we set $\mathcal{K}_{coll} = 0$ and $T^c_f = 9.26 K$ (Nb).](image)
can be expressed as

\[ P = I_C V_C \]

where \( I_C \) is the control current across the SINIS channel. The function \( P(V_C) \) is plotted in Fig. 4(a) for some ratios \( T_C^*/T_J^* \) at \( T = 0.01 T_J^* \), assuming \( R_T = 10^3 \) \( \Omega \) and \( T_J^* = 9.26 \) K. The impact of \( \Delta_C \) in controlling power dissipation is easily recognized. These effects clearly indicate that \( \Delta_C \ll \Delta_J \) is the condition to be fulfilled in order to minimize \( P \). In practice, the power dissipation for \( V_C > V_C^* \) constitutes an experimental problem as this energy needs to be carried out from the reservoirs. In a similar way the noise properties of the system are sensitive to the different \( T_C^*/T_J^* \) ratios. Assuming that the noise through one junction is essentially uncorrelated from the noise through the other, it follows that the input noise power \( S \) in the control line can be expressed as

\[ S = \frac{1}{R_T} \int_{-\infty}^{\infty} dE \mathcal{N}_I[f(E)(1 - \mathcal{F}_1) + \mathcal{F}_1[1 - f(E)]] \}

(9)

\( S(V_C) \) from Eq. (9) is shown in Fig. 4(b) for the same parameters of Fig. 4(a). For example, for \( T_C^*/T_J^* = 0.1 \) (corresponding roughly to the combination Al/Nb), \( P \) obtains values of the order of a few \( 10^{-15} \) W and \( S \) of some \( 10^{-30} \) A² Hz⁻¹ in the cooling regime, while these values are enhanced, respectively, to few tens of \( 10^{-12} \) W and \( 10^{-26} \) A² Hz⁻¹ for biases around the \( \pi \) transition.

In light of the possible use of this operational principle for device implementation, let us comment on the available gain and switching times. Input and output voltages are of the order of \( \Delta_C/e \) and \( E_{th}/e \), respectively, so that it seems hard to achieve voltage gain. On the other hand, differential current gain \( G_I = di_j/dl_j = (di_j/dV_C) \times (dl_C/dV_C)^{-1} \) can be very large. For \( V_C > V_C^* \) a simple estimate gives \( G_I \approx (E_{th}/\Delta_C)(R_T/R_N) \), meaning that with realistic ratios \( R_T/R_N \approx 10^3 \), \( G_I \) can exceed \( 10^5 \). \( G_I(V_C) \) calculated for \( T_C^*/T_J^* = 0.2 \) is plotted in Fig. 4(c) (the inset shows the gain in the \( \pi \)-state region). This calculation reveals that \( G_I \) can reach huge values, with some \( 10^9 \) for \( V_C < V_C^* \) [22] and several \( 10^8 \) in the opposite regime. Remarkably, gain is almost unchanged also in the presence of weak inelastic scattering (i.e., \( \mathcal{K}_{\text{coll}} = 1 \)).

The same holds for \( P \) and \( S \). As far as power gain is concerned, the Josephson junction has to be operated in the 

\( \text{dissipative} \)

regime in order to get out power. An estimate for the differential power gain gives \( G_P = dp_j/dp \approx (E_{th}/\Delta_C)G_I \approx 10^{-3} \) for \( V_C < V_C^* \) and \( \approx 10 \) for \( V_C > V_C^* \). The highest operating frequency \( \nu \) of the transistor is limited by the smallest energy in the system: \( \nu \leq \text{min} \{\Delta_C, \Delta_J, E_{th}/h(R_T C)^{-1}\} \), where \( C \) is the tunnel junction capacitance. For an optimized device, working frequencies of the order of \( 10^{11} \) Hz can be experimentally achieved in the high-voltage regime \( V_C > V_C^* \). For \( V_C < V_C^* \) conversely, the response is slower (somewhat below \( 10^8 \) Hz), owing to the long discharging time through the junctions.

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[12] At low control voltages there is a region of energies where \( \mathcal{N}_I = \mathcal{N}_J = 0 \). We assume that there the (otherwise weak) coupling to phonons makes the distributions at those energies equal to the equilibrium Fermi distribution. A more detailed discussion and another type of coupling is given in [13].
[21] This is straightforward assuming as typical parameters \( R_T = 10^3 \, \Omega \), \( D = 0.02 \, \text{m}^2/\text{s} \), and \( \Delta_C = 200 \, \mu\text{eV} \).
[22] In this calculation we chose to include depairing by a phenomenological but realistic parameter \( \Gamma = 10^{-4} \Delta_C \) [13]. Its omission would lead to extremely higher \( G_I \) values. For \( V_C \ll V_C^* \) and low temperature, \( \Gamma \) determines the differential resistance \( R = R_T \Delta_C/c \) of the SINIS line. Qualitatively, the large \( G_I \) stems from the fact \( R \gg R_N \).