AN EXPERIMENTAL INVESTIGATION
ON THE SUPERFLUIDITY OF $^3$He

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P.S.Kitän moni
motiveesta miellyttä-
mista yhteistyöstä.

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INTRODUCTION

In 1972 Osheroff, Richardson, and Lee\textsuperscript{1} at Cornell University reported the discovery of the two anomalous features, named "A" and "B", on the melting curve of $^3\text{He}$ at 2.6 and 2.0 mK, respectively. Today, after two years of intensive research in several laboratories, we know that there are two low-temperature superfluid phases of liquid $^3\text{He}$, the A-liquid and the B-liquid, in addition to the normal Fermi-liquid (cf. Fig. 1).

![Phase Diagram](image)

Fig. 1. The low-temperature phase diagram$^\text{a}$ of liquid $^3\text{He}$.

The acquired data$^2,^3$, together with many theoretical studies, show that the new phases of $^3\text{He}$ possess several extraordinary physical properties, some of which have never been encountered before.
This thesis consists of a series of measurements which verify superfluidity of liquid $^3$He in the A- and B-phases at the melting curve.

Superfluidity is a macroscopic manifestation of quantum statistics in nature. For such effects to occur, it is necessary, first, that the system under consideration is degenerate and second, that the particles may change places easily.

The first criterion requires that the thermal energy $kT$ should be much less than the characteristic quantum-mechanical energy $\hbar/mq^2_{av}$ of the system, which is determined by the uncertainty principle; $d_{av}$ is the average distance between particles. This implies low temperatures and high densities.

The second criterion, which is related to the indistinguishability of elementary particles and the consequent requirements on the symmetry of the wave function, implies that in solids the effects of quantum statistics are restricted by the fact the particles are more or less localized to the lattice sites. Thus the only atomic systems showing appreciable quantum statistical effects will be those that are liquids at very low temperatures, i.e. $^4$He and $^3$He. Of electronic systems, the Fermi-gas of conduction electrons in metals also fulfills the two criteria at sufficiently low temperatures.

Until now the two most dramatic manifestations of quantum statistics have been the low-temperature superfluid phase of liquid $^4$He and the superconductivity of some metals at low temperatures. The former is related to the Bose-Einstein condensation of $^4$He atoms and the latter represents a pairing type of condensation of Fermi-Dirac particles, the conduction electrons. The discovery of the A- and B-phases of liquid $^3$He and the consequent experimental verification of superfluidity$^4,5,6$ in them has now increased the number of such quantum systems to three. Since $^3$He is a fermion, its superfluid phases are to be explained theoretically$^7,8$ as a pairing type condensation of atoms, i.e. in terms of the BCS-theory.
No microscopic theory has yet been able to describe fully the strange properties characteristic of superfluidity. In this state $^4$He flows through narrow capillaries without apparent friction. It also shows the phenomenon of persistent flow: a flow once started in a ring does not decay.

Phenomenologically, however, superfluidity has successfully been described by a model which assumes that the liquid system consists of two interpenetrating fluids with independent velocity fields: the normal fluid of density $\rho_n$ and viscosity $\eta_n$ and the superfluid of density $\rho_s$ and zero viscosity and entropy. This is the two-fluid model of superfluidity, which has also been adopted in this thesis to explain the observed hydrodynamic effects in $^3$He.

CONTENTS

The purpose of this work has been to investigate experimentally the possible existence of superfluidity in the A- and B-phases of liquid $^3$He at the melting curve.

After a thermodynamic study of the nature of the A-transition, the first clear indications of superfluidity in the two phases were observed as drastic changes in the damping of a vibrating wire viscometer at the A- and B-transitions (cf. Fig. 2). Then, with improved resolution, the intercomparison of viscosity data in the normal Fermi-liquid region and the A- and B-phases has shown, within the frame- Fig. 2. work of the two-fluid
model, that viscous flow in $^3$He is accompanied by a flow of zero viscosity, i.e. the two phases behave as superfluids (cf. Fig. 3).

![Graphs showing reduced average viscosity and density](image)

Fig. 3. The reduced average viscosity and density of the normal component of superfluid $^3$He at the melting curve as a function of temperature.

The present investigation has been reported in a sequence of five papers which give a description of the various phases of the work, the construction of the refrigerators, the analysis of the measured data, and the results with discussions.

**PAPER 1.**


This report describes constructional details and performance characteristics of a cryogenic system for studying properties of $^3$He between 1 and 15 mK. $^3$He is refrigerated in two
steps. A dilution refrigerator, with a circulation rate of 70 μmoles/s, is employed to precool the specimen down to 15 mK. Below 15 mK ³He is self-cooled through adiabatic compression of a liquid/solid mixture, i.e. by Pomeranchuk’s method.

PAPER 2.


This article reports the determination of heat capacity of liquid ³He when it passes through the A-transition. The method involves the knowledge of a cell constant and the stray heat leak. The results, which must be considered as only qualitative, show a second order phase transition at 2.6 mK. Correction: The measured heat capacity corresponds to 1.3 cm³ of liquid ³He, not to 2.5 cm³ as calculated in the paper.

PAPER 3.


This Letter reports anomalous behavior of the damping of a vibrating wire viscometer in the A- and B-phases. The measurements show that as the Fermi-liquid cools from 15 mK, the viscometer signal first decreases, roughly in accordance with the expected 1/T² temperature dependence for η. At the A-transition the behavior is inverted. After a fast initial growth in the A-liquid the signal continues to increase more slowly while the temperature is further reduced. At the B-transition the vibration amplitude jumps discontinuously and then increases rapidly in the B-liquid. The lowest value of the effective viscosity was estimated to have been three
orders of magnitude smaller than at the A-transition. This was regarded as strong evidence for superfluidity in the A- and B-phases.

PAPER 4.


Based on the experience gained by the measurements reported in Paper 3, improvements were made in the apparatus and techniques to obtain better resolution. New results show that the viscosity of $^3$He in the normal Fermi-liquid region is accurately proportional to $1/T^2$. An intercomparison of the data in the Fermi-liquid region and the A- and B-phases allows the determination of the viscosity and the density of the normal component, $\eta_n$ and $\rho_n$, in the two phases; this is considered proof of superfluidity in $^3$He.

PAPER 5.


This paper reports measurements of the viscosity of liquid $^3$He down to 1 mK along the melting curve, and the corresponding methods of analysis, thus partially reviewing Papers 3 and 4. In addition, the data are employed to determine the melting curve of $^3$He below 10 mK down to the A-transition and the splitting of the A-transition in a magnetic field. The suppression of the B'-transition in a magnetic field is also studied.
The author first joined the $^3$He research group of the Low Temperature Laboratory in 1968 as an undergraduate student and participated during two years in the measurements of the specific heat of liquid $^3$He/$^4$He mixtures thus gaining the basic knowledge of low temperature physics. When the Pomeranchuk research project was started at the Low Temperature Laboratory in 1970 he joined the new group and has been working with this project throughout the years it has lasted. He has taken part in the construction of the apparatus and in all the measurements and their analysis.

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Otaniemi, April 1974

Tapio Alvesalo
REFERENCES


% In a recent private communication Ahonen, Haikala, and Krusius kindly provided the author with data on the phase diagram below 10 bar.
A CRYOGENIC SYSTEM FOR STUDYING THE PROPERTIES OF $^3\text{He}$ BETWEEN 1 AND 15 mK


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A CRYOGENIC SYSTEM FOR STUDYING THE PROPERTIES OF $^3$He BETWEEN 1 AND 15 mK


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ABSTRACT

A cryogenic system for studying the properties of $^3$He between 1 and 15 mK is described. The apparatus has been utilized for measurements of viscosity and heat capacity of liquid $^3$He along the melting curve. $^3$He is refrigerated in two steps. For temperatures down to about 15 mK a rather powerful dilution refrigerator, with a circulation rate of 70 μmoles/s, is used. This machine has sintered copper heat exchangers of novel design. Below 15 mK $^3$He is self-cooled through adiabatic compression of a liquid/solid mixture (Pomeranchuk's method). A description of constructional details and performance characteristics of the dilution refrigerator and the compressional cooling cell is given. Some aspects of thermometry are also discussed.

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1. INTRODUCTION

Dilution refrigeration is a well established technique for reaching temperatures in the vicinity of 10 mK\(^1\). However, the attainment of still lower temperatures by this method is difficult because of the intrinsic limitations of dilution refrigeration, such as the high viscosity and the good thermal conductivity of \(^3\)He. On the other hand, the cooling power of a refrigerator operating on the principle of adiabatic compression of a two-phase mixture of liquid and solid \(^3\)He exceeds that of a comparable size dilution cryostat by an order of magnitude below 10 mK.

The method of adiabatic compressional cooling was proposed by Pomeranchuk\(^2\), and first put into practice by Anufriyev\(^3\). After the technical refinements of Sites et al.\(^4\) and of Johnson et al.\(^5\) the method was known to produce a temperature in the vicinity of 2 mK. As \(^3\)He is its own cooling agent and as the difference between the specific entropies of liquid and solid \(^3\)He is of the order of R, it is clear that Pomeranchuk's method is the most efficient way of refrigerating both liquid and solid \(^3\)He to temperatures below 10 mK.

The present particular interest in compressional cooling dates back to the observation of the "A" and "B" anomalies on the melting curve of \(^3\)He by Osheroff et al.\(^6\), which soon turned out to be the discovery of the long-sought-for superfluid transition of \(^3\)He\(^7,8\).

Although descriptions of both the principle and experimental realization of compressional cooling apparatuses have already been published\(^1,4,5\), a description of our cryostat, so far used for studying the heat capacity\(^9\) and viscosity\(^8\) of superfluid \(^3\)He, should be of interest. In our apparatus the lowest temperatures are reached in two steps: 1) A continuously operating dilution refrigerator precools the compressional cell to about 15 mK and 2) temperatures
between 1 and 15 mK are achieved by compressional cooling. Although, unavoidably, both liquid and solid $^3$He are simultaneously present in the cell, our experience shows that meaningful results on the properties of liquid $^3$He can be obtained without undue interference caused by the presence of the solid.

We shall first describe the low temperature parts of our dilution refrigerator and its performance. We shall give constructional details of the Pomeranchuk cell, which in several respects differs from compressional devices built elsewhere. Finally, we shall discuss compressional cooling and thermometry in the temperature region below 10 mK.

2. THE DILUTION REFRIGERATOR

The lower part of our cryostat is shown in Fig. 1; the dilution refrigerator is inside the inner vacuum jacket below the $^4$He pot. Its constructional units are the still, the continuous heat exchanger spiral, three discrete heat exchangers, and the mixing chamber. The mechanical support for the heat exchangers and the mixing chamber is made of three fiber reinforced bakelite laths fastened with screws. Any one of them may easily be removed to give better access to the system. Heat exchangers and the mixing chamber are themselves integral parts of the supporting structure and give the system a good rigidity. The experimental set-up and the radiation shield are fastened to the mixing chamber. If more experimental space is needed, the supporting laths can be shortened.
Fig. 1. The dilution refrigerator and the low temperature parts of the experimental system for compressional cooling of $^3$He.
2.1. Pumping system

The pumping system for circulating the $^3$He gas consists of an oil diffusion pump (EDWARDS E04) backed by a mechanical pump (EDWARDS ES 200), which is capable of handling 200 l/min. Before the circulating $^3$He re-enters the cryostat a liquid nitrogen cooled charcoal trap removes traces of gaseous impurities and pump oil. Additional purification takes place in spirals of copper tubing in the $^4$He dewar. The incoming $^3$He is condensed in a copper tube located inside the $^4$He pot at 1.2 K. The circulation rate can be monitored with a thermocouple flow meter$^{10}$. If the pressure in the system rises above 1 bar, for example owing to blocking of the input tube, a security valve admits the $^3$He gas into storage tanks.

2.2. The still

The still consists of a copper segment which forms part of the inner vacuum flange; its volume is 45 cm$^3$. Within the still there is a copper tube (2 mm OD/1 mm ID) for precooling the incoming $^3$He to 0.8 K. Two constrictions in the $^3$He input tube, one before and the other after the still, build up the necessary pressure for condensing $^3$He. The flow impedances of the constrictions are $Z = 3 \times 10^{11}$ cm$^{-3}$ and $Z = 10^{11}$ cm$^{-3}$, respectively. The tube for pumping $^3$He out of the still has an electropolished orifice of 1.5 mm diameter, which is the only obstacle to prevent the $^4$He film from creeping up the pumping tube.

2.3. Heat exchangers

The performance of a dilution refrigerator is largely determined by the quality of its heat exchangers. Therefore, considerable effort was put into their construction. The system
consists of a continuous heat exchanger followed by three discrete exchangers. The need for such a construction is due to the $1/T^3$ temperature-dependence of the Kapitza thermal boundary resistance. Continuous exchangers can, therefore, only be used at above 0.1 K. Our continuous exchanger is made of two concentric Cu-Ni tubes (1 mm OD/0.85 mm ID and 2 mm OD/1.7 mm ID) and it is 1000 m long.

In addition to a large surface area, a discrete exchanger must have a low flow impedance because the viscous heating increases as $1/T^2$ with decreasing temperature. Fig. 2a shows the construction of the two upper exchangers and Fig. 2b the third and coldest exchanger. The two dilute compartments of the third exchanger are connected in parallel, in this way

Fig. 2. Fig. 2a illustrates the first and second discrete heat exchangers and Fig. 2b the third (i.e. the coldest) exchanger.
the flow impedance of the exchanger is about the same as the impedance of the interconnecting tubes between the exchangers. In all discrete exchangers we reduced the flow impedance by making the sintered copper sponges as flat as possible. In order to obtain a large area for heat exchange we used copper powder with a maximum particle size of 44 \( \mu \text{m} \) and with an average diameter of about 27 \( \mu \text{m} \). When sintered, with a packing factor of 40 \%, a surface area of about 2000 \( \text{cm}^2/(\text{cm}^3 \text{ of powder}) \) is obtained.\textsuperscript{11}

The thin and flat design of the sintered sponges results in a low thermal conductance in the radial direction and to the body of the heat exchanger. To eliminate this drawback a copper sheet (cf. Fig. 3) was sintered into the powder. As

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**Fig. 3.** Construction of the 0.2 mm thick copper sheets used to enhance thermal conductivity within the sintered copper sponges. See also Fig. 4.
shown in Fig. 4, the sheet extends outside the copper sponge. In the final assembly this extension protrudes into the joint between adjacent parts of the exchanger (cf. Fig. 2) and is joined by plasma welding to the body of the exchanger. The same extension sheet also prevents liquid from flowing round the edge of the sponge, thus eliminating the need for a tight fit between the sponge and the exchanger body.

Fig. 4. Sintered sponges. The copper sheets, the tubes that provide the right flow pattern within the heat exchangers, and the copper powder are sintered together simultaneously.

The sponges were sintered from copper powder in graphite moulds, which were machined so that the extension sheet was centered with respect to the powder when the cap was put on the mould. The powder was not compressed before sintering; the final packing factor was about 40%. The short copper tubes (cf. Figs. 2 and 4), which give the right flow pattern within the heat exchangers, were also inserted before sintering and kept in the right position by the mould. Before assembly into heat exchangers the sintered parts were not machined in any way.

Sintering was done in a vacuum ($<10^{-1}$ mbar) induction oven. The specimen was first heated at about 500°C for one
hour in order to purify the copper. The temperature was then raised to 850°C in 5 minutes after which the oven was turned off and left to cool. The temperature of the mould was measured with a thermocouple. Cooling to 70°C took about 7 hours. Table 1 gives numerical data on the heat exchangers.

**Table 1.**

<table>
<thead>
<tr>
<th>Discrete heat exchanger</th>
<th>( V_{\text{sint}} ) (cm(^3))</th>
<th>( \pi R^2/h ) (cm)</th>
<th>( Z \times 10^5 ) (cm(^{-3}))</th>
</tr>
</thead>
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<tr>
<td></td>
<td>concentr. dilute</td>
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<td>concentr. dilute</td>
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<tr>
<td>1.</td>
<td>5.1</td>
<td>14.2</td>
<td>20.2</td>
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<tr>
<td>2.</td>
<td>5.1</td>
<td>14.2</td>
<td>20.2</td>
</tr>
<tr>
<td>3.</td>
<td>14.2</td>
<td>39.2</td>
<td>56.5</td>
</tr>
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</table>

Table 1. Constructional data for the heat exchangers. \( V_{\text{sint}} \) is the total volume, \( R \) the radius and \( h \) the thickness of the sintered sponge. \( Z \) is the flow impedance measured after assembly.

2.4. Mixing chamber

The design of the mixing chamber was strongly influenced by the fact that the Pomeranchuk cell to be precooled was located outside it. In order to reduce the thermal resistance between liquid helium and the body of the mixing chamber the thermal contact area was again increased by means of sintered copper powder. Below 20 mK the phonon wave length becomes comparable with the size of the copper grains and this may give rise to an additional thermal resistance. Following Radebaugh et al.,\(^{11}\) we estimate that the effective surface area has a maximal value of about 600 cm\(^2\)/(cm\(^3\) of powder).
for an 0.5 mm thick layer when the average particle size is 30 μm and the packing factor is 40 %. Therefore, a 0.5 mm thick layer of copper powder was sintered on inner surfaces of the mixing chamber. The construction shown in Fig. 5 was adopted.

![Carbon Thermometer Diagram](image)

**Fig. 5.** The mixing chamber. Liquid helium occupies 36 cm³ and the volume of sintered powder is 17 cm³. The six horizontal compartments, the $^3$He inlet tube, and one of the two outlets are shown. The horizontal surfaces are covered with an 0.5 mm thick layer of copper sinter.

The different horizontal compartments in the mixing chamber were connected through holes in the adjacent walls. During assembly the walls were positioned so that these holes were not on top of each other, in order to force liquid streams to flow along the sintered surfaces. The concentrated $^3$He is fed into the dilute phase through the input tube ending at the bottom of the mixing chamber.
2.5. Performance of the dilution refrigerator

During the long time we have used our cryostat operational instabilities have never been seen. Such instabilities, occurring frequently when gas circulation is started, often necessitate changes in the circulation rate and in the composition of the gas mixture. We commenced cooling just by condensing the $^{3}$He-$^{4}$He mixture into the still and then starting the circulation. The good stability is apparently due to the fact that neither the concentrated nor the dilute side has wide-open continuous tubes, but are in a step-by-step manner closed by the copper sponges in the discrete heat exchangers. The cooling capacity, measured with a heater and a thermometer fastened externally to the mixing chamber, is shown in Fig. 6.

![Diagram](image)

**Fig. 6.** Cooling capacity of the dilution refrigerator at a circulation rate of 70 μmoles/s. The temperature was measured on the outer wall of the mixing chamber.
After the $^3\text{He}^4\text{He}$ mixture had been condensed into the still, circulation at a rather high rate was required, for about 2 or 3 hours, to attain the phase separation temperature ($\approx 0.7 \text{ K}$). The circulation speed was then gradually lowered to 70 $\mu$moles/s and the still heater was switched on. In about 6 additional hours the temperature of the liquid inside the mixing chamber was lowered below 15 mK.

3. **THE POMERANCHUK CELL**

The design of a Pomeranchuk cell depends on the purpose of the experiment and is not as universal as the construction of a dilution refrigerator. Our cell consists essentially of a $^4\text{He}$ pressurizer, two $^3\text{He}$ spaces interconnected via a 3 mm diameter tube, and a pressure-measuring capacitance gauge. A superconducting solenoid capable of producing a field of 50 kG is located at the lower $^3\text{He}$ space of the cell; the magnetic field in the upper part of the cell, where the $^4\text{He}$ pressurizer and the capacitance gauge are located, is less than 2 kG. This arrangement was designed in order to study the magnetic field dependence of the $^3\text{He}$ melting pressure. The top part of the cell was also equipped with a vibrating wire viscometer in order to study the hydrodynamic properties of liquid $^3\text{He}$.

3.1. **Construction**

A detailed cross sectional view of the compressional cell is illustrated in Fig. 7; Fig. 8 shows how the main parts of the cell were assembled together. The $^4\text{He}$ pressurizer is made from 0.5 mm thick Be-Cu sheet bent into U shape, and the side walls are made of brass. These parts, together with two copper spacers, were put tightly inside the solid copper body.
Fig. 7. Cross sectional view of the assembled Pomeranchuk cell.
(wall thickness 3 mm), and all parts were silver soldered together. The assembled lower part of the cell was then soft soldered to the upper part of the cell. A cold finger is attached to the massive body for making thermal contact with the mixing chamber of the dilution refrigerator.

Fig. 8. Constructional parts of the compressional cell before assembly. 1: side wall of the $^4$He pressurizer, 2: flexible Be-Cu wall of the pressurizer, 3: spacer, 4: solid copper body, 5: finger for thermal contact to the mixing chamber, 6: connecting tube between the upper and the lower $^3$He space, and 7: lower $^3$He volume.

The upper and the lower volume of the cell were connected by a copper tube (3 mm ID/5 mm OD) hard soldered to a hole in the copper body. At the lower end of this tube there is a cylindrical 0.7 cm$^3$ space. This volume, together with the tube and the space between the flexible Be-Cu wall and the solid copper body, and the upper space containing the wire viscometer (cf. Fig. 7) makes the $^3$He volume. With equal $^3$He and
$^3$He pressures the volume of the $^3$He space is 2.65 cm$^3$; if the $^3$He pressure is 16 bar higher than the $^4$He pressure, this volume is increased to 2.85 cm$^3$. Therefore, in compressional cooling experiments it is possible to solidify most of the $^3$He inside the cell. In an earlier version of the cell the flexible Be-Cu wall of the pressurizer was only 0.4 mm thick. This was too thin as evidenced by a sudden rupture after a number of pressurizations.

$^3$He is fed into the cell through a Cu-Ni tube at the lower end of the cell whereas $^4$He enters the pressurizer through the side wall. The $^3$He and the $^4$He tubes were made of 0.1 mm ID/0.3 mm OD Cu-Ni capillaries, and were thermally anchored to the $^4$He pot, the still and the discrete heat exchangers. Immediately above the cell the $^3$He tube was made into two parallel lines. One is a safety exit in case the tube used for pressurization should happen to become blocked.

The pressure sensitive gauge is located on the top of the cell (cf. Fig. 7). The flexible diaphragm of the pressure gauge was made of Be-Cu. It is 0.8 mm thick and has a diameter of 12 mm. Movements of the diaphragm are coupled to the lower capacitor plate above the diaphragm by a bakelite rod. This plate is made of carefully polished brass; its thickness is 0.5 mm and diameter 30 mm. The upper capacitor plate is also made of polished brass and is 2.5 mm thick. The two plates are separated by a narrow 20 μm thick annular Mylar spacer. The capacitor system is electrically insulated from the body of the cell.

The capacitance is measured with a General Radio 1620-A bridge using a three terminal connection. The sensitivity of the pressure measuring system, better than 0.1 mbar, was sufficient for observing fine details in the pressure vs. time curves during compressional cooling experiments. The vibrating wire viscometer was inserted as a single unit just below the
pressure gauge. It was sealed to the cell with a indium O-ring and could easily be removed or replaced as required.

3.2. Performance of the Pomeranchuk cell

A typical procedure of compressional cooling was as follows. The $^3$He and $^4$He volumes were first filled with liquid and the whole system was then cooled to about 1 K. At this temperature the pressure gauge was calibrated between 29 and 35 bar against a Texas Instruments Model 144 gauge having a fused quartz Bourdon tube. The $^3$He pressure was then adjusted to a value somewhat higher than the minimum of the $^3$He melting curve and the $^4$He pressure was set to about 1 bar. The cell was then ready for further cooling. After the dilution refrigerator had been in operation for 9 to 10 hours the temperature of $^3$He in the compression cell was lower than 20 mK. As a rule, compressions were not started at higher temperatures in order to minimize the amount of solid $^3$He inside the cell when reaching the A- and B-transitions.

The rate at which the $^4$He pressure increased determined the cooling speed. Because we had no heat switch between the mixing chamber and the Pomeranchuk cell, compressions were usually carried out rather fast. Typically it took about 30 minutes to cool from 15 mK to 2 mK or lower.

In the course of these experiments we observed that thermal coupling between the walls of the cell and the liquid inside appeared to be better than estimated for a copper-to-liquid $^3$He interface. This helped in reaching a low starting temperature for Pomeranchuk cooling but, during the compression, caused a considerable heat leak into cell. For this reason we often precooled for two days and then switched the dilution refrigerator into the single-cycle mode before starting the compression. In this way the lowest starting temperature
achieved was about 11 mK. We could then reach the superfluid transition region with as little as 10% of solid inside the cell. This quantity of solid is consistent with the assumption that solid $^3$He, once formed, is thermally relatively weakly coupled to the remaining liquid.

A typical compressional cooling run is illustrated in Fig. 9; the temperature $T(t)$ of liquid $^3$He inside the cell and

![Diagram](image)

**Fig. 9.** Illustration of a typical compressional cooling run. In this run the superfluid transition region of liquid $^3$He was traversed several times; the temperature of liquid $T(t)$ has been plotted as a function of time. The transitions between normal fluid and A-liquid are labelled A and A', those between A- and B-liquids by B and B'. Experiment was stopped at point Z. The time variation of $T(t)$ is effected by the changing pressure $P_4(t)$ in the $^4$He pressurizer; $P_4(t)$ is also given in the figure.
the pressure $P_4(t)$ of the $^4$He pressurizer have been plotted as functions of time. In this run the superfluid transition region was traversed several times; the transition from normal $^3$He to superfluid A-liquid is labelled as A and the transition from A-liquid back to normal liquid as A'. Similarly, the transition from A-liquid to B-liquid is labelled as B and the transition from B-liquid to A-liquid as B'. The A- and A'-transitions occur reproducibly at $T_A = T_A' = 2.6$ mK and are characterized by a discontinuity in $dT/dt$. Because the B- and B'-transitions are first order in Ehrenfest's sense, the reproducible transition temperature $T_{B'} = 2$ mK is characterized by a short period with $T(t) = \text{constant}$, whereas $T_B$, depending on the degree of supercooling, is situated somewhere between 1.5 and 2 mK and is usually accompanied by a sudden small increase in temperature. It is also observed from Fig. 9 that upon successively passing through the B and B'-transitions, without a deep passage into the A-liquid phase, the B-transition starts to resemble a mirror image of B'. At point Z the experiment was deliberately stopped and the pressure in the $^4$He pressurizer rapidly released because the viscometer ceased to operate properly. Its motion was hindered by solid $^3$He.

Using the experimentally determined cell constant of our Pomeranchuk cell $dV/\delta(P_3-P_4)$, the molar volumes and compressibilities of liquid and solid $^3$He, and the measured $^3$He and $^4$He pressures, $P_3(t)$ and $P_4(t)$, respectively, we can calculate the amount of solid inside the cell. For example, in the run illustrated in Fig. 9, there was 6.1 % of solid inside the cell when the first A-transition occurred. When the experiment was stopped at point Z, the amount of solid had increased to 13.8 %.
4. THERMOMETRY

4.1. The melting pressure temperature scale

Compression cooling is a paragon of the principle that a material property suitable for refrigeration is often very useful for thermometry also. The natural temperature standard inside a Pomeranchuk cell is the melting pressure of $^3$He. Because of the minimum on the melting curve the pressure must be measured in situ. This is usually done, as in our cell, by means of a flexible diaphragm capacitance gauge$^{12}$. The precision is very high. For example, a pressure resolution of 0.1 mbar is easily achieved; this corresponds to a precision of 2.5 μK in temperature.

Absolute measurements of pressure, however, are more difficult. We estimate that, in practice, an error of the order of 20 mbar is easily introduced into the absolute determination of the melting pressure; in the vicinity of 3 mK this corresponds to a temperature interval of 0.5 mK. This fact makes the intercomparison of different melting curve measurements$^{13,14}$ somewhat difficult and thus devalues melting pressure thermometry below 10 mK.

Melting pressure thermometry below 10 mK was at once put on firmer ground when the A-transition was found; this point provides a convenient fixed reference temperature. On the melting curve and in a small external magnetic field the temperature of the A-transition is now known to be 2.6 mK with an estimated uncertainty of less than 0.1 mK$^{15,16}$.

With the above mentioned facts in mind we constructed our temperature scale as follows:

1) The pressure $P_A$ of the A-transition was determined.

2) The melting curve $P_M(T)$ determined experimentally by Johnson et al.$^{14}$ was shifted by a constant amount, upwards or downwards, so that it coincided at 2.6 mK with our deter-
mination of $P_A$. The shifted melting curve then provided our temperature scale between 2.6 and 45 mK.

3) The melting curve $P_M(T)$ calculated by Goldstein\textsuperscript{17}, which essentially reproduces the smoothed data of Scribner et al.\textsuperscript{13}, was lifted upwards so that it joined smoothly on to the shifted melting curve of Johnson et al. The shifted melting curve of Goldstein was then employed for thermometry between 45 and 100 mK.

We estimate that the temperature scale obtained in this way is accurate to 4% over the whole range from 2.6 to 100 mK. Although it appears that the temperatures given by Johnson et al.\textsuperscript{14} are a fraction of a millikelvin too high and the absolute value of the derivate $dP_M/dT$ a bit too low, this does not seriously affect the accuracy of the temperature determination.

Below 2.6 mK the melting pressure is presently not well known. The best we can do is to extend the melting curve downwards with a constant slope of not less than 35 bar/K\textsuperscript{16}. Although it is expected that $|dP_M/dT|$ should decrease towards lower temperatures, we are not much guided by theory in guessing the actual shape of this decrease, because calculations\textsuperscript{17,18} seem to fail rapidly below 2.6 mK. This is most clearly indicated by the fact that the experimentally determined pressure difference between the maximum melting pressure, $P_{max}$, and $P_A$ is 58 mbar\textsuperscript{19}, whereas theoretical estimates for $P_{max} - P_A$ are 35 mbar\textsuperscript{17} and 14 mbar\textsuperscript{18}.

4.2. Secondary thermometry

At an early stage of these experiments we were interested in following the temperature inside the compressional cell when off the melting curve. For this purpose we used a nominal 100Ω Speer carbon resistor which was ground to a thin slab 0.5 mm thick, so that its resistance at room temperature was
about 300Ω. The resistor was immersed in the liquid inside the cell. By grinding off the insulating epoxy we had hoped to enhance the thermal contact between the carbon and the liquid helium. The resistor was calibrated against the melting pressure of ³He using the \( P_M - T \) relation obtained as described above. A typical curve is shown in Fig. 10. Down to about 5 mK the resistor was found to be in a fairly good contact with the

![Graph](image)

**Fig. 10.** Calibration curve for a nominal 100 Ω Speer carbon resistor ground to a thin slab having a 300 Ω resistance at room temperature. The low temperature calibration was carried out with the resistor immersed in liquid ³He inside the Pomeranchuk cell, and using the melting pressure \( P_M(T) \) of ³He as the reference.
liquid $^3$He. At lower temperatures it gradually lost the contact, often because solid $^3$He apparently formed an insulating layer on the surface of the resistor. Nevertheless, the characteristic change in $dT/dt$ at the $\alpha$-transition (cf. Fig. 9) was often discernible. The resistance was measured with an AC bridge; the power dissipated was about 10 pW at 10 mK and roughly proportional to temperature. The magneto-resistance of this thermometer was negative, in agreement with earlier measurements.²⁰

After the vibrating wire viscometer had been installed in our cell, we found that the viscosity $\eta$ of liquid $^3$He provides a convenient and simple thermometric parameter. The measurements showed that, from 16 to 100 mK, $\eta = 1/T^2(1.17-3.10T)$, where $\eta$ is given in $\mu$Poise and $T$ in K. Because this relation was derived from measurements between 16 mK and 100 mK, the actual value assigned to $T_\alpha$ was not important. However, extrapolating this relation towards lower temperatures, the value 2.6 mK with an uncertainty < 0.1 mK could, indeed, be assigned to $T_\alpha$. A single temperature measurement can easily be done with a precision of 3%. It is important to note that in this kind of thermometer there is no Kapitza thermal boundary resistance and that the magnetic field dependence is that of the viscosity of normal liquid $^3$He. We expect that accurate measurements of the $P^*_M$ vs. $T$-curve of $^3$He in high magnetic fields will be possible with this thermometer.

A few comments about the establishment of thermal equilibrium inside our compressional cooling cell are appropriate. At 15 mK the viscosity of normal liquid $^3$He is comparable to the viscosity of water at room temperature. Although $\eta$ varies as $1/T^2$, it is small enough for hydrostatic equilibrium to be established within the sample in a reasonable time down to 2.6 mK, as long as there are fairly open liquid paths inside the cell. As a result, all liquid-solid interfaces are at the
same pressure and, being on the melting curve, at the same
temperature. This does not mean, however, that thermal gra-
dients cannot exist in the liquid. This is especially true
if we expose the cell to a non-homogeneous magnetic field.
Because the melting pressure of $^3$He is depressed by the field,
solid has a tendency to grow first at a location where the
field is strongest. In this case the remaining liquid is
cooled only via thermal conduction and, if compression is
carried out quickly, considerable temperature inhomogeneities
are to be expected.

In the majority of our compressional cooling runs we
found that the formation of solid $^3$He apparently started
in the lower or in the middle part of the cell, even when
the tail was not intentionally exposed to a magnetic field.
This was evident by the fact that the temperature, as measured
by the vibrating wire viscometer, lagged behind the tempera-
ture determined from the measured melting pressure. These
two temperatures were coupled together with a time constant
varying as $T^2$; at 10 mK this constant was of the order of 1 to
2 minutes. We calculate that this corresponds to a situation
where the path for thermal conduction between the liquid
around the vibrating wire and the liquid-solid interface,
where the primary cooling took place, was 2 to 3 cm. In many
runs we could identify the moment at which solid $^3$He abruptly
started to form in the vicinity of the wire.

Our analysis of the thermal gradients showed that it was
possible to supercool liquid $^3$He as much as 0.4 mK or, equi-
valently, overpressurize it about 16 mbar, before solidi-
fication started. From this we conclude that solid $^3$He con-
tinues to grow preferably at locations where solidification has
started. This is in agreement with earlier observations$^{21}$ and
suggests that with special effort it should not be too diffi-
cult to prevent solid $^3$He from forming an insulating layer
around metallic samples to be cooled by the Pomeranchuk method.
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THE HEAT CAPACITY OF LIQUID $^3$He ALONG THE MELTING CURVE BETWEEN 1.8 AND 4.2 mK

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The heat capacity of liquid $^3$He has been measured along the melting curve between 1.8 and 4.2 mK in an apparatus employing compressional cooling techniques. The results show a second order phase transition at 2.6 mK.

Fig. 1. The compressional cooling run from which the heat capacity was determined. Point B' of Osheroff et al. [1] is not visible, probably due to the short time the cell stayed at temperatures below $T_B$ and because of the rapid warming. The resaturation was stopped at M above A. This is thought to be the result of lack of thermal equilibrium inside the cell, caused by our fast pressurization and the rather complicated cell geometry; it is not a reproducible feature. The B point is located at $2.6 \times 10^{-2}$ atm above A, in accordance with Osheroff et al. [1].

Figure 1 caption:

After the pressurization was stopped the cell was allowed to warm under an approximately constant heat leak of 0.4 erg/s.

By observing that $dP/dt = \left(\frac{dP_M}{dT}\right)\left(dT/dt\right)$, where $dP_M/dT$ is the temperature derivative of the

$$ C_{\text{total}} = \frac{dP_M/dT}{dP/dt} \left(\dot{q} - \Delta V \cdot T \frac{dP_M}{dT} \right), \quad (1) $$

where $\dot{q}$ is the heat input.
Fig. 2. The heat capacity determined from the warm-up curve of fig. 1 using eq. (1). The transition temperature $T_A$ is chosen equal to 2.60 mK [5]. The straight line above 2.6 mK represents a heat capacity $C_{\text{total}} = 19$ T erg/mK, which corresponds to about 2.5 cm$^3$ of liquid $^3$He [6].

melting pressure, $dP/dt$ the measured time derivative of the pressure, $\dot{q}$ the heat leak, $\Delta V$ the volume change at solidification and $\dot{n}$ is the rate at which liquid is transformed into solid. The term containing $\dot{n}$ may be treated as a correction to the heat leak. In our case it was approximately 10% of $\dot{q}$. As the pressurization was stopped at point M in fig. 1, $\dot{n}$ can be calculated simply from $\dot{n} = K(dP/dt)$, where $K$ is a constant characterizing the cell.

The heat capacity obtained by graphically differentiating $P(T)$ and by using eq. (1) is shown in fig. 2. It is observed that at 2.6 mK a second order transition occurs. The estimated amount of liquid $^3$He in our cell after the pressurization was stopped was 2.3 cm$^3$. Using the data of Abel et al. [6] we find that above 2.6 mK the linearly changing heat capacity corresponds to 2.5 cm$^3$ of liquid $^3$He. The measured heat capacity is thus believed to represent liquid $^3$He alone.

For the temperature scale and for $dP_{\text{M}}/dT$ we have used the melting curve as calculated theoretically by Goldstein [5], with an arbitrary constant shift to make $T_A = 2.60$ mK.

the amount of which diminished only 0.3% in warming from 1.8 to 4.2 mK. The shape of the heat capacity anomaly may suggest a superfluid transition with BCS type of pairing.

Although our work does not give any explanation as to the origin of the B transition, we want to put forward the following suggestion: When passing through point B the solid being formed undergoes a transition to a lattice of lower symmetry and smaller specific volume. The exchange coupling constant $J$ may change its sign, so as to make the newly formed solid layer ferromagnetic. The ferromagnetic surface, in contact with liquid, could break the pairs of liquid $^3$He atoms (assumed to be paired antiparallel). This would seem to explain qualitatively four characteristic features of the B point [1, 2]: 1) the small pressure drop at B, 2) the disappearance of the NMR frequency shift at B and reappearance at $B'$, 3) the "delayed" recovery at $B'$, and 4) the delay of the $P(T)$ curve at $B'$. Experiments to test this hypothesis have been started.

The authors want to acknowledge discussions with O.V. Loukas and would like to thank E.D. Adams for teaching us the technique of the capacitive pressure gauge.

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Evidence for Superfluidity in the Newly Found Phases of $^3$He

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The relative viscosity $\eta$ of liquid $^3$He has been measured along the melting curve from 10 mK to about 1 mK with a vibrating-string viscometer. As the temperature is reduced, $\eta$ first increases as expected until the pressure anomaly "A" is reached. The viscosity then starts to decrease, has a discontinuous drop at the pressure anomaly "B," and then continues to diminish rapidly. The lowest measured value of $\eta$ is 1000 times smaller than that at point A. This must be regarded as strong evidence for superfluidity.

After the original discovery by Osheroff, Richardson, and Lee\textsuperscript{1} a considerable amount of experimental and theoretical work\textsuperscript{2-14} has been done in trying to explain the nature and the origin of the "A" and "B" transitions in $^3$He below 3 mK. We have measured the relative viscosity $\eta$ of liquid $^3$He from 10 mK to well below these points. Our first results show several interesting features and strongly suggest that two superfluid transitions occur in the liquid below 3 mK.

The $^3$He specimen was refrigerated inside a Pomeranchuk cell which had first been precooled to 15 mK by means of a dilution refrigerator. The cell has a total volume of about 3 cm$^3$ and it is made of two chambers connected via a 3-cm-long tube which has a diameter of 3 mm. The upper chamber, with a volume of about 2 cm$^3$, is equipped with a capacitive pressure gauge\textsuperscript{15} and a vibrating-string viscometer.\textsuperscript{16} The string, 2.5 cm long, 0.25 mm thick, and made of NbTi wire, is under a tension of 16 N which gives the system a natural oscillation frequency of 1245 Hz and an intrinsic $Q$ of about 1000. An ac current of 2 mA is passed through the string.

It was expected that frictional heating and the magnetic field of 0.1 T required to operate the viscometer might promote the formation of solid $^3$He around the string which, in turn, would prevent the viscometer from working. For this reason a superconducting solenoid, capable of producing a sharply profiled field up to 4.5 T, was fitted around the lower part of the $^3$He cell. It was hoped in this way to restrict the solid to the lower chamber only, at least at the beginning of a compressional cooling experiment.

Runs were made in the conventional way by reducing the $^3$He volume at a steady rate with the aid of a $^4$He pressurizer. During an experiment the $^3$He pressure was continuously measured with a General Radio 1620-A capacitance bridge with phase-sensitive detection. The viscosity was monitored by sweeping the viscometer ac excitation current semicontinuously through the resonant frequency. Since the signal amplitude, frequency shift at the resonance, and the width of the resonance line are all related to the viscosity and the density of the normal fluid, a considerable amount of information can be deduced from this type of measurement.\textsuperscript{17}

Figure 1 illustrates the data obtained during one run, together with the simultaneously recorded pressure inside the $^3$He cell. It is seen that, as the cooling proceeds after the melting curve has been reached at $X$, the signal amplitude decreases continuously and the resonant frequency is reduced, roughly in accordance with the expected $1/T^2$ temperature dependence for $\eta$. Immediately after the A transition this behavior is inverted. The amplitude starts to increase and the resonant frequency shifts upwards; this shows that damping of the viscometer string is decreasing. Exactly simultaneously with the pressure drop at B the vibration amplitude jumps discontinuously. After this has happened the amplitude increases very quickly and then remains constant as the temperature no longer decreases. Finally, at point Z the signal suddenly, in about 0.1 sec, completely disappears, presumably because solid $^3$He starts forming on the wire.

So far we have made a dozen runs of the type illustrated in Fig. 1. The specific run shown was selected as an example because both the A and B transitions happen to occur near a resonance. In the other runs the overall behavior was very similar. During some experiments the viscometer string stuck soon after the A point was passed and in a few early experiments solid $^3$He started to form inside the viscometer chamber before the A point had been reached.

Once the viscometer had stuck because of solid formation, the signal could be recovered by reducing the pressure; this apparently melted a
sufficient amount of solid in the upper chamber. When a second compressional cooling run was then made it was invariably found, however, that successful measurements of η through even the A transition could be performed only if the cell pressure had first been reduced below the melting curve and approximately 10 h had been spent on precooling. The importance of a low starting temperature, in the vicinity of 15 mK, was clear from our latest runs. They showed that measurements of η could be carried out up to the highest attainable pressure if during the last 4 h of precooling and during the compression itself the dilution refrigerator was operated in the single-cycle mode.

With a magnetic field on in the lower chamber the A transition was characteristically rounded and often seemed to occur with two discontinuities in the slope of the 3He pressure-versus-time curve. This behavior is closely similar to that found by Osheroff, Guilly, Richardson, and Lee who recorded a splitting of the A transition in a magnetic field. Somewhat surprisingly, however, we noticed that the point at which the viscometer eventually became stuck could not be shifted in a reproducible manner by varying the field in the lower chamber. The main effect caused by an increase of the field was only a reduction in the cooling capacity and in the highest attainable pressure. As a rule, point B, with a sudden pressure drop, could not be reached unless the field was less than 0.9 T. The magnitude of the pressure drop ΔP_B was found to be about 0.6 mbar, and apparently was not dependent on the magnetic field. It is noteworthy that this is in disagreement with the data of Osheroff and co-workers and of Halperin, Bhurman, Webb, and Richardson who found ΔP_B = 0.3 mbar and ΔP_B = 0.1 mbar, respectively.

Results of several runs, with a 0.1-T field on the viscometer and with 0 and 0.5-T fields in the lower chamber, are shown in Fig. 2. As the melting pressure versus temperature relationship is inaccurately known at these temperatures, we present our data as functions of the relative 3He pressure, referred to the A point. An analysis of the results shows, by assuming that the A transition always occurs at 2.6 mK and by em-
FIG. 2. Normalized viscometer signal amplitudes from six runs of the $^3$He melting pressure, with the zero set at the $A$ transition. Note that the pressure scale is different on both sides of $A$ and that the temperature decreases towards the right. To a reasonable approximation the signal amplitude should be proportional to $(1/T)^{1/2}$; the straight line to the left of $A$ shows the expected behavior if $\eta T^2 = \text{const}$. The double hysteresis loop, corresponding to compression through the $A$ and $B$ transitions and decompression through $B'$ and $A'$, is marked with bent arrows. The pressure interval at which the $B$ transition occurs during the first compression of a run is also shown. Open circles, 0.6-T field in the lower chamber; closed circles, zero field in the lower chamber.

ploying the temperature derivative of the melting pressure as calculated by Goldstein, that the data between 2.6 and 6 mK can be only roughly fitted with a $1/T^2$-dependent viscosity. All compressions were performed relatively fast compared with the thermal time constant $\tau (\propto T^2)$ of the liquid in the cell. Consequently, the rather large deviations from the $1/T^2$ behavior at the highest temperatures could be caused by the slow thermal relaxation, combined with inaccuracies of our temperature scale.

After point $A$ (cf. Fig. 2) the signal increases with pressure. When point $B'$, about 20 mbar above $A$, has been passed, the signal continues to increase relatively slowly until at point $B$ the vibration amplitude jumps discontinuously and then increases rapidly to a value at which the signal somewhat later suddenly disappears. The location of point $B$ varied considerably for different runs. The largest measured signal corresponds to a viscosity which is 1000 times smaller than that at point $A$.

During our latest runs we stopped compression soon after point $B$ was passed, then reduced the pressure slowly, after which a new pressurization was started. By this procedure it was possible to traverse the hysteresis loop $B-B'-A-B$ (cf. Fig. 2) repeatedly before the viscometer string eventually stuck. If the decompression was not carried much below $B'$, then the pressure at which the $B$ transition occurred was each time closer to $B'$ than before whereby the hysteresis $B-B'-A$-B finally almost disappeared. The drop of $\eta$ at $B$ simultaneously decreased. Within the limits of our resolution it seems that $\eta$ changes rather steeply but continuously at $B'$. A surprising feature was that during decompression, between $B'$ and $A$, the measured values of $\eta$ fell consistently above the values recorded during the preceding (or subsequent) compression, and did not coincide with the upper curve (cf. Fig. 2) until the pressure was close to point $A$. We conclude that some kind of slow thermal relaxation within the liquid, and not simple supercooling, is responsible for the hysteresis $B'-A-B'$.

With our current knowledge of the physical properties of the newly found phases in liquid $^3$He, it is not possible to attempt a detailed analysis of the results below 2.6 mK. Therefore, we adopt the simple two-fluid model of superfluidity which suggests the following conclusions: At point $A$ the superfluid fraction of liquid $^3$He starts to grow from zero as the temperature is lowered. Liquid states from $B'$ to $B$ correspond to supercooling as suggested by Osheroff, Gully, Richardson, and Lee. At $B$, condensation into an energetically more favorable phase takes place and the superfluid density changes discontinuously. Finally, in the phase below point $B$ the normal fluid disappears rapidly as the temperature is further lowered.

Although the viscosity of liquid $^3$He changes profoundly at $B$ it is not clear whether the associated pressure anomaly could occur in the liquid alone or whether it is brought about by ordering in the solid and/or by solid-liquid interactions.

The NMR resonance shift found by Osheroff et
al. and the discontinuity in the specific heat of liquid $^3$He, together with several theoretical predictions, support the existence of an anisotropic superfluid $^3$He below 2.6 mK due to a pairing type of condensation. Our viscosity measurements seem to give the most direct evidence for this hypothesis. The observed continuous viscosity with a discontinuity in $d\eta/dT$ at A and a rapid decrease of $\eta$ below B seem to fit the suggestion by Anderson that between A and B the liquid is a gapless superfluid (Anderson-Morel type) and that the transition at B leads to a superfluid with an energy gap (Balian-Werthamer type).

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EXPERIMENTAL DETERMINATION OF THE VISCOSITY AND DENSITY OF THE NORMAL COMPONENT OF SUPERFLUID $^3$He AT THE MELTING CURVE

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ABSTRACT

The viscosity and the density of the normal component of liquid $^3$He in the A- and B-phases have been determined at the melting curve with a vibrating wire viscometer. The results show that resistive flow of the liquid is accompanied by a flow of zero viscosity. The data thus prove superfluidity both in the A- and in the B-phase.
The first clear indication of superfluidity in liquid $^3$He was the drastic change in the damping of a vibrating wire viscometer at the A- and B-transitions. Using the same technique with improved resolution, we have now been able to establish quantitative values for $\eta_n$ and $\rho_n$, the viscosity and the density of the normal component of liquid $^3$He, as a function of temperature. Our results show that viscous flow in the A- and B-phases is accompanied by frictionless flow, i.e. the two phases of $^3$He behave as superfluids.

Several vibrating wire experiments\textsuperscript{2,3,4} in $^4$He and in normal $^3$He have successfully been interpreted in terms of Stokes'\textsuperscript{5} theory of an infinite wire of circular cross section oscillating in an unbound fluid. He obtained

$$\ddot{F} = -\pi a^2 \rho \left[ k(m) \frac{d\dot{v}}{dt} + \omega k'(m) \dot{v} \right]$$

(1)

for the force per unit length of a wire vibrating with a small amplitude at an angular frequency $\omega$ (2\pi·1900 s\(^{-1}\) in our case). Here $a$ is the radius of the wire and $\rho$ is the density of the fluid; $\dot{v} = \Re \ddot{v} e^{i\omega t}$ is the velocity of the wire. The parameter $m$ is given by $m = a/2\delta$, where $\delta = (\eta/\rho \omega)^{1/2}$ is the characteristic length of decay for the velocity field around the wire, i.e. the viscous penetration depth. Functions $k(m)$ and $k'(m)$, as introduced and tabulated by Stokes, determine the hydrodynamic mass and the damping of the wire.

Stokes' model can be modified to describe a two-fluid system with an isotropic superfluid component.\textsuperscript{3} In this case we obtain from Eq. (1)

$$\ddot{F} = -\pi a^2 \left[ \rho_n k(m) + \rho_s \frac{d\dot{v}}{dt} + \omega \rho_n k'(m) \dot{v} \right]$$

(2)

where $\rho_s$ is the density of the superfluid component. The total density of the fluid $\rho$ is the sum of $\rho_n$ and $\rho_s$; now
we have \( \delta = (\eta_n / \rho_n \omega)^{1/2} \). Eq. (2) shows that the non-viscous component of the fluid influences the hydrodynamic mass of the wire only but cannot cause damping of the oscillations. We assume that this description is also valid in the case of an anisotropic superfluid, provided that \( \eta_n', \rho_n' \) and \( \rho_s' \) are replaced by the suitably averaged quantities \( \bar{\eta}_n', \bar{\rho}_n' \) and \( \bar{\rho}_s' \).

An approximate solution for the shift of the resonance line of a vibrating wire viscometer is then given by

\[
\Delta \omega_R = \frac{\omega_0 \bar{\rho}_n k(m) + \bar{\rho}_s}{2 \rho_V},
\]

(3)

where \( \omega_0 \) is the resonant frequency in vacuum and \( \rho_V \) is the density of the viscometer wire (N\( \text{b}\)\( \text{zr} \), \( \rho_V = 7.79 \, \text{g/cm}^3 \)). In the same approximation the width of the resonance at half maximum becomes

\[
\Delta \omega_W = \omega_0 \frac{\bar{\rho}_n k'(m)}{\rho_V}.
\]

(4)

After combining Eqs. (3) and (4) we find

\[
\frac{k-1}{k'} = \frac{2 \Delta \omega_R - \frac{\rho}{\rho_V} \omega_0}{\Delta \omega_W}.
\]

(5)

If both \( \Delta \omega_R \) and \( \Delta \omega_W \) are measured, Eq. (5) can be used to obtain \( m \) and, hence \( \bar{\eta}_n' / \bar{\rho}_n' \); Eq. (4) may then be employed to solve for \( \bar{\eta}_n' \) and \( \bar{\rho}_n' \) separately. We emphasize that Eqs. (3),
(4), and (5) are not exact and were not used in our final calculations; iterative processes were actually employed.

If the oscillation amplitude $r_0$ of the wire is of the same order of magnitude as the radius $a$ of the wire, Stokes' solution may fail due to the omission of quadratic term in the Navier-Stokes equation. In our experiment, however, $r_0$ is always several orders of magnitude smaller than $a$. In this case Fisk has shown, by using the relevant Reynolds and Strouhal numbers, that Stokes' solution remains valid as long as the inequality $r_0/L < 1$ is fulfilled. Here $L$ is a characteristic length, either the radius $a$ (0.15 mm in our case) of the wire or the viscous penetration depth $\delta$, whichever is smaller. At our lowest experimental temperatures, below 1.3 mK, $r_0/a \lesssim 10^{-4}$ and $r_0/\delta \lesssim 10^{-5}$; Fisk's condition is thus amply fulfilled. Further, the correction to Stokes' solution due to the finite size of the $^3$He sample is estimated to be small in the experiment; the radial distance of the wire to the cell wall is 2 mm in our case.

We refrigerated the $^3$He specimen by means of Pomeranchuk's method. After $^3$He had been brought to a temperature well below the B-point, pressurization was stopped and the sample was allowed to warm through the B' and A-transitions under the influence of the stray heat leak. Warming-up rates of about $10 \mu$K s$^{-1}$ were typical. Sometimes the pressure was slowly released during warming. The cycle was then repeated. Resonance ources were measured while traversing the A and B-phases and the normal Fermi-liquid region in both directions; $\Delta \omega_R$ and $\Delta \omega_W$ were then determined from the data. The measuring technique required that the sample be exposed to a magnetic field of 0.149 T during the experiment; other experimentalists as well as ourselves have shown that a field of this magnitude does not alter the conclusions derived from these results.

The values of $\Delta \omega_R$ and $\Delta \omega_W$ from our measurements in the normal Fermi-liquid region were first analyzed to find the
viscosity of $^3$He from about 15 mK to the A-transition. The data are consistent with the $T^{-2}$-law expected for a normal Fermi-liquid.\textsuperscript{3,4,11} We obtain for the viscosity at the A-point $\eta_A = 0.127$ poise.\textsuperscript{10}

![Diagram]

**Fig. 1.** The width of the resonance curve at half maximum as a function of the relative resonance shift between the resonance curves of equal damping in the normal Fermi-liquid region and the superfluid phases of $^3$He. Several lines of constant $\tilde{\eta}_n$ and $\tilde{\rho}_n$ have also been plotted in the figure. Open symbols represent points taken upon cooling and filled ones during warming; circles are in the A-phase and triangles in the B-phase.
Next, the measured resonance shifts, both in the normal $^3$He and in the A- and B-phases, were plotted against the corresponding resonance widths. The difference in $\Delta \omega_R$ between the normal Fermi-liquid phase and the superfluid phases at equal damping, $\delta \Delta \omega_R$, was then used to determine $\bar{\eta}_n$ and $\bar{\rho}_n$ as described above; the maximum difference, observed at $\sim 1.2$ mK, is about 2.5 Hz. The data were also found self-consistent, within the framework of Stokes' model, through an examination of the relation connecting the amplitude and the width of the resonance curves.

In Fig. 1 we have plotted $\Delta \omega_W$ as a function of $\delta \Delta \omega_R$. The data were measured during a run, in which the A- and B-phases were traversed three times. Lines of constant $\bar{\eta}_n$ and $\bar{\rho}_n$ are also shown in the figure. Fig. 2 illustrates the results on $\bar{\eta}_n/\eta_A$ and Fig. 3 our data on $\bar{\rho}_n/\rho$.

We observe from Fig. 2 that, as the sample cools through the A-transition, $\bar{\eta}_n$ decreases rapidly, within 0.3 mK, to approximately 25% of $\eta_A$ and then becomes essentially constant. The decrease is much faster than $(T/T_A)^8$, estimated by Greytak et al. for the B-phase from heat flow measurements at lower pressures. At the B-transition, below 1.7 mK, $\bar{\eta}_n$ drops discontinuously to about 20% of $\eta_A$ but starts to increase again as the temperature is reduced. Upon warming back to the A-phase $\bar{\eta}_n$ returns to 25% of $\eta_A$ via the lower set of points which thus seem to characterize the B-phase. We note in passing that the viscosity of both $^3$He and $^4$He behaves in the superfluid region in a rather similar way.

A calculation of viscosity for an isotropic Fermi-liquid by Shumeiko is in fair agreement with our results. His theory gives $\eta = \text{constant} \times (\eta_C)$ for $T < T_C$ and $\eta = \eta_C (1 - \alpha \Delta / T)$ in the vicinity of $T_C$. Here $\alpha$ is a constant and $\Delta$ is the energy gap; in our comparison we assumed $T_C = T_A$ and
Fig. 2. The reduced average viscosity $\frac{\eta_n}{\eta_A}$ of the normal component of superfluid $^3$He at the melting curve as a function of temperature; $\eta_A$ is the viscosity of normal $^3$He at the A-point. The tentative temperature scale was plotted by assuming $T_A = 2.6 \text{ mK}^{10,12}$ and $dP/dT = -35 \text{ bar/K}$ on the melting curve at $T < T_A$. The pressure difference relative to the A-transition is also indicated. For an explanation of the data symbols we refer to Fig. 1. The curves and arrows have been drawn as visual aids only. The error bars reflect the estimated uncertainty of $\pm 0.15 \text{ Hz}$ in the determination of the shifts of the resonance curves.
\( \eta_c = \eta_A \). Seiden's calculation\(^{14}\) for \( T < \theta_c \) predicts \( \eta \) to be a weak function of temperature. Quantitative comparisons between these theories and our measurements are, however, difficult to make.

Fig. 3 shows that in the A-phase \( \tilde{\rho}_n \) decreases relatively slowly with temperature, approximately as \( (T/T_A) \). This is in fair agreement with the fourth sound measurements of Kojima et al.\(^{15}\) However, at the B-transition \( \tilde{\rho}_n \) drops abruptly from

![Graph](image)

Fig. 3. The reduced average density \( \tilde{\rho}_n/\rho_A \) of the normal component of superfluid \(^3\)He at the melting curve as a function of temperature; \( \rho_A \) is the total density of \(^3\)He at the A-point. The slight variation of \( \rho \), as a function pressure, was taken into account in the normal Fermi-liquid region by extrapolating the data of Grilly.\(^{21}\) For an explanation of the data symbols we refer to Fig. 1.
about 60 % to 25 % of \( \rho_A \) and then decreases rapidly upon
further cooling; at our lowest experimental temperatures,
below 1.3 mK, \( \rho_n \) is < 1 % of \( \rho_A \). The jump in \( \rho_n \), if any,
in warming through the B'-transition is small.

The weak coupling theories predict that \( \rho_n \) decreases
most rapidly for the isotropic states. In this case \( \rho_n/\rho \)
can be described by the Yoshida function\(^{16}\) which gives the
same value for \( \rho_n \) at \( T/T_C \approx 0.3 \) as we find experimentally
at 1.3 mK. However, our experimental temperature corresponds
to \( T/T_C \geq 0.5 \), assuming for \( T_C \) a maximum temperature of the
A-transition, 2.6 mK. The agreement between theory and exper-
iment becomes even poorer if the Fermi corrections, through
Leggett's formula,\(^{17}\) are introduced.

If it is possible that the wire has an orientating effect
on the liquid, which tends to rotate the orbital angular
momentum perpendicular to the wire's surface, a proper
angular average could conceivably emphasize the directions
where the gap is largest. However, it is not probable that
this effect could be so large as to explain the discrepancy,
because e.g. in the Anderson-Morel state \( \rho_n/\rho \), even in the
most favorable direction,\(^{18}\) does not decrease more rapidly
than in the isotropic states.

A more likely explanation follows from the experimental
observation\(^{19}\) that the specific heat below the A-transition
is larger than the weak coupling theory would predict. If
this is true also in the B-phase, the gap decreases actually
faster than predicted by conventional BCS-type calculations
and consequently also \( \rho_n \) vanishes more rapidly.

The rapid variation of \( \rho_n \) as a function of temperature
below the B-transition seems to be in disagreement with the
data of Kojima et al.\(^{15}\); they observed no appreciable change
in the temperature dependence of \( \rho_n \) near B'. The experimental
discrepancies may indicate that the bulk properties of the
\(^3\)He superfluids, particularly those of the B-phase, are
changed when the liquid is contained inside a porous medium necessary for the observation of fourth sound; this could easily lead to spurious conclusions. Major changes in the bulk properties of the superfluid phases have been observed in NMR-measurements\textsuperscript{20} when $^3$He was intermixed with fine platinum powder.

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THE VISCOSITY AND DERIVED PROPERTIES OF LIQUID $^3$He BETWEEN 100 AND 1 mK ALONG THE MELTING CURVE

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ABSTRACT

The viscosity $\eta$ of liquid $^3$He, cooled by Pomeranchuk's method, has been measured along the melting curve over the temperature range from 100 to 1 mK with a vibrating wire viscometer. In the normal Fermi-liquid region the viscosity is given by $\eta = 1/T^2(1.17 - 3.1OT) \mu P$ for $T$ in Kelvin. The superfluid A-phase at the transition temperature $T_A = 2.6$ mK is characterized by a rapid decrease in the viscosity $\eta_n$ of the normal fluid component; $d\eta_n/dT = \infty$ at $T_A$. Within about 0.3 mK below $T_A$ $\eta_n$ decreases with temperature to about 25% of $\eta_n$ at $T_A$, but then becomes essentially constant. In the $B$-phase $\eta_n$ starts to increase again below 1.4 mK. Data on the behavior of the density $\rho_n$ of the normal fluid component in A- and B-superfluid phases are also presented.

The splitting of the A-transition as a function of external magnetic field $B$ has been determined up 0.9 T. The data yield $P_{A2}(B) - P_{A1}(B) = 2.10(B/T) \text{ mbar}$. Measurements of the pressure difference between the $B'$-transition and the $A_1$-transition as a function of $B$ up to 0.36 T give $P_{B'}(B) - P_{A1}(B) = [20.4 + 171(B/T)^2] \text{ mbar}$. 
1. INTRODUCTION

The main result presented in this paper is the determination of the viscosity and the normal density of liquid $^3\text{He}$ as a function of temperature in the A- and B-phases$^1$ at the melting pressure. We have, in addition, measured the viscosity between 100 and 2.6 mK, and by employing these data, made an attempt to determine an improved P,T-relationship at the melting curve below 10 mK. Further, we have studied the splitting of the A-transition in a magnetic field by means of changes in the effective viscosity of $^3\text{He}$ at the $A_1$- and $A_2$-transitions. Measurements of the suppression of the $B'$-transition in a magnetic field have also been carried out.

In our earlier experiments$^2$ with a vibrating wire we were able to show that the effective viscosity, i.e. damping of the oscillations, was greatly lowered in the new phases; reduction factors of about $10^3$ in the apparent viscosity, as compared with the expected behavior of a normal Fermi-liquid, were observed at the lowest temperature of approximately 1.0 mK. This was regarded as clear evidence for superfluidity in the A- and B-phases of liquid $^3\text{He}$.

In order to prove superfluidity in the new phases it is necessary to show that the results could be explained only if viscous flow of liquid $^3\text{He}$ is accompanied by a flow of zero viscosity. Adopting the concepts of the two-fluid model, this is equivalent to establishing separately the values of $\eta_n$ and $\rho_n$, the viscosity and the density of the normal component, respectively, in the A- and B-phases as a function of temperature.

In principle, the vibrating wire technique makes it possible to determine both $\eta_n$ and $\rho_n$, if the measured data can be analyzed to yield, in addition to the resonant amplitudes or the widths of the resonance curves at half maximum,
the shifts of the resonance curves. If amplitudes (and widths) only are measured the product \( \eta_n \rho_n \) is obtained. Owing to noise problems in our earlier measurements, obscuring the zero level of the resonance curves and thus preventing us from determining the amplitudes or widths of the curves with sufficient accuracy, the separation of \( \eta_n \) and \( \rho_n \) could not be carried out. However, in the measurements reported here, which have highly improved resolution, the determination of the two quantities has been accomplished and the data thus prove superfluidity in the A- and B-phases.

The measurements of the viscosity of \( ^3\text{He} \) in the normal Fermi-liquid region in the temperature range from 100 to 2.6 mK were performed for three reasons. First, our data reduction scheme in the A- and B-phases depends in an important way on the viscosity of normal \( ^3\text{He} \) between 16 and 2.6 mK. Second, these measurements were regarded as a useful preliminary study of our experimental technique and could also be used to determine the melting curve below 10 mK, and hence the temperature of the \( \lambda \)-transition. And finally, there are only two earlier measurements of this type. Black, Hall and Thompson\(^3\) have measured the viscosity of liquid \( ^3\text{He} \) at saturated vapor pressure between 3.0 and 0.05 K with a vibrating wire viscometer made of a superconducting loop with the magnetic field parallel to its plane. Lawson, Gully, Goldstein, Reppy, Lee and Richardson\(^4\) used a similar technique to determine the viscosity of liquid \( ^3\text{He} \) between 110 and 30 mK along the melting curve; their measurement could not be extended to lower temperatures, because solid \( ^3\text{He} \) prevented the motion of the viscometer wire at early stages of the compressional cooling runs.

The splitting of the \( \lambda \)-transition as a function of the external magnetic field has previously been studied by Osheroff\(^5\) and by Gully, Osheroff, Lawson, Richardson and Lee.\(^6\) They observed the appearance of two separate kinks, \( A_1 \) and
A\textsubscript{2}, in the P = P(t) curve of \textsuperscript{3}He in a compressional cooling cell. Later measurements of the splitting were performed through the detection of ultrasonic attenuation peaks at the transitions by Lawson, Gully, Goldstein, Richardson and Lee.\textsuperscript{7}

In our experiments both transitions were clearly observed as a sudden change in the slope of the curve of the viscometer amplitude vs. pressure at A\textsubscript{1} and A\textsubscript{2}. For these measurements an electronic feedback circuit was designed; this allowed us to observe the viscosity continuously at resonance even when η varied rapidly as a function of temperature. This device was useful also for determining the pressure difference between the B\textsuperscript{'}- and A\textsubscript{1}-transitions as a function of magnetic field, although our results in this respect are based mainly on the corresponding P(t)-data.

In Section 2 we describe our apparatus, the emphasis being on the viscometer and associated electronics. We next discuss the theory of a vibrating wire viscometer and the applicability of Stokes’ model to our measurements. Our data reduction scheme is explained in Section 4. We then report our results in the normal Fermi-liquid region. Our data in the A- and B-phases are presented and discussed in the last Section.

2. APPARATUS AND EXPERIMENTAL PROCEDURE

Our \textsuperscript{3}He specimen was refrigerated inside a Pomeranchuk cell which was precooled by means of a dilution refrigerator to about 16 mK.\textsuperscript{8} The \textsuperscript{3}He cell, having a volume of 2 cm\textsuperscript{3}, is a modification of the one used in our earlier viscosity measurements.

It was originally assumed that in a Pomeranchuk cell consisting of two interconnected chambers the place where
solid forms could be controlled by applying a strong magnetic field on one of the chambers only. Solid should start to grow where the magnetic field is strongest and thus the viscometer, located in the upper low field chamber, could be operated long enough to obtain meaningful results.

We observed, however, that this arrangement was not necessary, because measurements could be carried out satisfactorily independent of whether the lower chamber was exposed to a magnetic field or not. It was concluded that a proper thermal distribution is more important than the inhomogeneity of the applied field in determining the location of solid formation in the design employed.

On the other hand, the viscometer signal amplitude showed curious hysteresis phenomena in the A-phase. This was believed to arise from thermal relaxation effects within the $^3$He sample and from the inhomogeneity of the magnetic field employed to operate the viscometer. For the new measurements we removed the lower chamber and made the magnetic field homogenous throughout the cell. The experimental arrangement is illustrated in Fig. 1.

Another important change in our apparatus for the present work was the introduction of phase sensitive detection in the measurement of the viscometer resonance curves. This improved the signal to noise ratio by a factor of 50 and allowed us to determine the zero levels of the signals with better accuracy. Previously we had used a simple AC to DC converter for the detection of the resonance curves.

Our viscometer is of the type first employed by Tough, McCormick and Dash in measuring the viscosity of liquid $^4$He below the $\lambda$-transition. They used a straight tungsten wire as the vibrating sensor. When immersed in $^4$He and exposed to a transverse magnetic field the wire was momentarily displaced from its equilibrium position by application of
Fig. 1. The Pomeranchuk cell and the lower part of the dilution refrigerator.

A short current pulse through the wire. The viscosity of the fluid was then deduced from the decay times of the oscillations of the wire which were detected by measuring the Faraday voltage across the wire’s terminals.

We employed a straight superconducting NbZr wire. Oscillations were excited by passing alternating current through
the wire; the frequency of the current was swept semicontinuously through the resonance. The in-phase component of the induced voltage was then measured with a phase sensitive detector and recorded. The viscosity was obtained from the widths of the resonance curves at half maximum. We also determined the shifts of the resonance lines in order to perform self-consistency tests on the data as a whole and especially for the determination of $\eta_n$ and $\rho_n$ in the A- and B-phases.

Our vibrating wire is 25 mm long and it has a radius of 0.15 mm. Its effective density is 7.79 g/cm$^3$. The Formvar insulation was removed, because the thin copper layer ($\approx 0.03$ mm) around the superconducting core had to be exposed for soldering the ends of the wire with Wood's metal to a supporting brass piece. A low-melting-point solder was employed in order not to spoil the superconducting and elastic properties of the wire through excessive heating.

A tension of 30 N, as measured at room temperature, gives the system a resonant frequency of 1900 Hz and a Q-value of about 5500 in vacuum at 4 K. As a precaution against misalignment between a possible easy direction of bending and the plane of forced oscillations, which would cause degeneration of the vibrational plane of the viscometer, the wire was twisted through about 180° along its axis before soldering.

The viscometer system is shown in Fig. 2. The unit is inserted into the Pomeranchuk cell and sealed with an indium O-ring. The distance of the center of the wire to the supporting piece in the horizontal vibrational plane is 2 mm. The nearest obstacles in the vertical direction are 1.5 mm below and 2.5 mm above.

The superconducting leads for measuring the induced voltage, and the current carrying leads, were soft soldered to the viscometer wire 2 cm and about 10 cm outside the cell,
Fig. 2. The viscometer unit. The vibrating wire is of copper-coated superconducting NbZr wire with an effective density of 7.79 g/cm$^3$; the radius of the wire is 0.151 mm. A tension of 30 N, as measured at room temperature, gives the system a resonant frequency of 1900 Hz and a Q-value of about 5500 in vacuum at 4 K. The unit is inserted into the Pomeranchuk cell and sealed with an indium O-ring. The fiber glass piece is used in order to eliminate short circuits across the supporting brass fork. For the same reason one lead-through of the wire is prepared by using a 50-50% Araldite and chalk powder mixture as a seal. The magnetic field is perpendicular to the plane of the figure.
respectively. The junctions were made 1 cm long and the copper layer on the superconducting core of the wire was removed between the current and voltage junctions as a precaution against possible leakage of the excitation signal into the viscometer signal.

The strength of the applied magnetic field was varied between 0.1 and 0.9 T according to experimental requirements. The measurement of the viscosity and the density of the normal component in the A- and B-phases was performed in a field of 0.149 T using an rms excitation current of 0.4 mA. The same field and current were also employed in the experiments between 100 and 2.6 mK. In this way the applicability of Stokes' solution to our viscometer was guaranteed over the whole temperature range, while still sustaining a sufficiently good signal to noise ratio. However, in order to investigate the effect of the magnitude of the vibration amplitude and velocity of the wire to the reproducibility of the data, the current was varied by a factor of ten and no abnormal changes in the behavior of the viscometer were observed. For the same reason the field was also changed while the current was kept constant; the viscometer was again found to function consistently.

For unknown reasons the measurements of \( \eta_n \) and \( \rho_n \) became progressively more difficult when the field was lowered to 0.1 T and below. The same difficulties were also met with rather high fields (\( \gtrsim 0.25 \) T). In these cases formation of solid \(^3\)He in the middle and/or at the ends of the vibrating wire spoiled the measurements either before or soon after the A-transition was reached. The formation of solid on the wire was noticed from the behavior of the resonance frequency and the vibration amplitude; solid in the middle of the wire lowers the resonant frequency while solid at the ends increases it; the amplitude also decreases more rapidly than expected by the changing viscosity alone.
In our measurements of the splitting of the A-transition the resonant frequency of the wire increased sometimes by 2000 - 3000 Hz before the signal disappeared completely. However, because it was only necessary to observe sudden changes in the amplitude at the transitions, formation of solid on the wire during a run was not a serious drawback in these experiments.

While measuring the resonance curves in the A- and B- phases for determining $\eta_n$ and $\rho_n$ we tried very carefully to eliminate all effects caused by solid $^3$He. Resonance curves were recorded both when the sample cooled and when it warmed through the A- and B'-transitions. The warming was allowed to take place either under the influence of the stray heat leak at constant volume, with the amount of solid increasing in the cell, or after a new pressurization, by slowly releasing the pressure and thus by partly melting the solid. Solid on the wire should then manifest itself as an irreproducibility of the amplitude at a given pressure or, which is by far a more sensitive test, as an irreproducibility in the shifts of the resonance curves. We discarded the $\eta_n$ and $\rho_n$ data in which any anomalous behavior of the viscometer was observed. On the whole the presence of solid turned out to be a more serious problem with the modified Pomeranchuk cell than with our earlier construction.

In the measurements above 16 mK the formation of solid on the wire was used for determining the moment at which the sample either arrived or departed from the melting curve. This information, together with simultaneous recordings of the $P(t)$-curve, was employed in fixing the relevant temperatures.

Below 45 mK our temperature scale was established by shifting and extrapolating the melting curve data of Johnson, Lounasmaa, Rosenbaum, Symko and Wheatley in such a way that the absolute pressure which we measured for the A-transition
corresponded to a temperature of 2.6 mK. At higher temperatures we used the melting curve calculated by Goldstein\textsuperscript{11} and reproduced experimentally by Scribner, Panczyk and Adams.\textsuperscript{12} The latter data were shifted to join smoothly to the lower temperature melting curve at 45 mK.

A schematic illustration of the electronics employed in our measurements of the resonance curves, is shown in Fig. 3.

\begin{center}
\includegraphics[width=\textwidth]{diagram.png}
\end{center}

\textbf{Fig. 3.} The electronic measuring system. The vibrating wire viscometer is illustrated as a voltage source.

The driving signal $V_D$ and the reference signal $V_R$ for the phase-sensitive detector (Ithaco Dynatrac 391A) were taken from a voltage-controlled oscillator (Wavetek 146) which sweeps the frequency repeatedly through the resonance of the wire. The viscometer current loop is galvanically isolated from the feed circuit by a pot-core transformer $M_1$. Another transformer $M_2$ is used at the potential leads, mainly for impedance matching to the preamplifier (Princeton Applied Research 113).

In the continuous resonance measurements we used a feedback circuit, which was especially built for this purpose. A phase sensitive detector produced a signal proportional
to the out-of-phase component of the induced voltage. This signal was fed back to the driving circuit, which is a voltage controlled oscillator, so as to readjust the frequency of the input signal to the resonant frequency. Another phasesensitive detector was then employed to indicate the resonance amplitude.

3. HYDRODYNAMICS OF THE VISCOMETER

Viscosity measurements employing vibrating wire techniques are interpreted in terms of Stokes' theory of an infinite cylinder oscillating with a small amplitude in an unbound fluid. The question of the applicability of Stokes' solution to measurements which cover a wide range of viscosity values is, however, far from trivial and a few details on this subject are appropriate here.

Let us consider the velocity field $\mathbf{v}_n(x,y,z)$ which is created around a wire whose axis is parallel to the $z$-direction as it vibrates with a velocity $\mathbf{v} = \mathbf{v}_0 e^{i\omega t}$ in liquid helium; $\omega$ is the angular frequency of the oscillation. If the displacement amplitude $r_0$ of the wire, assumed to be in the plane $\theta = 0$, is small compared with the length $l$ of the wire and, hence, the velocity gradients are small in the $z$-direction, we may treat the flow problem as two-dimensional. Then $\mathbf{v}_n = \mathbf{v}_n(x,y)$ and Stokes' theory for a straight, infinite cylinder is applicable. Over the whole range of our measurements $r_0/l$ is of order $10^{-5}$ so that the assumption of two-dimensionality of the velocity field is well justified.

It can be shown (cf. APPENDIX I) that the vorticity, or the curl of the velocity field $\mathbf{v}_n$ obeys the equation

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{v}_n) = - \nabla \times (\nabla \mathbf{v}_n) + \frac{n}{\rho} \nabla^2 (\nabla \times \mathbf{v}_n).$$  (1)
The left hand side of this equation represents the total rate of change of vorticity in a differential volume element fixed in space. The first term on the right gives the rate at which vorticity enters or leaves the volume element by convection; this term causes Eq. (1) to be nonlinear in $v_n$.

The second term on the right hand side is the rate of change of vorticity in the volume element due to diffusion, which takes place via viscous interactions in the non-uniform velocity field. Stokes' solution is based on the assumption that the convective term in Eq. (1) can be neglected. This is the case if the rate of change of vorticity due to diffusion is large compared to its rate of change caused by convection.

Let us assume that vorticity moves outward from the surface of the wire and, as the situation develops in time and space, a characteristic length can be assigned which describes the radial decay of the non-uniformity of the velocity field. Since vorticity originates with motion of the wire, the total rate of change of vorticity must be of order $\omega$ or larger. The diffusion term is of order $\eta_n/\rho_n \delta^2$ and the convective term order $u_o/\delta$ or smaller. If diffusion predominates over convection we have $\eta_n/\rho_n \delta^2 >> u_o/\delta$ which implies $\eta_n/\rho_n \delta^2 \approx \omega$. The latter equation gives for the viscous penetration depth $\delta = (\eta_n/\rho_n \omega)^{1/2}$. Further, by writing $u_o = \omega r_o$, the inequality requires

$$r_o/\delta << 1 \ .$$

We thus see that the linearization of Eq. (1) is justified if the amplitude of oscillations is small compared to the viscous penetration depth. A more rigorous investigation by Fisk$^{14}$ of the omission of the quadratic term shows that
Stokes' solution is valid as long as, in addition to the condition of Eq. (2), we have simultaneously

\[ \frac{r_o}{a} \ll 1 \, , \quad (3) \]

where \( a \) is the radius of the wire. Thus the amplitude of the oscillation must be much smaller than \( a \).

In our measurements \( r_o \) was always less than 0.1 \( \mu m \) and \( \delta \) varied from about 5 to 100 \( \mu m \) in the normal Fermi-liquid region. In the A- and B-phases \( \delta \) first decreased to about 50 \( \mu m \) below the A-transition and then increased to about 1500 \( \mu m \) at our lowest temperatures below the B-transition. Since \( a = 0.15 \, mm \) in our case, the two requirements are amply fulfilled.

Corrections to Stokes' solution caused by the finite size of the specimen have been discussed by Segel\textsuperscript{15} in an investigation of fluid flow between two circular cylinders when the outer cylinder is fixed and the inner one vibrates with small amplitude about its axis. He shows that corrections become important when the viscous penetration depth is large compared with the radius of the outer cylinder. In our case the most critical situation arises at the lowest temperatures in the B-phase. However, since \( \delta \), as evaluated from our data, remains smaller than 1.5 \( mm \), the finite size of the sample should not affect our results.

A possible source of nonlinear effects is the so-called fluid separation or surface slip. This occurs when collisions between \(^3\)He quasiparticles and the wire are no longer totally diffuse and a specularly reflected component appears. However, if the mean free path of quasiparticles is small compared with the viscous penetration depth, the specular component will be brought into momentum equilibrium with the diffuse
component. The problem thus reduces essentially to a semi-microscopic examination of the validity of the hydrodynamic equations in terms of Fermi-liquid theory.

A comparison can be made between the mean free path of viscous excitations and the viscous penetration depth $\delta$ defined by the hydrodynamic equations. This requires $v_F \tau_\eta \ll \left( \frac{\eta_n}{\rho_n} \omega \right)^{1/2}$, where $v_F$ is the Fermi velocity and $\tau_\eta$ is the viscous relaxation time. Using the relation $\eta_n = \frac{1}{5} \frac{m^*}{m} \tau_\eta v_F^2 \rho_n$ from Fermi-liquid theory, with $m^*/m \approx 6$, the condition may be written in the form $\omega \tau_\eta \ll 1$. Since in our case $\omega = 2\pi \cdot 1900 \text{ s}^{-1}$ and $\tau_\eta \tau_T^2 \approx 1$, the inequality is amply fulfilled at 2.6 mK and above this temperature. In the superfluid phases the above equations are not meaningful.

In deriving Eq. (1) it is assumed that the liquid is incompressible. This requires $\nabla \cdot \vec{V}_n = 0$ and considering the $A$- and $B$-phases, for which the two-fluid model and the correspondingly modified Stokes solution have been adopted in this work, we must also have $\nabla \cdot \vec{V}_S = 0$. It can be shown that incompressibility of the normal fluid requires $(u_0/c_1)^2 \ll 1$, where $c_1$ ($\approx 400 \text{ m/s}$ in our case) is the velocity of the first sound. The velocity of the wire never exceeded 5 mm/s in our measurements above 2.6 mK. In the superfluid phases, where the creation of second sound, and critical velocity effects, should be taken into account, the velocity was always less than 0.5 mm/s. It thus seems plausible that nonlinearities of this type do not occur with the magnitude of velocities involved in our experiments.

In view of the possible sources of errors discussed above we performed self-consistency tests on our data through an examination of the relation connecting the resonance amplitude $A_R$ and the width $\Delta \omega_W$ of the resonance curves (cf. Section 4); the measured $A_R$ values were plotted against the measured $\Delta \omega_W$ values and a comparison was made with the theoretically expected behavior. No signs of nonlinearities were observed within the scatter of the data.
Surface roughness and a non-circular cross section of the wire may cause errors in the viscosity calculated from Stokes' solution for an ideal circular cylinder. If the characteristic dimension of the surface roughness is small compared with the viscous penetration depth, as in our experiment, departures from ideal behavior should be negligible.

As discussed by Black et al., the error associated with the possible non-circularity of the cross section of the wire is of order \((\Delta a/a)^2\) for viscous penetration depths large compared to the radius of the wire, and of order \((\Delta a/a)\) for penetration depths small compared to the radius; here \(\Delta a\) represent the deviation of the radius of the wire in the direction of the oscillation from the average radius \(a\). This effect is estimated to be small in our case, especially because of a slight twisting of the wire.

4. DATA REDUCTION

By solving the linearized Eq. (1) for the velocity field \(\vec{v}_n\) of the moving liquid with the appropriate boundary conditions and by employing the Navier-Stokes equation to solve for the pressure in the fluid, Stokes obtained for the force per unit length of the oscillating cylinder

\[
\vec{F} = -\pi a^2 \rho_n \left[ k(m) \frac{d\vec{u}}{dt} + \omega k'(m) \vec{u} \right],
\]

(4)

where \(k(m)\) and \(k'(m)\) are expressed in terms of Hankel functions. The parameter \(m\) is given by \(m = a/2\delta\) (cf. APPENDIX 1).

Eq. (4) is then inserted to the equation of motion of the wire and, by solving for the induced voltage, one obtains
\[ V = A(V_1 + iV_2)e^{i\omega t}, \]

where

\[ V_1 = \frac{\rho_n k' \omega^3}{[\rho_V \omega_o^2 - (\rho_V + \rho_n k) \omega^2]^2 + \rho_n^2 k' \omega^4} \]

and

\[ V_2 = \frac{[\rho_V \omega_o^2 - (\rho_V + \rho_n k) \omega^2] \omega}{[\rho_V \omega_o^2 - (\rho_V + \rho_n k) \omega^2]^2 + \rho_n^2 k' \omega^4}. \] (5)

In Eq. (5) \( \omega_o \) is the resonant frequency in vacuum and \( \rho_V \) is the density of the viscometer wire. \( A \) is a constant which includes the driving current, the magnitude of the external magnetic field, and geometrical factors of the system.

The resonant frequency in the presence of the fluid is then

\[ \omega_R = \omega_o \sqrt{\frac{1}{(1 + \frac{\rho_n k}{\rho_V})}}; \]

the shift of the resonant frequency is thus given by

\[ \Delta \omega_R = \omega_o [1 - \frac{1}{\sqrt{(1 + \frac{\rho_n k}{\rho_V})}}]. \] (6)
The width $\Delta \omega_W$ of the in-phase component $V_1$ may be derived by equating $V_1$ to one half of $V_1$ at resonance:

$$V_1(\omega) = \frac{1}{2} V_1(\omega_R) = \frac{1}{2 \rho_n k' \omega_R}$$  \hspace{1cm} (7)$$

(in regard to our method of analysis — to measure the width at half maximum of the resonance curve — we should, in fact, equate $V_1(\omega)$ to the maximum value of $V_1 = V_{1\text{max}}$; however, the above analysis may be shown to be a very good approximation over the range of our measurements). The solutions $\omega_+$ and $\omega_-$ of Eq. (7) at given $\eta_n$ and $\rho_n$ have to be found by iteration; $\Delta \omega_W$ is then obtained by subtracting $\omega_-$ from $\omega_+$.

Our data reduction for the normal Fermi-liquid region was carried out by employing Eq. (7). First, $\rho_n$ for a measured resonance curve was obtained from the $\rho_n(T)$ data reported by Grilly; $T$ was fixed by the pressure at the moment when the resonance amplitude reached its maximum. Next, a proper set of $\eta_n$ values were chosen and a computer was used to calculate and store the corresponding $\Delta \omega_W$ values and the measured $\Delta \omega_W$ was compared with the calculated values. Then, around the $\eta_n$ value, which corresponded to a calculated $\Delta \omega_W$ value closest to the measured one, a new set of $\eta_n$ values was chosen and the corresponding $\Delta \omega_W$'s were again calculated. The iteration was continued until $\eta_n$ was obtained with desired accuracy. In these calculations the functions $k$ and $k'$ were evaluated by employing the series expansions given by Stokes. Fig. 4 shows a recording of a resonance curve.

For data reduction in the A- and B-phases we adopted the two-fluid modification of Stokes' solution. It can be shown (cf. APPENDIX I) that in this case the force per unit length of the wire is given by

$$\vec{F} = -\pi a^2 \left[ \rho_n k(m) + \rho_s \left( \frac{d\vec{u}}{dt} + \omega \rho_n k'(m) \vec{u} \right) \right]$$ \hspace{1cm} (8)$$
Fig. 4. A recording of a resonance curve at $T = 28$ mK at the melting curve. The step in the pressure recording corresponds to a temperature change of about 0.05 mK. The crosses represent a theoretical fit to the measured resonance curve employing Eq. (5) for $V_1$.

Comparing Eq. (8) with Eq. (4) we note that Eqs. (5), (6), and (7) are then modified simply by replacing $\rho_n k$ everywhere by $\rho_n k + \rho_s$; we have to remember also that in this case the bulk density $\rho$ of the fluid is given by the sum of $\rho_n$ and $\rho_s$. On the other hand, since we may be dealing with anisotropic superfluids, we must now make the assumption that our model can be applied to such systems provided that $\eta_n$, $\rho_n$, and $\rho_s$ are replaced with suitably averaged quantities $\bar{\eta}_n$, $\bar{\rho}_n$, and $\bar{\rho}_s$.

For clarity we give first a description of a method of data reduction in the A- and B-phases in which we use approximation to the two-fluid counterparts of Eq. (6) and (7); our actual data reduction scheme for the superfluid phases is described later.
For the shift of the resonance line we obtain in this case

\[ \Delta \omega_R = \frac{\omega_o}{2} \frac{\overline{\rho_n} k(m) + \overline{\rho}_s}{\overline{\rho}_V}. \]  \hspace{1cm} (9)

The width of the resonance line is given by

\[ \Delta \omega_W = \omega_o \frac{\overline{\rho_n} k'(m)}{\overline{\rho}_V}. \]  \hspace{1cm} (10)

The approximations are obtained from series expansions retaining the leading terms only. By combining these equations we thus obtain

\[ \frac{k(m) - 1}{k'(m)} = \frac{2 \Delta \omega_R - \frac{\overline{\rho}_n \omega_o}{\overline{\rho}_V}}{\Delta \omega_W}. \]  \hspace{1cm} (11)

Eq. (11) shows that if both \( \Delta \omega_R \) and \( \Delta \omega_W \) are measured for a superfluid, \( m \) may be determined and hence \( \overline{\eta}_n / \overline{\rho}_n \). These quantities may then be separated by using Eq. (10).

However, in order to utilize the information we had obtained from the viscosity measurements in the normal Fermi-liquid region, which showed that the data were self-consistent within Stokes' model and agreed with the Fermi-liquid theory, we determined \( \overline{\eta}_n \) and \( \overline{\rho}_n \) from intercomparison of the resonance curves above and below \( T_A \); in this way the effects of any non-idealities in the behavior of the viscometer could be minimized.

We observed that at equal damping or at equal amplitude the resonance curves in the A- and B-phases were less shifted...
from \( \omega_0 \) than in the normal Fermi-liquid. This relative shift
\( \delta \Delta \omega_R \) was used as a parameter in solving for \( \bar{n}_n \) and \( \bar{\rho}_n \).

If we write \( \Delta \omega_R^F \) and \( \Delta \omega_S^F \) for the measured Fermi-liquid
data and correspondingly \( \Delta \omega_R^S \) and \( \Delta \omega_S^S \) for the superfluid data,
our method of analysis is in principle as follows (cf. Figs. 5 and 6).

Fig. 5. An illustration of the measured quantities; \( \Delta \omega_W \) is
measured at half maximum and \( \omega_R \) as \( \omega \) at the maximum
of the resonance curve.

Fig. 6. A schematic drawing of the quantities employed in the
analysis of the data in the A- and B-phases.
At equal damping we have \( \Delta \omega_W^F = \Delta \omega_W^S = \Delta \omega_W^{F,S} \). By writing Eq. (11) for the Fermi-liquid and the superfluid separately, and subtracting, we obtain

\[
\left( \frac{k-1}{k'} \right)^F - \left( \frac{k-1}{k'} \right)^S = \frac{2(\Delta \omega_R^F - \Delta \omega_R^S)}{\Delta \omega_W^{F,S}} = \frac{2\delta \Delta \omega_R}{\Delta \omega_W^{F,S}} \tag{12}
\]

The first term on the left hand side can be evaluated from the Fermi-liquid data and the term on the right hand side is known from a comparison between the resonance shifts at equal damping. We can thus find the value of \( (k-1)/k' \) for the superfluid and hence \( m \), which gives \( \bar{\eta}_n^{\bar{\rho}_n} \); \( \bar{\eta}_n \) and \( \bar{\rho}_n \) are then separated employing Eq. (10).

In practice, our final data reduction for finding \( \bar{\eta}_n \) and \( \bar{\rho}_n \) was carried out by employing Eqs. (6) and (7) iteratively in order to obtain a parametric representation of \( \Delta \omega_W = \Delta \omega_W(\delta \Delta \omega_R) \) with \( \bar{\eta}_n \) and \( \bar{\rho}_n \) as parameters and then by graphical interpolation. We plotted the measured resonance widths \( \Delta \omega_W \) against the relative resonance shifts \( \delta \Delta \omega_R \) and obtained the corresponding \( \bar{\eta}_n \) and \( \bar{\rho}_n \) values by interpolating between curves of constant \( \bar{\eta}_n \) and \( \bar{\rho}_n \) which had been drawn on the same plane.\(^{18}\)

The curves of constant \( \bar{\eta}_n \) and \( \bar{\rho}_n \) were calculated by using Eqs. (6) and (7) for both a normal fluid and two-fluid case. We chose first a suitable set of \( \bar{\eta}_n, \bar{\rho}_n, \) and \( \bar{\rho}_S(=\rho-\bar{\rho}_n) \) values and used a computer to calculate and store the corresponding \( \Delta \omega_W^S \) and \( \Delta \omega_R^S \) values. Then, by employing Eq. (7) for a normal fluid, we calculated the \( \eta_n \) values which gave the same \( \Delta \omega_W \)'s as the above mentioned set \( \bar{\eta}_n', \bar{\rho}_n', \) and \( \bar{\rho}_S' \); in this calculation \( \rho_n \) was obtained from the value of \( \eta_n \) using the data reported by Grilly\(^{17}\), and also our viscosity measurements in the Fermi-liquid region, in which values of \( \eta_n \) and \( \rho_n \) are given as a function of \( T \). Next, employing Eq. (6), we calculated the corresponding \( \Delta \omega_R^F \) values. Finally \( \delta \Delta \omega_R \)'s were found
by subtracting $\Delta \omega^S_R$ from $\Delta \omega^F_R$. Now, with the parametric relation $\Delta \omega^F_S = \Delta \omega^F_S(\delta \Delta \omega_R(\eta_n, \rho_n))$ we were able to plot the curves of constant $\eta_n$ and $\rho_n$.

Next we discuss the corrections which were applied in data reduction. The effect of the finite Q-value of the viscometer in vacuum was taken into account by the principle of superposition, i.e. the width of the resonance line ($\approx 0.3$ Hz) in vacuum was subtracted from the measured $\Delta \omega_W$ values before the viscosity was determined from the data. A rigorous calculation was carried out, in which, damping term in the equation of motion of the wire corrected for the finite Q of the wire in vacuum, and a correction to Eq. (5) was obtained. It was thus verified, that our simple procedure of subtracting the widths of the resonance lines, introduced negligible error.

Because of the relatively high cooling rates ($\approx 10$ $\mu$K/s), which we had to use for most of our viscosity runs, the normal Fermi-liquid data below about 5 mK had to be corrected for the temperature change occurring during the measurement of a resonance curve. The corrections to the measured widths and shifts of the resonant curves, $\Delta \Delta \omega_W$ and $\Delta \Delta \omega_R$, were obtained from the relations

\[
\frac{\Delta \Delta \omega_W}{\Delta \omega_W} = \frac{[\omega_+(m_R \pm \Delta m) - \omega_-(m_R \mp \Delta m)] - [\omega_+(m_R) - \omega_-(m_R)]}{\omega_+(m_R) - \omega_-(m_R)}
\]

(13)

and

\[
\frac{\Delta \Delta \omega_R}{\Delta \omega_W} = \frac{[\omega_+(m_R \pm \Delta m) + \omega_-(m_R \mp \Delta m)] - [\omega_+(m_R) + \omega_-(m_R)]}{2[\omega_+(m_R) - \omega_-(m_R)]}
\]

(14)

where $m_R$ is the value of the parameter m at the maximum of resonance curve; the latter correction assumes that the
resonance occurs at \((\omega_+ + \omega_-)/2\) and this may be shown to be a good approximation. The correction terms are based on the plausible assumption that \(m\) changes linearly during the recording of a resonance curve \((m\) is proportional to \(T\)). The choice of the sign of \(\Delta m\) depends on the direction of the frequency sweep; the absolute values of \(\Delta m\) were obtained from the change of temperature which had occurred during the recording of the widths of the resonance curves. The corrections to \(\Delta \omega_\text{W}\) were maximally about 2 \%; corrections to \(\Delta \omega_\text{R}\) would have been so small that they were neglected.

We examined the validity of these corrections by making a run in which the pressurization was stopped in the vicinity of the A-transition; resonance data were then taken while the sample warmed slowly. For the same purpose we occasionally reversed the direction of the frequency sweep for every second resonance curve. These results, and the data measured during cooling, were found consistent. In the superfluid phases this kind of correction was unnecessary because of the degree of reproducibility of the data taken upon cooling, or alternatively on warming; both types of results were equal within the scatter.

A tentative correction was also applied to the viscosity data obtained immediately below the temperature \((15 - 16 \text{ mK})\) at which the melting curve was reached in Pomeranchuk cooling runs. We observed frequently that the temperature sensed by the viscometer lagged behind the temperature determined from the melting pressure. However, as the cooling proceeded to about 10 mK, the temperature difference, which was typically about 0.3 mK at its maximum, diminished quickly suggesting that thermal relaxation was taking place between different parts of the sample. This was observed as a deviation from the expected behavior of the viscosity immediately below 16 mK, which we found by extrapolating to lower temperatures the constant temperature measurements at higher temperatures.
We introduced a simple model for the relaxation effect, which connected the spatial temperature distribution of the liquid to the temperature of a possible solid-liquid interface through the thermal time constant \( t \) of \(^3\)He \((\kappa \propto T^2)\) and were able to explain the phenomenon satisfactorily. However, we also made measurements in which the sample was allowed to warm very slowly along the melting curve below 16 mK, and accepted only this type of data for the final data reduction.

5. THE FERMI-LIQUID REGION

Our results on the viscosity of \(^3\)He between 100 and 2.6 mK along the melting curve are presented in Fig. 3. In the same figure we have also plotted the data of Lawson et al.\(^4\) along the melting curve, and Black et al.\(^3\) at the saturated vapor pressure.

It is customary to fit the measured viscosity data to formulae of the form \( (\eta T^2)^{-1} = a - \sum b_i T^i \), where \( i = 1, 2, 3 \). Powers of \( T \) on the right hand side then present corrections to Fermi-liquid theory in the zero temperature limit. Emery\(^19\) has shown that a term in \( T^2 \) and a small term in \( T \) must be present. A calculation by Rice\(^20\) introduced a leading term of order \( T^3 \).

First and second order regression analysis fits to our data between 100 and 16 mK yield \( (\eta T^2)^{-1} = 1.170 - 3.10T \) and \( (\eta T^2)^{-1} = 1.097 - 26.57T^2 \), for \( \eta \) in \( \mu P \) and \( T \) in Kelvin, respectively (cf. Fig. 8 and Table 1).
Fig. 7. The viscosity of liquid $^3$He between 100 and 2.6 mK along the melting curve.
In Fig. 8 we have plotted our data in the form $(\eta T^2)^{-1}$ as a function of temperature. The quantities $a-b_1 T$ and $a-b_2 T^2$, to which we fitted our data between 16 and 100 mK (dots) are also shown; the results of the regression analysis are given in Table 1.

![Graph](image)

Fig. 8. Deviation of the low-temperature viscosity of liquid $^3$He from a $T^{-2}$-dependence at the melting pressure. The fits to the data (dots) are shown; the circles were not included in the regression analysis (cf. the text below).

We did not include our results below 16 mK (circles) in the analysis because of the relatively strong, systematic temperature dependence of the data below about 10 mK, which was interpreted as being due to the inaccuracy of our temperature scale in that region. We return to this subject in connection with the determination of the melting curve from our viscosity measurements below 10 mK.
TABLE 1.

Regression analysis of the viscosity data between 100 and 16 mK; the numbers are for $T$ in Kelvin and $\eta$ in $\mu P$.

<table>
<thead>
<tr>
<th></th>
<th>$a - b_1T$</th>
<th>$a - b_2T^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$1.170 \pm 0.007$</td>
<td>$1.097 \pm 0.006$</td>
</tr>
<tr>
<td>$b_1$</td>
<td>$3.10 \pm 0.12$</td>
<td>$26.57 \pm 1.6$</td>
</tr>
<tr>
<td>rms deviation</td>
<td>$0.012$</td>
<td>$0.018$</td>
</tr>
<tr>
<td>correlation</td>
<td>$0.977$</td>
<td>$0.947$</td>
</tr>
</tbody>
</table>

According to Table 1 our experiments down to 16 mK suggest that the correction to the limiting $T^{-2}$-dependence of $\eta$ is of order $T$. However, statistically the difference between the first order and second order fits is insignificant.

Table 2 gives coefficients $a$ and $b_1$ from our work, measurements of Lawson et al., Black et al., and a calculation of Pethick and Dy$^\text{II}$)

TABLE 2.

Comparison of coefficients $a$ and $b_1$ as given by various authors.

<table>
<thead>
<tr>
<th>this work</th>
<th>Lawson et al.</th>
<th>Black et al.</th>
<th>Pethick and Dy (theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>pressure melting</td>
<td>melting</td>
<td>vapor</td>
<td>27 atm</td>
</tr>
<tr>
<td>$a$</td>
<td>1.17</td>
<td>0.816</td>
<td>0.471</td>
</tr>
<tr>
<td>$b_1$</td>
<td>3.10</td>
<td>4.43</td>
<td>1.08</td>
</tr>
</tbody>
</table>

$^\text{II}$) as reported by Lawson et al.$^4$
If we reduce our results to constant pressure by using the data of Paulson, Johnson and Wheatley\textsuperscript{11}) on the pressure dependence of viscosity, we find \( b_1 = 2.34 \) at 33.6 atm in agreement with the theoretical calculation at 27 atm. The agreement may, however, be fortuitous. Our result \( a^{-1} = 0.855 \) is in good accord with the measurement of Paulson et al. at the melting pressure. They found \( \eta T^2 = 0.88 \). An extrapolation of their results to zero pressure yields \( a^{-1} = 2.1 \) in agreement with the value \( a^{-1} \approx 2.12 \) of Black et al. The three experiments are thus compatible. On the other hand, the measurement of Lawson et al. suggests \( a^{-1} = 1.23 \).

By employing zero sound attenuation data, measured near the melting curve, Lawson et al.\textsuperscript{4} estimated that the viscosity of \( ^3\text{He} \) at 50 mK is between 400 and 500 \( \mu \)P. We find \( \eta = 400 \mu \)P at 50 mK, and the measurement of Paulson et al. gives \( \eta = 350 \mu \)P at the same temperature; the latter experiment does not give estimates for \( b_1 \) and \( b_2 \) so that an extrapolation yields too low a value for \( \eta \). However, the calculation of Dy and Pethick for the low temperature viscosity of \( ^3\text{He} \) at 0.28\textsuperscript{21} and 27 atm indicate that the viscosity at the higher pressure should be about a factor of four smaller than at the lower pressure. In view of the measurement of Black et al. this suggests \( \eta \approx 250 \mu \)P at 50 mK and at the melting pressure. It thus seems clear that the theoretically predicted pressure dependence of the low temperature viscosity of \( ^3\text{He} \) is somewhat too strong.

We return now to the determination of the melting curve below 10 mK from our viscosity data. We made the assumption that our temperature scale is accurate within a few percent at least down to 15 mK. This belief is supported by the observation that an extrapolation of our viscosity data between 100 and 16 mK and the lower temperature results (cf. Fig. 8) were consistent down to about 10 mK. The systematic deviation from the extrapolation below this temperature was then inter-

\textsuperscript{11}) as reported by Wheatley\textsuperscript{16}
interpreted as an indication of the failure of the temperature scale rather than as a stronger temperature dependence of the viscosity. By forcing the data below 10 mK on the higher temperature extrapolation, we could correct the temperature scale accordingly and find the improved P,T-relation. The result is seen in Fig. 9.

![Graph showing melting curve of $^3$He between 10 and 2.6 mK as determined from viscosity data.]

Fig. 9. The melting curve of $^3$He between 10 and 2.6 mK as determined from viscosity data.
A third degree polynomial fit to our melting curve data yields for the inflection point $T^\dagger = 7.4 \text{ mK}$ and for the temperature of the A-transition $T_A = 2.53 \text{ mK}$. Second and first order fits yield $T_A = 2.55 \text{ mK}$ and $T_A = 2.57 \text{ mK}$, respectively. If the $a - b_2 T^2$ fit and the corresponding extrapolation of the viscosity data measured above 15 mK are employed as the basis of temperature corrections below 10 mK, we obtain about 0.1 mK higher values for $T_A$. Measurements of Johnson, Paulson, Pierce and Wheatley$^{22}$ along the melting curve suggest $T_A = 2.5 - 2.6 \text{ mK}$ and $|dP/dT| = 33.9 - 37.3 \text{ bar/K}$ at the A-transition. In our case the third order fit yields $|dP/dT| = 39.4 \text{ bar/K}$ at $T_A = 2.53 \text{ mK}$.

Our melting curve data are in fair agreement with the calculation of Thompson and Meyer$^{23}$, and Goldstein$^{11}$. However, a more recent paper by Goldstein$^{24}$ seems to disagree with our measurements as well as with the data of Johnson et al., this calculation yields $T_A = 4 \text{ mK}$ in order to make the pressure difference between the maximum of the melting curve and the A-transition correspond to the largest observed pressure difference 58 mbar.$^{25}$

6. SUPERFLUID PHASES

The behavior of the viscometer amplitude below 15 mK along the melting curve is shown in Fig. 10.

The measurement was made in a magnetic field of 0.149 T using an rms excitation current of 0.4 mA. The vibrating wire was swept through the resonance once in about every twenty seconds. The A- and B-phases were traversed three
Fig. 10. Resonance amplitude as a function of a provisional temperature; the scale below $T_A = 2.6 \text{ mK}$ was plotted by assuming $|dP/dT| = 35 \text{ bar/K}$ at $T<T_A$. Note the different pressure scales above and below the $A$-transition.

times before the anomalous resonance shift due to solid formation on the wire occurred. The average viscous heating due to the measuring technique was calculated to be smaller than 0.01 erg/s.
Our results on $\bar{\eta}_n$ and $\bar{\rho}_n$ below 2.6 mK are illustrated in Figs. 11 and 12.

Fig. 11. The reduced average viscosity $\bar{\eta}_n/\eta_A$ of the normal component of superfluid $^3$He at the melting curve as a function of temperature; $\eta_A$ is the viscosity of normal $^3$He at $T_A = 2.6$ mK. The temperature scale has been plotted by assuming $|dP/dT| = 35$ bar/K $T < T_A$. The circles are in the A phase and triangles in the B-phase; open symbols were taken on cooling and filled ones upon warming. The $(T/T_A)^8$-temperature dependence suggested by the heat flow measurements of Greytak, Johnson, Paulson and Wheatley\textsuperscript{26}, and the theoretical prediction of Shumeiko\textsuperscript{27} with $c = 6$ (cf. the text below) are also shown in the figure (dashed lines). The solid line and arrows are visual aids only. The error bars represent the estimated uncertainty of $\pm 0.15$ Hz in the determination of the shifts of the resonance curves. The pressure difference relative to the A-transition is also indicated.
It is observed from Fig. 11 that $\bar{\eta}_n$ decreases within 0.3 mK of $T_A$ to about 25% of $\eta_A$, the normal Fermi-liquid viscosity at the A-transition, and then becomes essentially constant. In the B-phase $\bar{\eta}_n$ seems to be somewhat less than 25% of $\eta_A$ down to about 1.4 mK; at still lower temperatures $\bar{\eta}_n$ shows a tendency to grow. The change in $\bar{\eta}_n$ at the B-transition appears to take place discontinuously.

We have no data points in the immediate vicinity of the A-transition due to the rapid change of the amplitude near $T_A$. It is then very difficult to obtain reliable results in this region because of the relatively slow repetition rate of the measurement of the resonance curves, which we must employ in order to minimize our problems with solid $^3$He. For this reason also the observation of the splitting of the A-transition in an external magnetic field was practically impossible with the resonance curve method. However, it must be kept in mind that the origin of these kinds of difficulties lies mainly in the comparatively high cooling rates which we employed because of the limited measuring time available during a run; solid-free functioning of the viscometer lasted maximally about an hour after the melting curve had been reached, and about twenty minutes in the A- and B-phases. We believe that reducing the stray heat leak to a minimum would make this type of experiments considerably easier to perform; installation of a superconducting heat switch has accordingly been undertaken.

The calculation of the viscosity for an isotropic superfluid $^3$He by Shumeiko and our experimental results are qualitatively in good agreement. His theory predicts that $\eta_n$ becomes constant for $T < T_A$ and that this constant value is less than $\eta_A$. Near $T_A$ the calculation gives $1 - \eta_n/\eta_A = c(1 - T/T_A)^{1/2}$ and hence, $d\eta_n/dT = \infty$ at $T_A$; $c$ is a constant. The rapid decrease of $\bar{\eta}_n$ below $T_A$ in our measurements is consistent with this prediction. The behavior of $\bar{\eta}_n$ immediately
below $T_A$ is more clearly seen from the viscosity curves obtained during our continuous resonance measurements (cf. Fig. 16).

Seiden's calculation of the viscosity for an isotropic superfluid $^3$He at $T < T_A$ predicts that $\eta_n$ varies slowly with temperature. However, a quantitative estimate gives $\eta_n \approx 2\eta_A$ which seems to be in disagreement with our results.

Fig. 12. The reduced average density $\bar{\rho}_n/\rho_A$ of the normal component of superfluid $^3$He at the melting curve as a function of temperatures; $\rho_A$ is the viscosity of normal $^3$He at $T_A$. For details of the temperature scale and an explanation of the data symbols and error bars see Fig. 11. The $\bar{\rho}_n/\rho_A = T/T_A$ line is shown for comparison only.
We observe from Fig. 12 that $\bar{\rho}_n$ varies slowly, approximately as $T/T_A$, with temperature in the A-phase. At the B-transition, however, $\bar{\rho}_n$ drops discontinuously from about 60% of $\rho_A$, the total density of $^3$He at $T_A$, to 25% of $\eta_A$ and continues to decrease rapidly as the temperature is reduced further. Below 1.3 mK $\bar{\rho}_n$ is less than 1% of $\rho_A$. On warming through the B'-transition $\bar{\rho}_n$ increases very rapidly, yet continuously within our resolution. This cannot be seen from the data points of Fig. 12, but we conclude this on the basis of the smooth behavior of the viscometer amplitude (roughly proportional to $(\bar{\eta}_n \bar{\rho}_n)^{1/2}$) at B' in other runs; occasionally the B'-transition has occurred in the middle of a resonance line. In the continuous resonance measurements the amplitude was also found to behave smoothly at B'.

Our results on $\bar{\rho}_n$ in the A-phase are in fair agreement with the measurements of Kojima, Paulson and Wheatley\textsuperscript{29}; these authors determined $\bar{\rho}_S$ as a function of temperature from fourth sound measurements at pressures below the melting curve and found that $\bar{\rho}_S/\rho$ varies as $(1 - T/T_C)\alpha$, where $\alpha$ is in the range 1.1 - 1.3. On the other hand, there seems to be a discrepancy in the behavior of $\bar{\rho}_n$ in the vicinity of the B'-transition. Kojima et al. report that changes, if any, in $\bar{\rho}_S/\rho$ at $T_B$, are less than 5-10% as measured in their experiment. According to our results, however, the temperature dependence of $\bar{\rho}_n$ is relatively strong below B'; this qualitative feature should have been noticeable also in their experiment. An explanation of the discrepancy could be that the B-phase, or rather the bulk liquid B-phase, was never actually present in the experiment of Kojima et al. owing to the measuring geometry; for fourth sound measurements $^3$He was packed with fine CMN powder. NMR-measurements of Ahonen, Haikala and Krusius\textsuperscript{30} have shown that the properties of the B-phase are altered when the liquid $^3$He is confined within fine platinum powder.
The splitting of the A-transition in a magnetic field is clearly observable in our continuous resonance measurements. A trace of the viscometer signal amplitude and a simultaneous pressure recording are shown in Fig. 13.

![Diagram](image)

Fig. 13. The splitting of the A-transition into $A'_1$ and $A'_2$ in a magnetic field of 0.7 T as observed by the vibrating wire viscometer; the primes indicate that the sample warmed during the measurement.

Our results on the splitting are illustrated in Fig. 14. The pressure difference between the $A'_1$- and $A'_2$-transitions was obtained from the $P(t)$-curve by employing the two
discernible changes in the resonance amplitude curve in fixing the transition pressures. Measurements were made both upon cooling and upon warming.

Fig. 14. The splitting of the A-transition, expressed as the pressure difference between A₁⁻ and A₂⁻ transitions, in an external magnetic field.

We found \( P_{A1}(B) - P_{A2}(B) = 2.10(B/T) \) mbar; by assuming \( |dP/dT| = 39 \) bar/K at the A-transition we may write \( P_{A1}(B) - P_{A2}(B) = 54(B/T) \) \( \mu \)K. Gully et al.⁶ have determined the splitting from the change of pressure alone at A₁ and A₂. They found \( P_{A1}(B) - P_{A2}(B) = 2.0(B/T) \) mbar. Lawson et al.⁷ obtained for the splitting also 2.0(B/T) mbar from ultrasonic attenuation measurements.

Our results on the suppression of the B'-transition in a magnetic field are given in Fig. 15. The data are based on an analysis of the P(t)-curves.

We obtained \( P_{B'}(B) - P_{A1}(B) = [20.4 + 171(B/T)^2] \) mbar in fair agreement with the measurement of Osheroff.⁵ He found
Fig. 15. The suppression of the B'-transition, expressed as the pressure difference between the B' and A1'-transitions, in a magnetic field.

\[ P_{B'}(B) - P_{A1'}(B) = [19.8 + 202(B/T)^2] \text{ mbar} \]

Halperin, Buhrman, Webb and Richardson\(^{25}\) have reported \( P_{B'}(0) - P_{A1'}(0) = 20.4 \text{ mbar} \).

We have also made an attempt to determine \( \eta_n \) in various magnetic fields as a function of temperature from our continuous resonance data. The results are illustrated in Fig. 16.

In performing these measurements the excitation current was reduced by a factor of ten below that used in the resonance curve measurements; this was done in order to minimize viscous heating of \(^3\text{He}\). We could not, however, measure the actual shift of the resonance reliably in these measurements.
Fig. 16. The average reduced viscosity $\bar{\eta}_n/\eta_{A_1}$ of the normal component of superfluid $^3$He at the melting curve as a function of the pressure difference relative to the $A_1$-transition. The reduced temperature $T/T_{A_1}$ is also indicated in the figure; the temperature scale below $T_{A_1} = 2.6$ mK was plotted by assuming that the variation of the melting pressure as a function of magnetic field is negligible and $|dP/dT| = 35$ bar/K at $T<T_{A_1}$. The broken curve and the dashed curve represent fits to the theory of Shumeiko with $c = 1$ and $c = 3$, respectively. The $\bar{\rho}_n/\rho_{A_1} = T/T_{A_1}$ line shows the assumed behavior of the density of the normal component. The magnitude of the field is indicated for each curve.
owing to formation of solid on the wire. This was especially so in the case of high magnetic fields.

In order to obtain \( \bar{n}_n \) from the amplitude data alone we assumed that \( \bar{\rho}_n \) does not depend on the magnetic field and that it is given by \( \bar{\rho}_n / \rho_{Al} = T / T_{Al} \). We observed that the anomalously shifted resonance frequency \( \omega_R \) and the resonance amplitude \( A_R \) behaved roughly as \( A_R \omega_R = \text{constant at a given temperature, which is to be expected from a simple model for solidification at the ends of the wire. The present data were obtained before the anomalous shift exceeded 50 Hz so that the accuracy of the results should be better than 5% provided that our assumptions on the temperature and magnetic field dependence of \( \bar{\rho}_n \) are valid.}

It is seen from Fig. 16 that \( \bar{n}_n \) starts to decrease with an infinite slope at both the \( A_1 \) - and \( A_2 \)-transitions towards lower temperatures but levels then off a constant value of about 25% of \( n_{Al} \) within 0.2 - 0.3 mK. The viscosity seems to be somewhat higher for higher fields, i.e. \( (\partial \bar{n}_n / \partial B)_T \) is positive.

![Graph showing the relation between pressure, viscometer amplitude, and temperature.](image)

**Fig. 17.** The resonance amplitude of the viscometer in the A- and B-phases \( (B = 0.149 \, T) \). The pressure recording is also shown in the figure.
We observed some interesting features in the behavior of the viscometer amplitude in the A-phase. Fig. 17 illustrates a recording of the signal as the sample cooled through the $A_1^-$, $A_2^-$, and B-transition in a field of 0.149 T.

In the A-phase it seems as if something tried to attach itself to the wire but becomes shaken off thus causing scatter in damping. Fluctuations are very strong in the supercooled A-phase immediately before the B-transitions occurs. Since the roughness of the signal disappears in the B-phase it is tempting to assign this observation to the possible anisotropic flow properties of the A-phase.

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REFERENCES


APPENDIX 1

THE TWO-FLUID MODEL AND THE BASIC EQUATIONS FOR STOKES' SOLUTION

According to the two-fluid model of superfluidity the densities of the normal and superfluid components, \( \rho_n \) and \( \rho_s \), respectively, are connected by the relation

\[
\rho_n + \rho_s = \rho ,
\]

where \( \rho \) is the total, bulk density of the liquid. In addition, the model postulates that the superfluid component carries zero entropy and experiences no flow resistance, i.e. it exhibits neither viscosity nor turbulence. The latter condition for the superfluid velocity field \( \mathbf{v}_s \) is satisfied if

\[
\nabla \times \mathbf{v}_s = 0 .
\]

The velocity fields \( \mathbf{v}_n \) and \( \mathbf{v}_s \) as well as the corresponding mass current densities \( \rho_n \mathbf{v}_n \) and \( \rho_s \mathbf{v}_s \), are assumed to be independent. The conservation of mass then requires

\[
\frac{\partial}{\partial t} (\rho_s + \rho_n) + \nabla \cdot (\rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n) = 0 .
\]
Conservation of entropy gives

\[ \frac{\partial}{\partial t}(\rho S) + \nabla \cdot (\rho S \dot{v}_n) = 0, \quad (4) \]

where \( S \) is the entropy per unit mass of the liquid. Viscosity actually leads to generation of entropy, but since this is a second order effect in Eq. (4), the corresponding term is neglected.

For the present purposes other simplifications can be made, which make the mathematical treatment easier. We assume that \( \rho_n, \rho_s \), and \( S \) are constant in time and uniform throughout the liquid. These assumptions are satisfied if the temperature and density of the liquid are constant and uniform. In this case Eq. (4) requires that

\[ \nabla \cdot \dot{v}_n = 0 \quad (5) \]

and it then follows from Eq. (3)

\[ \nabla \cdot \dot{v}_s = 0. \quad (6) \]

\( \dot{v}_s \) is now determined for a complete set of boundary conditions by Eq. (2) and (6). As the flow of superfluid is assumed to be irrotational, we can write \( \dot{v}_s = \nabla \varphi_s \), where \( \varphi_s \) is a scalar field. Hence, as \( \nabla \cdot \dot{v}_s = 0 \), it follows that the motion of the superfluid is always governed by Laplace's equation

\[ \nabla^2 \varphi_s = 0. \quad (7) \]
In order to find \( \dot{\nu}_n \) we write the equation of motion for the fluid as a whole and obtain

\[
\rho_n \frac{\partial \dot{\nu}_n}{\partial t} + \rho_s \frac{\partial \dot{\nu}_s}{\partial t} + \rho_n (\dot{\nu}_n \cdot \nabla) \dot{\nu}_n + \rho_s (\dot{\nu}_s \cdot \nabla) \dot{\nu}_s = -\nabla P + \eta_n \nabla^2 \dot{\nu}_n .
\] (8)

This relation corresponds to the Navier-Stokes equation of classical fluid dynamics.

The equation of motion for the superfluid alone can be written

\[
\frac{d\dot{\nu}_s}{dt} = -\nu G = \nabla \nu T - \frac{1}{\rho} \nabla P ,
\] (9)

where \( G \) is the Gibbs free energy per unit mass of the fluid. Because we assumed \( \nabla T = 0 \), Eq. (9) yields

\[
\frac{d\dot{\nu}_s}{dt} = -\frac{1}{\rho} \nabla P .
\] (10)

Multiplying Eq. (10) by \( \rho_s \) and subtracting from Eq. (8) we obtain

\[
\rho_n \frac{\partial \dot{\nu}_n}{\partial t} + \rho_n (\dot{\nu}_n \cdot \nabla) \dot{\nu}_n = -\frac{\rho_n}{\rho} \nabla P + \eta_n \nabla^2 \dot{\nu}_n ,
\] (11)

which is the general equation of motion for the normal fluid. Taking the curl of both sides and keeping in mind the two-dimensionality of our flow problems, we obtain the vorticity equation for the normal liquid.
\frac{\partial}{\partial t}(\nu \nu_n) = - \left( \nu_n \cdot \nabla \right) (\nu \nu_n) + \frac{n}{\rho} \nu \nabla^2 (\nu \nu_n) \quad (12)

This relation together with Eq. (5) and a given set of boundary conditions completely specify the normal fluid velocity field \( \nu_n \).

When \( \nu_n \) and \( \nu_s \) are known, Eq. (8) may be employed to solve for \( P \) and hence, forces exerted on objects in the fluid can be obtained by integrating over surfaces.

Stokes' solution for the force per unit length of a circular cylinder vibrating with a small amplitude in an unbound fluid, neglecting the convective, second order term in Eq. (12), can be expressed by

\[ \tilde{F} = -\pi a^2 \rho \left[ k(m) \frac{d\tilde{u}}{dt} + \omega k'(m) \tilde{u} \right] , \quad (13) \]

where

\[ k(m) = 1 - \frac{2}{m} \text{Im} \frac{\sqrt{i} H^{(1)}_1(2\sqrt{i}m)}{H^{(1)}_0(2\sqrt{i}m)} \]

and

\[ k'(m) = \frac{2}{m} \text{Re} \frac{\sqrt{i} H^{(1)}_1(2\sqrt{i}m)}{H^{(1)}_0(2\sqrt{i}m)} \]

Here \( \tilde{u} \) is the velocity of the cylinder, given by \( \tilde{u} = \tilde{u}_0 e^{i\omega t} \); \( \omega \) is the angular frequency of the oscillations and \( a \) is the radius of the cylinder. The parameter \( m \) is given by \( m = a/2\delta \).
where $\delta$ is the viscous penetration depth defined by $\delta = (n_n/\rho_n \omega)^{1/2}$. $H_0$ and $H_1$ are Hankel functions.

In order to find $\tilde{F}$ for the two-fluid system we write $P$ in Eq. (8) in the form $P = P_o + P_s + P_n$, where $P_o$ is the pressure at infinity and $P_s$ and $P_n$ are fictitious superfluid and normal fluid components of the pressure. By adding a term $n_s\nu_s^2v_s^2 (= 0)$ on the right hand side, it is easily seen that Stokes' solution can be modified for the two-fluid system by superimposing the effect of the superfluid on Eq. (13). Since $P_o$ has no net effect on the cylinder we obtain

$$\tilde{F} = -\pi a^2 \rho_n [k(m)\frac{du}{dt} +\omega k'(m)\hat{u}] -\pi a^2 \rho_s [k(m)\frac{du}{dt} +\omega k'(m)\hat{u}] . \ (14)$$

Now, for the superfluid fraction $m = \infty$; $k(\infty) = 1$ and $k'(\infty) = 0$ so that Stokes' solution, when applied to the two-fluid model, yields

$$\tilde{F} = -\pi a^2\{[\rho_n k(m) + \rho_s \frac{du}{dt} + \omega \rho_n k'(m)\hat{u}] \} . \ (15)$$