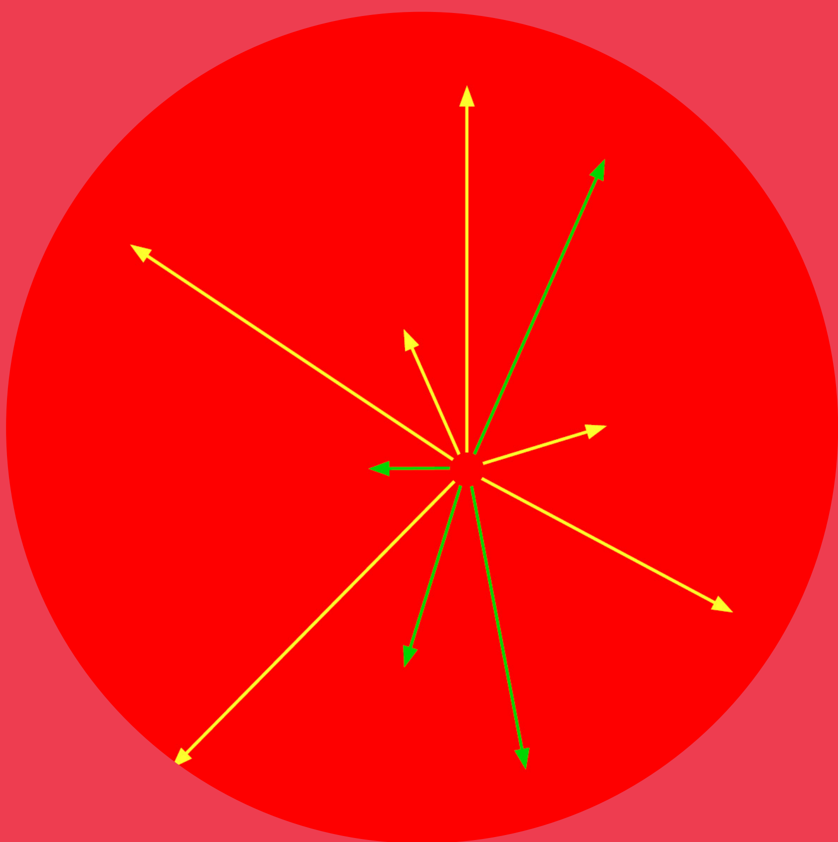
Department of Mathematics and Systems Analysis

Real Linear Operators on Complex Hilbert Spaces with Applications

Santtu Ruotsalainen



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Real Linear Operators on Complex Hilbert Spaces with Applications

Santtu Ruotsalainen

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Aalto University
School of Science
Department of Mathematics and Systems Analysis

Supervising professor
Prof. Olavi Nevanlinna

Thesis advisor
Prof. Marko Huhtanen

Preliminary examiners
Prof. Stephan Ramon Garcia, Pomona College, California, USA

Prof. Seppo Hassi, University of Vaasa, Finland

Opponent
Univ.-Prof. Dr. Jussi Behrndt, Technische Universität Graz, Austria

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Abstract

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Abstract

This dissertation studies real linear operators on complex Hilbert spaces. The focus is particularly on questions concerning spectral properties. Real linear operators arise naturally in applications such as the mathematical description of planar elasticity or the inverse conductivity problem utilized in electrical impedance tomography. Challenges in real linear operator theory stem from the observation that some foundational properties for complex linear operators may not hold for real linear operators in general.

The findings in this dissertation include basic spectral properties of real linear operators. An analogue of the Weyl-von Neumann theorem is proved concerning the diagonalizability of self-adjoint antilinear operators. Properties of the characteristic polynomial of a finite rank real linear operator are studied. Addressing invariant subspaces, an analogue of Lomonosov's theorem is proved for compact antilinear operators. With respect to the Beltrami equation, real linear multiplication operators are discussed. A factorization of symplectic and metaplectic operators is presented in connection to optics.

Keywords real linear operator, spectrum, antilinear operator, compact operator, invariant subspace, Weyl-von Neumann theorem, conjugation, factorization, symplectic matrix

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Tiivistelmä

Tekijä
Santtu Ruotsalainen

Väitöskirjan nimi
Reaalilineaariset operaattorit kompleksisilla Hilbertin avaruuksilla ja sovelluksia

Julkaisija Perustieteiden korkeakoulu

Yksikkö Matematiikan ja systeemianalyysin laitos

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Tiivistelmä

Tässä väitöskirjassa tutkitaan reaalilineaarisia operaattoreita kompleksisilla Hilbertin avaruuksilla. Kohteena on erityisesti spektrin ominaisuuksia koskevat kysymykset. Reaalilineaarisia operaattoreita ilmenee luonnollisesti sovelluksissa kuten tasoelastisuuden matemaattisessa kuvauksessa tai sähköisessä impedanssitomografiassa käytetyssä johtuvuuden käänteisongelmassa. Haasteet reaalilineaarisessa operaattoriteoriassa kumpuavat huomiosta, että jotkin kompleksilineaaristen operaattoreiden perusominaisuudet eivät välttämättä päde reaalilineaarisille operaattoreille yleisesti.

Väitöskirjan tulokset sisältävät reaalilineaaristen operaattoreiden spektrien perusominaisuuksia. Weylin-von Neumannin lauseen vastine todistetaan koskien itse-adjungoitujen antilineaaristen operaattoreiden diagonalisoituvuutta. Tutkitaan äärellistä rangia olevien reaalilineaaristen operaattoreiden karakterististen polynomien ominaisuuksia. Lomonosovin lauseen vastine koskien invariantteja aliavaruuksia todistetaan kompakteille antilineaarisille operattoreille. Beltramin yhtälöön liittyen käsitellään reaalilineaarisia kerroinoperaattoreita. Symplektisten ja metaplektisten operaattoreiden tekijöihinjako esitellään niiden yhteydensä optiikkaan puolesta.

Avainsanat reaalilineaarinen operaattori, spektri, antilineaarinen operaattori, kompakti operaattori, invariantti aliavaruus, Weyl-von Neumannin lause, konjugaatio, faktorointi, symplektinen matriisi

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Preface

This dissertation was started at the Institute of Mathematics of Helsinki University of Technology and finished at the Department of Mathematics and Systems Analysis of Aalto University School of Science with a brief hiatus at the Naval Research Center of the Finnish Defence Forces. To those organizational entities and the people constituting them I am indebted. Financially the research was partly supported by the Academy of Finland and The Research Foundation of Helsinki University of Technology.

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The mechanics ball grilling society MPGS, with the Anttis, Helena, Kurt, Mika and especially my five-year cell-mate Juho, have provided both a suitable diversion and an instrumental safety net of help. I have greatly enjoyed the deep and shallow, the scientific and non-scientific discussions with Dr. Jaakko Nissinen.

Finally, I thank my family for their continuous support and Ting-Ting for her loving pressure to finish.

Oulu, April 2, 2013,

Santtu Ruotsalainen

Contents

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I Huhtanen, Marko and Ruotsalainen, Santtu. Real linear operator theory and its applications. *Integral Equations and Operator Theory*, Volume 69, Number 1, pages 113–132, doi:10.1007/s00020-010-1825-4, January 2011.

II Ruotsalainen, Santtu. On a Weyl–von Neumann type theorem for antilinear self-adjoint operators. *Studia Mathematica*, Volume 213, Number 3, pages 191–205, doi:10.4064/sm213-3-1, December 2012.

III Ruotsalainen, Santtu. Compact real linear operators. *arXiv:1212.3292v2*, 15 pages, March 2013.

IV Huhtanen, Marko and Ruotsalainen, Santtu. Factoring in the metaplectic group and optics. *Operators and Matrices*, Volume 5, Number 1, pages 173–181, doi:10.7153/oam-05-12, March 2011.

Author's Contribution

Publication I: "Real linear operator theory and its applications"

Substantial part of writing and analysis are due to the author.

Publication II: "On a Weyl–von Neumann type theorem for antilinear self-adjoint operators"

This represents independent work by the author.

Publication III: "Compact real linear operators"

This represents independent work by the author.

Publication IV: "Factoring in the metaplectic group and optics"

Parts of writing and analysis in Sections 1–3 are due to the author.

1. Introduction

Theory of operators and their spectra can be seen as the fruitful axiomatization of the study of integral and differential operators and equations. In its past and present advances, this theory interacts with a variety of scientific fields ranging from technological improvements in computing to the theoretical description of nature. In the totality of this dissertation, real linear operators and their various properties have been studied. This is, somewhat loosely, called real linear operator theory. Here it means investigating real linear operators on complex linear spaces. The purpose of this overview is to expose some connections to existing operator theory and provide a few examples of applications.

From a historical viewpoint, real linear operator theory falls into a gap in the axiomatization process of linear operators on linear spaces. As can be seen, for example, from Banach's seminal work [5], in early operator theory continuous and additive mappings between vector and normed spaces over the real number field were studied. By continuity, additivity is then synonymous to real linearity. The subsequent operator theoretical development consisted of replacing the real number field with the complex number field throughout the axiomatization. In other words, complex linear operators on complex linear spaces became the focus of studies. This, of course, led to all the beautiful, fruitful, and effective theory of operators and their spectra. However, in doing so, the theoretically intermittent study of continuous additive mappings on complex linear spaces, i.e, of real linear operator theory, is leaped over.

Naturally, real linear operator theory has a bottomless source of problems in the gargantuous complex linear theory, and from this source the questions of this dissertation have emerged. Any property for complex linear operators entails the question whether, or possibly in what form, the similar statement holds for real linear operators. The arising real linear problems may be subjected to the following trivial trichotomy. Some concepts and theorems carry through with hardly any change, if at all. An example of this class is provided by compactness of the spectrum. Some require slight modification but finally can be transformed into the real linear domain. This happens for example in proving some of the properties of antilinear operators. Some, however, remain surprisingly elusive. An example of this could be merely determining the nonemptiness of the spectrum in the case of a real linear operator on a finite dimensional space.

From the viewpoint of applicability, real linear operators arise in various contexts in a more or less highlighted fashion. Let us mention in a deeply selective way some historical and some modern apparitions. In his lessons on projective complex geometry [8], Cartan writes about 'transformations antiprojectives'. A classical example from physics is the antilinear operator of time-reversal in quantum mechanics. More generally in quantum mechanics, Wigner studied antiunitary operators as symmetries of the Hilbert space [41, 42]. Partly inspired by this, antilinear operators have been used to study the variational objects of the so-called Hartree–Bogoliubov self-consistent theory in nuclear physics [20, 19].

More presently, antilinear operators arise in questions related to planar elasticity in the form of the Friedrichs operator [11, 33, 34, 27, 35, 30]. The general Beltrami equation in the plane entails a spectral question for real linear operators [3]. For example, in solving Calderón's inverse conductivity problem in the plane invertibility and compactness properties of a real linear operator are utilized [4]. Real linear operators are embedded in related studies of inverse scattering and non-linear evolution equations that involve solving the so-called $\overline{\partial}$-equation in the plane.

The present dissertation is a continuation of the work of Eirola, Huhtanen, Nevanlinna, and von Pfaler done in real linear matrix analysis [9], [23], [24], [22], [25]. In this sense it follows naturally that operators dealt with here are mainly on separable complex Hilbert spaces, though some results could be stated for real linear operators on complex Banach spaces.

The focal points of this dissertation can be depicted as follows. In Publication I, the general theory of real linear operators and their spectra is studied. Motivated by the Beurling transform and its use in solving the Beltrami equation, real linear multiplication operators are investigated. Their unitary approximation is discussed. In Publication II, the main goal is to prove a Weyl–von Neumann -type theorem for antilinear self-adjoint operators. Related properties of antilinear operators are studied. In Publication III, the main focus is on compact real linear operators including, naturally, finite rank operators. Related to invariant subspaces, an analogue of Lomonosov's theorem is proved for compact antilinear operators. The eigenvalue problem for real linear operators of finite rank is discussed in terms of the characteristic polynomial. In Publication IV, a factorization for symplectic matrices is presented. Using the so-called metaplectic representation, it is then seen as a factorization for metaplectic operators with applicability to optics.

Chapter 2 constitutes the main body of this overview. In Section 2.1, basic notions of real linear operators are discussed. Section 2.2 concentrates on features of real linear and antilinear self-adjointness. In Section 2.3, compact operators and the characteristic polynomial of finite rank operators are the objects of interest. In Section 2.4, symplectic matrices and metaplectic operators are introduced. Sections 2.5 and 2.6 briefly outline some examples of real linear methods used in studying the Friedrichs operator and the Beltrami equation, respectively. Chapter 3 summarizes the main results of the included articles.

2. Real linear operators

2.1 General notions

Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{C}$. A real linear operator A on $\mathcal{H}$ satisfies

$$A(x+y) = Ax + Ay, \quad A(rx) = rAx, \quad \forall x, y \in \mathcal{H},\ r \in \mathbb{R}.$$

A real linear operator A is called complex linear, if $Ai = iA$, or antilinear, if $Ai = -iA$. A real linear operator A can be split uniquely as the sum of its complex linear part $A_{\mathbb{C}} = \frac{1}{2}(A - iAi)$ and its antilinear part $A_{\mathbb{A}} = \frac{1}{2}(A + iAi)$:

$$A = A_{\mathbb{C}} + A_{\mathbb{A}}. \tag{2.1}$$

Related to this, the complexification of A with respect to the unitary conjugation κ is the complex linear operator

$$A^{\mathbb{C}} = \begin{pmatrix} A_{\mathbb{C}} & A_{\mathbb{A}}\kappa \\ \kappa A_{\mathbb{A}} & \kappa A_{\mathbb{C}}\kappa \end{pmatrix} \tag{2.2}$$

on $\mathcal{H} \oplus \mathcal{H}$. A unitary conjugation κ is an antilinear isometric involution, i.e., $\|\kappa x\| = \|x\|$ for all $x \in \mathcal{H}$ and $\kappa^2 = I$. Any unitary conjugation κ is naturally connected with an orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ of $\mathcal{H}$ such that $\kappa e_j = e_j$ for all $j \in \mathbb{N}$.

Spectral sets are defined exactly as for complex linear operators. The spectrum $\sigma(A)$ of a real linear operator A consists of those $\lambda \in \mathbb{C}$ for which $A - \lambda$ has no bounded inverse. The approximate point spectrum $\sigma_a(A)$ consists of those $\lambda \in \mathbb{C}$ for which there exists a sequence of vectors $\{e_j\}_{j=1}^{\infty}$ of unit length for which $\|(A-\lambda)e_j\| \to 0$ as $j \to \infty$. The point spectrum $\sigma_p(A)$ consists of eigenvalues of A. The residual spectrum $\sigma_r(A)$ consists of those $\lambda \in \mathbb{C}$ for which the range of $A - \lambda$ is not dense in $\mathcal{H}$. What is crucial about the definition of the spectral sets is that, firstly, they are defined as subsets of the complex plane, and secondly, they are not defined through the complexification. However, a key feature of the different spectral sets is the identity of real spectra

$$\sigma_X(A) \cap \mathbb{R} = \sigma_X(A^{\mathbb{C}}) \cap \mathbb{R}, \tag{2.3}$$

where $\sigma_X(A)$ may stand for $\sigma(A)$, $\sigma_a(A)$, $\sigma_p(A)$, or $\sigma_r(A)$. An often utilized property of the spectrum of a purely antilinear operator is that it is circularly symmetric with respect to the origin.

2.2 Self-adjointness

A real linear operator A is self-adjoint if $A = A^*$. The adjoint is defined through (2.1) as

$$A^* = A_{\mathbb{C}}^* + A_{\mathbb{A}}^*,$$

where $(A_{\mathbb{C}}^* x, y) = (x, A_{\mathbb{C}} y)$ and $(A_{\mathbb{A}}^* x, y) = (A_{\mathbb{A}} y, x)$ for all $x, y \in \mathcal{H}$. This definition is equivalent with the condition $\mathrm{Re}(A^* x, y) = \mathrm{Re}(x, Ay)$ for all $x, y \in \mathcal{H}$ taken in [9], [23], [24], [22], [25]. It seems the most natural for operators on Hilbert spaces. Similar and alternative definitions for additive continuous operators on complex Banach spaces have been studied in [32, 31].

Unlike for self-adjoint complex linear operators, the spectrum of a self-adjoint real linear operator is not necessarily contained in $\mathbb{R}$. However, it is symmetric with respect to the real line. Using the complexification (2.2) and the identity of real spectra (2.3), the spectrum of a self-adjoint real linear operator is seen to be nonempty.

Antilinear self-adjointness is equivalent with complex symmetry in the following sense. Let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Then for the antilinear self-adjoint operator A on $\mathcal{H}$, it holds

$$(Ae_j, e_i) = (Ae_i, e_j) \qquad \text{for all } i, j \in \mathbb{N}.$$

Defining $S_{ij} = (Ae_j, e_i)$, the antilinear self-adjoint operator can be represented as $A = S\kappa$, where κ is the unitary conjugation on $\mathcal{H}$ for which $\kappa e_j = e_j$ for all $j \in \mathbb{N}$. The complex linear operator S is symmetric in the sense that the, possibly infinite, matrix of S with respect to the basis $\{e_j\}_{j\in\mathbb{N}}$ is symmetric.

In view of this equivalence with complex symmetry, the Takagi factorization (cf. [36], [21, Section 4.4]) solves the eigenvalue problem for a purely antilinear self-adjoint operator A on $\mathbb{C}^n$. Namely A splits to the composition $A = S\tau$, where $S = S^T \in \mathbb{C}^{n\times n}$ and τ denotes the componentwise complex conjugation $\tau x = \overline{x}$ on $\mathbb{C}^n$. For the symmetric matrix S, the Takagi factorization provides a unitary matrix U and a diagonal matrix D with non-negative diagonal entries such that $S = UDU^T$. For the antilinear operator A it holds then that $A = UD\tau U^*$.

For complex linear operators, unitary diagonalization is one of the many properties equivalent with normality. For antilinear operators this equivalence no longer holds. As it was seen above through the Takagi factorization, self-adjointness is the concept equivalent with unitary diagonalization. However, other type of canonical forms have been studied. In [40], the authors define a conjugate normal matrix N by the condition $NN^* = \overline{N^*N}$ and show that it is unitary congruent, $N_0 = UNU^T$, to a 2-by-2 block diagonal matrix N_0. In the antilinear setting, this is translated to unitary similarity of the antilinear operator $M = N\tau$ with the antilinear operator $M_0 = N_0\tau$. Namely, $N_0\tau = UN\tau U^*$, where M is star-commuting, $MM^* = M^*M$.

For a general complex linear self-adjoint operator on an infinite dimensional Hilbert space it would be too much to ask for a diagonalization with respect to some orthonormal basis. However, it has been seen that any such operator is not too far from one. In [39], Weyl proved that a self-adjoint operator A can be perturbed by an arbitrarily small compact operator K, with $\|K\| < \varepsilon$, for the operator $A+K$ to have only pure point spectrum. This was generalized in [38] by von Neumann to cover unbounded operators and to make K small in Hilbert–Schmidt norm, $\|K\|_2 < \varepsilon$. Kuroda [26] noted that the use of Hilbert–Schmidt norm was not essential. This led to the following form of the Weyl–von Neumann theorem. For any self-adjoint operator A on a Hilbert space and any $\varepsilon > 0$ there is a compact operator K such that $\|K\|_p < \varepsilon$ and $A+K$ is diagonalizable. Here by diagonalizability it is meant that there exists an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of $A + K$.

The analogous theorem for antilinear self-adjoint operators on a separable complex Hilbert space is studied in Publication II. The method of proof is appropriately translated from the complex linear case with suitable care for antilinearity. Similarly, the polar decomposition of the antilinear self-adjoint operator A reads $A = |A|\kappa = \kappa|A|$. Here $|A| = (A^*A)^{1/2}$ is complex linear but the polar factor κ is now antiunitary. However, what is crucial is that κ can be chosen to be at the same time self-adjoint. Self-adjointness and antiunitarity imply that κ is a unitary conjugation on $\mathcal{H}$. A lemma similar to von Neumann's in [38] provides a finite-rank orthogonal projection and an arbitrarily small compact self-adjoint antilinear operator such that the projection reduces the perturbed operator. This lemma is suitably iterated with respect to the particular basis $\{e_j\}$ of $\mathcal{H}$ for which $\kappa e_j = e_j$. In this manner, a block diagonalization with finite dimensional blocks of the iteratively perturbated operator is attained. In addition, the desired smallness condition on the Schatten p-norm of the perturbed self-adjoint operator is gained. Finally, on each finite dimensional block the Takagi factorization ensures the existence of an orthonormal basis of eigenvectors.

Complex symmetric operators are by definition connected to antilinear self-adjoint operators. A complex linear operator $A_{\mathbb{C}}$ on $\mathcal{H}$ is called complex symmetric if there is a unitary conjugation κ on $\mathcal{H}$ such that $A_{\mathbb{C}} = \kappa A_{\mathbb{C}}^* \kappa$. This is equivalent with the antilinear operator $A_{\mathbb{C}}\kappa$ being self-adjoint. Conversely, every antilinear self-adjoint operator $A_{\mathbb{A}}$ defines a complex symmetric operator $A_{\mathbb{A}}\kappa$ with any unitary conjugation κ. Complex symmetric operators have been under recent investigations and they have been shown to include a variety of interesting complex linear operators [13, 14, 15] such as normal and Hankel operators. The antilinear viewpoint is advantageous in some cases. For example, it is known that the class of complex symmetric operators is not closed in operator norm [43], whereas the set of antilinear self-adjoint operators clearly is. Moreover, the basis independence of self-adjointness makes it appealing.

2.3 Compact and finite rank operators

The compactness of a real linear operator A is defined through the compactness of its complex linear part $A_{\mathbb{C}}$ and its antilinear part $A_{\mathbb{A}}$. A compact real linear operator is clearly a norm limit of finite rank operators. A real linear operator is of finite rank n if the complex dimension n of the closed complex linear span of $A\mathcal{H}$ is finite. In [23] this is called the left-rank $\dim((A\mathcal{H})^{\perp})^{\perp}$ of A.

The spectral theory of the two extremes are rather well understood. The eigenvalue theory of compact complex linear operators is classical. The eigenvalue problem for a compact antilinear operator $A = A_{\mathbb{A}}$ reduces to the complex linear case. The spectrum of $A_{\mathbb{A}}$ consists of circles centered at the origin with radius r_i, $i = 1, 2, \ldots$, where r_i is the square root of a non-negative eigenvalue of $A_{\mathbb{A}}^2$. The genuinely real linear case, where $A_{\mathbb{C}} \neq 0 \neq A_{\mathbb{A}}$, is more challenging.

Some of the facts for compact complex linear operators have easily accessible counterparts for real linear operators. Consider the intersection $L \cap \sigma(A)$ of any straight line through the origin L and the spectrum $\sigma(A)$ of a compact real linear operator A. Using a suitable rotation and the identity of real spectra (2.3), the intersection can be seen to consist of isolated points that accumulate at most to the origin, cf. Publication I, Theorem 2.21. Likewise, every point in the spectrum outside the origin is an eigenvalue. When $\mathcal{H}$ is infinite dimensional, the origin is in the spectrum, i.e., the spectrum is nonempty.

Investigations into the existence of nontrivial invariant subspaces are classical. In the complex linear case, von Neumann proved in apparently unpublished form that a compact operator has such a subspace. The subsequent developments leading to the current textbook form and proof of Lomonosov's theorem can be summarized as the papers [1], [6], [17], [28], and [29]. In Publication III, an analogue of Lomonosov's theorem is proven for compact antilinear operators. Instead of the commutant, it is stated for the norm closure of complex polynomials in a compact antilinear operator. This enables to prove the theorem by using a modification of Hilden's proof of the complex linear case [29].

The spectrum of a real linear operator of finite rank is the zero set of a characteristic polynomial $p(\lambda, \overline{\lambda}) = \det(A - \lambda)^{\mathbb{C}}$ defined as the $2n$-by-$2n$ determinant of the complexification of $A - \lambda$. Polynomials of this type were introduced and studied in [25], [9], and [23]. In [25], for example, it is shown that there is an abundance of operators with an empty spectrum. Namely, let $T = \frac{1}{2}(A_{\mathbb{A}} - A_{\mathbb{A}}^*)$ be the antilinear skew-adjoint part of A. Then if $\|T^{-1}\|\|A - T\| < 1$, the spectrum of A is empty. On the other hand, if $\sigma(A)$ is nonempty, it is the union of algebraic plane curves, i.e., not necessarily a discrete set of points.

The second focal point in Publication III is on the study of the characteristic polynomial p of finite rank real linear operator. Represent p as $p(\lambda, \overline{\lambda}) = v^* H v$, where $v = [\lambda^j]_{j=0}^n \in \mathbb{C}^{n+1}$ and $H = [h_{ij}]_{i,j=0}^n \in \mathbb{C}^{(n+1)\times(n+1)}$. The simple observation that p is real valued leads then to the coefficient matrix H being Hermitian. Using the unitary diagonalization of H, p can be cast as

$$p(\lambda, \overline{\lambda}) = \sum_{i=0}^{n} d_i |p_i(\lambda)|^2, \tag{2.4}$$

where each d_i is an eigenvalue of H and p_i is a complex polynomial. The representation (2.4) is attractive as it implies a possibility of using studies of sums of squares in connection with the finite dimensional eigenvalue problem for real linear operators. For such studies, [18] is a thorough exposition.

2.4 Symplectic matrices, the metaplectic representation and optics

A matrix $S \in \mathbb{R}^{2n\times 2n}$ is called symplectic if

$$S^T JS = J, \qquad \text{where} \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}. \tag{2.5}$$

Symplectic matrices form a group which is denoted by $\mathrm{Sp}(2n, \mathbb{R})$. A unitary operator U on $L^2(\mathbb{R}^n)$ is called metaplectic if there is a symplectic matrix $S \in \mathbb{R}^{2n\times 2n}$ such that

$$W[Uf] = W[f]S \tag{2.6}$$

for all $f \in L^2(\mathbb{R}^n)$. Here $W[f]$ denotes the Wigner distribution of f defined by

$$W[f](x,\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-iy\cdot\xi} f(x + \tfrac{1}{2}y)\overline{f(x - \tfrac{1}{2}y)}\, dy, \quad (x,\xi) \in \mathbb{R}^{2n} \tag{2.7}$$

with the notation $y \cdot \xi = \sum_{i=1}^n y_i\xi_i$. Metaplectic operators on $L^2(\mathbb{R}^n)$ form a group denoted by $\mathrm{Mp}(2n, \mathbb{R})$ which is connected to the group of symplectic matrices on $\mathbb{R}^{2n}$ through the so-called metaplectic representation. It is a one-to-two map $\mu : \mathrm{Sp}(2n, \mathbb{R}) \to \mathrm{Mp}(2n, \mathbb{R})$ such that

$$\mu(ST) = \pm\mu(S)\mu(T) \tag{2.8}$$

for all $S, T \in \mathrm{Sp}(2n, \mathbb{R})$, cf. [10, Chapter 4]. The representation μ maps matrices of the form

$$S_P = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \qquad \text{with } P = P^T \in \mathbb{R}^{n\times n} \tag{2.9}$$

to metaplectic operators of the form

$$\mu(S_P)f(x) = c_K e^{-\frac{i}{2}Px\cdot x} f(x), \tag{2.10}$$

and the matrix J to

$$\mu(J) = c_J \mathcal{F}^{-1}. \tag{2.11}$$

Here c_K and c_J are constants such that $|c_K| = |c_K| = 1$. In general, metaplectic operators can be cast as constant multiples of integral operators of the form [37]

$$f \mapsto e^{-\frac{i}{2}A^T Cx\cdot x} \int_{R(S_{12}^T)} e^{-\frac{i}{2}S_{12}^T S_{22}y\cdot y - iS_{12}^T S_{21}x\cdot y} f(S_{11}x + S_{12}y)\, dy, \tag{2.12}$$

where $R(S_{12}^T)$ stands for the row-space of the matrix S_{12}

The symplectic and metaplectic groups are related to optics in different approximations; cf. [16, Chapter I]. Linear optics is equivalent to $\mathrm{Sp}(4,\mathbb{R})$. Symplectic matrices of the form (2.9) correspond to refractions at the surface between two regions of constant index of refraction. Symplectic matrices of the form

$$L_d = \begin{bmatrix} I & dI \\ 0 & I \end{bmatrix} \qquad \text{with } d > 0 \tag{2.13}$$

correspond to motion in a medium of constant index of refraction of distance d along the optical axis. On the other hand, Fresnel optics is equivalent to the study of $\mathrm{Mp}(4,\mathbb{R})$ with the key operators being $\mu(S_P)$ and $\mu(L_d)$.

In Publication IV, the metaplectic representation and a factorization of symplectic matrices are used to factor a general metaplectic operator of the form (2.12) into consecutive applications of the inverse Fourier transform (2.11) and the multiplication operator (2.10). The number of factors is not dependent on the dimension n. The factorization of a symplectic matrix is based on an improvement of methods in [16, Chapter I] using the fact that any real matrix can be split into a product of two real symmetric matrices [12, 7].

Symplectic matrices can be represented by real linear operators on $\mathbb{C}^n$ as follows. With the identification $\mathbb{R}^{2n} \longleftrightarrow \mathbb{C}^n$, $(u_1, u_2) \longleftrightarrow u_1 + iu_2$, a symplectic matrix S can be restated as

$$C_1 + C_2\tau \tag{2.14}$$

on $\mathbb{C}^n$ with

$$\begin{aligned} C_1 &= \tfrac{1}{2}(S_{11} + S_{22} + iS_{21} - iS_{12}), \\ C_2 &= \tfrac{1}{2}(S_{11} - S_{22} + iS_{21} + iS_{12}), \\ S &= \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}. \end{aligned}$$

Again, τ denotes the componentwise complex conjugation on $\mathbb{C}^n$. Then J corresponds to the real linear operator of multiplication by $-i$ and matrices of the form (2.9) correspond to real linear operators of the form $I + \frac{i}{2}P + \frac{i}{2}P\tau$.

2.5 The Friedrichs operator

The so-called Friedrichs operator of a planar domain is an example of a much studied antilinear operator [11, 33, 34, 27, 35]. Let Ω be a connected open subset of the complex plane $\mathbb{C}$. Denote by $AL^2(\Omega)$ the Hilbert space of analytic square-integrable functions on Ω with the inner product $(f,g) = \int_\Omega f\overline{g}\,dA$, where dA denotes the area measure. Then the Friedrichs operator F_Ω of the planar domain Ω is the antilinear operator on $AL^2(\Omega)$ defined by

$$F_\Omega f = P_\Omega \overline{f} \tag{2.15}$$

for all $f \in AL^2(\Omega)$. Here P_Ω is the orthogonal projection from $L^2(\Omega)$ onto $AL^2(\Omega)$ which can be cast in the form

$$P_\Omega f(z) = \int_\Omega K_\Omega(z,w) f(w)\,dA(w) \tag{2.16}$$

using the Bergman kernel $K_\Omega(z,w)$ of Ω.

From the vantage point of real linear operator theory the Friedrichs operator is interesting in two ways. Firstly, it is always self-adjoint. This follows from the identity $(f, F_\Omega g) = \int_\Omega fg\,dA$ entailing

$$(f, F_\Omega g) = (g, F_\Omega f) \tag{2.17}$$

for all $f, g \in AL^2(\Omega)$ [35]. Secondly, the nature of the Friedrichs operator with respect to compactness depends on the geometry of the domain Ω, or more precisely on the boundary of Ω. In [33], it is shown that, when the boundary of Ω consists of finitely many continua, the Friedrichs operator is of finite rank if and only if Ω is a quadrature domain. A domain Ω is a quadrature domain if there exist non-negative integers r_j, complex numbers c_{ij} and points $z_j \in \Omega$ such that

$$\int_\Omega f\,dA = \sum_{j=1}^{n} \sum_{i=0}^{r_j} c_{ij} f^{(i)}(z_j) \tag{2.18}$$

for all $h \in AL^2(\Omega)$.

A Jordan curve Γ is of class $C^{n,\alpha}$ where n is a natural number and $0 < \alpha < 1$ if it has a parametrization $w : [0, 2\pi] \to \mathbb{C}$ which is n times continuously differentiable, $w'(t) \neq 0$, and $|w^{(n)}(t_1) - w^{(n)}(t_2)| \leq C|t_1 - t_2|^\alpha$ for some C. In [11], Friedrichs shows that if Ω is a bounded domain with $C^{1,\alpha}$ boundary, F_Ω is compact. In [27], Lin and Rochberg prove that assuming Ω is a bounded simply connected domain, if the boundary is $C^{2,\alpha}$, F_Ω is in every Schatten p-class for $1 < p < \infty$, or if the boundary is in $C^{3,\alpha}$, then F_Ω is in trace-class.

The applicability of the Friedrichs operator to problems in planar elasticity is discussed already in [11]. A more thorough exposition of the formulation in terms of the Friedrichs operator is given in [30]. Let Ω be now a simply connected domain the boundary of which is a rectifiable Jordan curve. Then the two main problems in planar elasticity can be formulated as follows. First, find $u \in AL^2(\Omega)$ such that

$$(I + F_\Omega)u = f \tag{2.19}$$

for given $f \in AL^2(\Omega)$ containing information about the tractions on the boundary $\partial\Omega$. Second, find $v \in AL^2(\Omega)$ such that

$$(k - F_\Omega)v = g \tag{2.20}$$

for given $g \in AL^2(\Omega)$ containing information about the displacements on the boundary $\partial\Omega$. Here k is the so-called Muskhelishvili constant for which it is proved in the theory of elasticity that $k > 1$.

2.6 The Beltrami equation and the two dimensional Caldéron problem

Real linear operators can be exemplified by the study of the general Beltrami equation in the complex plane $\mathbb{C}$

$$\overline{\partial} f - (\mu + \nu\tau)\partial f = 0. \tag{2.21}$$

Here the differential operators are defined as $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ and $\partial = \frac{1}{2}(\partial_x - i\partial_y)$ for $z = x + iy \in \mathbb{C}$. The coefficients are functions $\mu, \nu \in L^\infty(\mathbb{C})$. The operator τ denotes the pointwise complex conjugation $f \mapsto \overline{f}$. The Beurling transform is the singular integral operator

$$Sf(z) = -\tfrac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|w-z|>\varepsilon} \frac{f(w)}{(z-w)^2}\, dA(w) \tag{2.22}$$

and the Cauchy transform is

$$Pf(z) = \tfrac{1}{\pi} \int_{\mathbb{C}} \frac{f(w)}{z-w}\, dA(w), \tag{2.23}$$

where A is the Lebesgue measure on $\mathbb{C}$. These satisfy $Sf = \partial P f$ and $S\overline{\partial} f = \partial f$ for $f \in L^p(\mathbb{C})$ in the sense of weak derivatives. For a recent comprehensive exposition, see [2].

Using the Cauchy transform the solvability of the nonhomogenous equation $\overline{\partial} f - (\mu + \nu\tau)\partial f = h$, for given $h \in L^p(\mathbb{C})$, is transformed into invertibility of the real linear operator $I - (\mu + \nu\tau)S$ on $L^p(\mathbb{C})$, or evidently, into a spectral question for $(\mu + \nu\tau)S$. In [3] for example, this line of thought is followed to prove that $I - (\mu + \nu\tau)S$ is invertible on $L^p(\mathbb{C})$ precisely for all $p \in (1 + k, 1 + k^{-1})$ with $k = \|\mu + \nu\tau\| < 1$.

Motivated by the richness of real linear operators of the form $(\mu+\nu\tau)S$, real linear multiplication operators were studied in Publication I. For example, the condition of uniform ellipticity of (2.21) can be cast as a condition on the operator norm of the real linear multiplication operator $\mu+\nu\tau$,

$$\|\mu+\nu\tau\| = \operatorname*{ess\,sup}_{z\in\mathbb{C}} |\mu(z)|+|\nu(z)| \le k < 1. \tag{2.24}$$

The special case of (2.21) with $\mu = 0$ is central to the Caldéron's inverse conductivity problem in the plane. The inverse conductivity problem arises naturally in geophysical prospecting. In medical imaging it is known for its usefulness in electrical impedance tomography. In mathematical terms it is given as solving for $u \in H^1(\Omega)$ in the so-called conductivity equation

$$\nabla\cdot\sigma\nabla u = 0 \quad \text{in } \Omega \tag{2.25}$$

$$u|_{\partial\Omega} = \phi \quad \in H^{1/2}(\Omega). \tag{2.26}$$

Here, let Ω be the unit disk $\mathbb{D}\subset\mathbb{C}$, for simplicity. The measurable function $\sigma:\Omega\to(0,\infty)$ is bounded away from zero and infinity. The Sobolev space $W^{k,p}(\Omega)$ consists of functions that have distributional derivatives up to order k in $L^p(\Omega)$, and $H^k(\Omega)$ denotes $W^{k,2}(\Omega)$. The space $H^{1/2}(\partial\Omega)$ is defined as the quotient $W^{1,2}(\Omega)/W_0^{1,2}(\Omega)$, where $W_0^{1,2}(\Omega)$ is the closure in $W^{k,p}(\Omega)$ of the space of compactly supported smooth functions on Ω.

In [4], the observation is made that a function $f = u + iv$, with $u, v \in H^1(\Omega)$, satisfies

$$(\overline{\partial}+\nu\tau\partial)f = 0 \tag{2.27}$$

with $\nu = (1-\sigma)/(1+\sigma)$ if and only if

$$\nabla\cdot\sigma\nabla u = 0 \quad \text{and} \quad \nabla\cdot\sigma^{-1}\nabla v = 0. \tag{2.28}$$

Furthermore, the authors utilize the real linear operator

$$K = P(I-\nu\tau S)^{-1}\alpha\tau \tag{2.29}$$

on $L^p(\mathbb{C})$ to construct complex geometric optics solutions to the Beltrami equation. Here $\alpha\in L^\infty(\mathbb{C})$ with support in the unit disk $\mathbb{D}$, $\nu\in L^\infty(\mathbb{C})$ with $|\nu(z)|\le k\chi_{\mathbb{D}}(z)$ for almost all $z\in\mathbb{C}$, and $0<k<1$ is a constant.

3. Summary of findings

The main findings of this thesis can be summarized as follows.

I In this article general theory of real linear operators on a complex separable Hilbert space is studied. The identity of real spectra of a real linear operator and its complexification is proven. Spectral issues are addressed for antilinear, self-adjoint, compact, and unitary real linear operators. Basic properties of real linear multiplication operators on self-conjugate function spaces are investigated. The best unitary approximation of a real linear multiplication operator is obtained.

II In this artice an analogue of the Weyl–von Neumann theorem for antilinear self-adjoint operators is proved. Properties of unitary conjugations are presented. Using the polar decomposition, a representation of an antilinear self-adjoint operator as a spectral integral is introduced. The connection between self-adjoint antilinear operators and complex symmetric operators is explained. Unitary diagonalization of genuinely real linear finite rank operators is discussed.

III In this article compact and finite rank real linear operators are studied. Invariant subspaces of compact real linear operators are discussed. An analogue of Lomonosov's theorem for antilinear compact operators is proved. The characteristic polynomial of a finite rank real linear operators is represented as a weighted sum of squared moduli of complex polynomials. The so-called numerical function related to the characteristic polynomial is introduced and its relation to the coefficient matrix of the characteristic polynomial is studied.

IV In this article a factorization of symplectic matrices is presented in such a form that in the product a unique symplectic matrix and matrices from a subgroup alternate. The number of factors is independent of the dimension. Through the metaplectic representation, this factorization can be seen as one for metaplectic operators. The approach is constructive and numerically stable.

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