Paper [IV]

Salo, A., Punkka, A. (2011). Ranking intervals and dominance relations for ratio-based efficiency analysis. *Management Science* **57** 200–214.

MANAGEMENT SCIENCE
Vol. 57, No. 1, January 2011, pp. 200–214
ISSN 0025-1909 | EISSN 1526-5501 | 11 | 5701 | 0200

informs®

DOI 10.1287/mnsc.1100.1265

Ranking Intervals and Dominance Relations for Ratio-Based Efficiency Analysis

Ahti Salo, Antti Punkka
Systems Analysis Laboratory, Aalto University School of Science, 00076 Aalto, Finland
{ahti.salo@tkk.fi, antti.punkka@tkk.fi}

We develop comparative results for ratio-based efficiency analysis (REA) based on the decision-making units' (DMUs') relative efficiencies over sets of feasible weights that characterize preferences for input and output variables. Specifically, we determine (i) *ranking intervals*, which indicate the best and worst efficiency rankings that a DMU can attain relative to other DMUs; (ii) *dominance relations*, which show what other DMUs a given DMU dominates in pairwise efficiency comparisons; and (iii) *efficiency bounds*, which show how much more efficient a given DMU can be relative to some other DMU or a subset of other DMUs. Unlike conventional efficiency scores, these results are insensitive to outlier DMUs. They also show how the DMUs' efficiency ratios relate to each other for *all* feasible weights, rather than for those weights only for which the data envelopment analysis (DEA) efficiency score of *some* DMU is maximized. We illustrate the usefulness of these results by revisiting reported DEA studies and by describing a recent case study on the efficiency comparison of university departments.

Key words: efficiency analysis; data envelopment analysis; preference modeling
History: Received April 2, 2009; accepted September 6, 2010, by Teck-Hua Ho, decision analysis.

1. Introduction

The seminal paper of Charnes et al. (1978) has spawned a growing literature on data envelopment analysis (DEA) which offers numerous methods for examining the efficiency of decision-making units (DMUs) (see, e.g., Cooper et al. 2007). These methods are often employed in contexts where information about the unit prices of input and output variables is not readily available, but where it is still possible to elicit subjective information about how valuable these variables are relative to each other (Thompson et al. 1986, Allen et al. 1997; cf. Thanassoulis et al. 2004). This is the case in contexts such as higher education, health care, and technology management, among others (see, e.g., Sarrico and Dyson 2000).

Technically, the Charnes-Cooper-Rhodes-DEA (CCR-DEA; Charnes et al. 1978) computes efficiency scores for the DMUs relative to an efficient frontier, characterized by the DMUs that have the highest efficiency ratio between the aggregate value of their outputs and aggregate value of their inputs for *some* feasible input/output weights. By definition, an efficient DMU will have a score of one. For inefficient DMUs, the score is typically less than one and serves as a measure of how close to the efficient frontier a DMU can be when its inputs and outputs are aggregated with weights that are *most favorable* to this DMU. However, a concern with these scores is that they do not convey information about how the efficiency ratio of the DMU compares with the efficiency ratios of other DMUs for *other* input/output weights even though other weights reflect relevant preference information. This recognition has motivated the development of cross-efficiency (CE) methods where the efficiency score for every DMU is computed based on the weights for which the efficiency of *some* DMU is maximized (see Sexton et al. 1986, Doyle and Green 1994). Yet the consideration of these weights *only* does not show how the DMUs' efficiency ratios change relative to each other for *all* feasible weights.

A second concern with conventional efficiency scores is that they can be sensitive to which DMUs are included in or excluded from the analysis: for instance, the introduction or removal of a single outlier may shift the efficient frontier drastically and thus disrupt efficiency scores, which may be perplexing to users (see, e.g., Seiford and Zhu 1998a, b; Zhu 1996). A third concern with conventional scores is that they call for returns-to-scale assumptions, which may be difficult to justify. In effect, these three concerns can be addressed by focusing on pairwise one-on-one comparisons of efficiency ratios among DMUs because such comparisons (i) account for all feasible input/output weights, (ii) are less sensitive to the presence of outlier DMUs, and (iii) do not necessitate assumptions about what the set of production possibilities is beyond the DMUs that are included in the analysis (see, e.g., Galagedera and Silvapulle 2003, Dyson et al. 2001).

Motivated by the above considerations, we develop efficiency results in response to the following questions:

• What are the best and worst rankings that a given DMU can attain in comparison with other DMUs based on the comparison of DMUs' efficiency ratios for all feasible weights?

• Given a pair of DMUs, does the first DMU dominate the second one (in the sense that the efficiency ratio of the first DMU is higher than or equal to that of the second for all feasible weights and strictly higher for some weights)?

• How much more/less efficient can a given DMU be relative to some other DMU or, more generally, relative to the most and least efficient DMU in some subset of DMUs?

The first question is partly motivated by the popularity of ranking lists, as exemplified by the ranking of "best" universities by the Shanghai Jiao Tong University (cf. Liu and Cheng 2005, see also Köksalan et al. 2010). The resulting *ranking intervals*—defined by the DMUs' best/worst rankings over all feasible input/output weights—are robust, because the integer-valued bounds of these intervals can change at most by one when a single DMU is introduced or removed. The second question establishes *dominance relations* based on pairwise comparisons between two DMUs at a time. The third question (which is related to superefficiency; see, e.g., Andersen and Petersen 1993) yields *efficiency bounds* that provide information about the relative efficiency differences among the DMUs. All of these results can be employed in the specification of performance targets. With more preference information, the results become usually more conclusive in terms of narrower ranking intervals, additional dominance relations, and tighter efficiency bounds. Furthermore, ratio-based results can be presented even when the number of DMUs is small, because the results are not computed relative to an efficient frontier for the reliable estimation of which the number of DMUs would have to large compared with the number of input and output variables.

The rest of this paper is organized as follows. Section 2 discusses earlier methods for ratio-based efficiency analysis and their applications in selected application domains. Section 3 formulates ratio-based efficiency results, considers their uses in target setting, and contrasts them with cross-efficiency analysis. Section 4 illustrates these results in the context of reported DEA studies and describes a case study where they were employed in the comparison of university departments. Section 5 concludes.

2. DEA Methods and Their Applications

In the DEA literature, there are numerous methods for analyzing the relative efficiencies of DMUs that transform multiple inputs into multiple outputs (see, e.g., Cooper et al. 2007). Early approaches for incorporating preference information in these methods include the specification of assurance regions (Thompson et al. 1990) and cone ratios (Charnes et al. 1990). Subsequently, relationships between DEA models and multicriteria decision-making methods have been explored extensively (Stewart 1996, Joro et al. 1998, Bouyssou 1999). These relationships also underpin the value efficiency analysis method (Halme et al. 1999, Halme and Korhonen 2000, Korhonen et al. 2002), which makes inferences about the DMUs' value efficiencies with the help of an implicit value function. Recent advances include approaches based on the explicit construction of the decision maker's (DM's) value function (Gouveia et al. 2008) and the specification of context-sensitive assurance regions for input/output weights (Cook and Zhu 2008).

Instead of seeking to survey DEA applications (see, e.g., Cooper et al. 2007, Emrouznejada et al. 2008, Avkiran and Parker 2010), we only provide some pointers to selected DEA models in the three decision contexts—higher education, technology management, and health care—for which we provide numerical efficiency results in §4.

First, higher education is an attractive domain for DEA, because universities consume many inputs and produce multiple outputs to which prices may be difficult to attach. As a result, DEA has been employed extensively in higher education by treating universities, departments, research units, or even students as DMUs. For instance, Ahn et al. (1988) analyze the production behavior of higher education institutions and compare the relative efficiencies of public and private doctoral-granting universities in the United States. Johnes (2006) discusses the role of DEA in higher education and analyzes more than 100 higher educational institutions in the United Kingdom. Tauer et al. (2007) examine the efficiencies of the 26 academic departments at Cornell University when specifying performance targets. Korhonen et al. (2001) establish efficiency scores for research units at the Helsinki School of Economics and present an approach for allocating resources to support the attainment of higher aggregate efficiency. Sarrico and Dyson (2000) describe a DEA-based planning tool for the formulation of strategic options at the University of Warwick.

Second, comparative analyses in technology management involve subjective preferences about the inputs that are needed to develop and deploy technologies with the aim of generating desired outputs.

For example, Shafer and Bradford (1995) compare alternative machine group solutions based on DEA efficiencies. Baker and Talluri (1997) provide decision support for screening robots based on cross-efficiency analysis. Talluri and Yoon (2000) evaluate robots using an extended cone-ratio DEA approach. Eilat et al. (2008) integrate DEA models with a balanced scorecard approach and evaluate research and development projects in different stages of their life cycle. Farzipoor (2009) supports technology selection decision by developing a framework that captures preferences through assurance regions and accommodates both cardinal and ordinal information about the DMUs.

Third, DEA models in health care give insights into which DMUs are more efficient than others when health indicators are viewed as outputs and when inputs consist of health-care investments and possibly contextual factors as well. For instance, Garcia et al. (2002) analyze the efficiency of primary health units and explore how sensitive the DEA results are to the selection of output variables. Hollingsworth et al. (1999) review DEA applications in health care with a particular emphasis on the efficiency evaluation of hospitals. In his comprehensive book, Ozcan (2008) discusses uses of DEA models across a broad range of health-care planning problems.

3. Comparative Results for Ratio-Based Efficiency Analysis

3.1. Efficiency Ratios

Assume that there are K DMUs that consume M types of inputs and produce N types of outputs. The kth DMU (DMU$_k$ for short) consumes $x_{mk} \geq 0$ units of the mth input and produces $y_{nk} \geq 0$ units of the nth output. The input consumption and output production vectors are $x_k = (x_{1k}, \ldots, x_{Mk})^T$ and $y_k = (y_{1k}, \ldots, y_{Nk})^T$, respectively.

Preference information about the relative values of inputs and outputs is captured by nonnegative weights $v = (v_1, \ldots, v_M)^T$ and $u = (u_1, \ldots, u_N)^T$, respectively. These weights are assumed to satisfy homogeneous linear constraints (cf. Podinovski 2001, 2005)

$$S_v = \{v = (v_1, \ldots, v_M)^T \neq 0 \mid v \geq 0, A_v v \leq 0\}, \quad (1)$$

$$S_u = \{u = (u_1, \ldots, u_N)^T \neq 0 \mid u \geq 0, A_u u \leq 0\}, \quad (2)$$

where A_v and A_u are coefficient matrices derived from the DM's preference statements about how valuable different amounts of inputs and outputs are. These statements can be elicited with well-known techniques for the specification of assurance regions (see, e.g., Thompson et al. 1986, 1990); for instance, if the DM states that one unit of output 1 is at least as valuable as a unit of output 2 but not more valuable than two units of output 2, then the constraints $u_2 \leq u_1 \leq 2u_2$ must hold. If such statements are elicited from several DMs, a group preference representation for these DMs can be built by forming convex combinations of those weights that satisfy the constraints of some DM (Salo 1995).

For any feasible input weights $v \in S_v$, the virtual input of DMU$_k$ is $v^T x_k = \sum_{m=1}^{M} v_m x_{mk}$. Similarly, the virtual output for $u \in S_u$ is $u^T y_k = \sum_{n=1}^{N} u_n y_{nk}$. We assume that the virtual inputs and the virtual outputs are strictly positive for all feasible weights (i.e., $\sum_m v_m x_{mk} > 0$, $\forall\, v \in S_v$, and $\sum_n u_n y_{nk} > 0$, $\forall\, u \in S_u$, for all $k = 1, \ldots, K$). This assumption holds, for example, if all inputs and outputs have strictly positive weights and if there is at least one input (output) that is consumed (produced) by every DMU. It also holds if all DMUs consume/produce some positive amounts of all inputs/outputs. The assumption of positive virtual inputs/outputs implies that the (absolute) efficiency ratio (cf. Podinovski 2001) of DMU$_k$, defined as

$$E_k(u, v) = \frac{\sum_n u_n y_{nk}}{\sum_m v_m x_{mk}}, \quad (3)$$

is well defined for any $u \in S_u$, $v \in S_v$ (see also Dyson et al. 2001).

3.2. Ranking Intervals

For any feasible input/output weights, the DMUs can be ranked based on their efficiency ratios (3). The resulting rankings can change relative to each other for different weights. We first determine what is the best (highest) efficiency ranking that a DMU can attain relative to other DMUs over the set of input/output weights (1) and (2). For instance, this ranking is three for a DMU if the least number of other DMUs with a strictly higher efficiency ratio is two. Similarly, we compute the worst (lowest) ranking for a DMU. These two bounds establish a *ranking interval*, which conveys information about the relative efficiencies of the DMUs.

Toward this end, we define the sets

$$R_k^{>}(u, v) = \{l \in \{1, \ldots, K\} \mid E_l(u, v) > E_k(u, v)\},$$

$$R_k^{\geq}(u, v) = \{l \in \{1, \ldots, K\} \backslash \{k\} \mid E_l(u, v) \geq E_k(u, v)\},$$

which contain the indexes of those other DMUs whose efficiency ratios are either strictly higher than that of DMU$_k$ (for $R_k^{>}(u, v)$) or at least as high as that of DMU$_k$ (for $R_k^{\geq}(u, v)$). By construction, $R_k^{>}(u, v) \subseteq R_k^{\geq}(u, v)$.

The corresponding efficiency rankings are defined as $r_k^{>}(u, v) = 1 + |R_k^{>}(u, v)|$ and $r_k^{\geq}(u, v) = 1 + |R_k^{\geq}(u, v)|$ (here, $|R|$ denotes the cardinality of the set R). For example, if the efficiency ratio of DMU$_k$ is strictly higher than the efficiency ratios of all other DMUs for some $(u, v) \in (S_u, S_v)$, then $r_k^{>}(u, v)$ and $r_k^{\geq}(u, v)$ equal one, because $R_k^{>}(u, v) = R_k^{\geq}(u, v) = \varnothing$. Yet these rankings treat ties differently: if exactly two DMUs have same highest efficiency ratio at $(u', v') \in (S_u, S_v)$, then $r^{>}(u', v')$ ranks them both as first, but $r^{\geq}(u', v')$ ranks them as second.

The *ranking interval* for DMU_k is now defined as $[r_k^{\min}, r_k^{\max}]$, where the best and worst rankings for DMU_k are given by

$$r_k^{\min} = \min_{u,v} r_k^{>}(u, v),$$

$$r_k^{\max} = \max_{u,v} r_k^{\geq}(u, v),$$

and where the optimization problems are solved over $(u, v) \in (S_u, S_v)$. Both optimum solutions exist, because $r_k^{>}(u, v)$ and $r_k^{\geq}(u, v)$ assume values in the set $\{1, \ldots, K\}$.

Based on Theorems 1 and 2, the ranking interval $[r_k^{\min}, r_k^{\max}]$ can be determined from mixed integer linear programming problems where the weight sets are closed and bounded by constraints (5) and (7), respectively. In these and also later theorems, C denotes a large positive constant. The proofs are in the appendix.

If DMU_k is CCR-DEA efficient, then for some feasible weights its efficiency ratio is higher than or equal to the efficiency ratio of any other DMU, and thus its best ranking in Theorem 1 will be one.

THEOREM 1. *The optimum of the minimization problem*

$$\min_{u,v,z} \; 1 + \sum_{l \neq k} z_l$$

$$\text{subject to } \sum_n u_n y_{nl} \leq \sum_m v_m x_{ml} + C z_l, \quad l \in \{1, \ldots, K\}, \quad l \neq k, \tag{4}$$

$$\sum_n u_n y_{nk} = \sum_m v_m x_{mk} = 1, \tag{5}$$

$$z_l \in \{0, 1\}, \quad l \neq k,$$

$$(u, v) \in (S_u, S_v)$$

is $r_k^{\min}$, the best (highest) efficiency ranking of DMU_k.

THEOREM 2. *The optimum of the maximization problem*

$$\max_{u,v,z} \; 1 + \sum_{l \neq k} z_l$$

$$\text{subject to } \sum_m v_m x_{ml} \leq \sum_n u_n y_{nl} + C(1 - z_l), \quad l \in \{1, \ldots, K\}, \quad l \neq k, \tag{6}$$

$$\sum_n u_n y_{nk} = \sum_m v_m x_{mk} = 1, \tag{7}$$

$$z_l \in \{0, 1\}, \quad l \neq k,$$

$$(u, v) \in (S_u, S_v)$$

is the $r_k^{\max}$, the worst (lowest) efficiency ranking of DMU_k.

With the introduction of additional preference information, the constraints on the feasible input/output weights become tighter. In view of Theorems 1 and 2, such information may lead to narrower (but not wider) ranking intervals.

In general, DMUs that are outliers in the sense that their input/output profiles differ considerably from what is consumed/produced by most DMUs are likely to have wider ranking intervals. This is because these outlier DMUs can have either good (high) or bad (low) rankings at the extreme points of S_u and S_v. Conversely, DMUs whose profiles are more typical are likely to have narrower ranking intervals.

3.3. Efficiency Dominance

Although ranking intervals provide information about the relative efficiencies of the DMUs, they are not well suited for the comparison of *pairs* of DMUs. Specifically, even if two DMUs have overlapping ranking intervals, it is possible that one of them has a higher efficiency ratio (3) for all feasible input/output weights.

To compare the efficiency ratios of DMUs on a one-on-one basis, we build on concepts from preference programming (see, e.g., Salo and Hämäläinen 1992, 2001) and define *efficiency dominance* between DMUs as follows.

DEFINITION 1. DMU_k dominates DMU_l (denoted by $\mathrm{DMU}_k \succ \mathrm{DMU}_l$) if and only if

$$E_k(u, v) \geq E_l(u, v) \quad \text{for all } (u, v) \in (S_u, S_v), \tag{8}$$

$$E_k(u, v) > E_l(u, v) \quad \text{for some } (u, v) \in (S_u, S_v). \tag{9}$$

If $\mathrm{DMU}_k \succ \mathrm{DMU}_l$, the efficiency ratio of DMU_k is at least as high as that of DMU_l for all feasible weights, and moreover, there exist some weights for which its efficiency is strictly higher. By construction, Definition 1 establishes a strict partial order that is an irreflexive, asymmetric, and transitive binary relation among the DMUs. This relation, however, may not be total (i.e., it may be that neither $\mathrm{DMU}_k \succ \mathrm{DMU}_l$ nor $\mathrm{DMU}_l \succ \mathrm{DMU}_k$).

The dominance relation in Definition 1 can be determined based on the pairwise efficiency ratio

$$D_{k,l}(u, v) = \frac{E_k(u, v)}{E_l(u, v)}. \tag{10}$$

By Lemma 1, this ratio is invariant subject to multiplication of input/output weights by positive constants.

LEMMA 1. *Take any $(u, v) \in (S_u, S_v)$, and let (u', v') be vectors that are obtained from (u, v) by multiplying them componentwise so that $u' = c_u u$, $v' = c_v v$ for some $c_u > 0$, $c_v > 0$. Then, $(u', v') \in (S_u, S_v)$ and $D_{k,l}(u, v) = D_{k,l}(u', v')$.*

In view of Lemma 1, the ratio (10) remains invariant even if weights are normalized through constraints such as $\sum_n u_n = 1$ and $\sum_m v_m = 1$. After the introduction of such constraints, the feasible sets S_u and S_v become closed and bounded. Because the ratio $D_{k,l}(u,v)$ is continuous in input/output weights, it therefore achieves its maximum and minimum values, denoted by $\overline{D}_{k,l}$ and $\underline{D}_{k,l}$, respectively.

The relative efficiency ratio (10) is nonlinear in weights (u, v). Yet, by Theorem 3, this ratio can be maximized and minimized through linear programming.

Theorem 3. *The optimum of the maximization* (*minimization*) *problem*

$$\max_{u,v} \; (\min_{u,v}) \; \sum_n u_n y_{nk} \tag{11}$$

$$\text{subject to } \sum_n u_n y_{nl} = \sum_m v_m x_{ml}, \tag{12}$$

$$\sum_m v_m x_{mk} = 1, \tag{13}$$

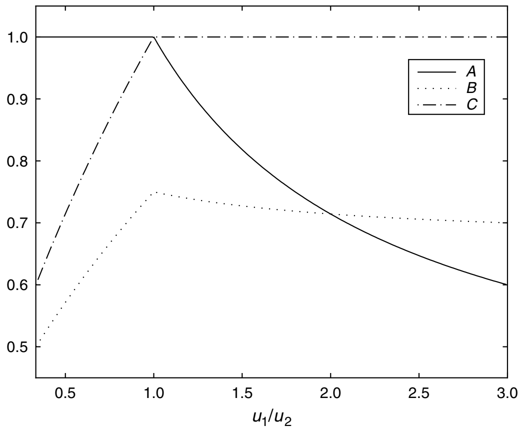
$$(u, v) \in (S_u, S_v) \tag{14}$$

is the maximum (*minimum*) *of* $D_{k,l}(u,v)$ *over* (S_u, S_v).

The optimization problems in Theorem 3 provide upper and lower bounds on how efficient DMU_k can be relative to DMU_l across feasible weights. For example, if $\overline{D}_{k,l} = 1.42$, the efficiency ratio of DMU_k can be *at most* 42% greater than that of DMU_l. Conversely, if $\underline{D}_{k,l} = 1.10$, the efficiency ratio of DMU_k is *at least* 10% higher than that of DMU_l. Thanks to Theorem 3, the dominance structure can be computed efficiently with linear programming. First, if the minimum $\underline{D}_{k,l}$ is greater than one, DMU_k dominates DMU_l. Second, if it is less than one, (8) is violated and dominance does not hold. Third, if the minimum is exactly one, the sufficiency condition (9) can be checked by maximizing (11) subject to (12)–(14). If the resulting maximum $\overline{D}_{k,l}$ exceeds one, dominance does hold; but if not, then DMU_k and DMU_l have the same efficiency ratio (3) for all feasible weights, and there is no dominance. Also, the transitivity and asymmetricity properties of $\succ$ can be exploited to further reduce the number of pairs for which the dominance relation must be explicitly computed.

A DMU is not necessarily dominated by another DMU that has a higher CCR-DEA efficiency score. For example, consider three DMUs, A, B, and C, that all consume one unit of a single input and produce two outputs so that $A = (1, 3)$, $B = (2, 1)$, and $C = (3, 1)$. For the weight information $1/3u_1 \le u_2 \le 3u_1$, there are two CCR-DEA-efficient DMUs, A and C. Yet, for feasible output weights such that $u_1/u_2 > 2$, the virtual output of DMU_B is higher than that of DMU_A so that DMU_A does not dominate DMU_B. Figure 1

Figure 1 The Efficiency Ratios for DMUs A, B, and C



shows how the DMUs' efficiency ratios change relative to each other for feasible output weights when the highest efficiency ratio is normalized to one. For example, the efficiency ratio of DMU_C is higher than that of DMU_B for all weights so that DMU_C dominates DMU_B. The ranking intervals are $[1, 3]$ for A, $[2, 3]$ for B, and $[1, 2]$ for C.

With the introduction of additional preference information, new dominance relations are often established. Furthermore, existing dominance relations are preserved, except in the unlikely case where both DMUs have the same efficiency ratio for all weights in the revised feasible set.

3.4. Efficiency Bounds

The analysis of relative efficiencies can be extended to situations where the efficiency DMU_k is benchmarked simultaneously with a group $\text{DMU}_L = \{\text{DMU}_l \mid l \in L \subseteq \{1, \ldots, K\}\}$ consisting of several DMUs. In this case, the ratios

$$D_{k,\bar{L}}(u,v) = \frac{E_k(u,v)}{\max_{l\in L} E_l(u,v)} = \min_{l\in L} \frac{E_k(u,v)}{E_l(u,v)}, \tag{15}$$

$$D_{k,\underline{L}}(u,v) = \frac{E_k(u,v)}{\min_{l\in L} E_l(u,v)} = \max_{l\in L} \frac{E_k(u,v)}{E_l(u,v)} \tag{16}$$

indicate how efficient DMU_k is relative to the most and least efficient DMUs in the benchmark group for different input/output weights. The maximum and minimum values of (15) over feasible weights are denoted by $\overline{D}_{k,\bar{L}}$ and $\underline{D}_{k,\bar{L}}$, whereas $\overline{D}_{k,\underline{L}}$ and $\underline{D}_{k,\underline{L}}$ are the corresponding maximum and minimum values of (16). By Theorems 4 and 5, these bounds can be solved with linear programming. It is worth noting that if the benchmark set L contains all DMUs, then $\overline{D}_{k,\bar{L}}$ is equal to the CCR-DEA score. If DMU_k is not contained in the benchmark set L, the maximum $\overline{D}_{k,\bar{L}}$

gives the superefficiency of DMU_k relative to this set of DMUs (see, e.g., Zhu 1996).

Theorem 4. $\underline{D}_{k,\bar{L}} = \min_{l\in L} \underline{D}_{k,l}$. *$\bar{D}_{k,\bar{L}}$ is the optimum of the maximization problem*

$$\max_{u,v} \sum_n u_n y_{nk} \tag{17}$$

$$\text{subject to } \sum_n u_n y_{nl} \le \sum_m v_m x_{ml}, \quad l \in L, \tag{18}$$

$$\sum_n v_m x_{mk} = 1,$$

$$(u, v) \in (S_u, S_v).$$

Theorem 5. $\bar{D}_{k,\underline{L}} = \max_{l\in L} \bar{D}_{k,l}$. *$\underline{D}_{k,\underline{L}}$ is the optimum of the minimization problem*

$$\min_{u,v} \sum_n u_n y_{nk} \tag{19}$$

$$\text{subject to } \sum_n u_n y_{nl} \ge \sum_m v_m x_{ml}, \quad l \in L, \tag{20}$$

$$\sum_n v_m x_{mk} = 1,$$

$$(u, v) \in (S_u, S_v).$$

3.5. Specification of Performance Targets

All of the above results can be employed to specify performance targets. For example, one can introduce targets such that DMU_k will be among (i) the R_k^* most efficient DMUs for *some* feasible weights or (ii) the R_k° most efficient DMUs for *all* feasible weights. These two cases are addressed by Theorems 6 and 7 for the case where efficiency improvements are sought through radial increases in output production.

Theorem 6. *Assume that $R_k^* < r_k^{\min}$. Then, the maximization problem*

$$\max_{u,v,z} \sum_n u_n y_{nk}$$

$$\text{subject to } 1 + \sum_{l\ne k} z_l \le R_k^*, \tag{21}$$

$$\sum_n u_n y_{nl} \le \sum_m v_m x_{ml} + C z_l, \quad l \ne k, \tag{22}$$

$$\sum_m v_m x_{mk} = 1,$$

$$z_l \in \{0, 1\}, \quad l \ne k,$$

$$(u, v) \in (S_u, S_v)$$

has an optimum $\zeta^* < 1$ such that $\rho^* = 1/\zeta^*$ gives the least radial output increase for which the best ranking of DMU_k is R_k^* or better.

Theorem 7. *Assume that $R_k^\circ < r_k^{\max}$. Then, the minimization problem*

$$\min_{u,v,z} \sum_n u_n y_{nk}$$

$$\text{subject to } 1 + \sum_{l\ne k} z_l \le K - R_k^\circ, \tag{23}$$

$$\sum_m v_m x_{ml} \le \sum_n u_n y_{nl} + C z_l, \quad l \ne k, \tag{24}$$

$$\sum_m v_m x_{mk} = 1,$$

$$z_l \in \{0, 1\}, \quad l \ne k,$$

$$(u, v) \in (S_u, S_v)$$

has an optimum $\zeta^* \le 1$ such that $\rho^* = 1/\zeta^*$ is the infimum of those radial output increases for which the worst ranking of DMU_k is R_k° or better.

Differences in Theorems 6 and 7 reflect asymmetric discontinuities when rankings improve. For instance, let there be three DMUs, A, B, and C, which all consume one unit of a single input and produce $y_A = 1$ and $y_B = y_C = 2$ units of a single output. Then, when A doubles its production, its best possible ranking jumps to one, but its worst possible ranking remains three until its production is strictly greater than two. Neither the best nor the worst ranking of A will be *exactly* two, no matter how much it increases its production.

An increase in the production of outputs by a factor of $\rho > 1$ corresponds to a decrease in the use of inputs by a factor of $1/\rho < 1$, because

$$\frac{\sum_n u_n[\rho y_{nk}]}{\sum_m v_m x_{mk}} = \frac{\sum_n u_n y_{nk}}{\sum_m v_m[1/\rho]x_{mk}}.$$

Thus, radial output targets can be translated into corresponding requirements on the input side. Similarly, the overall target ρ^* can be factored into a radial output target ρ_u and a radial input target ρ_v such that $\rho_u\rho_v = \rho^*$, $y'_{nk} = \rho_u y_{nk}$, and $x'_{mk} = [1/\rho_v]x_{mk}$.

Dominance relations, too, can be employed in target setting. For example, one may ask by how much a DMU_k that *does not* dominate DMU_l should increase its output to reach the threshold level beyond which it starts to dominate DMU_l. Based on $\underline{D}_{k,l} \le 1$ in Theorem 3, DMU_k achieves the efficiency level of DMU_l for all weights when it increases its production by $\rho_l^* = 1/\underline{D}_{k,l}$. If the target is to ensure that DMU_k begins to dominate several DMUs contained in the index set L, the threshold level for the required increase is $\rho^* = \max_{l\in L} \rho_l^*$. One may also ask by how much DMU_k that *is* dominated by DMU_l needs to increase its production so as not to be dominated. In this case, $1 \le \underline{D}_{l,k}$, and the threshold level for the required increase is $\rho_l^* = \underline{D}_{l,k}$. Even bounds for benchmark sets in §3.4 can be used in target specification.

3.6. Comparisons with Cross-Efficiency Analysis

In cross-efficiency analysis, every DMU is assigned a single CE score using those weights for which the efficiency of some DMU is maximized. By design, this approach recognizes that different weights are relevant in efficiency evaluation, in contrast to the standard CCR-DEA approach where the efficiency score for a DMU is determined using only those weights that are most favorable to it (see, e.g., Doyle and Green 1994).

Specifically, the DMUs' cross-efficiencies are computed from a square matrix $\theta_{l,k}$, $k, l = 1, \ldots, K$, whose lth row $\theta_{l,k} = [E_1(u^l, v^l), \ldots, E_K(u^l, v^l)]$ contains the efficiencies of DMUs with weights (u^l, v^l), which maximize the efficiency of DMU_l subject to the constraint that the maximum efficiency of any DMU is one. If there are multiple optima, alternative rules may be applied in weight selection. In the aggressive formulation, for example, weights are chosen by minimizing the relative efficiency of the aggregate DMU, which is formed by summing the inputs and outputs of all the other DMUs. In the benevolent formulation, the relative efficiency of the same aggregate DMU is maximized. Once the weights (u^l, v^l), $l = 1, \ldots, K$ have been chosen, the cross-efficiency of DMU_k is computed as

$$CE_k = \frac{1}{K}\sum_{l=1}^{K}\theta_{l,k} = \frac{1}{K}\sum_{l=1}^{K}E_k(u^l, v^l). \tag{25}$$

DMUs can be ranked based on their cross-efficiencies. If there are no ties, DMU_k has a unique CE ranking that is equal to one plus the number of those DMUs that have a strictly higher cross-efficiency, i.e., $r_k^{CE,>} = 1 + |\{CE_l \mid CE_l > CE_k\}|$. In the case of ties, the CE ranking can drop to $r_k^{CE,\geq} = 1 + |\{CE_l \mid CE_l \geq CE_k, l \neq k\}|$ if DMU_k is assigned the worst ranking among all the DMUs that have the same cross-efficiency. Except for the possibility of ties, a major difference between CE rankings and ranking intervals in §3.2 is that CE rankings typically assign a single ranking to each DMU. In contrast, ranking intervals show *all* the rankings that DMUs can attain across the full set of feasible input/output weights.

We draw attention to three concerns with cross-efficiency analysis. First, the CE rankings of any two DMUs may depend on what *other* DMUs are included in the analysis. Indeed, Theorem 8 shows that whenever there are two DMUs that do not dominate each other and whose efficiency ratios differ for some input/output weights, then it is possible to introduce additional DMUs so that the CE ranking of the first DMU becomes better than that of the second. By Theorem 9, the nondominance assumption is necessary so that a DMU that dominates some other DMU will have a higher CE ranking than the DMU that it dominates.

Theorem 8. *Assume that* $\text{DMU}_k \not\succ \text{DMU}_l$ *and* $\text{DMU}_l \not\succ \text{DMU}_k$ *and* $\exists(u, v)$ *such that* $E_l(u, v) \neq E_k(u, v)$. *There then exist* DMU_i, $i = K+1, \ldots, K+K'$ *such that* $CE_k > CE_l$ *in the augmented set* $\{\text{DMU}_i \mid i = 1, \ldots, K+K'\}$.

Theorem 9. *If* $\text{DMU}_k \succ \text{DMU}_l$, *then* $CE_k \geq CE_l$.

The phenomenon addressed by Theorem 8 is problematic because it means that cross-efficiency analyses are, in principle, susceptible to purposeful manipulation where the relative CE ranking of a nondominated DMU is altered by introducing appropriately chosen DMUs. Here, there are parallels to the contested rank reversal phenomenon where the introduction of a new alternative to a multicriteria decision problem changes the relative rankings of previously analyzed alternatives. Rank reversals have aroused plenty of controversy, and, for instance, they have been widely regarded as a shortcoming of the analytic hierarchy process (Belton and Gear 1983; see also Dyer 1990, Salo and Hämäläinen 1997).

A second concern is that the inequality in Theorem 9 may not be strict, meaning that a dominated DMU may have as high a CE score as the DMU it is dominated by. For example, consider three DMUs, A, B, and C, that consume one unit of a single input and produce three outputs according to profiles $A = (3, 3, 2)$, $B = (3, 2, 2)$, and $C = (0, 2, 3)$. Clearly, A dominates B. If there are no constraints on output weights, the efficiencies of DMUs B and C are maximized for weights $u^B = (1/3, 0, 0)$ and $u^C = (0, 0, 1/3)$, whereas DMU_A achieves its maximum efficiency for all convex combinations of weights $u^{A,1} = (1/3, 0, 0)$ and $u^{A,2} = (0, 1/3, 0)$. If the selection among the alternative optima is based on the aggressive formulation, the value of the aggregate output vector $u^{B+C} = (3+0, 2+2, 2+3) = (3, 4, 5)$ is minimized using output weights $u^{A,1} = (1/3, 0, 0)$. In this case, the case the cross-efficiency matrix becomes

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 2/3 & 2/3 & 1 \end{pmatrix}.$$

Here, the three rows contain the DMUs' efficiencies for weights $u^{A,1}$, u^B, and u^C, respectively. The DMUs' cross-efficiencies (25) are now obtained as column averages $[1+1+2/3]/3 = 8/9 = CE_A = CE_B$ and $CE_C = [0+0+1]/3 = 1/3 = CE_C$, which show that DMUs A and B have the same cross-efficiency although DMU_A dominates DMU_B. An analogous conclusion can be reached for the benevolent formulation by replacing the output vector of DMU_C by $C' = (0, 0, 3)$ and by choosing the output weights $u^{A,1} = (1/3, 0, 0)$ to maximize the virtual value of the aggregate output vector $u^{B+C'} = (3, 2, 5)$.

A third concern is that the CE ranking of a DMU may lie outside the ranking interval $[r_k^{\min}, r_k^{\max}]$. For example, consider three DMUs, A, B, and C, that produce a single output, $y_A = 26$, $y_B = 19$, and $y_C = 16$, and consume two inputs, $x_A = (13, 9)$, $x_B = (7, 9)$, and $x_C = (16, 1)$. If there are no weight constraints, all three DMUs are efficient and achieve a DEA efficiency of one for weights $u^k = 1/y_k, k = A, B, C$ and $v^A = (1/26, 1/18)$, $v^B = (1/7, 0)$, and $v^C = (0, 1)$, respectively. With these weights, the cross-efficiency matrix becomes

$$\begin{pmatrix} 1 & \frac{19}{20} & \frac{144}{157} \\ \frac{14}{19} & 1 & \frac{7}{19} \\ \frac{13}{72} & \frac{19}{144} & 1 \end{pmatrix},$$

which yields the cross-efficiencies $CE_A \approx 0.639 < CE_B \approx 0.694 < CE_C \approx 0.762$. Thus, DMU_A has the worst CE ranking.

However, DMU_A has the smallest efficiency ratio only for weights (u, v) such that

$$\begin{aligned} E_A(u,v) \le E_C(u,v) &\iff \frac{26u}{13v_1 + 9v_2} \le \frac{16u}{16v_1 + v_2} \\ &\iff -\frac{1}{2}v_1 + \frac{59}{208}v_2 \ge 0, \quad \text{and} \\ E_A(u,v) \le E_B(u,v) &\iff \frac{26u}{13v_1 + 9v_2} \le \frac{19u}{7v_1 + 9v_2} \\ &\iff \frac{5}{38}v_1 - \frac{63}{494}v_2 \ge 0. \end{aligned}$$

Multiplying the first inequality by 5/19 and summing up the inequalities gives $-(11/208)v_2 \ge 0$, which implies that v_2 must be zero. But then the first inequality gives $v_1 \le 0$, violating the assumption that $v_1 + v_2 > 0$. This proves that DMU_A has the worst CE ranking among the three DMUs although it is either the most or the second most efficient DMU for *all* feasible weights.

4. Applications of Ratio-Based Efficiency Measures

We next illustrate uses of ratio-based efficiency measures by revisiting two reported studies and also by describing a real case study where they supplied useful insights.

4.1. Ratio-Based Efficiency Results for the Evaluation of Robots

Baker and Talluri (1997) present an efficiency model for screening 27 robots using velocity and load capacity as outputs and cost and repeatability as inputs. They do not elicit preference information about the relative values of these input/output variables.

In Table 1, the CCR-DEA score of a robot is in the second column, followed by its best and worst efficiency rankings, a list of those robots it is dominated by, and lower and upper bounds for the robot's efficiency ratio relative to the highest efficiency ratio among all *other* robots over the set of feasible weights. Here, the worst efficiency ranking $r^{\max}$

Table 1 Efficiency Results for the Comparison of Robots

Robot	Eff.	$r^{\min}$	$r^{\max}$	Dominated by	$[\underline{D}_{k,\bar{l}}, \overline{D}_{k,\bar{l}}]$	CE	FPI(%)
1	1	1	21	—	[0.038, 1.012]	0.58	72.41
2	0.90	3	24	14	[0.024, 0.904]	0.48	88.28
3	0.53	7	23	11, 15, 19	[0.038, 0.529]	0.30	76.28
4	1	1	27	—	[0.004, 1.100]	0.31	222.58
5	0.59	3	27	1, 14, 19	[0.001, 0.592]	0.19	211.76
6	0.48	11	25	7, 8, 10, 13, 14, 19, 23, 24	[0.017, 0.482]	0.28	72.28
7	1	1	17	—	[0.055, 1.322]	0.70	42.86
8	0.78	5	15	—	[0.063, 0.783]	0.56	39.74
9	0.38	11	25	1, 7, 8, 10, 13, 14, 19	[0.029, 0.378]	0.27	40.14
10	1	1	17	—	[0.049, 1.043]	0.70	42.86
11	0.67	3	19	19	[0.063, 0.671]	0.42	59.84
12	0.10	18	27	1, 3, 7, 8, 10, 11, 13, 14, 15, 16, 19, 23, 25, 26, 27	[0.004, 0.102]	0.06	70.61
13	1	1	15	—	[0.061, 1.091]	0.73	36.99
14	1	1	13	—	[0.060, 1.769]	0.82	21.95
15	0.61	3	22	—	[0.038, 0.613]	0.36	70.14
16	0.60	3	24	—	[0.029, 0.604]	0.34	77.50
17	0.40	17	26	3, 7, 8, 10, 11, 13, 14, 15, 19, 23, 25	[0.013, 0.405]	0.19	112.92
18	0.37	12	25	1, 7, 8, 10, 13, 14, 19, 25	[0.031, 0.365]	0.26	40.47
19	1	1	10	—	[0.064, 1.021]	0.66	51.52
20	1	1	27	—	[0.001, 8.265]	0.34	194.12
21	0.85	2	25	—	[0.023, 0.852]	0.34	150.45
22	0.83	4	26	10, 13, 14	[0.005, 0.829]	0.46	80.19
23	0.69	3	22	7, 10	[0.039, 0.694]	0.44	57.79
24	0.64	5	22	7, 10, 13, 23	[0.036, 0.636]	0.41	55.15
25	0.55	10	18	7, 8, 13, 14, 19	[0.054, 0.553]	0.38	45.62
26	0.58	2	22	—	[0.037, 0.581]	0.36	61.40
27	1	1	25	—	[0.014, 3.880]	0.59	69.49

and the bounds $\underline{D}_{k,\bar{L}}$ show that for some weights the efficiency ratios of even CCR-DEA-efficient robots are quite low relative to the other robots. Moreover, the bounds $\overline{D}_{k,\bar{L}}$ show that CCR-DEA-efficient robots are superefficient, meaning that for any one of them it is possible to find feasible weights such that its efficiency ratio is strictly higher than that of all other robots. For example, the efficiency ratio of robot 4 can be 1.1 times as high as the maximum efficiency ratio of other robots.

The last two columns show the robots' cross-efficiency and so-called false positive index (FPI). The FPI index (Baker and Talluri 1997) is an indicator of how much the efficiency of the robot improves when its efficiency ratio is evaluated using weights that are most favorable to it rather than using also weights that favor other robots. Thus, the smaller the FPI, the less sensitive the efficiency of a robot is to the selection of weights.

There are 13 dominated robots that can be eliminated. Among the remaining 14 nondominated robots, 4 and 20 can be the least efficient of all for some weights, although they are efficient in the CCR-DEA sense. In the same vein, robots 1, 15, 16, 21, 26, and 27—which have large FPI values in excess of 60%—can be among the seven least efficient robots for some weights. Robots 7, 8, 10, 13, 14, and 19, in contrast, are more robust and belong to the 17 most efficient robots for all weights; they also have low FPI values below 50%. Robots 14 and 19 have the best ranking intervals. Robot 14 has a higher superefficiency value 1.769, whereas the ranking of 19 is never below 10. In this way, dominance structures and ranking intervals help identify nondominated DMUs like robots 14 and 19 that are more efficient than others across a broad range of weights. These results complement cross-efficiencies and FPI indices, yet they are based on a rigorous dominance concept and, in particular, the consideration of *all* feasible weights instead of only those weights for which the efficiency of some DMU is maximized.

4.2. Efficiency Comparison of Hospitals

Here, we revisit the example of Cooper et al. (2007, p. 155) with 14 hospitals whose inputs consist of nurses (x_1) and doctors (x_2), and whose outputs are outpatients (y_1) and inpatients (y_2). In the first phase, there is no preference information about the relative values of these variables. In the second phase, assurance regions for weights are introduced by stating that (i) neither input can be more than five times as valuable as the other and that (ii) neither output can be more than five times as valuable as the other. These statements correspond to the constraints $0.2v_1 \le v_2 \le 5v_1$ and $0.2u_1 \le u_2 \le 5u_2$.

Table 2 shows how the efficiency results change because of this preference information. Initially, hospitals H2, H3, H6, H8, and H10 are efficient in view of their CCR-DEA scores. DMU H8 becomes dominated when preference information is introduced. In view of Table 2, DMU H10 appears more efficient than others on several accounts, because (i) it is among the three most efficient hospitals for all feasible weights, (ii) all the dominated DMUs are dominated by it, (iii) it has the highest superefficiency ($\overline{D}_{k,\bar{L}} = 1.04$, i.e., for some weights it is up to 4% more efficient than the next most efficient DMU), and (iv) the bound $\underline{D}_{k,\bar{L}} = 0.98$ means its efficiency ratio is for all weights at least 98% of the highest efficiency ratio among all DMUs.

4.3. A Case Study on the Comparison of University Departments

This case study was carried out at a large technical university consisting of 12 departments responsible for research activities and educational degree programmes. The impetus for the study came from the board, which asked the resource committee of the university to consider alternative models for efficiency analysis and resource allocation.

The outputs consisted of three-year departmental averages in the university's reporting system that contained 44 outputs, structured under seven classes (degrees and credits awarded, international publications, domestic publications, international mobility of staff, other international scientific activities, other domestic scientific activities, and student exchanges). Statements about the relative values of these outputs were elicited from 10 members of the resource committee using a spreadsheet tool. First, in each output class, 10 points were associated with a reference output (e.g., an MS degree), whereafter each respondent was asked to assign points to the other outputs in the same class. For instance, by giving 80 points to a PhD degree the respondent could state that a single PhD degree is as valuable as eight MS degrees. Second, the respondent was asked to provide statements about the values of these seven reference outputs through similar point allocations. From these statements, the corresponding vector of normalized weights was derived for every respondent. The feasible output weights consisted of convex combinations of these weights, and thus contained the viewpoints of all respondents.[1]

The two input variables were basic funding, which is provided by the government and allocated to the departments by the rector, and external funding, which is acquired by research groups from external sources. Only these two inputs were chosen because most other inputs (e.g., annual person-years, office space) are ultimately financed through these two

[1] The original data are available from the authors upon request.

Table 2 **Results for Hospitals H1–H14 Without Preference Information (First Row) and With Preference Information** $0.2 \le u_1/u_2, v_1/v_2 \le 5$ **(Second Row)**

DMU	Eff.	$r^{\min}$	$r^{\max}$	Dominated by	$[\underline{D}_{k,\bar{L}}, \overline{D}_{k,\bar{L}}]$	$[\underline{D}_{k,\underline{L}}, \overline{D}_{k,\underline{L}}]$
H1	0.95	3	13	6	[0.75, 0.95]	[1.20, 2.56]
	0.93	6	11	2, 3, 6, 9, 10	[0.78, 0.93]	[1.32, 2.15]
H2	1	1	10	—	[0.80, 1.06]	[1.21, 2.91]
	1	1	7	—	[0.83, 1.02]	[1.36, 2.41]
H3	1	1	5	—	[0.91, 1.02]	[1.45, 2.72]
	1	1	4	—	[0.95, 1.01]	[1.58, 2.36]
H4	0.7	11	14	1, 2, 3, 6, 7, 8, 9, 10, 11, 12	[0.34, 0.70]	[0.71, 1.27]
	0.63	14	14	1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14	[0.40, 0.63]	[0.74, 0.92]
H5	0.83	7	13	3, 6, 9, 10	[0.59, 0.83]	[1.01, 2.37]
	0.82	7	12	2, 3, 6, 9, 10	[0.73, 0.82]	[1.16, 2.03]
H6	1	1	6	—	[0.91, 1.08]	[1.44, 2.67]
	1	1	5	—	[0.91, 1.00]	[1.55, 2.30]
H7	0.84	7	12	3, 6, 10, 12	[0.56, 0.84]	[1.22, 1.64]
	0.8	10	12	1, 2, 3, 6, 8, 9, 10, 11, 12	[0.61, 0.80]	[1.27, 1.53]
H8	1	1	12	—	[0.66, 1.00]	[1.25, 2.21]
	0.87	6	11	3, 6, 9, 10	[0.69, 0.87]	[1.31, 1.81]
H9	0.99	2	10	10	[0.79, 0.99]	[1.26, 2.87]
	0.98	2	5	10	[0.90, 0.98]	[1.42, 2.44]
H10	1	1	6	—	[0.88, 1.04]	[1.41, 2.88]
	1	1	3	—	[0.98, 1.04]	[1.56, 2.48]
H11	0.91	5	11	3, 6	[0.69, 0.91]	[1.26, 2.08]
	0.85	8	10	2, 3, 6, 9, 10, 12	[0.71, 0.85]	[1.32, 1.79]
H12	0.97	3	10	3	[0.70, 0.97]	[1.40, 2.03]
	0.93	4	9	3, 6, 10	[0.75, 0.93]	[1.47, 1.86]
H13	0.79	10	14	2, 3, 6, 7, 9, 10, 12	[0.38, 0.79]	[0.95, 1.40]
	0.74	13	13	1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 14	[0.50, 0.74]	[1.09, 1.35]
H14	0.97	3	14	3, 10	[0.43, 0.97]	[0.90, 2.26]
	0.93	4	12	3, 9, 10	[0.63, 0.93]	[1.16, 2.00]

sources. Because the management of external funding involves more work, and because such funding places constraints on its use, the respondents were asked to state how much more "valuable" basic funding is compared with external funding. Most respondents noted that basic funding is 1.25–2.00 times as valuable as funding from external sources (e.g., the value of $100,000 of basic funding would be the same as that of $125,000–$200,000 of external funding).

Based on Theorem 4, the efficiency bounds in Figure 2 indicate the ranges within which the departments' efficiency ratios vary relative to the highest

Figure 2 **Efficiency Intervals for the 12 Departments**

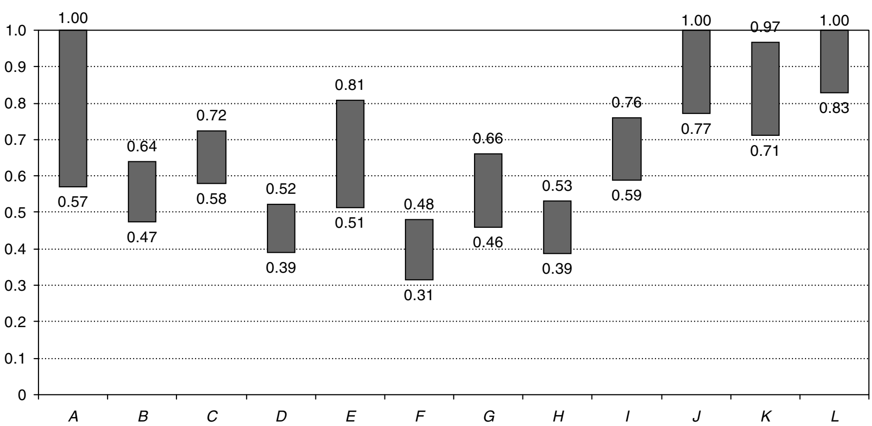
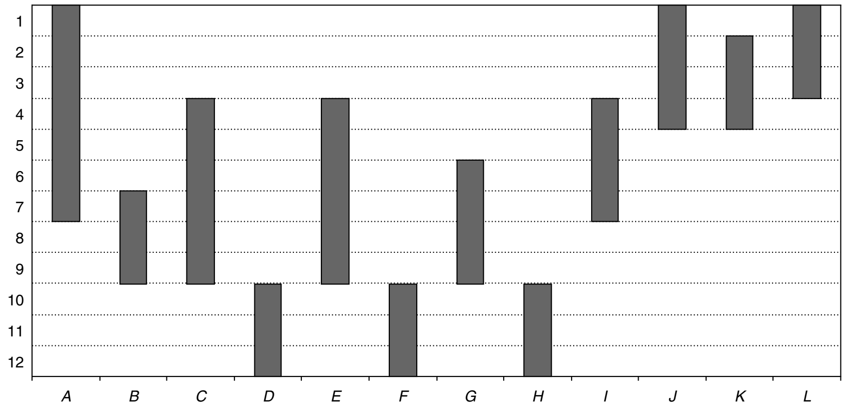


Figure 3 **Best and Worst Efficiency Rankings for the 12 Departments**



efficiency ratio among all departments. Specifically, the upper bounds are the usual CCR-DEA efficiency scores according to which there are three efficient departments (A, J, and L), followed by the "nearly" efficient department K (with an efficiency score of 0.97), then five departments with efficiency scores in the range 0.60–0.90, and, finally, three inefficient departments with scores less than 0.60. The lower bounds show how low the departments' efficiency ratios can be relative to the highest efficiency ratio over the set of feasible weights. Thus, for instance, the efficiency ratio of department L is for all weights at least 83% of the efficiency ratio of the most efficient department.

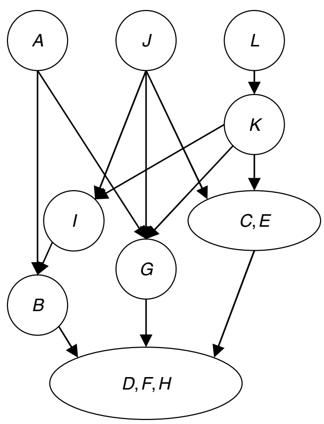
The ranking intervals in Figure 3 complement efficiency bounds. For instance, department L is among the three most efficient departments for all feasible weights, whereas J and the CCR-DEA-inefficient department K are among the four most efficient ones. Department A is efficient, but its ranking drops to 7 for some weights, indicating that its efficiency is sensitive to what input/output weights are employed. Departments D, F, and H are the three least efficient ones. Their ranking intervals show, for instance, that for all weights these three departments are less efficient than department G, although their efficiency intervals overlap with that of G.

Dominance relations are shown in Figure 4, where department X dominates Y if and only if there is a directed path from X to Y. Thus, department L dominates K, but K is not dominated by departments A and J. Also, A does not dominate I, meaning that for some weights the efficiency ratio of I is higher than that of A even though its CCR-DEA efficiency is lower than that of A. Moreover, department A dominates fewer departments (5) than K (8), which also indicates that the relative efficiency of A is more sensitive to the choice of weights. Departments D, F, and H do not dominate each other, but they are dominated by all other departments.

The results in §3.5 can be applied to specify performance targets. First, consider the three "midtier" departments C, E, and I, whose rankings are in the range from the fourth to the ninth most efficient. If department C is challenged to become one of the *three* most efficient departments for some feasible weights, it needs to increase its output by 8.80%; and if it is to be ranked as one of three most efficient departments for all weights, it must increase it output by more than 53.35%. Corresponding targets for departments E and I are 6.80% and 10.72% (for some weights) and 42.65% and 47.97% (for all weights).

Similarly, the least efficient departments D, F, and H could be required to achieve a position among the six most efficient departments. In this case, department D would have to increase its output by 25.97% to achieve such a position for some

Figure 4 **Efficiency Dominance Relations Among the Departments**



weights. Moreover, it would have to increase its output by more than 54.40% to secure this position for all weights. Corresponding results for departments F and H are 32.33% and 31.54% (for some weights) and 94.21% and 62.89% (for all weights).

5. Conclusion

We have developed ratio-based efficiency results (ranking intervals, dominance relations, and efficiency bounds) for comparing the relative efficiencies of DMUs for all feasible input/output weights. Unlike conventional DEA efficiency scores or cross-efficiencies, these results are robust in the sense that they (i) reflect how the DMUs' efficiency ratios change relative to each other over the entire feasible set of weights, (ii) tend to be insensitive to the introduction/removal of outlier DMUs, and (iii) do not necessitate particular assumptions about what production possibilities there are beyond the DMUs that are included in the analysis. Furthermore, these results do not exhibit rank reversals that may arise when ranking DMUs with cross-efficiency analysis. These results can also be employed to specify performance targets for DMUs.

We have illustrated the usefulness of these efficiency results by revisiting reported DEA studies and by describing a case study on the comparison of university departments. The encouraging feedback from this case study, together with the applicability of our efficiency results in many other contexts, leads us to believe that these results are helpful across the full range of decision contexts where ratio-based efficiency comparisons are appropriate.

Acknowledgments

This research was supported financially by the Academy of Finland. Parts of this paper were written when Ahti Salo was on a sabbatical at the London Business School and at the Paris Dauphine University. The authors are grateful to Juuso Liesiö, Pekka Korhonen, Srini Parthasarthy, Alexis Tsoukias, and the referees for their comments on an earlier version of this paper. Jussi Kangaspunta and Samuli Leppänen contributed to the case study of §4.3. Reda Guerfi, Matti Ollila, and Antti Silvast developed the REA-Solver(c) software for ratio-based efficiency analyses.

Appendix

Proof of Theorem 1. Let the best ranking of DMU_k be attained at $(u, v) \in (S_u, S_v)$. Then there exists $L = R_k^{>}(u, v) \subset \{1, \ldots, K\}$ so that $E_l(u, v) > E_k(u, v), l \in L$ and $E_k(u, v) \geq E_l(u, v), l \notin L$. Let $v'_m = v_m/[\sum_m v_m x_{mk}]$ and $u'_n = u_n/[\sum_n u_n y_{nk}]$. Then $(u', v') \in (S_u, S_v)$ and $\sum_m v'_m x_{mk} = \sum_n u'_n y_{nk} = 1$.

For any $l \neq k$, let $z_l = 1$ if $l \in L$, and $z_l = 0$ if $l \notin L$. Then, for any $l \notin L$, we have

$$1 \leq \frac{E_k(u, v)}{E_l(u, v)} = \frac{E_k(u', v')}{E_l(u', v')} = \frac{\sum_m v'_m x_{mk}}{\sum_n u'_n y_{nk}} \frac{\sum_m v'_m x_{ml}}{\sum_n u'_n y_{nl}} = \frac{\sum_m v'_m x_{ml}}{\sum_n u'_n y_{nl}},$$

which gives $\sum_n u'_n y_{nl} \leq \sum_m v'_m x_{ml}$. For $l \in L$, multiplying $z_l = 1$ by the large positive constant C implies that the constraint (4) is satisfied for $l \in L$ too. Because $1 + \sum_{l \neq k} z_l = 1 + |L| = 1 + |R_k^{>}(u, v)| = r_k^{\min}(u, v)$, the solution to the minimization problem is not larger than the best ranking.

Conversely, let (u, v, z) be a solution to the minimization problem. Let $L = \{l \mid l \neq k, z_l = 1\}$. Then introducing $z_l = 0, l \notin L$ into the first constraint in (4) gives $\sum_n u_n y_{nl} \leq \sum_m v_m x_{ml}$ so that

$$\frac{E_k(u, v)}{E_l(u, v)} = \frac{\sum_m v_m x_{ml}}{\sum_n u_n y_{nl}} \geq 1,$$

because $E_k(u, v) = 1$ due to (5). Thus, any $l \notin L$ cannot belong to $R_k^{>}(u, v)$. For $l \in L$, the inequality $\sum_n u_n y_{nl} \leq \sum_m v_m x_{ml} \iff E_k(u, v) \geq E_l(u, v)$ cannot hold because z is at optimum (otherwise, any such $z_l = 1$ could be changed to $z_l = 0$ without violating (4) while reducing the value of the objective function); hence $l \in L \subseteq R_k^{>}(u, v)$. It follows that $R_k^{>}(u, v) = L$ and $r_k^{\min} \leq 1 + |R_k^{>}(u, v)| = 1 + |L| = 1 + \sum_{l \neq k} z_l$. □

Proof of Theorem 2. If the worst ranking of DMU_k is attained at $(u, v) \in (S_u, S_v)$, there exists a subset $L = R_k^{\geq}(u, v) \subset \{1, \ldots, K\}, k \notin L$ such that $E_l(u, v) \geq E_k(u, v), l \in L$ and $E_k(u, v) > E_l(u, v), l \notin L$. Let $v'_m = v_m/[\sum_j v_m x_{mk}]$ so that $\sum_m v'_m x_{mk} = 1$ and $u'_n = u_n/[\sum_n u_n y_{nk}]$ so that $\sum_n u'_n y_{nk} = 1$.

For any $l \neq k$, let $z_l = 1$ if $l \in L$, and let $z_l = 0$ if $l \notin L$. Then, for any $l \in L$,

$$1 \leq \frac{E_l(u, v)}{E_k(u, v)} = \frac{E_l(u', v')}{E_k(u', v')} = \frac{\sum_m u'_n y_{nl}}{\sum_m v'_m x_{ml}} \;\Rightarrow\; \sum_m v'_m x_{ml} \leq \sum_n u'_n y_{nl},$$

and thus (6) holds. For $l \notin L$, multiplying $(1 - z_l) = 1$ by the positive constant C implies that (6) is satisfied in this case too. Now, $1 + \sum_{l \neq k} z_l = 1 + |L| = 1 + |R_k^{\geq}(u, v)| = r_k^{\max}$. Thus, the solution to the maximization problem is at least as large as the worst ranking.

Conversely, assume that (u, v, z) is a solution to the maximization problem, and let $L = \{l \mid l \neq k, z_l = 1\}$. For any $l \in L$ with $z_l = 1$, the constraint $\sum_m v_m x_{ml} \leq \sum_n u_n y_{nl}$ implies

$$\frac{E_l(u, v)}{E_k(u, v)} = \frac{\sum_n u_n y_{nl}}{\sum_m v_m x_{ml}} \geq 1,$$

because u and v satisfy (6)–(7); thus, $L \subseteq R_k^{\geq}(u, v)$. Because z is at optimum, the inequality $E_k(u, v) \leq E_l(u, v)$ cannot hold for $l \notin L$ (otherwise, any such $z_l = 0$ could be changed to $z_l = 1$ without violating constraints while increasing the objective function). Thus, $R_k^{\geq}(u, v)$ does not contain elements that are outside of L. It follows that $L = R_k^{\geq}(u, v)$ and $r_k^{\max} \geq 1 + |R_k^{\geq}(u, v)| = 1 + \sum_{l \neq k} z_l$. □

Proof of Lemma 1. To prove that $u' \in S_u$, note that $u \in S_u$ implies $A_u u \leq 0$ and, hence, $A_u u' = A_u c_u u = c_u (A_u u) \leq 0$; similarly, $v' \in S_v$. The last claim follows from

$$\begin{aligned} D_{k,l}(u', v') &= \frac{E_k(u', v')}{E_l(u', v')} \\ &= \frac{\sum_n u'_n y_{nk}}{\sum_m v'_m x_{mk}} \frac{\sum_m v'_m x_{ml}}{\sum_n u'_n y_{nl}} \\ &= \frac{c_u \sum_n u_n y_{nk}}{c_v \sum_m v_m x_{mk}} \frac{c_v \sum_m v_m x_{ml}}{c_u \sum_n u_n y_{nl}} \\ &= \frac{\sum_n u_n y_{nk}}{\sum_m v_m x_{mk}} \frac{\sum_m v_m x_{ml}}{\sum_n u_n y_{nl}} = D_{k,l}(u, v). \quad \square \end{aligned}$$

Proof of Theorem 3. Choose $(u^*, v^*) \in (S_u, S_v)$ such that $D_{k,l}(u^*, v^*) \geq D_{k,l}(u, v)$, $\forall (u, v) \in (S_u, S_v)$. Define v' so that $v'_m = v^*_m/[\sum_i v^*_i x_{ik}]$. By construction, $v' \in S_v$ and $\sum_m v'_m x_{mk} = 1$. Define $u'_n = u^*_n[\sum_m v'_m x_{ml}]/[\sum_j u^*_j y_{jl}]$. Then, $\sum_n u'_n y_{nl} = \sum_m v'_m x_{ml}$. The weights (u', v') satisfy constraints (12)–(14), and the repeated application of Lemma 1 gives $D_{k,l}(u^*, v^*) = D_{k,l}(u^*, v') = D_{k,l}(u', v') = \sum_n u'_n y_{nk}$, proving that the maximum of (11) over (12)–(14) is at least as high as $D_{k,l}(u^*, v^*)$.

Assume that the maximum of (11) is attained at (u^o, v^o). For these weights $(u^o, v^o) \in (S_u, S_v)$, we have

$$D_{k,l}(u^o, v^o) = \frac{E_k(u^o, v^o)}{E_l(u^o, v^o)} = \frac{\sum_n u^o_n y_{nk}}{\sum_m v^o_m x_{mk}} \frac{\sum_m v^o_m x_{ml}}{\sum_n u^o_n y_{nl}} = \sum_n u^o_n y_{nk},$$

because the weights (u^o, v^o) satisfy (12)–(13). Thus, the maximum of $D_{k,l}(u, v)$ over (S_u, S_v) cannot be smaller than the solution to the maximization problem in Theorem 3. The minimization case can be shown analogously. $\square$

Proof of Theorem 4.

$$\begin{aligned}\min_{u,v} D_{k,\bar{L}}(u, v) &= \min_{u,v} \frac{E_k(u, v)}{\max_{l\in L} E_l(u, v)}\\ &= \min_{u,v}\min_{l\in L} \frac{E_k(u, v)}{E_l(u, v)}\\ &= \min_{l\in L}\min_{u,v} D_{k,l}(u, v)\\ &= \min_{l\in L} \underline{D}_{k,l}(u, v).\end{aligned}$$

Let the maximum of (15) be ζ^* so that this optimum is attained at (u^*, v^*). There then exists some $l^* \in L$ such that $E_{l^*}(u^*, v^*) \geq E_l(u^*, v^*)$ $\forall l \in L$. Choose $v' = v^*/[\sum_m v^* x_{mk}]$ so that $\sum_m v'_m x_{mk} = 1$. Also, choose a constant $c_u > 0$ so that $\sum_n u'_n y_{nl^*} = \sum_m v'_m x_{ml^*}$ for $u' = c_u u^*$. For any $l \in L$, we have

$$1 \geq D_{l,l^*}(u^*, v^*) = D_{l,l^*}(u', v') = \frac{E_l(u', v')}{E_{l^*}(u', v')} = \frac{\sum_n u'_n y_{nl}}{\sum_m v'_m x_{ml}}$$

so that the constraint (18) is satisfied by (u', v'). By construction, $\zeta^* = \max_{u,v} D_{k,\bar{L}}(u, v) = D_{k,l^*}(u', v') = \sum_n u'_n y_{nk}$, which shows that the maximum of (17) is at least as high as that of (15).

Conversely, assume that the maximum of (17), ζ', is attained at (u', v'), and choose $l' \in L$ so that the constraint in (18) is binding (such l' exists, for otherwise u' could be increased to improve the value of the objective function, which would be in violation of the optimality assumption). Now,

$$\max_{u,v} D_{k,\bar{L}}(u, v) \geq \frac{E_k(u', v')}{E_{l'}(u', v')} = \zeta'$$

so that the maximum (15) must be at least as high as that of (17). $\square$

Proof of Theorem 5.

$$\begin{aligned}\max_{u,v} D_{k,\underline{L}}(u, v) &= \max_{u,v} \frac{E_k(u, v)}{\min_{l\in L} E_l(u, v)}\\ &= \max_{u,v}\max_{l\in L} \frac{E_k(u, v)}{E_l(u, v)}\\ &= \max_{l\in L}\max_{u,v} D_{k,l}(u, v)\\ &= \max_{l\in L} \bar{D}_{k,l}(u, v).\end{aligned}$$

Let the minimum of (16), ζ^*, be attained at (u^*, v^*). There then exists some l^* such that $E_{l^*}(u^*, v^*) \leq E_l(u^*, v^*)$, $\forall l \in L$, and $\zeta^* = \min_{u,v} D_{k,\underline{L}}(u, v) = E_k(u^*, v^*)/E_{l^*}(u^*, v^*)$. As in the proof of Theorem 4, use (u^*, v^*) in defining normalized valuation vectors (u', v') such that $\sum_m v'_m x_{mk} = 1$ and $E_{l^*}(u', v') = 1$. The choice of l^* guarantees that $1 \leq E_l(u', v')$ so that constraint (20) holds for all $l \in L$. Because

$$\zeta^* = \frac{E_k(u^*, v^*)}{E_{l^*}(u^*, v^*)} = \frac{E_k(u', v')}{E_{l^*}(u', v')} = \sum_n u'_n y_{nk},$$

the minimum of (19) is at least as small as the minimum of (16).

Assume that ζ', the minimum of (19), is obtained at (u', v'). Choose l' such that the constraint in (20) is binding (such l' must exist, for otherwise the assumption of optimality would be violated). Then $E_{l'}(u', v') = 1$, whereas constraint (20) implies that $E_l(u', v') \geq 1$ for any other $l \in L$; hence, $E_{l'}(u', v') \leq E_l(u', v')$. It follows that

$$\min_{u,v} D_{k,\underline{L}}(u, v) \leq D_{k,\underline{L}}(u', v') = \frac{E_k(u', v')}{\min_{l\in L} E_l(u', v')} = \frac{E_k(u', v')}{E_{l'}(u', v')} = \zeta',$$

proving that the minimum of (16) is at least as small as the optimum of (19). $\square$

Proof of Theorem 6. Because $u \in S_u \Rightarrow c_u u \in S_u$ for any $c_u > 0$, there exists $u \in S_u$ so that the constraints (21) and (22) are satisfied. The optimum ζ^* is attained at some weights (u^*, v^*), because v fulfills the normalization constraint and assumes values in a compact set $\{v \in S_v \mid \sum_m v_m x_{mk} = 1\}$, and u is maximized but bounded from above by constraint (22). If the optimum ζ^* were equal to one, then according to Theorem 1, DMU_k could only reach ranking $r_k^{\min}$ and constraint (21) would be violated. Thus, $\zeta^* < 1$.

For any feasible (u, v, z) that satisfy the constraints, the constraint (22) gives $z_l = 0 \Rightarrow 1 \geq E_l(u, v)$ so that $E_l(u, v) > 1 > E_k(u, l) \Rightarrow z_l = 1$. By (21), there are therefore at most $\sum_{l\neq k} z_k$ other DMUs whose efficiency is higher than that of DMU_k. By (21), the best ranking of DMU_k is therefore R_k^* or better. By construction, $1/\zeta^*$ is the revised efficiency ratio of DMU_k.

For any $\zeta' > \zeta^*$ and any feasible (u, v), the optimality of ζ^* implies that the constraint (21) will be violated when z_l are chosen by minimizing them so that (22) holds. But then there will be more than $R_k^* - 1$ other DMUs with an efficiency ratio that is strictly higher than that of DMU_k, meaning that the best ranking of DMU_k is worse than R_k^*.

Similarly, if constraint (21) holds and $\zeta' > \zeta^*$, constraint (22) is violated for some $l' \neq k$ such that $z_{l'} = 0$ and $E_{l'}(u, v) > E_k(u, v)$. We can assume that (21) holds with equality, for else the violation of (22) for l' could be eliminated by setting $z_{l'} = 1$. Because $E_l(u, v) > E_k(u, v) \Rightarrow z_l = 1$ for the constraints that are satisfied, there are again more than $R_k^* - 1$ other DMUs with a strictly higher efficiency ratio, and thus DMU_k does not attain the target ranking R_k^*. $\square$

Proof of Theorem 7. Because $u \in S_u \Rightarrow c_u u \in S_u$ for any $c_u > 0$, there exists $u \in S_u$ so that the constraints (23) and (24) are satisfied. The optimum ζ^* is attained at some weights (u^*, v^*), because v fulfills the normalization constraint and thus assumes values in a compact set $\{v \in S_v \mid \sum_m v_m x_{mk} = 1\}$, and u is minimized but bounded from below by constraints (24).

According to Theorem 2, there exists a solution u, v, z' such that $\sum_m v_m x_{mk} = \sum_n u_n y_{nk} = 1$, $\sum_{l\neq k} z'_l = r_k^{\max} - 1$, and $\sum_m v_m x_{ml} \leq \sum_n u_n y_{nl} + C(1 - z'_l)$. Solution u, v, z such that $z_l = 1 - z'_l \; \forall l \neq k$ is feasible because this substitution yields directly the constraints (24) and also the constraint $\sum_{l\neq k} z_l = K - r_k^{\max} \leq K - R_k^\circ - 1$, whereby (23) is fulfilled. Thus, $\zeta^* \leq 1$.

For any feasible (u, v, z), the constraint (24) gives $z_l = 0 \Rightarrow E_l(u, v) \geq E_k(u, v)$ so that $E_k(u, v) > E_l(u, v) \Rightarrow z_l = 1$. By (23), there are at most $\sum_{l\neq k} z_k$ other DMUs whose efficiency is lower than that of DMU_k, and hence the worst possible ranking is $R_k^\circ + 1$ or worse.

We show that $\rho^* = 1/\zeta^*$ is the maximum increase in the outputs of DMU_k such that $R_k^\circ + 1$ belongs to the ranking interval of the revised DMU, DMU_{k^ρ} with $x_{k^\rho} = x_k$, $y_{k^\rho} = \rho y_k$. For any increase greater than ρ^*, only better rankings belong to the interval.

For any $\zeta < \zeta^*$ and feasible (u, v), the optimality of ζ^* implies that constraint (23) must be violated if constraints (24) hold for all $l \neq k$. But then the worst ranking of DMU_{k^ρ} will be R_k° or better.

Conversely, if (23) holds and (24) is violated for $\text{DMU}_{l'}$, then $E_{l'}(u, v) < E_{k^\rho}(u, v)$ and $z_{l'} = 0$. Furthermore, the constraint (23) can be assumed to hold with equality, because otherwise we could set $z_{l'} = 1$, and the constraint would not be violated. This implies that the number of DMUs p for which $E_p(u, v) < E_{k^\rho}(u, v)$ is at least $|\{l \in \{1, \ldots, K\} \mid l \neq k, z_l = 1\}| + 1 = K - R_k^\circ - 1 + 1 = K - R_k^\circ$, and the ranking of DMU_{k^ρ} must be R_k° or better.

Thus, for any $\rho > 1/\zeta^*$, the ranking of DMU_{k^ρ} is R_k° or better for all feasible (u, v). The formulation thus provides the infimum of the radial increases for which the worst ranking is R_k° or better. □

Proof of Theorem 8. By assumption, $E_k(u', v') > E_l(u', v')$ for some $(u', v') \in (S_u, S_v)$. Let constant $M > \max_{i=k,l}[1/\underline{D}_{i,\bar{L}}]$, where $L = \{1, \ldots, K\}$. Define $\text{DMU}_{k'}$ and $\text{DMU}_{l'}$ so that $y_{k'} = M y_k$, $y_{l'} = \bar{D}_{k,l} M y_l$, $x_{k'} = x_k$, $x_{l'} = x_l$. Then, (i) $E_{l'}(u, v) \geq E_{k'}(u, v) > E_i(u, v)$ for all $i \in \{1, \ldots, K\}$, and (ii) there exist $(u', v') \in (S_u, S_v)$ such that $E_{l'}(u', v') = E_{k'}(u', v')$ and for any such weights, $E_k(u', v') > E_l(u', v')$.

Consider DMUs DMU_i, $i \in \{1, \ldots, K\} \cup \{l'\} \cup \{K + 2, \ldots, K + K'\}$ so that DMU_i, $i = K+2, \ldots, K+K'$ are equal to $\text{DMU}_{k'}$. Among these DMUs, $\theta_{k',k} > \theta_{k',l}$. Then, for a sufficiently large K',

$$
\begin{aligned}
&CE_k - CE_l \\
&= \frac{1}{K+K'} \sum_{i=1}^{K+K'} [\theta_{i,k} - \theta_{i,l}] \\
&= \frac{1}{K+K'} \left[\sum_{i=1}^{K} [\theta_{i,k} - \theta_{i,l}] + [\theta_{l',k} - \theta_{l',l}] + (K'-1)[\theta_{k',k} - \theta_{k',l}] \right]
\end{aligned}
$$

is positive, because $\sum_{i=1}^{K}[\theta_{i,k} - \theta_{i,l}] + \theta_{l',k} - \theta_{l',l}$ is bounded from above by $K+1$. □

Proof of Theorem 9. Let (u^i, v^i), $i = \{1, \ldots, K\}$ be the weights that maximize the efficiency of DMU_i in the specification of the cross-efficiency matrix. Because DMU_k dominates DMU_l, we have $E_k(u^i, v^i) \geq E_l(u^i, v^i)$ so that $\theta_{i,k} \geq \theta_{i,l}$. Summing this inequality over $i = 1, \ldots, K$ gives $CE_k \geq CE_l$. □

References

Ahn, T., A. Charnes, W. W. Cooper. 1988. Some statistical and DEA evaluations of relative efficiencies of public and private institutions of public and private institutions of higher learning. *Socio-Econom. Planning Sci.* **22**(6) 259–269.

Allen, R., A. Athanassopoulos, R. G. Dyson, E. Thanassoulis. 1997. Weight restrictions and value judgements in data envelopment analysis: Evolution, development and future directions. *Ann. Oper. Res.* **73** 13–34.

Andersen, P., N. C. Petersen. 1993. A procedure for ranking efficient units in DEA. *Management Sci.* **39**(10) 1261–1264.

Avkiran, N. K., B. R. Parker. 2010. Pushing the DEA research envelope. *Socio-Econom. Planning Sci.* **44**(1) 1–7.

Baker, R. C., S. Talluri. 1997. A closer look at the use of data envelopment analysis for technology selection. *Comput. Indust. Engrg.* **32**(1) 101–108.

Belton, V., T. Gear. 1983. On a shortcoming of Saaty's method of analytic hierarchies. *Omega* **11**(3) 228–230.

Bouyssou, D. 1999. Using DEA as a tool for MCDM: Some remarks. *J. Oper. Res. Soc.* **50**(9) 974–978.

Charnes, A., W. W. Cooper, E. Rhodes. 1978. Measuring the efficiency of decision making units. *Eur. J. Oper. Res.* **2**(6) 429–444.

Charnes, A., W. W. Cooper, Z. M. Huang, D. B. Sun. 1990. Polyhedral cone-ratio DEA models with an illustrative application to large commercial banks. *J. Econometrics* **46**(1–2) 73–91.

Cook, W. D., J. Zhu. 2008. CAR-DEA: Context-dependent assurance regions in DEA. *Oper. Res.* **56**(1) 69–78.

Cooper, W. W., M. Seiford, K. Tone. 2007. *Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References and DEA-Solver Software*, 2nd ed. Springer, New York.

Doyle, J., R. Green. 1994. Efficiency and cross-efficiency in DEA: Derivations, meanings and uses. *J. Oper. Res. Soc.* **45**(5) 567–578.

Dyer, J. S. 1990. Remarks on the analytic hierarchy process. *Management Sci.* **36**(3) 249–258.

Dyson, R. G., R. Allen, A. S. Camanho, V. V. Podinovski, C. S. Sarrico, E. A. Shale. 2001. Pitfalls and protocols in DEA. *Eur. J. Oper. Res.* **132**(2) 245–259.

Eilat, H., B. Golany, A. Shtub. 2008. R&D project evaluation: An integrated DEA and balanced scorecard approach. *Omega* **36**(5) 895–912.

Emrouznejada, A., B. R. Barnett, G. Tavares. 2008. Evaluation of research in efficiency and productivity: A survey and analysis of the first 30 years of scholarly literature in DEA. *Socio-Econom. Planning Sci.* **42**(3) 151–157.

Farzipoor, S. R. 2009. Technology selection in the presence of imprecise data, weight restrictions, and nondiscretionary factors. *Internat. J. Adv. Manufacturing Tech.* **41**(7–8) 827–838.

Galagedera, D. U. A., P. Silvapulle. 2003. Experimental evidence on robustness of DEA. *J. Oper. Res. Soc.* **54**(6) 654–660.

Garcia, F., C. Marcuello, D. Serrano, O. Urbina. 2002. Evaluation of efficiency in primary health care centres: An application of data envelopment analysis. *Financial Accountability Management* **51**(1) 67–83.

Gouveia, M. C., L. C. Dias, C. H. Antunes. 2008. Additive DEA based on MCDA with imprecise information. *J. Oper. Res. Soc.* **59**(1) 54–63.

Halme, M., P. Korhonen. 2000. Restricting weights in value efficiency analysis. *Eur. J. Oper. Res.* **126**(1) 175–188.

Halme, M., T. Joro, P. Korhonen, S. Salo, J. Wallenius. 1999. A value efficiency approach to incorporating preference information in data envelopment analysis. *Management Sci.* **45**(1) 103–115.

Hollingsworth, B., P. J. Dawson, N. Maniadakis. 1999. Efficiency measurement of health care: A review of non-parametric methods and applications. *Health Care Management Sci.* **2**(3) 161–172.

Johnes, J. 2006. Data envelopment analysis and its application to the measurement of efficiency in higher education. *Econom. Ed. Rev.* **25**(3) 273–288.

Joro, T., P. Korhonen, J. Wallenius. 1998. Structural comparison of data envelopment analysis and multiple objective linear programming. *Management Sci.* **44**(7) 962–970.

Korhonen, P., M. Soismaa, A. Siljamäki. 2002. On the use of value efficiency analysis and some further developments. *J. Productivity Anal.* **17**(1–2) 49–64.

Korhonen, P., R. Tainio, J. Wallenius. 2001. Value efficiency analysis of academic research. *Eur. J. Oper. Res.* **130**(1) 121–132.

Köksalan, M., T. Büyükbaşaran, Ö. Özpeynirci, J. Wallenius. 2010. A flexible approach to ranking with an application to MBA programs. *Eur. J. Oper. Res.* **201**(2) 470–476.

Liu, N. C., Y. Cheng. 2005. The academic ranking of world universities. *Higher Ed. Eur.* **30**(2) 127–316.

Ozcan, Y. A. 2008. *Health Care Benchmarking and Evaluation: An Assessment Using Data Envelopment Analysis*. Springer, New York.

Podinovski, V. V. 2001. DEA models for the explicit maximization of relative efficiency. *Eur. J. Oper. Res.* **131**(3) 572–586.

Podinovski, V. V. 2005. The explicit role of weight bounds in models of data envelopment analysis. *J. Oper. Res. Soc.* **56**(12) 1408–1481.

Salo, A. 1995. Interactive decision aiding for group decision support. *Eur. J. Oper. Res.* **84**(1) 134–149.

Salo, A., R. P. Hämäläinen. 1992. Preference assessment by imprecise ratio statements. *Oper. Res.* **40**(6) 1053–1061.

Salo, A., R. P. Hämäläinen. 1997. On the measurement of preferences in the analytic hierarchy process. *J. Multi-Criteria Decision Anal.* **6**(6) 309–319.

Salo, A., R. P. Hämäläinen. 2001. Preference ratios in multiattribute evaluation (PRIME)—Elicitation and decision procedures under incomplete information. *IEEE Trans. Systems, Man, Cybernetics* **31**(6) 533–545.

Sarrico, S. C., R. G. Dyson. 2000. Using DEA for planning in UK universities—An institutional perspective. *J. Oper. Res. Soc.* **51**(7) 789–800.

Seiford, L. M., J. Zhu. 1998a. Sensitivity analysis of DEA models for simultaneous changes in all of the data. *J. Oper. Res. Soc.* **49**(10) 1060–1071.

Seiford, L. M., J. Zhu. 1998b. Stability regions for maintaining efficiency in data envelopment analysis. *Eur. J. Oper. Res.* **108**(1) 127–139.

Sexton, T. R., R. H. Silkman, A. Hogan. 1986. Data envelopment analysis: Critique and extensions. R. H. Silkman, ed. *Measuring the Efficiency: An Assessment of Data Envelopment Analysis*. American Evaluation Association, Jossey Bass Inc., San Francisco, 73–104.

Shafer, S. M., J. W. Bradford. 1995. Efficiency measurements of alternate machine component grouping solutions via data envelopment analysis. *IEEE Trans. Engrg. Management* **42**(2) 159–165.

Stewart, T. J. 1996. Relationships between data envelopment analysis and multicriteria decision analysis. *J. Oper. Res. Soc.* **47**(5) 654–665.

Talluri, S., K. P. Yoon. 2000. A cone-ratio DEA approach for AMT justification. *Internat. J. Production Econom.* **66**(2) 119–129.

Tauer, L. W., H. O. Fried, W. E. Fry. 2007. Measuring efficiencies of academic departments within a college. *Ed. Econom.* **15**(4) 473–489.

Thanassoulis, E., M. C. Portela, R. Allen. 2004. Incorporating value judgements in DEA. W. W. Cooper, L. M. Seiford, J. Zhu, eds. *Handbook on Data Envelopment Analysis*. Kluwer, Boston, 99–137.

Thompson, R. G., F. Singleton, R. Thrall, B. Smith. 1986. Comparative site evaluations for locating a high-energy physics lab in Texas. *Interfaces* **16**(6) 35–49.

Thompson, R. G., L. N. Langemeier, C. T. Lee, E. Lee, R. M. Thrall. 1990. The role of multiplier bounds in efficiency analysis with application to Kansas farming. *J. Econometrics* **46**(1–2) 93–108.

Zhu, J. 1996. Robustness of the efficient DMUs in data envelopment analysis. *Eur. J. Oper. Res.* **90**(3) 451–460.