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Decision Aiding

Rank inclusion in criteria hierarchies

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Abstract

This paper presents a method called Rank Inclusion in Criteria Hierarchies (RICH) for the analysis of incomplete preference information in hierarchical weighting models. In RICH, the decision maker is allowed to specify subsets of attributes which contain the most important attribute or, more generally, to associate a set of rankings with a given set of attributes. Such preference statements lead to possibly non-convex sets of feasible attribute weights, allowing decision recommendations to be obtained through the computation of dominance relations and decision rules. An illustrative example on the selection of a subcontractor is presented, and the computational properties of RICH are considered.

Keywords: Multiple criteria analysis; Decision analysis; Hierarchical weighting models; Incomplete preference information

1. Introduction

Methods of multiple criteria decision making (MCDM) are widely employed in problems characterized by incommensurate objectives. Numerous successful MCDM applications have been developed in fields such as energy policy, environmental decision making and comparison of industrial investment opportunities (see, e.g., Corner and Kirkwood, 1991; Hämäläinen, 2004; Keefer et al., 2004). In MCDM applications, the decision problem is structured by associating measurable attributes with the objectives that are relevant to the decision maker (DM). In most methods—such as the Analytic Hierarchy Process (AHP; Saaty, 1980) and value tree analysis (Keeney and Raiffa, 1976)—the DM is also requested to supply weights as a measure for the relative importance of attributes.

In practice, the elicitation of precisely specified attribute weights may be difficult. This may be due to the urgency of the decision, lack of resources for completing the elicitation process, or conceptual difficulties in the interpretation of intangible objectives (see, e.g., Weber, 1987). In group settings, difficulties in determining attribute weights for the group's joint preference model may arise from differences in the group members' level of knowledge or their interpretation of what the relevant objectives mean (Hämäläinen et al., 1992).

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However, complete information about attribute weights is not always necessary in order to produce a decision recommendation. Together with the difficulties of producing a complete model specification, this realization has motivated the development of methods for dealing with incomplete information in hierarchical weighting models (see, e.g., Kirkwood and Sarin, 1985; Hazen, 1986; Weber, 1987; Salo and Hämäläinen, 1992; Salo, 1995; Kim and Han, 2000). Even though these methods differ in their details, they all (i) accommodate incomplete information about attribute weights and possibly other model parameters as well and (ii) provide more or less conclusive dominance relations concerning which alternatives are preferred to others.

In this paper, we extend the earlier literature on incomplete preference information by allowing the DM to specify subsets of attributes which contain the most important attribute or, more generally, to associate several rankings with a given set of attributes. Resulting method—called Rank Inclusion in Criteria Hierarchies (RICH)—generalizes the use of ordinal preference information in attribute weighting. In view of our theoretical and computational results, we believe that the RICH method is especially suitable for decision contexts where only rather few and easily elicited preference statements can be obtained before preliminary decision recommendations must be produced. Also, inspired by positive experiences from the deployment of internet-based decision aiding tools (e.g., Web-HIPRE; see Mustajoki and Hämäläinen, 2000; Lindstedt et al., 2001), we have already proceeded with the development of a user-friendly decision support tool for the RICH method. This tool—entitled *RICH Decisions*—is available free of charge to academic users (see `http://www.decisionarium.hut.fi`; Liesiö, 2002).

The remainder of this paper is structured as follows. Section 2 reviews earlier approaches to the analysis of incomplete information in hierarchical weighting models. Section 3 considers the use of incomplete ordinal information in the elicitation of attribute weights and the properties of resulting feasible weight regions. Section 4 presents a measure for the size of feasible regions, and Section 5 discusses the development of decision recommendations. Section 6 summarizes results from a simulation study on the computational properties of RICH. An illustrative example is given in Section 7, followed by concluding remarks in Section 8.

2. Earlier approaches to the analysis of incomplete information

In an early contribution on the modeling of incomplete information, Arbel (1989) discusses how the precise articulation of preferences through ratio statements can be extended to capture incomplete information about the relative importance of attributes. He models incomplete preference information through lower and upper bounds on the relative importance of attributes. These bounds correspond to linear constraints of linear programming (LP) problems from which the lower and upper bounds on the weight of each attribute can be obtained.

The PAIRS method (Preference Assessment by Imprecise Ratio Statements; Salo and Hämäläinen, 1992) extends Arbel's concepts to attribute hierarchies in which lower and upper bounds on the relative importance of attributes define a region of feasible weights at each higher-level attribute. Combined with possibly incomplete score information, such ratio-based information is processed by solving a series of hierarchically structured LP problems, in order to obtain bounds on the alternatives' overall values. The decision recommendations are based on the (*pairwise*) *dominance criterion* according to which alternative x_i is preferred to x_j if the overall value of x_i is higher than that of x_j, no matter how the weights are chosen from the feasible regions. If the available preference information does not lead to sufficiently conclusive dominance relations, the DM is requested to supply additional preference statements. PAIRS supports the consistency of the preference model through so-called *consistency bounds* which are presented to the DM before the elicitation of each new preference statement.

Analogous to PAIRS in many ways, the preference programming approach of Salo and Hämäläinen (1995) provides an *ambiguity index* which measures the incompleteness of a preference model. Salo (1995) extends the preference programming approach to group decision settings where several decision makers can supply incomplete preference information about (i) how the alternatives perform on the lowest-level attributes and (ii) how important the attributes are to the different DMs. These statements lead to linear constraints so that value intervals and dominance relations for the alternatives can be computed from LP problems. The potential of this approach has been explored in a study on traffic planning by Hämäläinen and Pöyhönen (1996), for instance.

The PRIME method (Preference Ratios in Multi-Attribute Evaluation; Salo and Hämäläinen, 2001) allows the DM to provide preference statements through holistic comparisons between alternatives, ordinal strength of preference judgments or ratios of value differences. Like PAIRS, PRIME provides information about the consistency of the DM's preference statements and dominance relations. Full support for PRIME is provided by the decision support tool *PRIME Decisions* which is available at `http://www.decisionarium.hut.fi`. PRIME Decisions employs value intervals and dominance structures to show intermediate results to the DM. It has been applied to the valuation of a high-technology firm, among others (Gustafsson et al., 2001).

Park and Kim (1997) give an extensive taxonomy of alternative ways to the elicitation of incomplete preference information in hierarchical weighting models. In particular, they distinguish between the following statements:

1. weak ranking: $\{w_i \geqslant w_j\}$,
2. strict ranking: $\{w_i - w_j \geqslant \alpha_i\}$,
3. ranking with multiples: $\{w_i \geqslant \alpha_i w_j\}$,
4. interval form: $\{\alpha_i \leqslant w_i \leqslant \alpha_i + \epsilon_i\}$,
5. ranking of differences: $\{w_i - w_j \geqslant w_k - w_l\}$ for $j \neq k \neq l$,

where $\alpha_i, \epsilon_i \geqslant 0\ \forall i$. Furthermore, they consider more general multi-criteria problems with incomplete probabilities, utilities and attribute weights. Although these problems may involve non-convex objective functions, approximate or even exact solutions can often be obtained by solving a series of LP problems.

Mármol et al. (1998) present an algorithm for computing the extreme points of the region of feasible attribute weights in two highly relevant cases (i.e., linear inequalities and weight intervals). They also examine the computational properties of their algorithm and establish conditions for introducing further linear relations which preserve the structure of the feasible region. A similar approach is taken by Puerto et al. (2000) who utilize the extreme points of the set of feasible weights in the implementation of three decision criteria (i.e., Laplace criterion, Wald's optimistic/pessimistic criterion, Hurwicz criterion).

Kim and Han (2000) extend the methods of Park and Kim (1997) to hierarchically structured attribute trees. In their model, the DM can place several kinds of linear constraints at any level of the attribute tree. These constraints are processed by an algorithm which can be invoked to obtain upper and lower bounds for the value of an alternative with regard to any attribute, subject to the assumption that the DM's preference statements remain consistent.

Fig. 1 presents a schematic diagram on the consecutive phases of the RICH method. In effect, this method is analogous to many others (e.g., PRIME; Salo and Hämäläinen, 2001) in that the DM can (i) interactively introduce new preference statements or revise earlier ones, and (ii) obtain tentative decision recommendations and information about the completeness of the currently available preference information. The key difference lies in the elicitation of attribute weights which are in the RICH method characterized through incomplete ordinal preference statements. At any phase of the process, results on (i) the alternatives' possible overall values, (ii) (pairwise) dominance structure of the alternatives, (iii) decision recommendations and (iv) information about the possible rankings of the attributes can be obtained from LP problems. After examining these results, the DM may either choose to accept one of the decision recommendations or continue with the specification of further preference information.

Except for the work of Park and Kim (1997)—in which combinations of incompletely specified

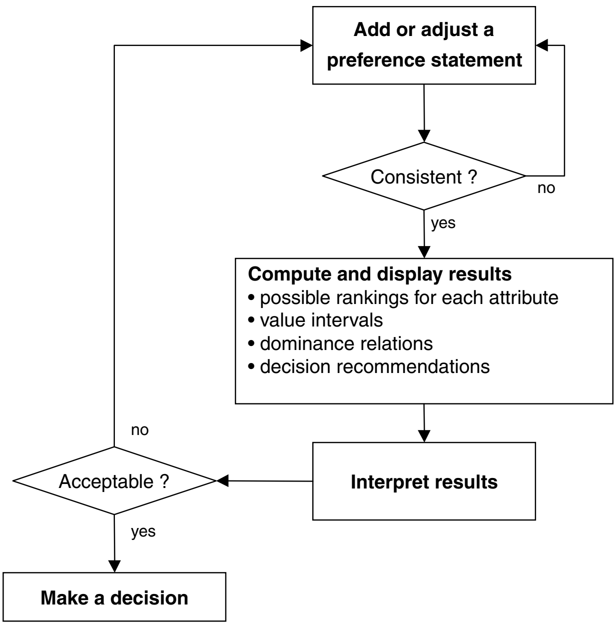

Fig. 1. Phases in the RICH method.

weights, probabilities and utilities are considered—a common feature of all earlier approaches is that the region of feasible attribute weights is convex and bounded by linear constraints. As we next move to the consideration of incomplete ordinal preference information, there is a significant difference in that the resulting feasible region may not be convex.

3. Formalization of incomplete ordinal information

Let $A = \{a_1, \ldots, a_n\}$ be the set of relevant attributes in the decision problem. The importance of attribute a_i is measured by its *weight* $w_i \in [0, 1]$. By convention, the attribute weights are normalized so that they add up to one, i.e., $\sum_{i=1}^{n} w_i = 1$. *Alternatives* are denoted by x_j, $j = 1, \ldots, m$. The performance of the jth alternative with regard to attribute a_i is measured by its *score* $v_i(x_j) \in [0, 1]$. The *overall value* of alternative x_j is given by $V(x_j) = \sum_{i=1}^{n} w_i v_i(x_j)$.

3.1. Weak orders, linear orders and rankings

Following several other approaches, we assume that the DM makes statements about the relative importance of attributes. These preferences are captured through a relation $\succeq$ on the set $A \times A$, in the understanding that $a_i \succeq a_j$ if and only if attribute a_i is at least as important as attribute a_j.

The relation is a *weak order* if it is comparable (i.e., $\forall a_i, a_j \in A$ either $a_i \succeq a_j$ or $a_j \succeq a_i$, or both) and transitive (i.e., if $a_i \succeq a_j$ and $a_j \succeq a_k$, then $a_i \succeq a_k$). If this relation is also antisymmetric (i.e., $\nexists a_i, a_j \in A, a_i \neq a_j$ such that $a_i \succeq a_j$ and $a_j \succeq a_i$), it is a *linear order*. In this case, each attribute $a \in A$ can be assigned a unique ranking $r(a) \in N = \{1, \ldots, n\}$ such that $a_i \succeq a_j$ if and only if $r(a_i) < r(a_j)$. Thus, the ranking of the most important attribute is one, that of the second most important is two, and so on, until the least important attribute is reached, the ranking of which is n.

If $\succeq$ is a weak order, there is a possibility that two or more attributes are equally important. In this case, the attributes A can be partitioned into sets $A(1),\dots,A(k)$ such that (i) $a_i \succeq a_j$, $a_j \succeq a_i$ if attributes a_i, a_j are in the same subset (i.e., $\exists A(l)$ such that $a_i, a_j \in A(l)$) and (ii) $a_i \succeq a_j$, $a_j \not\succeq a_i$ for any $a_i \in A(l)$, $a_j \in A(l+1)$. Nevertheless, the attributes can still be given rankings in $r(a_i)$, $r(a_j)$ such that $a_i \succeq a_j$ whenever $r(a_i) < r(a_j)$; but these rankings are not necessarily unique because permuting the rankings of attributes which belong to the same partition would lead to different rankings which still fulfil the above condition. Whatever the case, the ranking $r(a)$ implies that $r(a)-1$ attributes are at least as important as the attribute a.

Formally, a rank-ordering r is a function from the set of attributes $A = \{a_1,\dots,a_n\}$ onto the set N. The set of all possible rank-orderings r is denoted by R. Because each rank-ordering r is a bijection, the attribute with the ranking k is given by the inverse function r^{-1}, i.e., $a_i = r^{-1}(k) \Longleftrightarrow r(a_i) = k$. For example, if attribute a_3 is the second most important attribute, the ranking of a_3 is $r(a_3) = 2$ and $r^{-1}(2) = a_3$.

While linear and weak orders correspond to rank-orderings as indicated above, rank-orderings can be used directly in the elicitation of incomplete preference information. This can be helpful in situations where the DM does not provide a linear or weak order when considering the relative importance of the attributes: for example, if there are three attributes, the DM may state that the most important one is either the first or the second attribute, without taking a stance on which one of the two is the most important one. Among the six possible rank-orderings, four (i.e., $r = (r(a_1), r(a_2), r(a_3)) = (1,2,3), (1,3,2), (2,1,3)$ or (3,1,2)) are compatible with this statement which rules out the remaining two (i.e., (2,3,1) and (3,2,1)).

The above approach to preference elicitation can be formalized through (i) an attribute set $I \subseteq A$ and (ii) a set of rankings $J \subseteq N$ such that the rankings of attributes in I belong to J (subject to some qualifications discussed below). For instance, the example above corresponds to $I = \{a_1, a_2\}$ and $J = \{1\}$. Moreover, if I contains several attributes while the only ranking in J is one, it follows that the most important attribute must belong to I.

The attribute set I and the set of rankings J need not be equal in size. If the number of attributes is at least as large as that of possible rankings (i.e., $|I| \geqslant |J|$), the specification of these two sets is interpreted as the requirement that all attributes whose rankings belong to J are in the attribute set I. On the other hand, if there are fewer attributes than rankings (i.e., $|I| < |J|$), we require that for each attribute in I, the corresponding ranking is in the set J.

If a rank-ordering meets the above requirements, it is said to be *compatible* with the sets I and J. For example, if there are three attributes and the DM states that attribute a_2 is either the most important or the second most important attribute, then we have $I = \{a_2\}$ and $J = \{1,2\}$. The four rank-orderings that are compatible with these two sets are (2,1,3), (3,1,2), (1,2,3) and (3,2,1). Formally, rank-orderings that are compatible with an attribute set I and a set of rankings J are defined as follows:

Definition 1. If $I \subseteq A = \{a_1,\dots,a_n\}$ and $J \subseteq N$, the set of compatible rank-orderings is

$$R(I,J) = \begin{cases} \{r \in R \mid r^{-1}(j) \in I \ \forall j \in J\}, & \text{if } |I| \geqslant |J|, \\ \{r \in R \mid r(a_i) \in J \ \forall a_i \in I\}, & \text{if } |I| < |J|. \end{cases}$$

3.2. Feasible regions

Because the attribute weights are non-negative and add up to one, they belong to the set

$$S_w = \left\{ w = (w_1,\dots,w_n) \,\middle|\, \sum_{i=1}^{n} w_i = 1, \; w_i \geqslant 0 \ \forall i \in N \right\}. \tag{1}$$

The weight vector $w \in S_w$ is *consistent* with the rank-ordering r if $w_i \geqslant w_j$ whenever $r(a_i) < r(a_j)$. Thus, the feasible region associated with $r \in R$ can be defined as

$$S(r) = \{ w \in S_w \mid w_i \geqslant w_j \text{ for any } i,j \text{ such that } r(a_i) < r(a_j)\}. \tag{2}$$

For example, the feasible region implied by $r = (4,2,1,3)$ is $S(r) = \{w \in S_w \mid w_3 \geqslant w_2 \geqslant w_4 \geqslant$

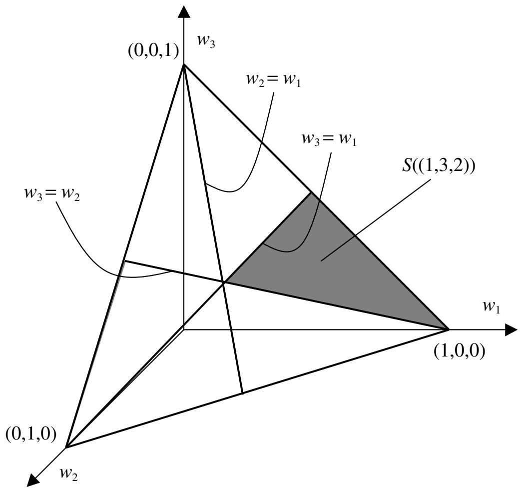

Fig. 2. The feasible region for $r = (1, 3, 2)$.

$w_1\}$. Fig. 2 illustrates the feasible region for $r = (1, 3, 2)$.

The region that corresponds to $R(I, J)$ is defined as the union of feasible regions that are associated with compatible rank-orderings, i.e.,

$$S(I,J) = \bigcup_{r \in R(I,J)} S(r).$$

In general, for a given $R' \subseteq R$, the corresponding feasible region is defined as $S(R') = \bigcup_{r \in R'} S(r)$. For example, Fig. 3 shows the feasible region associated with $R' = \{(3, 1, 2), (1, 3, 2)\}$, based on the statement that a_3 is the second most important one among three attributes (i.e., $I = \{a_3\}$, $J = \{2\}$).

An important special case is obtained when the DM specifies an attribute set I which contains the $p \leqslant |I|$ most important attributes. For brevity, we use $S_p(I)$ to denote the corresponding feasible region, $S_p(I) = S(I, \{1, \ldots, p\})$. In view of (1) and (2), this region is

$$S_p(I) = \{w \in S_w | \exists I' \subseteq I, |I'| = p, \\ \text{such that } w_k \geqslant w_i \; \forall a_k \in I', \quad a_i \notin I'\}. \tag{3}$$

It immediately follows that $S_p(I)$ can be written as $S_p(I) = \bigcup_{\{I' | I' \subseteq I \wedge |I'| = p\}} S_p(I')$. For example, Fig. 4 illustrates that in a case with three attributes, $S_1(\{a_1, a_2\})$ can be built as the union of $S_1(\{a_1\})$ and $S_1(\{a_2\})$.

3.3. Properties of feasible regions

We next examine several interesting properties of the feasible region based on attribute set I and the rankings J. Proofs are in Appendix A, unless otherwise stated.

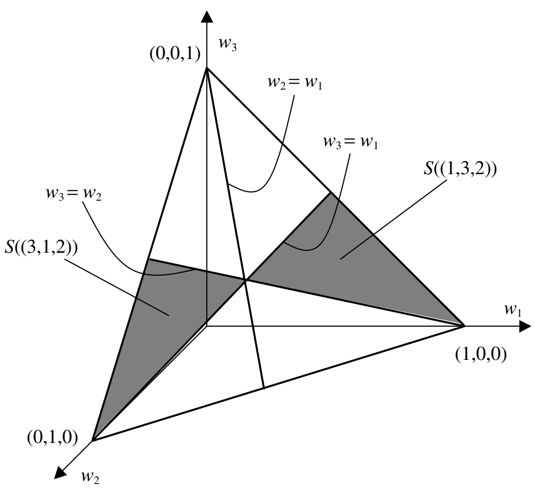

Fig. 3. The third attribute as the second most important one.

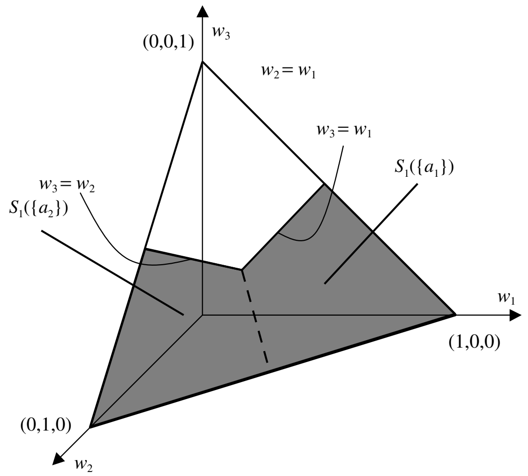

Fig. 4. A non-convex feasible region.

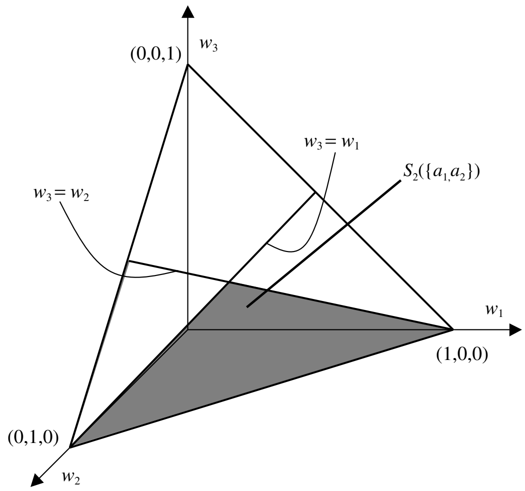

Fig. 5. A convex feasible region.

The feasible region $S_p(I)$ may not be convex (see Fig. 4). In fact, the feasible region is convex if and only if the number of rankings p is equal to the number of attributes in the set I; this result is stated in Theorem 1. Fig. 5 gives an example of the set $S_2(\{a_1, a_2\})$ in the case of three attributes. Here (and throughout this paper) '$\subset$' denotes a proper subset.

Theorem 1. *Let $I \subset A$ and $p \leqslant |I|$. Then $S_p(I)$ in* (3) *is convex if and only if $|I| = p$.*

Theorem 1 holds also when $I = A$. In this trivial case, $S_p(I) = S_w$ for $p \leqslant n$, because knowing that the p most important attributes come from the set of all attributes does not contain any preference information.

If two attribute sets I_1, I_2 are different but contain equally many attributes (p), the two feasible regions $S_p(I_1)$, $S_p(I_2)$—based on the requirement that the attributes in the sets I_1, I_2 are the p most important ones—have disjoint interiors.

Lemma 1. *If $I_1, I_2 \subset A$ such that $|I_1| = |I_2| = p$ and $I_1 \neq I_2$, then*
$\mathrm{int}(S_p(I_1)) \cap \mathrm{int}(S_p(I_2)) = \emptyset$.

For a given attribute set I and a set of rankings J, the resulting feasible region is the same as that defined by the complement sets of I and J. Taking the feasible region in Fig. 4 as an example, the statement that the most important attribute is a_1 or a_2 (i.e., $S(\{a_1, a_2\}, \{1\})$) is equivalent to stating that attribute a_3 is either the second or third most important one (i.e., $S(\{a_3\}, \{2, 3\})$).

Theorem 2. *Assume that I and J are non-empty proper subsets of $A = \{a_1, \ldots, a_n\}$ and N, respectively. Then*
$S(I, J) = S(\overline{I}, \overline{J})$,
where $\overline{I} = A \setminus I$ and $\overline{J} = N \setminus J$ are the complement sets of I and J.

Several comparative results about feasible regions can be obtained. If there are more rankings in J than attributes in I, then, as stated in Theorem 3, increasing the number of attributes that are associated with these rankings reduces the size of the feasible region. Conversely, if there are more attributes in I than rankings in J, reducing the number of attributes in I makes the feasible region smaller.

Theorem 3. *Let I_1 and I_2 be non-empty attribute sets such that $|I_1|$, $|I_2| < n$ and $|J| < n$.*

(a) *If $|I_1|, |I_2| \leqslant |J|$, then $I_1 \subset I_2 \Longleftrightarrow S(I_2, J) \subset S(I_1, J)$.*

(b) *If* $|I_1|, |I_2| \geqslant |J|$, *then* $I_2 \subset I_1 \Longleftrightarrow S(I_2, J) \subset S(I_1, J)$.

If there are fewer rankings in J than attributes in I, increasing the number of rankings leads to a feasible region that is a proper subset of the original one. Conversely, if there are more rankings in J than attributes in I, the feasible region becomes smaller if rankings are removed from the set J.

Theorem 4. *Let J_1 and J_2 are non-empty sets such that $|J_1|, |J_2| < n$ and $|I| < n$.*

(a) *If* $|J_1|, |J_2| \leqslant |I|$, *then* $J_1 \subset J_2 \Longleftrightarrow S(I, J_2) \subset S(I, J_1)$.
(b) *If* $|J_1|, |J_2| \geqslant |I|$, *then* $J_2 \subset J_1 \Longleftrightarrow S(I, J_2) \subset S(I, J_1)$.

The above results can be applied to examine how the feasible region $S_p(I)$—based on the requirement that the $p \leqslant |I|$ most important attributes are in the set I—changes due to incremental changes in the set I or the number p. That is, the feasible region $S_p(I)$ becomes smaller if

1. the attribute set I is extended to contain a larger number of the most important attributes; this means that p becomes larger (i.e., in Theorem 4, the set J_1 is extended to its proper superset $J_2 \supset J_1$), or
2. some attributes are removed from I without changing the number p; this means that the attributes that are removed from I are not among the p most important ones (i.e., in Theorem 3, the set I_1 is reduced to its proper subset $I_2 \subset I_1$).

The above results do not provide information on how 'large' the feasible regions are. We next turn to this issue, in order to provide guidance for eliciting statements which help reduce the size of the feasible region.

4. Measuring the completeness of information

Definition 1 and Eq. (2) suggest that a measure for the size of the feasible region $S(I, J)$ can be based on the number of compatible rank-orderings in the set $R(I, J)$. An appealing property of such a measure is that this number can be readily computed, as shown by Lemma 2 (here, we use the convention $0! = 1$).

Lemma 2. *The number of rank-orderings that are compatible with sets I and J is*

$$|R(I,J)| = \begin{cases} \frac{|I|!(n-|J|)!}{(|I|-|J|)!}, & \text{if } |I| \geqslant |J|, \\ \frac{|J|!(n-|I|)!}{(|J|-|I|)!}, & \text{if } |I| < |J|. \end{cases}$$

Proof. If $|I| \geqslant |J|$, there are $\binom{|I|}{|J|} = \frac{|I|!}{|J|!(|I|-|J|)!}$ different ways of choosing $|J|$ attributes from I. These $|J|$ attributes can be arranged in $|J|!$ ways while the remaining ones can be arranged in $(n - |J|)!$ ways, implying that there is a total of $\frac{|I|!(n-|J|)!}{(|I|-|J|)!}$ different rank-orderings. If $|I| < |J|$, the proof is similar, with the roles of I and J interchanged. □

The above lemma suggests a measure which is formally defined in the following theorem.

Theorem 5. *Let $\mathscr{P}(R)$ be the power set which contains all subsets of R. Then the function $\varphi(\cdot)$, defined for any $R' \in \mathscr{P}(R)$ as $\varphi(R') = \frac{|R'|}{n!}$, is a measure which maps the elements of $\mathscr{P}(R)$ onto the range* [0,1].

Table 1 shows the size of the feasible region (as measured by $\varphi(\cdot)$) for 10 attributes as a function of possible combinations of $|I|$ and $|J|$. The feasible region is smallest when (i) the attribute set and the ranking set are of equal size and (ii) they both contain (about) half as many elements as there are attributes (i.e., $|I| = |J| \approx \frac{n}{2}$). This means that bisecting the attributes into two sets—one which contains the $\frac{n}{2}$ most important attributes and one which contains the remaining $\frac{n}{2}$ less important attributes—effectively reduces the size of the feasible region.

Lemma 2 and Theorem 5 can be combined to obtain the following expression for the size of the feasible region $S_p(I)$.

Table 1
Size of the feasible region ($n = 10$)

$\|I\|$	$\|J\|$									
	1	2	3	4	5	6	7	8	9	10
1	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000	0.7000	0.8000	0.9000	1.0000
2	0.2000	0.0222	0.0667	0.1333	0.2222	0.3333	0.4667	0.6222	0.8000	1.0000
3	0.3000	0.0667	0.0083	0.0333	0.0833	0.1667	0.2917	0.4667	0.7000	1.0000
4	0.4000	0.1333	0.0333	0.0048	0.0238	0.0714	0.1667	0.3333	0.6000	1.0000
5	0.5000	0.2222	0.0833	0.0238	0.0040	0.0238	0.0833	0.2222	0.5000	1.0000
6	0.6000	0.3333	0.1667	0.0714	0.0238	0.0048	0.0333	0.1333	0.4000	1.0000
7	0.7000	0.4667	0.2917	0.1667	0.0833	0.0333	0.0083	0.0667	0.3000	1.0000
8	0.8000	0.6222	0.4667	0.3333	0.2222	0.1333	0.0667	0.0222	0.2000	1.0000
9	0.9000	0.8000	0.7000	0.6000	0.5000	0.4000	0.3000	0.2000	0.1000	1.0000
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Corollary 1. *For any $I \subset A$ such that $|I| \geqslant p$,*

$$\varphi_n(R(I,\{1,\ldots,p\})) = \frac{|I|!(n-p)!}{(|I|-p)!n!} = \frac{\binom{|I|}{p}}{\binom{n}{p}}.$$

The measure $\varphi(\cdot)$ can be used for the purpose of analyzing how the size of $S_p(I)$ changes when attributes are removed from the set I. It turns out that the resulting comparative change is larger, the more attributes there are in I.

Lemma 3. *Assume that the attribute set I_2 is obtained from the set I_1, $|I_1| = k > p$ by removing one of the attributes in I_1 (i.e., $I_2 = I_1 \setminus \{a_k\}$ for some a_k). Then the size of the revised feasible region $S_p(I_2)$, relative to the initial feasible region $S_p(I_1)$, is*

$$Q(1,k,p) = \frac{\varphi(R(I_2,\{1,\ldots,p\}))}{\varphi(R(I_1,\{1,\ldots,p\}))} = \frac{k-p}{k}.$$

Proof. Corollary 1 leads to the quotient

$$\begin{aligned} Q(1,k,p) &= \frac{\varphi(R(I_2,\{1,\ldots,p\}))}{\varphi(R(I_1,\{1,\ldots,p\}))} \\ &= \frac{(k-1)!(n-p)!}{(k-1-p)!n!}\,\frac{n!(k-p)!}{k!(n-p)!} \\ &= \frac{k-p}{k}. \qquad \square \end{aligned}$$

Lemma 3 can also be applied to examine changes in the size of the feasible region when several attributes are removed from the initial set of attributes. That is, if the DM chooses to remove $l \leqslant k-p$ attributes from I, consecutive application of Lemma 3 gives

$$\begin{aligned} Q(l,k,p) &= \prod_{i=0}^{l-1} \frac{k-p-i}{k-i} = \frac{(k-p)!(k-l)!}{(k-p-l)!k!} \\ &= \binom{k-p}{l} \Big/ \binom{k}{l}. \end{aligned}$$

For example, if $p = 2$ and the DM removes four attributes from an initial set of seven attributes, the revised feasible region is $[(7-2)!(7-4)!]/[(7-2-4)!7!] = [5!3!]/[1!7!] = 1/7$ of the size of the initial feasible region.

5. Computation of decision recommendations

The elicitation of preferences through the specification attributes and corresponding rankings would usually take place iteratively so that each new statement is combined with the earlier ones to obtain a reduced feasible region (see Fig. 1). The feasible regions implied by these statements, i.e. intersections of the sets $S(I,J)$, are unions of elementary sets that correspond to complete rank-orderings (i.e., $S(r)$ for some $r \in R$). This has implications for the computational analysis of incomplete ranking information.

The development of decision recommendations based on dominance relations and decision rules does not presume that the feasible regions are unions of the elementary sets $S(r), r \in R$. Thus, in Sections 5.1 and 5.2 we only assume that the feasible region S is some non-empty subset of S_w in (1).

5.1. Dominance structures

Following Salo and Hämäläinen (1992), dominance relations for the alternatives can be established on the basis of (i) the value intervals that the alternatives can assume, subject to the requirement that the attribute weights belong to the feasible region, and (ii) the minimization of value differences between pairs of alternatives, as computed from the *pairwise bounds*

$$\mu_0(x_k, x_l) = \min_{w \in S} [V(x_k) - V(x_l)] = \min_{w \in S} \sum_{i=1}^{n} w_i [v_i(x_k) - v_i(x_l)]. \quad (4)$$

If the minimum in (4) is non-negative, the value of alternative x_k is greater than or equal to that of alternative x_l, no matter how the feasible weights are chosen. In this case, alternative x_k dominates x_l in the sense of pairwise dominance.

The computation of dominance relations does not presume that precise score information is available. For instance, if incomplete information about scores is available as intervals, the minimization problem (4) can be solved by first determining attribute-specific pairwise bounds $\mu_i(x_k, x_l)$ from the minimization problems

$$\mu_i(x_k, x_l) = \min[v_i(x_k) - v_i(x_l)].$$

These bounds can be inserted into (4) to replace the bracketed differences. In hierarchically structured value trees with attributes on several levels, the computation of pairwise bounds proceeds from the lower levels towards the topmost attribute (for details see Salo and Hämäläinen, 1992).

In many problems, it is plausible to require that all attributes are essential in the sense that they influence the alternatives' overall values. This can be modelled by requiring that the weight of each attribute is greater than some fixed lower bound $\epsilon < \frac{1}{n}$ (i.e., $w_i \geqslant \epsilon \ \forall i \in N$): for example, if—for the sake of convenience—the weight of each attribute is required to be at least one third of the average weight of an attribute, then ϵ would be $1/[3n]$. With the requirement of lower bounded attribute weights, the set S_w in (1) becomes

$$S_w(\epsilon) = \{w = (w_1, \ldots, w_n) \in S_w | w_i \geqslant \epsilon \ \forall i \in N\}. \quad (5)$$

These constraints help reduce the size of the feasible region so that dominance results for alternatives are more likely obtained. In a somewhat different but analogous setting, Cook and Kress (1990, 1991) consider the use of lower bounds on weight differences so that the weight of attribute a_i with ranking $r(a_i) = k$ exceeds the weight of the attribute a_j with ranking $k + 1$ by a certain gap ϵ; in this case, the inequality $w_i - w_j \geqslant \epsilon$ must hold.

5.2. Decision rules

Throughout the analysis, the DM can be offered tentative decision recommendations based on different *decision rules* (Salo and Hämäläinen, 2001). These rules are procedures for extrapolating a decision recommendation from a preference specification which is not complete enough to establish dominance results. Alternative decision rules include, among others, (i) the choice of an alternative with the largest possible overall value (i.e., *maximax* rule), (ii) the choice of an alternative for which the smallest possible value is largest (i.e., *maximin* rule), (iii) the choice of an alternative such that the maximum value difference to some other alternative is minimized (i.e., *minimax regret*), and (iv) the comparison of central values, computed for each alternative as the average of its smallest and largest possible values. Formally, these decision rules can be defined as follows:

$$\text{maximax}: \ \arg\max_{x_i} \left[\max_{w \in S} V(x_i) \right],$$

$$\text{maximin}: \ \arg\max_{x_i} \left[\min_{w \in S} V(x_i) \right],$$

$$\text{minimax regret}: \ \arg\min_{x_i} \left[\max_{x_k \neq x_i} \max_{w \in S} [V(x_k) - V(x_i)] \right],$$

$$\text{central values}: \ \arg\max_{x_i} \left[\max_{w \in S} V(x_i) + \min_{w \in S} V(x_i) \right].$$

Because these decision rules are based on the analysis of alternatives' overall values, the recommendations depend on the scores (i.e., $v_i(x_j)$). It is also possible to offer decision recommendations by choosing representative vectors from the feasible region S without considering scores. Here, possibilities include the computation of (i) central weights, defined by normalizing the vector $w_i' = \max_{w\in S} w_i + \min_{w\in S} w_i$, and (ii) the point of gravity of feasible region S. Also, the use of equal weights (i.e., $w_i = 1/n$) offers a benchmark against which the performance of any decision rule can be contrasted (Salo and Hämäläinen, 2001).

5.3. Computational issues

Because the feasible region $S(I,J)$ is not necessarily convex, the computation of dominance results may lead to linear optimization problems over non-convex sets. In principle, these problems can be solved by branch-and-bound algorithms or other suitable approaches (see, e.g., Taha, 1997). In particular, if the DM states that the p most important attributes are in the attribute set I, Lemma 1 implies that $S_p(I)$ can be decomposed into $|I|!/[p!(|I|-p)!]$ convex subsets with disjoint interiors. Each of these subsets could be dealt with as a separate subproblem, allowing dominance structures and decision recommendations to be derived by combining results from these subproblems.

Because the objective functions in the computation of value intervals, dominance structures and decision rules are linear, solving these optimization problems over the convex hull of the feasible region $S(I,J)$ leads to the same result as solving these problems over the feasible region. This approach is not attractive, however, because the determination of a minimal set of constraints through which this convex hull is characterized entails an additional computational effort.

An efficient approach to the determination of dominance relations and decision rules can be based on the realization that each feasible region $S(I,J)$ is the union of the sets $S(r)$, $r \in R(I,J)$. By construction, each such set is convex, and its extreme points are related to the rank-orderings r as stated in the following lemma (for the proof, see, e.g., Carrizosa et al., 1995).

Lemma 4. *Let $r \in R$ be a rank-ordering. Then the extreme points of the feasible region $S(r)$ in* (2), *$X(r)$, are*

$$X(r) = \mathrm{ext}(S(r)) = \Big\{ w \in S_w \,|\, \exists k \in \{1,\dots,n\} \text{ s.t. } w_i = \frac{1}{k}\ \forall r(a_i) \leqslant k, \quad w_i = 0\ \forall r(a_i) > k \Big\}.$$

Lemma 4 can be adapted to obtain the extreme points of $S_\epsilon(r) = S(r) \cap S_w(\epsilon)$:

$$\mathrm{ext}(S(r) \cap S_w(\epsilon)) = \Big\{ w \in S_w \,|\, \exists k \leqslant n \text{ s.t. } w_i = \frac{1-(n-k)\epsilon}{k}\ \forall r(a_i) \leqslant k, \quad w_i = \epsilon\ \forall r(a_i) > k \Big\}.$$

Based on this result, the extreme points can be enumerated at the outset (on condition that the number of attributes is not too large). Then, as the DM supplies preference statements, the resulting list can be shortened by removing those extreme points that are not compatible with the DM's statements. At any stage of the analysis, value intervals, dominance structures and decision rules can be computed by inspection. For instance, the pairwise bound for alternatives x_k, x_l is obtained from $\mu_0(x_k,x_l) = \min_{r\in R'} \min_{w\in X(r)} \sum_{i=1}^n w_i[v_i(x_k) - v_i(x_l)]$, where R' is the set of rank-orderings that are compatible with the DM's preference statements.

6. A simulation study on the computational properties of RICH

To examine the computational properties of RICH, we carried out a simulation study in which the number of attributes was $n = 5$, 7, 10 and the number of alternatives was $m = 5$, 10, 15. The attribute weights were generated by assuming a uniform distribution over the set S_w. Because in many cases it is realistic to assume that the weight of each attribute is greater than some lower bound, simulation results are presented for the case where this bound was $\epsilon = 1/[3n]$, which seemed plausible enough.

Following Salo and Hämäläinen (2001), scores for the alternatives were defined under each attribute by (i) generating random numbers from a uniform distribution over [0,1] and by (ii) normalizing the resulting numbers under each attribute. This normalization was carried out through a linear mapping in which the random number of the best performing alternative was set to one and that of the worst performing alternative was set to zero.

Five thousand problem instances were generated for each problem type (as characterized by the number of attributes (n) and alternatives (m)). Each problem instance consisted of a full combination of weights and scores in an additive preference model. The alternative with the highest overall value, i.e.,

$$\arg\max_x V(x) = \sum_{i=1}^{n} w_i v_i(x)$$

will be referred to as the *correct choice*.

The simulation study was based on the following preference statements:

- A: The DM specifies the most important attribute only.
- B: The DM specifies the two most important attributes (without taking a stance on which one is more important than the other).
- C: The DM specifies a set of three attributes which contains the two most important attributes.

Even though other kinds of preference statements are also worth studying, these three statements are nevertheless indicative of different ways of expressing incomplete preference information through rank inclusion. The sizes of the respective feasible regions (see Table 2) indicate that statement B leads to a preference specification which is more informative than statement A or statement C. This is also in keeping with the theoretical results of Sections 3 and 4.

The preference statements—and corresponding feasible regions of attribute weights—were derived from the randomly generated weights as follows.

Table 2
Size of the feasible region

n	A	B	C	Complete rank-ordering
5	0.200	0.100	0.300	8.33×10^{-3}
7	0.143	0.048	0.143	1.98×10^{-4}
10	0.100	0.022	0.067	2.76×10^{-7}

Starting from the randomly generated weight vector w, the corresponding rank-ordering r was first derived. For instance, if the simulated weights of the five attributes were $w_1 = 0.09$, $w_2 = 0.30$, $w_3 = 0.18$, $w_4 = 0.20$ and $w_5 = 0.23$, the resulting rank-ordering was $r = (5, 1, 4, 3, 2)$.

For preference statement A, the feasible region was set equal to $S_1(\{r^{-1}(1)\})$. For preference statement B, the feasible region was defined analogously as $S_2(\{r^{-1}(1), r^{-1}(2)\})$. For the third preference statement C, the set of three attributes was defined by taking the union of the two most important attributes (i.e., $r^{-1}(1)$, $r^{-1}(2)$) and a third attribute from the remaining $n-2$ attributes. This third attribute a_i was selected at random by assuming a uniform distribution over the set $N \setminus \{r^{-1}(1), r^{-1}(2)\}$, whereafter the feasible region was defined as $S_2(\{r^{-1}(1), r^{-1}(2), a_i\})$.

Results based on the above preference statements were compared to those obtained on the use of (i) equal weights (i.e. $w_i = 1/n$, $\forall i \in N$) and (ii) complete rank-ordering (where the feasible region was set equal to $S(r) \cap S_w(1/[3n])$). The comparisons were made using four decision rules (maximax, maximin, central values and minimax regret) in conjunction with two measures of efficiency, i.e., (i) the average expected loss of value relative to the correct choice and (ii) the percentage of problem instances in which the decision rule lead to the identification of the correct choice. We also computed the average number of non-dominated alternatives that would remain after (i) the specification of the above three statements A, B, and C and (ii) the use of complete rank-ordering information.

Among alternative measures of efficiency, expected loss of value is arguably the most important as it indicates how great a loss of value the DM would incur, on the average, if he or she were to follow a particular decision rule (Salo and

Hämäläinen, 2001). For a given problem instance, the corresponding loss of value is obtained from

$$\mathrm{LV} = \sum_{i=1}^{n} w_i[v_i(x^*) - v_i(x')],$$

where w_i is weight of attribute a_i, x^* is the correct choice and x' is the alternative that is recommended by a particular decision rule. In our simulation study, averaging these terms over the entire sample lead to an estimate for the expected loss of value.

In the simulation results, the use of central values as a decision rule outperformed the other decision rules, wherefore the results are presented using this decision rule only (see also Salo and Hämäläinen, 2001). In particular, an analysis of the results in Table 3 supports the following conclusions:

- Among the three statements, statement B is the most efficient and C is the least efficient one with regard to all measures of efficiency. All the three preference statements A, B and C give better results than the use of equal weights.
- Changes in the number of attributes or alternatives do not reveal consistent trends in the expected loss of value. In comparative terms, statement A performs best when there are few attributes and alternatives, while the opposite holds for statement C. For preference statement B and complete rank-ordering information, changes in the expected loss of value are relatively small across the full range of problems.
- The percentage of problem instances in which the application of decision rules leads to the identification of the correct choice tends to decrease as the number of alternatives or attributes grows; this is because there is a higher chance that some other alternative (i.e., other than the correct choice) will be favored. The share of problem instances where the correct choice is identified increases with about 5% units when complete rank-ordering information is used instead of information about the two most important attributes only (i.e., statement B). For statement C, the corresponding difference is about 15% units.
- The percentage of non-dominated alternatives decreases as the number of alternatives increases. Increasing the number of attributes leads to a larger number of non-dominated alternatives. Statement B has the smallest percentage of non-dominated alternatives across the entire spectrum of problems, because the size of the feasible region is smallest for this statement.

7. An illustrative example

To further exemplify the application of RICH, we assume there is a main contractor who is about to choose a subcontractor for an engineering project at a construction site. The contractor chooses among competing subcontractors on the basis of five attributes: (i) ability to finish the project on schedule (i.e., *punctuality*), (ii) *quality* of work, (iii) overall *cost* of the contract, (iv) *references* from earlier engagements with the respective subcontractor, and (v) possibilities for introducing *changes* into the subcontract. These attributes are essential in the sense that the weight of each is greater than a positive lower bound ϵ, which in this example is set equal to $1/[3n] = 1/15 \approx 0.0667$.

The main contractor invites tenders from three potential subcontractors. Among these, the first (x_1) is a *large firm* which is punctual and offers its services at a reasonable cost. The second one (x_2) is *a small entrepreneur* who has had difficulties in completing the project tasks on schedule. The third subcontractor (x_3) is *a medium-sized firm* which is in many ways similar to the entrepreneur, except that it is more punctual.

Score information for the three subcontractors is generated as follows. Using the first attribute (i.e., punctuality) as a benchmark, the main contractor assigns 1.00 points to the best performance level and 0.00 to the worst performance level. Then, scores reflecting incomplete information about the subcontractors are generated using these ranges as a point of reference: thus, for the first attribute, the score of the large firm is given by the interval [0.80,1.00] while the score interval for the entrepreneur is [0.00,0.20]. For the other subcontractors and attributes, scores in the [0.00,1.00]

Table 3
Simulation results

n	m	Equal weights	A	B	C	Complete rank-ordering
Expected loss of value						
5	5	0.065	0.021	0.025	0.050	0.013
	10	0.062	0.024	0.023	0.048	0.014
	15	0.059	0.027	0.022	0.043	0.015
7	5	0.060	0.024	0.021	0.045	0.013
	10	0.061	0.027	0.023	0.041	0.015
	15	0.060	0.029	0.022	0.042	0.014
10	5	0.054	0.025	0.021	0.038	0.015
	10	0.054	0.030	0.023	0.039	0.015
	15	0.056	0.031	0.023	0.038	0.016
Percentage of correct choices						
5	5	61%	76%	76%	64%	81%
	10	53%	67%	70%	57%	76%
	15	50%	62%	66%	55%	72%
7	5	60%	72%	75%	64%	81%
	10	50%	63%	67%	57%	73%
	15	47%	59%	64%	53%	72%
10	5	58%	70%	72%	64%	77%
	10	49%	58%	65%	55%	71%
	15	44%	54%	60%	51%	66%

n	m	A	B	C	Complete rank-ordering
Percentage of non-dominated alternatives					
5	5	54%	53%	65%	41%
	10	37%	36%	49%	26%
	15	30%	27%	40%	19%
7	5	69%	62%	74%	46%
	10	52%	46%	60%	30%
	15	45%	37%	51%	23%
10	5	84%	75%	85%	51%
	10	71%	61%	75%	35%
	15	64%	53%	68%	28%

range are generated in the same way, recognizing that this range is used in interpreting the attribute weights (see Table 4). The subcontractors' scores of are assumed independent, i.e., the performance of a given subcontractor may assume all the scores within its respective interval, regardless of the other subcontractors' scores.

Assume that the DM confirms that the two most important attributes are among the three first attributes, i.e., punctuality (a_1), quality (a_2) and cost (a_3). Using the notation of Section 2, we have $p = 2$ and $I = \{a_1, a_2, a_3\}$ so that the feasible region is $S_2(\{a_1, a_2, a_3\})$. According to Lemma 2 and Theorem 5, the size of this region is $\varphi(S_2(\{a_1, a_2, a_3\}) = [3!(5-2)!]/[1!5!] = 3/10$, i.e., it covers 30% of the entire weight space $S_w(\epsilon)$ in (5).

To derive dominance results, the pairwise bounds $\mu_0(x_i, x_j)$ in (4) are computed. Towards this end, the pairwise bounds $\mu_i(\cdot,\cdot)$ are first computed

Table 4
Score intervals for the alternatives

	a_1	a_2	a_3	a_4	a_5
$v_i(x_1)$	[0.80,1.00]	[0.70,0.90]	0.80	0.40	0.70
$v_i(x_2)$	[0.00,0.20]	[0.50,0.70]	[0.40,0.60]	[0.20,0.60]	[0.30,0.90]
$v_i(x_3)$	0.60	[0.50,0.70]	0.60	[0.20,0.40]	[0.30,0.90]

with regard to each attribute (see Table 5). For example, because the score intervals of the first two subcontractors on the first attribute are [0.80,1.00] and [0.00,0.20], respectively, the pairwise bound $\mu_1(x_1, x_2)$ is $0.80 - 0.20 = 0.60$.

Next, for each pair of subcontractors, the weighted sum of pairwise bounds (4) is minimized over the feasible region $S_2(\{a_1, a_2, a_3\})$ which consists of three convex sub-regions $S_2(\{a_1, a_2\})$, $S_2(\{a_1, a_3\})$ and $S_2(\{a_2, a_3\})$. The results indicate that the first alternative (large firm) is better than the third (medium-sized enterprise), because the value difference $\sum_{k=1}^{5} w_k[v_k(x_1) - v_k(x_3)]$ is positive over the entire feasible region (see Table 6). No dominance relations are obtained for the two first subcontractors because the pairwise bounds $\mu_0(x_1, x_2)$, $\mu_0(x_2, x_1)$ are negative. Thus, the DM would be asked to supply further preference information, or to accept one of the recommendations based on decision rules.

Further insights can be obtained by examining the recommendations of three decision rules, i.e., maximax, maximin, and maximization of central values. For the maximax criterion, the decision recommendation is based on the comparison of largest possible values for each subcontractor, obtained as solutions to the linear problems $V_{\max}(x_i) = \max \sum_{k=1}^{5} w_k v_k^{\max}(x_i)$ subject to the requirement that $w \in S_2(\{a_2, a_3\})$ and $w_i \geqslant 1/15$, $i = 1, \ldots, 5$. The analysis is based on $S_2(\{a_2, a_3\})$, since elsewhere in the feasible region dominance relations are already obtained. Similarly, the minimum possible values are computed from $V_{\min}(x_i) = \min \sum_{k=1}^{5} w_k v_k^{\min}(x_i)$ subject to the same

Table 5
Attribute-specific pairwise bounds

	a_1	a_2	a_3	a_4	a_5
$\min[v_i(x_1) - v_i(x_2)]$	0.60	0.00	0.20	−0.20	−0.20
$\min[v_i(x_2) - v_i(x_1)]$	−1.00	−0.40	−0.40	−0.20	−0.40
$\min[v_i(x_1) - v_i(x_3)]$	0.20	0.00	0.20	0.00	−0.20
$\min[v_i(x_3) - v_i(x_1)]$	−0.40	−0.40	−0.20	−0.20	−0.40
$\min[v_i(x_2) - v_i(x_3)]$	−0.60	−0.20	−0.20	−0.20	−0.60
$\min[v_i(x_3) - v_i(x_2)]$	0.40	−0.20	0.00	−0.40	−0.60

Table 6
Pairwise bounds μ_0

	$S_2(\{a_1, a_2\})$	$S_2(\{a_1, a_3\})$	$S_2(\{a_2, a_3\})$	Min
$\mu_0(x_1, x_2)$	0.060	0.093	−0.007	−0.007
$\mu_0(x_2, x_1)$	−0.827	−0.827	−0.560	−0.827
$\mu_0(x_1, x_3)$	0.040	0.013	0.013	0.013
$\mu_0(x_3, x_1)$	−0.373	−0.373	−0.373	−0.373
$\mu_0(x_2, x_3)$	−0.520	−0.520	−0.387	−0.520
$\mu_0(x_3, x_2)$	−0.187	−0.160	−0.253	−0.253

Table 7
Maximax, maximin and central values

Alternative	V_{max}	V_{min}	V_{ave}
x_1	0.920	0.649	0.784
x_2	0.689	0.093	0.391
x_3	0.702	0.413	0.558

constraints. Finally, central values for the three subcontractors are obtained as the average $V_{ave} = [V_{max}(x_i) + V_{min}(x_i)]/2$.

Table 7 indicates that the maximum possible value for the large firm is greater than that for the small entrepreneur (0.920 > 0.689): thus, the large firm would be recommended by the maximax rule. The application of the maximin rule leads to the same conclusion (0.649 > 0.093). Because both maximax and maximin rules support it, the large firm outperforms the small entrepreneur according to the maximization of central values as well. Thus, it would be offered as a tentative decision recommendation.

Finally, we illustrate sensitivity analyses by assuming that (i) the DM states that quality and cost are the two most important attributes and that (ii) the DM wishes to know how large a weight the first attribute (i.e., punctuality) should have to establish a dominance relationship between the first two alternatives. The revised feasible region thus becomes $S_2(\{a_1, a_2\})$, in which no dominance relations between the two first alternatives were obtained. The feasible region is now defined by $I = \{a_2, a_3\}$, $J = \{1, 2\}$, the size of which is $\varphi(S_2(\{a_2, a_3\})) = 1/10$, i.e., one third of the original feasible region $S_2(\{a_1, a_2, a_3\})$. The question about the lower bound for the weight of the first attribute can be answered by maximizing w_1, subject to the constraint that the value of the large firm is not smaller than that of the small entrepreneur, i.e., $\mu_0(x_1, x_2) = 0$. Thus, we have a maximization problem $\max w_1$ subject to the constraints $\mu_0(x_1,x_2) = 0.6w_1 + 0.2w_3 - 0.2w_4 - 0.2w_5 = 0$, $w \in S_2(\{a_2, a_3\})$, and $w_i \geqslant 1/15\ \forall i = 1, \ldots, 5$. The solution to this problem is $w_1 \approx 0.0769$, which indicates that even a small increase in the lower bound for the weight of the first attribute (i.e., punctuality) would ensure that the large firm becomes preferred to the small entrepreneur.

8. Conclusion

The elicitation of precise statements about the relative importance of attributes can pose difficulties in the development of multi-attribute decision models. To some extent, these difficulties can be alleviated by allowing the DM(s) to provide incompletely specified rank-ordering information. In a natural way, such information constrains the attribute weights so that partial dominance results can be obtained even in the absence of complete preference information. An essential feature of such an approach is that the application of decision rules makes it possible to offer decision recommendations even when dominance concepts do not allow the most preferred alternative to be inferred.

One of the features of the proposed RICH method is that the DMs need not submit preference statements that would be more explicit than what they feel confident with. Thus, the DMs may remain ambiguous about their 'true' preferences. This, in turn, may lead to a decision support process which is more acceptable from the viewpoint of group dynamics than approaches where full preference information is solicited and communicated among the group members. For example, the decision recommendation can be produced under the assumption that the most important attribute in the group's aggregate preference model is an attribute that is regarded as the most important one by some group member.

From the viewpoint of applied work, the decision support tool *RICH Decisions* ©—which is available free-of-charge for academic users at `http://www.decisionarium.hut.fi`—is important because it provides full support for the RICH method and thus enables the development of case studies based on the proposed method. Such studies will be instrumental in assessing the benefits and disadvantages of incomplete ordinal preference information in challenging decision contexts, which in turn helps set directions for further theoretical research.

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Appendix A

Proof of Theorem 1. '$\Leftarrow$': Let $\lambda \in [0,1]$, choose any $w^1 = (w_1^1, w_2^1, \ldots, w_n^1), w^2 = (w_1^2, w_2^2, \ldots, w_n^2) \in S_p(I)$, and define $w^\lambda = \lambda w^1 + (1-\lambda)w^2$. We need to show that $w^\lambda \in S_p(I)$. Because $w_i^1, w_i^2 \geqslant 0$, it follows that $w_i^\lambda = \lambda w_i^1 + (1-\lambda)w_i^2 \geqslant 0$. Likewise, $\sum_{i=1}^n w_i^1 = 1$, $\sum_{i=1}^n w_i^2 = 1$ imply that $\sum_{i=1}^n w_i^\lambda = \sum_{i=1}^n (\lambda w_i^1 + (1-\lambda)w_i^2) = \lambda \sum_{i=1}^n w_i^1 + (1-\lambda)\sum_{i=1}^n w_i^2 = 1$. Finally, because $w_k^1 \geqslant w_i^1$ and $w_k^2 \geqslant w_i^2$ for all $a_k \in I$, $a_i \notin I$, it follows that $w_k^\lambda = \lambda w_k^1 + (1-\lambda)w_k^2 \geqslant \lambda w_i^1 + (1-\lambda)w_i^2 = w_i^\lambda$ if $a_k \in I$, $a_i \notin I$. Thus, $w^\lambda \in S_p(I)$.

'$\Rightarrow$': Assume that $p < |I|$ and choose some $I' \subset I$ such that $|I'| = p$. Since $|I| < n$, there exists an attribute $a_k \notin I$. Put $I_1 = I' \cup \{a_k\}$ define the weight vector w^1 by letting $w_l^1 = 1/(p+1), a_l \in I_1, w_l^1 = 0, a_l \notin I_1$. Next, choose attributes $a_i \in I \setminus I'$, $a_j \in I'$ and define the attribute set $I_2 = (I' \setminus \{a_j\}) \cup \{a_i\} \cup \{a_k\}$ and define w^2 by letting $w_l^2 = 1/(p+1)$, $a_l \in I_2$, $w_l^2 = 0$, $a_l \notin I_2$. By construction, $w^1, w^2 \in S_p(I)$ because they contain p elements in I that are greater than or equal to all the other elements. However, this is not true for the vector $w^3 = (1/2)w^1 + (1/2)w^2$ where $w_l^3 = 1/(p+1), a_l \in \{a_k\} \cup (I' \setminus \{a_i, a_j\}), w_l^3 = 1/(2(p+1))$, $a_l \in \{a_i, a_j\}, w_l^3 = 0$, otherwise. Thus, w^3 contains only $p-1$ elements in I that are larger than the other elements so that it does not belong to $S_p(I)$, which implies that $S_p(I)$ is not convex. $\square$

Proof of Lemma 1. By assumption, there exist attributes a_k, a_l such that $a_k \in I_1 \setminus I_2$, $a_l \in I_2 \setminus I_1$. Let $\alpha > 0$ and define the vector δ so that $\delta_k = \alpha$, $\delta_l = -\alpha$, $\delta_i = 0$, $i \neq k, l$. If $\exists w \in \mathrm{int}(S_p(I_1)) \cap \mathrm{int}(S_p(I_2))$, then for some $\varepsilon > 0$ the weight vectors $w^1 = w + \varepsilon\delta$ and $w^2 = w - \varepsilon\delta$ are also in $\mathrm{int}(S_p(I_1)) \cap \mathrm{int}(S_p(I_2))$. In particular, since $w^1 \in S_p(I_2)$, it follows that $w_l - \varepsilon\alpha \geqslant w_k + \varepsilon\alpha \Rightarrow w_l - w_k \geqslant 2\varepsilon\alpha > 0$. Also, since $w^2 \in S_p(I_1)$, it follows that $w_k - \varepsilon\alpha \geqslant w_l + \varepsilon\alpha \Rightarrow w_k - w_l \geqslant 2\varepsilon\alpha > 0$, in contradiction with the earlier inequality $w_l - w_k > 0$. $\square$

Proof of Theorem 2. The cases $|I| \geqslant |J|$ and $|I| < |J|$ can be dealt with separately. First, if $|I| \geqslant |J|$ and $w \in S(I,J)$, then $w \in S(r)$ for some $r \in R(I,J)$. Thus, for some $I' \subseteq I, |I'| = |J|$ we have $r(I') = J$. But then $r(\overline{I'}) = \overline{J}$. Since $\overline{I} \subseteq \overline{I'}$, we have $r(\overline{I}) \subseteq \overline{J}$ and $r \in R(\overline{I}, \overline{J})$. Second, if $|I| < |J|$ and $w \in S(I,J)$, then $w \in S(r)$ for some $r \in R(I,J)$ such that $r(I) \subset J$. Because $|I| < |J|$, there exists a set I' such that $I \subset I', |I'| = |J|$ and $r(I') = J$. Thus we have $r(\overline{I'}) = \overline{J}$. By construction, $|\overline{I'}| = |\overline{J}|$ and $\overline{I'} \subset \overline{I}$ so that $r \in R(\overline{I}, \overline{J})$. This far it has been shown that $w \in S(I,J) \Rightarrow w \in S(\overline{I}, \overline{J})$. Since $\overline{\overline{I}} = I$, $w \in S(\overline{I}, \overline{J}) \Rightarrow w \in S(I,J)$. $\square$

Proof of Theorem 3. Item (a). '$\Rightarrow$': If $w \in S(I_2, J)$, there exists a rank-ordering $r \in R(I_2, J)$ such that $w \in S(r)$ and $r(I_2) \subseteq J$. But since $I_1 \subset I_2$, it follows that $r(I_1) \subset J$. In addition, by Definition 1 this implies that $r \in R(I_1, J)$ and $w \in S(I_1, J)$. To prove that $S(I_2, J)$ is a *proper* subset of $S(I_1, J)$, we construct a rank-ordering r' such that $r'(I_1) \subseteq J$ and $r'(a_k) \notin J$ for some $a_k \in I_2$, $a_k \notin I_1$ (such an order exists because $|J| < n$). Then $r' \in R(I_1, J)$, but $r' \notin R(I_2, J)$.

'$\Leftarrow$': Take any $w \in S(I_2, J)$. Then there exists some $r \in R(I_2, J)$ such that $w \in S(r)$ and $r(I_2) \subseteq J$. Because $S(I_2, J) \subset S(I_1, J)$, it follows that $r \in R(I_1, J)$ and hence $r(I_1) \subseteq J$. Now, if $I_1 \not\subset I_2$, there exists some $a_k \in I_1$, $a_k \notin I_2$. Because $a_k \in I_1$, we have $i_k = r(a_k) \in J$. Also, since $|J| < n$, there is some a_l such that $i_l = r(a_l) \notin J$. By construction, this a_l is not in I_1 or I_2. Next, construct the rank-ordering r' so that $r'(a_k) = i_l$, $r'(a_l) = i_k$ and $r'(a_i) = r(a_i)$, $\forall i \neq k, l$. Then $r' \in R(I_2, J)$ but $r' \notin R(I_1, J)$. But this violates the assumption $S(I_2, J) \subset S(I_1, J)$, leading to a contradiction.

Item (b): By Theorem 2, $S(I,J) = S(\overline{I}, \overline{J})$. Thus, we have to prove that $I_2 \subset I_1 \iff S(\overline{I_2}, \overline{J}) \subset S(\overline{I_1}, \overline{J})$, which is equal to $\overline{I_1} \subset \overline{I_2} \iff S(\overline{I_2}, \overline{J}) \subset S(\overline{I_1}, \overline{J})$; but this follows from item (a) above. $\square$

Proof of Theorem 4. Item (a). '$\Rightarrow$': If $w \in S(I, J_2)$, then there is a rank-ordering $r \in R(I, J_2)$ and an attribute set $I' \subseteq I$ such that $|I'| = |J_2|$ and $r(I') = J_2$. Next, define the set $I'' = \{a_i \in I' | r(a_i) \in J_1\}$. Because $J_1 \subset J_2$, we have $|I''| = |J_1|$ so that $r \in R(I, J_1)$; hence $w \in S(I, J_1)$ as well.

'$\Leftarrow$': Assume that $S(I,J_2) \subseteq S(I,J_1)$. By Theorem 2, this is equivalent to $S(\overline{I},\overline{J_2}) \subseteq S(\overline{I},\overline{J_1})$. From the assumptions it also follows that $|\overline{I}| \leqslant |\overline{J_1}|, |\overline{J_2}|$. Choose a $w \in S(\overline{I},\overline{J_2})$. There then exists a rank-ordering r such that $w \in R(\overline{I},\overline{J_2})$, i.e. $r(\overline{I}) \subseteq J_2$. Since $S(\overline{I},\overline{J_2}) \subseteq S(\overline{I},\overline{J_1})$, $r \in R(\overline{I},\overline{J_1})$ so that $r(\overline{I}) \subset J_1$, too. Contrary to the claim $J_1 \subseteq J_2$, assume that there is an i_k such that $i_k \in J_1, i_k \notin J_2$. Then the rank-ordering associates i_k with an attribute $a_k \in I$ (because $r(\overline{I}) \in \overline{J_1}$ and $r(\overline{I}) \in \overline{J_2}$). Also, choose an $a_l \in \overline{I}$ and define $i_l = r(a_l)$; by construction, $i_l \notin J_1$, $i_l \notin J_2$. Next, define a rank-ordering r' so that $r'(a_k) = i_l$, $r'(a_l) = i_k$ and $r'(a_i) = r(a_i)$, $\forall i \neq k, l$. Then $r' \in R(\overline{I},\overline{J_2})$, but $r' \notin R(\overline{I},\overline{J_1})$, which violates the assumption $S(I,J_2) \subseteq S(I,J_1)$.

Item (b): According to Theorem 2, $S(I,J) = S(\overline{I},\overline{J})$. Thus, we have to show that $J_2 \subset J_1 \iff S(\overline{I},\overline{J_2}) \subset S(\overline{I},\overline{J_1})$, which is equal to $\overline{J_1} \subset \overline{J_2} \iff S(\overline{I},\overline{J_2}) \subset S(\overline{I},\overline{J_1})$. This follows directly from item (a) above. $\square$

Proof of Theorem 5. Clearly, $\varphi(\emptyset) = 0$. Assume that $R' \in \mathscr{P}(R)$ and that $R_1, \ldots, R_M \in R$ are disjoint sets of rank-orderings such that $R' = \biguplus_{i=1}^{M} R_i$. By construction, the intersection of any R_i and $R_j, i \neq j$ is empty; thus, $|R'| = \sum_{i=1}^{M} |R_i|$, which implies that $\varphi(R') = \frac{|R'|}{n!} = \sum_{i=1}^{M} \frac{|R_i|}{n!} = \sum_{i=1}^{M} \varphi(R_i)$. Finally, since the total number of different rank-orderings is $n!$, we have $\varphi(R) = 1$. $\square$

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