Finite-discrete element modelling of sea ice sheet elasticity, sea ice sheet fracture, and ice-structure interaction

A three-dimensional, lattice-based approach

Ville-Pekka Lilja

Finite-discrete element modelling of sea ice sheet elasticity, sea ice sheet fracture, and ice-structure interaction

A three-dimensional, lattice-based approach

Ville-Pekka Lilja

A doctoral dissertation completed for the degree of Doctor of Science (Technology) to be defended, with the permission of the Aalto University School of Engineering, via a remote connection link: <https://aalto.zoom.us/j/63875924913> on 9.12.2020 at 12 o'clock.

**Aalto University
School of Engineering
Department of Mechanical Engineering
Solid Mechanics**

Supervising professor

Assistant professor Arttu Polojärvi (D.Sc. Tech.), Aalto University, Finland

Thesis advisor

Professor Jukka Tuhkuri (D.Sc., Tech.), Aalto University, Finland

Preliminary examiners

Professor Holm Altenbach (Dr.-Ing. habil., Dr.h.c. mult.), Otto von Guericke University Magdeburg, Germany
Associate professor Raed Lubbad (Ph.D.), Norwegian University of Science and Technology, Norway

Opponent

Professor Holm Altenbach (Dr.-Ing. habil., Dr.h.c. mult.), Otto von Guericke University Magdeburg, Germany

Aalto University publication series

DOCTORAL DISSERTATIONS 188/2020

© 2020 Ville-Pekka Lilja

ISBN 978-952-64-0139-3 (printed)

ISBN 978-952-64-0140-9 (pdf)

ISSN 1799-4934 (printed)

ISSN 1799-4942 (pdf)

<http://urn.fi/URN:ISBN:978-952-64-0140-9>

Images: The cover figure has been adapted from V. P. Lilja, A. Polojärvi, J. Tuhkuri, and J. Paavilainen: "Effective Tensile Strength of an Ice Sheet Using a Three-Dimensional FEM-DEM Approach," Figure 6 (in Proceedings of the 24th International Conference on Port and Ocean Engineering under Arctic Conditions Held in Busan, South Korea, 11-16 June 2017).

Unigrafia Oy
Helsinki 2020

Finland

Author

Ville-Pekka Lilja

Name of the doctoral dissertation

Finite-discrete element modelling of sea ice sheet elasticity, sea ice sheet fracture, and ice-structure interaction - A three-dimensional, lattice-based approach

Publisher School of Engineering

Unit Department of Mechanical Engineering

Series Aalto University publication series DOCTORAL DISSERTATIONS 188/2020

Field of research Mechanical Engineering

Manuscript submitted 25 August 2020

Date of the defence 9 December 2020

Permission for public defence granted (date) 13 October 2020

Language English

☐ **Monograph**

☒ **Article dissertation**

☐ **Essay dissertation**

Abstract

In this thesis, the elastic and inelastic properties of an ice sheet modelled by a new hybrid, three-dimensional finite-discrete element (FE-DE) method were examined. Ice-structure interaction between an ice sheet and a conical offshore structure was studied as well. By this new method, an ice sheet is modelled with undeformable, i.e. rigid, discrete elements. The mass centroids of the discrete elements connect then via an in-plane beam lattice of co-rotational, viscously damped, de-cohesive Timoshenko beam finite elements. A centroidal-Voronoi-tessellation-based iterative scheme (CVT) was applied in creating the studied FE-DE meshes, i.e. the modelled ice sheets. Due to the internally damped, de-cohesive, lattice-based construction, the mechanical response of a modelled ice sheet turns out to be both strain rate- and size-dependent (dependent on both the absolute and relative sizes), the investigation of which formed an integral part of the present study.

A general objective of this thesis was to study the applicability of the new, hybrid FE-DE method in modelling the elasticity and fracture of sea ice sheets. In order to understand the effects of scale and to demonstrate the feasibility of the approach in studying ice mechanics applications in general, i.e. the ice-structure interaction, several conceptually simple constitutive tests with square FE-DE sheet samples of varying side lengths, thicknesses, and discrete element sizes were performed.

The results presented gave a partial guideline for choosing the microscale material parameters of a CVT-tessellated, lattice-based FE-DE model of an ice sheet in order to achieve a desired macroscale response, both elastic and inelastic. Furthermore, the results provided substantial insight into the functional dependencies each studied physical quantity has. In addition, the elastic bending tests showed that the model is able to emulate a free Kirchhoff-Love plate in bending on a Winkler-type foundation with a good accuracy. Finally, a new in-direct approach to compute ice-breaking loads on a conical offshore structure was proposed. The novelty of the approach lies in its simplicity. It provides a device not only to compute cone ice loads but also to investigate the ability of a numerical method to produce both radial and circumferential cracking.

While the contents of this thesis were strictly restricted to applications that are closely related to ice mechanics, i.e. the modelling of intact sea ice sheets, their fracture, and ice-structure interaction, the results presented should apply to other cohesive, lattice-based models on other application areas as well, in the *ab-initio*-type constitutive modelling of ceramic matrix composites or concrete for instance.

Keywords Ice, ice-structure interaction, plates, beam lattice networks, fracture mechanics, size effect, hybrid finite-discrete element methods, numerical algorithms, centroidal Voronoi tessellation

ISBN (printed) 978-952-64-0139-3

ISBN (pdf) 978-952-64-0140-9

ISSN (printed) 1799-4934

ISSN (pdf) 1799-4942

Location of publisher Helsinki

Location of printing Helsinki **Year** 2020

Pages 278

urn <http://urn.fi/URN:ISBN:978-952-64-0140-9>

Tekijä

Ville-Pekka Lilja

Väitöskirjan nimi

Finite-discrete element modelling of sea ice sheet elasticity, sea ice sheet fracture, and ice-structure interaction - A three-dimensional, lattice-based approach

Julkaisija Insinööritieteiden korkeakoulu**Yksikkö** Konetekniikka**Sarja** Aalto University publication series DOCTORAL DISSERTATIONS 188/2020**Tutkimusala** Konetekniikka**Käsikirjoituksen pvm** 25.08.2020**Väitöspäivä** 09.12.2020**Väittelyluvan myöntämispäivä** 13.10.2020**Kieli** Englanti☐ **Monografia**☒ **Artikkeliväitöskirja**☐ **Esseeväitöskirja****Tiivistelmä**

Tässä opinnäytetyössä tutkittiin uudella kolmidimensionaalisella, yhdistetyllä diskreetti-elementtimenetelmällä (FE-DE) mallinnetun jäälautan elastisia ja vauriomekaanisia ominaisuuksia. Niin ikään työssä tutkittiin FE-DE-menetelmällä mallinnetun jäälautan ja kartiomaisen offshore-rakenteen välisen vuorovaikutuksen mekaniikkaa. Yhdistettyä FE-DE-menetelmää käytettäessä jäälautta diskretoidaan deformaattomattomin diskreetein elementein, joiden massakeskipisteet yhdistyvät tasomaisen palkkielementtiverkon välityksellä. Palkkielementtiverkon elementit ovat korotationaalisia, lineaariviskoosisesti vaimennettuja ja kohesiivisiä Timoshenko-palkkielementtejä. Työssä tutkitut jäälautat generoitiin keskeis-Voronoi-tessellointiin perustuvalla iteratiivisella menetelmällä. Palkkielementtiverkon topologiasta, palkkielementtien sisäisestä vaimennuksesta ja vaurioitumisen kuvaamisesta käytettävän konstitutiivisen mallin kohesiivisuudesta johtuen mallinnetun jäälautan efektiivinen vaste osoittautuu sekä venymänopeudesta että tutkittavan kappaleen koosta riippuvaksi.

Opinnäytetyössä tutkittiin, tiivistetysti, onko uusi yhdistetty FE-DE-menetelmä soveltuva kuvaamaan merijäälauttojen elastisia ja vauriomekaanisia ominaisuuksia. Mallinnettavan jäälautan absoluuttisen ja suhteellisen koon vaikutuksen ymmärtämiseksi, kuten myös menetelmän yleisen toimivuuden osoittamiseksi jäämekaniikan alaan kuuluvien ilmiöiden numeerisessa mallintamisessa, joista keskeisin on jää-rakenne-vuorovaikutus, toteutettiin työssä useita verrattain yksinkertaisia numeerisia kokeita neliömäisillä jäälautoilla, joiden sivunpituutta, paksuutta ja käytettyä elementtikokoa varioitiin.

Työn tuloksena yhdistetyllä FE-DE-menetelmällä mallinnettavan jäälautan mikromekaanisten materiaaliparametrien valitseminen siten, että lautan efektiivinen konstitutiivinen vaste, sekä elastinen että vauriomekaaninen, on toivotunlainen, helpottuu. Mallinnetun jäälautan mekaanisten ominaisuuksien keskinäisiä riippuvuussuhteita kuvaavia relaatioita löydettiin useita. Tehdyt elastiset taivutustestit osoittivat lisäksi, että menetelmä on kykenevä kuvaamaan vapaan Kirchhoff-Love-laatan taivutusta Winklerin alustalla hyvällä tarkkuudella. Työssä esiteltiin lopuksi uusi epäsuora menetelmä jääkuormien laskemiseksi kartiomaisella offshore-rakenteella. Menetelmä tarjoaa lisäksi keinon sen osoittamiseksi, että kehitetty numeerinen menetelmä on kykenevä kuvaamaan sekä radiaalisten että kehämäisten säröjen synnyn.

Työn sisältö rajoittui jäämekaniikan sovellutuksiin. Työn tulosten tulisi kuitenkin olla sovellettavissa myös muunlaisten kohesiivisina palkkiverkkorakenteina mallinnettavissa olevien materiaalien tutkimuksessa, esimerkiksi keraamikkomposiittien tai betonin *ab initio*-tyyppisessä konstitutiivisessa mallinnuksessa.

Avainsanat Jää, jää-rakenne-vuorovaikutus, laatat, palkkiverkkorakenteet, murtumismekaniikka, kokoeffekti, yhdistetty diskreetti-elementtimenetelmä, numeeriset algoritmit, keskeis-Voronoi tessellaatio

ISBN (painettu) 978-952-64-0139-3**ISBN (pdf)** 978-952-64-0140-9**ISSN (painettu)** 1799-4934**ISSN (pdf)** 1799-4942**Julkaisupaikka** Helsinki**Painopaikka** Helsinki**Vuosi** 2020**Sivumäärä** 278**urn** <http://urn.fi/URN:ISBN:978-952-64-0140-9>

Preface

This thesis was prepared in Aalto University between the years of 2015 and 2020. I was financed by the Academy of Finland through the project “Discrete Numerical Simulation and Statistical Analysis of the Failure Process of a Non-Homogeneous Ice Sheet Against an Offshore Structure,” acronym “DICE”; National Graduate School in Engineering Mechanics; and the Doctoral Program of the School of Engineering. I acknowledge all of my funders with gratitude.

I would like to first thank my advisor and supervisor, assistant professor Arttu Polojärvi (D.Sc., Tech.), for all of his support and guidance. Without his expertise, the completion of this thesis would not have been possible. He has been a very helpful and nice boss throughout my entire time here in Otaniemi. He also provided me with truly pleasant working conditions that enabled me to concentrate on the present investigation. Thank you Arttu.

Next, I would like to thank my co-advisor, professor Jukka Tuhkuri (D.Sc., Tech.), for the numerous insightful discussions we had during the preparation of this thesis. I would like to thank him also for all of his efforts while applying for funding and, most importantly, for providing me an opportunity to work on the present project. Thank you Jukka. My sincere thanks go also to my friend and colleague Jani Paavilainen (D.Sc., Tech.) for all of his support and guidance he has provided me during the preparation of this thesis. In conjunction, I wish to express my thanks to Laura Mendoza (M.A.) as well for her excellent work in checking the language of the compilation part of my dissertation.

My special thanks go to professor John Dempsey (a FiDiPro, Ph.D., and a Kiwi!) from Clarkson University (Potsdam, New York, U.S.A.) for all of his support, guidance, and friendship throughout these years. I had both the pleasure and privilege of becoming friends with him while he was on a sabbatical leave from Clarkson University in Aalto University from 2015 to 2016. He is an endless source of information and a truly nice guy,¹ always

¹John once kindly told me that he has probably forgotten more than what I now, or ever will, know about mechanics. I cannot but agree. Hats off to John!

willing to help and support. No matter the problem, he had an answer or at least an idea on how to proceed. Thank you John. I appreciate it.

Next, I would like to thank all of my other colleagues for the creation of the pleasant atmosphere in our group. We have had fun. At least I have! I would like to mention especially Hanyang Gong (M.Sc., Tech.), Iman El Gharamti (M.Sc., Tech.), Ida Lemström (M.Sc., Tech.), Kari Santaoja (D.Sc., Tech.), Jouni Freund (D.Sc., Tech.), Tommi Mikkola (D.Sc., Tech.), Janne Ranta (D.Sc., Tech.), and Jani Romanoff (D.Sc., Tech.). My thanks go also to Mr. Kauko Laajanaho for the many friendly discussions we had during my time here in Otaniemi.

Finally, I would like to thank my family. My wife Riina Lilja (M.A. and M.Sc., Econ.) and our precious Liia and Vilho (plus Väinö and Kusti) have supported me in so many ways. My sincere thanks go also to my parents, my mother Saimi Lilja (M.A.) and my father Pekka Lilja (Ph.D., a professor emeritus), for all of their caring and support, not just during the preparation of this thesis, but throughout my whole life in general. Lastly, I would like to thank my mother-in-law, Mrs. Irja Ikonen, for all of her kindness and support she has provided our family over these years.

I wish to express my sincere gratitude to you all.

In Espoo, November 3, 2020,

Ville-Pekka Lilja

Contents

Preface	7
Contents	9
List of Publications	11
Author's Contribution	13
Original Features	15
1. Introduction	17
1.1 Background	17
1.2 Research approach	19
1.3 Scope and objectives	21
1.4 State-of-the-art	23
1.5 Structure of the compendium	26
2. Methods	27
2.1 Description of the numerical model	27
2.2 Centroidal Voronoi tessellation	29
2.3 Equations of motion	31
2.4 Description of the numerical experiments	31
3. Results	43
3.1 Elastic in-plane response	43
3.2 Elastic out-of-plane response	49
3.3 Inelastic in-plane response – tensile fracture	61
3.4 Inelastic out-of-plane response – penetration fracture	68
3.5 Ice-structure interaction response	71
4. Discussion	75
4.1 On the contents of Publication I	75
4.2 On the contents of Publication II	81

4.3	On the contents of Publication III	83
5.	Conclusions	89
5.1	Summary and concluding remarks	89
5.2	Suggestions for future research	92
	References	95
	Errata	105
	Publications	107

List of Publications

This thesis consists of an overview and the following publications, which are referenced in the text by their Roman numerals. It is emphasised that the overview *per se* contains no new scientific results or, for the most part, other new material, but is an *adapted compendium* of the following publications and thus their contents.

- I** Ville-Pekka Lilja, Arttu Polojärvi, Jukka Tuhkuri, Jani Paavilainen. Effective material properties of a finite element-discrete element model of an ice sheet. *Computers & Structures*, 224, 106107, September 2019.
- II** Ville-Pekka Lilja, Arttu Polojärvi, Jukka Tuhkuri, Jani Paavilainen. A free, square, point-loaded ice sheet: A finite element-discrete element approach. *Marine Structures*, 68, 102644, November 2019.
- III** Ville-Pekka Lilja, Arttu Polojärvi, Jukka Tuhkuri, Jani Paavilainen. Finite-discrete element modelling of sea ice sheet fracture. Submitted to *International Journal of Solids and Structures*, June 2020.

Author's Contribution

Publication I: “Effective material properties of a finite element-discrete element model of an ice sheet”

The author conceived the presented ideas, planned the numerical experiments, performed the simulations, gathered and analysed the data, and wrote the manuscript. The FE-DE code applied was originally developed, in parts, by Arttu Polojärvi and Jani Paavilainen. The author assisted Arttu Polojärvi in developing the code further. All authors discussed the results and commented on the manuscript. John Dempsey proofread and commented on the manuscript. Jukka Tuhkuri supervised the project.

Publication II: “A free, square, point-loaded ice sheet: A finite element-discrete element approach”

The author conceived most of the presented ideas, planned the numerical experiments, performed the FE-DE and Abaqus simulations, gathered and analysed the data, and wrote the manuscript. The FE-DE code applied was originally developed, in parts, by Arttu Polojärvi and Jani Paavilainen. The author assisted Arttu Polojärvi in developing the code further. All authors discussed the results and commented on the manuscript. John Dempsey proofread and commented on the manuscript. Jukka Tuhkuri supervised the project.

Publication III: “Finite-discrete element modelling of sea ice sheet fracture”

The author conceived the presented ideas (except of the in-direct approach, which was due to John Dempsey), planned the numerical experiments,

performed the simulations, gathered and analysed the data, and wrote the manuscript. The FE-DE code applied was originally developed, in parts, by Arttu Polojärvi and Jani Paavilainen. The author assisted Arttu Polojärvi in developing the code further. All authors discussed the results and commented on the manuscript. Jukka Tuhkuri supervised the project.

Language check

The language of the compilation part of my dissertation has been checked by Laura Mendoza (M.A.). I have personally examined and accepted/rejected the results of the language check one by one. This has not affected the scientific content of my dissertation.

Original Features

The following features and/or themes in this thesis are believed to be original:

In **Publication I**:

- i*) The in-plane size and rate effects of a viscoelastic, cellular, plate-like structure were analysed by a hybrid, three-dimensional finite-discrete element (FE-DE) method,
- ii*) a centroidal-Voronoi-tessellation-based meshing strategy applicable with a hybrid FE-DE method was introduced.

In **Publication II**:

- i*) The deflection of a free, square, point-loaded plate on a tensionless Winkler-type foundation was investigated by a hybrid, three-dimensional FE-DE method,
- ii*) the size effects as regards to the deflection were analysed as well as size estimates for an FE-DE sheet sample to represent either an infinite ice sheet or a semi-infinite ice sheet with a free edge sought for.

In **Publication III**:

- i*) The in-plane and out-of-plane inelastic responses – more specifically size and rate effects – of a cellular, viscoelastic, quasi-brittle, plate-like structure floating on a buoyant foundation were examined by a hybrid, three-dimensional FE-DE method,
- ii*) an in-direct approach to compute ice-breaking loads on a conical offshore structure, as well as to investigate the ability of a numerical method to produce both radial and circumferential cracking, was proposed.

1. Introduction

This chapter provides the background relevant to the thesis. In order to put things into a context, a big picture is contemplated upon first. Next, the research approach, scope, and the objectives are discussed. The chapter ends with a section on the state-of-the-art.

1.1 Background

The ice-covered areas in both the Arctic and Antarctic polar regions are shrinking in size. A consensus exists that the cause is the climate change, which in turn is widely recognised to be a consequence of detrimental human activity – excess greenhouse gas emissions in particular. From both an environmental and a societal point of view, the sea ice loss is a global disaster. Most coastal regions around the world, for instance, are densely populated. These regions may in the not-too-distant future become inhabitable because of the rising sea level. Certain sectors of the economy may, on the other hand, benefit. New shipping routes will most probably open, for example. Navigation on these still ice-infested waters necessarily requires vessels that have been ice strengthened. In order to be able to construct such vessels, knowledge about ice-structure interaction loads, as well as of ice mechanics in general, is required.

A major global challenge is to be able to produce electricity in an environmentally sustainable manner. The use of fossil fuels will decrease, whereas the renewable energy sources, such as solar and wind power, will gain in significance. The most potential regions for the new, prospective wind farms are generally offshore. In order to build and operate these farms, it is necessary to understand the local environmental conditions and thus also the loads imposed on offshore structures by sea ice.

Transportation of goods by sea is commonplace. The national interest of Finland in sea ice physics, for example, stems from the fact that about 78 % of its imports and roughly 92 % of its exports pass via the Baltic Sea [Finnish Customs, 2020]. It is not unprecedented that during winter,

the whole sea up to the Danish straits freezes. The estimates for the probability of occurrence of such a severe ice winter range from about 3, [Myrberg et al., 2006], to around 5-10 % [Luomaranta et al., 2010]. On an average, approximately 40 % of the whole sea is covered in ice every winter [Haapala et al., 2015]. Because of the climate change, ice winters can, paradoxically, turn out also even more severe [Tang et al., 2013]. Ice mechanics is thus both a relevant and a topical field of research.

Data on ice loads can be obtained both through experimental campaigns and by using analytical or numerical methods. *In-situ* experiments provide the most authentic results but are usually either overly expensive or technically too demanding to organise. The local conditions can, in addition, be so severe that conducting measurements cannot be done safely. Under such circumstances, use is made either of laboratory tests, or of analytical or numerical methods. Frequently, the analytical and numerical methods are able to provide insight into phenomena that would be unamenable to a scientific study otherwise. A problem can, for example, be easily parametrised, unlike what is the case with experimental methods. This thesis focuses on numerical methods.

Figure 1.1. A rubble pile formed in front of Kemi 1 lighthouse in the Gulf of Bothnia.
Photo: Mauri Määttänen.

1.2 Research approach

The contents of this thesis belong to the field of computational mechanics. A primary objective of the Aalto University Arctic Technology and Ice Mechanics research group is to be able to numerically simulate the three-dimensional ice-structure interaction processes in order to better understand the phenomena that occur. These processes are generally complex and involve all types of non-linearities, i.e. material, geometric, and contact. During a typical process, a sea ice sheet first collides with an offshore structure, then deforms, and finally breaks into fragments. These fragments may next form a rubble pile in front of the structure and thus affect the subsequent processes, Figure 1.1. Within the rubble pile, individual ice blocks may undergo further finite translations and rotations, collide with the adjacent blocks and the structure, and break to even

Figure 1.2. An FE-DE sheet sample with a side length of $L = 160$ m, a thickness of $h = 1.5$ m, and a discrete element size of $l = 3h$. After an extrusion in the out-of-plane direction, the polygons in cyan become the discrete elements. The red lines denote the longitudinal axes of the Timoshenko beam finite elements. The figure has been reproduced from Publication III, Figure 3.

smaller fragments. Because of the complexity of such strongly non-linear processes, analytical modelling approaches fail. Numerical modelling is, in turn, computationally expensive. The computational complexity of an often used explicit finite element (FE) method, for example, scales as $O(n, 1/L_e)$, where n is the number of nodes and L_e the element size. A three-dimensional problem making use of solid/plate finite elements, with sheets possibly having in-plane dimensions of hundreds of meters, appears thus intractable. The development of applicable methods is necessarily required.

In this thesis, the elastic and inelastic properties of an ice sheet modelled by a new hybrid, three-dimensional finite-discrete element (FE-DE) method are examined. Ice-structure interaction between an ice sheet and a conical offshore structure is studied as well. By this new method, an ice sheet is modelled with rigid discrete elements. The mass centroids of the discrete elements connect then via an in-plane beam lattice of co-rotational, viscously damped, de-cohesive Timoshenko beam finite elements, Figure 1.2. While an ice-structure interaction simulation is in progress, the kinematics and the post-fracture contacts get resolved by the discrete elements, the elasticity and the description of fracture by the beam finite elements. The FE-part of the method is an explicit finite element scheme but is computationally less expensive than a solid/plate model because of the reduced number of degrees of freedom and the larger element sizes.

Because of the internally damped, de-cohesive, lattice-based construction, the mechanical response, i.e. stiffness and strength, of a modelled ice sheet turns out to be both strain rate- and size-dependent (dependent on both the absolute and relative sizes), the investigation of which forms an integral part of the present study. Note that as regards to the stiffness, the size effect is to be taken as an artefact of the developed numerical model. It is *not* claimed that such an effect would exist in ice on size scales presently studied. As regards to the fracture, a deterministic (energetic) size effect, on the contrary, appears to exist [Dempsey et al., 1999; Bažant, 2002]. Both of these size effects, in addition to the rate effect, are examined in this thesis. An interested reader can find an account of the origins of these effects in, for example, [Bažant et al., 1990; Bažant and Planas, 1997; Gibson and Ashby, 1999], and [Ostoja-Starzewski, 2002].

Previously, a two-dimensional analogue of the technique has been applied in studying ice-structure interaction [Paavilainen et al., 2009]. While enabling a study of an ice-structure interaction process for a very wide structure, a two-dimensional model is limited for not allowing simulations with slender or conical structures that are often encountered in the Arctic. The present three-dimensional concept was therefore, originally, outlined in [Paavilainen, 2010, pp. 25-27]. It is believed that the meshing technique introduced in this thesis has not been, in the context of a hybrid FE-DE method, treated elsewhere. For a more conventional Voronoi-tessellation-

based model, see, e.g., [van den Berg, 2016].

1.3 Scope and objectives

While it is thus known that lattice-based models introduce scale effects and that de-cohesive damage models exhibit deterministic size effects, there has not been an in-depth study of such effects for a finite, three-dimensional, plate-like, internally damped, quasi-brittle structure floating on a buoyant foundation. Such a study is performed in this thesis. Note that the adjective *buoyant* is here used because of the fact that the modelled foundation is not a pure Winkler-type foundation. The discrete elements are able to both submerge and to rise above the water level.

A general intent of this thesis was to study the applicability of the new hybrid FE-DE method in modelling the elasticity and fracture of sea ice sheets. In order to achieve the objective of understanding the effects of scale and the loading rate, as well as to demonstrate the feasibility of the approach in studying ice mechanics applications in general, i.e. the ice-structure interaction, the main objectives of the work presented were set as follows:

- i*) quantify the effect of scale regarding elastic deformations both in the in-plane and the out-of-plane directions,
- ii*) relate the microscale material parameters to the macroscale constitutive behaviour of a modelled ice sheet where applicable,
- iii*) demonstrate the applicability of the method in describing sea ice sheet fracture both in the in-plane and the out-of-plane directions,
- iv*) quantify the effect of scale regarding both the tensile and the vertical breakthrough strengths of a modelled ice sheet,
- v*) quantify the effect of loading rate regarding both the elastic moduli and the tensile strength of a modelled ice sheet, and
- vi*) show that the method can be applied in studying ice-structure interaction.

In order to achieve these objectives, the contents of the individual publications were defined as follows:

In **Publication I**, the in-plane elastic response of a cellular, viscously damped, plate-like structure was studied. The effective, in-plane Young's, shear, and bulk moduli of a modelled ice sheet were computed. The publication focused on establishing a connection between the microscale material parameters and the observed, effective macroscale response of a modelled

ice sheet. Both the scale and the rate effects were examined. The publication presented, in addition, a new centroidal-Voronoi-tessellation-based meshing technique (CVT) applicable with a hybrid, FE-DE-based modelling strategy.

In **Publication II**, the out-of-plane elastic response of a similar cellular, plate-like structure was investigated. A primary objective was to examine whether a CVT-tessellated, unstructured, in-plane beam lattice (driven by a DE scheme) of varying physical sizes is able to emulate a free Kirchhoff-Love plate in bending on a Winkler-type foundation. In conjunction, size estimates for an FE-DE sheet sample to represent either an infinite ice sheet or a semi-infinite ice sheet with a free edge were sought for. Benchmark computations with plate finite elements were performed using [Abaqus, 2016].

In **Publication III**, the inelastic response of a cellular, viscously damped, plate-like, quasi-brittle structure was then examined. Both the in-plane and the out-of-plane directions were considered. First, the uniaxial tensile strengths were computed. Next, the breakthrough loads and strengths were investigated. In the vertical breakthrough tests, the sheets were penetrated by a rigid, truncated, flat-ended cylinder. The publication focused, similarly to **Publication I**, on establishing a connection between the microscale material parameters and the observed, effective macroscale response of a modelled ice sheet. Both the size and the rate effects were examined. The publication presented, in addition, a new in-direct technique to study ice-structure interaction, to compute ice-breaking loads on a conical offshore structure, and to investigate the ability of a numerical method to produce both radial and circumferential cracking. The technique involved a truncated, rigid cone surfacing through a circular hole in a stationary ice sheet. To conclude, the observed cracking characteristics were discussed in each of the above cases.

The contents of this thesis are strictly restricted to applications that are closely related to ice mechanics, i.e. the modelling of intact sea ice sheets, their fracture, and ice-structure interaction. While the model and the simulation code applied have been developed for such purposes, the results presented should apply to other cohesive, lattice-based models on other application areas as well, in the *ab initio*-type constitutive modelling of ceramic matrix composites or concrete for instance.

1.4 State-of-the-art

This section introduces the state-of-the-art first on a general level and then in relation to the contents of each publication. All the material in this section has been adapted from the corresponding publications.

1.4.1 A general overview

The use of an in-plane beam lattice (a grid in civil engineering vocabulary) as a substitute structure to mimic plate- or sheet-like behaviour is by no means a new idea. Such an approach has been followed to treat problems within plane elasticity [Klein and Wieghardt, 1904; Hrennikoff, 1941; McHenry, 1943; Griffiths and Mustoe, 2001], to compute deflections of plate-like, load carrying members [Hrennikoff, 1941; Lightfoot, 1964; Renton, 1965; Salonen, 1969], to model the fracture of concrete/rock or other engineering/geological materials [Burt and Dougill, 1977; Plesha and Aifantis, 1983; Zubelewicz and Bažant, 1987; Bažant et al., 1990; Schlangen and van Mier, 1992; Bolander and Saito, 1998; D’Addetta et al., 2002; Ibrahimbegovic and Delaplace, 2003; Liu et al., 2007], and, for instance, in studying the fracture of a two-dimensional, disordered medium [Herrmann et al., 1989]. Before and shortly after the advent of the finite element method (FEM), the approach was commonplace. The developments were to aid in customary civil engineering, in the design of bridges for instance [Lightfoot, 1964]. A somewhat natural area of application – not as a substitute structure, that is to say – is that of cellular solids, i.e. foams, cancellous bones, cork, composites, snow, nano, and other intrinsically cellular natural or man-made materials [Gibson and Ashby, 1999]. For a detailed survey and three accounts of reticulated plate and shell structures in general, see the review paper by Ostoja-Starzewski [2002] and the monographs of Renton [2002, Ch. 15 & App. 5.4], Pshenichnov [1993, Ch. 3.2], and Woźniak [1966b, Ch. 6], respectively. A more concise fracture-based overview is given in [Bažant and Planas, 1997, Ch. 14.4].

The discrete element method (DEM) was first presented by Cundall and Strack [1979]. Today, DEM is routinely applied in studying the mechanics of granular materials. The contemporary areas of application range from, e.g., powder flow simulations to the modelling of landslides. A review of the usage of DEM in the context of ice-structure interaction simulations can be found in [Tuhkuri and Polojärvi, 2018]. The hybrid FE-DE method is then treated in [Munjiza, 2004], see also [Munjiza et al., 2015].¹

A body of literature explores the use of lattice-, hybrid FE-DE-, and FE-based numerical methods to model a floating ice sheet, its fracture, and/or interaction with an offshore structure [Hocking, 1992; Jirásek and Bažant, 1995a; Hopkins, 1998; Sayed and Timco, 1998; Selvadurai and

¹Alternative acronyms include, at least, those of FEM-DEM and FDEM.

Sepehr, 1999; Sand, 2008; Dorival et al., 2008; Konuk et al., 2009; Gürtner, 2009; Paavilainen et al., 2009; Lu et al., 2014, 2015, 2016; Herman, 2016; van den Berg, 2016]. Lattices of either axial/rotational spring-, truss-, or beam-type were exploited in [Jirásek and Bažant, 1995a; Sayed and Timco, 1998; Dorival et al., 2008; Paavilainen et al., 2009; Herman, 2016], and [van den Berg, 2016]. Of these, the models of Jirásek and Bažant [1995a; Sayed and Timco [1998]; Dorival et al. [2008]; Paavilainen et al. [2009], and Herman [2016] were formulated in a two-dimensional space and only that of van den Berg [2016] in a three-dimensional space. This thesis focuses on a three-dimensional method.

1.4.2 An overview regarding the contents of **Publication I**

Size-dependent behaviour of foam-like materials has been investigated in abundance, both experimentally and theoretically [Onck et al., 2001; Diebels and Steeb, 2002; Teköglu and Onck, 2005; Teköglu et al., 2011; Liebenstein et al., 2018]. A classic general treatise is that of Gibson and Ashby [1999]. Albeit in a different context, a lattice-based model of an ice sheet and, for instance, of an industrial honeycomb-like foam share numerous analogous features. In **Publication I**, the findings were thus compared with the works just cited due to the similarities between the models used therein and in this thesis. It is to be noted, however, that all the previous non-experimental studies employ an elastostatic solution scheme and that the models include only beams or beam finite elements. In this thesis, the beam finite elements are displaced by the movements of discrete elements. In addition, because of the dynamic nature of an ice-structure interaction simulation, it was considered to be of an interest to examine the in-plane material properties of a viscously damped ice sheet also as functions of strain rate by a purely dynamic, explicit time marching procedure.

It is an established fact that a Timoshenko beam lattice is a discrete equivalent of a generalised (higher order) continuum of a micropolar type [Woźniak, 1966a; Askar and Cakmak, 1968; Banks and Sokolowski, 1968; Tauchert, 1970; Bažant and Christensen, 1972; Noor, 1988; Ostoja-Starzewski, 2002; Karihaloo et al., 2003; Kumar and McDowell, 2004]. A micropolar continuum is a continuous distribution of oriented, i.e. directed, rigid material particles. Note that a classical Cauchy-type continuum is, on the contrary, non-oriented, viz. introduces no couple stresses. The computed in-plane, elastic material moduli were thus compared also with respect to the corresponding moduli of an in-plane, elastic medium of a micropolar type. For a concise yet comprehensive treatment of micropolar mechanics, see [Eremeyev et al., 2013].

1.4.3 An overview regarding the contents of **Publication II**

Studies concerning the out-of-plane bending of a free, finite, rectangular ice sheet have been infrequent [Li and Dempsey, 1988; Lu et al., 2015, 2016; van den Berg, 2016]. A closed-form analytical solution for the deflection of a free, finite, rectangular plate on a Winkler-type foundation – either conventional or tensionless – appears not to exist. The method of solution referred to in [Lu et al., 2016] – for a conventional Winkler foundation – generates a set of truncated, infinite series with unknown coefficients. With the aid of the boundary conditions, a system of algebraic equations is then arrived at and next solved to find the coefficients. The method as such, akin to a numerical scheme, yields a solution that is not in a closed form. Other approximate solutions of either a Rayleigh-Ritz- or Galerkin-type (for a conventional Winkler foundation) can be found in [Happel, 1920; Murphy, 1937], and [Vlasov and Leont’ev, 1966, Ch. 3]. Note that Vlasov and Leont’ev [1966] studied a free, rectangular plate on an *elastic* foundation, reducible to a foundation with no shearing forces, i.e. a Winkler foundation.

The first analytic solution regarding the deflection of a point-loaded, infinite plate on a Winkler-type foundation was due to Hertz [1884]. Another solution, in terms of Kelvin functions, was communicated by Wyman [1950]. Deflection of a point-loaded, semi-infinite plate with a free edge resting on a Winkler-type foundation was studied by the method of images in [Kerr and Kwak, 1993], see also [Westergaard, 1926]. In **Publication II**, the solutions of Wyman [1950] and Kerr and Kwak [1993] were referred to in assessing the size needed for an FE-DE sheet sample to resemble either an infinite ice sheet or a semi-infinite ice sheet with a free edge, respectively. On other past treatments of computing the deflections of both an infinite and a semi-infinite ice sheet, see the review papers by Kerr [1976, 1996].

1.4.4 An overview regarding the contents of **Publication III**

The de-cohesive damage model presented in **Publication III** has its roots in the mid 1970’s. The kind of an approach was coined, apparently, by Dougill [1976]. Dougill developed his theory for a “progressively fracturing solid.” Other exemplary works of a similar type are, for example, those of Carol et al. [1997]; Gálvez et al. [2002] and Schreyer et al. [2006]. In [Carol et al., 1994], such developments were called as “elastic-degrading” models. What these models have in common is the fact that the basic machinery of small strain hardening plasticity is readily applicable [Carol et al., 1994, sec. 3]. Softening can be modelled as well; an adequate element length, however, is required in order to avoid a snapback instability. Note that the “crack band model” of Bažant and Oh [1983] and the “fictitious crack model” of Hillerborg et al. [1976], of which the latter is in this

thesis partly adopted, are conceptually equivalent. The main difference is that Bažant and Oh [1983] assume the existence of a finite width “crack band,” whereas Hillerborg et al. [1976] postulate the fracture to localise onto a plane.

A concise historical survey on the uniaxial fracture tests of cellular composites is given in [Bažant et al., 1990], see also [Jirásek and Bažant, 1995b]. A number of papers, both analytical and numerical, have been published about the vertical penetration fracture/failure of a plate [Bažant and Li, 1994; Bažant and Kim, 1998a,b; Bažant, 2002; Beltaos, 2002; Dempsey et al., 1995; Dempsey and Vasileva, 2006a,b; Dempsey et al., 2006; Kerr, 1976, 1996; Li and Bažant, 1994; Lu et al., 2015, 2016; McGilvary et al., 1990; Pushkin et al., 1991; Slepian, 1990; Sodhi, 1989, 1995, 1998; Vasileva and Dempsey, 2006]. A hybrid FE-DE method has not been, to the author’s best knowledge, applied before. No papers appear to exist that study the breaking of a floating plate containing a circular hole by a surfacing cone. It is believed that the approach proposed in this thesis is new. Nevel [1992] gives a clear account of the subject but with a direct approach. Other well known works of a similar kind are those of Ralston [1977] and Croasdale and Cammaert [1994]. The method of Ralston was suggested, in fact, by J. R. Rice. In this thesis, the computed ice-breaking loads were compared with those predicted by the model of Croasdale and Cammaert [1994]. The model of Croasdale and Cammaert [1994] is one of the ice load models implemented in [ISO 19906, 2019], which is the standard to be followed in designing offshore structures. No historical account is given as regards to the experimental work done on the vertical penetration fracture/failure of a plate. In that matter, see the review papers by Kerr [1976, 1996]. Often cited treatises are, however, those of Black [1958]; Gold et al. [1958]; Frankenstein [1963, 1966]; Meyerhof [1960], and Lichtenberger et al. [1974].

1.5 Structure of the compendium

The rest of the compendium is organised in four chapters. Chapter 2 gives a description of both the FE-DE model of an ice sheet and the numerical experiments performed. Chapter 3 states the results, Chapter 4 provides a discussion, while Chapter 5 presents the conclusions and speculates on some future research directions.

2. Methods

This chapter gives an outline of both the model features and the numerical experiments performed. First, the numerical FE-DE model of an ice sheet is described, after which the test setups follow. The FE-DE sheet samples examined are depicted last. All the material presented in this chapter has been adapted from the respective publications.

2.1 Description of the numerical model

This section discusses the modelling of an ice sheet with an FE-DE approach in general terms. **Publication I** gives a more detailed treatment without any reference to fracture, except in the context of mesh design. **Publication III** addresses then fracture. The whole formulation has been given in the annexed publications and is thus not here repeated in full.

As has already been stated, a sheet is modelled with rigid discrete elements. The mass centroids of the discrete elements connect then via an in-plane beam lattice of Timoshenko beam finite elements, Figures 1.2 and 2.1. The beam formulation adopted follows mainly [Crisfield, 1997, Ch. 17.1-2]: a local, elemental triad of vectors $\mathbf{T}_{p,q}$, associated with a beam finite element p, q , tracks the average, incremental motions of the discrete elements p and q and deforms. The strains in a beam element p, q are given by *i*) the stretch of the axial vector component and *ii*) the changes between the mutual orientations of the axial and the two in-plane vector components of $\mathbf{T}_{p,q}$. The curvatures and twist in a beam element p, q are due to the differing nodal orientations of the discrete elements p and q . Geometrical non-linearity is taken into account, due to the finite displacements and rotations of the discrete elements, yet the deformations in a beam finite element are assumed to be small. In [Belytschko et al., 1977], a very similar geometrically non-linear formulation for the transient analysis of space frames was developed.

The material of the beam finite elements follows Hooke's law. To dissipate energy, a viscous damping model is used. A rheological equivalent of the

Figure 2.1. Discrete elements p and q connected with a Timoshenko beam finite element p, q . For an explanation of the nomenclature used, see [Publication I](#). The figure has been adapted from [Lilja et al., 2017a, Fig. 1].

constitutive model implemented would thus be that of Kelvin-Voigt (a viscoelastic solid). Due to the Timoshenko beam finite elements following a co-rotational kinematical description (a co-rotational description extracts the rigid body displacement components from the total displacements), the application of a linearised stress-strain relationship – within the co-rotational, elemental frame – is justified, i.e. the formulation can be interpreted to be objective.

To model irreversible damage, the Timoshenko beam finite elements are able to fracture, Figure 2.2. A cohesive crack (a fictitious crack of Hillerborg et al. [1976]) is allowed to propagate along an interface separating adjacent discrete elements. The moment the cracking initiates, a rate-independent, linearly softening traction-separation law is activated in order to allow the discrete elements to separate. The activation is done independently at each integration point along the cross-section of the beam finite element. Because of the change of the constitutive behaviour once the damage begins, the model can be categorised as “extrinsic” [Seagraves and Radovitzky, 2010]. No new cohesive finite elements are introduced, only the constitutive behaviour is changed.

The model is, in essence, a three-dimensional extension of the earlier two-dimensional FE-DE model by Paavilainen et al. [2009]. In their model, a crack was able to propagate only vertically due to the modelling space being two-dimensional. Since the space is here three-dimensional, a crack is able to propagate both horizontally and vertically. A mixed mode fracture is modelled by first reducing, via a change of variables, the original three-component state of stress at an integration point to a one-dimensional “effective” state of stress and then representing that state of stress with respect to a loading surface (a failure surface), which degrades (evolves) as a function of an effective crack opening displacement measure. In this thesis, a failure criterion proposed by Schreyer et al. [2006] is adopted. For the details, see [Publication III](#). Once a beam has fully eroded, i.e. all

Figure 2.2. A schematic illustration of an advancing cohesive (fictitious) crack. A traction-free (true) crack tip advances along the cross-section as the integration points get fully damaged. For an explanation of the description of fracture, see [Publication III](#). The figure has been adapted from [Lilja et al., 2017b, Fig. 1].

the integration points have damaged fully, the adjacent discrete elements separate. A frictional contact scheme, i.e. a DEM scheme, between the separated discrete elements is finally activated. The scheme follows that presented in [Feng et al., 2005]. The buoyant-, gravity-, drag-, and the frictional contact forces are then computed as in [Polojärvi et al., 2012].

2.2 Centroidal Voronoi tessellation

A centroidal-Voronoi-tessellation-based iterative scheme with a random generating point set and Lloyd’s algorithm, see [Du et al., 1999], is followed in order to produce an unstructured mesh, Figure 2.3. Lloyd’s algorithm contains an iteration loop during which the seeds acting as the generators of a Voronoi diagram are incrementally displaced to merge with the centroids of the contiguous Voronoi cells, see [Talischi et al., 2012, Fig. 5, p. 313]. The iteration continues until a chosen termination criterion is met. A criterion based on the minimisation of an associated energy functional, with a cut-off tolerance of $\epsilon = 10^{-5}$ or 500 iteration cycles, is in this thesis adopted [Talischi et al., 2012, sec. 3].

As an “end product,” CVT produces an in-plane tessellation of highly regular polygons. The polygons have the shapes of convex hexagons, pentagons, and quadrilaterals and are next extruded in the out-of-plane direction to become prismatic polyhedrons. These prismatic polyhedrons denote the discrete elements. Finally, a network of lines that denote the Timoshenko beam finite elements is created by Delaunay triangulation. A Delaunay-triangulated lattice is a “dual” graph of a CVT. On a large enough relative size scale L_{rel} ($L_{\text{rel}} = L/l$), the beam lattice network has a nearly equilaterally triangular topology. For the definitions of L and l , see Figures 1.2 and 2.3.

Figure 2.3. CVT-tessellated FE-DE sheet samples with a side length of $L = 10$ m. On the top row, from left to right, samples with the thicknesses of $h = 0.5$, 1.0 , and 1.5 m and a discrete element size of $l = 2h$. On the bottom row, from left to right, samples with the thicknesses of $h = 0.5$, 1.0 , and 1.5 m and a discrete element size of $l = 3h$. After an extrusion in the out-of-plane direction, the polygons become the discrete elements. The red lines denote the longitudinal axes of the Timoshenko beam finite elements. The cross-sectional area of each beam finite element gets determined by the area of the joint surface separating the two, adjacent discrete elements. The figure has been adapted from [Publication I](#), Figure 3.

On a large enough relative size scale L_{rel} , a CVT-tessellated in-plane beam lattice automatically satisfies the requirements for a lattice to behave as an in-plane, isotropic medium [Bolander and Saito, 1998]. This is because the nuclei of the Voronoi cells define the beam end nodes and the beam cross-sectional areas are equal to the areas of the interfaces separating adjacent discrete elements. A CVT-tessellated in-plane beam lattice is, in addition, relatively imperfection insensitive, [Symons and Fleck, 2008; Vural and Alkhader, 2008], and stretching dominated, [Ashby et al., 2001], because the mesh consists mostly of nearly equilateral triangles.

If a mesh is structured, crack growth with cohesive elements in between the bulk finite elements is prone to preferred directions because the artificial toughness induced by the mesh varies with direction [Rimoli et al., 2012]. If a crack cannot evolve freely, is bound to the interelement boundaries, a method is – by its very nature – mesh dependent. Crack growth can be rendered approximately direction independent, i.e. isotropic (yet still mesh dependent), if a CVT-tessellated, unstructured mesh is used [Leon et al., 2014; Spring et al., 2014]. In [Leon et al., 2014; Spring et al., 2014], the meshes were polygonal finite element meshes, whereas the fracture was described by cohesive finite elements. CVT-tessellated

meshes exhibit approximately isotropic crack growth behaviour although the mesh induced toughness is still a bit high. For an excellent overview of the use of cohesive finite elements in order to describe dynamic fracture, see [Seagraves and Radovitzky, 2010].

2.3 Equations of motion

The equations to be solved, for each discrete element (discretely in time), are the three translational and rotational equations of motion, i.e. the Newton-Euler equations. In [Publication III](#), the semi-discretised equations of motion of an ice sheet modelled by a hybrid FE-DE method were written as:

$$\vec{F}_{\mathbf{g},\mathbf{de}}^{\text{int}}(\vec{x}(t), \dot{\vec{x}}(t), t) + \vec{F}_{\mathbf{g},\mathbf{de}}^{\text{ext}}(\vec{x}(t), \dot{\vec{x}}(t), t) = \mathbf{M}_{\mathbf{g},\mathbf{de}} \ddot{\vec{x}}(t), \quad (2.1)$$

where $\vec{F}_{\mathbf{g},\mathbf{de}}^{\text{int}}$ denotes an internal force vector containing forces and moments, both viscous and non-viscous, due to the deforming and fracturing beam finite elements; $\vec{F}_{\mathbf{g},\mathbf{de}}^{\text{ext}}$ an external force vector containing forces and moments due to the contacts, buoyancy, drag, and gravity; $\vec{x}(t)$ a vector containing the translational and the angular positions of the discrete elements (t denotes time); $\dot{\vec{x}}(t)$ and $\ddot{\vec{x}}(t)$ the first and the second time derivatives of $\vec{x}(t)$, respectively; and $\mathbf{M}_{\mathbf{g},\mathbf{de}}$ a diagonal matrix containing the translational masses and the mass moments of inertia of the discrete elements. The subindexes “**g**” and “**de**” (**g**lobal and **de**crete element, respectively) refer here to the bases the translational and the rotational equations of motion have been written with respect to, respectively, see [Figure 2.1](#).

The translational equations of motion are integrated in time with a 2nd order accurate central difference time integration scheme, whereas the rotational equations of motion – with three-dimensional, finite, non-commutative rotations – with a, modified, 4th order accurate Runge-Kutta scheme, see [Munjiza et al., 2003].

2.4 Description of the numerical experiments

This section gives a brief overview of the simulation setups for each numerical experiment performed. Full details can be found in the respective publications. First, the simulation setups are described, after which the sheet samples examined are presented. Because of the objective of understanding the effects of scale, ice sheets of varying absolute and relative sizes were studied. Note that the original naming convention established in each respective publication as regards to the different load cases is here retained. The convention is somewhat inconsistent, but is nevertheless followed in order to be able to refer to the original publications

unambiguously.

2.4.1 Elastic in-plane experiments

Publication I studied the in-plane elastic response of a modelled ice sheet. This subsection gives an overview of the tests conducted there. Three in-plane tests were performed in order to compute the effective, in-plane Young's, shear, and bulk moduli E , G , and K , respectively, of a modelled ice sheet. All the sheets shown in Figure 2.11 were examined.

2.4.1.1 Uniaxial tension and compression

The effective Young's modulus E of a sheet sample was computed by carrying out uniaxial tension and compression tests, Figure 2.4. The effect of load initiation was, in tension, examined by applying two different sets of initial and boundary conditions. In *Load case I*, the nodes on the left boundary of a specimen were fixed against a translation in the global x co-ordinate direction, whereas the nodes on the right boundary were displaced at a constant displacement rate, $|v_x|$, into the positive, global x co-ordinate direction. In *Load case II* (studied only with the samples having the side lengths of $L = 10$ and 160 m), a linearly changing initial velocity field, $\vec{v}_x(x, y, 0) = 2|v_x|x\vec{i}/L$, was initiated in each specimen so as to yield an approximately constant rate of initial longitudinal strain. The nodes on the right and left boundaries of a specimen were then displaced into the positive and negative global x co-ordinate directions, respectively, until a sheet got properly strained. In *Load case III*, the effect of changing the loading direction was studied by reversing that in *Load case I*. The forced displacements were, in *Load case I* and *Load case III*, initiated abruptly. To examine the effect of the applied strain rate, all the computations were repeated at three different displacement rates. The rates

Figure 2.4. Free-body diagrams of a uniaxially loaded sheet sample. In *Load case III*, the loading direction was reversed from that in *Load case I*. The figure has been reproduced from **Publication I**, Figure 5.

had the magnitudes of $|v_x| = 0.1, 0.01, \text{ and } 0.001 \text{ m/s}$. These velocities are representative ice sheet velocities encountered in nature.

The effective in-plane Young's modulus of a sheet sample was then evaluated through Hooke's law as $E = F_x / (A_0 \varepsilon_{xx}) \equiv \sigma_{xx} / \varepsilon_{xx}$, where F_x denotes the scalar component of a computed resultant reaction force vector in the global x co-ordinate direction, $A_0 = Lh$ is equal to the initial cross-sectional area of the sheet sample, and ε_{xx} is the applied axial strain (a value of $\varepsilon_{xx} = \delta / L = 0.001$ was chosen). The reaction force component F_x was computed (in *Load case I* and *Load case III*) by summing up the global x co-ordinate direction components of the internal nodal force vectors of the beams having nodes on the left boundary of a specimen. In *Load case II*, the component F_x was computed by summing up the global x co-ordinate direction components of the internal nodal force vectors of the beams having nodes, in turn, both on the left and right boundaries.

2.4.1.2 Simple shear

The effective in-plane shear modulus G of a sheet sample was computed by conducting a test in simple shear, Figure 2.5. The nodes both on the left and right boundaries of a specimen were fixed against a translation in the global x co-ordinate direction, whereas the nodes on the left boundary were fixed against a translation in the global y co-ordinate direction as well. The nodes on the right boundary were then displaced at a constant displacement rate, $|v_y|$, into the positive, global y co-ordinate direction until a sample got suitably sheared. The forced displacements were initiated abruptly. To examine the effect of the applied strain rate, all the computations were repeated, as above, at three different displacement rates. The rates had the magnitudes of $|v_y| = 0.1, 0.01, \text{ and } 0.001 \text{ m/s}$, as before. The effect of constraining the global drilling degrees of freedom, $\theta_z \vec{k}$, on the nodes both on the left and right boundaries of a specimen (simultaneously) was studied separately.

Figure 2.5. A free-body diagram of a sheet sample loaded in simple shear. The figure has been reproduced from [Publication I](#), Figure 6.

The effective in-plane shear modulus of a sheet sample was then eval-

uated through Hooke's law as $G = F_y/(A_0\gamma) \equiv \tau_{xy}/\gamma$, where F_y denotes the scalar component of a computed resultant reaction force vector in the global y co-ordinate direction and γ the applied angle of shear (a value of $\gamma \approx \delta = 0.001$ was chosen). The reaction force component F_y was evaluated by summing up the global y co-ordinate direction components of the internal nodal force vectors of the beams having nodes, in turn, both on the left and right boundaries of a specimen.

2.4.1.3 Equi-biaxial tension

The effective in-plane bulk modulus K of a sheet sample was computed by performing an equi-biaxial tensile test, Figure 2.6. The nodes on the left and bottom boundaries of a specimen were fixed against translations in the global x and y co-ordinate directions, respectively, while the nodes on the right and top boundaries were left as free. The nodes on the right boundary were then displaced at a constant displacement rate, $|v_x|$, into the positive, global x co-ordinate direction, whereas the nodes on the top boundary were, simultaneously, displaced at a constant displacement rate, $|v_y|$, into the positive, global y co-ordinate direction. The forced displacements were initiated abruptly. Unlike with the uniaxial test above – which was carried out both in tension and compression – the effective in-plane bulk modulus was evaluated in tension only. In contrast to the two previous cases, the computations were here repeated at the two highest rates only.

Figure 2.6. A free-body diagram of a sheet sample loaded in equi-biaxial tension. The figure has been reproduced from [Publication I](#), Figure 7.

The effective in-plane bulk modulus of a sheet sample was then evaluated through Hooke's law as $K = (F_x + F_y)/2A_0(\varepsilon_{xx} + \varepsilon_{yy}) \equiv \sigma_{\text{hyd}}/\varepsilon_{\text{hyd}}$, where σ_{hyd} denotes the mean, biaxial, hydrostatic stress and ε_{hyd} the mean, biaxial, hydrostatic strain within the sample. The reaction force components

F_x and F_y were computed by summing up the global x and y co-ordinate direction components, respectively, of the internal nodal force vectors of the beams having nodes on the left and bottom boundaries of a specimen, respectively. Each sheet sample examined was loaded so that the applied axial strains reached the values of $\varepsilon_{xx} = \varepsilon_{yy} = \delta/L = 0.001$.

2.4.2 Elastic out-of-plane experiments

In [Publication II](#), the elastic out-of-plane (bending) response of a modelled ice sheet was studied. This subsection gives a brief overview of the tests conducted there. All the other sheets shown in [Figure 2.11](#) were examined, except those with a side length of $L = 10$ m. These smallest sheets were excluded from the simulations because of their dimensions. They are essentially three-dimensional solids, not plates, and were thus left out.

Two load cases were studied. In Load case *i*), a vertical point load, $|F_z| = 10$ kN, acting in the negative, global z co-ordinate direction, [Figure 2.7a](#), was applied to a node closest to the centre of a sheet sample through a statically equivalent force system (i.e. as if the loading would act in the actual centre). This load magnitude was chosen so that the samples would not submerge and that the kinematic assumptions of the Kirchhoff-Love plate theory, with small deflections and rotations, would still hold. In Load case *ii*), the load was applied, similarly through a statically equivalent force system, to a node closest to the midpoint of the edge on which $x = 0$, [Figure 2.7b](#). The loads were applied instantaneously and the simulations continued until the vertical oscillations of the loaded nodes had settled. Each sheet sample was in an initial equilibrium, gravitation was taken into account, no displacement constraints were applied, nor were the beams allowed to fracture. Stoke's drag was then used to reach a quasi-static equilibrium. Stoke's drag is a drag model yielding a drag force that is linearly proportional to the velocity, see [Publication II](#).

The deflections of the sheet samples were evaluated along the paths shown in [Figure 2.7](#). These deflection profiles were next compared with the corresponding deflection profiles of the plates modelled by plate finite elements and of either an infinite or a semi-infinite ice sheet with a free edge – depending on the load case considered – as the sheet size increased. The benchmark computations with plate finite elements were performed using [[Abaqus, 2016](#)]. The plates were meshed with perfectly square S4R elements with the sizes of 0.5×0.5 m. For the other details regarding the simulations with Abaqus, see [Publication II](#), sec. 4.3.

2.4.3 Inelastic in-plane experiments

[Publication III](#) studied fracture. This and the next two subsections give a brief overview of the tests conducted there. First, the uniaxial tensile frac-

ture test (denoted as Load case A) is described, after which a description of the out-of-plane (vertical) fracture test follows. The out-of-plane fracture test was denoted as Load case B. The simulation setup for the ice-structure interaction experiments, denoted as Load case C, is depicted last.

The effective tensile strength of a sheet sample was computed by conducting a uniaxial tensile test, Figure 2.8. All the sheets shown in Figure 2.11 were examined. The nodes on the right and left boundaries of a specimen were pulled into the positive and negative global x co-ordinate directions,

(a) A sheet sample subjected to a vertical, central point load. Only one quarter of the sample is shown. The deflections were evaluated along the path highlighted in red. The symbol $w_{\max}^I$ denotes the maximum deflection of an infinite ice sheet of a thickness h corresponding to the sheet sample considered. In this particular case, $h = 0.5$ m.

(b) A sheet sample subjected to a vertical point load on an edge. Only one half of the sample is shown. The deflections were evaluated along the paths highlighted in red and blue. The symbol $w_{\max}^{II}$ denotes the maximum deflection of a semi-infinite ice sheet with a free edge and of a thickness h corresponding to the sheet sample considered. In this particular case, $h = 0.5$ m.

Figure 2.7. (a) Load case i), (b) Load case ii). The figure has been reproduced from **Publication II**, Figure 5.

respectively, until a sheet was completely fractured. To prevent early fracture near the boundaries (and to start the simulations with an approximately constant rate of initial longitudinal strain), a linearly changing initial velocity field, $\vec{v}_x(x, y, 0) = 2|v_x|x\vec{i}/L$, was established in each specimen. In order to examine the effect of the applied loading rate, the computations were repeated at two different displacement rates. The rates had the magnitudes of $|v_x| = 0.1$ (Load case A1) and 0.01 m/s (Load case A2). Post-fracture contacts were not computed, nor were the buoyant-, drag-, or gravity forces considered, but the Timoshenko beam finite elements were allowed to fracture. Note that including buoyant and gravity forces is not necessary in an in-plane test because the method is explicit, viz. there is no stiffness matrix that would turn out rank-deficient, i.e. singular, if not properly constrained. Not including post-fracture contacts was due to computational reasons. For an explanation, see subsection 4.3.1 below.

Figure 2.8. A schematic illustration of a simulation setup in Load case A. The figure has been adapted from [Lilja et al., 2017b, Fig. 3].

The effective tensile strength of a sheet sample was then evaluated through $\sigma_{\text{cr,eff}} = F_x^{\text{max}}/A_0$, where F_x^{max} denotes the scalar component of a computed resultant reaction force vector in the global x co-ordinate direction, and was taken equal to the recorded maximum value. This force component was computed for each sheet sample by summing up the global x co-ordinate direction components of the internal nodal force vectors of the beams having nodes either on the right or left boundary of the specimen.

2.4.4 Inelastic out-of-plane experiments

Two load cases, denoted as B1 and B2, were studied. In Load case B1, the samples were let to float freely, whereas in Load case B2, the outer boundaries of a sheet sample were pinned. In Load case B1, all the sheets shown in Figure 2.11 were examined, whereas in Load case B2, only the

largest sheets were studied. The purpose of Load case B2 was to examine whether the largest sheets are large enough to mimic an “infinite” ice sheet. If so, the boundary conditions should have no significant effects.

At the beginning of each simulation, a rigid, flat-ended, cylindrical indenter was positioned directly underneath the geometrical midpoint of a specimen, Figure 2.9. There was no initial vertical gap between the bottom surface of the sheet sample and the upper surface of the indenter. The indenter had a diameter of $D = 3h$. Once a simulation got started, the indenter was linearly accelerated up to a constant speed of $|v_z| = 0.025$ m/s. As a result, the indenter displaced vertically upwards and penetrated through the sheet. Each simulation was continued up until a smooth, horizontal post-fracture loading plateau was reached. In each simulation, a sheet sample examined was in an initial equilibrium; the gravity-, buoyant-, drag-, and the contact forces were taken into account; and the Timoshenko beam finite elements were allowed to fracture.

Figure 2.9. A schematic illustration of a simulation setup in Load case B. A rigid, flat-ended, cylindrical indenter is penetrating through a sheet sample. The figure has been reproduced from Publication III, Figure 11.

The vertical breakthrough load, F_{cr} , of a sheet sample was then computed as a maximum resultant force of the vertical contact forces experienced by the indenter during a simulation. The reported breakthrough strengths were next computed as $\sigma_F = F_{cr}/h^2$. Note that σ_F is not a true stress but a nominal stress with a correct dimension, see [Bažant, 2002].

2.4.5 Ice-structure interaction experiments – ice-breaking loads

Two load cases, denoted as C1 and C2, were studied. In both load cases, only the largest sheets were examined, Figure 2.11. The sheets contained, in addition, centrally-located circular holes with effective diameters of five meters, see Figure 2.10.

The boundary conditions applied were similar to those in Load case B. In Load case C1, the samples were let to float freely, whereas in Load case C2, the outer boundaries of a sheet sample were pinned.

At the beginning of each simulation, a rigid, truncated cone was positioned directly underneath the geometrical midpoint of a specimen. The cone had an upper diameter of five meters and a cone angle of 45 degrees. There was an initial vertical gap of 20 mm between the bottom surface of the sheet sample and the upper surface of the cone. Once a simulation got started, the cone was displaced at a constant speed, $|v_z| = 0.1$ m/s, directly upwards and through the hole. In each simulation (in both load cases considered), a sheet sample examined was in an initial equilibrium; the gravity-, buoyant-, drag-, and the contact forces were taken into account; and the Timoshenko beam finite elements were allowed to fracture.

Figure 2.10. A schematic illustration of a simulation setup in Load case C. A surfacing, rigid, truncated cone is breaking a sheet sample containing a circular hole. The figure has been reproduced from Publication III, Figure 13.

The scheme is probably best illustrated by an example: the attached animations, Animation 1 and Animation 2, depict the scheme, but via a generic case; a sheet with a side length of $L = 50$ m and a thickness

of $h = 0.1$ m is interacting with a cone. It can be seen that with an indirect approach, a cone is surrounded by ice and the contact zone extends around the whole circumference. The problem is thus an axisymmetric contact problem in contrast to the conventional case in which the contact is unilateral. The axisymmetry is to simplify the numerical problem and to aid in controlling an actual physical experiment – if organized.

The maximum ice load on the cone, $F_{V,FE-DE}$, was then computed as a maximum resultant force of the vertical contact forces experienced by the cone during a simulation. Each simulation was continued until a clear drop in the recorded load was observed. This maximum load was finally divided with the effective circumference of the cone, πD_w , in order to get a load per unit circumference. The diameter D_w was computed at the time instant the maximum load occurred at. For an explanation, see Figure 25 in [Publication III](#).

2.4.6 Sheet samples examined and the simulation parameters

Figure 2.11 presents the sheet samples examined. The samples were square; had a side length of either $L = 10, 20, 40, 80$, or 160 m; a thickness of either $h = 0.5, 1.0$, or 1.5 m; and a discrete element size of either $l = 2h$ or $3h$. The discrete element size l was defined as an average diameter of the circumscribed circles of the polygonal discrete elements in each mesh.

Figure 2.11. The FE-DE sheet samples examined, in scale. From left to right: $L = 10, 20, 40, 80$, and 160 m. The three thin horizontal lines atop each sample denote the thicknesses studied. From top to bottom: $h = 1.5, 1.0$, and 0.5 m. The largest sample on the right shows, as an example, sections of the two meshes that have the most and least amount of discrete elements for that particular sheet “type.” Mesh “A” has 29561 discrete elements ($L = 160$ m, $h = 0.5$ m, and $l = 2h$), whereas mesh “B” has 1460 discrete elements ($L = 160$ m, $h = 1.5$ m, and $l = 3h$). The figure has been reproduced from [Publication I](#), Figure 4.

In the case of a fracture, each broken fragment should thus have, at the minimum, a size comparable with actual block sizes measured from ridge sails [Kankaanpää, 1988; Høyland, 2007; Kulyakhtin, 2014]. Because of the properties of a centroidal Voronoi tessellation, the Timoshenko beam finite elements were of approximately the same “lengths,” i.e. $L_e \approx l$.

The results presented in Chapter 3 below are, for most of the cases, averages computed by 10 randomised CVT meshes created for each specific sheet “type” considered (i.e. a sheet with specific L , h , and l). For the exceptions, see the respective publications and the comments in Chapter 3.

To conclude the present section, Table 2.1 states the main simulation parameters. Note that not all parameters are relevant in each experiment. See the respective publications for further details.

Table 2.1. Main simulation parameters.

	Parameter	Symbol	Unit	Value or range
General	Standard gravity	g	m/s^2	9.81
	Drag coefficient ^a	c_d		1.0
	Damping coefficient ^b	c		critical
	Coefficient of friction ^c	$\mu_{i,s}$		0.05
	Coefficient of friction ^d	$\mu_{i,i}$		0.05
	Contact stiffness	k_c	GPa	4
	Contact damping	d_c		0.95
	Cone angle	α		45°
	Cone top width	D	m	5
	Time step	Δt	s	$1 \dots 5.0 \times 10^{-5}$
Ice sheet	Side length	L	m	10, 20, 40, 80, 160
	Thickness	h	m	0.5, 1.0, 1.5
	Young’s modulus ^e	E_b	GPa	4
	Poisson’s ratio	ν_b		0.3
	Density	ρ_i	kg/m^3	920
	Critical axial stress	σ_{cr}	kPa	125
	Critical shear stress	τ_{cr}	kPa	125
	Fracture energy	G_{eff}	J/m^2	15
Water	Density	ρ_w	kg/m^3	1010

^aBoth in translation and rotation. ^b $c = 2\sqrt{m_{eff}E_b}$, where the “effective” mass, m_{eff} , is taken to be the average of the translational masses of the two discrete elements a beam finite element connects. ^cCoefficient of friction between ice and structure. ^dCoefficient of friction between two pieces of ice. ^e“Apparent” or “effective” elastic modulus [Timco and Weeks, 2010, sec. 13]. The table has been reproduced from Publication III, Table 1.

3. Results

This chapter presents the results. The material, including the illustrations and tables, have all been adapted from the corresponding publications. This compendium *per se* presents no new scientific results but summarizes the findings of the publications. Sections 3.1 and 3.2 discuss the in-plane and out-of-plane elastic responses, respectively; sections 3.3 and 3.4 the in-plane and out-of-plane inelastic responses, respectively; while section 3.5 discusses the ice-structure interaction. Each section gives an overview of only the main findings. For additional results and explanations, consult the respective publications.

3.1 Elastic in-plane response

This section gives an overview of the computed Young's, shear, and bulk moduli E , G , and K , respectively, of the modelled ice sheets. In each case, both the size and the rate effects were examined. The moduli presented below are averages computed with the aid of 10 different, randomised CVT meshes, except in the case of the sheets with $L = 160$ m, $h = 0.5$ m, and $l = 2h$ for which only six meshes were produced.

3.1.1 Effective in-plane Young's moduli

This subsection gives an overview of the computed in-plane Young's moduli E .

3.1.1.1 Size effect

Figure 3.1 shows the moduli, with their standard deviations, for each side length L , thickness h , and discrete element size l considered. The results have been arranged, within each side length group L , in an order of ascending h and l (the latter for each h), and then normalised with respect to the Young's modulus E_b given to the Timoshenko beam finite elements. The results correspond to *Load case I*, $|v_x| = 0.1$ m/s, see Figure 2.4.

Figure 3.1. Averaged, effective, in-plane Young's moduli E . The symbols $E^\dagger$ and E^* denote the corresponding moduli of an infinite, in-plane Euler-Bernoulli beam lattice and of an infinite, in-plane micropolar medium, respectively. Both of these reference media have equilaterally triangular, periodic topologies. The figure has been reproduced from [Publication 1](#), Figure 9.

The modulus grows as a function of h and decreases as a function of L . The samples that are smaller in L , thicker, or meshed with a discrete element size of $l = 3h$ have a higher effective Young's modulus E .

Figure 3.2 gives the moduli as functions of the relative sheet size parameter L_{rel} on a linear chart. On the abscissa, the symbol n denotes the number of discrete elements in each mesh. A logarithm is taken only to better distinguish between the data points for the small L_{rel} .

The size effect is found to approximately vanish for $L_{\text{rel}} \gtrsim 25$ ($\ln \sqrt{n} \approx 3.25$). Without the logarithms, the result is evident, see [Lilja et al., 2017a]. With an increase in the relative sheet size, the moduli tend to a value of about $0.8E_0$. Of the elastic reference moduli, E tends closest to E^* with a value of around $1.1E^*$.

3.1.1.2 Rate effect

Table 3.1 presents the differences between the moduli computed at the displacement rates $|v_x| = 0.1$ and 0.01 m/s in *Load case I*. The differences are small. The largest difference is approximately 5 %. The two rightmost columns give the differences between the moduli computed in tension and compression, *Load case I* and *Load case III*, respectively. No signifi-

Figure 3.2. Ratios $E/E^\dagger$, E/E^* , and E/E_b as functions of a relative sheet size parameter $L_{\text{rel}} (\approx \sqrt{n})$, *Load case I*, $|v_x| = 0.1$ m/s. The figure has been reproduced from [Publication I](#), Figure 10.

cant differences were found, the model behaves similarly in tension and compression. The differences between the moduli computed at the displacement rates $|v_x| = 0.01$ and 0.001 m/s in *Load case I* were found to be negligible and so left out from the table.

Comparing the moduli of those different-sized samples that undergo approximately equal strain rates yields results similar to those above; the samples that are smaller in L , thicker, or meshed with a discrete element size of $l = 3h$ have a higher effective, in-plane Young's modulus E .

In *Load case II*, an approximately constant rate of initial longitudinal strain was initiated in each specimen so as to estimate the effect of ramping up the load more smoothly. Only the samples with the side lengths of $L = 10$ and 160 m were examined. With the moderate displacement rates applied, no practical differences were found; *Load case I* and *Load case II* produced virtually equal results. In conclusion, no significant rate effects were found.

3.1.2 Effective in-plane shear moduli

This subsection gives an overview of the computed in-plane shear moduli G .

3.1.2.1 Size effect

Figure 3.3 gives the moduli, with their standard deviations, for each side length L , thickness h , and discrete element size l considered, as above. The moduli have been computed at a displacement rate $|v_y| = 0.1$ m/s, Figure 2.5. Each modulus has then been normalised with respect to the shear modulus G_b given to the Timoshenko beam finite elements.

A similar trend to that already discussed is found. The modulus grows as a function of h and decreases as a function of L . The samples that are smaller in L , thicker, or meshed with a discrete element size of $l = 3h$ show a higher effective, in-plane shear modulus G . If the moduli are plotted as functions of the relative sheet size parameter L_{rel} , as above, the size effect is found to approximately vanish, again, for $L_{\text{rel}} \gtrsim 25$, see Figure 12 in Publication I.

What is especially striking – and to be noticed – is the major drop as the relative sheet size is increased. The moduli tend to a value of about

Table 3.1. The differences (% , to the nearest digit) between the effective, in-plane Young’s moduli E computed at the displacement rates $|v_x| = 0.1$ and 0.01 m/s in *Load case I*. The two rightmost columns, indicated with the superscripts “ $\dagger$ ”, give the differences between the moduli computed in tension and compression, $|v_x| = 0.1$ m/s, *Load case I* and *Load case III*, respectively. The table has been adapted from Publication I, Table 2.

L [m]	h [m]	[%]			
		$l = 2h$	$l = 3h$	$l = 2h^\dagger$	$l = 3h^\dagger$
10	0.5	1	1	0	0
	1.0	2	3	0	1
	1.5	3	5	0	3
20	0.5	0	0	0	0
	1.0	1	2	0	0
	1.5	2	2	0	0
40	0.5	0	0	0	0
	1.0	0	1	0	0
	1.5	1	1	0	0
80	0.5	0	0	0	0
	1.0	0	0	0	0
	1.5	1	1	0	0
160	0.5	0	0	0	0
	1.0	0	0	0	0
	1.5	0	0	0	0

$0.55G_b$. Of the elastic reference moduli, G tends closest to $G^\dagger$ with a value of approximately $0.8G^\dagger$.

The effect of constraining the global drilling degrees of freedom, i.e. $\theta_z \vec{k}$, on the nodes both on the left and right boundaries of a specimen, simultaneously, was studied separately and found as insignificant. The result is apparently due to the smallness of the applied shear angle γ . The effect ought to be more pronounced for larger angles.

3.1.2.2 Rate effect

Table 3.2 displays the differences between the moduli computed at the displacement rates $|v_y| = 0.1$ and 0.01 m/s. The differences are small. The largest difference is approximately 8 %. The differences between the moduli computed at the displacement rates $|v_y| = 0.01$ and 0.001 m/s were – as for the effective, in-plane Young’s modulus E above – negligible and so left out from the table. With the modest displacement rates applied, no significant rate effects were thus found.

Figure 3.3. Averaged, effective, in-plane shear moduli G . The symbols $G^\dagger$ and G^* denote the corresponding moduli of an infinite, in-plane Euler-Bernoulli beam lattice and of an infinite, in-plane micropolar medium, respectively. Both of these reference media have equilaterally triangular, periodic topologies. The figure has been reproduced from [Publication I](#), Figure 11.

Table 3.2. The differences (% , to the nearest digit) between the effective, in-plane shear moduli G computed at the displacement rates $|v_y| = 0.1$ and 0.01 m/s. The table has been adapted from [Publication I](#), Table 3.

L [m]	h [m]	$l = 2h$ [%]	$l = 3h$ [%]
10	0.5	1	1
	1.0	2	4
	1.5	5	8
20	0.5	0	0
	1.0	1	2
	1.5	2	4
40	0.5	1	1
	1.0	1	1
	1.5	1	2
80	0.5	0	1
	1.0	1	1
	1.5	0	0
160	0.5	-2	-1
	1.0	0	1
	1.5	1	0

3.1.3 Effective in-plane bulk moduli

This subsection gives an overview of the computed in-plane bulk moduli K .

3.1.3.1 Size effect

Figure 3.4 presents the moduli, with their standard deviations, for each side length L , thickness h , and discrete element size l considered, as before. The moduli have been computed at a displacement rate $|v_x| = |v_y| = 0.1$ m/s, Figure 2.6, and then normalised with respect to the in-plane bulk modulus K^{2D} of a linearly elastic, continuous medium in plane stress.

The findings are, again, very similar to those already discussed. The modulus grows as a function of h and decreases as a function of L . The samples that are smaller in L , thicker, or meshed with a discrete element size of $l = 3h$ exhibit a higher effective, in-plane bulk modulus K . If the results are plotted as functions of the relative sheet size parameter L_{rel} , the size effect is found to vanish, as before, for $L_{\text{rel}} \gtrsim 25$, see Figure 14 in [Publication I](#). With an increase in the relative sheet size parameter L_{rel} , the moduli tend to a value of about $0.7K^{2D}$. Of the elastic reference moduli,

Figure 3.4. Averaged, effective, in-plane bulk moduli K . The symbols $K^{\dagger}$ and K^* denote the corresponding moduli of an infinite, in-plane Euler-Bernoulli beam lattice and of an infinite, in-plane micropolar medium, respectively. Both of these reference media have equilaterally triangular, periodic topologies. The figure has been reproduced from [Publication I](#), Figure 13.

K is approximately equal to both $K^{\dagger}$ and K^* .

3.1.3.2 Rate effect

Due to the negligible differences between the effective, in-plane Young's moduli E computed at the displacement rates $|v_x| = 0.01$ and 0.001 m/s , the effective bulk moduli K were computed at the two highest rates only. Table 3.3 lists their differences, which are small. The largest difference is approximately 4 %. With the modest displacement rates applied, no significant rate effects were thus found.

3.2 Elastic out-of-plane response

This section gives an overview of the computed out-of-plane elastic response. Subsection 3.2.1 discusses a centrally-loaded, whereas subsection 3.2.2 an edge-loaded ice sheet. The deflections computed with the FE-DE approach are compared, in both cases, with respect to the known analytical solutions as the sheet size is increased and with the deflections computed by plate finite elements. Recall that no closed-form analyti-

cal solutions exist as regards to the deflection of a free, finite plate on a Winkler-type foundation.

3.2.1 Deflection of a free square sheet subjected to a vertical, central point load

In this subsection, the deflections of the sheet samples along the path shown in Figure 2.7a are described.

3.2.1.1 Relative deflections

Figure 3.5 depicts the deflections of all the studied sheets as well as of the plates modelled by plate finite elements using [Abaqus, 2016]. The deflection of an infinite ice sheet for each sheet thickness h considered is shown for a reference. Each deflection has then been normalised with respect to the maximum deflection of an infinite ice sheet of a corresponding thickness h . For an explanation of the other details and the parameters shown in the figure, consult Publication II.

The smaller and thicker samples tend to displace more, Figure 3.5a. The samples that are thicker tend to deform less. As a sample grows in its

Table 3.3. The differences (% , to the nearest digit) between the effective, in-plane bulk moduli K computed at the displacement rates $|v_x| = |v_y| = 0.1$ and 0.01 m/s. The table has been adapted from Publication I, Table 4.

L [m]	h [m]	$l = 2h$ [%]	$l = 3h$ [%]
10	0.5	0	1
	1.0	2	2
	1.5	3	4
20	0.5	0	0
	1.0	1	1
	1.5	1	2
40	0.5	0	0
	1.0	0	1
	1.5	0	1
80	0.5	0	0
	1.0	0	0
	1.5	0	0
160	0.5	0	0
	1.0	0	0
	1.5	0	0

Figure 3.5. The averaged, relative deflection profiles of the centrally-loaded sheet samples. The figure has been adapted from [Publication II](#), Figures 7 and 8.

in-plane size or thins, the deflection profile of an infinite ice sheet of a corresponding thickness h is gradually approached. The samples with a side length of $L = 160$ m, a thickness of $h = 0.5$ m, and a discrete element size of $l = 3h$ perform virtually identically to that of an infinite ice sheet, see Figure 3.5b.

The deflections of the sheet samples tend to agree well with those computed by plate finite elements. This is especially so for the samples with $L = 20$ m, viz. the samples that deform the least and displace the most, Figure 3.5a. For the larger samples, or the samples that deform more and displace less, the scatter is more pronounced since the number of degrees of freedom, and thus the deformability, of the sheet samples examined is significantly lower than that of the plates modelled with the plate finite elements, Figures 3.5a (on the right) and 3.5b.

Table 3.4 lists the maximum, relative deflections of the sheet samples. The maximum deflections of the samples with a discrete element size of $l = 2h$ are greater than of those with $l = 3h$. With a few exceptions, the sheet samples with $l = 3h$ deflect more than the plates modelled with FEM. The maximum deflections, if computed as an average of the maximum deflections with both $l = 2h$ and $3h$, exceed those produced by FEM. The errors are small. The maximum error is about 6 %.

Table 3.4. The averaged, maximum, relative deflections of the sheet samples and of the plates modelled with plate finite elements – Load case i).[†] The deflections shown have been normalised with respect to the maximum deflection of an infinite ice sheet of a corresponding thickness h . The table has been reproduced from Publication II, Table 3.

L [m]	h [m]	$l = 2h$	$l = 3h$	Average	FEM	Error, %
20	0.5	1.7300	1.7010	1.7155	1.6610	3
	1.0	3.9820	3.9680	3.9750	3.9630	0
	1.5	7.1650	7.1510	7.1580	7.1350	0
40	0.5	1.2040	1.1450	1.1745	1.1070	6
	1.0	1.4580	1.4210	1.4395	1.3720	5
	1.5	2.0640	2.0410	2.0525	2.0100	2
80	0.5	1.0810	1.0060	1.0435	1.0110	3
	1.0	1.1580	1.0930	1.1255	1.0810	4
	1.5	1.2380	1.1830	1.2105	1.1370	6
160	0.5	1.0740	1.0010	1.0375	1.0070	3
	1.0	1.0810	0.9998	1.0404	1.0110	3
	1.5	1.0900	1.0110	1.0505	1.0260	2

[†] The maximum, relative deflection of a point/node located at $x = y = 0$, Figure 2.7a. The “Error” (to the nearest digit) is computed as: $\text{Error} = ((\text{Average} - \text{FEM})/\text{FEM}) \times 100$.

Figure 3.6. Example global deflection fields of the sheet samples with $L = 40$ m – Load case *i*), Figure 2.7a. The red dots denote the mass centroids of the discrete elements. The figure has been reproduced from [Publication II](#), Figure 22. For similar plots of the other sheet sizes, consult Appendix A of [Publication II](#).

Figure 3.6 shows – as an example – global, relative deflection fields of the sheet samples with $L = 40$ m. For similar plots of the other sheet sizes, consult Appendix A of [Publication II](#).

3.2.1.2 Maximum, non-dimensionalised, absolute deflections

Figure 3.7a shows a graph of averaged, non-dimensionalised, maximum deflections $\bar{w}_{\max}$ as functions of h and for each L on a log-log graph. These deflections are not true deflections, but follow from $\bar{w}_{\max} = q/F_z = (F_z/kw_{\max}(L, h, l_{\text{ave}})L^2)^{-1}$, where q is an approximate resultant reaction force due to the buoyancy. Note that the deflections $w_{\max}$ denote here the maximum, absolute deflections of the sheet samples, not deflections relative to the maximum deflection of an infinite ice sheet, as previously. The results have then been averaged over both $l = 2h$ and $3h$ ($\sim l_{\text{ave}}$).

The data settles on nearly straight lines; linear regression is used for fitting. It is found that the deflection $\bar{w}_{\max}$ of the sheet samples with a side length of $L = 160$ m scales as $h^{-3/2}$ and with $L = 80$ m approximately as $h^{-7/5}$. The exponent m is found to decrease, and the corresponding deflection curves thereby to flatten, as the side length L decreases, implying that the smaller samples tend to displace as rigid bodies irrespective of the thickness h considered. If, instead of $\bar{w}_{\max}$, the maximum, absolute deflections $w_{\max}$ are plotted on a similar log-log graph, lines with identical slopes m emerge. Recall that the deflection of an infinite ice sheet scales as $h^{-3/2}$.

Figure 3.7b presents the deflections $\bar{w}_{\max}$, contrary to the foregoing, as functions of L and for each h on a log-lin graph. The results have been averaged over both $l = 2h$ and $3h$, as before. The deflections appear to settle, again, on nearly straight lines. On such a graph, an exponential function, i.e. $\bar{w}_{\max}(L) = b10^{mL}$, appears linear. It is found that the deflections of the sheet samples with a thickness of $h = 0.5$ m scale as $10^{L/93}$ and that the exponent m decreases as the samples become thicker.

Archimedes' principle shows that the deflections $\bar{w}_{\max}$ should attain a value of $\bar{w}_{\max} = 1$ for each side length L and thickness h considered if the sheets are to act as perfectly rigid bodies. Here, the values range from approximately $\bar{w}_{\max} = 1$ for the samples with $L = 20$ m and $h = 1.5$ m to about $\bar{w}_{\max} = 50$ for the samples with $L = 160$ m and $h = 0.5$ m. Any deviation from the value of $\bar{w}_{\max} = 1$ is thus an indicator of elastic deformation. In other words, the averaged, non-dimensionalised, maximum deflection $\bar{w}_{\max}$ of a sheet sample with a side length of $L = 160$ m, a thickness of $h = 0.5$ m, and loaded by a unit force F_z is roughly fifty times the deflection $\bar{w}_{\max}$ of a perfectly rigid plate. On the contrary, a sheet sample with $L = 20$ m and $h = 1.5$ m displaces approximately as a rigid body.

3.2.1.3 An infinite ice sheet

Figure 3.8 shows, in order to facilitate comparison between the deflection profiles of the two largest sheet sizes and that of an infinite ice sheet, the corresponding averaged deflection profiles as functions of characteristic

Figure 3.7. The averaged, maximum, non-dimensional deflections of the centrally-loaded sheet samples – Load case i), Figure 2.7a. The figure has been reproduced from Publication II, Figure 10.

length. A free square sheet with a side length of $L = 160$ m and a thickness of $h = 0.5$ m well approximates an infinite ice sheet. The deformations, and thereby also the stresses, are found to be negligible for $y/l_{ch} \gtrsim 5$.

Figure 3.8. The averaged deflections of the centrally-loaded sheet samples with $L = 80$ and 160 m as functions of characteristic length – Load case *i*), Figure 2.7a. The figure has been reproduced from Publication II, Figure 9.

3.2.2 Deflection of a free square sheet subjected to a vertical point load on an edge

In this subsection, the deflections of the sheet samples along the paths shown in Figure 2.7b are described.

3.2.2.1 Relative deflections

Figures 3.9 and 3.10 depict the deflections of all the studied sheets as well as of the plates modelled by the plate finite elements using [Abaqus, 2016]. Figure 3.9 gives the deflections along the free edge, i.e. along the path shown in red in Figure 2.7b, whereas Figure 3.10 along the path perpendicular to the free edge, i.e. along the path shown in blue in Figure 2.7b. The deflection of a semi-infinite ice sheet with a free edge, for each sheet thickness h considered, is shown in both figures for a reference. Each deflection shown has then been normalised with respect to the maximum deflection of a semi-infinite ice sheet of a corresponding thickness h . For an explanation of the other details, consult Publication II.

Figure 3.9. Averaged, relative deflection profiles of the edge-loaded sheet samples along the free edge, i.e. along the path shown in red in Figure 2.7b. The figure has been adapted from Publication II, Figures 12 and 13.

(a) $L = 20$ and 40 m (the first 10 and 20 m of a semi-infinite ice sheet due to symmetry).

(b) $L = 80$ and 160 m (the first 40 and 80 m of a semi-infinite ice sheet due to symmetry).

Figure 3.10. Averaged, relative deflection profiles of the edge-loaded sheet samples along the path perpendicular to the free edge, i.e. along the path shown in blue in Figure 2.7b. The figure has been adapted from Publication II, Figures 17 and 18.

Similarly to above, the smaller and thicker samples tend to displace more, as is shown in Figures 3.9a and 3.10a. The samples that are thicker tend to deform less. As a sample grows in its in-plane size or thins, the deflection profile of a semi-infinite ice sheet of a corresponding thickness h is gradually approached. The samples with a side length of $L = 160$ m, a thickness of $h = 0.5$ m, and a discrete element size of $l = 2h$ give a nearly identical averaged deflection profile to that of a semi-infinite ice sheet with a free edge, except of the point located at $x = y = 0$, Figures 3.9b and 3.10b.

The differences between the deflection profiles of the sheet samples and of the plates modelled with plate finite elements are small. Most scatter is produced by the samples with $L = 80$ m. The sheets with a side length of $L = 20$ m and the corresponding plates modelled with plate finite elements produce virtually identical results, see Figures 3.9a and 3.10a.

Table 3.5 lists the maximum, relative deflections of the sheet samples. The maximum, relative deflections of the samples with a discrete element size of $l = 2h$ are greater than of those with $l = 3h$. With a single exception, the samples with $l = 3h$ deflect less than the plates modelled with FEM. The maximum, relative deflections of the sheet samples, if computed as an average of the maximum deflections with both $l = 2h$ and $3h$, are less than those by FEM. The errors are moderate, with a maximum of about 12 %.

Table 3.5. The averaged, maximum, relative deflections of the sheet samples and of the plates modelled with plate finite elements – Load case *ii*).[†] The deflections shown have been normalised with respect to the maximum deflection of a semi-infinite ice sheet of a corresponding thickness h . The table has been reproduced from Publication II, Table 5.

L [m]	h [m]	$l = 2h$	$l = 3h$	Average	FEM	Error, %
20	0.5	1.7370	1.6800	1.7085	1.7700	−3
	1.0	4.2670	4.2510	4.2590	4.2780	0
	1.5	7.7700	7.7930	7.7815	7.7140	1
40	0.5	1.0460	0.9368	0.9914	1.0970	−10
	1.0	1.3990	1.3230	1.3610	1.4430	−6
	1.5	2.1250	2.0800	2.1025	2.1520	−2
80	0.5	0.9708	0.8596	0.9152	1.0120	−10
	1.0	1.0090	0.8971	0.9531	1.0650	−11
	1.5	1.0830	0.9738	1.0284	1.1420	−10
160	0.5	0.9269	0.8514	0.8892	1.0100	−12
	1.0	0.9683	0.8525	0.9104	1.0140	−10
	1.5	0.9714	0.8548	0.9131	1.0240	−11

[†] The maximum, relative deflection of a point/node located at $x = y = 0$, Figure 2.7b. The “Error” (to the nearest digit) is computed as: $\text{Error} = ((\text{Average} - \text{FEM})/\text{FEM}) \times 100$.

Figure 3.11 shows – as an example – global, relative deflection fields of the sheet samples with $L = 40$ m. For similar plots of the other sheet sizes, consult Appendix B of [Publication II](#).

Figure 3.11. Example global deflection fields of the sheet samples with $L = 40$ m – Load case *ii*), Figure 2.7b. The red dots denote the mass centroids of the discrete elements. The figure has been reproduced from [Publication II](#), Figure 26. For similar plots of the other sheet sizes, consult Appendix B of [Publication II](#).

3.2.2.2 Maximum, non-dimensionalised, absolute deflections

Figure 3.12a shows a graph of averaged, non-dimensionalised, maximum deflections $\bar{w}_{\max}$ as functions of h and for each L on a log-log graph, obtained similarly as above. The data settles, again, on nearly straight lines. It is found that the deflection $\bar{w}_{\max}$ of the sheet samples with a side length of $L = 160$ m scales as $h^{-3/2}$ and with $L = 80$ m approximately as $h^{-7/5}$. The exponent m is found to decrease as the samples become smaller in L , and the corresponding deflection curves thereby to flatten, as before. If only the maximum, absolute deflections $w_{\max}$ are plotted on a similar log-log graph, lines with identical slopes m emerge. Recall that the deflection of a semi-infinite ice sheet with a free edge scales as $h^{-3/2}$, see Publication II.

Figure 3.12b displays the deflections $\bar{w}_{\max}$ as functions of L for each sheet thickness h considered on a log-lin graph. The deflections $\bar{w}_{\max}$ appear to settle on nearly straight lines, as previously. An exponential function, i.e. $w_{\max}(L) = b10^{mL}$, thus appears to describe the data well. It is observed that the deflections $\bar{w}_{\max}$ of the sheet samples with a thickness of $h = 0.5$ m scale as $10^{L/95}$ and that the exponent m decreases as the samples become thicker.

3.2.2.3 A semi-infinite ice sheet with a free edge

Figure 3.13 shows, in order to facilitate comparison between the deflection profiles of the two largest sheet sizes and that of a semi-infinite ice sheet with a free edge, the corresponding averaged deflection profiles as functions of characteristic length. A free square sheet with a side length of $L = 160$ m and a thickness of $h = 0.5$ m well approximates a semi-infinite ice sheet with a free edge. The deformations, and thereby also the stresses, are found to be negligible for $y\beta \gtrsim 5$ and $x\beta \gtrsim 5 \dots 7$, see Figures 3.13a and 3.13b, respectively.

3.3 Inelastic in-plane response – tensile fracture

In this section, the inelastic in-plane response is overviewed. The computed tensile strengths are presented after which the observed cracking characteristics are briefly discussed.

3.3.1 Size and rate effects – effective tensile strength

Figure 3.14 shows the computed strengths as well as their standard deviations. The results have been given in an order of ascending sheet side length L , as before. For each L , the results have been arranged in an order of ascending sheet thickness h . For each h , the results are averages over 20 simulation results (10 randomised CVT meshes with both l), except the case

(a) Non-dimensional deflections as functions of h for each L .

(b) Non-dimensional deflections as functions of L for each h .

Figure 3.12. Averaged, maximum, non-dimensional deflections of the edge-loaded sheet samples – Load case *ii*), Figure 2.7b. The figure has been reproduced from Publication II, Figure 15.

(a) Deflections along the free edge.

(b) Deflections along the path perpendicular to the free edge.

Figure 3.13. The averaged deflections of the edge-loaded sheet samples with $L = 80$ and 160 m as functions of characteristic length – Load case *ii*), Figure 2.7b. The figure has been adapted from Publication II, Figures 14 and 19.

(a) $|v_x| = 0.1$ m/s (Load case A1). (b) $|v_x| = 0.01$ m/s (Load case A2).**Figure 3.14.** Averaged, normalised, effective tensile strengths. The figure has been reproduced from [Publication III](#), Figure 15.

with $L = 160$ m, $h = 0.5$ m, and $l = 2h$ for which 16 meshes were produced (10 with $l = 3h$ and six with $l = 2h$). Each result has then been normalised with respect to the critical axial stress parameter σ_{cr} , see Table 2.1. For the definition of σ_{cr} and the other associated parameters, consult Publication III. Figure 3.14a depicts results at the higher displacement rate $|v_x| = 0.1$ m/s, Load case A1, whereas Figure 3.14b at the lower displacement rate $|v_x| = 0.01$ m/s, Load case A2.

In Load case A1, the strength grows as a function of h and decreases as a function of L , Figure 3.14a. The sheets that are smaller in L or thicker have, in general, a higher effective tensile strength than those that are thinner or larger. At the largest sheet size, the strength, however, appears to saturate. Somewhat similar conclusions can be drawn from the results of Load case A2 as well, except that for the two largest sheet sizes ($L = 80$ and 160 m), the strength decreases as a function of h and that the sheets with $L = 40$ m produce practically equal results, see Figure 3.14b.

Figure 3.15 presents the results in a slightly different, in a fully non-dimensional, format. The logarithms of the ratios $\sigma_{cr,eff}/\sigma_{cr}$ have been plotted as functions of the logarithms of L/I_{ch} . The parameter I_{ch} denotes here Irwin's, or Hillerborg's, characteristic length and reads as $I_{ch} = EG_{eff}/\sigma_{cr}^2$. Note that the Young's modulus E is not here equal to the Young's modulus E_b given to the Timoshenko beam finite elements, but is the effective Young's modulus of a particular sheet "type," i.e. a sheet with specific L , h , and l . These were computed above. Strengths so computed have then been given separately for each sheet thickness h considered.

The non-dimensional, effective strength decreases as the non-dimensional sheet size increases, whereas the thickness h has an opposite effect. These observations apply at the higher displacement rate $|v_x| = 0.1$ m/s, Load case A1, Figure 3.15a. At the lower displacement rate, Load case A2, Figure 3.15b, the non-dimensional strength is a similar decreasing function of the non-dimensional sheet size, but as regards to the effect of the thickness h , the situation is more complex. It appears that an inflection point is located somewhere near the point $L/I_{ch} = 1$, and that for $L/I_{ch} > 1$, the thicker sheets give lower strengths. This last "branch" yields results typical for quasi-brittle materials: the thicker sheets exhibit a lower strength.

Figure 3.15 depicts also linear regression lines (with slopes m and coefficients of determination R^2) fitted to the data. The fits correlate reasonably well, given the parabolic appearance of the data curves. This is especially so in Figure 3.15a and is because of the higher strain rates for the sheets with the smaller L . A "truly" linear response is found only for the red data points in Figure 3.15b. It is observed that $|m|$ is a monotonically increasing function of h and that the trend continues, uninterruptedly, from Figure 3.15b to Figure 3.15a – rate effects dominate. The slope of the blue regression line in Figure 3.15a yields eventually a scaling law approximately equal to a LEFM-type scaling law of $(\cdot)^{-1/2}$. In other words,

(a) $|v_x| = 0.1$ m/s (Load case A1).

(b) $|v_x| = 0.01$ m/s (Load case A2).

Figure 3.15. Averaged, non-dimensionalised, effective tensile strengths. The figure has been reproduced from [Publication III](#), Figure 16.

for a thick enough sheet and at a large enough displacement rate, the response appears to mimic a LEFM-type response. Notice, however, that the strain rates were not kept as constants, and that at higher rates, the slopes may have got even steeper.

3.3.2 Cracking characteristics

Figure 3.16 portrays, as an example, a completely fractured ice sheet with a side length of $L = 160$ m, a thickness of $h = 0.5$ m, and a discrete element size of $l = 2h$. The mesh has in total 29561 discrete elements. Broad areas are softening and are highlighted in purple. Several fully grown cracks appear and are highlighted in grey. Their orientation appears to be, for the most part, approximately perpendicular to the loading direction. A usual assumption is that cracks grow in a direction perpendicular to the maximum principal stress trajectory. The areas remaining still undamaged are then highlighted in blue. The cracks tend to branch, bridge and have, in general, a rather tortuous pattern. The cracks, in short, bifurcate.

Figure 3.16. A fractured ice sheet with $L = 160$ m, $h = 0.5$ m, and $l = 2h$. The completely fractured beams are highlighted in grey, damaged (softening) beams in purple, and the beams that are still virginal in blue. The figure has been adapted from [Lilja et al., 2017b, Figure 6].

3.4 Inelastic out-of-plane response – vertical penetration fracture

This section gives an overview of the computed breakthrough loads and strengths. In Load case B1, a sheet penetrated by a rigid, flat-ended, cylindrical indenter was examined. All the sheets shown in Figure 2.11 were studied. In Load case B2, only the largest sheets ($L = 160$ m) were considered and had pinned boundary conditions. All the results in this and the next section are averages over 20 simulation results (10 with both l).

3.4.1 Breakthrough loads

Figure 3.17 presents the loads as well as their standard deviations for each side length L and thickness h considered. For each L , the results have been arranged in an order of ascending sheet thickness h , as before. Each load has then been normalised with respect to the vertical breakthrough load, $F_{cr, Wyman}$, of an infinite ice sheet of a corresponding thickness h . For the definition of $F_{cr, Wyman}$, see Publication III.

Figure 3.17. Averaged, normalised breakthrough loads. The figure has been reproduced from Publication III, Figure 19.

The breakthrough load is a “concave up” function of L for each sheet thickness h considered. The location of the relative minimum appears to depend on h . For the thinnest sheets, a minimum is located at about $L = 20$ m, whereas for the thicker sheets at around $L = 40$ m. Note that for the

sheets smallest in L , the response is strongly affected by the dimensions of the indenter. The thickest sheets tend to rise as rigid bodies and then suddenly fail when the buoyant (supporting) forces have nearly vanished. The rising trend there, as regards to the thickness h , is explained by the increased mass due to the thicker sheets. For the larger sheets, the trend appears, on the contrary, to be reversed; the thinner sheets exhibit higher relative loads. These two observations, i.e. the reversed effect of h and the concavity as $L = L_{\min} \mapsto L_{\max}$, can be interpreted to be due to a change in the mode of failure and inertia. The sheets with a side length of $L \leq 20$ m exhibit mostly radial cracking (the thinnest sheets show also some circumferential cracking, see Figure 3.20), whereas for $L \geq 40$ m, the failure is nearly always accompanied by circumferential cracking as well. Circumferential cracking necessarily means higher loads, which explains the rising trend. The other contributor to the higher observed loads as L increases is the fact that the larger sheets have a higher inertia. A successful completion of a failure mechanism via circumferential cracking requires a lateral restraint. Such a restraint is provided by the inertia of a large specimen.

3.4.2 Breakthrough strengths – size effect

Figure 3.18 presents the averaged breakthrough strengths as well as their standard deviations for each side length L and thickness h considered. The strengths have been normalised here with respect to the effective tensile strength, $\sigma_{\text{cr,eff}}$, of a sheet sample having the corresponding L and h . These were computed above and correspond to Load case A1, $|v_x| = 0.1$ m/s. For the effects of the different data sets on the normalisation, see [Publication III](#). A line $\sigma_F/\sigma_{\text{cr,eff}} = 1$ is plotted, in addition, to indicate that the breakthrough strengths are not equal to the effective tensile strengths, and because for most of the samples $\sigma_{\text{cr,eff}}/\sigma_{\text{cr}} \gtrsim 1$, not necessarily equal to the critical axial stress either.

A clear trend of thicker sheets exhibiting a lower relative breakthrough strength is found. A strong dependence on the side length L emerges as well. The larger and thinner the sheet, the higher the relative breakthrough strength. The strength, however, appears to saturate for the sheets with $L = 160$ m and $h = 0.5$ m; the boundary conditions have no significant effects. For the thicker sheets, the strength, on the contrary, keeps on increasing from Load case B1 to B2. The sheets with a side length of $L = 160$ m and a thickness of $h = 0.5$ m thus appear to resemble an infinite ice sheet.

Figure 3.19 shows the strengths in a fully non-dimensional format. The logarithms of the averaged, normalised breakthrough strengths, $\sigma_F/\sigma_{\text{cr,eff}}$, have been plotted as functions of the logarithms of h/I_{ch} . Both $\sigma_{\text{cr,eff}}$ and E ($I_{\text{ch}} = EG_{\text{eff}}/\sigma_{\text{cr,eff}}^2$) have been here computed at the higher displacement

Figure 3.18. Averaged, normalised breakthrough strengths. The figure has been reproduced from [Publication III](#), Figure 21.

rate $|v_x| = 0.1$ m/s, Load case A1 and Load case I, respectively.

The data settles on nearly straight lines. It is found that the slope m , and thereby the scaling rule, changes from one L to the other and that the slope of the line connecting the data points of the two thickest sheet sizes for $L = 160$ m is close to $-1/2$. Recall that the breakthrough strength of a point-loaded, infinite ice sheet scales as $h^{-1/2}$, see [Bažant, 2002]. For the effects of the different data sets on the normalisation, see [Publication III](#).

3.4.3 Cracking characteristics

Figure 3.20 depicts, as an example, a sequence of snapshots taken from a simulation in Load case B1. Several cracks nucleate on the top surface of the specimen and start to propagate towards the free edges. A star-shaped pattern of radial cracks next emerges. A failure mechanism completes the moment the cracks reach the free boundaries. For the thinnest sheets, and the sheets with $L \geq 40$ m, a failure is nearly always accompanied by circumferential cracking as well, as was noted above.

Figure 3.19. Averaged, non-dimensionalised breakthrough strengths. The figure has been reproduced from [Publication III](#), Figure 23.

3.5 Ice-structure interaction response

This section gives an overview of the computed cone ice loads. The typical cracking characteristics observed are shortly described as well. In Load case C1, the breaking of a freely-floating ice sheet containing a circular hole by a rigid, truncated cone was examined. Only the sheets with a side length of $L = 160$ m were studied, Figure 2.11. In Load case C2, the same sheets, but with pinned boundary conditions applied, were investigated.

3.5.1 Ice-breaking loads on a cone

Figure 3.21 presents the loads, with their standard deviations, for both load cases. The left-hand side is for Load case C1, whereas the right-hand side is for Load case C2.

In both load cases, the computed breaking loads are, if taken as an average over all h , approximately equal to those predicted by the model of Croasdale and Cammaert [1994]. For the definition of $F_{V,Croasdale}$, see [Publication III](#). At a rough estimate, the presented loads occurred at the time instants the circumferential cracks started to form.

An interesting result is that the boundary conditions have no significant effects. The two load cases produced approximately equal results. It is interpreted to indicate that the sheets studied are large enough, for the

Figure 3.20. A sequence of snapshots from a simulation in Load case B1 ($L = 20$ m, $h = 0.5$ m, and $l = 2h$). The indenter surfaces at the centre. The figure has been reproduced from [Publication III](#), Figure 20.

Figure 3.21. Averaged, normalised cone ice loads. The figure has been reproduced from [Publication III](#), Figure 24.

current load cases, to resemble an infinite ice sheet. The outer boundaries are far enough from the process zones as regards to the computation of the breaking loads.

3.5.2 Cracking characteristics

Figure 3.22 displays, as an example, two fractured sheets studied in Load case C1. Several radial cracks propagate outwards from the contact zones and approximately perpendicular to the cone surfaces. Circumferential cracks joining the radial cracks emerge and complete the failure mechanisms.

An interesting observation is that in a thick sheet, the cracks tend to propagate all the way to the free boundaries, Figure 3.22a, whereas in a thin sheet, Figure 3.22b, they tend to arrest. This looks like a clear size effect; the size and number of elements may have an effect though. In a thick sheet, cracks tend to grow longer due to the greater amount of stored potential energy being released from the water-ice sheet system, i.e. the crack driving force is larger for a thicker sheet. Otherwise, the responses appear quite similar. The effective “sizes” of the broken off wedges, for example, are in both cases approximately equal – about $3l \dots 5l$ – and the failure patterns near the cones seem to be coincident. Cracking

characteristics should be similar for geometrically similar structures.

(a) $L = 160$ m, $h = 1.5$ m, and $l = 3h$.

(b) $L = 160$ m, $h = 0.5$ m, and $l = 3h$.

Figure 3.22. Sheet samples interacting with rigid, truncated cones (not shown). Radial cracks propagate from the contact zones towards the free edges approximately perpendicular to the cone surfaces. Circumferential cracks joining the radial cracks emerge and complete the failure mechanisms. The figure has been reproduced from [Publication III](#), Figure 26.

4. Discussion

In this chapter, a discussion on the results and some other features observed is given. To aid in reading, the chapter is divided in three sections, each concentrating on the contents of a single publication. With the exception of the third paragraph in subsection 4.1.2, all the material presented has been adapted from the corresponding publications, mainly from the discussions therein. For further discussions and explanations, consult the respective publications.

4.1 On the contents of Publication I

This section gives a discussion on the topics considered in the first publication.

4.1.1 On the size effect

In each elastic in-plane test, a clear declining trend as regards to the effective in-plane moduli was found, see Figures 3.1-3.4 as well as Figures 12 and 14 in Publication I. The samples with a smaller relative size appeared stiffer than those that had a larger relative size. In addition, the moduli appeared to asymptote to values reasonably close to the reference moduli of the lattices with perfectly triangular topologies. These observed trends are, apparently, due to the CVT-tessellated cell topology, beam dimensions, and the applied boundary conditions. A lattice meshed with either perfectly regular or perturbed hexagons is a bending-dominated lattice (with a nodal connectivity $Z = 3$, the stiffness $\propto E_b I_{p,q}$), whereas a lattice meshed with triangles is a stretching-dominated lattice (with a nodal connectivity $Z = 6$, the stiffness $\propto E_b A_{p,q}$) [Ashby et al., 2001]. The axial stiffness, $E_b A_{p,q}$, is in general much greater than the in-plane bending stiffness $E_b I_{p,q}$. In the foregoing, $A_{p,q}$ and $I_{p,q}$ denote the cross-sectional area and the (in-plane) second moment of area of a beam finite element p, q , respectively. A CVT-tessellated lattice contains, for the most part, approximately equilateral

triangles and is thus a stretching-dominated lattice. As a consequence, the computed moduli are in a reasonably good agreement with the reference moduli of the lattices with perfectly triangular topologies. The smallest samples tend to stiffen due to the overly thick, overlapping beam finite elements and the close proximity of the boundaries. The displacements of the nodes on the boundaries are constrained, which affects approximately the width of an element layer, i.e. the whole sheet, see Figure 2.3. Studying sheets with such boundary conditions was nonetheless intentional. While an ice-structure interaction simulation is in progress, an initially intact ice sheet gets fractured into fragments which may eventually evolve into a rubble. Strong boundary disturbances will most probably prevail in an actual simulation as well.

Other similar studies, to be shortly discussed, have shown alike or contrasting trends depending on the type of the lattice topology examined and the boundary conditions applied. These are the two major contributing factors, i.e. the lattice topology and the boundary conditions. Most of the other investigations have focused on studying hexagonal topologies due to their areas of application being industrial honeycomb-like foams. Because of the characteristics of the manufacturing process, most such foams exhibit a nearly hexagonal internal structure [Gibson and Ashby, 1999]. For a comparison, some of these earlier results will next be reviewed. For further details, see Publication I.

Before proceeding with the comparison, it is briefly noted that no appropriate experimental results of tests with sea ice on a large enough size scale exist so as to assess whether a size trend as regards to the stiffness would occur in nature. The closest are probably those of [Dempsey et al., 1999]. They found, while carrying out *in-situ* fracture tests with square samples having an in-plane size range of 1 : 160, that the *elastic modulus* of an ice sheet varies substantially as a function of the sample size. There was, however, no observable trend in the results, which may have been due to the varied loading rates. No similar results exist of the in-plane shear and bulk moduli. On grain scale, on the other hand, results do exist, see [Elvin, 1996], suggesting that the effects to be seen occur, quite naturally, on scales much smaller than the ones examined in this thesis. It is emphasised, again, that the size effect is, in the present context, to be taken as an artefact of the developed numerical model. It is *not* claimed that such an effect would exist in ice on size scales presently studied.

4.1.1.1 *Effective Young's modulus*

It is noteworthy that the declining trend, depicted in Figures 3.1 and 3.2, is opposite to the findings of [Onck et al., 2001; Teköglu and Onck, 2005], and [Teköglu et al., 2011]. In [Gurtner and Durand, 2014, app. B], results partially similar to those in this thesis were found. In [Onck et al., 2001] and [Teköglu and Onck, 2005], the loaded boundaries had periodic

boundary conditions, whereas the other two boundaries were traction-free. In [Gurtner and Durand, 2014], the samples were loaded and supported as here in *Load case III*, but with periodic boundary conditions on the other two edges. In Teköglu et al. [2011], the top and bottom boundaries were clamped and the vertical boundaries traction-free.

In [Onck et al., 2001; Teköglu and Onck, 2005], and [Teköglu et al., 2011], the lattice topologies consisted either of perfectly regular or perturbed hexagons, whereas in [Gurtner and Durand, 2014] of perfect or disordered Delaunay-triangulated triangles. The lattices were discretised either with Euler-Bernoulli [Gurtner and Durand, 2014] or Timoshenko beam elements [Onck et al., 2001; Teköglu and Onck, 2005; Teköglu et al., 2011]. In all of the studies, the samples were rectangular and their relative densities $\bar{\rho} \ll 1$. For the definition of $\bar{\rho}$, see p. 19 in [Publication I](#).

4.1.1.2 Effective shear modulus

The found declining trend, see Figure 3.3 and Figure 12 in [Publication I](#), is now similar to the findings of Onck et al. [2001]; Teköglu and Onck [2005], and [Teköglu et al., 2011]. In [Gurtner and Durand, 2014, app. B], results partially similar to those in this thesis were found.

In [Teköglu et al., 2011], a δ -Voronoi-tessellated, rectangular lattice of a gradually increasing macroscopic size, but with a fixed aspect ratio of $W/H = 12$ ($W \sim$ Width, $H \sim$ Height), was studied. The boundary conditions applied were comparable to those in this thesis. It is reported in [Teköglu et al., 2011, p. 137], that:

“For shear, low rotational zones of ~ 1 or 2 cell sizes form at the boundaries where the samples are bonded to rigid loading platens, and act as strong boundary layers with a larger resistance to deformation. The area fraction of these strong boundary layers increases with decreasing sample size, leading to an increase in both the stiffness and strength of the samples.”

This is a physically plausible explanation. In [Onck et al., 2001], an analytical study of an infinitely wide lattice with clamped top and bottom boundaries was performed. The height H was then increased incrementally. In [Teköglu and Onck, 2005], a somewhat similar problem was investigated but numerically. In [Gurtner and Durand, 2014], a rectangular lattice with a hinged bottom boundary, a shear-loaded top boundary, and with periodic boundary conditions on the other two vertical boundaries was studied.

In [Diebels and Steeb, 2002], results partially opposite to those in this thesis were found. If the height H of a rectangular specimen with traction-free vertical boundaries, a fixed bottom boundary, and a clamped top boundary was held fixed and the width W was then increased, the effective shear stiffness G increased. If, on the contrary, the width W was held fixed

and the height H was increased, the stiffness decreased. The stiffness of an infinitely wide lattice, with the height H being increased, was found to decrease as in this thesis. The samples examined had a topology of perturbed hexagons, the cell walls were discretised with Euler-Bernoulli beam finite elements, and their relative densities $\bar{\rho} \ll 1$.

In [Liebenstein et al., 2018], a Voronoi-tessellated, rectangular, cellular solid with both a regular and an irregular topology was examined. The cell walls were discretised with Timoshenko beam finite elements and the relative density $\bar{\rho} \ll 1$. The two vertical boundaries were traction-free, whereas the bottom boundary was fixed and the top boundary clamped. With a regular mesh, increasing the width W – while keeping the height H fixed – led to an increase in the effective shear stiffness G . Increasing both the height H and the width W simultaneously, but in proportions, was found to either decrease or increase the stiffness depending on whether the aspect ratio W/H was below or above a critical threshold limit $(W/H)_{\text{critical}}$. The stiffness was found to increase for the samples with an aspect ratio $W/H \lesssim 2$, whereas for a larger ratio, the stiffness decreased. The effect of mesh irregularity was investigated with samples having a fixed aspect ratio of $2/\sqrt{3}$. An increase in the mesh irregularity increased the effective shear stiffness G for all specimen sizes. For a specimen with a fixed “irregularity index” β , a transition from “a smaller is less stiff” to “a smaller is stiffer” type of a behaviour occurred as a specimen grew in size. A transition from a declining to a rising trend thus appeared to occur when a critical mesh “irregularity index” β_{critical} was reached. A somewhat similar observation was done already in [Gurtner and Durand, 2014]; a transition from a declining to a nearly horizontal ($\sim$ no size effect) trend seemed to occur if a perfectly triangular mesh got sufficiently perturbed. This finding may provide an avenue to attenuate the size effect, discussed neither in [Liebenstein et al., 2018] nor [Gurtner and Durand, 2014].

4.1.1.3 *Effective bulk modulus*

With regard to the effective, in-plane bulk modulus K , no earlier studies with similar boundary conditions, as in this thesis applied, were found. In [Symons and Fleck, 2008], a brief discussion of the problem with periodic boundary conditions was given. A Kagome-type unit cell with an aspect ratio of $\sqrt{3}/2$ was examined by increasing the system size in proportions. The cell walls were discretised with Euler-Bernoulli beam finite elements and the relative density $\bar{\rho} \ll 1$. Similarly to the present results, the effective, in-plane bulk modulus K was found to be a decreasing function of the system size.

4.1.1.4 Poisson's ratios

The in-plane Poisson's ratios of the sheet samples examined were not explicitly computed. The in-plane Poisson's ratio of an infinite, in-plane Euler-Bernoulli beam lattice of an equilaterally triangular topology is equal to $1/3$, [Gibson and Ashby, 1999], whereas of an infinite, in-plane Timoshenko beam lattice of an equilaterally triangular topology equal to about 0.26 [Karihaloo et al., 2003]. For $\bar{\rho} \ll 1$, the in-plane Poisson's ratio of a Timoshenko beam lattice is equal to that of an Euler-Bernoulli beam lattice. Due to the effective, in-plane moduli E and K being close in their values to those of an in-plane beam lattice modelled both with Euler-Bernoulli and Timoshenko beams (for $L_{\text{rel}} = L/l \gtrsim 25$), it can be argued that an in-plane Poisson's ratio of about the same order of magnitude results, i.e. close to $1/3$. If, on the other hand, the equations of plane stress elasticity are directly resorted to, the equation for the shear modulus G gives an in-plane Poisson's ratio of about 0.87, whereas of the bulk modulus K around 0.21 (these both results have been averaged over the results computed with the sheets having $L = 160$ m). For the definitions of these equations, see Eqs. 10 in [Publication I](#). The equations of plane stress elasticity clearly do not apply; the effective macroscale response of a CVT-tessellated lattice cannot be predicted by the equations of the classical in-plane elasticity. It is emphasised that the in-plane Poisson's ratio of an FE-DE sheet sample ought to exhibit a similar size dependence as E , G , and K and is thus also understood to be an effective material property. Note that Poisson's ratios exceeding the limit of incompressibility (or in that matter, negative) for classical solids, i.e. $\nu = 0.5$, are not uncommon for cellular solids. For instance, for a lattice with a hexagonal topology and a relative density $\bar{\rho} \ll 1$, the in-plane Poisson's ratio is equal to one [Gibson and Ashby, 1999].

4.1.2 On the unstructuredness of a mesh

The main motivation for employing an unstructured lattice mesh is its ability to describe isotropic cracking. The application of an unstructured lattice mesh to ensure isotropic crack growth compromises, however, the predictability of the respective elastic properties and thus the susceptibility to failure. These two are clearly mutually coupled. The effective stiffnesses, the values of which depend on the relative sheet size parameter L_{rel} , affect the computed strains, which are, then, driving the damage. In other words, damage will generally not initiate on strain levels one would expect. The problem belongs, therefore, to the realm of multiscale problems. A size effect exists, of course, also for a structured lattice. In order to get an idea of how an unstructured lattice behaves elastically, constitutive tests such as those reported in this thesis should be performed. Of the previous works that used lattice-based meshes, the meshes in [Jirásek and Bažant, 1995a;

Sayed and Timco, 1998; Dorival et al., 2008; Herman, 2016], and [van den Berg, 2016] were unstructured, with no test results given, however.

Material properties of sea ice can vary quite considerably [Timco and Weeks, 2010]. It may thus be safe to say that the existence of a size effect in a numerical model of an ice sheet is tolerable as long as the bounds within which the effective moduli can vary are at least approximately known. In order to be able to reliably estimate the loads (i.e. the total loads, not just the breaking loads) imposed by sea ice on an offshore structure, the variability of the material properties must be taken into account. A lattice-based model may provide such a functionality. Furthermore, since the ice-structure interaction processes appear to exhibit deterministic chaos [Räty, 1992; Daley et al., 1998; Ranta et al., 2018a,b],¹ it may be speculated that a slight difference in the effective material properties of a lattice, from one lattice to another, can act as a triggering initial perturbation. Recall that lattices of equal in-plane sizes did not exhibit identical in-plane responses; small yet finite differences between the effective moduli existed.

A brief note about the previous paragraph: a number of questions may arise on the basis of the results found by Ranta et al. [2018a,b]. The most immediate relate to the well-posedness/well-conditionedness of the problem as well as to the stability of the applied numerical method. The so-called Hadamard's postulates state that for a problem to be well-posed, a solution must exist, be unique, and depend continuously on the data, i.e. the initial conditions, boundary conditions, and the problem parameters. Such is not the case for an ill-posed problem. A problem may, on the other hand, be well-posed, yet the solution ill-conditioned because of an instability due to, e.g., finite precision arithmetics, unstable time stepping, or a loss of hyperbolicity/ellipticity (material instability in a dynamic/quasi-static problem, respectively). To be more precise, it may, in this last case, be more appropriate to say that the problem becomes ill-posed because of the non-uniqueness of the solution, i.e. the mesh non-objectivity in describing a strain-softening response. A deterministically chaotic system, on the contrary, is ill-conditioned *per se*. For such a system, the solution trajectories diverge vastly even though the problem characteristics are deterministic and even if the solutions have been computed by a stable numerical method. An extreme sensitivity to the data exists, a phenomenon coined as a “butterfly effect” by Lorenz [1972]. Because of the characteristics of the model of Ranta et al. [2018a,b], i.e. the conditional stability of the time stepping algorithm and the stability of the material model – the latter is

¹In [Ranta et al., 2018a,b], a vanishingly small initial vertical perturbation (a velocity in the order of $v_0 = 10^{-12}$ m/s) of a node located at the free edge of an ice sheet led to hugely diverging solutions. Such a finding is not yet sufficient to declare chaos rigorously, but is in any case a strong indicator of it. Note that neither in [Ranta et al., 2018a] nor [Ranta et al., 2018b] was the phrase “deterministic chaos” explicitly stated or the root cause to the observed instability explained.

regularised via the introduction of an adequate element length prohibiting a snapback instability, it may be that the modelled processes truly are deterministically chaotic. Note, however, that the conditional stability of the time stepping algorithm holds, to be exact, only for a linear problem and that errors due to the finite precision arithmetics cannot be avoided, but add to the ill-conditionedness of the possibly inherently ill-conditioned solution.² If the processes nevertheless are chaotic, it may be that the findings of Ranta et al. [2018a,b] in two dimensions could be replicated by the present method in three dimensions – qualitatively – just by running simulations with randomised tessellations of similar-sized sheets due to their slightly varying material properties, both elastic and inelastic.

4.2 On the contents of Publication II

This section gives a discussion on the topics considered in the second publication.

4.2.1 On the internal stress resultant fields

While the deflection of an ice sheet modelled with an FE-DE approach was investigated quite thoroughly, it was out of the scope of the publication to study the spatial distributions of the internal stress resultants. It can, nevertheless, be argued that the spatial distributions of the internal stress resultants of an FE-DE sheet sample and the corresponding tensor fields of a Kirchhoff-Love plate are *discretely* similar. There are three justifications for this proposition:

- i) the observed (nearly equal) deflection profiles of the FE-DE sheet samples, plates modelled with plate finite elements, and the infinite/semi-infinite Kirchhoff-Love plates, see Figures 3.5, 3.9, and 3.10,
- ii) the (nearly) equilaterally triangular topology of a CVT-tessellated (Delaunay-triangulated) mesh; and
- iii) the fact that the partial differential equation governing the bending of an equilaterally triangular assembly of Euler-Bernoulli beams is of a biharmonic-type (the effect of shear is here thought to be of a negligible importance) [Renton, 2002].

Furthermore, with shear and damping being neglected, the spatial distributions of the internal stress resultants ought to converge to those of

²Not just due to rounding in floating point arithmetics. For example, irrational numbers, say $\sqrt{2}$; e ; or π , cannot be represented exactly because their decimal expansions do not terminate.

a Kirchhoff-Love plate (excluding, perhaps, the fields along the boundaries) as l tends to zero, provided the mesh is perfectly periodic and of an equilaterally triangular topology.

4.2.2 On the effective sheet size regarding ice-structure interaction simulations

Offshore structures in Arctic waters interact with ice floes of different sizes, from small to giant floes [WMO, 1989]. In order to conduct a full-scale ice-structure interaction simulation with a large floe, the in-plane size of a modelled ice sheet should be large enough for it to resemble a semi-infinite ice sheet. A medium- to a giant-sized floe in contact with, for example, a conical offshore structure is well approximated by a semi-infinite ice sheet subjected to a vertical point load on the free edge. While this is generally true, the following remark should be, however, noted: in the context of a flexural failure, which supposedly is the predominant failure mode of an ice sheet interacting with a cone, the radial distance from the semicircular line of contact along the perimeter of the cone to the approximately semicircular line ahead and around the cone (along which the circumferential cracks start to nucleate prior to the final failure – and which is preceded by radial cracking), is – theoretically – in the order of $l_{bl} = 0.6 \dots 0.8 l_{ch}$ [Kerr and Kwak, 1993; Li et al., 2003]. The subindex “bl” stands here for “**breaking length**.” The estimate is based on approximating the radial location of the maximum, circumferential bending moment in an ice sheet applying either a beam, wedge, or a plate model. On the basis of model and full-scale tests even smaller values have been found, in the order of $l_{bl} = 0.1 \dots 0.5 l_{ch}$ [Li et al., 2003]. It may, therefore, be possible to use a sheet with a side length L much smaller than that found in **Publication II**, and yet conceivable as a semi-infinite ice sheet interacting with a cone, when conducting an actual ice-structure interaction simulation. It is to be noted that a failure process proceeds in cycles. First, a load is ramped up, which is followed by breaking, after which a new cycle starts. What shall the “effective” in-plane size of an ice sheet being pushed against a structure be so that the stress waves emanating from the process zone and propagating through the medium do not reflect from the boundaries and back to the process zone – and so that a wanted number of cycles is still attained – is the relevant question to be asked. Some adequate internal damping and non-reflecting boundaries would have to be applied, of course. Note also that the sheets examined in this thesis were square. The requirement for a square sheet to be able to act both as an infinite and a semi-infinite ice sheet then led to the conclusion that the sheet side length should satisfy $L/2 \gtrsim 5l_{ch}$, see Figures 3.8 and 3.13. Regarding only the case of edge loading, a free rectangular sheet with the dimensions of about $L \times L/2$ (with L being the width of the sheet along the loaded edge)

should, however, suffice. This has, indeed, shown to be the case [Lu et al., 2015].

4.2.3 On the scaling relationships

It is noteworthy that in both load cases the averaged, non-dimensionalised, maximum deflections $\bar{w}_{\max}$ scaled by a power-law, i.e. $\bar{w}_{\max} \propto h^{m_1(L)}$, if computed over h , for each L , and averaged over both l . A relationship of a form: $\lim_{L \rightarrow L_{\min}, h \rightarrow h_{\max}} \bar{w}_{\max} = b(L, h)h^{m_1(L)} \approx b$, was eventually found, see Figures 3.7a and 3.12a. With a side length of $L = 160$ m, the $\bar{w}_{\max}$ scaled, up to one digit, as $h^{-3/2}$. A similar relationship held for the absolute, maximum deflections $w_{\max}$ as well. If, instead, computed over L , for each h , and averaged over both l , the deflections $\bar{w}_{\max}$ scaled, in each load case, exponentially, i.e. $\bar{w}_{\max} \propto 10^{m_2(h)L}$. A relationship of a form: $\lim_{L \rightarrow L_{\min}, h \rightarrow h_{\max}} \bar{w}_{\max} = b(L, h)10^{m_2(h)L} \approx b$, was then found, see Figures 3.7b and 3.12b. The exponents $m_1(L)$ and $m_2(h)$ were observed to be approximately equal in both load cases; the same power-law- and exponential-type scaling relationships applied, except for the multiplying coefficients b . It is to be noticed, however, that the exponents of the exponential scaling rules found in both load cases were rather small, implying that the scaling rules are weak. On the other hand, the exponents appeared to grow in magnitude quite fast with the sheet thicknesses being decreased. The ratios of the exponents of the scaling rules of the thinnest and the thickest sheets were, in fact, close to 1.5 in both load cases, see Figures 3.7b and 3.12b. This may imply that the scaling rules turn out significant for thin sheets.

It is of an interest to note that the deflection of both an infinite ice sheet and a semi-infinite ice sheet with a free edge scales as $h^{-3/2}$, i.e. similarly to that found here with the largest sheets for both the absolute and the non-dimensionalised deflections. This is an intriguing analogy in view of the fact that there is no partial differential (field) equation that would describe the deflection of an unstructured, CVT-tessellated lattice.

While the scaling relationships were derived with the aid of a relatively small number of parameters, i.e. three thicknesses h and four side lengths L only, it is believed that they can be used to estimate the maximum deflection of an ice sheet in general. More accurate results would necessarily require larger simulation sets.

4.3 On the contents of Publication III

This section gives a discussion on the topics considered in the third publication.

4.3.1 On the uniaxial tensile strength – rate effect

A lucid *a priori* assumption would have been that all rate-related effects are of a minor importance: the de-cohesive damage model implemented was rate-independent and the displacement rates applied rather low. Such an assumption, however, would have proved to be false – a significant rate effect emerged. For example, the ratio between the effective tensile strengths of the smallest thickest sheets, for the two displacement rates considered, was approximately four, see Figure 3.14. It then reduced to an average of around 1.3 for the largest thickest sheets. For the thinner sheets, and so with more elements, the effect somewhat attenuated, but was still present. A possible explanation for the observed increase in the effective tensile strength at the higher loading rates is that of diffuse cracking. Instead of a single dominant crack, several smaller cracks (microcracks) appear, see Figure 3.16, which then produces an increase in the effective strength. This is not only because of an increase in the dissipated fracture energy but due to inertia as well. A plausible assumption would have been that inertia plays no role because of the initial (linearly changing) velocity field established in each specimen, see Figure 2.8. It may, however, be that as a result of elastic restoring forces; viscous damping forces; and microcracking, “microinertia,” i.e. the inertia of individual discrete elements, becomes significant. Microcrack nucleation, growth, interaction, and coalescence inevitably either accelerates or decelerates individual discrete elements, not just those with beams undergoing damage, but the nearby elements as well. Inertia, in conjunction with an intrinsic time scale, leads then to an apparent strength increase.

It has been shown that a cohesive finite element with a rate-independent traction-separation law exhibits not only a characteristic length scale but a characteristic time scale, t_{ch} , as well [Camacho and Ortiz, 1996]. The expression of the characteristic (relaxation) time t_{ch} is a linear function of both density (here, the inertia of a discrete element) and the longitudinal wave speed, c_s , in an element, see [Camacho and Ortiz, 1996, Eq. 69, p. 2916]. The effective stress amplitude to cause fracture, on the other hand, is an exponentially decaying function of τ/t_{ch} , where τ denotes pulse duration. An increase in the loading rate (so that the pulse duration decreases) results then in an increase in the apparent strength. The observed effective strength increase is thus interpreted to result from both the increased fracture energy dissipation and microinertia.

Diffuse cracking *per se* may be best explained by following lines similar to those set forth by Mott [1947]. Mott analysed detonation-driven wave propagation and fracture in a ring-shaped specimen made out of a ductile metal (a one-dimensional problem). At a sufficiently low strain rate, a stress release wave from a single crack is able to unload the rest of the specimen, thus resulting in no further damage. Conversely, at a sufficiently

high strain rate, the wave is too slow, which then leads to further damage. Whether Mott's theory holds also for a lattice, especially because the strain rates were here rather low (i.e. $c_s/L > \dot{\epsilon}$ should have held), is questionable, but the principle is plausible. Another explanation could be that of bifurcation instability. Schardin [1959] found experimentally that at higher loading rates, cracks in a brittle solid tend to bifurcate and the limit speed to be far less than c_s . Such an observation appears to support the current results. Instead of a single dominant crack, rather distributed damage occurs while $c_s/L > \dot{\epsilon}$ simultaneously holds.

Note that the omission of computation of the post-fracture contacts, as well as not considering drag forces, gave probably lower estimates for the effective tensile strengths by not taking into account effects due to friction, say interlocking. Not including contacts was due to computational reasons. A look in Figure 3.16 reveals that softening, fracturing, and thus contacts occur not only locally but globally. Running all the simulations on a Fujitsu Celsius W530 Power workstation with an Intel Xeon E3-1245V3 processor took about eight months.

4.3.2 On the vertical penetration fracture – cracking characteristics

There has been much discussion in the pertinent literature as to whether the radial cracks are, at maximum load, fully through or not [Dempsey et al., 1995; Bažant and Kim, 1998a,b; Bažant, 2002]. The last three results are for an infinite ice sheet, whereas the first one is for a finite, clamped plate not on an elastic foundation. Here, in Load case B1 (for a finite, freely-floating ice sheet), the cracks appeared to be fully through. The observation is based solely on visual inspection of animated crack growth, but appeared evident for the sheets with the side lengths of $L = 10, 20$, and 40 m. For the larger sheets, and especially for the sheets with the pinned boundary conditions in Load case B2, the situation was much more complex. A closer look was obscured by the fact that a beam was judged to be fully degraded as soon as a critical number of fully damaged integration points was reached. The bottommost row of integration points (i.e. those closest to the indenter) got thus probably destroyed when reached by the crack front. For an explanation of the criterion applied to judge whether a beam is fully degraded or not, as well as for the other details regarding the de-cohesive damage model implemented, see Publication III.

Another interesting question is that of crack closure. Some crack closure may have occurred in Load case B1, but only instantly, and is because of the boundary conditions. The outer boundaries of a sheet sample were free in Load case B1. To be more precise, no “Dirichlet”-type boundary conditions were applied. Because of the free boundary conditions, radial cracks tend to propagate all the way to the free boundaries. This is especially so in

thick sheets. Due to the momentum imparted by the penetrating indenter to the sheet, the broken pieces tend then to drift apart. There is nothing holding them back, which is necessary for the dome (or arching) effect – and thus the crack closure – to take place. The dome effect ought to dominate when the boundaries are not free and with samples that have a large enough inertia. For an infinite ice sheet, as well as the sheets in Load case B2, this, indeed, is the case. The attached animations, [Animation 3](#) and [Animation 4](#), illustrate these phenomena quite clearly. In the first animation, a sheet with a side length of $L = 40$ m and a thickness of $h = 1.0$ m is penetrated by an indenter from below (Load case B1). In the latter animation, a surfacing cone is breaking a sheet with a side length of $L = 160$ m and a thickness of $h = 1.5$ m (Load case C1). Similar conclusions can be drawn from both simulations: the broken pieces start to drift apart and crack closure takes place only instantly. It can be argued, of course, that at least partly the observed movement is because of force transmit through crack closure.

It is stated in [Bažant and Kim, 1998b] and [Bažant, 2002] that the breakthrough strength of a floating, point-loaded, infinite ice sheet scales as $h^{-1/2}$ if the cracks are partially through and as $h^{-3/8}$ if they are fully through. These results are, more specifically, for a notched Kirchhoff-Love plate resting on a Winkler-type foundation. Following Bažant and Guo [2002], a statically indeterminate in-plane frame with inelastic softening hinges, and which rests on a Winkler-type foundation, exhibits a similar strong, monotonic size effect of a type $h^{-1/2}$. It was in this thesis found, interestingly, see Figure 3.19, that the breakthrough strength of the largest sheets scales approximately as $h^{-1/2}$. Notice, however, that the result holds true only for the two thickest sheet sizes considered and was obtained by reading the slope of the line joining the two points. The observed saturation for the thinnest sheets causes the fit of the regression over the full data set to deteriorate. For more accurate results, a much larger simulation set with both thinner and thicker ice sheets would have been required. The found saturation may be interpreted to indicate that the sheets with $L = 160$ m and $h = 0.5$ m resemble an infinite ice sheet while the thicker sheets do not or that the thinnest sheets are thin enough for the breakthrough strengths to settle on the small size asymptotic tail of Bažant's generalized size effect law on which a failure is governed by a strength- or a yield-type criterion. The data points in Figure 3.19 for the largest sheets resemble, in fact, a typical size effect plot, cf. [Bažant, 2002, Figure 3g, p. 16]. Note that the experimental data of Frankenstein [1963, 1966] and Lichtenberger et al. [1974] clearly suggest that a size effect exists also in nature, see Bažant and Kim [1998b] and Bažant [2002].

A final remark: one may assume – based on intuition – that the vertical load a semi-infinite ice sheet with a free edge imposes on an inclined offshore structure should be about half the breakthrough load of an infinite

ice sheet if the loading “widths” are equal. For a narrow structure this, indeed, has been shown to be the case [Gold et al., 1958; Black, 1958; Meyerhof, 1960]. The results for the largest sheets with a thickness of $h = 0.5$ m should thus be applicable in approximating the vertical load imposed on an offshore structure by a semi-infinite ice sheet with a free edge and of the same thickness. The result should hold if the width of the structure is equal to that of the indenter.

4.3.3 On the ice-structure interaction simulations

While computing the analytic breaking load component $F_{V,Croasdale}$, it was assumed that (see [Publication III](#)):

- i) the flexural strength of an ice sheet, σ_{cr} , corresponds to the break-through strength ($\sim$ modulus of rupture $\sim$ flexural strength), σ_F , computed for each sheet in Load case B1;
- ii) the Young’s modulus of an ice sheet, E , corresponds to the effective Young’s modulus, E , computed for each sheet in [Lilja et al., 2019a] in *Load case I*, $|v_x| = 0.1$ m/s; and that
- iii) the characteristic length of an ice sheet, l_{ch} , corresponds to the effective characteristic length, $l_{ch,eff}$, computed for each sheet in [Lilja et al., 2019b].

The foregoing is emphasised because a direct usage of the data given in [Table 2.1](#) (as applied on the “microscopic” scale of a lattice, i.e. with the beams) would have yielded loads with the magnitudes of about $F_{V,FE-DE}/F_{V,Croasdale} \approx 4 \dots 5$ for each h and in both load cases considered. A situation of not knowing the values of the “real” constitutive parameters one – implicitly – applies while computing the loads may thus occur and is likely to lead to erratic interpretations. If, on the other hand, the model has been tested and the (effective) constitutive properties are known, the computed ice-structure interaction loads appear consistent. What data sets one uses with the items above affects, of course, the results. It was here estimated that because the vertical cone speed was 0.1 m/s and the effective in-plane speeds 0.05 m/s (due to the cone angle of 45°), the made selections ought to provide the best approximation.

Based on the computed loads, it thus appears that an ice-structure interaction scheme in which a stationary ice sheet containing a circular hole is interacting with a surfacing cone yields ice-breaking loads (per unit circumference) that are approximately equal to the case in which an ice sheet is moving, a structure is held stationary, and the contacts occur only unilaterally, i.e. one-sidedly. The found result ought to hold, at least, at moderately low penetration speeds. At higher speeds, adverse effects due to inertia may occur.

It is clear that with the proposed in-direct approach, no downdrift (wake) region forms. The applicability of the approach may thus be limited to the first occurrence of a flexural failure. Even so, the approach should provide a reasonable approximation of the breaking load as well as a means to investigate the failure as regards to the ability of a numerical method to produce both radial and circumferential cracking. It is not limited to the hybrid FE-DE method applied in this thesis, but should be generally applicable in examining the ability of a numerical method to describe the typical cracking characteristics observed in nature. Fracture characteristics of lab-scale model ice could probably be studied as well. Note that this statement implicitly regards model ice to be describable by fracture mechanics, which is not certain at all.

5. Conclusions

This short chapter first gives a summary of the whole thesis and its findings and then speculates on some possible future research directions. The made conclusions, as well as some of the future research suggestions, have been adapted from the corresponding publications.

5.1 Summary and concluding remarks

In this thesis, the elastic and inelastic properties of an ice sheet modelled by a new hybrid, three-dimensional finite-discrete element (FE-DE) method were examined. Ice-structure interaction between an ice sheet and a conical offshore structure was studied as well. By this new method, an ice sheet was modelled with rigid discrete elements. The mass centroids of the discrete elements were then connected via an in-plane beam lattice of co-rotational, viscously damped, de-cohesive Timoshenko beam finite elements. A centroidal-Voronoi-tessellation-based iterative scheme, with a random generating point set and Lloyd's algorithm, was then applied in creating the studied FE-DE meshes, i.e. the modelled ice sheets. Due to the internally damped, de-cohesive, lattice-based construction, the mechanical response of a modelled ice sheet turned out to be both strain rate- and size-dependent (dependent on both the absolute and relative sizes), the investigation of which formed a central part of the present study.

A general objective of this thesis was to study the applicability of the new, hybrid FE-DE method in modelling the elasticity and fracture of sea ice sheets. In order to understand the effects of scale and the loading rate, as well as to demonstrate the feasibility of the method in studying ice mechanics applications in general, i.e. the ice-structure interaction, several conceptually simple constitutive tests with square FE-DE sheet samples of varying side lengths L , thicknesses h , and discrete element sizes l were performed. Based on the results of these tests, the following conclusions/observations were then immediate (in the list below, the Roman numerals refer to the respective publications and thus to their contents):

- I
 - 1) In each constitutive in-plane test, a clear size effect was found. It was shown that the effective moduli of an in-plane beam lattice can deviate quite substantially from the moduli given to the beam finite elements and of a linearly elastic, isotropic, continuous medium in plane stress.
 - 2) The moduli varied rather significantly as functions of L , h , and l . The samples that were smaller in L or thicker were stiffer than those that were larger or thinner. In each constitutive test, the FE-DE sheet samples of a smaller relative size $L_{\text{rel}} = L/l$ appeared stiffer.
 - 3) In each constitutive test, the size effect approximately vanished for a relative sheet size parameter $L_{\text{rel}} \gtrsim 25$. In other words, the size of an RVE (representative volume element) appears to be in the order of $25l$ for the load cases presently studied.
 - 4) For such saturated responses, relationships between the microscale material parameters and the effective (macroscale) moduli could be established. The ratios E/E_b , G/G_b , and K/K^{2D} tended to values of about 0.8, 0.55, and 0.7, respectively.
 - 5) The effective moduli tended to values reasonably close to those both of an infinite, in-plane Euler-Bernoulli beam lattice and of an infinite, in-plane micropolar medium. Both of these reference media were periodic and had equilaterally triangular internal topologies.
 - 6) With the relatively low displacement rates applied, no significant rate effects were found.
- II
 - 1) In the out-of-plane elastic tests, the smaller and thicker samples tended to displace more and deform less. As a sample grew in its in-plane size or got thinner, the deflections gradually approached, in the load case *i*) those of an infinite ice sheet, and in the load case *ii*) those of a semi-infinite ice sheet with a free edge.
 - 2) In both load cases, a good agreement was found between the deflections computed with the FE-DE approach and with FEM.
 - 3) A free, square, point-loaded FE-DE sheet sample well approximated both an infinite ice sheet and a semi-infinite ice sheet with a free edge if $L/2 \gtrsim 5l_{\text{ch}}$.
 - 4) The maximum, non-dimensionalised deflections $\bar{w}_{\text{max}}$ scaled in each load case either by a power-law, i.e. $\bar{w}_{\text{max}} \propto h^{m_1(L)}$, or exponentially, i.e. $\bar{w}_{\text{max}} \propto 10^{m_2(h)L}$, depending on whether computed over h for each L or vice versa. The exponents m_1 and m_2 were not constants, but functions of L and h , respectively, and approximately equal in either load case.

- 5) The maximum, absolute deflections $w_{\max}$ scaled in both load cases by a power-law, i.e. $w_{\max} \propto h^{m_1(L)}$, if computed over h and for each L .
- III
- 1) Both the tensile and breakthrough strengths were strong functions of both L and h .
 - 2) The tensile strength was a strong function of the applied loading rate, and virtually mesh-independent for $L \geq 40$ m. Simulations with different tessellations gave strengths with vanishingly small standard deviations.
 - 3) The failure mode as regards to the vertical penetration fracture changed drastically as a function of L .
 - 4) The model was able to demonstrate both radial and circumferential cracking.
 - 5) The proposed in-direct approach, in which a stationary ice sheet containing a circular hole was interacting with a surfacing cone, yielded ice-breaking loads (per unit circumference) that were comparable to those of the conventional approach in which an ice sheet is moving, a structure is held stationary, and the contacts occur only unilaterally, i.e. one-sidedly.

Based on the above list, the results presented in this thesis gave thus a partial guideline for choosing the microscale material parameters of a CVT-tessellated, lattice-based FE-DE model of an ice sheet in order to achieve a desired macroscale response, both elastic and inelastic. Furthermore, the results provided substantial insight into the functional dependencies each studied physical quantity has. In addition, the presented work led to some suggestions for the proper choice of a model size. The out-of-plane elastic tests, for example, indicated minimum sheet sizes that should be employed in emulating the mechanical behaviour of either an infinite ice sheet or a semi-infinite ice sheet with a free edge. In conjunction, the same tests showed that the model is able to mimic a free Kirchhoff-Love plate in bending on a Winkler-type foundation with a good accuracy. Finally, a new in-direct approach to compute ice loads on a conical offshore structure was proposed. The approach yielded breaking loads comparable to those predicted by the model of Croasdale and Cammaert [1994]. The novelty of the approach lies in its simplicity. It provides a device not only to compute cone ice loads but also to investigate the ability of a numerical method to produce both radial and circumferential cracking.

While the contents of this thesis were strictly restricted to applications that are closely related to ice mechanics, i.e. the modelling of intact sea ice sheets, their fracture, and ice-structure interaction, the results presented should apply to other cohesive, lattice-based models on other application areas as well, in the *ab initio*-type constitutive modelling of ceramic matrix composites or concrete for instance.

5.2 Suggestions for future research

In forthcoming studies, it could be an interesting topic – from a purely theoretical point of view – to repeat the in-plane constitutive tests performed in [Publication I](#) by employing a two-scale mesh with each CVT-tessellated cell discretised into roughly 25 subcells. Such a scheme would probably aid in regularising the elastic in-plane response of a modelled ice sheet. The computational cost would naturally increase substantially.

The found size effect is, of course, heavily dependent on the applied boundary conditions. In loose ice floe fields, individual floes experience, while in compression, at least some lateral containment. The effect of lateral constraint could be straightforwardly studied by applying displacement boundary conditions along the presently free edges. Another interesting problem is that of indentation. A semi-infinite ice sheet in contact with an offshore structure is, in essence, an indentation problem. Such a study would require the use of non-reflecting, i.e. absorbing, boundary conditions.

It could be beneficial to repeat at least some of the simulations performed in both [Publication I](#) and [Publication III](#) at equal strain rates. The effective moduli would probably not show any significant changes, but the uniaxial tensile strengths may. It was clearly visible, see [Figure 3.15a](#), that the data curves tended to “curve up” for the small L . This was interpreted to be because of the higher strain rates.

All the analyses in [Publication II](#) were restricted to be geometrically linear. While an ice-structure interaction simulation is in progress, moderate to large displacements and rotations ought to occur. A co-rotational beam formulation, as is adopted in this thesis, is especially capable in describing arbitrarily large displacements and rotations, see [[Wasfy and Noor, 2003](#)]. The out-of-plane bending response with finite rotations and displacements enabled was here, however, not studied. Such an investigation should be taken, especially in order to investigate buckling, which is discussed below.

It was found in [Publication III](#) that an in-direct approach in which a stationary ice sheet containing a circular hole is interacting with a surfacing cone produces loads (per unit circumference) approximately equal to the conventional case of a moving ice sheet interacting with a one-sided, stationary structure. It may thus be possible to devise an experimental setting, for measuring ice loads imposed on a conical model structure by a model ice sheet – for instance, that is more easily controllable. A sheet may remain stationary while a model structure is driven through a hole in it.

One special feature of the proposed in-direct approach deserves a special mention: it should provide a device to compute bifurcation-type buckling loads. It is evident that a sheet is under an approximately axisymmetric, compressive, membrane state of stress (with other than free boundary conditions) through the contacts with a cone. A successful completion of

such an investigation is dependent on the ability of the model to describe finite displacements and rotations that were discussed above.

It was stated in the Introduction, see section 1.2, that a two-dimensional method – such as the one of Paavilainen et al. [2009] – is limited because it does not allow simulations with slender or conical structures that are often encountered in the Arctic. While this is generally true, it should be noted that if, on the contrary, the structure is very wide, modelling of the three-dimensional effects, say clearing, may not be necessary. Maximum ice loads, for example, may be well predictable without considering any three-dimensional effects. If so, a two-dimensional method would be highly preferable because of the significantly better computational efficiency. It could thus be an interesting topic to study how wide a structure should be so that the dimensionality of the problem does not affect the computed loads – or if such a limit can be found at all, i.e. that the three-dimensional effects can never be ignored.

References

- Abaqus. *Dassault Systèmes, Providence, Rhode Island, U.S.A.*, 2016.
- M. F. Ashby, V. S. Deshpande, and N. A. Fleck. Foam Topology Bending Versus Stretching Dominated Architectures. *Acta Materialia*, 49:1035–1040, 2001.
- A. Askar and A. S. Cakmak. A Structural Model of a Micropolar Continuum. *International Journal of Engineering Science*, 6:583–589, 1968.
- C. B. Banks and M. Sokolowski. On Certain Two-Dimensional Applications of the Couple Stress Theory. *International Journal of Solids and Structures*, 4:15–29, 1968.
- Z. P. Bažant. Scaling of Sea Ice Fracture – Part I: Vertical Penetration. *Journal of Applied Mechanics*, 69:11–18, 2002.
- Z. P. Bažant and M. Christensen. Analogy Between Micropolar Continuum and Grid Frameworks Under Initial Stress. *International Journal of Solids and Structures*, 8:327–346, 1972.
- Z. P. Bažant and Z. Guo. Size Effect on Strength of Floating Sea Ice under Vertical Line Load. *Journal of Engineering Mechanics*, 128:254–263, 2002.
- Z. P. Bažant and J. J. H. Kim. Size Effect in Penetration of Sea Ice Plate with Part-Through Cracks. I: Theory. *Journal of Engineering Mechanics*, 124:1310–1315, 1998a.
- Z. P. Bažant and J. J. H. Kim. Size Effect in Penetration of Sea Ice Plate with Part-Through Cracks. II: Results. *Journal of Engineering Mechanics*, 124:1316–1324, 1998b.
- Z. P. Bažant and Y. N. Li. Penetration Fracture of Sea Ice Plate: Simplified Analysis and Size Effect. *Journal of Engineering Mechanics*, 120:1304–1321, 1994.
- Z. P. Bažant and B. H. Oh. Crack Band Theory for Fracture of Concrete. *Materials and Structures*, 16:155–177, 1983.
- Z. P. Bažant and J. Planas. *Fracture and Size Effect in Concrete and Other Quasibrittle Materials*. CRC Press, 1997.
- Z. P. Bažant, M. R. Tabbara, M. T. Kazemi, and G. Pijaudier-Cabot. Random Particle Model for Fracture of Aggregate or Fiber Composites. *Journal of Engineering Mechanics*, 116:1686–1705, 1990.
- S. Beltaos. Collapse of Floating Ice Covers Under Vertical Loads: Test Data vs. Theory. *Cold Regions Science and Technology*, 34:191–207, 2002.

- T. Belytschko, L. Schwer, and M. J. Klein. Large Displacement, Transient Analysis of Space Frames. *International Journal for Numerical Methods in Engineering*, 11:65–84, 1977.
- L. D. Black. Relative Strengths of Plates on Elastic Foundation. *Transactions of the Engineering Institute of Canada*, 2:129–131, 1958.
- J. E. Bolander and S. Saito. Fracture Analyses Using Spring Networks with Random Geometry. *Engineering Fracture Mechanics*, 61:569–591, 1998.
- N. J. Burt and J. W. Dougill. Progressive Failure in a Model Heterogeneous Medium. *Journal of the Engineering Mechanics Division*, 103:365–376, 1977.
- G. T. Camacho and M. Ortiz. Computational Modelling of Impact Damage in Brittle Materials. *International Journal of Solids and Structures*, 33:2899–2938, 1996.
- I. Carol, E. Rizzi, and K. Willam. A Unified Theory of Elastic Degradation and Damage Based on a Loading Surface. *International Journal of Solids and Structures*, 31:2835–2865, 1994.
- I. Carol, P. C. Prat, and C. M. López. Normal/Shear Cracking Model: Application to Discrete Crack Analysis. *Journal of Engineering Mechanics*, 123:765–773, 1997.
- M. A. Crisfield. *Non-Linear Finite Element Analysis of Solids and Structures, Volume 2: Advanced Topics*. John Wiley & Sons, 1997.
- K. R. Croasdale and A. B. Cammaert. An Improved Method for the Calculation of Ice Loads on Sloping Structures in First-Year Ice. *Hydrotechnical Construction*, 28:174–179, 1994.
- P. A. Cundall and O. D. L. Strack. A Discrete Numerical Model for Granular Assemblies. *Geotechnique*, 29:47–65, 1979.
- G. A. D’Addetta, F. Kun, and E. Ramm. On the Application of a Discrete Model to the Fracture Process of Cohesive Granular Materials. *Granular Matter*, 4: 77–90, 2002.
- C. Daley, J. Tuhkuri, and K. Riska. The Role of Discrete Failures in Local Ice Loads. *Cold Regions Science and Technology*, 27:197–211, 1998.
- J. P. Dempsey, L. I. Slepyan, and I. I. Shekhtman. Radial Cracking with Closure. *International Journal of Fracture*, 73:233–261, 1995.
- J. P. Dempsey, R. M. Adamson, and S. V. Mulmule. Scale Effects on the In-Situ Tensile Strength and Fracture of Ice. Part II: First-Year Sea Ice at Resolute, NWT. *International Journal of Fracture*, 95:347–366, 1999.
- K. M. Dempsey and I. V. Vasileva. Ill-Posedness in Floating Plate Dynamics. In *Proceedings of the 18th International Conference on Port and Ocean Engineering Under Arctic Conditions Held in Potsdam, New York, U.S.A., 26-30 June 2005*, pages 1117–1130, 2006a.
- K. M. Dempsey and I. V. Vasileva. Dynamic Uplift Scenarios for Floating Ice. *Journal of Cold Regions Engineering*, 20:39–51, 2006b.
- K. M. Dempsey, N. Grossman, and I. V. Vasileva. Force Inversion in Floating Plate Dynamics. *Inverse Problems*, 22:381–397, 2006.
- S. Diebels and H. Steeb. The Size Effect in Foams and its Theoretical and Numerical Investigation. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 458:2869–2883, 2002.

- O. Dorival, A. V. Metrikine, and A. Simone. A Lattice Model to Simulate Ice-Structure Interaction. In *Proceedings of the 27th International Conference on Offshore Mechanics and Arctic Engineering Held in Estoril, Portugal, 15-20 June 2008*, pages 989–996, 2008.
- J. W. Dougill. On Stable Progressively Fracturing Solids. *Journal of Applied Mathematics and Physics (ZAMP)*, 27:423–437, 1976.
- Q. Du, V. Faber, and M. Gunzburger. Centroidal Voronoi Tessellations: Applications and Algorithms. *SIAM review*, 41:637–676, 1999.
- A. A. Elvin. Number of Grains Required to Homogenize Elastic Properties of Polycrystalline Ice. *Mechanics of Materials*, 22:51–64, 1996.
- V. A. Eremeyev, L. P. Lebedev, and H. Altenbach. *Foundations of Micropolar Mechanics*. Springer, 2013.
- Y. T. Feng, K. Han, and D. R. J. Owen. An Energy-Based Polyhedron-to-Polyhedron Contact Model. In *Proceedings of the Third M.I.T. Conference on Computational Fluid and Solid Mechanics Held in Cambridge, U.S.A., 14-17 June 2005*, pages 210–214, 2005.
- Finnish Customs. Ulkomaankaupan kuljetukset 2019 (in Finnish). Technical report, Tulli, 2020.
- E. G. Frankenstein. Load Test Data for Lake Ice Sheet, Rep. No. 89. Technical report, U.S. Army Corps of Engineers, Cold Regions Research & Engineering Laboratory, Hanover, New Hampshire, 1963.
- E. G. Frankenstein. Strength of Ice Sheets. In *Proceedings of the Conference on Ice Pressures Against Structures Held at Laval University, Quebec, 10-11 November 1966*, pages 79–87. National Research Council of Canada, 1966.
- J. C. Gálvez, J. Červenka, D. A. Cendón, and V. Saouma. A Discrete Crack Approach to Normal/Shear Cracking of Concrete. *Cement and Concrete Research*, 32:1567–1585, 2002.
- L. J. Gibson and M. F. Ashby. *Cellular Solids: Structure and Properties*. Cambridge University Press, 1999.
- L. W. Gold, L. D. Black, F. N. Trofimenkoff, and D. Matz. Deflections of Plates on Elastic Foundation. *Transactions of the Engineering Institute of Canada*, 2: 123–130, 1958.
- D. V. Griffiths and G. G. W. Mustoe. Modelling of Elastic Continua Using a Grillage of Structural Elements Based on Discrete Element Concepts. *International Journal for Numerical Methods in Engineering*, 50:1759–1775, 2001.
- A. Gurtner. *Experimental and Numerical Investigations of Ice-Structure Interaction*. PhD thesis, Norwegian University of Science and Technology, Trondheim, 2009.
- G. Gurtner and M. Durand. Stiffest Elastic Networks. *Proceedings of the Royal Society of London Series A: Mathematical, Physical and Engineering Sciences*, 470, 2014.
- J. Haapala, I. Ronkainen, N. Schmelzer, and M. Sztobryn. Recent Change – Sea Ice. In *Second Assessment of Climate Change for the Baltic Sea Basin*, pages 145–153. Springer International Publishing, 2015.
- H. Happel. Über das Gleichgewicht von Elastischer Platten Unter Einer Einzelast. *Mathematische Zeitschrift*, 6:203–218, 1920.

- A. Herman. Discrete-Element Bonded-Particle Sea Ice Model DESign, Version 1.3 a - Model Description and Implementation. *Geoscientific Model Development*, 9: 1219–1241, 2016.
- H. J. Herrmann, A. Hansen, and S. Roux. Fracture of Disordered, Elastic Lattices in Two Dimensions. *Physical Review B*, 39:637–648, 1989.
- H. Hertz. Über das Gleichgewicht Schwimmender Elastischer Platten. *Wiedemann's Annalen der Physik und Chemie*, 22:449–455, 1884.
- A. Hillerborg, M. Modéer, and P. E. Petersson. Analysis of Crack Formation and Crack Growth in Concrete by Means of Fracture Mechanics and Finite Elements. *Cement and Concrete Research*, 6:773–781, 1976.
- G. Hocking. The Discrete Element Method for Analysis of Fragmentation of Discontinua. *Engineering Computations*, 9:145–155, 1992.
- M. A. Hopkins. Four Stages of Pressure Ridging. *Journal of Geophysical Research: Oceans*, 103:21883–21891, 1998.
- K. V. Høyland. Morphology and Small-Scale Strength of Ridges in the North-Western Barents Sea. *Cold Regions Science and Technology*, 48:169–187, 2007.
- A. Hrennikoff. Solution of Problems of Elasticity by the Framework Method. *Journal of Applied Mechanics*, 8:169–175, 1941.
- A. Ibrahimbegovic and A. Delaplace. Microscale and Mesoscale Discrete Models for Dynamic Fracture of Structures Built of Brittle Material. *Computers and Structures*, 81:1255–1265, 2003.
- ISO 19906. Petroleum and Natural Gas Industries – Arctic Offshore Structures. Technical report, International Organization for Standardization, 2019.
- M. Jirásek and Z. P. Bažant. Particle Model for Quasibrittle Fracture and Application to Sea Ice. *Journal of Engineering Mechanics*, 121:1016–1025, 1995a.
- M. Jirásek and Z. P. Bažant. Macroscopic Fracture Characteristics of Random Particle Systems. *International Journal of Fracture*, 69:201–228, 1995b.
- P. Kankaanpää. Morphology of Sea Ice Pressure Ridges in the Baltic Sea. *Geophysica*, 24:15–44, 1988.
- B. L. Karihaloo, P. F. Shao, and Q. Z. Xiao. Lattice Modelling of the Failure of Particle Composites. *Engineering Fracture Mechanics*, 70:2385–2406, 2003.
- A. D. Kerr. The Bearing Capacity of Floating Ice Plates Subjected to Static or Quasi-Static Loads. *Journal of Glaciology*, 17:229–268, 1976.
- A. D. Kerr. Bearing Capacity of Floating Ice Covers Subjected to Static, Moving, and Oscillatory Loads. *Applied Mechanics Reviews*, 49:463–476, 1996.
- A. D. Kerr and S. S. Kwak. The Semi-Infinite Plate on a Winkler Base, Free Along the Edge, and Subjected to a Vertical Force. *Archive of Applied Mechanics*, 63: 210–218, 1993.
- F. Klein and K. Wieghardt. Über Spannungsflächen und Reziproke Diagramme, mit Besonderer Berücksichtigung der MAXWELLSchen Arbeiten. *Archiv der Mathematik und Physik*, 8:95–119, 1904.
- I. Konuk, A. Gürtner, and S. Yu. A Cohesive Element Framework for Dynamic Ice-Structure Interaction Problems – Part I: Review and Formulation. In *Proceedings of the 28th International Conference on Ocean, Offshore and Arctic Engineering Held in Honolulu, Hawaii, U.S.A., May 31-June 5 2009*, pages 33–41, 2009.

- S. Kulyakhtin. Distribution of Ice Block Sizes in Sails of Pressure Ice Ridges. In *Proceedings of the 22nd IAHR International Symposium on Ice Held in Singapore, 11-15 August 2014*, 2014.
- R. S. Kumar and D. L. McDowell. Generalized Continuum Modeling of 2-D Periodic Cellular Solids. *International Journal of Solids and Structures*, 41: 7399–7422, 2004.
- S. E. Leon, D. W. Spring, and G. H. Paulino. Reduction in Mesh Bias for Dynamic Fracture Using Adaptive Splitting of Polygonal Finite Elements. *International Journal for Numerical Methods in Engineering*, 100:555–576, 2014.
- F. Li, G. J. Yue, K. N. Shkhinek, and T. Kärnä. A Qualitative Analysis of Breaking Length of Sheet Ice Against Conical Structures. In *Proceedings of the 17th International Conference on Port and Ocean Engineering under Arctic Conditions Held in Trondheim, Norway, 16-19 June 2003*, 2003.
- H. Li and J. P. Dempsey. Unbonded Contact of a Square Plate on an Elastic Half-Space or a Winkler Foundation. *Journal of Applied Mechanics*, 55:430–436, 1988.
- Y. N. Li and Z. P. Bažant. Penetration Fracture of Ice Plate: 2D Analysis and Size Effect. *Journal of Engineering Mechanics*, 120:1481–1498, 1994.
- G. J. Lichtenberger, J. W. Jones, R. D. Stegall, and D. W. Zadow. Static Ice Loading Tests: Resolute Bay – Winter 1973/74, APOA Project No. 64, Report No. 745B-74-14. Technical report, Sunoco Science and Technology, 1974.
- S. Liebenstein, S. Sandfeld, and M. Zaiser. Size and Disorder Effects in Elasticity of Cellular Structures: From Discrete Models to Continuum Representations. *International Journal of Solids and Structures*, 146:97–116, 2018.
- E. Lightfoot. A Grid Framework Analogy for Laterally Loaded Plates. *International Journal of Mechanical Sciences*, 6:201–208, 1964.
- V. P. Lilja, A. Polojärvi, J. Tuhkuri, and J. Paavilainen. A Three-Dimensional FEM-DEM Model of an Ice Sheet. In *Proceedings of the 24th International Conference on Port and Ocean Engineering under Arctic Conditions Held in Busan, South Korea, 11-16 June 2017*, 2017a.
- V. P. Lilja, A. Polojärvi, J. Tuhkuri, and J. Paavilainen. Effective Tensile Strength of an Ice Sheet Using a Three-Dimensional FEM-DEM Approach. In *Proceedings of the 24th International Conference on Port and Ocean Engineering under Arctic Conditions Held in Busan, South Korea, 11-16 June 2017*, 2017b.
- V. P. Lilja, A. Polojärvi, J. Tuhkuri, and J. Paavilainen. Effective Material Properties of a Finite Element-Discrete Element Model of an Ice Sheet. *Computers and Structures*, 224, 2019a.
- V. P. Lilja, A. Polojärvi, J. Tuhkuri, and J. Paavilainen. A Free, Square, Point-Loaded Ice Sheet: A Finite Element-Discrete Element Approach. *Marine Structures*, 68, 2019b.
- J. X. Liu, S. C. Deng, J. Zhang, and N. G. Liang. Lattice Type of Fracture Model for Concrete. *Theoretical and Applied Fracture Mechanics*, 48:269–284, 2007.
- E. N. Lorenz. Predictability; Does the Flap of a Butterfly’s Wings in Brazil Set Off a Tornado in Texas? American Association for the Advancement of Science, 139th meeting, 1972.
- W. Lu, R. Lubbad, and S. Løset. Simulating Ice-Sloping Structure Interactions with the Cohesive Element Method. *Journal of Offshore Mechanics and Arctic Engineering*, 136, 2014.

- W. Lu, R. Lubbad, and S. Løset. Out-of-Plane Failure of an Ice Floe: Radial-Crack-Initiation-Controlled Fracture. *Cold Regions Science and Technology*, 119:183–203, 2015.
- W. Lu, R. Lubbad, S. Løset, and M. Kashafutdinov. Fracture of an Ice Floe: Local Out-of-Plane Flexural Failures Versus Global In-Plane Splitting Failure. *Cold Regions Science and Technology*, 123:1–13, 2016.
- A. Luomaranta, J. Haapala, H. Gregow, K. Ruosteenoja, K. Jylhä, and A. Laaksonen. Itämeren Jääpeitteen Muutokset Vuoteen 2050 Mennessä (in Finnish). Technical report, Finnish Meteorological Institute, 2010.
- W. R. McGilvary, D. S. Sodhi, and J. H. Lever. Dynamic analysis of a floating ice sheet undergoing vertical indentation. In *Proceedings of the Ninth International Conference on Offshore Mechanics and Arctic Engineering Held in Houston, Texas, U.S.A., 18-23 February 1990*, pages 195–203, 1990.
- D. McHenry. A Lattice Analogy for the Solution of Stress Problems. *Journal of the Institution of Civil Engineers*, 2:59–82, 1943.
- G. G. Meyerhof. Bearing Capacity of Floating Ice Sheets. *Journal of the Engineering Mechanics Division*, 86:113–146, 1960.
- N. F. Mott. Fragmentation of Shell Cases. *Proceedings of the Royal Society of London, Series A: Mathematical, Physical and Engineering Sciences*, 189:300–308, 1947.
- A. A. Munjiza. *The Combined Finite-Discrete Element Method*. John Wiley & Sons, 2004.
- A. A. Munjiza, J. P. Latham, and N. W. M. John. 3D Dynamics of Discrete Element Systems Comprising Irregular Discrete Elements – Integration Solution for Finite Rotations in 3D. *International Journal for Numerical Methods in Engineering*, 56:35–55, 2003.
- A. A. Munjiza, E. E. Knight, and E. Rougier. *Large Strain Finite Element Method: A Practical Course*. John Wiley & Sons, Ltd., England, 2015.
- G. Murphy. Stresses and Deflections in Loaded Rectangular Plates on Elastic Foundations. *Iowa Engineering Experiment Station Bulletin*, 36, 1937.
- K. Myrberg, M. Leppäranta, and H. Kuosa. *Itämeren fysiikka, tila ja tulevaisuus (in Finnish)*. Helsinki University Press, 2006.
- D. E. Nevel. Ice forces on cones from floes. In *Proceedings of the 11th IAHR Symposium on Ice Held in Banff, Alberta, Canada, 15-19 June 1992*, pages 1391–1404, 1992.
- A. K. Noor. Continuum Modeling for Repetitive Lattice Structures. *Applied Mechanics Reviews*, 41:285–296, 1988.
- P. R. Onck, E. W. Andrews, and L. J. Gibson. Size Effects in Ductile Cellular Solids. Part I: Modeling. *International Journal of Mechanical Sciences*, 43:681–699, 2001.
- M. Ostoja-Starzewski. Lattice Models in Micromechanics. *Applied Mechanics Reviews*, 55:35–60, 2002.
- J. Paavilainen. Jäälautan Murtuminen Kartiorakennetta Vasten (in Finnish). Technical report, Technical Research Center of Finland, 2010.
- J. Paavilainen, J. Tuhkuri, and A. Polojärvi. 2D Combined Finite-Discrete Element Method to Model Multi-Fracture of Beam Structures. *Engineering Computations*, 26:578–598, 2009.

- M. E. Plesha and E. C. Aifantis. On the Modeling of Rocks with Microstructure. In *Proceedings of the 24th U.S. Symposium on Rock Mechanics Held in College Station, Texas, U.S.A., 20-23 June 1983*, pages 27–35, 1983.
- A. Polojärvi, J. Tuhkuri, and O. Korkalo. Comparison and Analysis of Experimental and Virtual Laboratory Scale Punch Through Tests. *Cold Regions Science and Technology*, 81:11–25, 2012.
- G. I. Pshenichnov. *A Theory of Latticed Plates and Shells*. World Scientific Publishing, 1993.
- A. V. Pushkin, L. I. Slepyan, and A. N. Zlatin. Dynamical Problems of Ice Cover Fracture. *International Journal of Offshore and Polar Engineering*, 1:212–219, 1991.
- T. D. Ralston. Ice Force Design Considerations for Conical Offshore Structures. In *Proceedings of the Fourth International Conference on Port and Ocean Engineering under Arctic Conditions Held in St. John's, Newfoundland, Canada, 26-30 September 1977*, pages 741–752, 1977.
- J. Ranta, A. Polojärvi, and J. Tuhkuri. Scatter and Error estimates in Ice Loads – Results from Virtual Experiments. *Cold Regions Science and Technology*, 148: 1–12, 2018a.
- J. Ranta, A. Polojärvi, and J. Tuhkuri. Ice Loads on Inclined Marine Structures – Virtual Experiments on Ice Failure Process Evolution. *Marine Structures*, 57: 72–86, 2018b.
- R. Rätty. *Studies on Nonlinear Dynamics of Driven Vibrating Systems with Dissipation*. PhD thesis, Helsinki University of Technology, Espoo, 1992.
- J. D. Renton. On the Gridwork Analogy for Plates. *Journal of the Mechanics and Physics of Solids*, 13:413–420, 1965.
- J. D. Renton. *Elastic Beams and Frames*. Woodhead Publishing, 2002.
- J. J. Rimoli, J. J. Rojas, and F. N. Khemani. On the Mesh Dependency of Cohesive Zone Models for Crack Propagation Analysis. In *Proceedings of the 53rd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference Held in Honolulu, Hawaii, U.S.A., 23-26 April 2012*, 2012.
- E. M. Salonen. A Gridwork Method for Plates in Bending. *Acta Polytechnica Scandinavica, Civil Engineering and Building Construction Series*, 59:1–31, 1969.
- B. Sand. *Nonlinear Finite Element Simulations of Ice Forces on Offshore Structures*. PhD thesis, Luleå University of Technology, Luleå, 2008.
- M. Sayed and G. Timco. A Lattice Model of Ice Failure. Technical report, Canadian Hydraulics Centre, National Research Council of Canada, Ottawa, Ont. K1A 0R6, 1998.
- H. Schardin. Velocity Effects in Fracture. In B. L. Averbach, editor, *Fracture*, pages 297–330. MIT Press, 1959.
- E. Schlangen and J. G. M. van Mier. Simple Lattice Model for Numerical Simulation of Fracture of Concrete Materials and Structures. *Materials and Structures*, 25:534–542, 1992.
- H. L. Schreyer, D. L. Sulsky, L. B. Munday, M. D. Coon, and R. Kwok. Elastic-Decohesive Constitutive Model for Sea Ice. *Journal of Geophysical Research*, 111, 2006.

- A. Seagraves and R. Radovitzky. Advances in Cohesive Zone Modeling of Dynamic Fracture. In A. Shukla, G. Ravichandran, and Y. D. S. Rajapakse, editors, *Dynamic Failure of Materials and Structures*, pages 349–405. Springer Science, 2010.
- A. P. S. Selvadurai and K. Sepehr. Two-Dimensional Discrete Element Simulations of Ice-Structure Interaction. *International Journal of Solids and Structures*, 36: 4919–4940, 1999.
- L. I. Slepyan. Modeling of Fracture of Sheet Ice. *Rossiyskaya Akademiya Nauk Izvestiya, Mekhanika Tverdogo Tela (Transactions of the Academy of Sciences of the U.S.S.R., Mechanics of Solids)*, 25:151–157, 1990.
- D. S. Sodhi. Interaction Forces During Vertical Penetration of Floating Ice Sheets With Cylindrical Indentors. In *Proceedings of the Eighth International Conference of Offshore Mechanics and Arctic Engineering Held in The Hague, The Netherlands, 19-23 March 1989*, pages 377–382, 1989.
- D. S. Sodhi. Breakthrough Loads of Floating Ice Sheets. *Journal of Cold Regions Engineering*, 9:4–22, 1995.
- D. S. Sodhi. Vertical Penetration of Floating Ice Sheets. *International Journal of Solids and Structures*, 35:4275–4294, 1998.
- D. W. Spring, S. E. Leon, and G. H. Paulino. Unstructured Polygonal Meshes with Adaptive Refinement for the Numerical Simulation of Dynamic Cohesive Fracture. *International Journal of Fracture*, 189:33–57, 2014.
- D. D. Symons and N. A. Fleck. The Imperfection Sensitivity of Isotropic Two-Dimensional Elastic Lattices. *Journal of Applied Mechanics*, 75, 2008.
- C. Talischi, G. H. Paulino, A. Pereira, and I. F. M. Menezes. Polymesher: A General-Purpose Mesh Generator for Polygonal Elements Written in Matlab. *Structural and Multidisciplinary Optimization*, 45:309–328, 2012.
- Q. Tang, X. Zhang, X. Yang, and J. Francis. Cold Winter Extremes in Northern Continents Linked to Arctic Sea Ice Loss. *Environmental Research Letters*, 8: 1–6, 2013.
- T. R. Tauchert. A Lattice Theory for Representation of Thermoelastic Composite Materials. In A. C. Eringen, editor, *Recent Advances in Engineering Science*, pages 325–345. Gordon & Breach, 1970.
- C. Teköglu and P. R. Onck. Size Effects in the Mechanical Behavior of Cellular Materials. *Journal of Materials Science*, 40:5911–5917, 2005.
- C. Teköglu, L. J. Gibson, T. Pardoen, and P. R. Onck. Size Effects in Foams: Experiments and Modeling. *Progress in Materials Science*, 56:109–138, 2011.
- G. W. Timco and W. F. Weeks. A Review of the Engineering Properties of Sea Ice. *Cold Regions Science and Technology*, 60:107–129, 2010.
- J. Tuhkuri and A. Polojärvi. A Review of Discrete Element Simulation of Ice-Structure Interaction. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 376, 2018.
- M. van den Berg. A 3-D Random Lattice Model of Sea Ice. In *Proceedings of the Arctic Technology Conference Held in St. John's, Newfoundland, Canada, 24-26 October 2016*, 2016.
- I. V. Vasileva and K. M. Dempsey. Numerical Force Inversion in Floating Plate Dynamics. In *Proceedings of the 18th International Conference on Port and Ocean Engineering Under Arctic Conditions Held in Potsdam, New York, U.S.A., 26-30 June 2005*, pages 1103–1116, 2006.

- V. Z. Vlasov and N. N. Leont'ev. *Beams, Plates and Shells on Elastic Foundations (Balki, Plity i Obolochki Na Uprugom Osnovanii)*. Israel Program for Scientific Translations, 1966.
- M. Vural and M. Alkhader. Mechanical Response of Cellular Solids: Role of Cellular Topology and Microstructural Irregularity. *International Journal of Engineering Science*, 46:1035–1051, 2008.
- T. M. Wasfy and A. K. Noor. Computational Strategies for Flexible Multibody Systems. *Applied Mechanics Reviews*, 56:553–613, 2003.
- H. M. Westergaard. Stresses in Concrete Pavements Computed by Theoretical Analysis. *Public Roads*, 7:25–35, 1926.
- WMO. Sea-Ice Nomenclature, Rept. No. 259, Supplement No. 5. Technical report, World Meteorological Organization, Geneva, Switzerland, 1989.
- C. Woźniak. Bending and Stability Problems of Plates with Lattice Structure. *Archiwum Mechaniki Stosowanej*, 18:781–796, 1966a.
- C. Woźniak. *Siatkowe Dźwigary Powierzchniowe – Podstawy Teorii i Przykłady Obliczeń (in Polish) (Reticulated Surface Structures – Fundamentals of Theory and Examples of Calculations)*. Państwowe Wydawnictwo Naukowe, 1966b.
- M. Wyman. Deflections of an Infinite Plate. *Canadian Journal of Research*, 28, Sec. A.:293–302, 1950.
- A. Zubelewicz and Z. P. Bažant. Interface Element Modeling of Fracture in Aggregate Composites. *Journal of Engineering Mechanics*, 113:1619–1630, 1987.

Errata

The attached articles are the author's own, personal transcripts of the published articles. They have been typeset with $\text{\LaTeX}$ and contain graphics, tables, and equations of a much better visual appearance to those in the published articles. The list below depicts the other differences between the attached and the published articles. Only minor, typesetting-related changes have been done.

Publication I

Minor typographical errors and ambiguities have been fixed.

Publication II

Minor typographical errors and ambiguities have been fixed.

ISBN 978-952-64-0139-3 (printed)

ISBN 978-952-64-0140-9 (pdf)

ISSN 1799-4934 (printed)

ISSN 1799-4942 (pdf)

Aalto University
School of Engineering
Department of Mechanical Engineering
www.aalto.fi

**BUSINESS +
ECONOMY**

**ART +
DESIGN +
ARCHITECTURE**

**SCIENCE +
TECHNOLOGY**

CROSSOVER

**DOCTORAL
DISSERTATIONS**