Department of Mathematics and Systems Analysis

A Factorization Method for the Inverse Scattering of the Wave Equation

Pekka Tietäväinen

Aalto University

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Aalto University
School of Science
Department of Mathematics and Systems Analysis
Inverse Group

Supervisor
Professor Olavi Nevanlinna

Instructor
Professor Matti Lassas

Preliminary examiners
Professor Velery Serov, University of Oulu, Finland
Professor Bastian von Harrach, Technische Universität München, Germany

Opponent
Professor Fioralba Cakoni, University of Delaware, USA

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pekka@tietavainen.fi

Aalto University

Abstract

Aalto University, P.O. Box 11000, FI-00076 Aalto www.aalto.fi

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Pekka Tietäväinen

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Abstract

In this work the reconstruction problem of the scattering of the wave equation is considered. Our approach is to start from Colton-Kirch factorization method, derive a similar method for the time harmonic Robin problem, and use this to create a reconstruction for the wave equation.

The new results presented in this work are the factorization reconstruction method for the time harmonic Robin scattering; the error limits that predict the trustworthiness of this reconstruction for various frequencies; and the factorization for the scattering of the wave equation, which includes two tests to determine whether a point is located in the scatterer.

Keywords Inverse problems, factorization, Robin boundary conditions, wave equation, Helmholtz equation, scattering

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Tiivistelmä

Tässä työssä tutkitaan sirottajan rekonstruktio-ongelmaa aaltoyhtälön siroamisessa. Lähestymistapamme on tehdä ensin Colton-Kirsch:in faktorisaatiomenetelmän pohjalta rekonstruktio Robin reuna-arvoiselle yhden taajuuden ongelmalle ja sitten käyttää tätä aaltoyhtälön rekonstruktion tekemiseen.

Työssä on saavutettu seuraavanlaisia uusia tuloksia: Uusi faktorisaatiomenetelmä yhden taajuuden Robin-ongelmalle. Taajuusanalyysi edelliselle, jossa saadaan arvio menetelmän luotettavuudesta eri taajuuksilla. Faktorisaatio aaltoyhtälön siroamisongelmalle ja testiä sille, että piste kuuluu kappaleeseen.

Avainsanat Inversio-ongelmat, faktorisaatio, Robin-reuna-arvot, aaltoyhtällö, Helmholtz-yhtälö, siroaminen

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Pekka Tietäväinen

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Pekka Tietäväinen
Helsinki 25.8.2011

Contents

Chapter 1

Introduction

The subject matter of this study is inverse scattering, in which we know a certain set of waves that scatter from an obstacle and want to reconstruct the obstacle from this information. The scattering process can be presented in the time domain, in which we have waves that develop in time, or in the frequency domain, in which the waves have a constant frequency.

Our main interest in this work is to devise a reconstruction method that uses time domain data for the reconstruction of the obstacle which we will henceforth refer to as the "scatterer". Our approach is to use the single frequency factorization method as a basis and then to form a time domain method with the help of the partial "frequencies to time" Fourier transformation $\mathcal{F}_{k\to t}$. In order to facilitate the frequency analysis of the factorization method, we consider the scatterer to have the Robin boundary conditions

$$(\partial_\nu - \alpha(x)\partial_t + \beta(x))u = 0 \tag{1.1}$$

with $\alpha(x) \geq \alpha_0 > 0$ and also the Dirichlet energy decay property, namely that under the Dirichlet boundary conditions the waves decay exponentially in the vicinity of the scatterer.

Assuming these conditions we derive a factorization method which works for all wave numbers $k \in \mathbb{R} \setminus \{0\}$. We also explore how the method works for varying values of the wave number, which is useful in deriving the time domain reconstruction.

In Section 1.1 we provide a short introduction to inverse problems in general and linear sampling and factorization methods in particular. In Section 1.2 we set forth a more detailed introduction to the structure of this work.

1.1 Introduction to Inverse Problems

The bedrock of all sciences is the idea that there are causes and effects, relationship of which one can uncover through use of the intellect. A forward problem, then, in all generality, is one in which causes and the laws that connect the causes to their effects are known and therefore one infers the effects from their causes. In an inverse problem the chain of reasoning works counter to the direction of causation and we are faced with a situation in which some of the causes and effects are known, and we want to know the rest of the causes.

As an example of an inverse problem let us take the scattering of a time harmonic waves. In this example we have a vibration $u(t,x)$ in $\mathbb{R}^4$ that is governed by the wave equation

$$\Box u = (\partial_t^2 - \Delta)u = 0, \tag{1.2}$$

where $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$. In a scattering problem we have an impenetrable obstacle D, which has certain boundary conditions for the wave on ∂D. Let us assume that we have the Robin boundary conditions, that is

$$(\partial_\nu - \alpha(x)\partial_t + \beta(x))u = 0, \tag{1.3}$$

where $\alpha, \beta \in C^2(\partial D)$, $\alpha \geq 0$ and ν is the exterior surface normal of ∂D.

In order to specify the direction of time, we can consider the development of the wave only after a certain time $t = 0$. The necessary initial information to calculate the wave in this case is the values of u and $\partial_t u = u_t$ at $t = 0$. Altogether we require from u the following:

$$\begin{cases} \Box u(x,t) & = 0 \qquad\qquad \text{in } \mathbb{R}_+ \times D^+ \\ (\partial_\nu - \alpha(x)\partial_t + \beta(x))\, u & = 0 \qquad\qquad \text{on } (\mathbb{R}_+ \cup \{0\}) \times \partial D \\ u|_{t=0} = f_1, & u_t|_{t=0} = f_2. \end{cases} \tag{1.4}$$

We refer to (1.3) as the **time domain Robin problem**. This problem is solved for example in [45, Theorem 7.6.2]. In the case of a time harmonic wave $u(t,x) = v(x)e^{-ikt}$, the conditions (1.2) and (1.3) become

$$\begin{cases} -(\Delta + k^2)u & = 0, \\ (\partial_\nu + ik\alpha + \beta)u & = 0. \end{cases}$$

From here on we will denote $ik\alpha + \beta = \lambda$.

The scattering process for time harmonic waves is described as follows. Let us suppose that there is a free space incident wave u_i that satisfies

$$-(\Delta + k^2)u_i = 0$$

in $\mathbb{R}^3$. Let us further suppose that the total wave u_{tot} on $D^+ = \mathbb{R}^3 \setminus \overline{D}$

consists of the incident wave u_i and a scattered wave u_{sc}. Since the total wave satisfies

$$(\partial_\nu + \lambda)u_{tot} = 0$$

on ∂D, the scattered wave u_{sc} satisfies

$$\begin{cases} -(\Delta + k^2)u_{sc} &= 0, \qquad \text{in } D^+ \\ (\partial_\nu + \lambda)u_{sc} &= -(\partial_\nu + \lambda)u_i \quad \text{on } \partial D. \end{cases}$$

The above problem has multiple possible solutions for u_{sc}. We are able to arrive at a problem with unique solution if we demand that u_{sc} satisfies Sommerfeld's radiation condition

$$\lim_{r\to\infty} r(\partial_r - ik)u_{sc} = 0 \qquad \text{uniformly w.r.t. } \hat{x} \in \mathbb{S}^2,$$

where $r = |x|$. In the solutions that satisfy Sommerfeld's radiation condition, the energy flows outwards.

In the forward problem of time harmonic scattering we know the obstacle D, together with the parameter λ, and the incident wave u_i and want to determine the scattered wave u_{sc} that satisfies

$$\begin{cases} -(\Delta + k^2)u_{sc} &= 0 \qquad \text{in } \quad D^+ \\ (\partial_\nu + \lambda)u_{sc} &= -(\partial_\nu + \lambda)u_i \quad \text{on } \partial D \\ \lim_{r\to\infty} r\,(\partial_r - ik)\,u_{sc} &= 0 \qquad \text{uniformly} \end{cases} \tag{1.5}$$

One possible inverse problem would be to find the shape of the scatterer when we know a family of incident waves $\{u_i(\cdot, d)\} := \{e^{ikx\cdot d}\}_{d\in\mathbb{S}^2}$ and the far field asymptotics of the corresponding scattered waves $u_s(\cdot, d)$. We have solved this problem in Section 4.1.

Until the middle of the 20th century, inverse problems were not considered to be applicable to rigorous mathematical analysis, since they are ill-posed in the sense of J. Hadamard. Hadamard's criterion for a problem to be well-posed, as can be found e.g. in [21], are

(i) The problem has a solution.

(ii) The solution is unique.

(iii) The solution depends continuously on the data.

In general these conditions are not satisfied by inverse problems. This state of affairs can be seen to follow from the very nature of this class of problems, since widely different causes can lead to similar effects. The first condition can also be problematic because there is always some amount of error in our measurements and this error can change the data so that there is no solution to the problem given such data. For example, in the above case of time harmonic scattering we would measure values of the scattered wave $u_s(x,d)$ and would gain values $u_s(x,d)+\epsilon(x)$, which would not necessarily be compatible with any scatterer with Robin boundary conditions whatsoever.

On this subject a great contribution was made by A.N. Tikhonov in his paper [54]. His idea was to regularize the problem by setting constraints on what kind of solutions are to be considered. If the original ill-posed problem were to solve x in the equation $Ax = y$, we would find the minimizer of $\|Ax - y\|^2 + \alpha\|x\|^2$, where the parameter α is to be chosen using good judgement.

This regularized problem is in fact well-posed and experience has shown that the solutions to it provide relevant information about the original problem. An example of this, among numerous others, is the linear sampling method, which we discuss next.

Linear sampling was invented by D. Colton and A. Kirsch and published in the article [12]. In linear sampling one uses the far field expansion

$$u_s(x,d) = \frac{e^{ik|x|}}{|x|}\left(u_\infty(\hat{x},d) + \mathcal{O}\left(\frac{1}{|x|}\right)\right).$$

of the scattered waves of problem (1.5) that have plane wave $e^{ikx\cdot d}$ as incident waves. One forms the far field operator $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ defined by

$$Fh(\hat{x}) := \int_{\mathbb{S}^2} u_\infty(\hat{x},d)h(d)\mathrm{d}\mathcal{H}^2(d).$$

This operator has the property of mapping the density h of an incident Herglotz wave

$$u_h = \int_{\mathbb{S}^2} e^{ikx\cdot d}h(d)\mathrm{d}\mathcal{H}^2(d)$$

to the far field of the corresponding scattered wave.

The far field of a point source

$$\Phi(x,z) = \frac{e^{ik|x-z|}}{4\pi|x-z|}$$

is

$$\Phi_\infty(\hat{x},z) = e^{-ik\hat{x}\cdot z}. \tag{1.6}$$

The function Φ_∞ is in the range of the set of possible far fields of waves scattered from D if and only if $z \in D$. Hence if there is a $h \in L^2(\mathbb{S}^2)$ such that $Fh = \Phi_\infty(\hat{x}, z)$, we see that $z \in D$. The linear sampling method uses the Tikhonov regularization method to calculate a regularized approximation to $F^{-1}\Phi_\infty(\hat{x}, z)$. When $z \in D$ approaches the boundary, the norm of $F^{-1}\Phi_\infty(\hat{x}, z)$ approaches infinity and hence we are able to determine the boundary ∂D.

The linear sampling method has been quite successful and has applications in many areas, see e.g. [8], [15], [19], [44] and [22]. Nevertheless it is still not completely understood why the method works as well as it does. This aspect has been studied by M. Hanke in [24], T. Arens in [3] and also by A. Kirsch starting in the 1990's. Kirsch's research on the foundations for the linear sampling method led him to invent a variant of the method, nowdays known as the factorization method, which was first published in the article [34].

In the factorization method one uses the operator G which maps the boundary values of the scattered wave to its far field and a factorization of the type

$$F = GAG^* \tag{1.7}$$

where the middle operator A depends on the problem.

One uses the properties of the operators F, G and A to infer that

$$\operatorname{Ran}\left(F^{\frac{1}{2}}\right) = \operatorname{Ran}(G)$$

and that there exists constants c and C such that for $\varphi \in \operatorname{Ran}(G)$ one has

$$c\|G^{-1}\varphi\| \leq \|F^{-\frac{1}{2}}\varphi\| \leq C\|G^{-1}\varphi\|,$$

with appropriate norms.

Hence one infers that $z \in D$ if and only if $\Phi_\infty(\hat{x}, z) \in \operatorname{Ran}\left(F^{\frac{1}{2}}\right)$. This condition is amenable to a numerical calculation.

The factorization method has also found many applications and has been elaborated on different areas. In addition to the treatment of the Dirichlet and Neumann scattering problems of the article [34] A. Kirsch has also provided a factorization solution to the transmission problem, [33] and [31]; in time harmonic electromagnetic scattering, [30]; and in elastics, [32]. In addition N. Grinberg and A.Kirsch have applied the factorization, in a modified form, to the time harmonic Robin scattering, [20].

The time domain counterpart to the frequency domain factorization method has been invented by Q. Chen, H. Haddar, A. Lechleiter and P. Monk in the unpublished work [9].

The factorization was found to be applicable to electric impedance tomography (EIT) by M. Brühl in [4] and [5]. This area has been quite active ever since. Further elaboration was provided by M. Brühl and M. Hanke in [6] and [7]. The Complete electrode model was used by N. Hyvönen in [25] and the complete electrode model together with error limits was developed by A. Lechleiter, N. Hyvönen and H. Hakula in [38].

In their work [43] A. Nachmann, A Teirilä and L. Päivärinta made a variant of the factorization method for the anisotropic EIT problem and the obstacle scattering. The implications of this work are still unfolding.

The previous examples of the linear sampling and factorization methods are just a few of the most salient theoretical advances; numerous additional works on the subject exist.

1.2 Introduction to This Work

The main idea in this work is to use the partial Fourier transformation to take the frequency domain factorization method described in the previous section to the time domain. This idea has been converted to an actual time domain reconstruction method for the Robin Problem in Section 5.1, Theorems 5.7 and 5.8.

In order to derive the time domain reconstruction method, we study the single frequency Robin problem in Chapter 4 and provide a frequency dependent estimates for the operators F, G and A. We also develop a single frequency reconstruction method for Robin obstacles, which is presented in Section 4.5, Theorem 4.29. An important aspect of this part is the analysis of the time and frequency domain Dirichlet problems presented in Chapter 3.

We define and study the time domain far field operator F_{time} in Sections 5.2 and 5.3. Here we also define the time domain far field and make decay estimates for the far field and the operator F_{time}. These may be important for further study of the time domain methods, though we derive them for some less formal comments on the numerical applicability of the multi-frequency factorization method in Section 5.4.

In the time domain Robin reconstruction we assume the following condition:

point z is in the scatterer D if and only if

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}\times\mathbb{S}^2}\overline{\Big((\epsilon I+\widehat{L})^{-1}\widehat{\psi}_z\Big))(t,\widehat{x})}\widehat{\psi}_z(t,\widehat{x})\mathrm{d}t\mathrm{d}\mathcal{H}^2(\widehat{x})<\infty,$$

where $L : L^2(\mathbb{R}\times\mathbb{S}^2)\to L^2(\mathbb{R}\times\mathbb{S}^2)$ is defined by

$$Lu(k,\widehat{x}) := (k\mathrm{Im}(F_k)u(k,\cdot))(\widehat{x}),$$

$\widehat{L} := \mathcal{F}_{k\to t}L\mathcal{F}^{-1}_{t\to k}$ and $\widehat{\psi}_z(t,\widehat{x}) := \widehat{\psi}(t+z\cdot\widehat{x})$. Here ψ is the weight function with which we choose the frequencies that we use in the reconstruction and $\widehat{\psi}=\mathcal{F}_{k\to t}\{\psi\}$.

We use the imaginary part $\mathrm{Im}(F_k)$ in order to have a self-adjoint operator and also to provide frequency dependent estimates for the "middle operator" $\mathrm{Im}(A_k)$ of equation (4.68).

The function $\psi\in\mathcal{S}(\mathbb{R})$ needs to be such that $\widehat{\psi}\in C_0^\infty(\mathbb{R})$ with

$$\int_{\mathbb{R}}\widehat{\psi}(t)\mathrm{d}t = 0. \tag{1.8}$$

The $\widehat{\psi}$ function corresponds roughly to the time dependence profile of the waves that we use to probe our obstacle. It is enough to use an anti-symmetric profile $\widehat{\psi}(-t)=-\widehat{\psi}(t)$ in order to satisfy condition (1.8).

As background information, we have apparently developed a new class of distributions $\mathcal{S}'(\mathbb{R}\times U)$ for the partial Fourier transformation $\mathcal{F}_{k\to t}:\mathcal{S}'(\mathbb{R}\times U)\to\mathcal{S}'(\mathbb{R}\times U)$. This class is constructed in appendix C in a way that resembles strongly the construction of $\mathcal{D}'(U)$ of [55]. The connection $\mathcal{F}_{k\to t}:\mathcal{S}'(\mathbb{R}\times U)\to\mathcal{S}'(\mathbb{R}\times U)$ is our basic bridge between the time and frequency domains, which is augmented with the Bochner integrals in Section 3.3.

Chapter 2

Preliminaries

2.1 Distributions and Sobolev Spaces

In this section we fix our notation and introduce some basic concepts. We follow relatively closely the presentation of Sobolev spaces found in [40].

We denote the closure of a set U with $\overline{U}$ and the interior of a set U with $\operatorname{int}(U)$. If a set K is a compact subset of a set A we will use the notation $K \subset\subset A$.

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $r > 0$. We define

$$B(x, r) := \{y \in \mathbb{R}^n : |y - x| < r\},$$

where $|x| := (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$.

An open set $U \subset \mathbb{R}^n$ is C^k smooth if for any $x \in \partial U$ there exists $r_x > 0$ and a C^k function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that after relabelling and re-orienting the coordinate axis we have

$$U \cap B(x, r_x) = \{y \in B(x, r) : y_n > \gamma(y_1, \ldots y_{n-1}).$$

We notice that the coordinates $\varphi_x : \partial U \cap B(x, r_x) \to \mathbb{R}^{n-1}$ defined by

$$\varphi_x(y_1, \ldots, y_{n-1}, \gamma(y_1, \ldots, y_{n-1})) = (y_1, \ldots, y_{n-1})$$

give ∂U a manifold structure, which we will use later on. Mostly we will be dealing with a compact set $D \subset \mathbb{R}^3$, which is a C^2 domain, and its open complement $D^+ = \mathbb{R}^3 \setminus \overline{D}$, which is also C^2.

Let $U \subset \mathbb{R}^n$ be an open set. The test function space $\mathcal{D}(U) = C_0^\infty(U)$ consists of infinitely smooth functions φ whose support $\operatorname{supp}(\varphi)$ is a compact subset of U. A sequence $(\varphi_i)_{i\in\mathbb{N}} \subset \mathcal{D}(U)$ is said to converge to $\varphi \in \mathcal{D}(U)$ if there is a compact set $K \subset\subset U$ such that for all i the support $\mathrm{supp}(\varphi_i) \subset K$ and for all multi indexes $\alpha \in \mathbb{N}^n$ functions $\partial^\alpha \varphi_i$ converge to $\to \partial^\alpha \varphi$ uniformly. The distributions $\mathcal{D}'(U)$ are linear maps from $\mathcal{D}(U)$ to $\mathbb{C}$ that are sequentially continuous.

The space $\mathcal{E}'(U)$ consists of those distributions of $\mathcal{D}'(U)$ that have a compact support. These distributions can be seen to be the dual of $\mathcal{E}(U) = C^\infty(U)$.

The space of tempered functions, $\mathcal{S}(\mathbb{R}^n)$, consists of infinitely smooth functions φ that also satisfy for all $\alpha, \beta \in \mathbb{N}^n$

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)| < \infty.$$

A sequence of tempered functions (φ_i) converges to $\varphi \in \mathcal{S}(\mathbb{R}^n)$ if for all $\alpha, \beta \in \mathbb{N}$ we have

$$\lim_{i\to\infty} \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta (\varphi_i(x) - \varphi(x))| = 0.$$

The tempered distributions, $\mathcal{S}'(\mathbb{R}^n)$, are sequentially continuous linear maps from $\mathcal{S}(\mathbb{R}^n)$ to $\mathbb{C}$.

Let $t \in \mathbb{R}$, $\langle t \rangle = (1+t^2)^{\frac{1}{2}}$ and $U \subset \mathbb{R}^{n-1}$ be an open set. The tempered distributions $\mathcal{S}'(\mathbb{R}\times U)$ are studied in detail in Appendix C. We present here only the following working definition, which is sufficient for our purposes: Let $(K_j)_{j\in\mathbb{N}}$ be a sequence of sets $K_j \subset\subset U$ such that $K_j \subset \mathrm{int}\,(K_{j+1})$ and $U = \bigcup_{j\in\mathbb{N}} K_j$. We define the norms

$$\|\varphi\|_{k,i,j} := \sup_{0\le|\alpha|\le k} \sup_{\substack{t\in\mathbb{R} \\ x\in K_j}} |\langle t\rangle^i \partial^\alpha \varphi(t,x)|$$

and the set

$$S(\mathbb{R}\times U) = \{\varphi \in C^\infty(\mathbb{R}\times U) : \exists\, j \text{ s.t. } \mathrm{supp}(\varphi) \subset \mathbb{R}\times K_j,\ \forall i,k : \|\varphi\|_{k,i,j} < \infty\}.$$

We say that a sequence $(\varphi_n) \subset \mathcal{S}(\mathbb{R}\times U)$ converges to 0 if there exists a $K \subset\subset U$ such that for all n we have $\mathrm{supp}(\varphi_n) \subset \mathbb{R}\times K$ and for all $k,i,j \in \mathbb{N}$

$$\lim_{n\to\infty} \|\varphi_n\|_{k,i,j} = 0.$$

The tempered distributions $\mathcal{S}'(\mathbb{R}\times U)$ are linear maps from $\mathcal{S}(\mathbb{R}\times U)$ to $\mathbb{C}$ that are sequentially continuous.

Let $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$. We define the norms

$$\|\varphi\|_{k,i} := \sup_{|\alpha|\leq k} \sup_{\substack{t\in\mathbb{R} \\ x\in\mathbb{S}^2}} |\langle t\rangle^i \partial^\alpha \varphi(t,x)|$$

and the space

$$\mathcal{S}(\mathbb{R}\times\mathbb{S}^2) := \{\varphi \in C^\infty(\mathbb{R}\times\mathbb{S}^2) : \forall k,i : \|\varphi\|_{k,i} < \infty\}.$$

We say that $(\varphi_n) \subset \mathcal{S}(\mathbb{R}\times\mathbb{S}^2)$ converges to 0 if for all $k, i \in \mathbb{N}$ we have

$$\lim_{n\to\infty} \|\varphi_n\|_{k,i} = 0.$$

The distributions $\mathcal{S}'(\mathbb{R}\times\mathbb{S}^2)$ from $\mathcal{S}(\mathbb{R}\times\mathbb{S}^2)$ to $\mathbb{C}$ are sequentially continuous linear maps.

We denote the value of a distribution u at the test function φ with $(u,\varphi)_{\text{dist}}$. If there is possibility of confusion about the type of the distribution, we denote this value with $(u,\varphi)_{\mathcal{D}'\times\mathcal{D}}$, $(u,\varphi)_{\mathcal{S}'\times\mathcal{S}}$, etc. In case of possible confusion about the domain, we use notations $(u,\varphi)_{\mathcal{D}'\times\mathcal{D}(U)}$, $(u,\varphi)_{\mathcal{S}'\times\mathcal{S}(U)}$, etc.

All integrable functions can be seen as distributions in $\mathcal{D}'$, or per the definition: for $u \in L^1(U)$ we define

$$(u,\varphi)_{\mathcal{D}'\times\mathcal{D}(U)} := \int_U \overline{u(x)}\varphi(x)\mathrm{d}x \tag{2.1}$$

Throughout this work the brackets $(\cdot,\cdot)$ stand for a complex conjugated duality like the one in (2.1).

Let $f \in \mathcal{D}'(\mathbb{R}^n)$ and $g \in \mathcal{D}'(\mathbb{R}^m)$. We define the tensor product $f \otimes g \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^m)$ by

$$(f\otimes g,\varphi)_{\mathcal{D}'\times\mathcal{D}(\mathbb{R}^{n+m})} := \Big(f(x), (g(y),\varphi(x,y))_{\mathcal{D}'\times\mathcal{D}(\mathbb{R}^n)}\Big)_{\mathcal{D}'\times\mathcal{D}(\mathbb{R}^m)}. \tag{2.2}$$

For clarity we sometimes write $f\otimes g = f(x)\otimes g(y)$.

We next make some remarks about the convolution of distributions based on the exposition given by V.S. Vladimirov in [56]. By [56, Section I.4.3] the convolution of $f, g \in \mathcal{D}'(\mathbb{R}^n)$ exists if for any $R > 0$ the set

$$T_R := \{(x,y) \in \mathbb{R}^{2n} : x \in \operatorname{supp}(f), y \in \operatorname{supp}(g), |x+y| < R\}$$

is bounded. In this case the convolution can be defined, with functions $\chi_f, \chi_g \in C^\infty(\mathbb{R}^n)$ such that $\chi_f \equiv 1$ on $\operatorname{supp}(f) + B(0,\epsilon)$, $\chi_f \equiv 0$ on $\mathbb{R}^n \setminus$

$(\mathrm{supp}(f) + B(0, 2\epsilon))$, $\chi_g \equiv 0$ on $\mathrm{supp}(g) + B(0,\epsilon)$ and $\chi_g \equiv 0$ on $\mathbb{R}^n \setminus (\mathrm{supp}(g) + B(0, 2\epsilon))$, by setting for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$

$$(f * g, \varphi)_{\mathcal{D}' \times \mathcal{D}(\mathbb{R}^n)} := (f(x) \otimes g(y), \chi_f(x)\chi_g(y)\varphi(x+y))_{\mathcal{D}' \times \mathcal{D}(\mathbb{R}^{2n})}.$$

A practical way, especially suited for the wave equation, to check if the set T_R is bounded is by using cones. A **cone** is a set $\Omega \subset \mathbb{R}^n$ with the property that if $x \in \Omega$ then $\lambda x \in \Omega$ for all $\lambda > 0$. A **cone is acute** if there is an $n-1$ dimensional plane in $\mathbb{R}^n$ such that Ω is on one side of it and it intersects the closure of the convex hull of Ω only at 0. The **conjugate of a cone** Ω is the set

$$\Omega^* = \{y \in \mathbb{R}^n : y \cdot x \geq 0, \ \forall x \in \Omega\}.$$

Let Ω be a closed acute cone and $C = \mathrm{int}\,(\Omega^*)$. A surface $S \subset \mathbb{R}^n$ without an edge is C-like if for all $y \in \mathbb{R}^n$ and $x \in \Omega$ the straight line $y + tx$; $t \in \mathbb{R}$ intersects S at a unique point. By [56, Lemma I.4.4.5] for all $R > 0$ there exists a $R'(R) > 0$ such that

$$T_R = \{(x, y) \in \mathbb{R}^{2n} : x \in S, y \in \Omega, |x + y| < R\} \tag{2.3}$$

is contained in $B(0, R')$.

The surface S is **strictly** C**-like** if there are constants $a > 0$ and $\nu \geq 1$ such that for all $R > 0$ the set T_R of (2.3) satisfies $T_R \subset B(0, R'(R))$ with $R' < a(1 + R)^\nu$.

Example 2.1 *For all $y \in \Omega^* \setminus \{0\}$ the plane $\{x \in \mathbb{R}^n : (y, x) = 0\}$ is strictly C-like with $\nu = 1$.For details on this example, we refer to [56, Section I.4.4].*

Let Ω be a closed convex acute cone, $C = \mathrm{int}\,(\Omega^*)$, S a C-like surface and S_+ the region on the same side of S as Ω. By [56, Section I.4.5] the convolution of $f, g \in \mathcal{D}'(\mathbb{R}^n)$ exists if there is a compact set K and a cone Ω such that $\mathrm{supp}(f) \subset \Omega + K$ and $\mathrm{supp}(g) \subset S_+$. If $f, g \in \mathcal{S}'(\mathbb{R}^n)$ the same criterion for the existence of the convolution in $\mathcal{D}'(\mathbb{R}^n)$ applies, but there is the additional question if $f * g \in \mathcal{S}'(\mathbb{R}^n)$. By [56, Section I.5.6(b)] this is the case if the surface S is strictly C-like. We will use this result for $\Omega = \{(t, x) \in \mathbb{R}^4 : t \geq 0, \ |x| \leq t\}$ and $S = \{(t, x) \in \mathbb{R}^4 : t = 0\}$ in the proof of Lemma 2.7.

In addition to the cone criterion, the convolution of distributions $f, g \in \mathcal{S}'(\mathbb{R}^n)$ exists and $f * g \in \mathcal{S}'(\mathbb{R}^n)$ if the support of f or g is compact. For this result we refer to [56, Section I.5.6(a)].

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we define the **Fourier transformation**

$$\mathcal{F}\{\varphi\}(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x) \mathrm{d}x.$$

The Fourier transformation maps $\mathcal{S}(\mathbb{R}^n)$ to itself in a sequentially continuous manner, so we can define $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ through

$$(\mathcal{F}\{u\}, \varphi)_{\text{dist}} = \left(u, \mathcal{F}^{-1}\{\varphi\}\right)_{\text{dist}}.$$

Let $k \in \mathbb{N}$ and $1 \leq p < \infty$. We define the **Sobolev space**

$$W^{k,p}(U) := \{u \in \mathcal{D}'(U) : \forall |\alpha| \leq k : \partial^\alpha u \in L^p(U)\}.$$

The space $W^{k,p}(U)$ can be equipped with the norm

$$\|u\|_{W^{k,p}} := \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p(U)}$$

and the pair $(W^{k,p}(U), \|\cdot\|_{(W^{k,p}(U))})$ is a Banach space.

Another approach to the Sobolev spaces is through the Fourier transformation. Let $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$. For $s \in \mathbb{R}$ the Sobolev space $H^s(\mathbb{R}^n)$ consists of $u \in \mathcal{S}'(\mathbb{R}^n)$ for which the inner product

$$(u, u)_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} \overline{\widehat{u}(\xi)} \widehat{u}(\xi) \mathrm{d}\xi$$

is finite. The pair $(H^s(\mathbb{R}^n), (\cdot, \cdot)_{H^s(\mathbb{R}^n)})$ is a Hilbert space.

The dual space of $H^s(\mathbb{R}^n)$ is isomorphic to $H^{-s}(\mathbb{R}^n)$ through the dual pairing

$$(u, v)_{H^{-s} \times H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \overline{\widehat{u}(\xi)} \widehat{v}(\xi) \mathrm{d}\xi.$$

For a closed set $A \subset \mathbb{R}^n$ we define the subspace $H^s_A \subset H^s(\mathbb{R}^n)$ by

$$H^s_A := \{u \in H^s(\mathbb{R}^n) : \operatorname{supp}(u) \subset A\}.$$

The space H^s_A is a closed subspace of $H^s(\mathbb{R}^n)$, so we have an orthogonal projection $P : H^s(\mathbb{R}^n) \to H^s_A$.

For an open set $U \subset \mathbb{R}^n$ we define

$$H^s(U) := \{u \in \mathcal{D}'(U) : u = v|_U \text{ for some } v \in H^s(\mathbb{R}^n)\}.$$

With the orthogonal projection $P : H^s(\mathbb{R}^n) \to H^s_{\mathbb{R}^n \setminus U}$ we can give $H^s(U)$ an inner product; for any $u_1, u_2 \in H^s(U)$ let $v_1, v_2 \in H^s(\mathbb{R}^n)$ be such that $v_1|_U = u_1$ and $v_2|_U = u_2$. We define

$$(u_1, u_2)_{H^s(U)} = ((I - P)v_1, (I - P)v_2)_{H^s(\mathbb{R}^n)},$$

The pair $(H^s(U), (\cdot,\cdot)_{H^s(U)})$ is a Hilbert space.

The space $H^s_0(U)$ is the closure of $\mathcal{D}(U)$ in $H^s(U)$ with respect to the metric generated by the inner product $(\cdot,\cdot)_{H^s(U)}$.

For $k \in \mathbb{N}$ and U a Lipschitz domain $W^{k,2}(U) = H^k(U)$ with equivalent norms. For details of this result we refer to [40, Theorem 3.18].

Let $\Gamma \subset \mathbb{R}^n$ be a C^2 graph, i.e. there exists a C^2 map $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\Gamma = \{x \in \mathbb{R}^n : x = (x_1, \dots x_{n-1}, \gamma(x_1, \dots x_{n-1}))\}.$$

We denote $\varphi(x_1, \dots x_{n-1}) = (x_1, \dots x_{n-1}, \gamma(x_1, \dots x_{n-1}))$ and the Jacobian of φ with J_φ. We define for all $s \in [-2, 2]$ and $u, v \in C^2_0(\Gamma)$ the inner product

$$(u, v)_{H^s(\Gamma)} := (u \circ \varphi^{-1} \cdot J^{\frac{1}{2}}_{\varphi^{-1}}, v \circ \varphi^{-1} \cdot J^{\frac{1}{2}}_{\varphi^{-1}})_{H^s(\mathbb{R}^{n-1})} \tag{2.4}$$

and the Sobolev space $H^s(\Gamma)$ as the completion of $C^2_0(\Gamma)$ with respect to the topology induced by the inner product (2.4). More details on the Jacobian can be found e.g. in [18].

Let $b(k)$ be the volume of a k dimensional ball of radius one. For all $A \subset \mathbb{R}^n$, $k \in \mathbb{N}$ and $\delta > 0$ we define

$$\mathcal{H}^k_\delta(A) := \inf\{\sum_{j=1}^{\infty} b(k) r_j^k : A \subset \bigcup_{j=1}^{\infty} B(x_j, r_j), 2r_j > \delta\}.$$

The k dimensional **Hausdorff measure** $\mathcal{H}^k$ is defined by

$$\mathcal{H}^k(A) := \sup_{\delta>0} \mathcal{H}^k_\delta(A).$$

For $p \in [1, \infty)$ we define the spaces $L^p(\Gamma)$ on a k dimensional Lipschitz graph Γ with the measure $\mathcal{H}^k$. More details on Hausdorff measures can be found e.g. in [18].

Let $\Gamma \subset \mathbb{R}^n$ be a C^2-graph and $s \in [0, 2]$, $u \in H^{-s}(\Gamma)$, $v \in H^s(\Gamma)$ and $(u_k) \subset C^2_0(\Gamma)$ such that $u_j \to u$ in $H^{-s}(\Gamma)$. We define

$$(u, v)_{H^{-s} \times H^s(\Gamma)} := \lim_{j \to \infty} (u_j, v)_{L^2(\Gamma)}.$$

This gives an isometric realization of the dual $(H^s(\Gamma))^*$.

Let $D \subset \mathbb{R}^n$ be a bounded C^2 domain, $s \in [-2, 2]$, $\{(U_i, \varphi_i)\}_{i\in\{1,2,\dots,k\}}$ be an atlas of ∂D and $\{\psi_i^2\}_{i\in\{1,2,\dots,k\}}$ a partitioning of unity subordinate to the atlas $\{(U_i,\varphi_i)\}$. We define the inner product $(\cdot,\cdot)_{H^s(\partial D)}$ by setting for all $u,v\in C^2(\partial D)$

$$(u,v)_{H^s(\partial D)} := \sum_{i=1}^{k} (\psi_i u, \psi_i v)_{H^s(\Gamma_i)},$$

where Γ_i is a C^2 graph that contains the chart U_i. We notice that above we chose ψ_i^2 to be the elements of the partitioning of the unity in order to get

$$(\cdot,\cdot)_{H^0(\partial D)} = (\cdot,\cdot)_{L^2(\partial D)}.$$

The space $H^s(\partial D)$ is the completion of $C^2(\partial D)$ with respect to the topology induced by the inner product $(\cdot,\cdot)_{H^s(\partial D)}$. The pair $(H^s(\partial D),(\cdot,\cdot)_{H^s(\partial D)})$ is a Hilbert space. For $s\neq 0$ the inner product of this space depends on the choice of the atlas $\{(U_i,\varphi_i)\}$ and the partitioning of the unity $\{\psi_i^2\}$. However, any two such choices produce equivalent metrics, so the set $H^s(\partial D)$ does not depend on these choices.

Let $\{\Gamma_i\}$, $\{\psi_i\}$ be as before, $s\in[0,2]$ and $\Psi_i\in C_0^2(U_i)$ be such that $\Psi_i\equiv 1$ on $\operatorname{supp}(\psi_i)$. We define for all $u\in H^{-s}(\partial D)$ and $v\in H^s(\partial D)$ the dual action

$$(u,v)_{H^{-s}\times H^s(\partial D)} := \sum_{i,j=1}^{k} (\psi_i u, \Psi_i\psi_j v)_{H^{-s}\times H^s(\Gamma_i)}\,.$$

The map $j: H^{-s}(\partial D)\to (H^s(\partial D))^*$ defined by

$$j(u)(v) := (u,v)_{H^{-s}\times H^s(\partial D)}$$

is an isometry. If $u\in L^2(\partial D)$ then

$$(u,v)_{H^{-s}\times H^s(\partial D)} = (u,v)_{L^2(\partial D)}.$$

Let U be a Lipschitz domain. We will use the **trace operator** tr that maps an element $u\in C^\infty(\overline{U})$ to the boundary value $\operatorname{tr} u = u|_{\partial U}\in C(U)$. By [40, Theorem 3.38] this operator can be extended to $\operatorname{tr}: H^s(U)\to H^{s-\frac{1}{2}}(\partial U)$ for $s\in(\frac{1}{2},\frac{3}{2})$. If U is a C^k domain, then the trace can be defined on $H^s(U)$ for $s\in(\frac{1}{2},k]$. If there is ambiguity about the domain U we will use notation $\operatorname{tr}_{\partial U}$ for $\operatorname{tr}: H^s(U)\to H^{s-\frac{1}{2}}(\partial U)$. If u is defined on both sides of the boundary ∂U, we denote $\operatorname{tr}^- u$ for $\operatorname{tr}_{\partial U}(u|_U)$ and tr^+ for $\operatorname{tr}_{\partial(U^c)}(u|_{U^c})$.

The trace operator is surjective, so we can define one-sided inverse mapping tr^{-1} with the property $\operatorname{tr}\operatorname{tr}^{-1}\varphi=\varphi$. This operator is not uniquely defined, but for the applications in which we use it this will not be a problem.

If the domain U is C^1, then the surface ∂U has the normal vector $\nu(U)$, which we will take to be the exterior normal vector that points to the complement of U. For u in $C^1(\mathbb{R}^n \setminus U)$ we define the **normal derivative operator** $\partial^+_{\nu(U)}$ by setting for all $x \in \partial U$

$$\partial^+_{\nu(U)} u(x) := \lim_{h \to 0^+} \frac{1}{h} \left(u(x + h\nu(U)) - u(x) \right).$$

For $u \in C^1(\overline{U})$ we define

$$\partial^-_{\nu(U)} u(x) := \lim_{h \to 0^+} \frac{1}{h} \left(u(x) - u(x - h\nu(U)) \right).$$

If the domain U is clear from the context, we write ∂_ν instead of $\partial_{\nu(U)}$. If there is ambiguity about the variable on which the normal derivative operator operates, we will denote it by $\partial_{\nu(y)}$ in if operates with respect to the y variable. If there is no ambiguity as to whether ∂^+_ν or ∂^-_ν is meant, we will use the notation ∂_ν.

The notion of the normal derivative, ∂_ν, can be extended to the case where $u \in H^s(U)$ with $s = 1$ if some additional conditions are met. If $u \in H^1(U)$ and $\Delta u \in L^2(U)$, we define $\partial^-_\nu u \in H^{-\frac{1}{2}}(\partial U)$ by setting for all $\varphi \in H^{\frac{1}{2}}(\partial U)$

$$\left(\partial^-_\nu u, \varphi \right)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}(\partial U)} := (\Delta u, \operatorname{tr}^{-1} \varphi)_{L^2(U)} + \sum_{i=1}^{3} (\partial_{x_i} u, \partial_{x_i} \operatorname{tr}^{-1} \varphi)_{L^2(U)}. \quad (2.5)$$

This definition does not depend on the choice of the inverse map tr^{-1}. If v_1 and $v_2 \in H^1(U)$ are such that $\operatorname{tr}(v_1 - v_2) = 0$, it follows that $v = v_1 - v_2$ satisfies

$$(\Delta u, v)_{L^2(U)} + \sum_{i=1}^{3} (\partial_{x_i} u, \partial_{x_i} v)_{L^2(U)} = 0,$$

since we can approximate $v_1 - v_2$ with $C_0^\infty(U)$ functions.

If $u \in H^1(\mathbb{R}^n \setminus \overline{U})$ and $\Delta u \in L^2(\mathbb{R}^n \setminus \overline{U})$, we define $\partial^+_\nu u$ by setting for all $\varphi \in H^{\frac{1}{2}}(\partial U)$

$$-\left(\partial^+_\nu u, \varphi \right)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}(\partial U)} := (\Delta u, \operatorname{tr}^{-1} \varphi)_{L^2(\mathbb{R}^n \setminus \overline{U})} + \sum_{i=1}^{3} (\partial_{x_i} u, \partial_{x_i} \operatorname{tr}^{-1} \varphi)_{L^2(\mathbb{R}^n \setminus \overline{U})}.$$

We note that we have the minus sign on the right-hand side of the previous equation since the normal ν points into the interior of $\mathbb{R}^n \setminus \overline{U}$.

The space $L^2_{\ell oc}(U)$ consists of elements u such that for all $R > 0$ we have $u|_{B(0,R) \cap U} \in L^2(B(0,R) \cap U)$. Correspondingly $H^1_{\ell oc}(U)$ consists of such elements u that for all $R > 0$ we have $u|_{B(0,R)\cap U} \in H^1(B(0,R) \cap U)$. The normal derivative ∂_ν^+ can also be defined for $u \in H^1_{\ell oc}(\mathbb{R}^n \setminus \overline{U})$ with $\Delta u \in L^2_{\ell oc}(\mathbb{R}^n \setminus U)$. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi \equiv 1$ on a neighbourhood of $\overline{U}$ and define for all $\varphi \in H^{\frac{1}{2}}(\partial U)$

$$
\begin{aligned}
-\left(\partial_\nu^+ u, \varphi\right)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}(\partial U)} &:= (\Delta u, \chi \mathrm{tr}^{-1}\varphi)_{L^2(\mathbb{R}^n \setminus \overline{U})} + \\
&+ \sum_{i=1}^{3} (\partial_{x_i} u, \partial_{x_i} \chi \mathrm{tr}^{-1}\varphi)_{L^2(\mathbb{R}^n \setminus \overline{U})}. \qquad (2.6)
\end{aligned}
$$

As in (2.6), we have for all different functions χ_1 and χ_2

$$
(\Delta u, (\chi_1 - \chi_2)\mathrm{tr}^{-1}\varphi)_{L^2(\mathbb{R}^n \setminus \overline{U})} + \sum_{i=1}^{3} (\partial_{x_i} u, \partial_{x_i}(\chi_1 - \chi_2)\mathrm{tr}^{-1}\varphi)_{L^2(\mathbb{R}^n \setminus \overline{U})} = 0.
$$

Hence the definition of the normal derivative is well-defined. In the case of $C^2(\mathbb{R}^n \setminus \overline{U})$-functions, the above definition agrees with Green's First Theorem.

Let X be a normed space. We say that an operator $T : X \to H^1_{\ell oc}(U)$ is continuous if for all $R \geq 0$ there exists a constant C_R such that for all $x \in X$

$$
\|Tx\|_{H^1(B(0,R)\cap U)} \leq C_R \|x\|_X.
$$

2.2 The Free Space Solutions

We establish an existence and uniqueness result for the wave equation in the free space, in which the solution is expressed as a convolution with the fundamental solution E_+. This result is used in Section 5.3 to establish a formula for the far field of the solution u in the time domain.

The fundamental solution of the wave operator $\Box = \partial_t^2 - \Delta_x$, is defined by setting for all $\varphi \in \mathcal{S}(\mathbb{R}^4)$

$$
(E_+, \varphi)_{\mathcal{S}' \times \mathcal{S}(\mathbb{R}^4)} := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x|} \varphi(|x|, x) \mathrm{d}x. \qquad (2.7)
$$

For details we refer the reader to [26, Section 6.2].

Theorem 2.2 *Let $H \in \mathcal{D}'(\mathbb{R}^4)$ be supported in the half space $\{(t,x) \in \mathbb{R}^4 : t \geq 0\}$. The problem*

$$
\begin{cases} \Box u &= H \quad in\ \mathbb{R}^4 \\ u|_{t<0} &= 0 \end{cases} \qquad (2.8)
$$

has a unique solution $u \in \mathcal{D}'(\mathbb{R}^4)$ given by

$$
u = E_+ * H.
$$

Proof. The result is well known, see [26] for existence.

□

2.3 The Partial Fourier Transformation $\mathcal{F}_{k\to t}$

For practical reasons we take the partial Fourier transformation to map from the frequency domain, with variable k, to the time domain, with variable t. For $u \in \mathcal{S}(\mathbb{R} \times U)$ the definition of the transformation is straightforward.

Definition 2.3 *Let $U \subset \mathbb{R}^3$ be an open set. We define the partial Fourier transformation $\mathcal{F}_{k\to t}$ for $\varphi \in \mathcal{S}(\mathbb{R} \times U)$ by setting for all $t \in \mathbb{R}$ and $x \in U$*

$$\mathcal{F}_{k\to t}\{\varphi\}(t,x) := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-ikt}\varphi(k,x)dk.$$

We define the partial Fourier transformation for $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{S}^2)$ by setting for all $t \in \mathbb{R}$ and $\widehat{x} \in \mathbb{S}^2$

$$\mathcal{F}_{k\to t}\{\varphi\}(t,\widehat{x}) := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-ikt}\varphi(k,\widehat{x})dk.$$

The transformation $\mathcal{F}_{k\to t} : \mathcal{S}(\mathbb{R} \times U) \to \mathcal{S}(\mathbb{R} \times U)$ has the inverse

$$\mathcal{F}^{-1}_{t\to k}\{\varphi\}(k,x) := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{ikt}\varphi(t,x)dt.$$

Likewise for $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{S}^2)$

$$\mathcal{F}^{-1}_{t\to k}\{\varphi\}(t,\widehat{x}) := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{ikt}\varphi(t,\widehat{x})ds.$$

We can extend the partial Fourier transformation in a fashion similar to the extension of $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ to $\mathcal{F} : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$:

Definition 2.4 *For $u \in \mathcal{S}'(\mathbb{R} \times U)$ we define $\mathcal{F}_{k\to t}\{u\} \in \mathcal{S}'(\mathbb{R} \times U)$ as follows: for all $\varphi \in \mathcal{S}(\mathbb{R} \times U)$*

$$(\mathcal{F}_{k\to t}\{u\},\varphi)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times U)} = \left(u, \mathcal{F}^{-1}_{t\to k}\{\varphi\}\right)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times U)}.$$

For $u \in \mathcal{S}'(\mathbb{R} \times \mathbb{S}^2)$ we define $\mathcal{F}_{k\to t}\{u\} \in \mathcal{S}'(\mathbb{R} \times \mathbb{S}^2)$ as follows: for all $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{S}^2)$

$$(\mathcal{F}_{k\to t}\{u\},\varphi)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times\mathbb{S}^2)} = \left(u, \mathcal{F}^{-1}_{t\to k}\{\varphi\}\right)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times\mathbb{S}^2)}.$$

The restriction of $\mathcal{F}_{k\to t}$ on L^2 is an isometry, as we see in the following lemma.

Lemma 2.5 *Operators $\mathcal{F}_{k\to t} : L^2(\mathbb{R}\times U) \to L^2(\mathbb{R}\times U)$, $\mathcal{F}^{-1}_{t\to k} : L^2(\mathbb{R}\times U) \to L^2(\mathbb{R}\times U)$, $\mathcal{F}_{k\to t} : L^2(\mathbb{R}\times \mathbb{S}^2) \to L^2(\mathbb{R}\times \mathbb{S}^2)$ and $\mathcal{F}^{-1}_{t\to k} : L^2(\mathbb{R}\times \mathbb{S}^2) \to L^2(\mathbb{R}\times \mathbb{S}^2)$ are isometries.*

Proof. We prove the result only for $L^2(\mathbb{R}\times U)$ since the proof for $L^2(\mathbb{R}\times \mathbb{S}^2)$ is analogous. The restriction $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an isometry, so for all $u \in \mathcal{S}(\mathbb{R}\times U)$ we have by Fubini's Theorem

$$\begin{aligned}\|\mathcal{F}_{k\to t}\{u\}\|^2_{L^2(\mathbb{R}\times U)} &= \int_U \|\mathcal{F}_{k\to t}\{u\}(\cdot,x)\|^2_{L^2(\mathbb{R})}\mathrm{d}x \\ &= \int_U \|u(\cdot,x)\|^2_{L^2(\mathbb{R})}\mathrm{d}x \\ &= \|u\|^2_{L^2(\mathbb{R}\times U)}. \end{aligned} \tag{2.9}$$

Since $\mathcal{S}(\mathbb{R}\times U) \subset L^2(\mathbb{R}\times U)$ is dense the equality (2.9) holds for all $u \in L^2(\mathbb{R}\times U)$.

The proof for $\mathcal{F}^{-1}_{t\to k}$ is the same as for $\mathcal{F}_{k\to t}$.

□

If we have an operator $T : L^2(\mathbb{R}\times \mathbb{S}^2) \to L^2(\mathbb{R}\times \mathbb{S}^2)$, then we can define operator $\widehat{T} : L^2(\mathbb{R}\times \mathbb{S}^2) \to L^2(\mathbb{R}\times \mathbb{S}^2)$ on the Fourier side by

$$\widehat{T} := \mathcal{F}^{-1}_{t\to k} T \mathcal{F}_{k\to t}.$$

It follows from Lemma 2.5 that $\|\widehat{T}\| = \|T\|$. We will use this operator in Subsection 5.1 to form the scattering operator in the time domain.

An integral part of this work is the connection between the fundamental solution of the Helmholtz operator $-(\Delta + k^2)$, denoted by Φ_k, and the fundamental solution of the wave equation, E_+.

Lemma 2.6 *The Fourier transformation of the fundamental solution*

$$\Phi_k(x) = \frac{e^{ik|x|}}{4\pi|x|}$$

is

$$\mathcal{F}_{k\to t}\{\Phi_k\} = (2\pi)^{\frac{1}{2}} E_+.$$

Proof. By definition of $\mathcal{F}_{k\to t}$ we have for all $\varphi \in \mathcal{S}(\mathbb{R}^4)$

$$\begin{aligned} I &= (\mathcal{F}_{k\to t}\{\Phi_k\}, \varphi)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}^4)} \\ &= \left(\Phi_k, \mathcal{F}^{-1}_{t\to k}\{\varphi\}\right)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}^4)}. \end{aligned}$$

Since $\overline{\Phi_k}\mathcal{F}^{-1}_{t\to k}\{\varphi\} \in L^1(\mathbb{R}^4)$, we have by Fubini's Theorem

$$\begin{aligned} I &= \int_{\mathbb{R}^4} \frac{e^{-ik|x|}}{4\pi|x|}\mathcal{F}^{-1}_{t\to k}\{\varphi\}(k,x)\mathrm{d}k\mathrm{d}x \\ &= \int_{\mathbb{R}^3} \frac{1}{4\pi|x|}\left(\int_{\mathbb{R}} e^{-ik|x|}\mathcal{F}^{-1}_{t\to k}\{\varphi\}(k,x)\mathrm{d}k\right)\mathrm{d}x. \end{aligned}$$

We have

$$\begin{aligned} \int_{\mathbb{R}} e^{-ik|x|}\mathcal{F}^{-1}_{t\to k}\{\varphi\}(k,x)\mathrm{d}x &= (2\pi)^{\frac{1}{2}}\mathcal{F}_{k\to t}\{\mathcal{F}^{-1}_{t\to k}\{\varphi(\cdot,x)\}\}(|x|) \\ &= (2\pi)^{\frac{1}{2}}\varphi(|x|,x). \end{aligned}$$

Hence

$$\begin{aligned} I &= (2\pi)^{\frac{1}{2}}\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x|}\varphi(|x|,x)dx \\ &= (2\pi)^{\frac{1}{2}}\left(E_+,\varphi\right)_{\text{dist}}. \end{aligned}$$

□

In the following lemma we find the partial Fourier transformations of the distributions $E_+*(\delta\otimes f)$ and $E*h$, where $f \in L^2(\mathbb{R}^3)$ is compactly supported and $h \in L^1(\mathbb{R}^4)$ with $\mathrm{supp}(h) \subset \overline{\mathbb{R}_+}\times B(0,R)$ for some $R > 0$. We next make a few clarificatory comments on the existence of the Fourier transformations and explain notation used in the lemma.

By Fubini's Theorem for almost every $x \in \mathbb{R}^3$ the restriction $h(\cdot,x) \in L^1(\mathbb{R})$, so we see that for these x the Fourier transformation $\mathcal{F}^{-1}_{t\to k}\{h\}(k,x)$ exists. In addition

$$\int_{\mathbb{R}^3}|\int_{\mathbb{R}} e^{ikt}h(t,x)\mathrm{d}t|\mathrm{d}x \quad \leq \quad \int_{\mathbb{R}^4}|h(t,x)|\mathrm{d}t\mathrm{d}x,$$

so $\mathcal{F}^{-1}_{t\to k}\{h\}(k,\cdot)$ is in $L^1(\mathbb{R}^3)$ and for all $k \in \mathbb{R}$ we have

$$\|\mathcal{F}^{-1}_{t\to k}\{h\}(k,\cdot)\|_{L^1(\mathbb{R}^3)} \leq \|h\|_{L^1(\mathbb{R}^4)}.$$

Since Φ_k is also locally integrable in $L^1(\mathbb{R}^3)$, we see by Fubini's Theorem that for all k the convolution $\Phi_k *_y \mathcal{F}^{-1}_{t\to k}\{h\}(k,\cdot)$ is in $L^1_{\ell oc}(\mathbb{R}^3)$. Here $*_y$ refers to a convolution taken over the spatial variables in $\mathbb{R}^3$.

We use the notation $\Phi_k *_y \mathcal{F}^{-1}_{t\to k}\{h\}$ for a distribution in $\mathcal{S}'(\mathbb{R}^4)$ which maps $\varphi \in \mathcal{S}(\mathbb{R}^4)$ to

$$\left(\Phi_k *_y \mathcal{F}^{-1}_{t\to k}\{h\}, \varphi\right)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}^4)} := \int_{\mathbb{R}} \left(\Phi_k *_y \mathcal{F}^{-1}_{t\to k}\{h\}(k,\cdot), \varphi(k,\cdot)\right)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}^3)} \mathrm{d}k.$$

Lemma 2.7 *Let* $f \in L^2(\mathbb{R}^3)$ *be compactly supported,* $R < \infty$ *and* $h \in L^1(\mathbb{R}^4)$ *with* $\operatorname{supp}(h) \subset \overline{\mathbb{R}_+} \times B(0,R)$. *Then*

$$\mathcal{F}^{-1}_{t\to k}\{E_+ * (\delta(t)\otimes f)\}(k,x) = (2\pi)^{-\frac{1}{2}} \left(\Phi_k *_y f\right)(k,x),$$

$$\mathcal{F}^{-1}_{t\to k}\{E_+ * (\delta'(t)\otimes f)\}(k,x) = (2\pi)^{-\frac{1}{2}} ik \left(\Phi_k *_y f\right)(k,x)$$

and

$$\mathcal{F}^{-1}_{t\to k}\{E_+ * h\}(k,x) = \left(\Phi_k *_y \mathcal{F}^{-1}_{t\to k}\{h\}(k,\cdot)\right)(k,x).$$

Proof. We prove the result only for $E_+ * (\delta'(t)\otimes f)$ and provide only a few comments on the proof for $E_+ * h$ since the proofs are almost the same.

Example 2.1 on page 14 is applicable to the distributions E_+ and $\delta'\otimes f$, so the convolution $E_+ * (\delta'(t)\otimes f)$ is well-defined and in $\mathcal{S}'(\mathbb{R}^4)$ and we can take the Fourier transformation of this element. By the definition of the Fourier transformation $\mathcal{F}^{-1}_{t\to k}$ and the convolution

$$\begin{aligned}
I &= \left(\mathcal{F}^{-1}_{t\to k}\{E_+ * (\delta'\otimes f)\}, \varphi\right)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}^4)} \\
&= \left(E_+(t,x)\otimes(\delta'\otimes f)(s,y), \chi_1(t,x)\chi_2(s,y)\mathcal{F}_{k\to t}\{\varphi\}\left((t,x)+(s,y)\right)\right)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}^8)}, \\
&= \Big(E_+(t,x), \chi_1\left((\delta'\otimes f)(s,y), \chi_2\mathcal{F}_{k\to t}\{\varphi\}\left((t,x)+(s,y)\right)\right)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}^4)}\Big)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}^4)},
\end{aligned}$$

where $\chi_1, \chi_2 \in C^\infty(\mathbb{R}^n)$ are as specified in the definition of the convolution on page 14.

We use the definition (2.7) of E_+ and the fact that $\chi_1 \equiv 1$ in a neighbourhood of $\operatorname{supp}(E_+)$, which gives us

$$I = \int_{\mathbb{R}^3} \frac{1}{4\pi|x|} \left((\delta'\otimes f)(s,y), \chi_2\mathcal{F}_{k\to t}\{\varphi\}\left((|x|,x)+(s,y)\right)\right)_{\mathcal{S}'\times\mathcal{S}} \mathrm{d}x. \quad (2.10)$$

Since $\chi_2 \equiv 1$ on a neighbourhood of $\{0\}\times\operatorname{supp}(f)$ we see that

$$I = \int_{\mathbb{R}^6} \frac{1}{4\pi|x|} \overline{f(y)} \left((2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} -ike^{-ik(|x|+0)}\varphi(k,x+y)\mathrm{d}k\right) \mathrm{d}y\mathrm{d}x.$$

Here the function $(k,x,y) \to \frac{e^{-ik|x|}}{4\pi|x|}\overline{f(y)}\varphi(k,x+y)$ is in $L^1(\mathbb{R}^7)$, so we can use Fubini's Theorem, from which it follows that

$$\begin{aligned} I &= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}^7} \overline{\frac{e^{ik|x|}}{4\pi|x|} ikf(y)}\varphi(k,x+y)\mathrm{d}y\mathrm{d}x\mathrm{d}k \\ &= (2\pi)^{-\frac{1}{2}} \left((\Phi_k *_y (ikf)(k,x), \varphi\right)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}^4)} . \end{aligned}$$

The proof for $\delta \otimes f$ follows exactly the same steps as the one for $\delta' \otimes f$.

Example 2.1 is also applicable to the distributions E_+ and h and the proof is the same up to equation (2.10), after which we get

$$\begin{aligned} I &= \left(\mathcal{F}^{-1}_{t\to k}\{E_+ * h\}, \varphi\right)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}^4)} \\ &= \int_{\mathbb{R}^7} \frac{1}{4\pi|x|}\overline{h(s,y)} \left((2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{ik(|x|+s)}\varphi(k,x+y)\mathrm{d}k\right) \mathrm{d}y\mathrm{d}s\mathrm{d}x. \end{aligned}$$

For all $y \in B(0,R)$ we have an estimate

$$\int_{\mathbb{R}^3} \left|\frac{1}{|x|}\varphi(k,x+y)\right| \mathrm{d}x \le \frac{C}{\langle k\rangle^2}, \tag{2.11}$$

so the function $(k,x,s,y) \to \frac{e^{-ik|x|}}{4\pi|x|}\overline{h(s,y)}e^{-ik(|x|+s)}\varphi(k,x+y)$ is in $L^1(\mathbb{R}^8)$. Hence we can use Fubini's Theorem, which implies that

$$\begin{aligned} I &= \int_{\mathbb{R}^7} \overline{\frac{e^{ik|x|}}{4\pi|x|}\mathcal{F}^{-1}_{t\to k}\{h\}(k,y)}\varphi(k,x+y)\mathrm{d}k\mathrm{d}y\mathrm{d}x \\ &= \left(\left(\Phi_k *_y \mathcal{F}^{-1}_{t\to k}\{h\}(k,\cdot)\right)(k,x), \varphi(k,x)\right)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}^4)} . \end{aligned}$$

□

We will also need the following result to prove the multi-frequency reconstruction in Chapter 5.

Lemma 2.8 *Let* $z, x \in \mathbb{R}^3$, $r_{z,k} = e^{-ikz\cdot\widehat{x}}$ *and* $\psi \in C^\infty(\mathbb{R})$ *be integrable. Then*

$$\mathcal{F}_{k\to t}\{\psi r_{z,k}\}(t) = \mathcal{F}_{k\to t}\{\psi\}(t + z\cdot\widehat{x}).$$

Proof. We have

$$
\begin{aligned}
\mathcal{F}_{k\to t}\{\psi r_{z,k}\}(t) &= (2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}} e^{-ikt}\psi(k)e^{-ikz\cdot\widehat{x}}\mathrm{d}k \\
&= (2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}} e^{-ik(t+z\cdot\widehat{x})}\psi(k)\mathrm{d}k \\
&= \mathcal{F}_{k\to t}\{\psi\}(t+z\cdot\widehat{x}).
\end{aligned}
$$

□

The next two lemmas can be seen as continuation of Lemma 2.8. These are used at the very end of this work, when outline arguments for the numerical applicability of the multi-frequency method proposed in this work.

Lemma 2.9 *Let $U\subset\mathbb{R}^3$ be open or $U=\mathbb{S}^2$ and $h\in\mathcal{S}'(\mathbb{R}\times U)$. In both cases*

$$
\mathcal{F}_{k\to t}\{e^{-iky\cdot\widehat{x}}h\}(t,\widehat{x})=\mathcal{F}_{k\to t}\{h\}(t+\widehat{\ }\cdot y,\widehat{x}).
$$

Proof. By the definition of the partial Fourier transformation, we have for all $\varphi\in\mathcal{S}(\mathbb{R}\times\mathbb{S}^2)$

$$
\begin{aligned}
\left(\mathcal{F}_{k\to t}\{e^{-ik\widehat{x}\cdot y}h\},\varphi\right)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times\mathbb{S}^2)} &= \left(h, e^{ik\widehat{x}\cdot y}\mathcal{F}^{-1}_{t\to k}\{\varphi\}\right)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times\mathbb{S}^2)} \\
&= \left(h,\mathcal{F}^{-1}_{t\to k}\{\varphi(t-\widehat{x}\cdot y,\widehat{x})\}\right)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times\mathbb{S}^2)}
\end{aligned}
$$

The map $\Psi:\mathbb{R}\times\mathbb{S}^2\to\mathbb{R}\times\mathbb{S}^2$ defined by $\Psi(t,\widehat{x})=(t+\widehat{x}\cdot y,\widehat{x})$ is a diffeomorphism with $J_\Psi\equiv 1$, so with the change of variables $(t',\widehat{x}')=\Psi(t,\widehat{x})$ we see that

$$
\left(\mathcal{F}_{k\to t}\{e^{-ik\widehat{x}\cdot y}h\},\varphi\right)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times\mathbb{S}^2)}=(\mathcal{F}_{k\to t}\{h\}(t+\widehat{x}\cdot y,\widehat{x})),\varphi)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times\mathbb{S}^2)}
$$

□

Lemma 2.10 *Let U be an open and bounded set, $h\in L^1(\mathbb{R}^4),\mathrm{supp}(h)\subset\mathbb{R}\times U$ and*

$$
\widehat{w}_{sc,\infty}(k,\widehat{x})=\frac{1}{4\pi}\int_U e^{-ik\widehat{x}\cdot y}\mathcal{F}^{-1}_{t\to k}\{h\}(k,y)dy.
$$

Then

$$
\mathcal{F}_{k\to t}\{\widehat{w}_{sc,\infty}\}(t,\widehat{x})=\frac{1}{4\pi}\int_U h(t+\widehat{x}\cdot y,y)dy.
$$

Proof. By the definition of the partial Fourier transformation we have for all $\varphi \in S(\mathbb{R} \times \mathbb{S}^2)$

$$\begin{aligned} I &= (\mathcal{F}_{k\to t}\{\widehat{w}_{sc,\infty}\}(k,\widehat{x}),\varphi)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times\mathbb{S}^2)} \\ &= \int_{\mathbb{R}\times\mathbb{S}^2} \frac{1}{4\pi}\int_U e^{ik\widehat{x}\cdot y}\overline{\mathcal{F}^{-1}_{t\to k}\{h\}(k,y)}\mathrm{d}y \ldots \\ &\quad \ldots \mathcal{F}^{-1}_{t\to k}\{\varphi\}(k,\widehat{x})\mathrm{d}\mathcal{H}^2(\widehat{x})\mathrm{d}k. \end{aligned} \tag{2.12}$$

The integrand is in $L^1(\mathbb{R} \times \mathbb{S}^2 \times U)$, so we can use Fubini's Theorem, which gives us

$$\begin{aligned} I &= \frac{1}{4\pi}\int_U\int_{\mathbb{R}\times\mathbb{S}^2} e^{ik\widehat{x}\cdot y}\overline{\mathcal{F}^{-1}_{t\to k}\{h\}(k,y)}\mathcal{F}^{-1}_{t\to k}\{\varphi\}(k,\widehat{x})\mathrm{d}k\mathrm{d}\mathcal{H}^2(\widehat{x})\mathrm{d}y \\ &= \frac{1}{4\pi}\int_U \left(\mathcal{F}_{k\to t}\{e^{-ik\widehat{x}\cdot y}\mathcal{F}^{-1}_{t\to k}\{h\}(k,y),\varphi\right)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times\mathbb{S}^2)}\mathrm{d}y. \end{aligned}$$

By Lemma 2.9 we have

$$\mathcal{F}_{k\to t}\{e^{-ik\widehat{x}\cdot y}\mathcal{F}^{-1}_{t\to k}\{h\}\}(t,\widehat{x}) = h(t+\widehat{x}\cdot y, y),$$

so

$$\begin{aligned} I &= \frac{1}{4\pi}\int_U (h(t+\widehat{x}\cdot y,y),\varphi(t,\widehat{x}))_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times\mathbb{S}^2)}\mathrm{d}y \\ &= \frac{1}{4\pi}\int_U (h(t,y)\otimes\mathbf{1}(\widehat{x}),\varphi(t-\widehat{x}\cdot y,\widehat{x}))_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times\mathbb{S}^2)}\mathrm{d}y. \end{aligned}$$

This is expressible as an integral with an L^1 integrand, so we can use Fubini's Theorem, which gives us

$$\begin{aligned} I &= \frac{1}{4\pi}\int_U\int_{\mathbb{R}\times\mathbb{S}^2}\overline{h(t,y)}\varphi(t+\widehat{x}\cdot y,\widehat{x})\mathrm{d}t\mathrm{d}\mathcal{H}^2(\widehat{x})\mathrm{d}y \\ &= \frac{1}{4\pi}\int_{\mathbb{S}^2}\int_{\mathbb{R}\times U}\overline{h(t,y)}\varphi(t-\widehat{x}\cdot y,\widehat{x})\mathrm{d}y\mathrm{d}t\mathrm{d}\mathcal{H}^2(\widehat{x}). \end{aligned}$$

With the change of variables $\psi(t, y) = (t + \widehat{x} \cdot y, y)$ we see that

$$\begin{aligned} I &= \frac{1}{4\pi}\int_{\mathbb{S}^2}\int_{\mathbb{R}\times U}\overline{h(t+\widehat{x}\cdot y,y)}\varphi(t,\widehat{x})\mathrm{d}y\mathrm{d}t\mathrm{d}\mathcal{H}^2(\widehat{x}) \\ &= \left(\frac{1}{4\pi}\int_U h(t+\widehat{x}\cdot y,y)\mathrm{d}y,\varphi\right)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times\mathbb{S}^2)}. \end{aligned}$$

□

2.4 Spectral Theory

We define a few basic concepts from spectral theory for future reference. Spectral theory is used in Section 4.4 to analyze the single frequency factorization of the imaginary part of the far field operator $\mathrm{Im}(F_k)$ and more extensively in Section 5.1 to provide the multi-frequency factorization method.

Our operators will be mostly self-adjoint and the functions that we use to form new operators will be continuous, so the classical treatment of spectral theory found in [57] will be sufficient for our needs.

Let H be a Hilbert space and $\{P(\lambda)\}_{\lambda\in\mathbb{R}}$ be a family of operators from H to H. Let $T : H \to H$ be continuous. We use the notation

$$\text{s-}\lim_{\lambda\to\mu} P(\lambda) = T$$

if for all $x \in H$ we have

$$\lim_{\lambda\to\mu} \|P(\lambda)x - Tx\|_H = 0,$$

i.e. a limit in the strong sense.

Definition 2.11 *A family of projections $\{P(\lambda)\}_{\lambda\in\mathbb{R}}$ that map objects from a Hilbert space H to itself, is called a* **resolution of identity** *if they satisfy the following conditions:*

i) $P(\lambda)P(\mu) = P(\min(\lambda, \mu))$

ii) $s\text{-}\lim_{\lambda\to-\infty} P(\lambda) = 0$ *and* $s\text{-}\lim_{\lambda\to\infty} P(\lambda) = I$

iii) $P(\lambda + 0) := s\text{-}\lim_{\mu\searrow 0} P(\lambda + \mu) = P(\lambda)$.

For a resolution of identity $\{P(\lambda)\}_{\lambda\in\mathbb{R}}$ we use the notation

$$P(\lambda_1, \lambda_2] := P(\lambda_2) - P(\lambda_1).$$

Let function $f : \mathbb{R} \to \mathbb{C}$ be continuous. For all $x \in X$ and $\alpha, \beta \in \mathbb{R}$ the integral

$$\int_\alpha^\beta f(\lambda)\mathrm{d}P(\lambda)x$$

can be defined as a s-lim of Riemann sums $\sum_j f(\lambda_j)P(\lambda_j, \lambda_{j+1}]x$, see [57, Proposition XI.5.2]. In this way we can define an operator T, which is defined on the set

$$\mathrm{Dom}\,(H) := \{x \in X : \int_{-\infty}^{\infty} |f(\lambda)|^2 \mathrm{d}\|P(\lambda)x\|^2 < \infty\}.$$

See [57, Theorem XI.5.2]. The value Tx itself is defined by

$$Tx := \int f(\lambda)\mathrm{d}P(\lambda)x. \tag{2.13}$$

It is useful to notice here that

$$\|Tx\|^2 = \int_{-\infty}^{\infty} |f(\lambda)|^2 \mathrm{d}\|P(\lambda)x\|^2$$

for $x \in \mathrm{Dom}\,(T)$, see [57, Corollary XI.5.2].

If the function f is real valued, the operator T in (2.13) is self-adjoint. It turns out, see for example [57, Theorem XI.6.1], that all self-adjoint operators have a unique presentation as a spectral integral

$$T = \int_{-\infty}^{\infty} \lambda \mathrm{d}P(\lambda).$$

This is called the **spectral resolution** of the self-adjoint operator T.

These are the tools from spectral theory needed in this work. Spectral theory can be extended to more exotic functions and more general classes of operators, but these are not needed in our work. For further reading on this subject we recommend the books [57], [16], [42] and [50] for the interested reader.

2.5 Properties of the Single and Double Layer Operators

We will here give a short account of the layer operators in a Sobolev space setting. For a more detailed analysis we refer to [40].

Let $s \in \mathbb{R}$. We will use the **volume potential operator** that is defined for $u \in \mathcal{E}'(\mathbb{R}^3)$ by

$$\mathcal{G}_k u = \Phi_k * u.$$

We recall that throughout this work D is a bounded C^2 domain. The next lemma provides a continuity result for the volume potential operator which is instrumental in the proof of the continuity properties of the single and double layer operators introduced later on. The k dependency of the estimate given is an important aspect for this work.

Lemma 2.12 *Let R be such that $\overline{D} \subset B(0,R)$, $\chi \in C_0^\infty(B(0,R+1))$ and let χ also denote the multiplication operator $u \to \chi u$. Then for all $s \in [-2,0]$ the operator $T_k = \chi \mathcal{G}_k \chi : H^s(\mathbb{R}^3) \to H^{s+2}(\mathbb{R}^3)$ is continuous, with the estimate*

$$\|T_k\|_{H^s(\mathbb{R}^3) \to H^{s+2}(\mathbb{R}^3)} \leq C\langle k \rangle^2.$$

Proof. Let $R \in \mathbb{R}_+$ and $B_R = B(0,R)$. For $M = \int_{B_{2(R+2)}} \frac{1}{4\pi|x|} dx$ it holds that

$$\begin{aligned} \int_{B_{R+2}} |\Phi_k(x,y)| dy &< M \qquad \text{for all } k \text{ and } x \in B_{R+2} \\ \int_{B_{R+2}} |\Phi_k(x,y)| dx &< M \qquad \text{for all } k \text{ and } y \in B_{R+2}. \end{aligned}$$

Hence Schur's Test, [51], implies that $\mathcal{G}_k : L^2(B_{R+2}) \to L^2(B_{R+2})$ is continuous, with with

$$\|\mathcal{G}_k\| \leq M, \tag{2.14}$$

for all k.

Let $f \in L^2(\mathbb{R}^3)$ with $\operatorname{supp}(f) \subset B_{R+1}$ and $u = \mathcal{G}_k f$. We have

$$-(\Delta + k^2)u = f \qquad \Leftrightarrow \qquad -\Delta u = f + k^2 u,$$

in $\mathbb{R}^3$. By regularization theory for elliptic PDEs, [17, Theorem 6.3.1] , and (2.14) we have

$$\begin{aligned} \|u\|_{H^2(B_{R+1})} &\leq C\left(\langle k \rangle^2 \|u\|_{L^2(B_{R+2})} + \|f\|_{L^2(B_{R+2})}\right) \\ &\leq C\langle k \rangle^2 \|f\|_{L^2(B_{R+2})} \\ &= C\langle k \rangle^2 \|f\|_{L^2(B_{R+1})}. \end{aligned}$$

Hence $\|\mathcal{G}_k\|_{L^2(B_{R+1}) \to H^2(B_{R+1})} \leq C\langle k \rangle^2$ and $T_k := \chi \mathcal{G}_k \chi$ is continuous from $L^2(\mathbb{R}^3)$ to $H^2(\mathbb{R}^3)$ with the estimate

$$\|T_k\|_{L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3)} \leq C\langle k \rangle^2. \tag{2.15}$$

As the adjungate operator of T_k is

$$(T_k)^* = \chi \mathcal{G}_{-k} \chi = T_{-k},$$

we also see that

$$\|T_k\|_{H^{-2}(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} \leq C\langle k \rangle^2. \tag{2.16}$$

By interpolation between equations (2.15) and (2.16), see [39, Theorem 1.5.1.], we see that for $s \in [-2,0]$ the operator T_k is continuous from $H^s(\mathbb{R}^3)$ to $H^{s+2}(\mathbb{R}^3)$ and that

$$\|T_k\|_{H^s(\mathbb{R}^3) \to H^{s+2}(\mathbb{R}^3)} \leq C\langle k \rangle^2.$$

□

In the following definition we use the **dual trace** $\operatorname{tr}^* : H^s(\partial U) \to \mathcal{E}'(\mathbb{R}^3)$

and **dual normal derivative** $\partial_\nu^* : H^s(\partial D) \to \mathcal{E}'(\mathbb{R}^3)$ operators, which are defined through

$$(\operatorname{tr}^* u, \phi)_{\text{dist}} := (u, \operatorname{tr}\phi)_{H^s \times H^{-s}(\partial D)}$$

and

$$(\partial_\nu^* u, \phi)_{\text{dist}} := (u, \partial_\nu \phi)_{H^s \times H^{-s}(\partial D)} .$$

With these we define the single and double layer operators:

Definiton 2.13 *Let $k \in \mathbb{R}$, $D \subset \mathbb{R}^3$ be a bounded C^2 domain and $\operatorname{tr}^*, \partial_\nu^*$ be associated with ∂D. Then the* **single and double layer** *operators associated with $-(\Delta + k^2)$ are*

$$\mathrm{SL}_k = \mathcal{G}_k \operatorname{tr}^* \qquad \text{and} \qquad \mathrm{DL}_k = \mathcal{G}_k \partial_\nu^*.$$

Whenever $\mathrm{SL}_k\varphi$ is in $H^s(\partial D)$ for $s > \frac{1}{2}$ we define $\mathrm{S}_k\varphi := \operatorname{tr}\mathrm{SL}_k\varphi$.

In this work the k-dependency of the single and double layer operators are very important, so we keep the index k in their notations through the work.

For $\psi \in C(\partial D)$ the single and double layer operators exist as improper integrals. For the single layer operator we have

$$\mathrm{SL}_k\psi(x) := \int_{\partial D} \Phi_k(x,y)\psi(y)\mathrm{d}\mathcal{H}^2(y)$$

and for the double layer operator

$$\mathrm{DL}_k\psi(x) := \int_{\partial D} \left(\partial_{\nu(y)}\Phi_k(x,y)\right)\psi(y)\mathrm{d}\mathcal{H}^2(y).$$

For the details of this theory we refer to [13].

We need the classical representation for the single layer operator when we prove the k-estimates for $\mathrm{S}_k : L^2(\partial D) \to H^1(\partial D)$. The idea is to express

$$\mathrm{S}_k = (\mathrm{S}_k - \mathrm{S}_0) + \mathrm{S}_0,$$

use [14, Theorem 3.6], by which $\mathrm{S}_0 : L^2(\partial D) \to H^1(\partial D)$ is continuous, and then prove an estimate for $\mathrm{S}_k - \mathrm{S}_0$.

In the proof of the lemma we will use the **surface gradient** Grad. If U is a neighbourhood of ∂D and $u \in C^1(U)$, the surface gradient of u can be defined as a projection of the gradient of u to the surface ∂D. If $\varphi \in C^1(\partial D)$ and ∂D is C^2 we can extend φ to a function in $C^1(U)$ and then define the surface gradient as the projection of this extended function. The surface gradient can also be extended to elements in $H^1(\partial D)$ in which case the norm $\|\varphi\|_{H^1(\partial D)}$ is equivalent to $\|\varphi\|_{L^2(\partial D)} + \|\mathrm{Grad}\varphi\|_{L^2(\partial D)}$.

An alternative approach to the surface gradient is to use the intrinsic geometry of the surface and then to define the surface gradient with the help of the first fundamental tensor g. In local coordinates, (A, ψ), the surface gradient of a function $u \in C^1(\partial D)$ is

$$\mathrm{Grad}u := \sum_{i,k=1}^{2} g^{ik} \frac{\partial u}{\partial x_i} \frac{\partial \psi^{-1}}{\partial x_k},$$

where (g^{ik}) is the inverse of the first fundamental tensor $g = (g_{ik})$.

Lemma 2.14 *Let $k \in \mathbb{R}$. Then there exists a $C > 0$, that depends on ∂D but not on k, such that*

$$\|\mathrm{S}_k - \mathrm{S}_0\|_{L^2(\partial D)\to H^1(\partial D)} \leq \frac{C}{2\pi}(|k| + 2k^2)\mathcal{H}^2(\partial D)\|\varphi\|_{L^2(\partial D)}.$$

Proof. We prove first that

$$\|\mathrm{S}_k - \mathrm{S}_0\|_{L^2(\partial D)\to L^2(\partial D)} \leq C|k|.$$

Let $\varphi \in L^2(\partial D)$. Then for all $x \in \partial D$ we have

$$(\mathrm{S}_k - \mathrm{S}_0)\,\varphi(x) = \int_{\partial D} \frac{e^{ik|x-y|} - 1}{4\pi|x-y|}\varphi(y)\mathrm{d}\mathcal{H}^2(y).$$

We have

$$\begin{aligned} \left|e^{ikr} - 1\right| &\leq \sup_{r\geq 0} |\partial_r e^{ikr}| r \\ &\leq |k| r. \end{aligned}$$

Hence

$$\left|\frac{e^{ik|x-y|} - 1}{4\pi|x-y|}\right| \leq \frac{|k|}{4\pi}$$

for all $x, y \in \mathbb{R}^3$ with $x \neq y$. We thus see that

$$\begin{aligned} |(\mathrm{S}_k - \mathrm{S}_0)\,\varphi(x)| &\leq \int_{\partial D} \left|\frac{k}{4\pi}\varphi(y)\right| \mathrm{d}\mathcal{H}^2(y) \\ &\leq \frac{|k|}{4\pi}\mathcal{H}^2(\partial D)^{\frac{1}{2}}\|\varphi\|_{L^2(\partial D)}. \end{aligned}$$

Hence

$$\| \left(\mathrm{S}_k - \mathrm{S}_0 \right) \varphi \|_{L^2(\partial D)} \leq \frac{|k|}{4\pi} \mathcal{H}^2(\partial D) \|\varphi\|_{L^2(\partial D)}. \tag{2.17}$$

We prove next that

$$\|\mathrm{Grad} \left(\mathrm{S}_k - \mathrm{S}_0 \right) \varphi\|_{L^2(\partial D)} \leq \frac{k^2}{2\pi} \mathcal{H}^2(\partial D) \|\varphi\|_{L^2(\partial D)}.$$

To this end we notice that

$$\|\mathrm{Grad} \left(\mathrm{S}_k - \mathrm{S}_0 \right) \varphi\|_{L^2(\partial D)} \leq \|\nabla_x \left(\mathrm{SL}_k - \mathrm{SL}_0 \right) \varphi\|_{L^2(\partial D)}. \tag{2.18}$$

For all $x \in \mathbb{R}^3$ we have

$$\begin{aligned} \left(\mathrm{SL}_k - \mathrm{SL}_0 \right) \varphi(x) &= \int_{\partial D} \frac{e^{ik|x-y|} - 1}{4\pi |x-y|} \varphi(y) \mathrm{d}\mathcal{H}^2(y) \\ &= \int_{\partial D} \frac{e^{ik|x-y|} - 1 - ik|x-y|}{4\pi |x-y|} \varphi(y) \mathrm{d}\mathcal{H}^2(y) \\ &\quad + \frac{ik}{4\pi} \int_{\partial D} \varphi(y) \mathrm{d}\mathcal{H}^2(y). \end{aligned} \tag{2.19}$$

For

$$P_k(x,y) = \frac{e^{ik|x-y|} - 1 - ik|x-y|}{4\pi |x-y|}$$

we have

$$\nabla_x P_k(x,y) = \left(-\frac{e^{ik|x-y|} - 1 - ik|x-y|}{4\pi |x-y|^2} + \frac{ike^{ik|x-y|} - ik}{4\pi |x-y|} \right) \widehat{(x-y)},$$

where $\widehat{x-y} = \frac{x-y}{|x-y|}$.

By Taylor's Theorem we have for all $r \geq 0$

$$\begin{aligned} |e^{ikr} - 1 - ikr| &\leq \sup_{r \geq 0} |\mathrm{d}_r^2 e^{ikr}| r^2 \\ &\leq k^2 r^2, \end{aligned}$$

so we see that for all $x \neq y$ we have

$$|\nabla_x P_k(x,y)| \leq \frac{k^2}{4\pi} + \frac{k^2}{4\pi}. \tag{2.20}$$

Hence $|\nabla_x P_k(x,y)| \varphi(y)$ is integrable over $y \in \partial D$ and by using Lebesgue's Dominated Convergence Theorem we can take the gradient inside the integral in (2.19), which yields

$$\nabla_x \left(\mathrm{S}_k - \mathrm{S}_0 \right) \varphi(x) = \int_{\partial D} \nabla_x P_k(x,y) \varphi(y) \mathrm{d}\mathcal{H}^2(y).$$

With (2.20) we see that for all $x \in \partial D$

$$\begin{aligned} |\nabla_x (\mathrm{S}_k - \mathrm{S}_0) \varphi(x)| &\leq \int_{\partial D} \left| \frac{k^2}{2\pi} \varphi(y) \right| \mathrm{d}\mathcal{H}^2(y) \\ &\leq \frac{k^2}{2\pi} \mathcal{H}^2(\partial D)^{\frac{1}{2}} \|\varphi\|_{L^2(\partial D)}. \end{aligned}$$

Hence

$$\|\nabla_x (\mathrm{S}_k - \mathrm{S}_0) \varphi\|_{L^2(\partial D)} \leq \frac{k^2}{2\pi} \mathcal{H}^2(\partial D) \|\varphi\|_{L^2(\partial D)}.$$

We combine this with equations (2.17) and (2.18) and see that

$$\begin{aligned} \|(\mathrm{S}_k - \mathrm{S}_0) \varphi\|_{H^1(\partial D)} &\leq C \left(\|(\mathrm{S}_k - \mathrm{S}_0) \varphi\|_{L^2(\partial D)} + \|\mathrm{Grad}\,(\mathrm{S}_k - \mathrm{S}_0) \varphi\|_{L^2(\partial D)} \right) \\ &\leq \frac{C}{2\pi} (|k| + 2k^2) \mathcal{H}^2(\partial D) \|\varphi\|_{L^2(\partial D)}. \end{aligned}$$

□

Corollary 2.15 *For all $k \in \mathbb{R}$ and $s \in [-1, 0]$ the single layer operator $\mathrm{S}_k : H^s(\partial D) \to H^{s+1}(\partial D)$ is continuous and there exists a positive constant C that depends on ∂D but not on k nor s, such that*

$$\|\mathrm{S}_k\|_{H^s(\partial D) \to H^{s+1}(\partial D)} \leq C(1 + k^2).$$

In addition, for all k we have $\mathrm{S}_k^* = \mathrm{S}_{-k}$, where the dual is taken with respect to the $H^{-s} \times H^s$ duality.

Proof. By [14, Theorem 3.6] operator $\mathrm{S}_0 : L^2(\partial D) \to H^1(\partial D)$ is bounded. By Lemma 2.14 we have

$$\begin{aligned} \|\mathrm{S}_k\|_{L^2(\partial D) \to H^1(\partial D)} &\leq \|\mathrm{S}_k - \mathrm{S}_0\|_{L^2(\partial D) \to H^1(\partial D)} + \|\mathrm{S}_0\|_{L^2(\partial D) \to H^1(\partial D)} \\ &\leq C(1 + k^2). \end{aligned}$$

From this it follows that $\mathrm{S}_k^* : H^{-1}(\partial D) \to L^2(\partial D)$ is continuous and that

$$\|\mathrm{S}_k^*\|_{H^{-1}(\partial D) \to L^2(\partial D)} \leq C(1 + k^2). \tag{2.21}$$

We have

$$\begin{aligned} \mathrm{S}_k^* &= (\mathrm{tr}\, \mathcal{G}_k \mathrm{tr}^*)^* \\ &= \mathrm{tr}\, \mathcal{G}_{-k} \mathrm{tr}^* \\ &= \mathrm{S}_{-k}. \end{aligned}$$

Hence by inequality (2.21)

$$\begin{aligned}\|\mathrm{S}_k\|_{H^{-1}(\partial D)\to L^2(\partial D)} &= \|\left(\mathrm{S}_{-k}\right)^*\|_{H^{-1}(\partial D)\to L^2(\partial D)} \\ &\leq C(1+k^2). \qquad (2.22)\end{aligned}$$

By interpolation, [39, Theorem 1.5.1.], we see that for all $k \in \mathbb{R}$ and $s \in [-1,0]$

$$\|\mathrm{S}_k\|_{H^{-1}(\partial D)\to L^2(\partial D)} \leq C(1+k^2).$$

□

In the interior of the domain the single and double layer potentials are much smoother, as we will see in the next lemma.

Lemma 2.16 *Let $k \in \mathbb{R}$, $x \in D^+$ and $\epsilon < d(x, \partial D)$. Then for all $m \in \mathbb{N}$ the operators $\mathrm{SL}_k, \mathrm{DL}_k : L^2(\partial D) \to C^m(B(x,\epsilon))$ are bounded. In addition for bounded and open $U \subset D^+$ with $\overline{U} \subset D^+$, the operators $\mathrm{SL}_k, \mathrm{DL}_k : L^2(\partial D) \to C^m(U)$ are continuous.*

Proof. Let $\epsilon' > \epsilon$ be such that $\epsilon' < d(x, \partial D)$. The kernels of SL and DL, $\Phi_k(x,y)$ and $\partial_{\nu(y)}\Phi(x,y)$ respectively, are C^∞ from $\partial D \times B(x,\epsilon')$ to $\mathbb{C}$ and $\partial D \times \overline{B(x,\epsilon)}$ is compact. Hence we see that $\mathrm{SL}_k, \mathrm{DL}_k : L^2(\partial D) \to C^m(B(x,\epsilon))$ are continuous.

The continuity of $\mathrm{SL}_k, \mathrm{DL}_k : L^2(\partial D) \to C^m(U)$ follows from the compactness of $\overline{U}$ and from the first part.

□

Chapter 3

The Dirichlet Boundary Conditions

3.1 The Frequency Domain

In the frequency domain we consider oscillations with a fixed frequency. These waves satisfy the **Helmholtz equation**

$$-(\Delta + k^2)u = 0. \tag{3.1}$$

Under the Dirichlet boundary conditions it is assumed that the waves are zero on the boundary of the scattering obstacle, that is $u|_{\partial D} = 0$.

The scattering is described by assuming an incident wave u_i, which is an empty space solution and satisfies (3.1) on all of $\mathbb{R}^3$, and finding the scattered wave u_{sc} that satisfies the exterior problem

$$\begin{cases} -(\Delta + k^2)u_{sc} &= 0 \quad \text{in } D^+ \\ u_{sc}|_{\partial D} &= h \\ \lim_{r\to\infty} r\,(\partial_r u_{sc} - iku_{sc}) &= 0 \quad \text{uniformly w.r.t. } \widehat{x} \in \mathbb{S}^2, \end{cases} \tag{3.2}$$

where $h = -u_i|_{\partial D}$. The total wave $u_{\text{tot}} = u_i + u_{sc}$ satisfies $u_{\text{tot}} = 0$ on the boundary ∂D.

The the exterior solution of (3.2) is found as a weak solution, that is u_{sc} is an element in $H^1_{\ell oc}(D^+)$ that satisfies $-(\Delta + k^2)u_{sc} = 0$ on D^+ in the sense of distributions, and $\operatorname{tr}_{\partial D} u_{sc} = -\operatorname{tr}_{\partial D} u_i$. Since u_{sc} satisfies $-(\Delta + k^2)u = 0$ in D^+ it is in fact a real analytic function on this domain. Hence Sommerfeld's radiation condition

$$\lim_{r\to\infty} r\,(\partial_r u_{sc} - iku_{sc}) = 0$$

can be understood to refer to the point-wise values of u_{sc}.

Theorem 3.1 *For all $h \in H^{\frac{1}{2}}(\partial D)$ the problem (3.2) has a unique solution and the solution operator $U_k : H^{\frac{1}{2}}(\partial D) \to H^1_{\ell oc}(D^+)$ is continuous.*

Proof. We refer to [40, Theorem 9.11].

□

We also occasionally need the interior Dirichlet solutions, which satisfy

$$\begin{cases} -(\Delta + k^2)u &= 0 \quad \text{in } D \\ u|_{\partial D} &= h. \end{cases} \tag{3.3}$$

With certain values of the k there are non-zero solutions to the problem (3.3) with boundary values $h = 0$. These values of k are referred to as the **Dirichlet eigenvalues of $-\Delta$ in D**. If k is not one of these eigenvalues, then there exists a unique solution u_h that satisfies

$$\|u_h\|_{H^1(D)} \leq C\|h\|_{H^{\frac{1}{2}}(\partial D)},$$

where the constant C depends on k and the domain D. Since the the interior and exterior solutions satisfy $-(\Delta + k^2)u = 0$ in the interior or exterior domains, the normal derivatives can be defined; see page 18.

Definiton 3.2 *Let $h \in H^{\frac{1}{2}}(\partial D)$ and v and u be solutions of (3.3) and (3.2) with the boundary value h respectively. The* **Dirichlet-to-Neumann maps** *$\Lambda_{k,-}$ and $\Lambda_{k,+}$ are defined by*

$$\begin{aligned} \Lambda_{k,-}h &:= \partial_\nu^- v \\ \Lambda_{k,+}h &:= \partial_\nu^+ u. \end{aligned} \tag{3.4}$$

In the interior case we have for all $\psi, \varphi \in H^{\frac{1}{2}}(\partial D)$ by the definition of the normal derivative (2.5)

$$(\Lambda_{k,-}\psi, \varphi)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}(\partial D)} = \int_D \left((\overline{-k^2 v_\psi}) v_\varphi + \overline{\nabla u_\psi} \cdot \nabla u_\varphi \right) \mathrm{d}x,$$

where u_ψ and u_φ are solutions of (3.3) with $\operatorname{tr} u_\psi = \psi$ and $\operatorname{tr} u_\varphi = \varphi$ respectively. We see that

$$(\Lambda_{k,-}\psi, \varphi)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}(\partial D)} = (\psi, \Lambda_{k,-}\varphi)_{H^{\frac{1}{2}} \times H^{-\frac{1}{2}}(\partial D)}$$

and refer to this fact by saying that the map $\Lambda_{k,-}$ is self-adjoint.

3.2 The Time Domain – the Energy of a Solution

Let us consider the energy of a wave u. If $u \in C^\infty(\mathbb{R}^4)$, $\Box u = 0$ and U is a bounded domain with a C^1 boundary, we have

$$\begin{aligned} 0 &= \int_U \left((\partial_t^2 - \Delta)u\right) \partial_t u \mathrm{d}x \\ &= \int_U \left(\frac{1}{2}\partial_t(\partial_t u)^2 - \sum_{i=1}^{3}(\partial_{x_i}^2 u)\partial_t u\right) \mathrm{d}x. \end{aligned}$$

In order to write $(\partial_{x_i}^2 u)\partial_t u$ as a time derivative of something else we notice that

$$\partial_{x_i}(\partial_{x_i} u)\partial_t u = (\partial_{x_i}^2 u)\partial_t u + (\partial_{x_i} u)\partial_{x_i}\partial_t u \tag{3.5}$$

and

$$\partial_t(\partial_{x_i} u)^2 = 2(\partial_{x_i} u)\partial_t\partial_{x_i} u. \tag{3.6}$$

Hence

$$\begin{aligned} -\int_U (\partial_{x_i}^2 u)\partial_t u \mathrm{d}x &= -\int_U \left(\partial_{x_i}((\partial_{x_i} u)\partial_t u) - \frac{1}{2}\partial_t(\partial_{x_i} u)^2\right) \mathrm{d}x \\ &= \partial_t \frac{1}{2}\int_U (\partial_{x_i} u)^2 \mathrm{d}x - \int_{\partial U} \nu_i(\partial_{x_i} u \partial_t u)\mathrm{d}\mathcal{H}^2 \end{aligned}$$

and we see that

$$\partial_t \frac{1}{2}\int_U \left((\partial_t u)^2 + |\nabla u|^2\right) \mathrm{d}x = \int_{\partial U} (\partial_\nu u)\partial_t u \mathrm{d}\mathcal{H}^2.$$

If $u(t,\cdot)$ is compactly supported and $\mathrm{supp}(u(t,\cdot)) \subset U$ for all $t \in [t_1, t_2]$, then the quantity

$$\mathcal{E}(u, U, t) := \frac{1}{2}\int_U \left((\partial_t u)^2 + |\nabla u|^2\right) \mathrm{d}x \tag{3.7}$$

is a constant for $t \in [t_1, t_2]$ We call this constant the energy of the wave u in the volume U. If $\mathrm{supp}(u) \not\subset U$ then it is possible that there is a flux of energy

$$\partial_t \mathcal{E}(u, U, t) = \int_{\partial U} (\nu \cdot \nabla u)\partial_t u \mathrm{d}\mathcal{H}^2$$

through the boundary of U. We hence see that $-(\nabla u)\partial_t u$ is a vector that points where the energy of the wave u is going and whose length gives the density of the energy flux.

In the case of a Dirichlet problem the wave satisfies

$$u \equiv 0$$

on the boundary ∂D, so we see that the flux through ∂D is zero and hence the energy of the wave, $\mathcal{E}(u, D^+, t)$, is conserved. We will use the energy of a solution to establish certain norm estimates. For this reason we define

$$E(u, U, t) := \mathcal{E}(u, U, t)^{\frac{1}{2}}.$$

3.3 The Time Domain Dirichlet Problem

We consider the solutions of the **time domain Dirichlet Problem**

$$\begin{cases} \Box u &= 0 \quad \text{in } \mathbb{R}_+ \times D^+ \\ u|_{t=0} &= f_1 \\ u_t|_{t=0} &= f_2 \\ u|_{\partial D \times \{\mathbb{R}_+ \cup \{0\}\}} &= 0. \end{cases} \tag{3.8}$$

We need a wider class of solutions than the twice differentiable ones. A natural extension to the class of solutions is gained by using the energy (3.7) of the solution. Hence we define the H norm of the initial data as

$$\|(f_1, f_2)\|_H = \left(\int_{D^+} \left(|\nabla f_1|^2 + |f_2|^2 \right) dx \right)^{\frac{1}{2}}$$

and the space $H(D^+)$ to be the completion of $C_0^\infty(D^+) \times C_0^\infty(D^+)$ with respect to the norm $\|\cdot\|_H$.

The space $H(D^+) = H_1(D^+) \times H_2(D^+)$ is a Cartesian product of the spaces $H_1(D^+)$ and $H_2(D^+)$. The space $H_2(D^+)$ is the completion of $C_0^\infty(D^+)$ with respect to the L^2 norm, which produces $L^2(D^+)$.

Since we need the space H_1 in the case of the Robin problem in a slightly modified form, let us define the H_1 space in a general manner. Let U be an open set. The space $H_1(U)$ is the completion of $C_0^\infty(U)$ with respect to the Dirichlet norm $\|\cdot\|_D$ defined by

$$\|u\|_{D(U)} := \|\nabla u\|_{L^2(U)}.$$

Hence $H_1(U)$ is the set of Cauchy sequences

$$H_1(U) = \{(\varphi_n)_{n \in \mathbb{N}^+} : \varphi_n \in C_0^\infty(U), (\varphi_n) \text{ is a Cauchy seq. w.r.t. } \|\cdot\|_D\}$$

equipped with the norm

$$\|(\varphi_n)\|_D := \lim_{n\to\infty} \|\varphi_n\|_D.$$

There is a clearer and more compact way to look at $H_1(U)$, which we unfold in the next two lemmas.

Lemma 3.3 (Poincare's Inequality) *Let $U \subset \mathbb{R}^n$ be an open set and $\varphi \in C_0^\infty(U)$. Then for all $R > 0$*

$$\|\varphi\|_{L^2(B(0,R)\cap U)} \leq 2^{-\frac{1}{2}} R \|\nabla\varphi\|_{L^2(U)}. \tag{3.9}$$

Proof. The proof is analogous to the proof of [46, Lemma IV.1.1].

□

Lemma 3.3 gives us a way to embed $H_1(U)$ into $L^2_{\ell oc}(U)$:

Lemma 3.4 *Let $U \subset \mathbb{R}^3$ be an open set, $\varphi \in C_0^\infty(U)$, $(\varphi) = (\varphi, \varphi, \dots) \in H_1(U)$ and the map I defined by $I((\varphi)) = \varphi$. The map I can be continuously extended into a map $I : H_1(U) \to L^2_{\ell oc}(U)$. In addition for all $u \in H_1(U)$ we have $\nabla I(u) \in L^2_{\ell oc}(U)$, $I : H_1(U) \to (\mathrm{Ran}\,(I)\,, \|\cdot\|_D)$ is an isometry, and for all $u \in \mathrm{Ran}\,(I)$ and $R > 0$ the inequality*

$$\|u\|_{L^2(B(0,R)\cap U)} \leq 2^{-\frac{1}{2}} R \|\nabla u\|_{L^2(U)}.$$

holds.

Proof. Let $(\varphi_n) \in H_1(U)$. It follows from Lemma 3.3 that for all $R > 0$ the sequence (φ_n) is a Cauchy sequence in $L^2(B(0,R)\cap U)$ and hence converges to some $v \in L^2_{\ell oc}(U)$.

As a distribution

$$v = \lim_{n\to\infty} \varphi_n$$

That is, for all $\varphi \in C_0^\infty(U)$

$$(v, \varphi)_{\mathcal{D}'\times\mathcal{D}(U)} = \lim_{n\to\infty} (\varphi_n, \varphi)_{\mathcal{D}'\times\mathcal{D}(U)}\,.$$

We have for all $i \in \{1, 2, 3\}$

$$\begin{aligned}(\partial_{x_i} v, \varphi)_{\mathcal{D}'\times\mathcal{D}(U)} &= \lim_{n\to\infty} (\partial_{x_i}\varphi_n, \varphi)_{\mathcal{D}'\times\mathcal{D}(U)} \\ &= (v_i, \varphi)_{\mathcal{D}'\times\mathcal{D}(U)}\,,\end{aligned}$$

where $v_i \in L^2(U)$ since (φ_n) is a Cauchy sequence with respect to the norm $\|\cdot\|_D$. Hence $v \in L^2_{\ell oc}(U)$, $\nabla v \in L^2(U)$, $\varphi \to v$ with respect to the norm $\|\cdot\|_D$ and for all $R > 0$

$$\begin{aligned}\|v\|_{L^2(B(0,R)\cap U)} &= \lim_{n\to\infty} \|\varphi_n\|_{L^2(B(0,R)\cap U)} \\ &\le \lim_{n\to\infty} 2^{-\frac{1}{2}} R \|\varphi_n\|_{D(U)} \\ &= 2^{-\frac{1}{2}} R \|v\|_{D(U)}. \end{aligned} \tag{3.10}$$

In $L^2_{\ell oc}(U)$ the semi-norm $\|\cdot\|_D$ is not a norm since for a constant function c, $c(x) = c$, we have $\|c\|_D = 0$. In $\mathrm{Ran}\,(I)$, however, equation (3.10) holds and hence for all $v \in \mathrm{Ran}\,(T)$ the condition $\|v\|_D = 0$ implies that $\|v\|_{L^2(D_R)} = 0$ for all R. From this it follows that $v = 0$ and hence $\|\cdot\|_D$ is a norm on $\mathrm{Ran}\,(I)$.

To see that I is an isometry we notice that for all $(\varphi_n) \in H_1(D^+)$ we have

$$\begin{aligned}\|I((\varphi_n))\|_D &= \lim_{n\to\infty} \|\varphi_n\|_D \\ &= \|(\varphi_n)\|_{H_1(U)}.\end{aligned}$$

□

In the future we identify $u \in H_1(U)$ and $I(u)$ and state that by Poincare's inequality we have for all $u \in H_1(U)$ and $R > 0$

$$\|u\|_{L^2(D_R)} \le 2^{-\frac{1}{2}} R \|u\|_D.$$

In passing we notice that the above scheme fails in dimension 2, since the completion of $C_0^\infty(D^+)$ with respect to the Dirichlet norm is not contained in the class of distributions. An example of this phenomenon made by Soga can be found e.g. in [45, p.30].

We return now to the time domain problem (3.8). By [45, Theorem 7.1.2] the problem has a solution in the following sense:

Definiton 3.5 *Let $(f_1, f_2) \in H(D^+)$. Distribution $u(t,x) \in \mathcal{D}'(\mathbb{R} \times D^+)$ is a solution of (3.8) if*

(i) For all $t \in \mathbb{R}_+ \cup \{0\}$ we have $(u(t,\cdot), \partial_t u(t,\cdot)) \in H(D^+)$ and the functions

$$t \to \nabla_x u(t,\cdot), \quad t \to \partial_t u(t,\cdot)$$

are continuous from $\mathbb{R}_+ \cup \{0\}$ to $L^2(D^+)$.

(ii) $(u(0,\cdot), \partial_t u(0,\cdot)) = (f_1, f_2)$.

(iii) $\Box u = 0$ *in* $\mathbb{R}_+ \times D^+$ *in the sense of distributions.*

By Lemma 3.4 we know that for all $t \in [0,\infty)$ we also have $u(t,\cdot) \in L^2_{\ell oc}(D^+)$ and that the inequality

$$\|u(t,\cdot)\|_{L^2(D_R)} \leq 2^{-\frac{1}{2}} R \|u(t,\cdot)\|_D. \tag{3.11}$$

holds. This will be useful with the following class of scatterers.

Definition 3.6 *An obstacle D is said to have Dirichlet (respectively Robin)* **local energy decay** *if it has the property that every solution to the problem (3.8) (respectively (1.4)) with $E(u, D^+, 0) < \infty$ and compactly supported initial data f_1, f_2, satisfies for all $R > 0$*

$$E(u, D_R, t) \leq C(R) e^{-\delta t} E(u, D^+, 0), \tag{3.12}$$

for some positive constants $C(R)$ and δ which depend only on the shape of the obstacle D, R and the supports of f_1 and f_2.

It has been proved, [41, The Main Theorem], that D has Dirichlet local energy decay if the obstacle D is such that there exists a convex function on D^+ with a positive normal derivative on ∂D, and this function is equal to the distance from D sufficiently far from the origin. If D is convex and $0 \in D$ then the function $|x|$ satisfies these conditions.

In order to use the partial Fourier transformation $\mathcal{F}^{-1}_{t\to k}$ we need to extend the solution u to negative time.

Lemma 3.7 *Let $\widetilde{u}$ be a solution to the exterior Dirichlet problem (3.8). Then the extension u defined by*

$$u(t,\cdot) := \begin{cases} \widetilde{u}(t,\cdot) & ; t \geq 0, \\ 0 & ; t < 0 \end{cases} \tag{3.13}$$

satisfies as a distribution in $\mathcal{S}'(\mathbb{R} \times D^+)$

$$\Box u = \delta(t) \otimes f_2 - \delta'(t) \otimes f_1.$$

Proof. Let $(f_1, f_2) \in H(D^+)$ have bounded supports, $\widetilde{u}$ the solution of (3.8) with the initial data (f_1, f_2) and $(f_{1,i}, f_{2,i}) \in C_0^\infty(D^+) \times C_0^\infty(D^+)$ be such that $(f_{1,i}, f_{2,i}) \to (f_1, f_2)$ in $H(D^+)$. For $i \in \mathbb{N}_+$ let $\widetilde{u}_i$ be the solution of

(3.8) with initial data $(f_{1,i}, f_{2,i})$ and u_i be the corresponding zero extension. By [17, Theorem 7.2.7] we have $u_i \in C^\infty(\overline{\mathbb{R}_+} \times D^+)$ for all $i \in \mathbb{N}$ and $(u_i(t,\cdot), \partial_t u_i(t,\cdot)) \to (u(t,\cdot)\partial_t u(t,\cdot))$ in $H(D^+)$ uniformly with respect to t. It follows from Poincare's Inequality, inequality (3.11), and the conservation of energy that $u_i \to u$ in $\mathcal{S}'(\mathbb{R} \times D^+)$.

We integrate by parts and see that for all $\varphi \in \mathcal{S}(\mathbb{R} \times D^+)$

$$\begin{aligned}(\partial_t^2 u_i, \varphi)_{\mathcal{S}' \times \mathcal{S}(\mathbb{R}\times D^+)} &= (u_i, \partial_t^2 \varphi)_{\mathcal{S}' \times \mathcal{S}(\mathbb{R}\times D^+)} \\ &= -\int_{D^+} \overline{\widetilde{u}_i(0,x)} \partial_t \varphi(0,x) \mathrm{d}x + \int_{D^+} \overline{\partial_t \widetilde{u}_i(0,x)} \varphi(0,x) \mathrm{d}x + \\ &\quad + \int_0^\infty \int_{D^+} \overline{\partial_t^2 \widetilde{u}_i(t,x)} \varphi(t,x) \mathrm{d}x \mathrm{d}t. \end{aligned} \tag{3.14}$$

We have

$$\int_0^\infty \int_{D^+} \overline{\partial_t^2 \widetilde{u}_i(t,x)} \varphi(t,x) \mathrm{d}x \mathrm{d}t = (\Delta u_i, \varphi)_{\mathcal{S}' \times \mathcal{S}(\mathbb{R}\times D^+)} .$$

Hence it follows from (3.14) that

$$\Box u_i = \delta \otimes f_{2,i} - \delta' \otimes f_{1,i}$$

and

$$\begin{aligned}\Box u &= \lim_{i\to\infty} \Box u_i \\ &= \delta \otimes f_2 - \delta' \otimes f_1 .\end{aligned}$$

□

Hence by Lemma 3.7 the extended element u satisfies all in all

$$\begin{cases} \Box u(t,x) &= \delta'(t) \otimes f_1 + \delta(t) \otimes f_2 \quad \text{in } \mathbb{R} \times D^+ \\ u|_{\partial D \times \mathbb{R}_+} &= 0 \\ u &\equiv 0 \qquad\qquad t < 0, \end{cases} \tag{3.15}$$

where we understand the solution to have the properties (i) and (ii) of Definition 3.5 for $t \geq 0$ and the equality $\Box u(x,t) = \delta \otimes f_2 - \delta' \otimes f_1$ to be in the sense of distributions in $\mathbb{R} \times D^+$.

The Fourier transformation $\mathcal{F}^{-1}_{t\to k}\{u\}$ can also be expressed as a Bochner integral. This will enable us to refer to the values $\mathcal{F}^{-1}_{t\to k}\{u\}(k,\cdot)$ and it will also be useful later in deriving a boundary source problem in the frequency domain.

Lemma 3.8 *Let the obstacle D have the Dirichlet energy decay and u be a solution of (3.15). Then for all $R \in \mathbb{R}_+$, such that $\overline{D} \subset B(0,R)$, the map $T_k : [0,\infty) \to H^1(D_R); t \to e^{ikt}u(t,\cdot)$ is Bochner integrable. In addition*

$$v(k,\cdot) = (2\pi)^{-\frac{1}{2}} \int_0^\infty e^{ikt}u(t,\cdot)\,dt$$

satisfies $\mathcal{F}^{-1}_{t\to k}\{u\} = v$, when we understand v to be the distribution defined by

$$(v,\varphi)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times D^+)} = \int_{\mathbb{R}} \int_{D^+} \overline{v(k,x)}\varphi(k,x)\,dx\,dk. \tag{3.16}$$

Proof. Let $R \in \mathbb{R}$ be such that $\overline{D} \subset B(0,R)$. The map $T_k : [0,\infty) \to H^1(D_R); t \to e^{ikt}u(t,\cdot)$ is continuous and $H^1(D_R)$ is separable. In addition it follows from the decay estimate (3.12) and Poincare's Inequality, inequality (3.11), that for all R there exists constants $C(R)$ and c such that

$$\|u(t,\cdot)\|_{L^2(D_R)} \leq C(R)e^{-ct}. \tag{3.17}$$

Hence there exists a constant $C_{\text{int}}(R)$ depending on R such that

$$\int_0^\infty \|e^{ikt}u(t,\cdot)\|_{L^2(D_R)}\mathrm{d}t \leq C_{\text{int}}(R) \tag{3.18}$$

and by Lemma B.4 the map T_k is Bochner integrable.

It follows from the estimate (3.18) that for all k and R

$$\|v(k,\cdot)\|_{L^2(D_R)} \leq C_{\text{int}}(R)$$

Hence v defined by equation (3.16) belongs to $\mathcal{S}'(\mathbb{R}\times D^+)$.

Now let $\varphi \in \mathcal{S}(\mathbb{R}\times D^+)$, R be so large that for all $t \in \mathbb{R}$ we have $\operatorname{supp}(\varphi(t,\cdot)) \subset B(0,R)$ and χ_A be the characteristic function of set $A \subset \mathbb{R}$. We consider a sequence of simple functions; let $n \in \mathbb{N}_+, \ell \in \mathbb{N}$, $t_{n,\ell} = \frac{\ell}{n}$, $\Delta_{n,\ell} = [t_{n,\ell-1}, t_{n,\ell})$ and $P_n(t) = \sum_{\ell=1}^{2^n} t_{n,\ell}\chi_{\Delta_{n,\ell}}(t)$. The simple function

$$s_n(k,t,\cdot) := \sum_{\ell=1}^{2^n} e^{ikP_n(t)}u(P_n(t),\cdot)\chi_{\Delta_{n,\ell}}(t)$$

satisfies by (3.17)

$$\|s_n(k,t,\cdot)\|_{H^1(D_R)} \leq Ce^{-ct}.$$

It follows from the property (i) of Definition 3.5, and inequality (3.11), that $s_n(k,t,\cdot)$ converges pointwise to $e^{ikt}u(t,\cdot)$ in $L^2(D_R)$ as $n \to \infty$. Hence

Lebesgue's Dominated Convergence Theorem, in the form Theorem B.5, implies that the Bochner integral

$$\int_0^\infty e^{ikt}u(t,\cdot)\mathrm{d}t = \lim_{n\to\infty}\int_0^\infty s_n(k,t,\cdot)\mathrm{d}t.$$

Let

$$\begin{aligned} S_n(k,\cdot) &= \int_0^\infty s_n(k,t.\cdot)\mathrm{d}t \\ &= \sum_{\ell=1}^{2^n} e^{ik\frac{\ell}{n}}u(\frac{\ell}{n},\cdot)\frac{1}{n} \end{aligned} \tag{3.19}$$

and

$$S(k,\cdot) = \int_0^\infty e^{ikt}u(t,\cdot)\mathrm{d}t.$$

We prove next that for all $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that for all $n, m > n_\epsilon$ and $k \in \mathbb{R}$ we have

$$\|S_n(k,\cdot) - S_m(k,\cdot)\|_{L^2(D_R)} \le \epsilon\langle k\rangle. \tag{3.20}$$

Let $\epsilon > 0$. From equation (3.17) it follows that there is $t_\epsilon \in \mathbb{R}_+$ such that

$$\begin{aligned} \int_{t_\epsilon}^\infty \|u(t,\cdot)\|_{L^2(D_R)}\mathrm{d}t &\le \int_{t_\epsilon}^\infty C(R)e^{-ct}\mathrm{d}t \\ &< \frac{\epsilon}{4}. \end{aligned}$$

Since for all $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$ it holds that $P_n(t) \ge t$, we see that

$$\int_{t_\epsilon}^\infty \|s_n(k,t,\cdot)||_{L^2(D_R)}\mathrm{d}t < \frac{\epsilon}{4} \tag{3.21}$$

and we can concentrate to the interval $[0, t_\epsilon]$ in proving the estimate (3.20).

The function $t \to u(t,\cdot) : \mathbb{R} \to L^2(D_R)$ is continuous, so it is uniformly continuous on the interval $[0, t_\epsilon]$. Hence there exists δ_ϵ such that for all $t_1, t_2 \in [0, t_\epsilon]$ with $|t_1 - t_2| < \delta_\epsilon$ we have

$$\|u(t_1,\cdot) - u(t_2,\cdot)\|_{L^2(D_R)} < \frac{\epsilon}{4t_\epsilon}.$$

Let $n, m \in \mathbb{N}$ be such that $\frac{1}{m}, \frac{1}{n} < \delta_\epsilon$. Then for all $t \in [0, t_\epsilon]$ we have

$$\begin{aligned} \|s_n(k,t,\cdot) - s_m(k,t,\cdot)\|_{L^2(D_R)} &\le \|(e^{ikP_n(t)} - e^{ikP_m(t)})u(P_n(t),\cdot)\|_{L^2(D_R)} + \\ &\quad \|e^{ikP_m(t)}(u(P_n(t),\cdot) - u(P_m(t),\cdot))\|_{L^2(D_R)} \\ &\le 2|k|\delta_\epsilon C + \frac{\epsilon}{4t_\epsilon}, \end{aligned}$$

where C is the constant in inequality (3.17).
Let us demand that $\delta_\epsilon < \frac{\epsilon}{8Ct_\epsilon}$ so that

$$\|s_n(k,t,\cdot) - s_m(k,t,\cdot)\|_{L^2(D_R)} < \frac{\epsilon}{2t_\epsilon}\langle k\rangle$$

and

$$\int_0^{t_\epsilon} \|s_n(k,t,\cdot) - s_m(k,t,\cdot)\|_{L^2(D_R)}\mathrm{d}t \le \frac{\epsilon}{2}. \tag{3.22}$$

It follows from (3.22) and (3.21) that

$$\begin{aligned}
\|S_n(k,\cdot)\|_{L^2(D_R)} &\le \int_0^\infty \|s_n(k,t,\cdot) - s_m(k,t,\cdot)\|_{L^2(D_R)}\mathrm{d}t \\
&\quad \int_0^{t_\epsilon} \|s_n(k,t,\cdot) - s_m(k,t,\cdot)\|_{L^2(D_R)}\mathrm{d}t + \\
&\quad \int_{t_\epsilon}^\infty \|s_n(k,t,\cdot)\|_{L^2(D_R)}\mathrm{d}t + \int_{t_\epsilon}^\infty \|s_m(k,t,\cdot)\|_{L^2(D_R)}\mathrm{d}t \\
&\le \epsilon\langle k\rangle.
\end{aligned}$$

It follows from inequality (3.20) that for $n > n_\epsilon$

$$\|S_n(k,\cdot) - S(k,\cdot)\|_{L^2(D_R)} \le \epsilon\langle k\rangle \tag{3.23}$$

also holds for all $k \in \mathbb{R}$.
As $\varphi \in \mathcal{S}(\mathbb{R}\times D^+)$ there exists $C_\varphi \in \mathbb{R}_+$ such that

$$\|\varphi(k,\cdot)\|_{L^2(D_R)} \le C_\varphi\langle k\rangle^{-3}.$$

We have

$$\begin{aligned}
I_n &= \left|\int_{\mathbb{R}\times D^+} \overline{S_n(k,x)}\varphi(k,x)\mathrm{d}x\mathrm{d}t - \int_{\mathbb{R}\times D^+} \overline{S(k,x)}\varphi(k,x)\mathrm{d}x\mathrm{d}t\right| \\
&\le \int_{\mathbb{R}\times D^+} \left|(\overline{S_n(k,x) - S(k,x)})\varphi(k,x)\right| \mathrm{d}x\mathrm{d}k
\end{aligned}$$

Since $\mathrm{supp}(\varphi) \subset B(0,R)$ we have by Hölder's inequality

$$I_n \le \int_{\mathbb{R}} \|S_n(k,\cdot) - S(k,\cdot)\|_{L^2(D_R)}\|\varphi(k,\cdot)\|_{L^2(D_R)}\mathrm{d}k$$

Let now $\widetilde{\epsilon} = \frac{\epsilon}{C_\varphi\int_0^\infty\langle k\rangle^{-2}\mathrm{d}k}$. By equation (3.23) we have for $n > n_{\widetilde{\epsilon}}$

$$I_n \le \epsilon.$$

Hence

$$\begin{aligned}(v,\varphi)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times D^+)} &= (2\pi)^{-\frac{1}{2}}\lim_{n\to\infty}\int_{\mathbb{R}\times D^+}\overline{S_n(k,x)}\varphi(k,x)\mathrm{d}x\mathrm{d}k\\ &= (2\pi)^{-\frac{1}{2}}\lim_{n\to\infty}\int_{\mathbb{R}\times D^+}\sum_{\ell=1}^{2^n}\overline{e^{ik\frac{\ell}{n}}u(\frac{\ell}{n},x)}\frac{1}{n}\varphi(k,x)\mathrm{d}x\mathrm{d}k.\end{aligned}$$

By Fubini's Theorem

$$(v,\varphi)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times D^+)} = \lim_{n\to\infty}\sum_{\ell=1}^{2^n}\int_{D^+}\frac{1}{n}\overline{u(\frac{\ell}{n},x)}\mathcal{F}_{k\to t}\{\varphi\}(\frac{\ell}{n},x)\mathrm{d}x.$$

It follows from the energy decay (3.17) that for all $\epsilon>0$ there exists a $T_\epsilon\in\mathbb{R}_+$, that depends on the constants C and c of equation (3.17), such that for all $t>T_\epsilon$ and a $c_1>0$

$$\|\overline{u(t,\cdot)}\mathcal{F}_{k\to t}\{\varphi\}(t,\cdot)\|_{L^1(D_R)}\le\frac{\epsilon}{2}c_1e^{-c_1t}.$$

Hence for all n

$$\begin{aligned}\sum_{\ell>nT_\epsilon}^{\ell=2^n}\int_D\left|\frac{1}{n}\overline{u(\frac{\ell}{n},x)}\mathcal{F}_{k\to t}\{\varphi\}(\frac{\ell}{n},x)\right|\mathrm{d}x &\le \int_0^\infty\frac{\epsilon}{2}c_1e^{-c_1t}\mathrm{d}t\\ &= \frac{\epsilon}{2}.\end{aligned}\tag{3.24}$$

Likewise

$$\int_{T_\epsilon}^\infty\int_{D_R}\left|\overline{u(t,x)}\mathcal{F}_{k\to t}\{\varphi\}(t,x)\right|\mathrm{d}x\mathrm{d}t\le\frac{\epsilon}{2}.\tag{3.25}$$

On the interval $t\in[0,T_\epsilon]$

$$u_n(t,\cdot):=\sum_{\ell=1}^{2^n}\overline{u(P_n(t),\cdot)}\chi_{\Delta_{n,i}(t)}$$

converges uniformly to $u(t,\cdot)$ in the $L^2(D_R)$ norm, so for every $\eta>0$ there exists a m_η such that for all $n>m_\eta$ and $t\in[0,T_\epsilon+1]$ we have

$$\|u_n(t,\cdot)-u(t,\cdot)\|_{L^2(D_R)}<\frac{\eta}{(T_\epsilon+1)(1+\sup_{t\in[0,T_\epsilon+1]}\{\|\mathcal{F}_{k\to t}\{\varphi\}(t,\cdot)\|_{L^2(D_R)}\})}.\tag{3.26}$$

Hence for $n > \max\{n_{\widetilde{\epsilon}}, n_\eta\}$ we have by inequalities (3.24), (3.25) and (3.26)

$$\begin{aligned}
\widetilde{I}_n &= \left|\sum_{\ell=1}^{2^n} \frac{1}{n} \int_{D^+} \overline{u(\frac{\ell}{n}, x)} \mathcal{F}_{k\to t}\{\varphi\}(\frac{\ell}{n}, x)\mathrm{d}x - \int_0^\infty \int_{D^+} \overline{u(t,x)} \mathcal{F}_{k\to t}\{\varphi\}(t,x)\mathrm{d}x\right| \\
&\le \int_0^{T_\epsilon+1} \int_{D^+} \left|\overline{(u_n(t,x) - u(t,x))} \mathcal{F}_{k\to t}\{\varphi\}(t,x)\right| \mathrm{d}x + \\
&\quad + \sum_{\ell\in A} \frac{1}{n} \int_D \left|\overline{u(\frac{\ell}{n}, x)} \mathcal{F}_{k\to t}\{\varphi\}(\frac{\ell}{n}, x)\right| \mathrm{d}x + \\
&\quad + \int_{T_\epsilon}^\infty \int_{D_R} \left|\overline{u(t,x)} \mathcal{F}_{k\to t}\{\varphi\}(t,x)\right| \mathrm{d}x \\
&< \eta + \epsilon,
\end{aligned}$$

where $A = \{T_\epsilon + 1, T_\epsilon + 2, \dots, 2^n\}$.

Hence $\widetilde{I}_n$ converges to zero as n goes to infinity and

$$\begin{aligned}
(v, \varphi)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times D^+)} &= \int_0^\infty \int_{D^+} \overline{u(t,x)} \mathcal{F}_{k\to t}\{\varphi\}(t,x)\mathrm{d}x \\
&= \left(\mathcal{F}^{-1}_{t\to k}\{u\}, \varphi\right)_{\mathcal{S}'\times\mathcal{S}(\mathbb{R}\times D^+)}
\end{aligned}$$

□

Now we are ready to state the connection between the time and frequency domain solutions. First in Lemma 3.9 we deal with the smooth initial data and in Theorem 3.10 we prove the result for f_1 in the closure of $C_0^\infty(D^+)$ with respect to the norm $\|\cdot\|_D$ and $f_2 \in L^2(D^+)$.

Lemma 3.9 *Let $f_1, f_2 \in C_0^\infty(D^+)$ and u be the solution of (3.15) with the initial data (f_1, f_2). Then for all $k \in \mathbb{R}$ the function $v(k,\cdot) := \mathcal{F}^{-1}_{t\to k}\{u\}(k,\cdot)$ satisfies*

$$\begin{cases} -(\Delta + k^2)v(k,\cdot) &= (2\pi)^{-\frac{1}{2}}\,(f_2 - ikf_1) \quad \text{in } (D^+) \\ v(k,\cdot)|_{\partial D} &= 0 \\ \lim_{r\to\infty} r\,(\partial_r v - ikv) &= 0. \end{cases} \tag{3.27}$$

Proof. Since $f_1, f_2 \in C_0^\infty(D^+)$ there is $R \in \mathbb{R}_+$ such that $\mathrm{supp}(f_1), \mathrm{supp}(f_2) \subset D_R$. It follows from [17, Theorem 7.2.6] that $u \in C^\infty(\overline{\mathbb{R}_+} \times D^+)$ with

$$\|u|_{t=0}\|_{H^m(D_R)} \le C\left(\|f_1\|_{H^m(D_R)} + \|f_2\|_{H^{m-1}(D_R)}\right)$$

and

$$\|u_t|_{t=0}\|_{H^m(D_R)} \le C\left(\|f_1\|_{H^{m+1}(D_R)} + \|f_2\|_{H^m(D_R)}\right).$$

For all $\alpha \in \mathbb{N}^3$ the function $w = \partial_x^\alpha u$ satisfies

$$\begin{cases} \qquad\qquad \Box w &= 0 \qquad\qquad\qquad \text{in } \mathbb{R}_+ \times D^+, \\ \quad w|_{\partial D \times \{\mathbb{R}_+ \cup 0\}} &= 0 \\ w|_{t=0} = \partial^\alpha u|_{t=0}, & w_t|_{t=0} = \partial_t \partial^\alpha u|_{t=0}, \end{cases} \tag{3.28}$$

where $\partial_x^\alpha u|_{t=0}, \partial_x^\alpha u|_{t=0} \in C_0^\infty(D^+)$. Hence we can apply the energy estimate (3.12) and Poincare's inequality to w and see that for all R such that $\overline{D} \subset B(0,R)$

$$\|w(t,\cdot)\|_{L^2(D_R)} \le CE(w, D^+, 0)e^{-ct}.$$

Hence for all $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$ we have $u(t,\cdot) \in W_2^n(D_R)$ with the estimate

$$\|u(t,\cdot)\|_{W_2^n(D_R)} \le Ce^{-ct}\left(\|f_1\|_{H^{n+1}(D_R)} + \|f_2\|_{H^n(D_R)}\right),$$

where C depends on R and n. The Sobolev inequality, e.g. [17, Theorem 5.7.6], implies that $u(t,\cdot) \in C^{n-2}(D_R)$ with

$$\|u(t,\cdot)\|_{C^{n-2}(D_R)} \le Ce^{-ct}\left(\|f_1\|_{H^{n+1}(D_R)} + \|f_2\|_{H^n(D_R)}\right), \tag{3.29}$$

where C depends on R and n.

Hence the partial Fourier transformation of u can be expressed as an integral

$$\begin{aligned} v(k,x) &= \mathcal{F}^{-1}_{t\to k}\{u\}(k,x) \\ &= (2\pi)^{-\frac{1}{2}} \int_0^\infty e^{ikt} u(t,x)\mathrm{d}t. \end{aligned} \tag{3.30}$$

Because of the decay rate of equation (3.29) we can integrate by parts which yields

$$\begin{aligned} -k^2 v(k,x) &= (2\pi)^{-\frac{1}{2}} \int_0^\infty (\partial_t^2 e^{ikt}) u(t,x)\mathrm{d}t \\ &= (2\pi)^{-\frac{1}{2}} \left(f_2(x) - ikf_1(x) + \int_0^\infty e^{ikt}\partial_t^2 u(t,x)\mathrm{d}t\right). \end{aligned}$$

Hence

$$\begin{aligned} -(\Delta + k^2)v(k,\cdot) &= (2\pi)^{-\frac{1}{2}} \left(f_2 - ikf_1 + \int_0^\infty e^{ikt}(\partial_t^2 - \Delta)u(t,x)\mathrm{d}t\right) \\ &= (2\pi)^{-\frac{1}{2}} \left(f_2 - ikf_1\right). \end{aligned}$$

On $\mathbb{R} \times \partial D$ the function u vanishes, so for all $x \in \partial D$ and $k \in \mathbb{R}$ we have

$$\begin{aligned} v(k,x) &= \int_0^\infty e^{ikt} u(t,x)\mathrm{d}t \\ &= 0. \end{aligned}$$

Finally, we prove Sommerfeld's radiation condition. Let $R_2 < R$ be such that $\overline{D} \subset B(0, R_2)$ and $\chi \in C^\infty(\mathbb{R}^3)$ be such that $\chi \equiv 0$ in a neighbourhood of $\overline{D}$ and $\chi \equiv 1$ in $\mathbb{R}^3 \setminus B(0, R)$. We define $\widetilde{u}_2 := \chi u$ and extend it by zero inside D. The zero extension, denoted by u_2, satisfies

$$\begin{cases} \Box u_2 & = H \quad \text{in } \mathbb{R}^4 \\ u_2|_{t<0} & = 0 \end{cases} \tag{3.31}$$

where by Lemma 3.7

$$\begin{aligned} H &= \delta'(t) \otimes (\chi f_2) - \delta(t) \otimes (\chi f_1) + (\partial_t^2 \chi) u_2 + 2(\partial_t \chi) \partial_t u_2 + \\ &\quad + (\Delta \chi) u_2 + 2 \nabla \chi \cdot \nabla u_2. \end{aligned}$$

By Theorem 2.2 the unique solution of (3.31) is

$$u_2 = E_+ * H.$$

We denote

$$h = (\partial_t^2 \chi) u_2 + 2(\partial_t \chi) \partial_t u_2 + (\Delta \chi) u_2 + 2 \nabla \chi \cdot \nabla u_2.$$

Since χ is compactly supported, $\operatorname{supp}(h) \subset \mathbb{R} \times B(0, R)$ for some $R > 0$. The decay estimate (3.12) implies that $h \in L^1(\mathbb{R}^4)$ and we can apply Lemma 2.7, by which

$$\mathcal{F}^{-1}_{t \to k}\{E_+ * H\}(k, \cdot) = \Phi_k *_y \left(\mathcal{F}^{-1}_{t \to k}\{H\}(k, \cdot) \right).$$

Here

$$\mathcal{F}^{-1}_{t \to k}\{H\}(k, x) = (2\pi)^{-\frac{1}{2}} \chi (f_2 - ik f_1) + \mathcal{F}^{-1}_{t \to k}\{h\}(k, x).$$

Since for all k the function $\mathcal{F}^{-1}_{t \to k}\{H\}(k, \cdot)$ is in $C_0^\infty(\mathbb{R}^3)$, we see that for $|x| > R$

$$\begin{aligned} (\partial_r - ik)\, v(k, x) &= (\partial_r - ik)\, \mathcal{F}^{-1}_{t \to k} u_2(k, x) \\ &= \int_{\mathbb{R}^3} \left(\partial_{r(x)} - ik \right) \Phi_k(x - y) \mathcal{F}^{-1}_{t \to k} H(k, y) \mathrm{d}y. \end{aligned}$$

We have

$$\left(\partial_{r(x)} - ik \right) \Phi_k(x - y) = (\widehat{x} \cdot \widehat{(x - y)} - 1) \frac{ik e^{ik|x-y|}}{4\pi |x - y|} + \widehat{x} \cdot \widehat{(x - y)} \frac{e^{ik|x-y|}}{4\pi |x - y|^2}.$$

Here

$$(\widehat{x} \cdot \widehat{(x - y)} - 1) = \mathcal{O}\left(\frac{1}{|x|} \right),$$

as $|x| \to \infty$, so we see that

$$\left(\partial_{r(x)} - ik\right) \Phi_k(x-y) = \mathcal{O}\left(\frac{1}{|x|^2}\right). \tag{3.32}$$

Hence

$$\int_{\mathbb{R}^3} \left(\partial_{r(x)} - ik\right) \Phi_k(x-y) \mathcal{F}^{-1}_{t\to k} H(k,y) \mathrm{d}y = \mathcal{O}\left(\frac{1}{|x|^2}\right)$$

and

$$\lim_{r\to\infty} r(\partial_r - ik)v = 0$$

uniformly since the estimate (3.32) does not depend on the direction $\widehat{x}$.

$\square$

Lemma 3.10 *Let f_1 be in the closure of $C_0^\infty(D^+)$ with respect to the $\|\cdot\|_D$ semi-norm, f_2 be in $L^2(D^+)$ and u be the solution of (3.15) with the initial data (f_1, f_2). Then for all $k \in \mathbb{R}$*

$$v(k,\cdot) := \mathcal{F}^{-1}_{t\to k}\{u\}(k,\cdot)$$

satisfies

$$\begin{cases} -(\Delta + k^2)v(k,\cdot) &= (2\pi)^{-\frac{1}{2}} \left(f_2 - ikf_1\right) \quad \text{in } (D^+) \\ v(k,\cdot)|_{\partial D} &= 0 \\ \lim_{r\to\infty} r\left(\partial_r v - ikv\right) &= 0. \end{cases}$$

Proof. Let R be such that $\overline{D} \subset B(0,R)$. By Lemma 3.8 $\mathcal{F}^{-1}_{t\to k}\{u\}(k,\cdot)$ can be represented as the Bochner integral

$$\mathcal{F}^{-1}_{t\to k}\{u\}(k,\cdot) = (2\pi)^{-\frac{1}{2}} \int_0^\infty e^{ikt} u(t,\cdot) \mathrm{d}t.$$

Here $u(t,\cdot)$ is considered to be an element in $H^1_{\mathrm{tr}}(D_R)$, that is, an element of $H^1(D_R)$ whose trace on ∂D vanishes. It follows that $\mathcal{F}^{-1}_{t\to k}\{u\}(k,\cdot)$ is also in $H^1_{\mathrm{tr}}(D_R)$. From the energy estimate (3.12) it follows that

$$\|\mathcal{F}^{-1}_{t\to k}\{u\}(k,\cdot)\|_{H^1(D_R)} \leq CE(u, D^+, 0). \tag{3.33}$$

Now let $(f_{1,j}), (f_{2,j}) \subset C_0^\infty(D^+)$ be such that $f_{1,j} \to f_1$ with respect to the semi-norm $\|\cdot\|_D$, $f_{2,j} \to f_2$ in $L^2(D^+)$ and u_j be the solution of the problem

(3.15) with initial data $(f_{1,j}, f_{2,j})$. It follows from the energy decay (3.12) that for $t \geq 0$

$$E(u - u_j, D_{R+1}, t) \leq Ce^{-ct} E(u - u_j, D^+, 0).$$

Poincare's Inequality, Lemma 3.3, implies that

$$\|u(t,\cdot) - u_j(t,\cdot)\|_{H^1(D_R)} \leq Ce^{-ct} \left(\|f_{1,j} - f_1\|_{L^2(D^+)} + \|f_{2,j} - f_2\|_{L^2(D^+)} \right).$$

Hence we see that for all $R \in \mathbb{R}_+$

$$\|\mathcal{F}^{-1}_{t\to k}\{u_j\}(k,\cdot) - \mathcal{F}^{-1}_{t\to k}\{u\}(k,\cdot)\|_{H^1(D_R)} \to 0 \tag{3.34}$$

as $j \to \infty$. By Lemma 3.9, $\mathcal{F}^{-1}_{t\to k}\{u_j\}(k,\cdot)$ is the unique solution of

$$\begin{cases} -(\Delta + k^2)v(k,\cdot) &= F_j \quad \text{in } D^+ \\ v(k,\cdot)|_{\partial D} &= 0 \\ \lim_{r\to\infty} r\left(\partial_r v - ikv\right) &= 0 \end{cases} \tag{3.35}$$

with $F_j = (2\pi)^{-\frac{1}{2}} \left(f_{2,j} - ikf_{1,j}\right)$.
The solution operator of (3.35) is

$$\widetilde{U}_k = \mathcal{G}_k - U_k \mathrm{tr}_{\,\partial D} \mathcal{G}_k, \tag{3.36}$$

where U_k is the solution operator of the Dirichlet problem (3.2), referred to in Theorem 3.1. By Theorem 3.1 operator $U_k : H^{-\frac{1}{2}}(\partial D) \to H^1_{\ell oc}(D^+)$ is continuous and it follows from Lemma 2.12 that $\mathcal{G}_k : L^2(B(0,R)) \to H^1_{\ell oc}(D^+)$ is continuous, so the operator $\widetilde{U}_k : L^2(\partial D) \to H^1_{\ell oc}(D^+)$ is continuous.
Let $F = \lim_{j\to\infty}(2\pi)^{-\frac{1}{2}} \left(f_{2,j} - ikf_{1,j}\right)$. By equation (3.34)

$$\mathcal{F}^{-1}_{t\to k}\{u\}(k,\cdot) \;=\; \lim_{j\to\infty} \mathcal{F}^{-1}_{t\to k}\{u_j\}(k,\cdot). \tag{3.37}$$

Since operator $\widetilde{U}_k : L^2(\partial D) \to H^1_{\ell oc}(D^+)$ is continuous, we have

$$\begin{aligned} \lim_{j\to\infty} \mathcal{F}^{-1}_{t\to k}\{u_j\}(k,\cdot) &= \lim_{j\to\infty} \widetilde{U}_k F_j \\ &= \widetilde{U}_k F. \end{aligned} \tag{3.38}$$

That is, $\mathcal{F}^{-1}_{t\to k}\{u\}(k,\cdot)$ is the solution of (3.35) with $F = f_2 - ikf_1$.

$\square$

3.4 Dependency of U_k on the Parameter k

Theorem 3.11 *Let D have the Dirichlet local energy decay property. Then for all $R \in \mathbb{R}_+$ the frequency domain Dirichlet solution operator $U_k : H^{\frac{1}{2}}(\partial D) \to H^1_{\ell oc}(D^+)$ has the estimate*

$$\|U_k\|_{H^{\frac{1}{2}}(\partial D) \to H^1(D_R)} \leq C\langle k\rangle^2$$

where C depends on R and D.

Proof. Let $h \in H^{\frac{1}{2}}(\partial D)$, $k \in \mathbb{R} \setminus \{0\}$ and v be the unique solution of

$$\begin{cases} -(\Delta + k^2)v &= 0 \quad \text{in } D^+ \\ v|_{\partial D} &= h \\ \lim_{r\to\infty} r\,(\partial_r v - ikv) &= 0. \end{cases} \tag{3.39}$$

In order to achieve a connection to the time domain solution we form a corresponding source equation to (3.15) in the frequency domain, which is connected to (3.39) above. To this end let $R > 0$ be such that $\overline{D} \subset B(0,R)$ and w be the unique solution to

$$\begin{cases} \Delta w &= 0 \quad \text{in } D_R \\ w|_{\partial D} &= h \\ w|_{\partial B(0,R)} &= 0. \end{cases} \tag{3.40}$$

Furthermore let $\chi \in C_0^\infty(B(0,R))$ be such that $\chi \equiv 1$ on a neighbourhood of D. Then $\widetilde{v}(x) := v - \chi w$ is the unique solution of

$$\begin{cases} -(\Delta + k^2)\widetilde{v} &= g \quad \text{in } D^+ \\ \widetilde{v}|_{\partial D} &= 0 \\ \lim_{r\to\infty} r\,(\partial_r \widetilde{v} - ik\widetilde{v}) &= 0, \end{cases} \tag{3.41}$$

where $g := (\Delta\chi)w + 2\nabla\chi \cdot \nabla w + k^2\chi w$ is in $L^2(D^+)$.

By Theorem 3.10 the problem (3.41) is the Fourier transform of the problem (3.15) with Cauchy data $f_1 = 0, f_2 = (2\pi)^{\frac{1}{2}}g$. Hence $\widetilde{v} = \mathcal{F}^{-1}_{t\to k}\{u\}(k,\cdot)$ and estimate (3.33) yields

$$\begin{aligned} \|\widetilde{v}\|_{H^1(D_R)} &\leq CE(u, D^+, 0) \\ &= C\|g\|_{L^2(D^+)} \\ &\leq C\langle k\rangle^2 \|w\|_{H^1(D_R)} \\ &\leq C\langle k\rangle^2 \|h\|_{H^{\frac{1}{2}}(\partial D)}. \end{aligned}$$

In the last step we have used the fact that the solution operator of problem (3.40) is continuous from $H^{\frac{1}{2}}(\partial D)$ to $H^1(D_R)$.

Hence for all $h \in H^{\frac{1}{2}}(\partial D)$

$$\begin{array}{rcl} \|U_k h\|_{H^1(D_R)} & = & \|\widetilde{v} + \chi w\|_{H^1(D_R)} \\ & \leq & \|\widetilde{v}\|_{H^1(D_R)} + \|\chi w\|_{H^1(D_R)} \\ & \leq & C\langle k \rangle^2 \|h\|_{H^{\frac{1}{2}}(\partial D)}. \end{array}$$

That is,

$$\|U_k\|_{H^{\frac{1}{2}}(\partial D) \to H^1(D_R)} \leq C_R \langle k \rangle^2,$$

where C_R depends on R and D.

□

Chapter 4

The Robin Boundary Conditions and the Frequency Dependency of the Factorization Method

Throughtout this Chapter we assume that the domain D has a C^2 boundary if not otherwise stated.

4.1 Existence and Uniqueness in the Frequency Domain

We shall adopt the approach of using single and double layer operators and their jump relations to prove the existence of the solution to the time harmonic Robin problem. This approach has been applied to Dirichlet and Neumann problems in [14] and [40], the results and ideas of which we also make use here.

We note that one can find a solution to the Robin problem in the literature, e.g. [28, Theorem 2.2]. We include a proof in the following for the sake of completeness and also since the proof is different from the one in [28].

We recall first the continuity properties and the jump relations of the single and double layer operators.

Theorem 4.1 *Let $\chi \in C_0^\infty(\mathbb{R}^3)$. The following operators are continuous*

$$
\begin{array}{rlrl}
\chi \mathrm{SL}_k : H^{-\frac{1}{2}}(\partial D) \to H^1(\mathbb{R}^3), & & \chi \mathrm{DL}_k : H^{\frac{1}{2}}(\partial D) \to H^1(D^\pm), \\
\operatorname{tr} \mathrm{SL}_k : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D), & & \operatorname{tr}^\pm \mathrm{DL}_k : H^{\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D), \\
\partial_\nu^\pm \mathrm{SL}_k H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D), & & \partial_\nu \mathrm{DL}_k : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D).
\end{array}
$$

In addition, the single and double layer operators satisfy the jump relations

$$
\begin{aligned}
[\mathrm{SL}_k \psi] &= 0, \quad \textit{and} && (4.1)\\
[\partial_\nu \mathrm{SL}_k \psi] &= -\psi && (4.2)
\end{aligned}
$$

for $\psi \in H^{-\frac{1}{2}}(\partial D)$ and

$$
\begin{aligned}
[\mathrm{DL}_k \varphi] &= \varphi \quad \textit{and} && (4.3)\\
[\partial_\nu \mathrm{DL}_k \varphi] &= 0 && (4.4)
\end{aligned}
$$

for $\varphi \in H^{\frac{1}{2}}(\partial D)$.

Proof. See [40, Theorem 6.11].

□

In the case of Hölder continuous functions $\varphi \in C^{0,\alpha}(\partial D)$ and $\psi \in C^{1,\alpha}(\partial D)$ the integrals

$$
\int_{\partial D} \Phi_k(x,y)\varphi(y)\mathrm{d}\mathcal{H}^2(y) \quad \text{and} \quad \int_{\partial D} (\partial_{\nu(y)}\Phi_k(x,y))\psi(y)\mathrm{d}\mathcal{H}^2(y)
$$

exist as improper integrals and the boundary values of the single and double layer potentials can be expressed as

$$
\begin{aligned}
\partial^\pm \mathrm{SL}_k \varphi(x) &= \mp \frac{1}{2}\varphi(x) + \int_{\partial D} (\partial_{\nu(x)}\Phi_k(x,y))\psi(y)\mathrm{d}\mathcal{H}^2(y)\\
\operatorname{tr}^\pm \mathrm{DL}_k \psi(x) &= \pm \frac{1}{2}\psi(x) + \int_{\partial D} (\partial_{\nu(y)}\Phi_k(x,y))\psi(y)\mathrm{d}\mathcal{H}^2(y).
\end{aligned}
$$

For details we refer to [13].

We have adopted the Sobolev space approach according to [40], so we cannot refer to the operators such as

$$
T : \psi \to \int_{\partial D} (\partial_{\nu(y)}\Phi_k(x,y))\psi(y)\mathrm{d}\mathcal{H}^2(y) \quad x \in \partial D \tag{4.5}
$$

as easily as in the Hölder theory. Our approach is to use the jump relations and define

$$\begin{aligned} T_k &= \frac{1}{2}\left(\operatorname{tr}^+\mathrm{DL}_k + \operatorname{tr}^-\mathrm{DL}_k\right) \qquad \text{and} \\ R_k &= \frac{1}{2}\left(\partial_\nu^+\mathrm{SL}_k + \partial_\nu^-\mathrm{SL}_k\right). \end{aligned} \tag{4.6}$$

We state in the following lemma how the boundary values of the single and double layers can be expressed with T_k and R_k and prove some mapping properties of these operators for future reference.

Lemma 4.2 *The operators $T_k, R_k : H^{\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ are compact and for all $\psi \in H^{-\frac{1}{2}}(\partial D)$ and $\varphi \in H^{\frac{1}{2}}(\partial D)$ the following relations hold*

$$\partial^\pm\mathrm{SL}_k\psi = \left(\mp\frac{1}{2}I + R_k\right)\psi \tag{4.7}$$

$$\operatorname{tr}^\pm\mathrm{DL}_k\varphi = \left(\pm\frac{1}{2}I + T_k\right)\varphi. \tag{4.8}$$

Proof. For the mapping property we refer to [53, Proposition 1.11.2].

The equations (4.7) and (4.8) can be inferred from the jump relations (4.2) and (4.3) respectively.

□

Before the existence and uniqueness result we recall the definition of a Fredholm operator.

Definition 4.3 *Let X and Y be a Banach spaces and $T : X \to Y$ a linear operator. T is a* **Fredholm operator** *if following conditions hold:*

(i) Subspace $\operatorname{Ran}(T)$ *is closed*

(ii) spaces $\operatorname{Ker}(T)$ *and* $Y/\operatorname{Ran}(T)$ *are finite dimensional.*

The **index of a Fredholm operator** *is*

$$\operatorname{index}(T) := \dim(\operatorname{Ker}(T)) - \dim(Y/\operatorname{Ran}(T)).$$

With these preliminary observations we are ready to prove the existence and uniqueness for the time harmonic Robin problem.

Theorem 4.4 *Let $k \in \mathbb{R} \setminus \{0\}$, $\alpha, \beta \in C^2(\partial D)$, $\alpha(x) \geq \alpha_0 > 0$ and $\lambda(x) = ik\alpha(x) + \beta(x)$. Then the time harmonic Robin boundary problem*

$$\begin{cases} -(\Delta + k^2)u &= 0 \quad \text{in} \quad D^+ \\ (\partial_\nu + \lambda(x)\text{tr}^+)u &= h \quad \text{where} \quad h \in H^{-\frac{1}{2}}(\partial D) \\ \lim_{r\to\infty} r\,(\partial_r u - iku) &= 0 \quad \text{uniformly} \end{cases} \tag{4.9}$$

has a unique solution $u \in H^1_{\ell oc}(D^+)$ and the solution operator $U_{rob,k} : H^{-\frac{1}{2}}(\partial D) \to H^1_{\ell oc}(D^+)$ is continuous.

Proof. We prove first the uniqueness of the solution. To this end suppose that $u \in H^1_{\ell oc}(D^+)$ is a solution of (4.4) with $h \equiv 0$. Then we have $\partial_\nu u = -\lambda(x)\text{tr}^+u$ on the boundary ∂D and we see that

$$(\partial_\nu u, u)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}(\partial D)} = (-ik\alpha u, u)_{L^2(\partial D)} + (-\beta u, u)_{L^2(\partial D)}.$$

Hence for $k > 0$

$$\text{Im}\,(\partial_\nu u, u)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}(\partial D)} \geq k\alpha_0 \|u\|^2_{L^2(\partial D)} \tag{4.10}$$

and for $k < 0$

$$\text{Im}\,(\partial_\nu u, u)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}(\partial D)} \leq k\alpha_0 \|u\|^2_{L^2(\partial D)}. \tag{4.11}$$

From Sommerfeld's radiation condition it follows, see [14, Equation 2.10], that

$$\lim_{r\to\infty} \int_{\partial B(0,r)} \left(|\partial_\nu u|^2 + k^2|u|^2\right) \mathrm{d}\mathcal{H}^2 = -2k\text{Im}\,(\partial_\nu u, u)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}(\partial D)}. \tag{4.12}$$

Rellich's Lemma, [14, Lemma 2.11], states that if the solution to the Helmholtz equation, u, satisfies

$$\lim_{r\to\infty} \int_{\partial B(0,r)} |u|^2 \mathrm{d}\mathcal{H}^2 = 0, \tag{4.13}$$

then $u \equiv 0$ in $\mathbb{R}^3 \setminus B(0,R)$ for sufficiently large R. It follows from the unique continuation principle, [37, Section 4.3], that $u \equiv 0$ on D^+.

For $k > 0$, equations (4.10) and (4.12) imply (4.13) and hence $u \equiv 0$ on D^+. For $k < 0$, equations (4.11) and (4.12) imply (4.13), so again $u \equiv 0$ on D^+ and the solution of (4.4) is unique for $k \in \mathbb{R} \setminus 0$.

Next we establish the existence of the solution. We make an ansatz

$$u = (\text{DL}_k + \text{SL}_k M_\lambda)\,\varphi,$$

where $\varphi \in H^{\frac{1}{2}}(\partial D)$ and M_λ, defined by $M_\lambda \varphi := \lambda\varphi$, is continuous from $H^s(\partial D) \to H^s(\partial D)$ for all $s \in [-2, 2]$. With equations (4.7) and (4.8) we see that

$$\begin{aligned}
\left(\partial_\nu^+ + \lambda \mathrm{tr}^+\right) u &= \partial_\nu^+ \mathrm{DL}_k \varphi + \left(-\frac{1}{2} + R_k\right) M_\lambda \varphi \\
&\quad + \lambda \left(\frac{1}{2} + T_k\right) \varphi + \lambda \mathrm{S}_k M_\lambda \varphi \\
&= \partial_\nu^+ \mathrm{DL}_k \varphi + \left(R_k M_\lambda + M_\lambda T_k + M_\lambda \mathrm{S}_k M_\lambda\right) \varphi. \quad (4.14)
\end{aligned}$$

By [40, Theorem 2.8] the operator $\partial_\nu^+ \mathrm{DL}_k : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is Fredholm with index 0. By Lemma 4.2 operators $T_k, R_k : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ are compact and by Lemma 2.15 operator $\mathrm{S}_k : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is compact. Since $M_\lambda : H^s(\partial D) \to H^s(\partial D)$ is continuous for $s \in \{-\frac{1}{2}, \frac{1}{2}\}$ we see that the operator

$$K_k := R_k M_\lambda + M_\lambda T_k + M_\lambda S_k M_\lambda : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D) \quad (4.15)$$

is compact.

By [40, Theorem 2.26], the sum of a Fredholm operator with index 0 and a compact operator is a Fredholm operator with index 0, so we see that $\partial_\nu^+ \mathrm{DL}_k + K_k$ is a Fredholm operator with index 0. By the Fredholm alternative $\partial_\nu^+ \mathrm{DL}_k + K_k$ is bijective if and only if it is injective. We will prove this next.

Let us assume that for certain $\varphi \in H^{\frac{1}{2}}(\partial D)$ we have

$$\left(\partial_\nu^+ \mathrm{DL}_k + K_k\right) \varphi = 0. \quad (4.16)$$

Then $u = (\mathrm{DL}_k + \mathrm{SL}_k M_\lambda)\varphi$ satisfies

$$\left\{\begin{array}{rll}
-(\Delta + k^2) u &= 0 & \text{in } D^+ \\
(\partial_\nu^+ + \lambda \mathrm{tr}^+) u &= 0 & \\
\lim_{r\to\infty} r\left(\partial_r - ik\right) u &= 0 & \text{uniformly}
\end{array}\right. \quad (4.17)$$

and it follows from the uniqueness that $u \equiv 0$ on D^+.

We see from the jump relations (4.1) and (4.2) that

$$[u] = \varphi.$$

As $\mathrm{tr}^+ u = 0$ we see that

$$-\mathrm{tr}^- u = \varphi. \quad (4.18)$$

From the jump relations (4.3) and (4.4) we infer that

$$[\partial_\nu u] = -M_\lambda \varphi.$$

Since $\partial_\nu^+ u = 0$, we see that

$$\partial_\nu^- u = M_\lambda \varphi. \tag{4.19}$$

We use now Green's Theorem with (4.18) and (4.19), which gives

$$\begin{aligned}\int_{\partial D} \overline{\lambda\varphi}(-\varphi)\mathrm{d}\mathcal{H}^2 &= \left(\partial_\nu^- u, \mathrm{tr}^- u\right)_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}(\partial D)} \\ &= \int_D \left(|\nabla u|^2 - k^2|u|^2\right)\mathrm{d}x. \end{aligned} \tag{4.20}$$

Since for $k > 0$ we have $\mathrm{Im}(\lambda) \geq k\alpha_0 > 0$ and for $k < 0$ we have $\mathrm{Im}(\lambda) \leq k\alpha_0 < 0$ we see that

$$\begin{aligned}\mathrm{Im}\left(\int_{\partial D} \overline{\lambda\varphi}(-\varphi)\mathrm{d}\mathcal{H}^2\right) &\geq k\alpha_0\|\varphi\|^2_{L^2(\partial D)} \qquad ; k > 0 \\ \mathrm{Im}\left(\int_{\partial D} \overline{\lambda\varphi}(-\varphi)\mathrm{d}\mathcal{H}^2\right) &\leq k\alpha_0\|\varphi\|^2_{L^2(\partial D)} \qquad ; k < 0.\end{aligned}$$

It follows from equation (4.20) that

$$\|\varphi\|_{L^2(\partial D)} = 0. \tag{4.21}$$

Hence the equation (4.16) has only the zero solution and the operator

$$\mathcal{B}_k = \left(\partial_\nu^+ \mathrm{DL}_k + K_k\right) : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D) \tag{4.22}$$

is bijective. By the open mapping theorem, [57, Theorem II.5], the surjective inverse, $\mathcal{B}_k^{-1} : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is continuous, so the solution operator

$$U_{rob,k} = \left(\mathrm{DL}_k + \mathrm{SL}_k M_\lambda\right)\mathcal{B}_k^{-1} \tag{4.23}$$

is continuous from $H^{-\frac{1}{2}}(\partial D)$ to $H^1_{\ell oc}(D^+)$.

$\square$

We are able to establish a local k-dependence result from Theorem 4.4, which is needed later on in the proof of Theorem 4.17 to establish the single frequency factorization to all $k \in \mathbb{R} \setminus \{0\}$.

Corollary 4.5 *The operator $\mathcal{B}_k^{-1}$, where operator $\mathcal{B}_k$ is given in (4.22), depends continuously on k in the sense that for all $k_0 \in \mathbb{R} \setminus \{0\}$*

$$\lim_{k \to k_0} \|\mathcal{B}_k^{-1} - \mathcal{B}_{k_0}^{-1}\| = 0. \tag{4.24}$$

Also operator $U_{rob,k}$ depends continuously on k in the above sense.

Proof. The kernels of the operators SL_k and DL_k depend smoothly on k, so we see that operators $\mathcal{B}_k : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ and $(\mathrm{DL}_k + \mathrm{SL}_k M_\lambda) : H^{\frac{1}{2}}(\partial D) \to H^1_{\ell oc}(D^+)$ depend continuously on k.

Let $k_0 \in \mathbb{R} \setminus \{0\}$, $A = \mathcal{B}_{k_0}$ and $B_k = \mathcal{B}_k - \mathcal{B}_{k_0}$. Since

$$A(I + A^{-1}B_k) = A + B_k,$$

we have

$$\begin{aligned} \mathcal{B}_k^{-1} &= (A + B_k)^{-1} \\ &= (I + A^{-1}B_k)^{-1}A^{-1}. \end{aligned} \tag{4.25}$$

Operator $\mathcal{B}_k$ depends continuously on k in the sense of (4.24), so when $|k - k_0|$ is sufficiently small we have

$$\|A^{-1}B_k\| < 1$$

and

$$(I + A^{-1}B_k)^{-1} = \sum_{n=0}^{\infty} (A^{-1}B_k)^n. \tag{4.26}$$

The series on the right-hand side of (4.26) depends continuously on k, so we see from (4.25) that $\mathcal{B}_k^{-1}$ depends continuously on k.

Since both $\mathcal{B}_k^{-1}$ and $\mathrm{DL}_k + \mathrm{SL}_k M_\lambda$ depend continuously on k it follows from equation (4.23) that $U_{rob,k}$ depends continuously on k.

4.2 Sommerfeld's Radiation Condition

In the subsequent three sections we use the family of plane waves

$$P = \{u_i(x.d) = e^{ikx \cdot d} : d \in \mathbb{S}^2\} \tag{4.27}$$

rather intensively. This family gives a certain "window" to the scattering process and obstacle itself, a window which is summarised in the far field operator introduced at the end of section 4.3

In the following theorem the notation $\mathcal{O}_k\left(\frac{1}{|x|}\right)$ means that the term denoted by this is smaller than $\frac{C(k)}{|x|}$, where the constant $C(k)$ depends on k. In addition, the term $\mathcal{O}_S$ refers to the $\mathcal{O}$ term that is connected to the kernel $\Phi_k(x,y)$ of the single layer operator SL_k and $\mathcal{O}_D$ refers to the corresponding term that is connected to the kernel $\partial_{\nu(y)}\Phi_k(x,y)$ of the double layer operator DL_k. These terms do not depend on k, since we separated the k dependence in Lemma A.1. We will use this notation in the rest of the work.

Lemma 4.6 *For a fixed $k \in \mathbb{R} \setminus \{0\}$ the solution u_{sc} of the exterior Robin problem (4.9) has the far field asymptotics*

$$u_{sc}(x) = \frac{e^{ik|x|}}{|x|}\left(u_\infty(\widehat{x}) + \mathcal{O}_k\left(\frac{1}{|x|}\right)\right) \tag{4.28}$$

and satisfies

$$(\partial_r - ik)u_{sc} = \mathcal{O}_k\left(\frac{1}{|x|^2}\right) \tag{4.29}$$

In addition in the far field asymptotics of the scattered plane waves

$$u_s(x,d) = \frac{e^{ik|x|}}{|x|}\left(u_{k,\infty}(\widehat{x},d) + \mathcal{O}_{k,d}\left(\frac{1}{|x|}\right)\right),$$

where the $\mathcal{O}_{k,d}\left(\frac{1}{|x|}\right)$ terms have the estimate: for all $d \in \mathbb{S}^2$

$$|\mathcal{O}_{k,d}\left(\frac{1}{|x|}\right)| < \frac{C(k)}{|x|}. \tag{4.30}$$

Sommerfeld's radiation condition

$$\lim_{r\to\infty} r(\partial_r - ik)u_s(x,d) = 0$$

is satisfied uniformly with respect to $\widehat{x}, d \in \mathbb{S}^2$.

Proof. Let $R \in \mathbb{R}_+$ be such that $\overline{D} \subset B(0,R)$ and $x \in \mathbb{R}^3 \setminus B(0,2R)$ and $h = (\partial_\nu + \lambda \mathrm{tr}\,)u_{sc}$. By Theorem 4.4 the exterior solution is

$$u_{sc} = (\mathrm{DL}_k + \mathrm{SL}_k M_\lambda)\,\mathcal{B}_k^{-1}h,$$

where λ is the density of equation (4.9) and for each $k \in \mathbb{R}\setminus\{0\}$ the operator $\mathcal{B}_k^{-1} : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is bounded. By Lemma A.1 the kernels of the single and double layer operators have the following asymptotics for all $y \in B(0,R)$ and $x \in \mathbb{R}^3 \setminus B(0,2R)$:

$$\Phi_k(x,y) = \frac{e^{ik|x|}}{|x|}\left(\frac{1}{4\pi}e^{-ik\widehat{x}\cdot y} + \langle k\rangle \mathcal{O}_S\left(\frac{1}{|x|}\right)\right)$$

and

$$\partial_{\nu(y)}\Phi_k(x,y) = \frac{e^{ik|x|}}{|x|}\left(-\frac{\nu(y)\cdot\widehat{x}}{4\pi}ike^{-ik\widehat{x}\cdot y} + \langle k\rangle^2 \mathcal{O}_D\left(\frac{1}{|x|}\right)\right).$$

Hence

$$\begin{aligned} u_{sc}(x) &= \int_{\partial D}\left(\partial_{\nu(y)}\Phi_k(x,y) + \Phi_k(x,y)\lambda(y)\right)\mathcal{B}_k^{-1}h(y)\mathrm{d}\mathcal{H}^2(y) \\ &= \frac{e^{ik|x|}}{4\pi|x|}\int_{\partial D}\left(-\nu(y)\cdot\widehat{x}ike^{-ik\widehat{x}\cdot y} + e^{-ik\widehat{x}\cdot y}\lambda(y)\right)\mathcal{B}_k^{-1}h(y)\mathrm{d}\mathcal{H}^2(y) + \\ &\quad +\mathcal{O}_k\left(\frac{1}{|x|^2}\right) \end{aligned}$$

and the asymptotics (4.28) hold with

$$u_{k,\infty}(\widehat{x}) = \frac{1}{4\pi}\int_{\partial D}\left(-\nu(y)\cdot\widehat{x}ike^{-ik\widehat{x}\cdot y} + e^{-ik\widehat{x}\cdot y}\lambda(y)\right)\mathcal{B}_k^{-1}h(y)\mathrm{d}\mathcal{H}^2(y). \quad (4.31)$$

We next prove equation (4.29). We observe first that

$$(\partial_r - ik)u_{sc}(x) = \int_{\partial D}(\partial_r - ik)\left(\partial_{\nu(y)}\Phi_k(x,y) + \Phi_k(x,y)\lambda(y)\right)\mathcal{B}_k^{-1}h(y)\mathrm{d}\mathcal{H}^2(y).$$

By Lemma A.2

$$\left(\partial_{r(x)} - ik\right)\Phi(x,y) = \langle k\rangle\mathcal{O}\left(\frac{1}{|x|^2}\right)$$

and

$$\left(\partial_{r(x)} - ik\right)\partial_{\nu(y)}\Phi(x,y) = \langle k\rangle^2\mathcal{O}\left(\frac{1}{|x|^2}\right).$$

Hence we see that (4.29) holds.

We prove next the uniformity of the far field asymptotics of the scattered plane waves. The boundary values of plane waves $f_d := (\partial_\nu + \lambda\mathrm{tr})e^{ikd\cdot x}$ satisfy

$$\sup_{d\in\mathbb{S}^2}\|f_d\|_{H^{-\frac{1}{2}}(\partial D)} = C < \infty.$$

Hence there exists a constant $C(k)$ such that for all $d \in \mathbb{S}^2$

$$\left| \frac{1}{4\pi} \int_{\partial D} \left(\langle k \rangle \mathcal{O}_S \left(\frac{1}{|x|} \right) \lambda(y) + \langle k \rangle^2 \mathcal{O}_D \left(\frac{1}{|x|} \right) \right) \mathcal{B}_k^{-1} f_d(y) \mathrm{d}\mathcal{H}^2(y) \right| \leq \frac{C(k)}{|x|}$$

and the estimate (4.30) holds uniformly for $d \in \mathbb{S}^2$.

Next we prove the uniformity of Sommerfeld's radiation condition for the plane waves. We have by Lemma A.2

$$\begin{aligned} (\partial_r - ik) U_{rob,k} f_d(x) &= \int_{\partial D} \left[(\partial_r - ik) \left(\partial_{\nu(y)} \Phi_k(x,y) + \Phi_k(x,y) \lambda(y) \right) \right] \\ &\qquad \mathcal{B}_k^{-1} f_d(y) \mathrm{d}\mathcal{H}^2(y) \\ &= \int_{\partial D} \langle k \rangle^2 \mathcal{O} \left(\frac{1}{|x|^2} \right) \mathcal{B}_k^{-1} f_d(y) \mathrm{d}\mathcal{H}^2(y) \\ &= \mathcal{O}_k \left(\frac{1}{|x|^2} \right), \end{aligned}$$

since

$$\sup_{d \in \mathbb{S}^2} \| \mathcal{B}_k^{-1} f_d \|_{L^2(\partial D)} = C(k) < \infty.$$

This establishes an estimate for $(\partial_r - ik) U_{rob,k} f_d(x)$ that is uniform with respect to the d variable.

$\square$

We notice that in Lemma 4.6 we would get an explicit expression for the k dependence of the $\mathcal{O}_k$ terms if we had a k estimate for the operator $\mathcal{B}_k^{-1}$ which, however, is unavailable.

Corollary 4.7 *The far field kernel $u_{k,\infty}(\widehat{x}, d)$ is in $C^\infty(\mathbb{S}^2 \times \mathbb{S}^2)$.*

Proof. Let $f_d := (\partial_\nu + \lambda \mathrm{tr}\,) e^{ikd \cdot x}$ be the boundary values of a plane wave. By equation (4.31) the corresponding far field is

$$u_{k,\infty}(\widehat{x}, d) = \frac{1}{4\pi} \int_{\partial D} \left(-\nu(y) \cdot \widehat{x} ik e^{-ik\widehat{x} \cdot y} + e^{-ik\widehat{x} \cdot y} \lambda(y) \right) \mathcal{B}_k^{-1} f_d(y) \mathrm{d}\mathcal{H}^2(y). \tag{4.32}$$

The functions $-\nu(y) \cdot \widehat{x} ik e^{-ik\widehat{x} \cdot y}$ and $e^{-ik\widehat{x} \cdot y}$ are C^∞ with respect to the variables $\widehat{x}$ and y, so we see that $u_\infty(\widehat{x}, d)$ is C^∞ with respect to the $\widehat{x}$ variable.

The function f_d is C^∞ with respect to the variable d. In addition the operator $\mathcal{B}_k^{-1}$ is linear and bounded from $H^{-\frac{1}{2}}(\partial D)$ to $L^2(\partial D)$, so we can infer with Lebesque's Dominated Convergence Theorem that the derivatives with respect to the d variable can be taken inside the integral. With a local parametrisation $d = d(\phi_1, \phi_2)$ we see that for $i \in \{1, 2\}$ we have

$$\begin{aligned} \partial_{\phi_i} \mathcal{B}_k^{-1} f_d &= \mathcal{B}_k^{-1} \partial_{\phi_i} f_d \\ &= \mathcal{B}_k^{-1} (\partial_\nu + \lambda \mathrm{tr}\,) \partial_{\phi_i} e^{ikx \cdot d}. \end{aligned}$$

Since $e^{ikx \cdot d}$ is C^∞ with respect to the d variable, we see that $u_\infty(\widehat{x}, d)$ is C^1 with respect to the d variable. Higher derivatives can be taken care of using the same argument.

□

4.3 Herglotz Waves and Their Scattering

We now introduce the concept of a **Herglotz wave** and analyze its scattering from an obstacle with a Robin boundary condition. The Herglotz wave is a central concept in this work and our way of analyzing the scattering process.

As we will see below in Corollary 4.15, the Herglotz waves' scattering can be constructed from the scattering of plane waves and the far field of a scattered Herglotz wave can be obtained from the Herglotz waves' density by the far field operator F_k, which has attractive functional analytic properties. The novel aspect of our approach is that we provide frequency dependent estimates for $||F_k||$ and later on also in the reconstruction estimates, in equation (4.87). This is done in order to make possible the multi-frequency reconstruction later on in Chapter 5.

The Herglotz waves will also surface in Section 5.3, where we use their counterparts in the time domain in order to provide a way to calculate the time domain far fields and estimates for the time domain far field operator $F_{time} = \widehat{F}$, which, in the author's opinion, provide a way to estimate the numerical applicability of the multi-frequency reconstruction that will be in introduced in Section 5.1.

Definition 4.8 *Let $h \in L^2(\mathbb{S}^2)$. A Herglotz wave u_h is defined by setting*

$$u_h(x) = \int_{\mathbb{S}^2} e^{ikx \cdot d} h(d) \mathrm{d}\mathcal{H}^2(d),$$

for all $x \in \mathbb{R}^3$.

A Herglotz wave u_h is both incoming and outgoing as we see in the next theorem.

Lemma 4.9 *Let $k \in \mathbb{R} \setminus \{0\}$. The Herglotz wave*

$$u_h(x) = \int_{\mathbb{S}^2} e^{ikx\cdot d} h(d) \mathrm{d}\mathcal{H}^2(d)$$

with $h \in C^\infty(\mathbb{S}^2)$ has the asymptotics

$$u_h(x) - 2\pi i \frac{e^{ik|x|}}{k|x|} h(\widehat{x}) - 2\pi i \frac{e^{-ik|x|}}{k|x|} h(-\widehat{x}) = \mathcal{O}_h\left(\frac{1}{(k|x|)^2}\right) \tag{4.33}$$

uniformly with respect to $\widehat{x} = \frac{x}{|x|}$ as $|x| \to \infty$.

We note that in (4.33) the k dependence of the $\mathcal{O}$ term is stated explicitly.

Proof. Proof of this result is well know, see [1], [47]. We present the proof since it is needed for a further result in Lemma 4.10.

Let $x \in \mathbb{R}^3 \setminus \{0\}$ and the coordinate system $(e_{x,1}, e_{x,2}, e_{x,3})$ be such that $e_{x,3} = \widehat{x}$. We form charts: let $\delta = \pi/10$ and $U_1 = (0,\pi) \times (\frac{\pi}{2}, \frac{5\pi}{2}), U_2 = (0,\pi) \times (-\delta, \pi+\delta), U_3 = (0,\pi) \times (\pi, 2\pi)$ and

$$\begin{aligned}
p_{x,1}(\theta,\varphi) &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\
p_{x,2}(\theta,\varphi) &= (\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi) \\
p_{x,3}(\theta,\varphi) &= (\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi)
\end{aligned}$$

We note that the θ angle is between the $e_{x,3}$ axis and the vector pointing to $p_{x,1}(\theta,\varphi)$ in the first chart and that the θ angle is between the $e_{x,1}$ axis and the vector in the charts $p_{x,2}$ and $p_{x,3}$.

We have arranged things here so that the special points $\widehat{x}$ and $-\widehat{x}$ are only in a single charts, the $p_{x,2}$ and $p_{x,3}$ respectively, and so that they are interior points in the domain of the chart.

In addition we need a partition of unity $\{\psi_i\}_{i\in\{1,2,3\}}$ subordinate to the atlas $\{(U_i, p_{x,i})\}_{i\in\{1,2,3\}}$. With these we can express

$$\begin{aligned}
u_h(x) &= \sum_{i=1}^{3} \int_{U_i} e^{ik|x|\widehat{x}\cdot p_{x,i}(\theta,\varphi)} (h \circ p_{x,i})(\theta,\varphi)(\psi_i \circ p_{x,i})(\theta,\varphi) J_{p_{x,i}}(\theta,\varphi) \mathrm{d}\theta\mathrm{d}\varphi \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{4.34}$$

In I_1 the phase function is

$$\begin{aligned} f_1 &:= \widehat{x} \cdot p_{x,1}(\theta, \varphi) \\ &= \cos\theta. \end{aligned}$$

On the domain U_1 we have $\nabla f_1 \neq 0$, so it follows from the method of stationary phase, in the form of [26, Theorem 7.7.1], that

$$|I_1| \leq \frac{C}{(k|x|)^2}, \tag{4.35}$$

where the constant C depends on the derivatives of $f_1, J_{p_1}, p_1, h \circ p_1$ and $\psi \circ p_1$ of degree two or less on the support of $\psi \circ p_1$.

The set $A = \operatorname{supp}(\psi_i \circ p_{x,i})$ and $J_{p_{x,i}}$ do not depend on x. Moreover there exists a constant C such that for all two dimensional multi-indexes $|\alpha| \leq 2$ we have

$$\sup_{\substack{\widehat{x} \in \mathbb{S}^2 \\ (\theta,\varphi) \in A}} |D^\alpha (h \circ p_{x,1})| < C \qquad \text{and} \qquad \sup_{\substack{\widehat{x} \in \mathbb{S}^2 \\ (\theta,\varphi) \in A}} |D^\alpha (\psi_1 \circ p_{x,1})| < C,$$

so we see that the estimate (4.35) is uniform with respect to the direction $\widehat{x}$.

On the chart $p_{x,2}$ we have the phase function

$$\begin{aligned} f_2 &:= \widehat{x} \cdot p_{x,2}(\theta, \varphi), \\ &= \sin\theta \sin\varphi. \end{aligned}$$

The gradient of f_2

$$\nabla f_2(\theta, \varphi) = (\cos\theta \sin\varphi, \sin\theta \cos\varphi)$$

is zero only at $(\frac{\pi}{2}, \frac{\pi}{2})$. In addition

$$\det(D^2 f)(\theta, \varphi) = \sin^2\theta \sin^2\varphi - \cos^2\theta \cos^2\varphi$$

is 1 at the point $(\frac{\pi}{2}, \frac{\pi}{2})$, so we can use the method of stationary phase in the form of [26, Theorem 7.7.5]. This yields

$$\begin{aligned} &\left| I_2 - 2\pi i \frac{e^{ik|x| f_2(\frac{\pi}{2}, \frac{\pi}{2})}}{k|x|} \sum_{j<2} (k|x|)^{-j} L_j (h \circ p_{x,2} \psi \circ p_{x,2} J_{p_{x,2}}) \right| \leq \\ &\qquad \leq C (k|x|)^{-2} \sum_{|\alpha| \leq 4} \sup \left| D^\alpha (h \circ p_{x,2} \psi \circ p_{x,2} J_{p_{x,2}}) \right|, \end{aligned}$$

where α is a two dimensional multi-index, $L_0 u = u(\frac{\pi}{2}, \frac{\pi}{2})$; $L_j u$ is a linear combination of the terms $D^\alpha u(\frac{\pi}{2}, \frac{\pi}{2})$ with $|\alpha| < 2j$; and C depends on the partial derivatives of f_2 of degree less than 8. We move the L_1 term to the right-hand-side and see that

$$\left| I_2 - 2\pi i \frac{e^{ik|x|}}{k|x|} h(\widehat{x}) \right| = \mathcal{O}\left(\frac{1}{(k|x|)^2} \right). \tag{4.36}$$

In the same manner as with the estimate (4.35) we can see that this estimate does not depend on the direction $\widehat{x}$.

With the same arguments as for the integral I_2 and taking into account that $f_3(\frac{\pi}{2}, \frac{3\pi}{2}) = -1$ we can see that I_3 satisfies

$$\left| I_3 - 2\pi i \frac{e^{-ik|x|}}{k|x|} h(-\widehat{x}) \right| = \mathcal{O}\left(\frac{1}{(k|x|)^2} \right) \tag{4.37}$$

uniformly.

We sum equations (4.35), (4.36) and (4.37), which yieds

$$u_h(x) - 2\pi i \frac{e^{ik|x|}}{k|x|} h(\widehat{x}) - 2\pi i \frac{e^{-ik|x|}}{k|x|} h(-\widehat{x}) = \mathcal{O}\left(\frac{1}{(k|x|)^2} \right).$$

□

Later on we will need the following property of the $\mathcal{O}_h\left(\frac{1}{(k|x|)^2}\right)$ term.

Lemma 4.10 *The* $\mathcal{O}_h\left(\frac{1}{(k|x|)^2}\right)$ *term in equation (4.33) satisfies* $\partial_r \mathcal{O}_h\left(\frac{1}{(k|x|)^2}\right) = \mathcal{O}\left(\frac{1}{k|x|^2}\right)$.

Proof. In the proof of Lemma 4.9 the Herglotz wave was expressed with three integrals, I_1, I_2, I_3, each of which is of the form

$$I = \int_U e^{ik|x|\widehat{x}\cdot p(\theta,\varphi)} u(\widehat{x}, \theta, \varphi) \mathrm{d}\theta \mathrm{d}\varphi,$$

where u is smooth. Hence

$$\partial_r I = \int_U e^{ik|x|\widehat{x}\cdot p(\theta,\varphi)} ik\widehat{x} \cdot p(\theta, \varphi) u(\widehat{x}, \theta, \varphi) \mathrm{d}\theta \mathrm{d}\varphi. \tag{4.38}$$

In the case of the integral I_1 the phase function $f_1(\theta,\varphi) := \cos\theta$ does not have any critical points, where $\nabla f = 0$, so the method of the stationary phase, [26, Theorem 7.7.1] implies that there exists a constant C such that

$$|\partial_r I_1| \le \frac{C}{k|x|^2}.$$

The $\mathcal{O}\left(\frac{1}{(k|x|)^2}\right)$ term due to I_3 is

$$\mathcal{O}_{h,3}\left(\frac{1}{(k|x|)^2}\right) = I_3 - (2\pi i)\frac{e^{-ik|x|}}{k(|x|)}h(-\widehat{x}).$$

Hence

$$\begin{aligned}\frac{\partial\mathcal{O}_{h,3}}{\partial|x|} &= \int_U e^{ik|x|\widehat{x}\cdot p(\theta,\varphi)} ik\widehat{x}\cdot p(\theta,\varphi)u(\widehat{x},\theta,\varphi)\mathrm{d}\theta\mathrm{d}\varphi \\ &\quad +(2\pi i)ik\frac{e^{-ik|x|}}{(k|x|)}h(-\widehat{x}) + (2\pi i)\frac{e^{-ik|x|}}{k|x|^2}h(-\widehat{x}) \qquad (4.39)\end{aligned}$$

We notice that at the critical point $(\frac{\pi}{2},\frac{3\pi}{2})$ of the integral in (4.39), the term $\widehat{x}\cdot p(\theta,\varphi) = -1$. Hence we can use the method of the stationary phase, [26, Theorem 7.7.5] and see that the subtraction of the two first terms is $\mathcal{O}\left(\frac{1}{k|x|^2}\right)$. Since the last term in (4.39) is $\mathcal{O}\left(\frac{1}{k|x|^2}\right)$, we see that $\partial_{|x|}\mathcal{O}_{h,3}\left(\frac{1}{(k|x|)^2}\right)$ is $\mathcal{O}\left(\frac{1}{k|x|^2}\right)$.

The I_2 term can be handled in a similar fashion.

□

By Theorem 4.4 to the incident wave $u_i(x) = e^{ikx\cdot d}$ corresponds a scattered wave

$$u_s(s,d) := U_{rob,k}\left(-(\partial_\nu + \lambda\mathrm{tr}\,)e^{ikx\cdot d}\right).$$

We next use a Herglotz wave as an incident field and determine what the scattered field is. First we prove that our constructed solution is a solution and that it satisfies Sommerfeld's radiation condition.

Lemma 4.11 *Let $k \in \mathbb{R}\setminus\{0\}$, $h \in L^2(\mathbb{S}^2)$ and*

$$u_{sc}(x) = \int_{\mathbb{S}^2} u_s(x,d)h(d)\mathrm{d}\mathcal{H}^2, \qquad (4.40)$$

for $x \in D^+$. Then $-(\Delta + k^2)u_{sc} = 0$ and u_{sc} satisfies Sommerfeld's radiation condition (3.3).

Proof. We prove $-(\Delta+k^2)u_{sc}=0$ by showing that $-(\Delta+k^2)$ can be evaluated inside the integral in (4.40). Let $x\in D^+$. By Theorem 4.4

$$U_{rob,k}=(\mathrm{DL}_k+\mathrm{SL}_k M_\lambda)\,\mathcal{B}_k^{-1},$$

where $\mathcal{B}_k^{-1}:H^{-\frac{1}{2}}(\partial D)\to H^{\frac{1}{2}}(\partial D)$ and by Lemma 2.16 there is $\epsilon>0$ s.t. $\mathrm{DL}_k+\mathrm{SL}_k M_\lambda : L^2(\partial D)\to C^2(B(x,\epsilon))$.

As

$$\sup_{d\in\mathbb{S}^2}\|(\partial_\nu+\lambda\mathrm{tr}\,)e^{ikx\cdot d}\|_{H^{-\frac{1}{2}}(\partial D)}<\infty,$$

we have

$$\sup_{d\in\mathbb{S}^2}\|U_{rob,k}(\partial_\nu+\lambda\mathrm{tr}\,)e^{ikx\cdot d}\|_{C^2(B(x,\epsilon))}<\infty.$$

From this it follows, by repeated application of Lebesgue's Dominated Convergence Theorem and Hölder's Inequality, that we can take all derivatives of order 2 or less inside the integral in (4.40). Hence

$$\begin{aligned}-(\Delta+k^2)u_{sc}(x) &= \int_{\mathbb{S}^2}-(\Delta+k^2)u_s(x,d)h(d)\mathrm{d}\mathcal{H}^2(d)\\ &= 0,\end{aligned}$$

since $-(\Delta+k^2)u_s(x,d)=0$ in D^+ for all $d\in\mathbb{S}^2$.

As the derivatives can be taken inside the integral, we see that

$$(\partial_r-ik)u_{sc}=\int_{\mathbb{S}^2}((\partial_r-ik)u_s(x,d)h(d))\,\mathrm{d}\mathcal{H}^2(d).$$

By Lemma 4.6

$$\lim_{r\to\infty}r(\partial_r-ik)u_s(x,d)=0$$

uniformly with respect to $\widehat{x}$ and d. Hence we have

$$\lim_{r\to\infty}r(\partial_r-ik)u_{sc}(x)=0 \tag{4.41}$$

uniformly with respect to $\widehat{x}$, that is, u_{sc} satisfies Sommerfeld's radiation condition.

□

Next we prove that the constructed solution has the same boundary values as the incident Herglotz wave.

Lemma 4.12 *Let $h \in L^2(\mathbb{S}^2)$. The exterior solution*

$$u_{sc}(x) := \int_{\mathbb{S}^2} u_s(x,d)h(d)\mathrm{d}\mathcal{H}^2(d)$$

satisfies

$$(\partial_\nu + \lambda \mathrm{tr}\,)u_{sc} = -\int_{\mathbb{S}^2} (\partial_\nu + \lambda \mathrm{tr}\,)e^{ikx\cdot d}h(d)\mathrm{d}\mathcal{H}^2(d).$$

Proof. Let $h \in C^\infty(\mathbb{S}^2)$, $R \in \mathbb{R}$ be such that $\overline{D} \subset B(0,R)$ and $D_R = B(0,R) \setminus \overline{D}$. In Lemma B.6 we proved that the map $T_h : \mathbb{S}^2 \to H^1(D_R)$, defined by

$$\begin{aligned} T_h(d) &:= u_s(\cdot,d)h(d) \\ &= U_{rob,k}\left(-(\partial_\nu + \lambda \mathrm{tr}\,)e^{ikx\cdot d}\right)h(d), \end{aligned}$$

is Bochner integrable.

We form simple functions for approximating $T_h(d)$ as follows: let

$$v_n(d) = \sum_{i=1}^{i_n} (U_{\mathrm{rob},k}\left(-(\partial_\nu + \lambda \mathrm{tr}\,)\right) e^{ikx\cdot d_i}\chi_{\Delta_{n,i}},$$

where $(\Delta_{n,i})_{i\in\{1,2,\ldots,n\}}$ is a partitioning of $\mathbb{S}^2$ with $\mathrm{diam}(\Delta_{n,i}) < \frac{1}{n}$, $d_i \in \Delta_{n,i}$ and $\chi_{\Delta_{n,i}}$ is the characteristic function of $\Delta_{n,i}$. There exists a constant C such that

$$\sup_{\substack{d\in\mathbb{S}^2\\ n\in\mathbb{N}}} \|v_n(d)\|_{H^1(D_R)} < C. \tag{4.42}$$

Hence it follows from Lebesgue's Dominated Convergence Theorem, Theorem B.5, that

$$\int_{\mathbb{S}^2} T_h(d)\mathrm{d}\mathcal{H}^2(d) = \lim_{n\to\infty}\int_{\mathbb{S}^2} v_n(d)\mathrm{d}\mathcal{H}^2(d). \tag{4.43}$$

Let $X = \{u \in H^1(D_R) : -(\Delta + k^2)u = 0\}$. The normal derivative operator $\partial_{\nu(D)}$ can be defined on X by equation (2.5). We see that with this definition the operator $(\partial_\nu + \lambda \mathrm{tr}\,) : X \to H^{-\frac{1}{2}}(\partial D)$ is well-defined and continuous. As $(v_n) \subset X$, it follows from (4.43) that

$$\begin{aligned} I &= (\partial_\nu + \lambda \mathrm{tr}\,)\int_{\mathbb{S}^2} T_h(d)\mathrm{d}\mathcal{H}^2(d) \\ &= \lim_{n\to\infty}(\partial_\nu + \lambda \mathrm{tr}\,)\int_{\mathbb{S}^2} v_n\mathrm{d}\mathcal{H}^2 \\ &= \lim_{n\to\infty}\sum_{i=1}^{i_n}\left(-(\partial_\nu + \lambda \mathrm{tr}\,)e^{ikx\cdot d_i}\right)\mathcal{H}^2(\Delta_{n,i}) \\ &= \lim_{n\to\infty}\int_{\mathbb{S}^2}(\partial_\nu + \lambda \mathrm{tr}\,)v_n(d)\mathrm{d}\mathcal{H}^2(d). \end{aligned} \tag{4.44}$$

There exists C such that for all $d \in \mathbb{S}^2$ and $n \in \mathbb{N}$ we have

$$\|(\partial_\nu + \lambda \mathrm{tr}\,)v_n(d)\|_{H^{-\frac{1}{2}}(\partial D)} \leq C.$$

Hence by Lebesgue's Dominated Convergence Theorem, B.5, the limit in (4.44) can be taken inside the integral. In addition, for all $d \in \mathbb{S}^2$ the functions $v_n(d)$ converge to $T_h(d)$, so the elements $(\partial_\nu + \lambda \mathrm{tr}\,)v_n(d)$ converge in $H^{-\frac{1}{2}}(\partial D)$ to

$$(\partial_\nu + \lambda \mathrm{tr}\,)T_h(d) = (\partial_\nu + \lambda \mathrm{tr}\,)u_s(\cdot, d)h(d).$$

We continue the reasoning from formula (4.44) and see that

$$\begin{aligned} I &= \int_{\mathbb{S}^2} (\partial_\nu + \lambda \mathrm{tr}\,)u_s(\cdot, d)h(d)\mathrm{d}\mathcal{H}^2(d) \\ &= \int_{\mathbb{S}^2} -(\partial_\nu + \lambda \mathrm{tr}\,)e^{ikx\cdot d}h(d)\mathrm{d}\mathcal{H}^2(d). \end{aligned}$$

That is

$$(\partial_\nu + \lambda \mathrm{tr}\,)u_{sc}(x) = -\int_{\mathbb{S}^2} (\partial_\nu + \lambda \mathrm{tr}\,)e^{ikx\cdot d}h(d)\mathrm{d}\mathcal{H}^2(d).$$

Let $h \in L^2(\partial D)$. There is a sequence $(h_n) \subset C^\infty(\mathbb{S}^2)$ such that $h_n \to h$ in $L^2(\mathbb{S}^2)$. Hence it follows from equation (B.5) that in $H^1(D_R)$

$$\lim_{n\to\infty} \int_{\mathbb{S}^2} T_{h_n}(d)\mathrm{d}\mathcal{H}^2(d) = \int_{\mathbb{S}^2} T_h(d)\mathrm{d}\mathcal{H}^2(d).$$

Moreover in $H^{-\frac{1}{2}}(\partial D)$

$$\lim_{n\to\infty} \int_{\mathbb{S}^2} -(\partial_\nu + \lambda \mathrm{tr}\,)e^{ikx\cdot d}h_n(d)\mathrm{d}\mathcal{H}^2(d) = \int_{\mathbb{S}^2} -(\partial_\nu + \lambda \mathrm{tr}\,)e^{ikx\cdot d}h(d)\mathrm{d}\mathcal{H}^2(d),$$

so

$$(\partial_\nu + \lambda \mathrm{tr}\,)\int_{\mathbb{S}^2} T_h(d)\mathrm{d}\mathcal{H}^2(d) = \int_{\mathbb{S}^2} -(\partial_\nu + \lambda \mathrm{tr}\,)e^{ikx\cdot d}h(d)\mathrm{d}\mathcal{H}^2(d).$$

□

Next we prove that the far field of the scattered wave u_s is $\int_{\mathbb{S}^2} u_\infty(\widehat{x}, d)h(d)\mathrm{d}\mathcal{H}^2(d)$. This is a corollary of Lemma 4.6.

Corollary 4.13 *The far field of the exterior solution*

$$u_{sc}(x) = \int_{\mathbb{S}^2} u_s(x, d)h(d)\mathrm{d}\mathcal{H}^2(d)$$

is

$$u_\infty(\widehat{x}) = \int_{\mathbb{S}^2} u_\infty(\widehat{x}, d)h(d)\mathrm{d}\mathcal{H}^2(d)), \tag{4.45}$$

where $u_\infty(\widehat{x}, d)$ is the far field of $u_s(x, d)$.

Proof. By Lemma 4.6 the scattered plane waves have the far field expansion

$$u_s(x, d) = \frac{e^{ik|x|}}{|x|}\left(u_\infty(\widehat{x}, d) + \mathcal{O}_{k,d}\left(\frac{1}{|x|}\right)\right),$$

where the $\mathcal{O}_{k,d}\left(\frac{1}{|x|}\right)$ term is uniform with respect to $\widehat{x}$ and d. Hence we see that

$$\begin{aligned}
u_{sc}(x) &= \int_{\mathbb{S}^2} u_s(x, d)h(d)\mathrm{d}\mathcal{H}^2 \\
&= \frac{e^{ik|x|}}{|x|}\left(\int_{\mathbb{S}^2} u_{k,\infty}(\widehat{x}, d)h(d)\mathrm{d}\mathcal{H}^2(d) + \int_{\mathbb{S}^2} \mathcal{O}_{k,d}\left(\frac{1}{|x|}\right) h(d)\mathrm{d}\mathcal{H}^2(d)\right) \\
&= \frac{e^{ik|x|}}{|x|}\left(\int_{\mathbb{S}^2} u_{k,\infty}(\widehat{x}, d)h(d)\mathrm{d}\mathcal{H}^2(d) + \mathcal{O}_k\left(\frac{1}{|x|}\right)\right).
\end{aligned}$$

□

Now we introduce the **far field operator**, which is our main tool in solving the inverse problem and which will be our focus in the rest of the work. We will mainly use the imaginary part of this operator.

Definition 4.14 *Let $k \in \mathbb{R} \setminus \{0\}$. The far field operator $F_k : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ is defined by*

$$(F_k h)(\widehat{x}) = \int_{\mathbb{S}^2} u_{k,\infty}(\widehat{x}, d)h(d)\mathrm{d}\mathcal{H}^2(d).$$

We note that by Corollary 4.7 the kernel $u_{k,\infty}(\widehat{x}, d)$ is in $C^\infty(\mathbb{S}^2 \times \mathbb{S}^2)$. From this it follows that F_k is compact and that the inverse scattering problem is ill-posed.

As we saw in Corollary 4.13 the far field of a scattered wave of a Herglotz wave u_h is simply $F_k h$. For future reference, we state this as a corollary.

Corollary 4.15 *The far field operator F_k maps an element h in $L^2(\mathbb{S}^2)$ to the far field of the scattered wave*

$$u_{sc}(x) = \int_{\mathbb{S}^2} u_s(x, d)h(d)\mathrm{d}\mathcal{H}^2(d).$$

Next we prove an estimate for the norm of the far field operator F_k for different values of k.

Theorem 4.16 *Let $k \in \mathbb{R}\backslash\{0\}$. The far field operator $F_k : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ has the estimate*

$$\|F_k\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)} \leq \frac{4\pi}{|k|}. \tag{4.46}$$

Proof. Let our incident wave be a Herglotz wave

$$u_{in}(x) = \int_{\mathbb{S}^2} e^{ikx\cdot d} h(d) \mathrm{d}\mathcal{H}^2(d)$$

with $h \in C^\infty(\mathbb{S}^2)$.

By Lemma (4.9) we can decompose u_{in} as

$$u_{in}(x) = 2\pi i \frac{e^{ikr}}{kr} h(\widehat{x}) + 2\pi i \frac{e^{-ikr}}{kr} h(-\widehat{x}) + \mathcal{O}_h\left(\frac{1}{(kr)^2}\right),$$

Where $r = |x|$.

By Lemma 4.6 and Corollary 4.15 the scattered field can be expressed as

$$u_{sc} = \frac{e^{ikr}}{r}(F_k h)(\widehat{x}) + \mathcal{O}_{sc,k}\left(\frac{1}{r^2}\right).$$

Hence we can decompose the total field into incoming and outgoing parts

$$\begin{aligned} u_{tot} &= u_{in} + u_{sc} \\ &= \frac{e^{ikr}}{r} v_+(\widehat{x}) + \frac{e^{-ikr}}{r} v_-(\widehat{x}) + \mathcal{O}_{tot,k}\left(\frac{1}{r^2}\right), \end{aligned} \tag{4.47}$$

where

$$v_+(\widehat{x}) = \frac{2\pi i}{k} h(\widehat{x}) + (F_k h)(\widehat{x}) \qquad , \qquad v_-(\widehat{x}) = \frac{2\pi i}{k} h(-\widehat{x}).$$

and

$$\mathcal{O}_{tot,k}\left(\frac{1}{(r)^2}\right) = \mathcal{O}_h\left(\frac{1}{(kr)^2}\right) + \mathcal{O}_{sc,k}\left(\frac{1}{(r)^2}\right). \tag{4.48}$$

The total field satisfies $-(\Delta + k^2)u_{tot} = 0$ on D^+, so we have

$$\begin{aligned} 0 &= \lim_{R\to\infty} \int_{D_R} \left((\Delta u_{tot})\overline{u_{tot}} - u_{tot}\overline{\Delta u_{tot}}\right) dx \\ &= -\int_{\partial D} \left((\partial_\nu u_{tot})\overline{u_{tot}} - u_{tot}\overline{\partial_\nu u_{tot}}\right) \mathrm{d}\mathcal{H}^2 + \\ &\quad + \lim_{R\to\infty} \int_{\partial B(0,R)} \left((\partial_\nu u_{tot})\overline{u_{tot}} - u_{tot}\overline{\partial_\nu u_{tot}}\right) \mathrm{d}\mathcal{H}^2 \\ &= I_{\partial D} + \lim_{R\to\infty} I_R. \end{aligned} \tag{4.49}$$

Here ν denotes the exterior unit normal vector in the both integrals.

From the boundary condition $\partial_\nu u_{tot} + \lambda u_{tot} = 0$ it follows that

$$I_{\partial D} = -\int_{\partial D} \left(-\lambda |u_{tot}|^2 + \overline{\lambda} |u_{tot}|^2\right) \mathrm{d}\mathcal{H}^2.$$

As $\lambda = ik\alpha + \beta$ it follows that $\overline{\lambda} - \lambda = -2ik\alpha$ and hence

$$I_{\partial D} = 2ik \int_{\partial D} \alpha |u_{tot}|^2 \mathrm{d}\mathcal{H}^2. \tag{4.50}$$

We now turn to the integral I_R, in which the normal derivative ∂_ν is in fact the radial partial derivative ∂_r. From equation (4.48) we see that

$$\partial_r \mathcal{O}_{tot,k}\left(\frac{1}{r^2}\right) = \partial_r \mathcal{O}_h\left(\frac{1}{(kr)^2}\right) + \partial_r \mathcal{O}_{sc,k}\left(\frac{1}{r^2}\right). \tag{4.51}$$

By Lemma 4.10

$$\partial_r \mathcal{O}_h\left(\frac{1}{(kr)^2}\right) = \mathcal{O}\left(\frac{1}{kr^2}\right)$$

and by equations (4.30) and (4.28)

$$\partial_r \mathcal{O}_{sc,k}\left(\frac{1}{r^2}\right) = \mathcal{O}_k\left(\frac{1}{r^2}\right), \tag{4.52}$$

so we see that

$$\partial_r \mathcal{O}_{tot}\left(\frac{1}{r^2}\right) = \mathcal{O}_k\left(\frac{1}{r^2}\right). \tag{4.53}$$

Hence

$$\partial_r u_{tot}(r\widehat{x}) = ik\frac{e^{ikr}}{r} v_+(\widehat{x}) - ik\frac{e^{-ikr}}{r} v_-(\widehat{x}) + \mathcal{O}_k\left(\frac{1}{r^2}\right).$$

It follows that

$$\begin{aligned}
(\partial_r u_{tot})\overline{u_{tot}} &= \left(ik\frac{e^{ikr}}{r} v_+(\widehat{x}) - ik\frac{e^{-ikr}}{r} v_-(\widehat{x})\right)\left(\frac{e^{-ikr}}{r}\overline{v_+}(\widehat{x}) + \frac{e^{ikr}}{r}\overline{v_-}(\widehat{x})\right) + \\
&\quad + \mathcal{O}_k\left(\frac{1}{r^3}\right) \\
&= \frac{ik}{r^2}\left(v_+\overline{v_+} - v_-\overline{v_-} + e^{2ikr}\overline{v_-}v_+ - e^{-2ikr}v_-\overline{v_+}\right) + \mathcal{O}_k\left(\frac{1}{r^3}\right).
\end{aligned}$$

and

$$\begin{aligned}
(\overline{\partial_r u_{tot}})u_{tot} &= \overline{(\partial_r u_{tot})\overline{u_{tot}}} \\
&= \frac{-ik}{r^2}\left(v_+\overline{v_+} - v_-\overline{v_-} + e^{-2ikr}v_-\overline{v_+} - e^{2ikr}\overline{v_-}v_+\right) + \mathcal{O}_k\left(\frac{1}{r^3}\right).
\end{aligned}$$

We subtract these two terms:

$$(\partial_r u_{tot})\overline{u_{tot}} - (\overline{\partial_r u_{tot}})u_{tot} = \frac{2ik}{r^2}(v_+\overline{v_+} - v_-\overline{v_-}) + \mathcal{O}_k\left(\frac{1}{r^3}\right).$$

Hence

$$\lim_{R\to\infty} I_R = 2ik\int_{\mathbb{S}^2}(v_+\overline{v_+} - v_-\overline{v_-})\,\mathrm{d}\mathcal{H}^2$$

and it follows from equations (4.49) and (4.50) that

$$2ik\int_{\mathbb{S}^2}\left(|v_+|^2 - |v_-|^2\right)\mathrm{d}\mathcal{H}^2 = -2ik\int_{\partial D}\alpha|u|^2\mathrm{d}\mathcal{H}^2.$$

That is

$$\begin{aligned}\int_{\mathbb{S}^2}\left(|v_+|^2 - |v_-|^2\right)\mathrm{d}\mathcal{H}^2 &= -\int_{\partial D}\alpha|u|^2\mathrm{d}\mathcal{H}^2 \\ &\leq 0. \end{aligned} \tag{4.54}$$

We put $v_+ = \frac{2\pi i}{k}h(\widehat{x}) + (F_k h)(\widehat{x})$ and $v_-(\widehat{x}) = \frac{2\pi i}{k}h(-\widehat{x})$ in (4.54) and see that

$$\|\frac{2\pi i}{k}h + F_k h\|_{L^2(\mathbb{S}^2)} \leq \|\frac{2\pi i}{k}h\|_{L^2(\mathbb{S}^2)}.$$

Hence for all $h \in C^\infty(\mathbb{S}^2)$

$$\begin{aligned}\|F_k h\|_{L^2(\mathbb{S}^2)} &\leq \|\frac{2\pi i}{k}h\|_{L^2(\mathbb{S}^2)} + \|\frac{2\pi i}{k}h + F_k h\|_{L^2(\mathbb{S}^2)} \\ &\leq 2\|\frac{2\pi i}{k}h\|_{L^2(\mathbb{S}^2)} \\ &= \frac{4\pi}{|k|}\|h\|_{L^2(\mathbb{S}^2)}.\end{aligned}$$

Since $C^\infty(\mathbb{S}^2) \subset L^2(\mathbb{S}^2)$ is dense we have

$$\begin{aligned}\|F_k\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)} &= \sup_{h\in C^\infty(\mathbb{S}^2)}\frac{\|F_k h\|_{L^2(\mathbb{S}^2)}}{\|h\|_{L^2(\mathbb{S}^2)}} \\ &\leq \frac{4\pi}{|k|}.\end{aligned}$$

□

We summarise the scattering process of a Herglotz wave as follows:

By Lemma 4.9 a Herglotz wave u_h with density $h \in L^2(\mathbb{S}^2)$ has asymptotics

$$u_h(x) - 2\pi i \frac{e^{ik|x|}}{k|x|} h(\widehat{x}) - 2\pi i \frac{e^{-ik|x|}}{k|x|} h(-\widehat{x}) = \mathcal{O}_h\left(\frac{1}{(k|x|)^2}\right).$$

The corresponding scattered wave is

$$u_{sc}(x) = \frac{e^{ik|x|}}{|x|}\left(F_k h(\widehat{x}) + \mathcal{O}_k\left(\frac{1}{|x|}\right)\right),$$

so the total wave is

$$u_{tot} = 2\pi i \frac{e^{ik|x|}}{k|x|}\left(I - \frac{ik}{2\pi} F_k\right) h + 2\pi i \frac{e^{-ik|x|}}{k|x|} h(-\widehat{x}) + \mathcal{O}_k\left(\frac{1}{|x|^2}\right). \qquad (4.55)$$

We concentrate on the $\frac{1}{|x|}$ terms and describe the scattering process by the operator $S : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ which maps the Herglotz density h to

$$Sh := \left(I - \frac{ik}{2\pi} F_k\right) h.$$

4.4 Factorization and Properties of the Constituent Operators

In Grinberg's and Kirsch's article [20] the far field operator F of the Robin problem was given a factorization that we now present in the following Theorem, the proof of which follows the one found in the article. Our interest is mainly in the k dependence, so we are focusing on different aspects than in the Kirsch article, which can be seen in Theorem 4.19 where we derive k dependent coerciveness estimates for the operator A_k.

The G_k operator below maps the Robin boundary values of a solution u of the exterior problem to the far field u_∞ of the solution.

Theorem 4.17 *Let $k \in \mathbb{R} \setminus \{0\}$. The far field operator of the Robin problem $F_k : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ can be factorized as*

$$F_k = -G_k A_k^* G_k^*, \qquad (4.56)$$

where

$$A_k := (\Lambda_{k,+} + \lambda I)[\mathrm{S}_k(\Lambda_{k,+} + \overline{\lambda} I) + I]. \qquad (4.57)$$

If k^2 is not a Dirichlet eigenvalue on $-\Delta$ in D, the operator A_k can be written as

$$A_k = (\Lambda_{k,+} + \lambda I)\mathrm{S}_k(\Lambda_{k,-} + \lambda I)^*. \qquad (4.58)$$

Proof. When k^2 is not a Dirichlet eigenvalue on $-\Delta$ in D, we can express the far field of the scattered plane wave with operators G_k, $\Lambda_{k,-}$ as

$$u_{k,\infty}(\cdot, d) = -G_k(\Lambda_{k,-} + \lambda I)e^{ikx\cdot d}.$$

If the incoming wave is a Herglotz wave

$$H_k h(x) := \int_{\mathbb{S}^2} e^{ikx\cdot d} h(d) \mathrm{d}\mathcal{H}^2, \tag{4.59}$$

with $h \in L^2(\mathbb{S}^2)$, then by Corollary 4.15

$$-G_k(\Lambda_{k,-} + \lambda I)H_k h \;\; = \;\; F_k h.$$

Hence

$$F_k = -G_k(\Lambda_{k,-} + \lambda I)H_k. \tag{4.60}$$

For all $\psi \in L^2(\partial D)$

$$H_k^* \psi(d) = \int_{\partial D} e^{-ikd\cdot y} \psi(y) \mathrm{d}\mathcal{H}^2(y),$$

which is the far field of the single layer $\mathrm{SL}_k \psi$. Hence we see that

$$H_k^* = G_k(\Lambda_{k,+} + \lambda I)\mathrm{S}_k$$

and

$$H_k = \mathrm{S}_k^*(\Lambda_{k,+} + \lambda I)^* G_k^*.$$

It follows from (4.60) that

$$F_k = -G_k(\Lambda_{k,-} + \lambda I)\mathrm{S}_k^*(\Lambda_{k,+} + \lambda I)^* G_k^*,$$

which gives us the factorization with formula (4.58) for the operator A_k.

By the jump relation (4.2)

$$(\Lambda_{-k,-} - \Lambda_{-k,+})\mathrm{S}_{-k} = I,$$

where $(\Lambda_{-k,-} - \Lambda_{-k,+})\mathrm{S}_{-k} : H^{\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$. By taking adjungates on both sides we see that

$$\mathrm{S}_k(\Lambda_{k,-} - \Lambda_{k,+}) = I,$$

where $\mathrm{S}_k(\Lambda_{k,-} - \Lambda_{k,+}) : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$. Since $\Lambda_{k,-}$ is self-adjoint we see that

$$A_k = (\Lambda_{k,+} + \lambda I)[\mathrm{S}_k(\Lambda_{k,+} + \overline{\lambda} I) + I].$$

Operators S_k and $\Lambda_{k,+}$ are continuous with respect to the parameter k, so we see from equation (4.57) that operator A_k as well as its adjungate also depend continuously on k. By Corollary 4.5 the operator $\mathcal{B}_k^{-1}$ depends continuously on k, so it follows from equation 4.31 that operators G_k and G_k^* depend continuously on k. Since $F_k = G_k(\partial_\nu + \lambda \mathrm{tr}\,)$ we see that F_k also depends continuously on k.

The factorization

$$F_k = -G_k A_k^* G_k^*$$

holds for all $k \in \mathbb{R} \setminus \{0\}$ that are not Dirichlet eigenvalues of $-\Delta$ in D. All the operators in the factorization (4.61) depend continuously on k, so we see that the factorization holds for all $k \in \mathbb{R} \setminus \{0\}$.

$\square$

We prove next a coerciveness inequality for the middle operator A_k. This is inspired by Grinberg's Kirsch's article [20], though we will derive the coerciveness from a different term of equation (4.65) than was done in Grinberg's and Kirsch's article. We do it this way because our interest lies mainly in the k dependence and we want to have a k-dependent estimate for the coercivity.

Lemma 4.18 *Let $\psi \in H^{-\frac{1}{2}}(\partial D)$. Then for all $k \in \mathbb{R} \setminus \{0\}$*

$$-\mathrm{Im}\,(\psi, \mathrm{S}_k\psi)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}(\partial D)} = k \|G_{D,k}\mathrm{S}_k\psi\|^2_{L^2(\mathbb{S}^2)},$$

where $G_{D,k}$ is the G operator of the Dirichlet problem.

Proof. Let $v = \mathrm{SL}_k\psi$ in $\mathbb{R}^3$. By the jump relations (4.1) and (4.2) of the single layer operator

$$\psi = (\partial_\nu^- - \partial_\nu^+)v.$$

Hence by the definition of the normal derivatives ∂_ν^- and ∂_ν^+, equations (2.5) and (2.6) respectively, we have

$$\begin{aligned}
(\psi, \mathrm{S}_k\psi)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}(\partial D)} &= \big((\partial_\nu^- - \partial_\nu^+)v, \mathrm{tr}\,v\big)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}(\partial D)} \\
&= \int_{D \cup D_R} \big(|\nabla v|^2 - k^2|v|^2\big)\,\mathrm{d}x + \\
&\quad + \int_{\partial B(0,R)} \overline{(\partial_\nu v)} v \,\mathrm{d}\mathcal{H}^2. \qquad (4.61)
\end{aligned}$$

Function v satisfies

$$(\partial_\nu - ik)v(x) = \mathcal{O}_k\left(\frac{1}{R^2}\right),$$

so

$$\begin{aligned}
\overline{(\partial_\nu v)}v &= \overline{(\partial_\nu - ik)v}v - ik|v|^2 \\
&= -ik|v|^2 + \mathcal{O}_k\left(\frac{1}{R^3}\right),
\end{aligned} \tag{4.62}$$

since by the far field asymptotics, (4.6), we have $v = \mathcal{O}\left(\frac{1}{R}\right)$.
With the far field asymptotics

$$v = \frac{e^{ik|x|}}{|x|}\left(v_\infty(\widehat{x}) + \mathcal{O}_k\left(\frac{1}{R}\right)\right)$$

we also see that

$$|v(x)|^2 = \frac{1}{|x|^2}\left(|v_\infty(\widehat{x})|^2 + \overline{v_\infty(\widehat{x})}\mathcal{O}_k\left(\frac{1}{|x|}\right) + v_\infty(\widehat{x})\overline{\mathcal{O}_k\left(\frac{1}{|x|}\right)} + \left(\mathcal{O}_k\left(\frac{1}{|x|}\right)\right)^2\right).$$

Hence

$$\begin{aligned}
\lim_{R\to\infty}\int_{\partial B(0,R)} |v(x)|^2 \mathrm{d}\mathcal{H}^2(x) &= \lim_{R\to\infty}\frac{1}{R^2}\int_{\partial B(0,R)]} |v_\infty(\widehat{x})|^2 \mathrm{d}\mathcal{H}^2(\widehat{x}) \\
&= \int_{\mathbb{S}^2} |v_\infty(d)|^2 \mathrm{d}\mathcal{H}^2(d).
\end{aligned}$$

and we see that by equations (4.61) and (4.62) we have

$$\begin{aligned}
-\mathrm{Im}\,(\psi, \mathrm{S}_k\psi)_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}(\partial D)} &= k\lim_{R\to\infty}\int_{\partial B(0,R)} |v(x)|^2 \mathrm{d}\mathcal{H}^2(x) \\
&= k\int_{\mathbb{S}^2} |v_\infty(d)|^2 \mathrm{d}\mathcal{H}^2(d).
\end{aligned}$$

□

Next we prove the coerciveness result for the operator A_k.

Theorem 4.19 *Let $k \in \mathbb{R}\setminus\{0\}$. The middle operator A_k of decomposition (4.56) satisfies for $k > 0$ and all $\psi \in H^{\frac{1}{2}}(\partial D)$*

$$-\mathrm{Im}\,(A_k\psi, \psi)_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}(\partial D)} \geq k\alpha_0\|\psi\|^2_{L^2(\partial D)} \tag{4.63}$$

and for $k < 0$

$$-\mathrm{Im}\,(A_k\psi, \psi)_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}(\partial D)} \leq k\alpha_0\|\psi\|^2_{L^2(\partial D)}. \tag{4.64}$$

Proof. We first assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$. It follows from the jump relation (4.2) that

$$(\Lambda_{k,-} - \Lambda_{k,+})\,\mathrm{S}_k = I.$$

We see from equation (4.58) that

$$\begin{aligned} A_k &= (\Lambda_{k,-} + \lambda I)\mathrm{S}_k(\Lambda_{k,-} + \lambda I)^* - (\Lambda_{k,-} - \Lambda_{k,+})\mathrm{S}_k(\Lambda_{k,-} + \lambda I)^* \\ &= T\mathrm{S}_k T^* - (\Lambda_{k,-} + \overline{\lambda} I), \end{aligned} \tag{4.65}$$

where $T = (\Lambda_{k,-} + \lambda I)$.

By Lemma 4.18 for all $\psi \in H^{-\frac{1}{2}}(\partial D)$

$$-\mathrm{Im}\,(\psi, \mathrm{S}_k\psi)_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}(\partial D)} = k\|G_{D,k}\mathrm{S}_k\psi\|^2_{L^2(\mathbb{S}^2)}.$$

Operator $\Lambda_{k,-}$ is self-adjoint, in the sense of equation (3.5), so for all $\varphi \in H^{\frac{1}{2}}(\partial D)$ we have

$$\mathrm{Im}\,(\Lambda_{k,-}\varphi, \varphi)_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}(\partial D)} = 0.$$

Hence by (4.65) we have for $k > 0$

$$\begin{aligned} -\mathrm{Im}\,(A_k\varphi, \varphi)_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}(\partial D)} &= -\mathrm{Im}\,(\mathrm{S}_k T_k^*\psi, T_k^*\psi)_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}(\partial D)} + \\ &\quad +\mathrm{Im}\,(\Lambda_{k,-}\varphi, \varphi)_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}(\partial D)} + \mathrm{Im}\,(\overline{\lambda}\varphi, \varphi)_{L^2(\partial D)} \\ &\geq \int_{\partial D} k\overline{\alpha(x)}|\varphi(x)|^2 \mathrm{d}\mathcal{H}^2(x) \\ &\geq k\alpha_0\|\varphi\|^2_{L^2(\partial D)}. \end{aligned} \tag{4.66}$$

For $k < 0$ we have analogously

$$\begin{aligned} -\mathrm{Im}\,(A_k\varphi, \varphi)_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}(\partial D)} &\leq \int_{\partial D} k\overline{\alpha(x)}|\varphi(x)|^2 \mathrm{d}\mathcal{H}^2(x) \\ &\leq k\alpha_0\|\varphi\|^2_{L^2(\partial D)}. \end{aligned} \tag{4.67}$$

Since the operator A_k is continuous with respect to the k variable and since the Dirichelet eigenvalues of $-\Delta$ form a discrete set, we see that the inequalities (4.66) and (4.67) hold for all $k \in \mathbb{R} \setminus \{0\}$.

□

We multiply both sides of equation (4.56) with $\mathrm{sign}(-k)$ and take the imaginary parts which yields

$$\mathrm{Im}\,(\mathrm{sign}(-k)F_k) = G_k(\mathrm{sign}(-k)\mathrm{Im}\,A_k)G_k^*. \tag{4.68}$$

We will denote the coercive operator $(\mathrm{sign}(-k)\mathrm{Im}\,A_k)$ with B_k. With this notation for $k \in \mathbb{R} \setminus \{0\}$ the inequalities (4.63) and (4.64) are written as

$$(B_k \psi, \psi)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}} \geq |k| \alpha_0 \|\psi\|_{L^2}^2. \tag{4.69}$$

The appearence of the L^2 norm on the right-hand side of equation (4.69) requires some consideration. We solve this problem by first proving in Lemma 4.23 that B_k is continuous from $L^2(\partial D)$ to itself. Then we will proceed in Theorem 4.27 to prove the equality $\mathrm{Ran}\left(G_k|_{L^2(\partial D)}\right) = \mathrm{Ran}\left(F_k^{\frac{1}{2}}\right)$. In addition to these results we will prove norm estimates that depend on the wave number k in order to faciliate the frequency analysis of the inverse problem.

To prove the continuity of the operator B_k we need a few properties of the Dirichlet-to-Neumann map $\Lambda_{k,+} : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ and the operator $\mathrm{S}_k - \mathrm{S}_k^*$, which will be discussed in Lemmas 4.20 to 4.22. We start with a decomposition for the Dirichlet-to-Neumann map.

Lemma 4.20 *For all $k \in \mathbb{R} \setminus \{0\}$ the Dirichlet-to-Neumann map $\Lambda_{k,+} : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ satisfies $\Lambda_{k,+}^* = \Lambda_{-k,+}$ and has the decomposition*

$$\Lambda_{k,+} = \frac{1}{2}\left[(\Lambda_{k,+} + \Lambda_{-k,+}) + (\Lambda_{k,+} - \Lambda_{-k,+})\right],$$

where operator $L_k = \Lambda_{k,+} + \Lambda_{-k,+} : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is symmetric and the operator $K_k = \Lambda_{k,+} - \Lambda_{-k,+}$ is continuous $L^2(\partial D) \to L^2(\partial D)$ and satisfies

$$\|K_k\|_{L^2(\partial D) \to L^2(\partial D)} \leq C \langle k \rangle^2.$$

Proof. We first show the relation $\Lambda_{k,+}^* = \Lambda_{-k,+}$, where $\Lambda_{k,+}^*$ is the dual operator with respect to the $H^{-\frac{1}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D)$ -duality. Let $\varphi_1, \varphi_2 \in H^{\frac{1}{2}}(\partial D)$ and u_{φ_i} for $i \in \{1,2\}$ be solutions to the exterior Dirichlet problem (3.39) such that $\mathrm{tr}\, u_{\varphi_i} = \varphi_i$. We apply Green's Second Theorem in $D_R = B(0,R) \setminus \overline{D}$ and obtain

$$\begin{aligned}(\Lambda_{k,+}\varphi_1, \varphi_2)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}} - (\varphi_1, \Lambda_{-k,+}\varphi_2)_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}} &= \\ \int_{\partial B(0,R)} \left(\overline{(\partial_r u_{\varphi_1})} u_{\varphi_2} - \overline{u_{\varphi_1}} (\partial_r u_{\varphi_2})\right) \mathrm{d}\mathcal{H}^2.\end{aligned} \tag{4.70}$$

By Sommerfeld's radiation condition for $i \in \{1,2\}$

$$\partial_r u_{\varphi_i} = ik u_{\varphi_i} + v_i,$$

where v_i is of order $o(|x|^{-1})$. Hence

$$\left(\Lambda_{k,+}\varphi_1, \varphi_2\right)_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}} - \left(\varphi_1, \Lambda_{-k,+}\varphi_2\right)_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}} = \int_{\partial B(0,R)} \left(\overline{v_1} u_{\varphi_2} - \overline{u_{\varphi_1}} v_2\right) \mathrm{d}\mathcal{H}^2. \tag{4.71}$$

With Green's Second Theorem and Sommerfeld's radiation condition we can see, as is done in [14, Equation 2.9] for C^2 solutions, that for all R such that $\overline{D} \subset B(0,R)$ we have $u_{\varphi_i} \in L^2(\partial B(0,R))$ with $\|u_{\varphi_i}\|_{L^2(\partial B(0,R))} < C$, where C does not depend on R.

By Schwartz's Lemma

$$\int_{\partial B(0,R)} |v_j u_{\varphi_i}| \mathrm{d}\mathcal{H}^2 \leq \|v_j\|_{L^2(\partial B(0,R))} \|u_{\varphi_i}\|_{L^2(\partial B(0,R))}. \tag{4.72}$$

On the right-hand side of equation (4.72) the first term tends to zero as $R \to \infty$ and the second term is bounded, so the integral on the left-hand side tends to zero as $R \to \infty$. We apply this limit in (4.71) and see that

$$\left(\Lambda_{k,+}\varphi_1, \varphi_2\right)_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}} = \left(\varphi_1, \Lambda_{-k,+}\varphi_2\right)_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}}. \tag{4.73}$$

From (4.73) it follows that $\Lambda_{k,+}^* = \Lambda_{-k,+}$ and we see that the operator $\frac{1}{2}\left(\Lambda_{k,+} + \Lambda_{-k,+}\right)$ is symmetric.

We next prove that the operator $\frac{1}{2}\left(\Lambda_{k,+} - \Lambda_{-k,+}\right) : L^2(\partial D) \to L^2(\partial D)$ is continuous. To this end we consider the Dirichlet problem

$$\begin{cases} -(\Delta + k^2)u_k^f &= 0 \quad \text{in } D^+ \\ u_k^f|_{\partial D} &= f \\ \lim_{r\to\infty} r\left(\partial_r u_k^f - iku_k^f\right) &= 0, \quad \text{uniformly w.r.t. } \widehat{x} \in \mathbb{S}^2 \end{cases}$$

for $f \in H^{\frac{1}{2}}(\partial D)$ and $k \in \mathbb{R} \setminus \{0\}$.

Let $\chi \in C_0^\infty(\mathbb{R}^3)$ be such that $\chi \equiv 1$ in a neighbourhood of $\overline{D}$ and $w = \chi(u_k^f - u_{-k}^f)$. The element w satisfies

$$\begin{cases} -(\Delta + k^2)w &= \nabla\chi \cdot \nabla\left((u_k^f - u_{-k}^f\right) \\ &\quad +(\Delta\chi)\left((u_k^f - u_{-k}^f\right) \qquad \text{in } D^+ \\ w|_{\partial D} &= 0, \qquad w|_{B(0,R)} = 0. \end{cases}$$

We denote $F_k := \nabla\chi \cdot \nabla\left((u_k^f - u_{-k}^f\right) + (\Delta\chi)\left((u_k^f - u_{-k}^f\right)$.

We use elliptic regularity estimates, e.g. [17, Theorem 6.3.4], and see that

$$\begin{aligned}\|w\|_{H^2(D_R)} &\leq C\langle k\rangle^2\left(\|F_k\|_{L^2(D_R)}+\|w\|_{L^2(D_R)}\right)\\ &\leq C\langle k\rangle^2\|f\|_{H^{\frac{1}{2}}(\partial D)},\end{aligned}$$

since by Theorem 3.11 the solution operator U_k satisfies

$$\|U_k f\|_{H^1(D_R)} \leq C\|f\|_{H^{\frac{1}{2}}(\partial D)}.$$

As

$$\partial_\nu w = \left(\Lambda_{k,+}-\Lambda_{-k,+}\right) f,$$

we deduce that $K_k = \frac{1}{2}\left(\Lambda_{k,+}-\Lambda_{-k,+}\right) : H^{\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is bounded with $\|K_k\| \leq C\langle k\rangle^2$.

We have $-K_k = K_k^*$, which is by definition bounded from $H^{-\frac{1}{2}}(\partial D)$ to $H^{-\frac{1}{2}}(\partial D)$ with $\|K_k^*\| \leq C\langle k\rangle^2$. We interpolate, [39, Theorem 1.5.1], between $K_k : H^{\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ and $K_k : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ and we see that $K_k : L^2(\partial D) \to L^2(\partial D)$ is bounded with

$$\|K_k\|_{L^2(\partial D)\to L^2(\partial D)} \leq C\langle k\rangle^2.$$

□

Lemma 4.21 *Let D be a bounded C^2 domain. For $s \in [-\frac{1}{2}, \frac{3}{2}]$ the the Dirichlet-to-Neumann map $\Lambda_{k,+}$ has an extension $\Lambda_{k,+} : H^s(\partial D) \to H^{s-1}(\partial D)$, which satisfies*

$$\|\Lambda_{k,+}\|_{H^s(\partial D)\to H^{s-1}(\partial D)} \leq C\langle k\rangle^2.$$

Proof. Let $\varphi \in H^{\frac{3}{2}}(\partial D)$, $u_\varphi = U_k\varphi$, R be such that $\overline{D} \subset B(0,R)$ and $\chi \in C_0^\infty(B(0,R))$ with $\chi \equiv 1$ on a neighbourhood of $\overline{D}$. Now χu_φ satisfies

$$\begin{cases} \Delta(\chi u_\varphi) &= (\Delta\chi)u_\varphi + 2\nabla\chi\cdot\nabla u_\varphi - k^2\chi u_\varphi \quad \text{on } D_R \\ (\chi u_\varphi)|_{\partial D} = \varphi & (\chi u)|_{\partial B(0,R)} = 0. \end{cases}$$

It follows from [10, Theorem 1.5.2] that

$$\begin{aligned}\|\chi u_\varphi\|_{H^2(D_R)} &\leq C\left(\langle k\rangle^2\|u_\varphi\|_{L^2(D_R)}+\|u_\varphi\|_{H^1(D_R)}+\|\varphi\|_{H^{\frac{3}{2}}(\partial D)}\right)\\ &\leq C\langle k\rangle^2\|\varphi\|_{H^{\frac{3}{2}}(\partial D)},\end{aligned} \tag{4.74}$$

since by Theorem 3.11 we have

$$\|U_k\varphi\|_{H^1(D_R)} \leq C\|\varphi\|_{H^{\frac{3}{2}}(\partial D)}$$

for all $\varphi \in H^{\frac{3}{2}}(\partial D)$.

It follows from equation (4.74) that

$$\|\partial_\nu u_\varphi\|_{H^{\frac{1}{2}}(\partial D)} \leq C\langle k\rangle^2 \|\varphi\|_{H^{\frac{3}{2}}(\partial D)}.$$

Hence

$$\|\Lambda_k\|_{H^{\frac{3}{2}}(\partial D)\to H^{\frac{1}{2}}(\partial D)} \leq C\langle k\rangle^2. \tag{4.75}$$

From the definition of the adjoint $(\Lambda_k)^* : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{3}{2}}(\partial D)$ it follows that it is continuous and that $\|(\Lambda_k)^*\| = \|\Lambda_k\|$. On the other hand by equation (4.73) $\Lambda_k = (\Lambda_{-k})^*$ so we see that Λ_k has an extension $\Lambda_k : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{3}{2}}(\partial D)$, which satisfies

$$\begin{aligned}
\|\Lambda_k\|_{H^{-\frac{1}{2}}(\partial D)\to H^{-\frac{3}{2}}(\partial D)} &= \|(\Lambda_{-k})^*\|_{H^{-\frac{1}{2}}(\partial D)\to H^{-\frac{3}{2}}(\partial D)} \\
&= \|\Lambda_{-k}\|_{H^{\frac{3}{2}}(\partial D)\to H^{\frac{1}{2}}(\partial D)} \\
&\leq C\langle k\rangle^2
\end{aligned} \tag{4.76}$$

by equation (4.75). We interpolate, [39, Theorem 1.5.1], between formulas (4.75) and (4.76) and see that

$$\|\Lambda_k\|_{H^s\to H^{s-1}} \leq C\langle k\rangle^2,$$

for $s \in [-\frac{1}{2}, \frac{3}{2}]$.

□

We will prove next a continuity result for the operator $\mathrm{S}_k - \mathrm{S}_k^*$. This makes it possible to use the fact that the operator B_k is a subtraction of an operator and its adjungate in the proof of the continuity of $B_k : L^2(\partial D) \to L^2(\partial D)$.

Lemma 4.22 *Let D be a bounded C^2 domain and $k \in \mathbb{R}$. Then $\mathrm{S}_k - \mathrm{S}_k^* = \mathrm{S}_k - \mathrm{S}_{-k} : H^{-1}(\partial D) \to H^2(\partial D)$ with*

$$\|\mathrm{S}_k - \mathrm{S}_k^*\|_{H^{-1}(\partial D)\to H^2(\partial D)} \leq C(1+k^2)^2.$$

Proof. Let us first consider the operator $\mathcal{G}_k - \mathcal{G}_{(-k)} = (\Phi_k - \Phi_{-k})*$. To this end let R be such that $\overline{D} \subset B(0,R)$, $v \in L^2(B(0,R))$, $w_1 = \mathcal{G}_k v$ and $w_2 = \mathcal{G}_{-k} v$. Then in $\mathbb{R}^3$ we have

$$-(\Delta + k^2)w_1 = v = -(\Delta + k^2)w_2$$

and $u = w_1 - w_2$ satisfies

$$-(\Delta + k^2)u = 0.$$

Next we use elliptic regularity estimates. By [17, Theorem 6.3.2] we have

$$\begin{aligned}
\|u\|_{H^4(B(0,R))} &\leq C\langle k\rangle^2 \|u\|_{H^2(B(0,R+1))} \\
&\leq C\langle k\rangle^4 \|u\|_{L^2(B(0,R+2))} \\
&\leq C\langle k\rangle^4 \|v\|_{L^2(B(0,R))},
\end{aligned}$$

since by Schur's Test $\mathcal{G}_k : L^2(B(0,R)) \to L^2(B(0,R+2))$ as we saw in the proof of Theorem 2.12.

Hence $\mathcal{G}_k - \mathcal{G}_{-k} : L^2(B(0,R)) \to H^4(B(0,R))$ with

$$\|\mathcal{G}_k - \mathcal{G}_{-k}\|_{L^2(B(0,R))\to H^4(B(0,R))} \leq C\langle k\rangle^4. \tag{4.77}$$

Let us consider the operator $T_k = \chi(\mathcal{G}_k - \mathcal{G}_{-k})\chi$, where χ denotes multiplication by function $\chi \in C_0^\infty(B(0,R))$ with the property $\chi \equiv 1$ in a neighbourhood of $\overline{D}$. From equation (4.77) it follows that

$$\begin{aligned}
\|T_k\|_{L^2(\mathbb{R}^3)\to H^4(\mathbb{R}^3)} &\leq C\|\mathcal{G}_k - \mathcal{G}_{-k}\|_{L^2(B(0,R))\to H^4(B(0,R))} \\
&\leq C\langle k\rangle^4. \qquad (4.78)
\end{aligned}$$

We have $T_k^* = -T_k$, so we have an extension $T_k : H^{-4}(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ with

$$\begin{aligned}
\|T_k\|_{H^{-4}(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} &= \|T_k\|_{L^2(\mathbb{R}^3)\to H^4(\mathbb{R}^3)} \\
&\leq C\langle k\rangle^4. \qquad (4.79)
\end{aligned}$$

We interpolate, [39, Theorem 1.5.1], between equations (4.78) and (4.79) and see that $T_k : H^{-\frac{3}{2}}(\mathbb{R}^3) \to H^{\frac{5}{2}}(\mathbb{R}^3)$ with

$$\|T_k\|_{H^{-\frac{3}{2}}(\mathbb{R}^3)\to H^{\frac{5}{2}}(\mathbb{R}^3)} \leq C(1+k^2)^2.$$

Let us consider

$$\mathrm{S}_k - \mathrm{S}_{-k} = \mathrm{tr}\, T_k \mathrm{tr}^*\,.$$

Since $\mathrm{tr}^* : H^{-1}(\partial D) \to H^{-\frac{3}{2}}(\mathbb{R}^3)$ and $\mathrm{tr} : H^{\frac{5}{2}}(\mathbb{R}^3) \to H^2(\partial D)$ it follows that $\mathrm{S}_k - \mathrm{S}_{-k}$ is $H^{-1}(\partial D) \to H^2(\partial D)$ with

$$\|\mathrm{S}_k - \mathrm{S}_{-k}\|_{H^{-1}(\partial D)\to H^2(\partial D)} \leq C\langle k\rangle^4.$$

□

Our goal is to prove that the ranges of $\mathrm{Im}(F_k)$ and G_k are the same. Hence we need to study the operator

$$\begin{aligned} B_k &= \mathrm{sign}(-k)\mathrm{Im}(A_k) \\ &= \mathrm{sign}(-k)\frac{1}{2i}(A_k - A_k^*) \end{aligned}$$

more closely.

Since $A_k : H^{\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ and $A^* : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ we see that $B_k : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is continuous. In the next lemma we derive a stricter mapping property for B_k that is better suited for our designs regarding the ranges of $\mathrm{Im}(F_k)$ and B_k.

Lemma 4.23 *Operator B_k is continuous $L^2(\partial D) \to L^2(\partial D)$ with*

$$\|B_k\|_{L^2(\partial D)\to L^2(\partial D)} \le C\langle k\rangle^8. \tag{4.80}$$

Proof. In this proof $\Lambda_{k,+} = \Lambda_k$. We use the factorization (4.57) to see that

$$\begin{aligned} A_k &= (\Lambda_k + \lambda I)[\mathrm{S}_k(\Lambda_k + \overline{\lambda} I) + I] \\ A_k^* &= [I + (\Lambda_k + \overline{\lambda} I)^* \mathrm{S}_k^*](\Lambda_k + \lambda I)^*. \end{aligned}$$

The decomposition $\Lambda_k = K_k + L_k$ of Lemma 4.20 implies

$$\begin{aligned} A_k - A_k^* &= [(\Lambda_k + \lambda I) - (\Lambda_k + \lambda I)^*] + L_k(\mathrm{S}_k - \mathrm{S}_k^*)L_k + L_k\mathrm{S}_k(K + \overline{\lambda} I) + \\ &\quad + (K_k + \lambda I)\mathrm{S}_k(\Lambda_k + \overline{\lambda} I) + L_k\mathrm{S}_k^*(K_k + \lambda I)^* + \\ &\quad + (K_k + \lambda I)^*\mathrm{S}_k^*(\Lambda_k + \lambda I)^*. \end{aligned} \tag{4.81}$$

Next we will prove that each term in (4.81) is bounded from $L^2(\partial D)$ to $L^2(\partial D)$.

We have

$$(\Lambda_k + \lambda I) - (\Lambda_k + \lambda I)^* = 2K_k + (\lambda - \overline{\lambda})I,$$

where λ is a C^2 function, so by Lemma 4.20 we have

$$\|(\Lambda_k + \lambda I) - (\Lambda_k + \lambda I)^*\|_{L^2(\partial D)\to L^2(\partial D)} \le C\langle k\rangle^2.$$

By Lemma 4.21 we have $\|L_k\|_{L^2\to H^{-1}} \le C\langle k\rangle^2$ and $\|L_k\|_{H^1\to L^2} \le C\langle k\rangle^2$. By Lemma 4.22, $\|\mathrm{S}_k - \mathrm{S}_k^*\|_{H^{-1}\to H^2} \le C\langle k\rangle^4$, so we see that

$$L_k(\mathrm{S}_k - \mathrm{S}_k^*)L_k : L^2(\partial D) \xrightarrow{L_k} H^{-1}(\partial D) \xrightarrow{\mathrm{S}_k - \mathrm{S}_k^*} H^2(\partial D) \xrightarrow{L_k} H^1(\partial D) \hookrightarrow L^2(\partial D),$$

and that

$$\|L_k\left(\mathrm{S}_k - \mathrm{S}_k^*\right)L_k\|_{L^2(\partial D)\to H^1(\partial D)} \le C\langle k\rangle^8.$$

By Lemma 4.20, $\|K_k\|_{L^2\to L^2} \le C\langle k\rangle^2$, by Corollary 2.15, $\|\mathrm{S}_k\|_{L^2\to H^1} \le C\langle k\rangle^2$ and by Lemma 4.21, $\|L_k\|_{H^1\to L^2} \le C\langle k\rangle^2$, so we see that

$$L_k\mathrm{S}_k(K_k+\overline{\lambda}I) : L^2(\partial D) \xrightarrow{K_k+\overline{\lambda}I} L^2(\partial D) \xrightarrow{\mathrm{S}_k} H^1(\partial D) \xrightarrow{L_k} L^2(\partial D)$$

and that

$$\|L_k\mathrm{S}_k(K_k+\lambda I)\|_{L^2\to L^2} \le C\langle k\rangle^6.$$

By Lemma 4.21, $\|\Lambda_k\|_{L^2\to H^{-1}} \le C\langle k\rangle^2$, by Corollary 2.15, $\|\mathrm{S}_k\|_{H^{-1}\to L^2} \le C\langle k\rangle^2$ and by Lemma 4.20, $\|K_k\|_{L^2\to L^2} \le C\langle k\rangle^2$, so we see that

$$(K_k+\lambda I)\mathrm{S}_k(\Lambda_k+\overline{\lambda}I) : L^2(\partial D) \xrightarrow{\Lambda_k+\overline{\lambda}I} H^{-1}\partial D \xrightarrow{\mathrm{S}_k} L^2(\partial D) \xrightarrow{K_k+\lambda I} L^2(\partial D)$$

and that

$$\|(K_k+\lambda I)\mathrm{S}_k(\Lambda_k+\lambda I)\|_{L^2(\partial D)\to L^2(\partial D)} \le C\langle k\rangle^6.$$

By Lemma 4.20, $\|K_k\|_{L^2\to L^2} \le C\langle k\rangle^2$, by Theorem 2.15, $\|\mathrm{S}_k^*\|_{L^2\to H^1} \le C\langle k\rangle^2$ and by Lemma 4.21, $\|L_k\|_{H^1\to L^2} \le C\langle k\rangle^2$, so we have

$$L_k\mathrm{S}_k^*(K_k+\lambda I) : L^2(\partial D) \xrightarrow{K_k+\lambda I} L^2(\partial D) \xrightarrow{\mathrm{S}_k^*} H^1(\partial D) \xrightarrow{L_k} L^2(\partial D)$$

and

$$\|L_k\mathrm{S}_k^*(K_k+\lambda I)\|_{L^2(\partial D)\to L^2(\partial D)} \le C\langle k\rangle^6.$$

Finally, by Lemma 4.21, $\|\Lambda_k\|_{L^2\to H^{-1}} \le C\langle k\rangle^2$, by Corollary 2.15, $\|\mathrm{S}_k^*\|_{H^{-1}\to L^2} \le C\langle k\rangle^2$ and by Lemma 4.20, $\|K_k\|_{L^2\to L^2} \le C\langle k\rangle^2$, so we have

$$(K_k+\lambda I)^*\mathrm{S}_k^*(\Lambda_k+\lambda I)^* : L^2(\partial D) \xrightarrow{\Lambda_k+\lambda I} H^1(\partial D) \xrightarrow{\mathrm{S}_k^*} L^2(\partial D) \xrightarrow{(K_k+\lambda I)^*} L^2(\partial D)$$

and

$$\|(K_k+\lambda I)\mathrm{S}_k^*(\Lambda_k+\lambda I)^*\|_{L^2(\partial D)\to L^2(\partial D)} \le C\langle k\rangle^6.$$

Hence we have

$$\begin{aligned}\|B_k\|_{L^2(\partial D)\to L^2(\partial D)} &= \|\mathrm{Im}\, A_k\|_{L^2(\partial D)\to L^2(\partial D)} \\ &\le C\langle k\rangle^8.\end{aligned}$$

□

One of the ramifications of Lemma 4.23 is that in the factorization Im $(\mathrm{sign}(-k)F_k) = G_k B_k G_k^*$ we can take B_k to be the restriction $B_k|_{L^2(\partial D)}$ and the operator G_k^* to be the adjoint of $G_k|_{L^2(\partial D)}$. This approach is taken in section 4.5 and eases our use of the operator calculus.

With the continuity of $B_k : L^2(\partial D) \to L^2(\partial D)$ we are able to prove the coerciveness of the operator Im $(\mathrm{sign}(-k)F_k)$, which is part of the following lemma.

Theorem 4.24 *Operator* $\mathrm{Im}\,(\mathrm{sign}(-k)F_k) : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ *is self-adjoint, compact and positive.*

Proof. Operator $\mathrm{Im}\,(\mathrm{sign}(-k)F_k) : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ is bounded and symmetric, so it is self-adjoint.

We proved in Corollary 4.7 that the kernel $u_\infty(\widehat{x}, d)$ of the far field operator F_k is in $C^\infty(\mathbb{S}^2 \times \mathbb{S}^2)$, so we see that in fact

$$F_k : L^2(\mathbb{S}^2) \to C^\infty(\mathbb{S}^2) \hookrightarrow L^2(\mathbb{S}^2), \tag{4.82}$$

where the imbedding is compact. Hence F_k is compact. The kernel of F_k^* is $\overline{u_\infty(d, \widehat{x})}$, so we see with the same argument that F_k^* is also compact. Hence $\mathrm{Im}\,(\mathrm{sign}(-k)F_k)$ is compact.

Let $k > 0$. The factorization

$$\mathrm{Im}\,(\mathrm{sign}(-k)F_k) = G_k B_k G_k^* \tag{4.83}$$

and the coerciveness equation (4.69) imply that for all $h \in L^2(\mathbb{S}^2)$ we have

$$\begin{aligned} (\mathrm{Im}\,(\mathrm{sign}(-k)F_k)h, h)_{L^2} &= (G_k B_k G_k^* h, h)_{L^2} \\ &= (B_k G_k^* h, G_k^* h)_{L^2} \\ &\geq |k|\alpha_0 \|G_k^* h\|_{L^2}^2. \end{aligned}$$

□

We have seen that the operator $B_k : L^2(\partial D) \to L^2(\partial D)$ is continuous. As it also is self-adjoint, we can apply the spectral calculus, see e.g. [57, sections XI.5 and XI.12]. In spectral calculus we express functions of an operator T as an integral

$$f(T) = \int_{-\infty}^{\infty} f(\lambda) dP(\lambda),$$

where P is the unique resolution of identity that gives the spectral resolution of the operator T.

We gather together a few properties of the operator $B_k^{\frac{1}{2}}$ in the next lemma.

Lemma 4.25 *Let $k \in \mathbb{R} \setminus \{0\}$. The operator B_k is invertible and has an invertible square root $B_k^{\frac{1}{2}}$ which satisfies for all $\psi \in L^2(\partial D)$*

$$C\langle k\rangle^4 \|\psi\|_{L^2(\partial D)} \geq \|B_k^{\frac{1}{2}}\psi\|_{L^2(\partial D)} \geq (|k|\alpha_0)^{\frac{1}{2}}\|\psi\|_{L^2(\partial D)}. \tag{4.84}$$

Proof. Let $k \in \mathbb{R} \setminus \{0\}$. From equation (4.69) and the continuity of $B_k : L^2(\partial D) \to L^2(\partial D)$ it follows that for all $\psi \in L^2(\partial D)$

$$(B_k\psi, \psi)_{L^2(\partial D)} \geq k\alpha_0\|\psi\|^2_{L^2(\partial D)}. \tag{4.85}$$

Operator B_k is self-adjoint and

$$\|B_k\|_{L^2(\partial D)\to L^2(\partial D)} \leq C\langle k\rangle^8,$$

so it follows from (4.85) that

$$B_k = \int_{\alpha_0 k}^{C\langle k\rangle^8} s\mathrm{d}P(s).$$

Functions $s \to s^{-1}$, $s \to s^{\frac{1}{2}}$ and $s \to s^{-\frac{1}{2}}$ are bounded on $[\alpha_0 k, C\langle k\rangle^8]$, so we can form the operators B^{-1}, $B^{\frac{1}{2}}$ and $B^{-\frac{1}{2}}$. Since the function $s \to s^{\frac{1}{2}}$ has real values on $[\alpha_0 k, C\langle k\rangle^8]$, the operator $B_k^{\frac{1}{2}}$ is self-adjoint and we have

$$\begin{aligned}\|B_k^{\frac{1}{2}}\psi\|^2_{L^2(\partial D)} &= (B_k\psi, \psi)_{L^2(\partial D)}\\ &\geq k\alpha_0\|\psi\|^2_{L^2(\partial D)},\end{aligned}$$

by inequality (4.85).

From the spectral representation

$$B_k^{\frac{1}{2}} = \int_{\alpha_0 k}^{C\langle k\rangle^8} s^{\frac{1}{2}}\mathrm{d}P(s)$$

we can infer that

$$\|B_k^{\frac{1}{2}}\|_{L^2(\partial D)\to L^2(\partial D)} \leq C\langle k\rangle^4.$$

Hence $B_k^{\frac{1}{2}}$ satisfies (4.84).

□

The next result regarding the properties of $\mathrm{Im}(F_k)^{\frac{1}{2}}$ and G_k will be useful in proving that the ranges of these operators are the same in Theorem 4.27.

Theorem 4.26 *Let $k \in \mathbb{R} \setminus \{0\}$. Operators* $\mathrm{Im}\,(F_k) : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$, $G_k : L^2(\partial D) \to L^2(\mathbb{S}^2)$ *and* $G_k^* : L^2(\mathbb{S}^2) \to L^2(\partial D)$ *are injective and have dense ranges.*

Proof. Suppose that for some $\varphi \in L^2(\partial D)$ we have $G_k\varphi = 0$. To see what this implies, let u_φ be the solution $U_{k,rob}\varphi$. Equation $G_k\varphi = 0$ means that the far field $u_{\varphi,\infty} \equiv 0$, which implies by Rellich's Lemma, [14, Lemma 2.11], that $u_\varphi = 0$ in D^+. Hence also the Robin boundary values of u_φ are zero, that is $\varphi = 0$. Hence G_k is injective and $\overline{\mathrm{Ran}\,(G_k^*)} = N(G_k)^\perp = L^2(\partial D)$, that is the range of G_k^* is dense.

We next prove that the range of G_k is dense. To this end we define the operator $T : L^2(B) \to L^2(\mathbb{S}^2)$, which maps a source $h \in L^2(B)$ from a domain $B \subset\subset D$ to the far field of

$$u_h(x) = \int_B \Phi_k(x,y)h(y)dy.$$

That is,

$$\begin{aligned} Th &= u_{h,\infty}(\widehat{x}) \\ &= \int_B \Phi_{k,\infty}(\widehat{x},y)h(y)dy, \end{aligned}$$

where $\Phi_{k,\infty}(\widehat{x},y) = \frac{1}{4\pi}e^{-ik\widehat{x}\cdot y}$.

The adjoint of $T^* : L^2(\mathbb{S}^2) \to L^2(B)$ maps $p \in L^2(\mathbb{S}^2)$ to

$$\begin{aligned} T^*p(y) &= \int_{\mathbb{S}^2} \overline{\Phi_{k,\infty}(\widehat{x},y)}p(\widehat{x})\mathrm{d}\mathcal{H}^2(\widehat{x}) \\ &= \frac{1}{4\pi}\int_{\mathbb{S}^2} e^{-ik\widehat{x}\cdot y}p(\widehat{x})\mathrm{d}\mathcal{H}^2(\widehat{x}). \end{aligned} \tag{4.86}$$

The function T^*p is real analytic, so we see by analytic continuation that $T^*p = 0$ on B implies that $T^*p = 0$ on $\mathbb{R}^3$. By [14, Theorem 3.15] this implies that $p = 0$. Hence T^* is injective and $\mathrm{Ran}\,(T)$ is dense.

The restriction of u_h to D^+ is a solution of the Robin problem (4.4). Hence we see that $\mathrm{Ran}\,(T) \subset \mathrm{Ran}\,(G_k)$ and also that $\mathrm{Ran}\,(G_k) \subset L^2(\mathbb{S}^2)$ is dense.

The range of G_k is dense, so it follows that G_k^* is injective.

Next we will prove that $\mathrm{Im}\,(F_k)$ is injective and that it has a dense range. By equation (4.68)

$$\mathrm{Im}\,(F_k) = -G_kB_kG_k^*$$

By Lemma 4.25 the operator B_k is bijective, so $\mathrm{Im}\,(F_k)$ is injective as it is a combination of three injective operators.

To see that the range is dense, let $\varphi \in L^2(\mathbb{S}^2)$. There is a sequence $(G_k\varphi_n) \subset \mathrm{Ran}\,(G_k)$ such that $G_k\varphi_n \to \varphi$. The mapping B_k is bijective, so we can form the sequence $(B_k^{-1}\varphi_n)$. The range of G_k^* is also dense so for every n there exists a ρ_n such that $\|G_k^*\rho_n - B_k^{-1}\varphi_n\|_{L^2(\mathbb{S}^2)} < \frac{1}{n}$. We have

$$\begin{aligned}\|G_kB_kG_k^*\rho_n - G_k\varphi_n\|_{L^2(\mathbb{S}^2)} &\leq \|G_k\|\,\|B_k\|\,\|G_k^*\rho_n - B_k^{-1}\varphi_n\| \\ &\leq \|G_k\|\,\|B_k\|\frac{1}{n},\end{aligned}$$

so the sequence $(G_kB_kG_k^*\rho_n)$ converges to φ and the range of $\mathrm{Im}\,(F_k)$ is dense.

□

4.5 Single Frequency Reconstruction

In this section we derive the single frequency method, which is given in Theorem 4.29.

We start with k-dependent estimates for the operators G_k^{-1} and $\mathrm{Im}\,(F_k)^{\frac{1}{2}}$. It would perhaps make the calculations more concise if we used the operator $|\mathrm{Im}\,(F_k)|^{\frac{1}{2}}$ rather than $\mathrm{Im}\,(F_k)^{\frac{1}{2}}$ as we do below. We use $\mathrm{Im}\,(F_k)^{\frac{1}{2}}$ since this will avoid certain difficulties in sections 5.3 and 5.4. For purely single frequency purposes $|\mathrm{Im}\,(F_k)|^{\frac{1}{2}}$ is a viable option, perhaps even a better one.

Theorem 4.27 *Let $k \in \mathbb{R}\setminus\{0\}$. We have $\mathrm{Ran}\left(\mathrm{Im}\,(F_k)^{\frac{1}{2}}\right) = \mathrm{Ran}\,(G_k)$ and for all $\varphi \in \mathrm{Ran}\,(G_k)$ it holds that*

$$\frac{1}{C\langle k\rangle^4}\|G_k^{-1}\varphi\|_{L^2(\partial D)} \leq \|\mathrm{Im}\,(F_k)^{-\frac{1}{2}}\varphi\|_{L^2(\mathbb{S}^2)} \leq \sqrt{\frac{1}{\alpha_0|k|}}\|G_k^{-1}\varphi\|_{L^2(\partial D)}. \quad (4.87)$$

Proof. Let $k \in \mathbb{R}\setminus\{0\}$. We start with the factorization

$$\mathrm{Im}\,(\mathrm{sign}(-k)F_k) = \left(G_k|_{L^2(\partial D)}\right) B_k \left(G_k|_{L^2(\partial D)}\right)^*. \quad (4.88)$$

As we saw in Lemma 4.25 the operator B_k has a self-adjoint square root. We use it in the factorization (4.88), which gives

$$\begin{aligned}\mathrm{Im}\,(\mathrm{sign}(-k)F_k) &= G_kB_k^{\frac{1}{2}}B_k^{\frac{1}{2}}G_k^* \\ &= \left(G_kB_k^{\frac{1}{2}}\right)\left(G_kB_k^{\frac{1}{2}}\right)^*.\end{aligned}$$

Operator $\widetilde{G}_k^* = (G_k B_k^{\frac{1}{2}})^*$ is compact, so we can apply Picard's Theorem, [14, Theorems 4.7 and 4.8]. Picard's Theorem shows that $\widetilde{G}_k^*$ has a singular system:

$$\begin{aligned} \mathrm{Im}(\mathrm{sign}(-k)F_k)\varphi &= \sum_{j=1}^{\infty} s_j^2 (\varphi_j, \varphi)_{L^2(\mathbb{S}^2)} \varphi_j \\ \widetilde{G}_k^* \varphi &= \sum_{j=1}^{\infty} s_j (\varphi_j, \varphi)_{L^2(\mathbb{S}^2)} \psi_j, \end{aligned} \tag{4.89}$$

where $\{\varphi_j\}$ and $\{s_j^2\}$ are the eigenvectors and eigenvalues of the compact, positive and self-adjoint operator $\mathrm{Im}(\mathrm{sign}(-k)F_k)$ and $\{\psi_j\} = \{\frac{1}{s_j}\widetilde{G}_k^* \varphi_j\}$. The eigenvectors $\{\varphi_j\}$ form a basis in $L^2(\mathbb{S}^2)$ and since the range of $\widetilde{G}_k^*$ is dense in $L^2(\partial D)$, the set $\{\psi_j\}$ is an orthonormal basis in $L^2(\partial D)$.

From (4.89) we can deduce a singular value decomposition for $\widetilde{G}_k = G_k B_k^{\frac{1}{2}}$

$$\widetilde{G}_k \psi = \sum_{j=1}^{\infty} s_j (\psi_j, \psi)_{L^2(\partial D)} \varphi_j. \tag{4.90}$$

If $\varphi = \widetilde{G}_k \psi$ then

$$\begin{aligned} \sum_{j=1}^{\infty} s_j^{-1} (\varphi_j, \varphi)_{L^2(\mathbb{S}^2)} \psi_j &= \sum_{j=1}^{\infty} s_j^{-1} (\varphi_j, \widetilde{G}_k \psi)_{L^2(\mathbb{S}^2)} \psi_j \\ &= \sum_{j=1}^{\infty} s_j^{-1} (\widetilde{G}_k^* \varphi_j, \psi)_{L^2(\partial D)} \psi_j \\ &= \sum_{j=1}^{\infty} (\psi_j, \psi)_{L^2(\partial D)} \psi_j \\ &= \psi, \end{aligned}$$

since $\{\psi_j\}$ is an orthonormal basis of $L^2(\partial D)$. We also see that

$$\|\widetilde{G}_k^{-1} \varphi\|_{L^2(\partial D)}^2 = \sum_{j=1}^{\infty} |s_j^{-1} (\varphi_i, \varphi)_{L^2(\mathbb{S}^2)}|^2.$$

On the other hand, if we have

$$\sum_{j=1}^{\infty} |s_j^{-1} (\varphi_i, \varphi)_{L^2(\mathbb{S}^2)}|^2 < \infty \tag{4.91}$$

for $\varphi \in L^2(\mathbb{S}^2)$, then we can form an element $\psi = \sum s_j^{-1}(\varphi_j, \varphi)_{L^2}\psi_j$ in $L^2(\partial D)$ for which $\widetilde{G}_k\psi = \varphi$. Hence (4.91) is equivalent to $\varphi \in \mathrm{Ran}\left(\widetilde{G}_k\right)$.

We apply the singular value decomposition argument, which was just used for $\widetilde{G}_k$, to $\mathrm{Im}\,(\mathrm{sign}(-k)F_k)^{\frac{1}{2}}$ defined by

$$\mathrm{Im}\,(\mathrm{sign}(-k)F_k)^{\frac{1}{2}}\varphi = \sum_{j=1}^{\infty} s_j(\varphi_j, \varphi)_{L^2(\mathbb{S}^2)}\varphi_j. \tag{4.92}$$

We see from equations (4.92) and (4.90) that $\mathrm{Ran}\left(\widetilde{G}_k\right) = \mathrm{Ran}\left(\mathrm{Im}\,(F_k)^{\frac{1}{2}}\right)$ and from (4.91) that for all $\varphi \in \mathrm{Ran}\left(\widetilde{G}_k\right)$

$$\begin{aligned} \|\widetilde{G}_k^{-1}\varphi\|_{L^2(\partial D)} &= \|\mathrm{Im}\,(\mathrm{sign}(-k)F_k)^{-\frac{1}{2}}\varphi\|_{L^2(\mathbb{S}^2)} \\ &= \|\mathrm{Im}\,(F_k)^{-\frac{1}{2}}\varphi\|_{L^2(\mathbb{S}^2)}. \end{aligned} \tag{4.93}$$

By Lemma 4.25 the operator $B_k^{\frac{1}{2}}$ is invertible and hence surjective. We see that

$$\begin{aligned} \mathrm{Ran}\left(\widetilde{G}_k\right) &= \mathrm{Ran}\left(G_k B_k^{\frac{1}{2}}\right) \\ &= \mathrm{Ran}\,(G_k)\,, \end{aligned}$$

and by (4.93) $\mathrm{Ran}\left(\mathrm{Im}\,(F_k)^{\frac{1}{2}}\right) = \mathrm{Ran}\,(G_k)$.

From equation (4.84) it follows that

$$\frac{1}{C\langle k\rangle^4}\|\psi\|_{L^2(\partial D)} \le \|B_k^{-\frac{1}{2}}\psi\|_{L^2(\partial D)} \le \sqrt{\frac{1}{\alpha_0|k|}}\|\psi\|^2_{L^2(\partial D)}. \tag{4.94}$$

We use equation (4.93) in (4.94) and see that

$$\frac{1}{C\langle k\rangle^4}\|G_k^{-1}\varphi\|_{L^2(\partial D)} \le \|\mathrm{Im}\,(F_k)^{-\frac{1}{2}}\varphi\|_{L^2(\mathbb{S}^2)} \le \sqrt{\frac{1}{\alpha_0|k|}}\|G_k^{-1}\varphi\|_{L^2(\partial D)},$$

for all $\varphi \in \mathrm{Ran}\,(G_k)$.

□

We next form a criterion for the Robin single frequency construction and analyze the k-dependence of this construction. We start by recalling the function $r_{z,k}(\widehat{x}) = e^{-ikz\cdot\widehat{x}} = 4\pi\Phi_{k,\infty}(\widehat{x}, z)$, that is 4π times the far field of the fundamental solution $\Phi_k(\cdot, z)$, and reproduce [34, Theorem 3.7], which provides a way to find the obstacle when $\mathrm{Ran}\,(G_k)$ is known.

Lemma 4.28 *Let $k \neq 0$. Then $z \in D$ if and only if $r_{z,k} \in \mathrm{Ran}\,(G_k)$.*

Proof. Let $z \in D$. Then $4\pi\Phi_k(\cdot, z)$ is a solution to the exterior Robin problem with the far field $r_{z,k}$ and $G_k(\partial_\nu + \lambda \mathrm{tr}\,)4\pi\Phi_k(\cdot, z) = r_{z,k}$.

Let $z \notin D$. If for some exterior solution u we have $u_\infty = r_{z,k}$ then it follows from Rellich's Lemma, [14, Lemma 2.11] and the analyticity of u on D^+, that $u = 4\pi\Phi_k(\cdot, z)$ in $D^+ \setminus \{z\}$. This contradicts $u \in H^1_{\ell oc}(D^+)$, so we see that $r_{z,k} \notin \mathrm{Ran}\,(G_k)$.

□

We note that we have a degree of freedom in setting the test for $z \in D$. If $z \in D$ then the function $x \to (\partial_\nu + \lambda)\Phi_k(x, z)$ is in fact in $C^\infty(\partial D)$, so we can require that there exists $\psi \in H^s(\partial D)$ for any $s \in \mathbb{R}$ such that $G_k\psi = r_{z,k}$. Above we chose $s = 0$ and the test $\|G_k^{-1} r_{z,k}\|_{L^2(\partial D)} < \infty$ in order to obtain a result from which a multi-frequency test is easy to develop. We note that $s = 0$ is not necessarily an optimal value, but studying this point is beyond the scope of this work.

It is also interesting to notice that the inequality (4.87) provides a way to interpret the construction of the scatterer in a more precise manner. Suppose that we agree on a threshold value C_D and deem the point z to be in D if

$$\|G_k^{-1} r_{z,k}\| \leq C_D. \tag{4.95}$$

In our measurements we do not gain direct knowledge of the G_k-operator, but rather an approximation of F_k. If

$$\|\mathrm{Im}\,(F_k)^{-\frac{1}{2}} r_{z,k}\| \leq C_D \frac{1}{C\langle k\rangle^4},$$

then it follows from (4.87) that (4.95) is satisfied and the prediction would be that $z \in D$.

On the other hand if

$$\|\mathrm{Im}\,(F_k)^{-\frac{1}{2}} r_{z,k}\| > \frac{C_D}{\sqrt{\alpha_0 |k|}}$$

then we see again from the inequality (4.87) that

$$\|G_k^{-1} r_{z,k}\| > C_D$$

and the prediction would be that $z \notin D$.

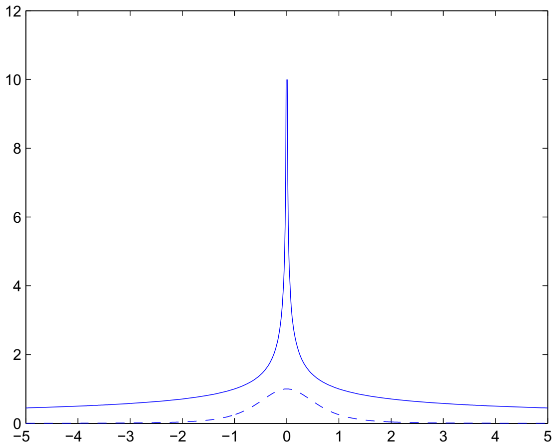

Figure 4.1: Values of k (x axis) vs. values of $(\frac{1}{|k|})^{\frac{1}{2}}$ (solid line) and values of $\frac{1}{\langle k\rangle^4}$ (dashed line).

When the value of $\|\mathrm{Im}\,(F_k)^{-\frac{1}{2}} r_{z,k}\|$ is in the interval $[\frac{C_D}{C\langle k\rangle^4}, \frac{C_D}{\sqrt{|\alpha_0 k|}}]$ we cannot say whether the point is in D or not. Hence this forms an uncertain boundary for D just because of the relation of G_k^{-1} and $\mathrm{Im}\,(F_k)^{-\frac{1}{2}}$.

The inequalities in (4.87) are not necessarily the best possible ones, so it might be possible to achieve a narrower gap interval.

In figure 4.5 we have plotted the function $k \to |k|^{-\frac{1}{2}}$ with a solid line and the function $k \to \frac{1}{\langle k\rangle^4}$ with a dashed line. We notice that the gap between the curves widens as k goes to 0. Hence we have a bigger uncertainty interval in this area. In addition, when k is large the test starts to lose effectivity since both of the borders are close to zero.

We finish this section with a theorem that provides a numerical test for the reconstruction of the scatterer. This is an analogous result to the original factorization method for Dirichlet and Neumann problems given in [34], which has also been derived in the case of the transmission problem in [33].

Theorem 4.29 *Let* (λ_i) *be a sequence that consists of the eigenvalues of* $\mathrm{Im}\,(F_k)$ *repeated by their multiplicity and* (φ_i) *be a corresponding sequence of* *the eigenvectors. The point z is in the obstacle D if and only if*

$$\sum_{i=1}^{\infty} \lambda_i^{-1} |(\varphi_i, r_{z,k})_{L^2(\mathbb{S}^2)}|^2 < \infty. \tag{4.96}$$

Proof. By Lemma 4.28 $z \in D$ if and only if $r_{z,k} \in \mathrm{Ran}\,(G_k)$ and by Theorem 4.27 $\mathrm{Ran}\left(\mathrm{Im}\,(F_k)^{\frac{1}{2}}\right) = \mathrm{Ran}\,(G_k)$. We use the fact that $\mathrm{Ran}\left(\mathrm{Im}\,(F_k)^{\frac{1}{2}}\right) = \mathrm{Dom}\left(\mathrm{Im}\,(F_k)^{-\frac{1}{2}}\right)$ and the characterisation of [57, section XI.2]:

$$\mathrm{Dom}\left(\mathrm{Im}\,(F_k)^{-\frac{1}{2}}\right) = \{\varphi \in L^2(\mathbb{S}^2) : \int_{\mathbb{R}} \lambda^{-1} d\|P(\lambda)\varphi\|^2 < \infty\}.$$

Here the integral is actually $\|\mathrm{Im}\,(F_k)^{-\frac{1}{2}}, \varphi\|^2_{L^2(\mathbb{S}^2)}$, when finite. By Theorem 4.24 the operator $\mathrm{Im}\,(F_k)$ is self-adjoint and compact, so by [16, Theorem II.5.1] its spectrum consists of eigenvalues $\{\lambda_k\}$. Hence we see that

$$\begin{aligned}
\mathrm{Ran}\,(G_k) &= \mathrm{Dom}\left(\mathrm{Im}\,(F_k)^{-\frac{1}{2}}\right) \\
&= \{u \in L^2(\mathbb{S}^2) : \sum_{i=1}^{\infty} \lambda_i^{-1} |(\varphi_i, \varphi)_{L^2(\mathbb{S}^2)}|^2 < \infty\},
\end{aligned}$$

and the test (4.96) holds.

$\square$

Chapter 5

Multi-Frequency Reconstruction

5.1 An Integral Method

We start by recalling the single frequency far field asymptotics for the total wave, which consist of an incident Herglotz wave and the corresponding scattered wave,

$$u_{tot} = 2\pi i \frac{e^{ik|x|}}{k|x|}\left(I - \frac{ik}{2\pi}F_k\right)h + 2\pi i \frac{e^{-ik|x|}}{k|x|}h(-\widehat{x}) + \mathcal{O}\left(\frac{1}{|x|^2}\right). \tag{5.1}$$

This was derived in section 4.3, equation (4.55).

We direct our attention to the operator kF_k and recall that in Theorem 4.16 we found the estimate

$$\|F_k\| \leq \frac{4\pi}{|k|}$$

for the norm of the single frequency far field operator F_k. Hence the k-dependence of the norm of the operator kF_k is very simple; it is bounded by a constant. We take this operator as the basis for the analysis of the multi-frequency scattering and introduce the notation

$$L_k := -k\mathrm{Im}(F_k). \tag{5.2}$$

Since $F_k = \mathrm{Re}(F_k) + i\mathrm{Im}(F_k)$ we see that for all $k \in \mathbb{R} \setminus \{0\}$

$$\begin{aligned} \|L_k\| &\leq \|kF_k\| \\ &\leq 4\pi. \end{aligned} \tag{5.3}$$

We can sum up the single frequency operators L_k by defining an operator L which maps an $h \in L^2(\mathbb{R} \times \mathbb{S}^2)$ to

$$(Lh)(k, \widehat{x}) := (L_k h(k\cdot))(\widehat{x}). \tag{5.4}$$

This operator is continuous from $L^2(\mathbb{R} \times \mathbb{S}^2)$ to $L^2(\mathbb{R} \times \mathbb{S}^2)$ as we will see in the following lemma.

Lemma 5.1 *Operator $L : L^2(\mathbb{R} \times \mathbb{S}^2) \to L^2(\mathbb{R} \times \mathbb{S}^2)$ is continuous and*

$$\|L\| \leq 4\pi.$$

Proof. Let $h \in L^2(\mathbb{R} \times \mathbb{S}^2)$. By Fubini's Theorem $h(k, \cdot) \in L^2(\mathbb{S}^2)$ for almost all $k \in \mathbb{R}$, so we see that $L_k h(k, \cdot)$ is defined for almost all $k \in \mathbb{R}$ and by equation (5.3)

$$\|L_k h(k, \cdot)\|_{L^2(\mathbb{S}^2)} \leq 4\pi \|h(k, \cdot)\|_{L^2(\mathbb{S}^2)}$$

for almost all $k \in \mathbb{R}$. By Fubini's Theorem

$$\begin{aligned} \|Lh\|^2_{L^2(\mathbb{S}^2)} &= \int_{\mathbb{R}} \int_{\mathbb{S}^2} \|L_k h(k, \widehat{x})\|^2 \mathrm{d}\mathcal{H}^2(\widehat{x}) \mathrm{d}k \\ &\leq \int_{\mathbb{R}} (4\pi)^2 \|h(k, \cdot)\|^2_{L^2(\mathbb{S}^2)} \mathrm{d}k \\ &= (4\pi)^2 \|h\|^2_{L^2(\mathbb{S}^2)}. \end{aligned} \tag{5.5}$$

We hence see that $L : L^2(\mathbb{R} \times \mathbb{S}^2) \to L^2(\mathbb{R} \times \mathbb{S}^2)$ is continuous and that $\|L\| \leq 4\pi$.

□

In the next lemma we prove that the spectral theory is well-behaved in the transition from L_k to L.

Lemma 5.2 *For every bounded and continuous function $f : \mathbb{R} \to \mathbb{C}$ and $h \in L^2(\mathbb{R} \times \mathbb{S}^2)$ we have*

$$(f(L)h)(k, \widehat{x}) = (f(L_k)h(k, \cdot))(\widehat{x}),$$

where the equality holds in $L^2(\mathbb{R} \times \mathbb{S}^2)$.

Proof. Operators L_k and L are self-adjoint, so by [57, Theorem XI.6.1] they have unique resolutions P_k and P respectively. By the definition of L and the spectral resolution of L_k we have

$$(Lh)(k, \widehat{x}) = \left(\lim_{n \to \infty} \sum_{j=1}^{j_n} \lambda_{n,j} P_k[\lambda_{n,j}, \lambda_{n,j+1}) h(k, \cdot) \right) (\widehat{x}). \tag{5.6}$$

We define a new resolution of identity $\widetilde{P}$ on $L^2(\mathbb{R}\times\mathbb{S}^2)$ by

$$(\widetilde{P}(\lambda)h)(k,\widehat{x}) = (P_k(\lambda)h(k,\cdot))(\widehat{x}).$$

With $\widetilde{P}$ the equation (5.6) can be expressed as

$$\begin{aligned} Lh &= \lim_{n\to\infty}\sum_{j=1}^{j_n}\lambda_{n,j}\widetilde{P}[\lambda_{n,j},\lambda_{n,j+1})h \\ &= \int_{\mathbb{R}}\lambda \mathrm{d}\widetilde{P}(\lambda)h. \end{aligned}$$

Since this is a spectral resolution, see Definition 2.11, of L and by [57, Corollary XI.5.2] L has an unique resolution, we see that $\widetilde{P}=P$. To finish the proof of the lemma we infer that

$$\begin{aligned} (f(L)h)(k,\widehat{x}) &= \left(\int_{\mathbb{R}} f(\lambda)\mathrm{d}\widetilde{P}(\lambda)h\right)(k,\widehat{x}) \\ &= \left(\lim_{n\to\infty}\sum_{j=1}^{j_n} f(\lambda_{n,j})\widetilde{P}[\lambda_{n,j},\lambda_{n,j+1})h\right)(k,\widehat{x}) \\ &= \left(\lim_{n\to\infty}\sum_{j=1}^{j_n} f(\lambda_{n,j})P_k[\lambda_{n,j},\lambda_{n,j+1})h(k,\cdot)\right)(\widehat{x}) \\ &= (f(L_k)h(k,\cdot))(\widehat{x}). \end{aligned}$$

□

We will also need the following connection between the Fourier transformation of $f(L)$, defined by

$$\widehat{f(L)} = \mathcal{F}_{k\to t}f(L)\mathcal{F}^{-1}_{t\to k}$$

and the operator $f(\widehat{L})$. We use the uniqueness of the spectral resolution in the same way as we did in Lemma 5.2.

Lemma 5.3 *Let $f:\mathbb{R}\to\mathbb{C}$ be continuous. Then*

$$\widehat{f(L)} = f(\widehat{L}).$$

Proof. By definition of $f(L)$ for all $h\in L^2(\mathbb{R}\times\mathbb{S}^2)$ we have

$$f(L)h = \lim_{n\to\infty}\sum_{j=1}^{j_n} f(\lambda_{n,j})P[\lambda_{n,j},\lambda_{n,j+1})h, \tag{5.7}$$

where P is a resolution of unity in the spectral resolution of L. Hence

$$\begin{aligned}\widehat{f(L)}h &= \mathcal{F}_{k\to t}f(L)\mathcal{F}^{-1}_{t\to k}h \\ &= \lim_{n\to\infty}\sum_{j=1}^{j_n} f(\lambda_{n,j})\mathcal{F}_{k\to t}P[\lambda_{n,j},\lambda_{n,j+1})\mathcal{F}^{-1}_{t\to k}h. \end{aligned} \tag{5.8}$$

On the other hand by the definition of the spectral resolution of L and the definition of $\widehat{L}$

$$\widehat{L}h = \lim_{n\to\infty}\sum_{j=1}^{j_n} \lambda_{n,j}\mathcal{F}_{k\to t}P[\lambda_{n,j},\lambda_{n,j+1})\mathcal{F}^{-1}_{t\to k}h. \tag{5.9}$$

With a direct calculation we see that $\mathcal{F}_{k\to t}P\mathcal{F}^{-1}_{t\to k}$ is a resolution of identity. By (5.9) it is the unique spectral resolution of $\widehat{L}$, so we see by equation (5.8) that

$$\widehat{f(L)} = f(\widehat{L}).$$

□

Lemma 5.4 *If $\epsilon_1 \geq \epsilon_2 > 0$, we have*

$$\|(I\epsilon_1 + L_k)^{-\frac{1}{2}}h\|_{L^2(\mathbb{S}^2)} \leq \|(I\epsilon_2 + L_k)^{-\frac{1}{2}}h\|_{L^2(\mathbb{S}^2)}$$

for all $h \in L^2(\mathbb{S}^2)$.

Proof. Let $\epsilon_1 \geq \epsilon_2 > 0$. We have for all $\epsilon > 0$

$$\|(I\epsilon + L_k)^{-\frac{1}{2}}h\|^2_{L^2(\mathbb{S}^2)} = \int_{[0,\infty)} (\epsilon+\lambda)^{-1}\mathrm{d}\|P(\lambda)h\|^2. \tag{5.10}$$

Hence the result follows from

$$(\epsilon_1+\lambda)^{-\frac{1}{2}} \leq (\epsilon_2+\lambda)^{-\frac{1}{2}}.$$

□

Lemma 5.5 *Let $k \in \mathbb{R}\setminus\{0\}$. We have for all $h \in \mathrm{Ran}\left(L_k^{\frac{1}{2}}\right) = \mathrm{Ran}\,(G_k)$*

$$\lim_{\epsilon\to 0}\|(I\epsilon + L_k)^{-\frac{1}{2}}h\|_{L^2(\mathbb{S}^2)} = \|L_k^{-\frac{1}{2}}h\|_{L^2(\mathbb{S}^2)}.$$

If $h \notin \mathrm{Ran}\left(L_k^{\frac{1}{2}}\right)$ then

$$\lim_{\epsilon\to 0}\|(I\epsilon + L_k)^{-\frac{1}{2}}h\|_{L^2(\mathbb{S}^2)} = \infty.$$

Proof. Let $\epsilon > 0$. Operator L_k is positive, self-adjoint and by Theorem 4.16 it is bounded. Since function $\lambda \to (\epsilon + \lambda)^{-\frac{1}{2}}$ is bounded on all of $\mathbb{R}$, the operator $(I\epsilon + L_k)^{-\frac{1}{2}}$ exists and

$$(I\epsilon + L_k)^{-\frac{1}{2}} = \int_{-\infty}^{\infty} (\epsilon + \lambda)^{-\frac{1}{2}} \mathrm{d}P_k(\lambda).$$

We also see from this that $(I\epsilon + L_k)^{-\frac{1}{2}}$ is a bounded operator.

We use Lebesgue's Monotone Convergence Theorem to see that for all $h \in L^2(\mathbb{S}^2)$

$$\begin{aligned} \lim_{\epsilon \to 0} \|(I\epsilon + L_k)^{-\frac{1}{2}} h\|^2_{L^2(\mathbb{S}^2)} &= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} (\epsilon + \lambda)^{-1} \mathrm{d}\|P_k(\lambda)h\|^2 \\ &= \int_{-\infty}^{\infty} \lim_{\epsilon \to 0} (\epsilon + \lambda)^{-1} \mathrm{d}\|P_k(\lambda)h\|^2. \end{aligned}$$

Hence if $h \in \operatorname{Ran}\left(L_k^{\frac{1}{2}}\right)$ then

$$\lim_{\epsilon \to 0} \|(I\epsilon + L_k)^{-\frac{1}{2}} h\|_{L^2(\mathbb{S}^2)} = \|L_k^{-\frac{1}{2}} h\|_{L^2(\mathbb{S}^2)}.$$

On the other hand if

$$\lim_{\epsilon \to 0} \|(I\epsilon + L_k)^{-\frac{1}{2}} h\|_{L^2(\mathbb{S}^2)} < \infty$$

then we have

$$\int_{-\infty}^{\infty} \lambda^{-1} \mathrm{d}\|P_k(\lambda)h\|^2 < \infty.$$

So in other words $h \in \operatorname{Dom}\left(L_k^{-\frac{1}{2}}\right) = \operatorname{Ran}\left(L_k^{\frac{1}{2}}\right)$. Hence $u \notin \operatorname{Ran}\left(L_k^{\frac{1}{2}}\right)$ implies

$$\lim_{\epsilon \to 0} \|(I\epsilon + L_k)^{-\frac{1}{2}} h\|_{L^2(\mathbb{S}^2)} = \infty.$$

□

Before forming the integral test for $z \in D$ we prove the following lemma on the property of the operator G_k.

Lemma 5.6 *Let $D \subset \mathbb{R}^3$ be a bounded C^2 domain, $k \in \mathbb{R} \setminus \{0\}$, $z \in D$ and $G_k : H^{-\frac{1}{2}}(\partial D) \to L^2(\mathbb{S}^2)$ be the map which takes the Robin boundary values $(\partial_\nu + \lambda \mathrm{tr}\,)_{\partial D} u$ to the far field of the solution u of (4.9). Then*

$$\|G_k^{-1} r_{z,k}\|_{L^2(\partial D)} \leq C(z)(1 + |k|) \max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \mathcal{H}(\partial D)^{\frac{1}{2}},$$

where $C(z)$ depends on $d(z, \partial D)$.

Proof. The point z is in the interior of D, so $\Phi_k(x,z) = \frac{e^{ik|x-z|}}{4\pi|x-z|}$ is C^∞ on ∂D and

$$\begin{aligned} b(x) &= (\partial_\nu + \lambda(x))\,\Phi(x,z) \\ &= \left(\nu(x)\cdot\widehat{(x-z)}(ik - \frac{1}{|x-z|}) + \lambda(x)\right)\frac{e^{ik|x-z|}}{4\pi|x-z|}. \end{aligned}$$

Function λ is smooth, so

$$\|G_k^{-1} r_{z,k}\|_\infty = \|b\|_\infty \le C(z)(1+|k|)\|\lambda\|_\infty,$$

where $C(z)$ depends on $\mathrm{d}(z,\partial D)$. By definition of the L^2 norm

$$\|G_k^{-1} r_{z,k}\|_{L^2(\partial D)} \ \le \ C(z)(1+|k|)\|\lambda\|_\infty \mathcal{H}(\partial D)^{\frac{1}{2}}.$$

□

The following theorem is one of the main results of this work.

Theorem 5.7 *Let $\psi \in \mathcal{S}(\mathbb{R})$ be such that $|\psi(k)| \le C|k|$ in a neighbourhood of 0. For all $z \in D$ and $\epsilon > 0$ we have*

$$\|(\epsilon I + L)^{-\frac{1}{2}}\psi(k) r_{z,k}(\widehat{x})\|^2_{L^2(\mathbb{R}\times\mathbb{S}^2)} = \|(\epsilon I + \widehat{L})^{-\frac{1}{2}}\widehat{\psi}(t + z\cdot\widehat{x})\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}$$

and $z \in D$ if and only if

$$\lim_{\epsilon\to 0}\|(\epsilon I + \widehat{L})^{-\frac{1}{2}}\widehat{\psi}(t + z\cdot\widehat{x})\|_{L^2(\mathbb{R}\times\mathbb{S}^2)} < \infty.$$

Proof. For any $\epsilon > 0$ the function $f : [0,\infty) \to \mathbb{R}$, $f(\lambda) = (\epsilon+\lambda)^{-\frac{1}{2}}$ is bounded, so operators $(\epsilon I + L_k)^{-\frac{1}{2}} : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ and $(\epsilon I + L)^{-\frac{1}{2}} : L^2(\mathbb{R}\times\mathbb{S}^2) \to L^2(\mathbb{R}\times\mathbb{S}^2)$ are also bounded. It follows from Lemma 5.3 that

$$\left((\epsilon I + L)^{-\frac{1}{2}}\right)^\wedge = (\epsilon I + \widehat{L})^{-\frac{1}{2}}.$$

By Lemma 2.8 we have $\mathcal{F}_{k\to t}\{\psi(k) r_{z,k}\} = \widehat{\psi}(t + z\cdot\widehat{x})$ and hence

$$\begin{aligned} \|(\epsilon I + \widehat{L})^{-\frac{1}{2}}\widehat{\psi}(t + z\cdot\widehat{x})\|^2_{L^2(\mathbb{R}\times\mathbb{S}^2)} &= \|\widehat{(\epsilon I + L)}^{-\frac{1}{2}}\widehat{\psi}(t + z\cdot\widehat{x})\|^2_{L^2(\mathbb{R}\times\mathbb{S}^2)} \\ &= \|\mathcal{F}_{k\to t}(\epsilon I + L)^{-\frac{1}{2}}\mathcal{F}^{-1}_{t\to k}\mathcal{F}_{k\to t}\{\psi(k) r_{z,k}\}\|^2 \\ &= \|(\epsilon I + L)^{-\frac{1}{2}}\psi(k) r_{z,k}\|^2_{L^2(\mathbb{R}\times\mathbb{S}^2)}. \end{aligned}$$

By Lemma 5.2

$$(\epsilon I + L)^{-\frac{1}{2}}\psi(k) r_{z,k}(\widehat{x}) = \left((\epsilon I + L_k)^{-\frac{1}{2}}\psi(k) r_{z,k}\right)(\widehat{x}).$$

Hence

$$\|(\epsilon I + \widehat{L})^{-\frac{1}{2}}\widehat{\psi}(t + z\cdot\widehat{x})\|^2_{L^2(\mathbb{R}\times\mathbb{S}^2)} = \int_{\mathbb{R}} \|(\epsilon I + L_k)^{-\frac{1}{2}}\psi(k) r_{z,k}\|^2_{L^2(\mathbb{S}^2)} dk.$$

By Corollary 5.4 for all $\epsilon_2 < \epsilon_1$

$$\|(\epsilon_1 I + L_k)^{-\frac{1}{2}} r_{z,k}\|^2_{L^2(\mathbb{S}^2)} \le \|(\epsilon_2 I + L_k)^{-\frac{1}{2}} r_{z,k}\|^2_{L^2(\mathbb{S}^2)},$$

so it follows from Lebesque's Monotone Convergence Theorem that

$$\lim_{\epsilon\to 0}\|(\epsilon I + \widehat{L})^{-\frac{1}{2}}\widehat{\psi}(t + z\cdot\widehat{x})\|^2_{L^2(\mathbb{R}\times\mathbb{S}^2)} = \int_{\mathbb{R}} \psi(k)^2 \lim_{\epsilon\to 0}\|(\epsilon I + L_k)^{-\frac{1}{2}} r_{z,k}\|^2_{L^2(\mathbb{S}^2)} dk. \tag{5.11}$$

If $z \in D$ then by Lemma 5.5

$$\lim_{\epsilon\to 0}\|(\epsilon I + L_k)^{-\frac{1}{2}} r_{z,k}\|^2_{L^2(\mathbb{S}^2)} = \|L_k^{-\frac{1}{2}} r_{z,k}(\widehat{x})\|^2_{L^2(\mathbb{S}^2)}.$$

As $L_k^{-\frac{1}{2}} = k^{-\frac{1}{2}}\mathrm{Im}(F_k)^{-\frac{1}{2}}$, it follows from Theorem 4.27 and Lemma 5.6 that there is a positive value $C(z)$ that depends on point z such that

$$\|L_k^{-\frac{1}{2}} r_{z,k}\|^2_{L^2(\mathbb{S}^2)} \le \frac{C(z)(1+|k|)^2}{\alpha_0 k^2}. \tag{5.12}$$

Hence

$$\begin{aligned}\lim_{\epsilon\to 0}\|(\epsilon I + \widehat{L})^{-\frac{1}{2}}\widehat{\psi}(t + z\cdot\widehat{x})\|^2_{L^2(\mathbb{R}\times\mathbb{S}^2)} &= \int_{\mathbb{R}} |\psi(k)|^2 \frac{C(1+|k|)^2}{k^2}\mathrm{d}k. \\ &< \infty.\end{aligned}$$

If $z \notin D$ then by Lemma 5.5

$$\lim_{\epsilon\to 0}\|(\epsilon I + L_k)^{-\frac{1}{2}} r_{z,k}\|^2_{L^2(\mathbb{S}^2)} = \infty$$

for all $k \neq 0$. Hence by equation (5.11)

$$\lim_{\epsilon\to 0}\|(\epsilon I + \widehat{L})^{-\frac{1}{2}}\widehat{\psi}(t + z\cdot\widehat{x})\|^2_{L^2(\mathbb{R}\times\mathbb{S}^2)} = \infty.$$

□

The condition $|\psi(k)| < |k|$ in the vicinity of 0 is satisfied by the functions $\psi \in C_0^\infty(\mathbb{R})$, whose Fourier transformation has a vanishing first moment, that is

$$\int_{\mathbb{R}} \widehat{\psi}(t)\mathrm{d}t = 0.$$

This condition is studied in more detail in Section 5.3 and is needed while considering the Fourier transformation $\widehat{F} = \mathcal{F}_{k\to t} F \mathcal{F}^{-1}_{t\to k}$ of the far field operator F.

We also provide a slightly different formulation of Theorem 5.7.

Theorem 5.8 *Let $\psi \in \mathcal{S}(\mathbb{R})$ be such that $|\psi(k)| \leq C|k|$ in a neighbourhood of 0. Then $z \in D$ if and only if*

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}\times\mathbb{S}^2} \overline{\left((\epsilon I + \widehat{L})^{-1}\widehat{\psi}_z\right)(t,\widehat{x})}\widehat{\psi}_z(t,\widehat{x})\,dt\mathrm{d}\mathcal{H}^2(\widehat{x}) < \infty, \tag{5.13}$$

where $\widehat{\psi}_z(t,\widehat{x}) = \widehat{\psi}(t + z\cdot\widehat{x})$.

Proof. Operator L is bounded and self-adjoint, so we see that so are $\widehat{L}$ and $(I\epsilon + \widehat{L})^{-\frac{1}{2}}$ for all $\epsilon > 0$. Hence for all $z \in \mathbb{R}^3$ we have

$$\|(I\epsilon + \widehat{L})^{-\frac{1}{2}}\widehat{\psi}_z\|^2_{L^2(\mathbb{R}\times\mathbb{S}^2)} = ((\epsilon I + \widehat{L})^{-1}\widehat{\psi}_z, \widehat{\psi}_z)_{L^2(\mathbb{R}\times\mathbb{S}^2)}$$

and the result follows from Theorem 5.7.

□

The condition (5.13) does not contain a square root of $\widehat{L}$ and is hence a bit easier to deal with. In order to obtain $\left(\epsilon I + \widehat{L}\right)^{-1}\widehat{\psi}_z$ we need to solve $\widehat{h}$ in the equation

$$\left(\epsilon I - \frac{1}{2}\partial_t(\widehat{F} - \widehat{F}^*)\right)\widehat{h} = \widehat{\psi}_z \tag{5.14}$$

in the time domain. Some comments on the numerical solution of this equation are given in section 5.4 after we have studied certain aspects of the time domain Robin problem in sections 5.2 and 5.3.

5.2 The Time Domain Robin Problem

The purpose of the current and the following sections is to provide a short discussion of the numerical applicability of the multi-frequency method derived in the previous section. In order to keep the presentation short we assume that the obstacle has the Robin energy decay property. This property is not proved for any subclass of obstacles here nor is it, to the author's knowledge, available in the literature.

What is readily available in the literature is for example a weaker energy decay result

$$\liminf_{t\to\infty} E(u, D_R, t) = 0$$

derived by Lax and Phillips in [36]. One approach would be to determine whether this weaker condition would be enough to make the time domain / frequency domain changes possible in a manner that could provide a framework for numerics.

Another approach would be to use the Quantum Non-Trapping Condition found e.g. in Tang and Zworski, [52], and to see if this would give a sufficient energy decay. Both approaches are, however, beyond the scope of this work.

We return to the time domain Robin problem mentioned in the introduction

$$\begin{cases} \Box u(x,t) &= 0 \quad \text{in } \mathbb{R}_+ \times D^+ \\ u|_{t=0} &= f_1 \\ u_t|_{t=0} &= f_2 \\ (\partial_\nu - \alpha\partial_t + \beta)u &= 0 \quad \text{on } (\mathbb{R}_+ \cup \{0\}) \times \partial D. \end{cases} \tag{5.15}$$

We assume that for all t the restriction $u(t,\cdot)$ is in $H^1(\overline{D^+})$. The space $H_1(\overline{D^+})$ is the completion of $C_0^\infty(\mathbb{R}^3 \setminus \overline{B(x_0,r)})$ with respect to the $\|\cdot\|_{D(\mathbb{R}^3\setminus\overline{B(x_0,r)})}$ norm, where x_0 and r are such that $\overline{B(x_0,r)} \subset D$. By Lemma 3.4 all $u \in H_1(\overline{D^+})$ satisfy $u \in L^2_{\ell oc}(D^+)$.

By [45, Theorem 7.6.2] the time domain Robin problem has a solution in the following sense:

Definition 5.9 *Let $(f_1, f_2) \in H(D^+)$. Function $u(t,x) \in H^1_{\ell oc}(\mathbb{R} \times D^+)$ is a solution of (5.15) if*

(i) For all $t \in \mathbb{R}_+ \cup \{0\}$ we have $(u(t,\cdot), \partial_t u(t,\cdot)) \in H(\overline{D^+})$ and the functions

$$t \to \nabla_x u(t,\cdot), \quad t \to \partial_t u(t,\cdot)$$

are continuous from $\mathbb{R}_+ \cup \{0\}$ to $L^2(D^+)$.

(ii) $(u(0,\cdot), \partial_t u(0,\cdot)) = (f_1, f_2)$.

(iii) *Let* $s_0 > 0$ *be so small that the map* $H : \partial D \times [0, s_0] \to D^+; H(x,s) = x + s\nu(D)$ *is injective. Then*

$$\widetilde{u}(t,z,s) := u(t, H(z,s)))$$

satisfies:

$$\begin{array}{rcl} \widetilde{u} & \in & C^\infty([0,s_0], \mathcal{D}'(\mathbb{R} \times \partial D) \\ \widetilde{u} & \in & H^1((-T,T) \times \partial D \times (0,s_0)) \end{array}$$

for all $T \in \mathbb{R}_+$.

(iv) $\Box u = 0$ *in* $\mathbb{R}_+ \times D^+$ *and* $u(t,x)$ *satisfies the Robin boundary condition in the sense of distributions.*

We will next prove the exponential decay for $\|u\|_{L^2(D_R)}$ from the exponential energy decay. First we prove a version of Poincare's Inequality and then use this together with the energy decay.

Lemma 5.10 *Let* $\varphi \in C_0^\infty(\mathbb{R}^3)$, $R > R_0 > 0$ *and* $U_R = B(0,R) \setminus B(0,R_0)$. *Then*

$$\|\varphi\|_{L^2(U_R)} \leq 2^{-\frac{1}{2}}(R^2 - R_0^2)^{\frac{1}{2}} \|\varphi\|_{D(\mathbb{R}^3 \setminus \overline{B(0,R_0)})}. \tag{5.16}$$

Proof. The proof of analogous to [46, Lemma IV.1.1].

□

This immediately yields a corresponding result in $H_1(\overline{D^+})$.

Corollary 5.11 *Let* $R > R_0 > 0$ *and* $U_R = B(0,R) \setminus B(0,R_0)$. *Then for all* $u \in H_1(\overline{D^+})$

$$\|u\|_{L^2(U_R)} \leq 2^{-\frac{1}{2}}(R^2 - R_0^2)^{\frac{1}{2}} \|u\|_{D(\mathbb{R}^3 \setminus \overline{B(0,R_0)})}. \tag{5.17}$$

Proof. Let $u \in H_1(\overline{D^+})$ and (φ_n) be the defining sequence of u. By Lemma 5.10

$$\begin{array}{rcl} \|u\|_{L^2(U_R)} & = & \lim_{n\to\infty} \|\varphi_n\|_{L^2(U_R)} \\ & \leq & \lim_{n\to\infty} 2^{-\frac{1}{2}}(R^2 - R_0^2)^{\frac{1}{2}} \|\varphi_n\|_{D(\mathbb{R}^3 \setminus \overline{B(0,R_0)})} \\ & = & 2^{-\frac{1}{2}}(R^2 - R_0^2)^{\frac{1}{2}} \|u\|_{D(\mathbb{R}^3 \setminus \overline{B(0,R_0)})}. \end{array}$$

□

Lemma 5.12 *Let u be a solution of (5.15) in the sense of Definition 5.9 and have the exponential energy decay property, that is for all $R > 0$ there exist constants C and c, which depend on R, D and the supports of $u(0,\cdot)$ and $\partial_t u(0,\cdot)$, such that*

$$E(u, D_R, t) \leq Ce^{-ct}E(u, D^+, 0). \tag{5.18}$$

Then for all $R > 0$ there exist constants C and c, which depend on R, D and the supports of $u(0,\cdot)$ and $\partial_t u(0,\cdot)$, such that for all $t \in [0,\infty)$

$$\|u(t,\cdot)\|_{L^2(D_R)} \leq Ce^{-ct}E(u, D^+, 0). \tag{5.19}$$

Proof. Let $U \subset D^+$ be an open bounded set and $u(t,x)$ be the solution of (5.15) in the sense of Definition 5.9. We define

$$u_U(t) = \frac{1}{\mathcal{L}(U)} \int_U u(t,x)\mathrm{d}x,$$

where $\mathcal{L}(U)$ is the Lebesgue measure of U. From Hölders inequality and the exponential energy decay of u it follows that

$$\begin{aligned} |\partial_t u_U(t)| &\leq \frac{1}{\mathcal{L}(U)} \int_U |u_t(t,x)|\mathrm{d}x \\ &\leq \frac{1}{\mathcal{L}(U)^{\frac{1}{2}}} \left(\int_U |u_t(t,x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq \frac{1}{\mathcal{L}(U)^{\frac{1}{2}}} CE(u, D^+, 0)e^{-ct}, \end{aligned}$$

where C and c depend on $\mathrm{diam}(U)$, D and the supports of $u(0,\cdot)$ and $\partial_t u(0,\cdot)$. Hence there is a limit

$$V_U = \lim_{t\to\infty} u_U(t).$$

It follows from Poincare's Inequality, [23, Theorem 1], and the energy decay that

$$\begin{aligned} \|u(t,x) - u_U(t)\|_{L^2(U)} &\leq C\|u(t,\cdot)\|_{D(U)} \\ &\leq CE(u, D^+, 0)e^{-ct}, \end{aligned} \tag{5.20}$$

where C and c depend on $\mathrm{diam}(U)$, D and the supports of $u(0,\cdot)$ and $\partial_t u(0,\cdot)$.

Now let $U_1 \subset U$ be an open set. We have

$$\begin{aligned} |u_{U_1}(t) - u_U(t)|\mathcal{L}(U_1) &= \|u(t,x) - u_U(t) - (u(t,x) - u_{U_1}(t))\|_{L^2(U_1)} \\ &\leq \|u - u_{U_1}(t)\|_{L^2(U_1)} + \|u - u_U(t)\|_{L^2(U_1)} \\ &\leq \|u - u_{U_1}(t)\|_{L^2(U_1)} + \|u - u_U(t)\|_{L^2(U)}. \end{aligned} \quad (5.21)$$

By equation (5.20) the right-hand side of equation (5.21) converges to zero, so we see that for all open $U_1 \subset U$ we have

$$\lim_{t\to\infty} \|u(t,\cdot) - V_U\|_{L^2(U_1)} = 0. \quad (5.22)$$

Let $R >> R_1 > 0$, R_1 be such that $\overline{D} \subset B(0,R_1)$, $U = D_R$ and $U_1 = B(0,R) \setminus \overline{B(0,R_1)}$. By Poincare's inequality on an annulus, Corollary 5.11, we have

$$\begin{aligned} \|u(t,\cdot)\|_{L^2(B(0,R)\setminus\overline{B(0,R_1)})} &\leq 2^{-\frac{1}{2}}(R^2 - R_1^2)^{\frac{1}{2}}\|u\|_{D(\mathbb{R}^3\setminus\overline{B(0,R)})} \\ &\leq Ce^{-ct}E(u,D^+,0), \end{aligned}$$

where C and c depend on R, D and the supports of $u(0,\cdot)$ and $\partial_t u(0,\cdot)$.

Hence $V = 0$ and

$$\begin{aligned} |u_{D_R}(t)| &\leq |\int_t^\infty \partial_t u_{D_R}(t')\mathrm{d}t' + 0| \\ &\leq \int_t^\infty Ce^{-ct'}E(u,D^+,0)\mathrm{d}t' \\ &\leq Ce^{-ct}E(u,D^+,0), \end{aligned}$$

where C and c depend on $\mathrm{diam}(U)$, D and the supports of $u(0,\cdot)$ and $\partial_t u(0,\cdot)$. It follows from the inequality (5.20) that

$$\begin{aligned} \|u(t,\cdot)\|_{L^2(D_R)} &\leq \|u(t,\cdot) - u_{D_R}(t)\|_{L^2(D_R)} + \|u_{D_R}(t)\|_{L^2(D_R)} \\ &\leq Ce^{-ct}E(u,D^+,0), \end{aligned}$$

where C and c depend on R, D and the supports of $u(0,\cdot)$ and $\partial_t u(0,\cdot)$.

$\square$

5.3 The Time Domain Far Field

In this section we derive a formula for the time domain far field and study the time domain counterparts of the frequency domain far field operator and its adjungate. We do this in order to use the asymptotics attained in Section 5.4 for providing grounds for the numerical application of the multi-frequency factorization method.

Let us consider a Herglotz wave

$$\widehat{u}_{\widehat{h}}(k,x) := \int_{\mathbb{S}^2} e^{ikx\cdot d}\widehat{h}(k,d)\mathrm{d}\mathcal{H}^2(d)$$

as an incident wave in the frequency domain and the corresponding incident wave

$$u_{in}(t,x) = \int_{\mathbb{S}^2} h(t - x\cdot d, d)\mathrm{d}\mathcal{H}^2(d),$$

with $h = \mathcal{F}_{k\to t}\{\widehat{h}\}$ in the time domain. In the following we assume that $h \in C_0^\infty(\mathbb{R}\times\mathbb{S}^2)$, which is equivalent to the assumption that $\widehat{\psi} \in C_0^\infty(\mathbb{R})$ in the previous section's notation.

The total wave is

$$u_{tot} = u_{in} + u_{sc}. \tag{5.23}$$

Let $R > 0$ be such that $\overline{D} \subset B(0,R)$ and T be such that $\mathrm{supp}(h) \subset [-T,T]\times\mathbb{S}^2$. Then for all $x \in B(0,R)$, $t \in \mathbb{R}\setminus[-T-R, T+R]$ and $d \in \mathbb{S}^2$, the function $h(t - x\cdot d, d) = 0$, from which we see that the support of the incident wave intersects the set D only for a finite time interval. Hence the scattered wave u_{sc} satisfies

$$\begin{cases} \Box u_{sc} &= 0 & \text{in } [T+R,\infty)\times D^+ \\ (\partial_\nu - \alpha(x)\partial_t + \beta(x))u_{sc} &= 0 & \text{on } [T+R,\infty)\times\partial D \\ u_{sc}|_{t=T+R} = f_1 & \partial_t u_{sc}|_{t=T+R} = f_2, & \end{cases} \tag{5.24}$$

where $\mathrm{supp}(f_1), \mathrm{supp}(f_2) \subset B(0, 2(T+R))$ because of the finite propagation speed of waves.

We assume that the obstacle D has the local Robin energy decay property stated in Definition 3.6, from which it follows that for all $R' > 0$ and $t > T+R$

$$E(u_{sc}, D_{R'}, t) \le Ce^{-ct}E(u_{sc}, D^+, T+R). \tag{5.25}$$

By Lemma 5.12 we have for $t > T+R$

$$\|u_{sc}(t,\cdot)\|_{H^1(D_R)} \le Ce^{-ct}E(u_{sc}, D^+, T+R), \tag{5.26}$$

where C and c depend on R, D and the supports of $u_{sc}(T+R,\cdot)$ and $\partial_t u_{sc}(T+R,\cdot)$.

We use this decay property to find an entity in the time domain which corresponds to the far field of the scattered wave in the frequency domain. Let $R_1 > R$, $\chi \in C^\infty(\mathbb{R}^3)$ be such that $\chi \equiv 0$ on a neighbourhood of $B(0, R)$ and $\chi \equiv 1$ outside $B(0, R_1)$. Let

$$w = \chi u_{tot} \tag{5.27}$$

be continued by zero inside D. The function w satisfies

$$\begin{cases} \Box w &= Q \qquad\qquad \text{in } \mathbb{R}^4 \\ w|_{t<-T-R} &= u_{in}|_{t<-T-R} \end{cases} \tag{5.28}$$

where $\mathrm{supp}(Q) \subset [-T - R, \infty) \times (B(0, R_1) \setminus \overline{B(0, R)})$.

We decompose w as

$$w = w_{in} + w_{sc}, \tag{5.29}$$

where $w_{in} = \chi u_{in}$ and $w_{sc} = \chi u_{sc}$. The Corresponding decomposition of the source term Q is

$$\begin{aligned} Q &= Q_{in} + Q_{sc} \\ &= \Box w_{in} + \Box w_{sc}. \end{aligned}$$

Since

$$\begin{aligned} Q_{sc} &= \Box(\chi u_{sc}) \\ &= 2(\partial_t \chi)\partial_t u_{sc} + 2\nabla\chi \cdot \nabla u_{sc} + (\Box\chi) u_{sc}, \end{aligned}$$

it follows from equations (5.25) and (5.26) that for $t > T + R$

$$\|Q_{sc}(t, \cdot)\|_{L^2(B(0,R))} \le C e^{-ct} E(u_{sc}, D^+, T + R), \tag{5.30}$$

where C and c depend on R_1, D and the supports of $u_{sc}(0, \cdot)$ and $\partial_t u_{sc}(0, \cdot)$. Hence $\mathrm{supp}(Q_{sc}) \subset [-T - R, \infty) \times (B(0, R_1) \setminus \overline{B(0, R)})$ and $Q_{sc} \in L^1(\mathbb{R}^4)$, which implies that we can apply Lemma 2.7, by which

$$\mathcal{F}^{-1}_{t\to k}\{E_+ * Q_{sc}\} = (\Phi_k *_y \mathcal{F}^{-1}_{t\to k}\{Q_{sc}\}(k, \cdot))(k, x). \tag{5.31}$$

The wave w_{sc} satisfies

$$\begin{cases} \Box w_{sc} &= Q_{sc} \quad \text{in } \mathbb{R}^4 \\ w_{sc}|_{t<-T-R} &= 0, \end{cases}$$

so by Theorem 2.2, we have

$$w_{sc} = E_+ * Q_{sc}$$

and by equation (5.31) the Fourier transformation

$$\mathcal{F}^{-1}_{t\to k}\{w_{sc}\} = (\Phi_k *_y \mathcal{F}^{-1}_{t\to k}\{Q_{sc}\}(k,\cdot))(k,x).$$

Since for all k the element $\mathcal{F}^{-1}_{t\to k}\{Q_{sc}\}(k,\cdot) \in L^1(\mathbb{R}^3)$ we see that by Lemma A.1 the far field of $\mathcal{F}^{-1}_{t\to k}\{w_{sc}\}(k,\cdot)$ is

$$\mathcal{F}^{-1}_{t\to k}\{w_{sc}\}_\infty(k,\widehat{x}) = \frac{1}{4\pi}\int_{B(0,R)} e^{-ik\widehat{x}\cdot y}\mathcal{F}^{-1}_{t\to k}\{Q_{sc}\}(k,y)\mathrm{d}y. \tag{5.32}$$

For all $k \in \mathbb{R}$ we have $\|\mathcal{F}^{-1}_{t\to k}\{Q_{sc}\}(k,\cdot)\|_{L^1(\mathbb{R}^3)} \le \|Q_{sc}\|_{L^1(\mathbb{R}^4)}$ so $\mathcal{F}^{-1}_{t\to k}\{w_{sc}\} \in \mathcal{S}'(\mathbb{R}\times\mathbb{S}^2)$ and we can define the time domain far field of w_{sc} to be

$$w_{sc,\infty}(t,\widehat{x}) = \mathcal{F}_{k\to t}\{\mathcal{F}^{-1}_{t\to k}\{w_{sc}\}_\infty\}(t,\widehat{x})$$

By Lemma 2.10

$$w_{sc,\infty}(t,\widehat{x}) = \frac{1}{4\pi}\int_{B(0,R_1)\setminus\overline{B(0,R)}} Q_{sc}(t+\widehat{x}\cdot y, y)\mathrm{d}y. \tag{5.33}$$

With Hölder's inequality, Fubini's Theorem, moving to the spherical coordinates with the z-axis always aligned in the same direction as y and by using the change of variables $f:(0,\pi)\times((B(0,R_1)\setminus\overline{B(0,R)}) \to (t-R_1,t+R_1)\times(B(0,R_1)\setminus\overline{B(0,R)})$; $f(\alpha,y) := (t+|y|\cos\alpha, y)$ we see that for all $t > T+2R_1$

$$\|w_{sc,\infty}(t,\cdot)\|^2_{L^2(\mathbb{S}^2)} \le C\int_{(t-R_1,t+R_1)\times(B(0,R_1)\setminus\overline{B(0,R)})} |Q_{sc}(t,y)|^2\mathrm{d}t\mathrm{d}y,$$

where C depends on R_1 and R. It follows from equation (5.30) that for $t > T+2R_1$

$$\|w_{sc,\infty}(t,\cdot)\|_{L^2(\mathbb{S}^2)} \le Ce^{-ct}E(u_{sc},D^+,T+R), \tag{5.34}$$

where C and c depend on R_1, R, D and the supports of $u_{sc}(T+R,\cdot)$ and $\partial_t u_{sc}(T+R,\cdot)$.

Since $\|w_{sc,\infty}(t,\cdot)\|_{L^2(\mathbb{S}^2)}$ is bounded for $t \le T+2R_1$ the estimate (5.34) holds for all t.

As $w_{sc} \equiv u_{sc}$ on $\mathbb{R}\times(\mathbb{R}^3\setminus\overline{B(0,R_1)})$ we make the following definition

Definition 5.13 *Let u_{sc} be the scattered wave of equations (5.23) and (5.24) and w_{sc} be the wave defined by (5.28) and (5.29). We define the time domain far field $u_{sc,\infty}$ of u_{sc} to be*

$$u_{sc,\infty} := w_{sc,\infty}$$

where $w_{sc,\infty}$ is defined by equation (5.32).
Moreover we define the operator F_{time} by

$$F_{time} h := u_{sc,\infty},$$

where $u_{sc,\infty}$ is the far field of the scattered wave of the incident wave (5.23).

We will next determine how F_{time} behaves for large t in order to facilitate the numerical solution of the equation $(\epsilon I + \widehat{L})h = \psi$.
By Taylor's Theorem for all $\widehat{h} \in \mathcal{S}(\mathbb{R} \times \mathbb{S}^2)$ we have

$$\widehat{h}(k, d) = \widehat{h}(0, d) + k \partial_k \widehat{h}(k', d), \tag{5.35}$$

where $k' \in [0, k]$. Let $k_0 > 0$. We have

$$\sup_{\substack{d \in \mathbb{S}^2 \\ k \in [-k_0, k_0]}} |\partial_k \widehat{h}(k', d)| = C_{k_0} < \infty \tag{5.36}$$

and it follows from equation (5.35) and Theorem 4.16 that for $\widehat{h}$ for which $\widehat{h}(0, d) = 0$ for all $d \in \mathbb{S}^2$ we have

$$\begin{aligned}
\int_{\mathbb{R}} \|F_k \widehat{h}(k, \cdot)\|^2_{L^2(\mathbb{S}^2)} \mathrm{d}k &= \int_{[-k_0,k_0]} \|F_k \widehat{h}(k, \cdot)\|^2_{L^2(\mathbb{S}^2)} \mathrm{d}k + \\
&= + \int_{\mathbb{R} \setminus [-k_0,k_0]} \|F_k \widehat{h}(k, \cdot)\|^2_{L^2(\mathbb{S}^2)} \mathrm{d}k \\
&\leq 4\pi^2 C_{k_0}^2 |\mathbb{S}^2| + \frac{4\pi}{k_0} \|\widehat{h}\|^2_{L^2(\mathbb{R} \times \mathbb{S}^2)}.
\end{aligned}$$

We define the set

$$\mathcal{S}_{k0}(\mathbb{R} \times \mathbb{S}^2) := \{\widehat{h} \in \mathcal{S}(\mathbb{R} \times \mathbb{S}^2) : \forall\, d \in \mathbb{S}^2 : \ \widehat{h}(0, d) = 0\}$$

and the operator $F : \mathcal{S}_{k0}(\mathbb{R} \times \mathbb{S}^2) \to L^2(\mathbb{R} \times \mathbb{S}^2)$, which maps a $\widehat{h} \in \mathcal{S}_{k0}(\mathbb{R} \times \mathbb{S}^2)$ to

$$F\widehat{h}(k, d) := \left(F_k \widehat{h}(k, \cdot)\right)(d).$$

Since

$$\begin{aligned}
\widehat{h}(0, d) &= \mathcal{F}^{-1}_{t \to k}\{h\}(0, d) \\
&= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} h(t, d) \mathrm{d}t,
\end{aligned} \tag{5.37}$$

the condition $\widehat{h}(0,d) = 0$ in the frequency domain corresponds in the time domain to the vanishing of the first moment of $h(\cdot, d)$. Hence $\mathcal{F}^{-1}_{t\to k}$ maps the set

$$\mathcal{S}_{m0}(\mathbb{R}\times\mathbb{S}^2) := \{h \in \mathcal{S}(\mathbb{R}\times\mathbb{S}^2) : \forall\, d \in \mathbb{S}^2 : \int_{\mathbb{R}} h(t,d)\mathrm{d}t = 0\} \tag{5.38}$$

to $\mathcal{S}_{k0}(\mathbb{R}\times\mathbb{S}^2)$ and the map $\widehat{F} := \mathcal{F}_{k\to t} F \mathcal{F}^{-1}_{t\to k} : \mathcal{S}_{m0}(\mathbb{R}\times\mathbb{S}^2) \to L^2(\mathbb{R}\times\mathbb{S}^2)$ is well- defined. In a similar way we see that operator $(F^*)^\wedge := \mathcal{F}_{k\to t} F^*_k \mathcal{F}^{-1}_{t\to k}$ is well- defined.

In the next lemma we establish the equality between the F_{time} and $\widehat{F}$ operators.

Lemma 5.14 *For all $h \in \mathcal{S}_{m0}(\mathbb{R}\times\mathbb{S}^2) \cap C_0^\infty(\mathbb{R}\times\mathbb{S}^2)$ we have*

$$F_{time}h = \widehat{F}h.$$

Proof. Let $h \in \mathcal{S}_{m0}(\mathbb{R}\times\mathbb{S}^2) \cap C_0^\infty(\mathbb{R}\times\mathbb{S}^2)$. We have

$$\begin{aligned}
\mathcal{F}^{-1}_{t\to k}\{F_{time}h\}(k,\widehat{x}) &= \mathcal{F}^{-1}_{t\to k}\{w_{sc,\infty}\}(k,\widehat{x}) \\
&= \left(\mathcal{F}^{-1}_{t\to k}\{w_{sc,\infty}(k,\cdot)\right)_\infty(\widehat{x}) \\
&= \left(\mathcal{F}^{-1}_{t\to k}\{u\}(k,\cdot)\right)_\infty(\widehat{x}).
\end{aligned}$$

On the other hand for each k it holds that

$$F_k \mathcal{F}^{-1}_{t\to k}\{h\}(k,\cdot) = \left(\mathcal{F}^{-1}_{t\to k}\{u\}(k,\cdot)\right)_\infty,$$

so we see that

$$\mathcal{F}^{-1}_{t\to k} F_{time} = F\mathcal{F}^{-1}_{t\to k},$$

that is

$$F_{time} = \widehat{F}.$$

$\square$

Next we derive the decay property of the operator $\widehat{L}$. First we provide a lemma on the commutativity of $\widehat{F}$ and the operators ∂_t and τ_s that is defined by

$$\tau_s h(t,d) := h(t-s,d).$$

Lemma 5.15 *Operators $\widehat{F}$, $(F^*)^\wedge : \mathcal{S}_{m0}(\mathbb{R}\times\mathbb{S}^2) \to L^2(\mathbb{R}\times\mathbb{S}^2)$ commute with the operators τ_s and ∂_t.*

Proof. We have

$$\begin{aligned}
\widehat{F}\tau_s &= \mathcal{F}_{k\to t}F\mathcal{F}^{-1}_{t\to k}\tau_s \\
&= \mathcal{F}_{k\to t}Fe^{-iks}\mathcal{F}^{-1}_{t\to k} \\
&= \mathcal{F}_{k\to t}e^{-iks}F\mathcal{F}^{-1}_{t\to k} \\
&= \tau_s\mathcal{F}_{k\to t}F\mathcal{F}^{-1}_{t\to k},
\end{aligned} \tag{5.39}$$

that is, $\widehat{F}$ commutes with τ_s. In the same way we see that $(F^*)^\wedge$ commutes with τ_s.

For all $h \in \mathcal{S}'(\mathbb{R}\times\mathbb{S}^2)$ it holds that

$$k\mathcal{F}^{-1}_{t\to k}\{h(t)\}(k) = \mathcal{F}^{-1}_{t\to k}\{i\partial_t h(t)\}(k) \tag{5.40}$$

and

$$\partial_t\mathcal{F}_{k\to t}\{h(k)\}(t) = \mathcal{F}_{k\to t}\{-ikh(k)\}(t). \tag{5.41}$$

Hence

$$\begin{aligned}
\partial_t\widehat{F} &= \partial_t\mathcal{F}_{k\to t}F\mathcal{F}^{-1}_{t\to k} \\
&= \widehat{F}\partial_t.
\end{aligned}$$

In the same way we see that

$$\partial_t(F^*)^\wedge = (F^*)^\wedge\partial_t.$$

□

We continue by making an estimate for the operator $F_{time} = \widehat{F}$.

Theorem 5.16 *Let* $h,\varphi \in \mathcal{S}_{m0}(\mathbb{R}\times\mathbb{S}^2)\cap C_0^\infty(\mathbb{R}\times\mathbb{S}^2)$ *and* T *be such that* $\mathrm{supp}(h),\mathrm{supp}(\varphi) \subset [-T,T]\times\mathbb{S}^2$. *There exist* $C,c\in\mathbb{R}_+$ *that depend on* R,R_1,T *and* D *such that*

$$\left|(F_{time}h,\tau_s\varphi)_{L^2(\mathbb{R}\times\mathbb{S}^2)}\right| \le Ce^{-cs}\|h\|_{C(\mathbb{R}\times\mathbb{S}^2)}\|\varphi\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}, \tag{5.42}$$

Proof. By equation (5.34) there are $C,c\in\mathbb{R}_+$ that depend on R,R_1,D and the supports of $u_{sc}(T+R,\cdot)$ and $\partial_t u_{sc}(T+R,\cdot)$ such that

$$\|F_{time}h(t,\cdot)\|_{L^2(\mathbb{S}^2)} \le Ce^{-ct}E(u_{sc},D^+,T+R). \tag{5.43}$$

Since the interaction of the incident wave and the obstacle starts after the time $-T-R$, the supports of $u_{sc}(T+R,\cdot)$ and $\partial_t u_{sc}(T+R,\cdot)$ are contained in

$B(0, 2(T+R)+R)$. Hence we can make the estimate (5.43) with constants C and c that depend on T instead of the supports of $u_{sc}(T+R, \cdot)$ and $\partial_t u_{sc}(T+R, \cdot)$.

For $t < -T-R$ the incident wave u_{in} is zero inside $\overline{B(0,R)}$. Let $\chi_2 \in C_0^\infty(\mathbb{R}^3)$ be such that $\chi_2 \equiv 1$ in $B(0, 4(T+R)+R)$ and $\chi_2 \equiv 0$ on $\mathbb{R}^3 \setminus B(0, 4(T+R)+R+1)$. Function $\chi_2 u_{in}$ is compactly supported, so there is an exterior Robin solution $\widetilde{u}_{tot}$ with the initial data $((\chi_2 u_{in})(-T-R, \cdot), \partial_t(\chi_2 u_{in})(-T-R, \cdot))$.

Because of the finite propagation speed of the waves, the waves u_{tot} and $\widetilde{u}_{tot}$ are identical inside the cone $\{(t,x) : |x| < 3(T+R)+R-t, t \geq -T-R\}$. Hence at $t = T+R$ we have inside the ball $B(0, 2(T+R)+R)$

$$\begin{aligned} u_{sc} &= u_{tot} - u_{in} \\ &= \widetilde{u}_{tot} - u_{in}. \end{aligned}$$

Since $\operatorname{supp}(u_{sc}(T+R, \cdot)) \subset B(0, 2(T+R)+R)$ we see that

$$\begin{aligned} E(u_{sc}, D^+, T+R) &\leq E(\widetilde{u}_{tot}, D_{2(T+R)+R}, T+R) + \\ &\quad + E(u_{in}, B(0, 2(T+R)+R), T+R) \\ &\leq 2E(u_{in}, B(0, 4(T+R)+R+1, -T-R), \end{aligned}$$

because of the conservation of energy and the finite propagation speed of the waves.

We have

$$\partial_{x_i} u_{in}(t,x) = \int_{\mathbb{S}^2} \partial_t h(t - x \cdot d, d)(-d_i) \mathrm{d}\mathcal{H}^2(d)$$

and

$$\partial_t u_{in}(t,x) = \int_{\mathbb{S}^2} \partial_t h(t - x \cdot d, d) \mathrm{d}\mathcal{H}^2(d).$$

Hence

$$E(u_{in}, B(0, 4(T+R)+R+1), -T-R) \leq C e^{-ct} \|\partial_t u\|_{C(\mathbb{R}\times\mathbb{S}^2)},$$

where C depends on T and R. It follows from equation (5.43) that

$$\|F_{time} h(t, \cdot)\|_{L^2(\mathbb{S}^2)} \leq C e^{-ct} \|\partial_t h\|_{C(\mathbb{R}\times\mathbb{S}^2)},$$

where C and c depend on R, R_1, T and D.

The estimate (5.42) follows from Hölder's inequality and the fact that $\operatorname{supp}(\varphi)$ is contained in $[-T, T] \times \mathbb{S}^2$.

□

Finally we get the estimate of the operator $\widehat{L}$.

Theorem 5.17 *Operator $\widehat{L} : C_0^\infty(\mathbb{R}\times\mathbb{S}^2) \to L^2(\mathbb{R}\times\mathbb{S}^2)$ commutes with the translation τ_s and satisfies for all $h,\varphi \in C_0^\infty(\mathbb{R}\times\mathbb{S}^2)$ with $\mathrm{supp}(h), \mathrm{supp}(\varphi) \subset [-T,T]\times\mathbb{S}^2$*

$$\begin{aligned}\left|(\widehat{L}h, \tau_s\varphi)_{L^2(\mathbb{R}\times\mathbb{S}^2)}\right| &\leq Ce^{-c|s|}\Big(\|\partial_t h\|_{C(\mathbb{R}\times\mathbb{S}^2)}\|\varphi\|_{L^2(\mathbb{R}\times\mathbb{S}^2)} + \\ &\qquad \ldots \|\partial_t\varphi\|_{C(\mathbb{R}\times\mathbb{S}^2)}\|h\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}\Big),\end{aligned}$$

where C and c depend on R, R_1, T and D.

Proof. We observe first that by (5.40)

$$\begin{aligned}\widehat{L} &= -\mathcal{F}_{k\to t} k \mathrm{Im}(F)\mathcal{F}^{-1}_{t\to k} \\ &= -\mathcal{F}_{k\to t}\frac{1}{2i}(F - F^*)\mathcal{F}^{-1}_{t\to k} i\partial_t.\end{aligned}$$

Since for all $\varphi \in C_0^\infty(\mathbb{R}\times\mathbb{S}^2)$ we have $\partial_t\varphi \in \mathcal{S}_{m0}(\mathbb{R}\times\mathbb{S}^2)$, we see that $\widehat{L} : C_0^\infty(\mathbb{R}\times\mathbb{S}^2) \to L^2(\mathbb{R}\times\mathbb{S}^2)$ is well-defined.

Since

$$\widehat{L} = -\frac{1}{2}\left(\widehat{F} - (F^*)^\wedge\right)\partial_t$$

and by Lemma 5.15 operators $\widehat{F}$ and $(F^*)^\wedge$ commute with τ_s we see that $\widehat{L}$ commutes with τ_s.

We have for all $h,\varphi \in C_0^\infty(\mathbb{R}\times\mathbb{S}^2)$ and $s \in \mathbb{R}$

$$\begin{aligned}(\widehat{L}h, \tau_s\varphi)_{L^2(\mathbb{R}\times\mathbb{S}^2)} &= -\frac{1}{2}((\widehat{F} - (F^*)^\wedge)\partial_t h, \tau_s\varphi)_{L^2(\mathbb{R}\times\mathbb{S}^2)} \\ &= \frac{1}{2}\left(\left(\widehat{F}\partial_t h, \tau_s\varphi\right) + \left(h, \widehat{F}\partial_t\tau_s\varphi\right)\right). \qquad (5.44)\end{aligned}$$

It follows from equation (5.42) that for all $s \geq 0$

$$\left|(\widehat{F}\partial_t h, \tau_s\varphi)_{L^2(\mathbb{R}\times\mathbb{S}^2)}\right| \leq Ce^{-cs}\|\partial_t h\|_{C(\mathbb{R}\times\mathbb{S}^2)}\|\varphi\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}, \qquad (5.45)$$

where C and c depend on R, R_1, T and D.

By Lemma 5.15 operators $\widehat{F}$ and τ_s commute, so

$$\begin{aligned}(h, \widehat{F}\partial_t\tau_s\varphi)_{L^2(\mathbb{R}\times\mathbb{S}^2)} &= (h, \tau_s\widehat{F}\partial_t\varphi)_{L^2(\mathbb{R}\times\mathbb{S}^2)} \\ &= (\tau_{-s}h, \widehat{F}\partial_t\varphi)_{L^2(\mathbb{R}\times\mathbb{S}^2)}.\end{aligned}$$

For $s < 0$ we have by equation (5.34)

$$\left|(\tau_{-s}h, \widehat{F}\partial_t\varphi)_{L^2(\mathbb{R}\times\mathbb{S}^2)}\right| \leq Ce^{-ct}\|h\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}\|\partial_t\varphi\|_{C(\mathbb{R}\times\mathbb{S}^2)}, \tag{5.46}$$

where C and c depend on R, R_1, T and D.

Since $\text{supp}(\varphi) \subset [-T, T] \times \mathbb{S}^2$ the Q_{sc} function corresponding to the incident wave kernel $\partial_t\varphi$ also satisfies $\text{supp}(Q_{sc}) \subset [-T - R, \infty) \times B(0, R)$. From equation (5.33) it follows that for $t < -T - R_1 - R$ we have

$$(\widehat{F}\partial_t\varphi)(t, d) = 0,$$

for all $d \in \mathbb{S}^2$. Hence for $s > 2T + R_1 + R$ we have

$$(\tau_{-s}h, \widehat{F}\partial_t\varphi)_{L^2(\mathbb{R}\times\mathbb{S}^2)} = 0$$

and by equations (5.44), (5.45) and (5.46)

$$\begin{aligned}\left|(\widehat{L}h, \tau_s\varphi)_{L^2(\mathbb{R}\times\mathbb{S}^2)}\right| \;\leq\; & Ce^{-c|s|}\Big(\|\partial_t h\|_{C(\mathbb{R}\times\mathbb{S}^2)}\|\varphi\|_{L^2(\mathbb{R}\times\mathbb{S}^2)} + \ldots \\ & \ldots \|\partial_t\varphi\|_{C(\mathbb{R}\times\mathbb{S}^2)}\|h\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}\Big),\end{aligned}$$

where C and c depend on R, R_1, T and D.

□

5.4 A Comment on Numerical Applicability

In this section we present few less formal comments on the implications of the previous rigorous analysis for the possible numerical implementation of the time domain method just presented. The numerical implementation and testing of the method are outside the scope of this work.

We emphasise that the treatment in this section is non-rigorous.

Let us recall the reconstruction condition

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}\times\mathbb{S}^2}\overline{\Big((\epsilon I + \widehat{L})^{-1}\widehat{\psi}_z\Big)}(t, \widehat{x})\widehat{\psi}_z(t, \widehat{x})\mathrm{d}t\mathrm{d}\mathcal{H}^2(\widehat{x}) < \infty$$

of Theorem 5.8. In solving this numerically, we first choose a sufficiently small $\epsilon > 0$ and consider the magnitude of the inner product

$$((\epsilon I + \widehat{L})^{-1}\widehat{\psi}_z, \widehat{\psi}_z)_{L^2(\mathbb{R}\times\mathbb{S}^2)}.$$

The function $\widehat{\psi}_z(t,\widehat{x}) = \widehat{\psi}(t+z\cdot\widehat{x},\widehat{x})$ is compactly supported, so we need to find the values of $(\epsilon I+\widehat{L})^{-1}\widehat{\psi}_z$ on a compact set $\mathrm{supp}(\widehat{\psi}_z)\subset\widetilde{I}\times\mathbb{S}^2$.

We wish to determine what kind of error is made if we truncate the domain $L^2(\mathbb{R}\times\mathbb{S}^2)$ to a bounded one, $I\times\mathbb{S}^2$, while solving the equation

$$(\epsilon I+\widehat{L})f = \widehat{\psi}_z. \tag{5.47}$$

To this end we discretise the equation (5.47) by approximating f and $\widehat{\psi}_z$ by the series

$$\begin{aligned} f &\sim \sum_{n=1}^{\infty}\sum_{j=-\infty}^{\infty} f_{j,n}\upsilon(t-\delta j)\Psi_n(\widehat{x}) \\ \widehat{\psi} &\sim \sum_{n=1}^{\infty}\sum_{j=-\infty}^{\infty} \psi_{j,n}\upsilon(t-\delta j)\Psi_n(\widehat{x}), \end{aligned}$$

where $\delta>0$ is a constant, $\upsilon\in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}(\upsilon)\subset(-\frac{\delta}{2},\frac{\delta}{2})$,

$$\int_{\mathbb{R}}|\upsilon|^2\mathrm{d}t = 1,$$

and $\{\Psi_k\}_{k\in\mathbb{Z}_+}\subset C_0^\infty(\mathbb{S}^2)$ is a orthonormal basis of $L^2(\mathbb{S}^2)$. Functions Ψ_k can for example be the spherical harmonics $\{Y_p^q\}$.

For fixed n_0 and n_1 the equation (5.47) has a discrete counterpart

$$(\epsilon[I]+[\widehat{L}]_{n_0,n_1})[f]_{n=n_0} = [\widehat{\psi}_z]_{n=n_1},$$

where

$$[f]_{n=n_0} = \begin{bmatrix} \vdots \\ f_{-1,n_0} \\ f_{0,n_0} \\ f_{1,n_0} \\ \vdots \end{bmatrix},$$

I is an identity matrix and $[\widehat{L}]$ is a matrix with the coefficients

$$\Big([\widehat{L}]_{n_0,n_1}\Big)_{\ell,j} = (\widehat{L}\upsilon(t-\delta j)\Psi_{n_0},\upsilon(t-\delta\ell)\Psi_{n_1})_{L^2(\mathbb{R}\times\mathbb{S}^2)}.$$

By Theorem 5.17 operator $\widehat{L}$ commutes with τ_s, so we see that

$$\begin{aligned} \Big([\widehat{L}]_{n_0,n_1}\Big)_{\ell,j} &= (\tau_{\delta j}\widehat{L}\upsilon\Psi_{n_0},\tau_{\delta\ell}\upsilon\Psi_{n_1})_{L^2(\mathbb{R}\times\mathbb{S}^2)} \\ &= (\widehat{L}\upsilon\Psi_{n_0},\tau_{\delta(\ell-j)}\upsilon\Psi_{n_1})_{L^2(\mathbb{R}\times\mathbb{S}^2)}. \end{aligned}$$

Both $\upsilon\Psi_{n_0}$ and $\tau_{\delta(\ell-j)}\upsilon\Psi_{n_1}$ are in $C_0^\infty(\mathbb{R}\times\mathbb{S}^2)$, so it follows from Theorem 5.17 that

$$\begin{aligned} \left|\left([\widehat{L}]_{n_0,n_1}\right)_{\ell,j}\right| \;\leq\; & Ce^{-c|\delta(\ell-j)|}\Big(\|\partial_t\upsilon\Psi_{n_0}\|_{C^1(\mathbb{R}\times\mathbb{S}^2)}\|\upsilon\Psi_{n_1}\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}+\dots \\ & \dots+\|\partial_t\upsilon\Psi_{n_1}\|_{C^1(\mathbb{R}\times\mathbb{S}^2)}\|\upsilon\Psi_{n_0}\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}\Big), \end{aligned} \tag{5.48}$$

where C and c depend on R, R_1, δ and D.

We speculate that the $e^{-c|\delta(\ell-j)|}$ factor dominates the overall behaviour of the entries of $[\widehat{L}]_{n_0,n_1}$ so that the entries of the matrix $(\epsilon[I]+[\widehat{L}]_{n_0,n_1})$ that are far off the diagonal are very small. Since the vector $[\widehat{\psi}_z]_{n=n_1}$ has non-zero values only for indices $\{-j_m,-j_m+1,\dots,j_m-1,j_m\}$ we speculate that the solution $[f]_{n_0}$ will be negligible for indices outside some interval $\{-M,-M+1,\dots,M-1,M\}$ with $M\geq j_m$. If this holds we can find a good solution by solving the finite matrix equation

$$(\epsilon[I]_M+[\widehat{L}]_{n_0,n_1M})[f]_{n=n_0,M}=[\widehat{\psi}_z]_{n=n_1,M}. \tag{5.49}$$

Here $[\cdot]_M$ indicates a truncated $(2M+1)\times(2M+1)$ or $(2M+1)$ matrix.

We expect that the decay of the off-diagonal elements in (5.48) is uniform enough so that a certain number M is large enough for all indices n_0 and n_1. In this case the equation

$$(\epsilon I+\widehat{L})f=\widehat{\psi}_z$$

can be successfully solved by limiting the calculations for some finite set $I_M\times\mathbb{S}^2$.

The numerical solution can be done better with some system other than the one we used. In the above we only wanted to highlight the causal connection between the different time slices $(\delta j-\frac{\delta}{2},\delta j+\frac{\delta}{2})$.

5.5 Conclusion

The aim of this work was to study the expansion of the factorization method which allows information coming from several fequencies to be analyzed at the same time. To this end a wave number analysis was made to operators U_k, F_k, G_k and others in Chapters 3 and 4. Along the way a new single frequency method with wave number dependent estimates was developed. This method is the basis of the multi-frequency method presented Section 5.1.

The multi-frequency method developed in this work is not directly applicable numerically. First steps in this direction were taken in Sections 5.2 and 5.3 in which the time domain far field operator was formed while assuming that the scatterer has a Robin energy decay property. Based on this a heuristic discussion of how to do numerics with the multi-frequency factorization was provided in Section 5.4.

The avenues of further research based on this work are, in the author's view, at least to threefold:

First, and perhaps of the highest priority, would be to complete the numerical arguments and check how well the reconstructions truly work using actual numerical and/or physical data.

Second one could find ways to bring rigour to the assumption of Robin energy decay. This can be done in at least in two ways; one can find a class of scatterers that have the Robin energy decay property or one can divide the scattered wave into two parts, one of which is decaying and the other of which related to the resonances of the scatterer. After that one can apply the multi-frequency method to the decaying part and perhaps some other method to the resonances.

Third one could study how other reconstruction methods could be subjected to a frequency domain - time domain analysis. Many tools in this work can be used in the other factorization methods and frequency domain methods. Our bridge between the frequency and time domains, $\mathcal{F}_{k\to t} : \mathcal{S}'(\mathbb{R}\times\mathbb{S}^2) \to \mathcal{S}'(\mathbb{R}\times\mathbb{S}^2)$ might have applications in an even wider class of methods.

Appendices

Appendix A

Expansion of the Kernels

Lemma A.1 *Let $k \in \mathbb{R}$ and $R > 0$. We have for all $y \in B(0,R)$ and $x \in \mathbb{R}^3 \setminus B(0,2R)$ the asymptotics*

$$\Phi_k(x,y) = \frac{e^{ik|x|}}{|x|}\left(\frac{1}{4\pi}e^{-ik\widehat{x}\cdot y} + \langle k\rangle \mathcal{O}_S\left(\frac{1}{|x|}\right)\right) \tag{A.1}$$

and

$$\nabla_y \Phi_k(x,y) = \frac{e^{ik|x|}}{|x|}\left(-\frac{\widehat{x}}{4\pi}ike^{-ik\widehat{x}\cdot y} + \langle k\rangle^2 \mathcal{O}_D\left(\frac{1}{|x|}\right)\right), \tag{A.2}$$

where the $\mathcal{O}\left(\frac{1}{|x|}\right)$ terms are C^∞ with respect to x and y and do not depend on k.

Remark: When we say that the $\mathcal{O}\left(\frac{1}{|x|}\right)$ terms do not depend on k we mean that there is an estimate: $\left|\mathcal{O}\left(\frac{1}{|x|}\right)\right| \leq \frac{C}{|x|}$. The derivatives of this term also have the same kind of asymptotics.

Proof. We use the mean value theorem on the function $t \rightarrow (1+t^2)^{\frac{1}{2}}$ and see that

$$\begin{aligned} |x-y| &= |x|(1-2\widehat{x}\cdot y + \frac{|y|^2}{|x|^2})^{\frac{1}{2}} \\ &= |x| - \widehat{x}\cdot y + \mathcal{O}\left(\frac{1}{|x|}\right). \end{aligned}$$

Hence

$$\begin{aligned}\frac{e^{ik|x-y|}}{4\pi|x-y|} &= \frac{e^{ik(|x|-\widehat{x}\cdot y+\mathcal{O}\left(\frac{1}{|x|}\right))}}{4\pi(|x|-\widehat{x}\cdot y+\mathcal{O}\left(\frac{1}{|x|}\right))} \\ &= \frac{1}{4\pi}\left(\frac{e^{ik|x|}}{|x|}+\left(\frac{e^{ik|x|}}{(|x|-\widehat{x}\cdot y+\mathcal{O}\left(\frac{1}{|x|}\right))}-\frac{e^{ik|x|}}{|x|}\right)\right)\cdot \\ &\quad \cdot e^{ik(-\widehat{x}\cdot y+\mathcal{O}\left(\frac{1}{|x|}\right))}.\end{aligned}$$

Here

$$\frac{e^{ik|x|}}{(|x|-\widehat{x}\cdot y+\mathcal{O}\left(\frac{1}{|x|}\right))}-\frac{e^{ik|x|}}{|x|}=\mathcal{O}\left(\frac{1}{|x|^2}\right)$$

and

$$e^{ik(-\widehat{x}\cdot y+\mathcal{O}\left(\frac{1}{|x|}\right))}=e^{-ik\widehat{x}\cdot y}+|k|\mathcal{O}\left(\frac{1}{|x|}\right).$$

Hence

$$\frac{e^{ik|x-y|}}{4\pi|x-y|}=\frac{e^{ik|x|}}{|x|}\left(\frac{1}{4\pi}e^{-ik\widehat{x}\cdot y}+(1+|k|)\mathcal{O}\left(\frac{1}{|x|}\right)\right). \tag{A.3}$$

For $y\in B(0,R)$ and $x\in\mathbb{R}^3\setminus B(0,R)$ the functions $\frac{e^{ik|x|}}{|x|}$, $e^{-ik\widehat{x}\cdot y}$ and $\frac{e^{ik|x-y|}}{4\pi|x-y|}$ are C^∞, so we see that the $(1+|k|)\mathcal{O}\left(\frac{1}{|x|}\right)$ term in (A.3) is C^∞ with respect to x and y.

For the other kernel we notice that

$$\nabla_y\frac{e^{ik|x-y|}}{4\pi|x-y|}=-\widehat{(x-y)}\left(\frac{ike^{ik|x-y|}}{4\pi|x-y|}-\frac{e^{ik|x-y|}}{4\pi|x-y|^2}\right).$$

Here

$$\frac{e^{ik|x-y|}}{4\pi|x-y|^2}=\mathcal{O}\left(\frac{1}{|x|^2}\right)$$

and

$$|\widehat{(x-y)}-\widehat{x}|=\mathcal{O}\left(\frac{1}{|x|}\right),$$

so by equation (A.3)

$$\nabla_y\frac{e^{ik|x-y|}}{4\pi|x-y|} = \frac{e^{ik|x|}}{|x|}\left(-\frac{\widehat{x}}{4\pi}ike^{-ik\widehat{x}\cdot y}+(1+k^2)\mathcal{O}\left(\frac{1}{|x|}\right)\right). \tag{A.4}$$

For $y\in B(0,R)$ and $x\in\mathbb{R}^3\setminus B(0,R)$ the functions $\frac{e^{ik|x|}}{|x|}$, $-\widehat{x}ike^{-ik\widehat{x}\cdot y}$ and $\nabla_y\frac{e^{ik|x-y|}}{4\pi|x-y|}$ are C^∞, so we see that the $(1+k^2)\mathcal{O}\left(\frac{1}{|x|}\right)$ term in (A.4) is C^∞.

$\square$

Lemma A.2 *Let $k \in \mathbb{R}$ and $R > 0$. We have for all $y \in B(0,R)$ and $x \in \mathbb{R}^3 \setminus B(0,2R)$ the asymptotics*

$$\left(\partial_{r(x)} - ik\right) \Phi(x,y) = \langle k \rangle \mathcal{O}\left(\frac{1}{|x|^2}\right)$$

and

$$\left(\partial_{r(x)} - ik\right) \nabla_y \Phi(x,y) = \langle k \rangle^2 \mathcal{O}\left(\frac{1}{|x|^2}\right).$$

Proof. We have

$$\left(\partial_{r(x)} - ik\right) \frac{e^{ik|x-y|}}{4\pi|x-y|} = \left(\widehat{x} \cdot \widehat{(x-y)} - 1\right) \frac{ike^{ik|x-y|}}{4\pi|x-y|} + \widehat{x} \cdot \widehat{(x-y)} \frac{e^{ik|x-y|}}{4\pi|x-y|^2}.$$

Here

$$\left(\widehat{x} \cdot \widehat{(x-y)} - 1\right) = \mathcal{O}\left(\frac{1}{|x|}\right),$$

so we see that

$$\left(\partial_{r(x)} - ik\right) \frac{e^{ik|x-y|}}{4\pi|x-y|} = \langle k \rangle \mathcal{O}\left(\frac{1}{|x|^2}\right).$$

For the other kernel we notice that

$$\left(\partial_{r(x)} - ik\right) \nabla_y \frac{e^{ik|x-y|}}{4\pi|x-y|} = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1 &= -(\partial_{r(x)} \widehat{(x-y)}) \left(\frac{ike^{ik|x-y|}}{4\pi|x-y|} - \frac{e^{ik|x-y|}}{4\pi|x-y|^2}\right) \\
&= \langle k \rangle \mathcal{O}\left(\frac{1}{|x|^2}\right) \\
I_2 &= -\widehat{(x-y)} \left(ik \left(\partial_{r(x)} - ik\right) \frac{e^{ik|x-y|}}{4\pi|x-y|}\right) \\
&= \langle k \rangle^2 \mathcal{O}\left(\frac{1}{|x|^2}\right) \\
I_3 &= -\widehat{(x-y)} \left(\partial_{r(x)} - ik\right) \frac{e^{ik|x-y|}}{4\pi|x-y|^2} \\
&= \langle k \rangle \mathcal{O}\left(\frac{1}{|x|^3}\right).
\end{aligned}$$

Hence

$$\left(\partial_{r(x)} - ik\right) \nabla_y \frac{e^{ik|x-y|}}{4\pi|x-y|} = \langle k \rangle^2 \mathcal{O}\left(\frac{1}{|x|^2}\right).$$

$\square$

Appendix B

Bochner Integrals

In this Appendix we present a definition of Bochner integrals and a small application of them in Lemma B.6. Our presentation of Bochner integrals closely follows the book [35]. We begin with the definitions of a strongly measurable function and a Bochner integrable function.

Definition B.1 *Let (Ω, Σ, μ) be a σ-finite measure space and let X be a Banach space. Then simple functions $T : \Omega \to X$ are of the form*

$$T(s) = \sum_{i=1}^{n} a_i \chi_{E_i}(s),$$

where $a_i \in X$, $\mu(E_i) < \infty$ and χ_{E_i} is the characteristic function. Function $T : \Omega \to X$ is **strongly measurable** *if there exists a sequence (T_n) of simple functions that converges pointwise to T.*

Definition B.2 *Function $T : \Omega \to X$ is* **Bochner integrable** *if there exists a sequence of simple functions (T_n) that converges to T pointwise and satisfies*

$$\lim_{m,n\to\infty} \int_\Omega \|T_n(s) - T_m(s)\|_X \, d\mu(s) = 0.$$

If T is Bochner integrable, we define

$$\int_\Omega T(s) d\mu := \lim_{n\to\infty} \int_\Omega T_n \, d\mu.$$

The following two Theorems are useful in proving that a function is Bochner integrable.

Theorem B.3 *Function $T : \Omega \to X$ is strongly measurable if and only if for all open $U \subset X$ we have $T^{-1}(U)$ is measurable and $T(\Omega)$ is separable.*

Proof. We refer to [35, Theorem 23.2].

□

Theorem B.4 *A function $T : \Omega \to X$ is Bochner integrable if and only if T is strongly measurable and*

$$\int_\Omega \|T(s)\|_X \, d\mu(s) < \infty.$$

In this case there exists a sequence of simple functions (T_n) that converges pointwise to T and for all $s \in \Omega$ and $n \in \mathbb{N}$

$$\|T_n(s)\|_X \leq 2\|T(s)\|_X \tag{B.1}$$

and

$$\lim_{n\to\infty} \int_\Omega \|T(s) - T_n(s)\|_X \, d\mu(s) = 0. \tag{B.2}$$

Proof. We refer to [35, Theorem 23.16].

□

Lebesgue's Dominated Convergence Theorem can be applied to the Bochner integrals. For the convenience of the reader we reproduce the Theorem here and refer to [35] for the proof.

Theorem B.5 *Let $T : \Omega \to X$ be strongly measurable and (T_n) be a sequence of functions such that for almost every $s \in \Omega$ we have $T_n(s) \to T(s)$. If there is $g \in L^1(\Omega)$ such that for all $n \in \mathbb{N}$ and almost every $s \in \Omega$ we have*

$$\|T_n(s)\|_X \leq g(s), \tag{B.3}$$

then T is Bochner integrable and

$$\int_\Omega T(s)\, d\mu = \lim_{n\to\infty} \int_\Omega T_n(s)\, d\mu$$

Proof. We refer to [35, Theorem 23.20]. We now provide a small application of the Bochner integral that was needed in the main text.

Lemma B.6 *Let $R \in \mathbb{R}$ be such that $\overline{D} \subset B(0,R)$ and $h \in L^2(\mathbb{S}^2)$. The map $T : \mathbb{S}^2 \to H^1(D_R)$ defined by*

$$T(d) = U_{rob,k}(-(\partial_{\nu(D)} + \lambda \mathrm{tr}\,)e^{ikx\cdot d})h(d)$$

is Bochner integrable.

Proof. By Theorem B.4 the map T is Bochner integrable if and only if it is strongly measurable and satisfies

$$\int_{\mathbb{S}^2} \|T(d)\|_{H^1(D_R)} \mathrm{d}\mathcal{H}^2(d) < \infty. \tag{B.4}$$

We prove first that T is strongly measurable. By Theorem B.3 the map T is strongly measurable if and only if 1^0 for all $U \subset H^1(D_R)$ open $T^{-1}(U)$ is measurable and 2^0 $T(\mathbb{S}^2)$ is separable.

1^0: We can decompose T as

$$T : \mathbb{S}^2 \xrightarrow{f} H^{-\frac{1}{2}}(\partial D) \xrightarrow{U_{rob,k}} H^1(D_R),$$

where

$$\begin{aligned} f(d) &= (-(\partial_{\nu(D)} + \lambda \mathrm{tr}\,)e^{ikx\cdot d})h(d) \\ &= A(d)h(d). \end{aligned}$$

Here $A : \mathbb{S}^2 \to H^{-\frac{1}{2}}(\partial D)$ is continuous and $h \in L^2(\mathbb{S}^2)$.

There exist simple functions $h_n : \mathbb{S}^2 \to \mathbb{C}$ such that for all $d \in \mathbb{S}^2$ we have $h(d) = \lim_{n\to\infty} h_n(d)$. For all $n \in \mathbb{N}$ the map Ah_n is measurable and

$$Ah = \lim_{n\to\infty} Ah_n.$$

By [11, Proposition 8.1.8] a limit of measurable functions Ah_n from a measure space $(\mathbb{S}^2, \mathcal{H}^2)$ to a metrizable topological space $H^{-\frac{1}{2}}(\partial D)$ is measurable, so we see that $f = Ah$ is measurable.

Since $U_{rob,k} : H^{-\frac{1}{2}}(\partial D) \to H^1(D_R)$ is continuous we see that for all $U \subset H^1(D_R)$ open, the set $U_{rob,k}^{-1}(U) \subset H^{-\frac{1}{2}}(\partial D)$ is open. Hence $T^{-1}(U)$ is measurable.

2^0: $H^1(D_R)$ is separable, so $T(\mathbb{S}^2)$ is separable as a subset of a separable set.

Now we turn to the condition (B.4). The map $B : \mathbb{S}^2 \to H^1(D_R)$ defined by

$$B(d) = U_{rob,k} \circ A$$

is continuous and $\mathbb{S}^2$ is compact, so we see that

$$\sup_{d\in\mathbb{S}^2} \|B(d)\|_{H^1(D_R)} = C < \infty.$$

We have

$$\begin{aligned}\int_{\mathbb{S}^2} \|T(d)\|_{H^1(D_R)} \mathrm{d}\mathcal{H}^2(d) &= \int_{\mathbb{S}^2} |h(d)| \|B(d)\|_{H^1(D_R)} \mathrm{d}\mathcal{H}^2(d) \\ &\leq C\|h\|_{L^2(\mathbb{S}^2)} \mathcal{H}^2(\mathbb{S}^2)^{\frac{1}{2}}, \qquad \text{(B.5)}\end{aligned}$$

so T is Bochner integrable.

$\square$

Appendix C

Frechet Spaces, LF Spaces and $\mathcal{S}'(\mathbb{R} \times U)$

We are able to form the space $\mathcal{S}'(\mathbb{R} \times U)$ in exactly the same way as the space $\mathcal{D}'(U)$ is formed in e.g. [55]. The only difference is that we start with a different family of semi-norms. For the sake of completeness we give the relevant definitions and proofs for forming the space $\mathcal{S}'(\mathbb{R} \times U)$. We follow the approach of [55] rather closely, though we take topologies to be the fundamental concept instead of filters, which are used in [55].

Let $\mathcal{C}$ be the Euclidean topology of the complex plane $\mathbb{C}$.

Definition C.1 *Let E be a vector space with a topology $\mathcal{T}$ and the spaces $E \times E$ and $\mathbb{C} \times E$ be equipped with the product topologies $\mathcal{T} \times \mathcal{T}$ and $\mathcal{C} \times \mathcal{T}$ respectively. Then $(E, \mathcal{T})$ is a* **topological vector space** *if the addition $+ : E \times E \to E$ and the scalar multiplication $: \mathbb{C} \times E \to E$ are continuous.*

We note that for all $x \in E$ the map $x+ : E \to E; y \to x + y$ is a homeomorphism and for all $\lambda \in \mathbb{C} \setminus \{0\}$ the map $\lambda : E \to E; y \to \lambda y$ is a homeomorphism.

Definition C.2 *Let $(E, \mathcal{T})$ be a topological space. A collection of sets $\mathcal{B} \subset \mathcal{T}$ is a basis of the topology $\mathcal{T}$ if for all $A \in \mathcal{T}$ and $x \in A$ there exists a $B \in \mathcal{B}$ such that $x \in B$ and $B \subset A$.*

We note that if $\mathcal{B}$ is a basis of the topology then for any $A \in \mathcal{T}$ there are elements $\{B_x\}_{x \in A} \subset \mathcal{B}$ such that $A = \bigcup_{x \in A} B_x$. The next result is very useful in determining if a collection is a basis of a topology.

Lemma C.3 *A collection $\mathcal{B}$ of sets is a basis of some topology if and only if for all $A, B \in \mathcal{B}$ and $x \in A \cap B$ there exists a $D \in \mathcal{B}$ such that $x \in D \subset A \cap B$.*

Proof. We refer to [27, Theorem 1.11]

Definition C.4 *A topological vector space $(E, \mathcal{T})$ is a locally convex space if $\mathcal{T}$ has a basis that consists of convex sets.*

The spaces of test functions are constructed as spaces induced by a set of semi-norms. These spaces will be locally convex topological vector spaces as we will see in the next lemma.

Lemma C.5 *Let E be a vector space, $\mathcal{P}$ a collection of semi-norms on E such that for all $p_1, p_2 \in \mathcal{P}$ there exists a $p \in \mathcal{P}$ such that $p_1, p_2 \leq p$. For all $p \in \mathcal{P}$ and $\epsilon > 0$ let*

$$B_p(\epsilon) = \{x \in E : p(x) < \epsilon\}.$$

Then the collection

$$\mathcal{A} := \{x + B_p(\epsilon) : x \in E, p \in \mathcal{P}, \epsilon > 0\}]$$

is a basis of a topology, denoted $\mathcal{T}$ on E and $(E, \mathcal{T})$ is a locally convex topological vector space.

Proof. We use the basis criterion of Lemma C.3. Hence let $x_1, x_2 \in E$, $\epsilon_1, \epsilon_2 > 0$, $p_1, p_2 \in \mathcal{P}$ and

$$z \in (x_1 + B_{p_1}(\epsilon_1)) \cap (x_2 + B_{p_2}(\epsilon_2)) = A.$$

Let p be such that $p_1, p_2 \leq p$ and

$$\epsilon = \min\{\epsilon_1 - p_1(x - z), \epsilon_2 - p_2(y - z)\}.$$

Then $z + B_p(\epsilon) \subset A$ and by Lemma C.3 the collection $\mathcal{A}$ is a basis.

For all $x \in E$, $p \in \mathcal{P}$ and $\epsilon > 0$ the set $x + B_p(\epsilon)$ is convex, so we see that the space $(E, \mathcal{T})$ is locally convex.

We prove next that the arithmetical operations are continuous. Let $\lambda \in \mathbb{C}$, $x, y \in A$, $f(x) = \lambda x$, $g(x, y) = x + y$ and $U \subset E$ be open. Let $z \in f^{-1}(U)$. There exist $p \in \mathcal{P}$ and $\epsilon > 0$ such that $\lambda z + B_p(\epsilon) \subset U$. Hence

$$z + \lambda^{-1} B_p(\epsilon) = z + B_p(|\lambda|^{-1}\epsilon) \subset \lambda^{-1} U = f^{-1}(U)$$

and $f^{-1}(U)$ is open.

Let $U \subset E$ be open and $(x, y) \in g^{-1}(U)$. Hence there is $p \in \mathcal{P}$ and an $\epsilon > 0$ such that

$$(x + y) + B_p(\epsilon) \subset U.$$

It follows from the sub-additivity of p that

$$B_p(\frac{\epsilon}{2}) + B_p(\frac{\epsilon}{2}) \subset B_p(\epsilon),$$

so we see that

$$(x + B_p(\frac{\epsilon}{2}), y + B_p(\frac{\epsilon}{2})) \subset g^{-1}(U).$$

Hence $g^{-1}(U)$ is open and g continuous.

□

Definition C.6 *A Topological vector space $(E, \mathcal{T})$ is a Frechet space if*

1. *It is locally convex.*
2. *It is metrizable.*
3. *It is complete.*

Definition C.7 *Let $\{E_j\}_{j\in\mathbb{N}}$ be Frechet spaces, $E_j \subset E_{j+1}$ for all $j \in \mathbb{N}$ and the inclusions $i_j : E_j \hookrightarrow E_{j+1}$ be an isomorphism from E_j to $i_j(E_j)$, where $i_j(E_j)$ is endowed with the relative topology from E_{j+1}. Then the* **numerable strict inductive limit of the Frechet spaces** *$\{E_j\}$ is the set*

$$E := \bigcup_{j=1}^{\infty} E_j$$

together with the topology $\mathcal{T}$ having a basis of neighbourhoods of 0, which consist of those convex sets V for which $0 \in V$ and for all $j \in \mathbb{N}$ the intersection $V \cap E_j$ is open in E_j. We call the space $(E, \mathcal{T})$ an LF space and the sequence $\{E_j\}_{j\in\mathbb{N}}$ a **sequence of definition** *of E.*

In the above definition it is indeed enough to specify the neighbourhoods of zero, as we will see in the next lemma

Lemma C.8 *Let $\{E_j\}_{j\in\mathbb{N}}$, E and the neighbourhoods V of zero be as in Definition C.7. Then the collection*

$$\mathcal{B} := \{x + V : x \in E,\ V \text{ neighborhood of } 0\}$$

is a basis of a topology in E.

Proof. We will use the basis criterion of Lemma C.3. To this end, let $x, y \in E$ and V_x, V_y be open neighbourhoods of zero and

$$z \in (x + V_x) \cap (y + V_y).$$

Let j be so large that $x, y, z \in E_j$. Then

$$((x - z) + V_x) \cap E_j = x - z + (V_x \cap E_j),$$

which is an open set of E_j since $V_x \cap E_j \subset E_j$ is open and the map $p \to p + x$ is a homeomorphism in E_j. In the same way we see that $((y-z)+V_y) \cap E_j \subset E_j$ is open, so

$$(((x - z) + V_x) \cap E_j) \cap (((y - z) + V_y) \cap E_j) \subset E_j$$

is a open neighbourhood of 0 in E_j.

Let $k < j$. The inclusion $i_{k,j} : E_k \to i_{k,j}(E_k) \hookrightarrow E_j$ is an isomorphism from E_k to $i_{k,j}(E_k)$, so we see that the pre-image

$$\begin{aligned} P &= i_k^{-1} \left(((x - z) + V_x) \cap E_j) \cap ((y - z) + V_y) \cap E_j) \right) \\ &= ((x - z) + V_x) \cap E_k) \cap (((y - z) + V_y) \cap E_k) \end{aligned}$$

is an open set of E_k. Hence by the criterion of the open neighbourhoods of 0 in E the set $((x - z) + V_x) \cap ((y - z) + V_y)$ is an open neighbourhood of 0 in E. Hence $(x + V_x) \cap (y + V_y) \in \mathcal{B}$ and by the basis criterion $\mathcal{B}$ is a basis of a topology.

□

Now we are able to move to the definition of the test functions $\mathcal{S}(\mathbb{R} \times U)$.

Let $U \subset \mathbb{R}^3$ be an open set. By [55, Lemma 10.1] there exist compact sets $\{K_j\}_{j \in \mathbb{N}}$ of U such that for all $j \in \mathbb{N}$ the set K_j is in the interior of K_{j+1} and $U = \bigcup_{j \in \mathbb{N}} K_j$. We define the semi-norms

$$\|\varphi\|_{k,i,j} := \sup_{|\alpha| \leq k} \sup_{\substack{t \in \mathbb{R} \\ x \in K_j}} |\langle t \rangle^i \partial^\alpha \varphi(t, x)|,$$

where $\alpha \in \mathbb{N}^4$, and the space

$$\widetilde{\mathcal{S}}(\mathbb{R} \times U) := \{\varphi \in C^\infty(\mathbb{R} \times U) : \|\varphi\|_{k,i,j} < \infty \ \forall k, i, j \in \mathbb{N}\}.$$

By Lemma C.5 the semi-norms $\{\| \cdot \|_{i,j,k}\}$ induce a topology $\mathcal{T}$ on the space $\widetilde{\mathcal{S}}(\mathbb{R} \times U)$, which makes it a locally convex topological vector space. By

[55, Proposition 8.1] a topological space induced by a numerable set of semi-norms is metrizable, so we see that $\widetilde{\mathcal{S}}(\mathbb{R} \times U)$ is metrizable. To see that $\widetilde{\mathcal{S}}(\mathbb{R} \times U)$ is also complete, let $(\varphi_n)_{n\in\mathbb{N}} \subset \widetilde{\mathcal{S}}(\mathbb{R} \times U)$ be a Cauchy sequence. With the norms $\|\cdot\|_{0,0,j}$ we can see that there is a $\varphi \in C^\infty(\mathbb{R} \times U)$ such that for all $(t,x) \in \mathbb{R} \times U$ we have $\varphi_n(t,x) \overset{n\to\infty}{\longrightarrow} \varphi(t,x)$ and that the convergence is uniform over compact sets $K \subset\subset U$. In the same way we see that for all $i \in \mathbb{N}$ and $\alpha \in \mathbb{N}^4$ there exists a f_α such that $\langle t\rangle^i \partial^\alpha \varphi_n \to \langle t\rangle^i f_\alpha$ uniformly on $\mathbb{R} \times K$ for all $K \subset\subset U$. Hence we see that $\varphi \in C^\infty(\mathbb{R} \times U)$ and that for all k, i, j

$$\|\varphi\|_{k,i,j} < \infty.$$

Hence $\varphi \in \widetilde{\mathcal{S}}(\mathbb{R} \times U)$, $\varphi_n \to \varphi$ in $\widetilde{\mathcal{S}}(\mathbb{R} \times U)$ and $\widetilde{\mathcal{S}}(\mathbb{R} \times U)$ is complete. Thus $\widetilde{\mathcal{S}}(\mathbb{R} \times U)$ is a Frechet space.

Let the space

$$\mathcal{S}(\mathbb{R} \times K_j) := \{\varphi \in \widetilde{\mathcal{S}}(\mathbb{R} \times U) : \mathrm{supp}(\varphi) \subset K_j\}$$

be equipped with the induced topology from $\widetilde{\mathcal{S}}(\mathbb{R} \times U)$, that is the topology induced by the inclusion $i : \mathcal{S}(\mathbb{R} \times K_j) \hookrightarrow \widetilde{\mathcal{S}}(\mathbb{R} \times U)$. This is the same topology as the one induced by the semi-norms $\{\|\cdot\|_{k,i,j}\}_{k,i\in\mathbb{N}}$ on $\mathcal{S}(\mathbb{R} \times K_j)$. If a sequence $(\varphi_n) \subset \mathcal{S}(\mathbb{R} \times K_j)$ converges to an $\varphi \in \widetilde{\mathcal{S}}(\mathbb{R} \times U)$ then for all $x \in U \setminus K_j$ and $t \in \mathbb{R}$ we have $\varphi_n(t,x) \to 0$ as $n \to \infty$. Hence we see that in fact $\varphi \in \mathcal{S}(\mathbb{R} \times K_j)$ and that $\mathcal{S}(\mathbb{R} \times K_j)$ is closed. Hence $\mathcal{S}(\mathbb{R} \times K_j)$ also is a Frechet space.

The inclusions $i_j : \mathcal{S}(\mathbb{R} \times K_j) \hookrightarrow \mathcal{S}(\mathbb{R} \times K_{j+1})$ are also isomorphisms. Hence we can define by Definition C.7

$$\mathcal{S}(\mathbb{R} \times U) := \bigcup_{j\in\mathbb{N}} \mathcal{S}(\mathbb{R} \times K_j)$$

to be the countable strict inductive limit of Frechet spaces $(\mathcal{S}(\mathbb{R} \times U), \mathcal{T})$.

Now we can define the corresponding distributions.

Definition C.9 *The distributions $\mathcal{S}'(\mathbb{R}\times U)$ are continuous maps from $\mathcal{S}(\mathbb{R}\times U)$ to $\mathbb{C}$.*

The following result is analogous to the one for distributions in $\mathcal{D}'(U)$ that can be found e.g. in [55, Proposition 21.1.]. We reproduce this result with a proof.

Theorem C.10 *A linear map $f : \mathcal{S}(\mathbb{R} \times U) \to \mathbb{C}$ is continuous if and only if one of the following equivalent conditions hold:*

(i) For every compact set $K \subset U$ there are indexes $k, i \in \mathbb{N}$ and a constant $C > 0$ such that for all $\varphi \in \mathcal{S}(\mathbb{R} \times U)$ with $\operatorname{supp}(\varphi) \subset K$ we have

$$|f(\varphi)| \leq \sup_{|\alpha| \leq k} \sup_{\substack{t \in \mathbb{R} \\ x \in K}} |\langle t \rangle^i \partial^\alpha \varphi(t,x)|.$$

(ii) For all sequences $(\varphi_n) \subset \mathcal{S}(\mathbb{R} \times U)$ with $\operatorname{supp}(\varphi_n) \subset \mathbb{R} \times K$ for some compact $K \subset U$ and with the property: for all $k, i \in \mathbb{N}$

$$\lim_{n \to \infty} \sup_{|\alpha| \leq k} \sup_{\substack{t \in \mathbb{R} \\ x \in K_j}} |\langle t \rangle^i \partial^\alpha \varphi(t,x)| = 0$$

we have

$$\lim_{n \to \infty} |f(\varphi_n)| = 0.$$

Proof. By [55, Proposition 13.1] a linear map L from a LF space E to $\mathbb{C}$ is continuous if and only if for all E_j in a sequence of definition of E the restriction $L|_{E_j}$ is continuous. Hence a linear map $f : \mathcal{S}(\mathbb{R} \times U) \to \mathbb{C}$ is continuous if and only if for all $j \in \mathbb{N}$ the restriction $f|_{\mathcal{S}(\mathbb{R} \times K_j)}$ is continuous. By [55, Proposition 7.7] a map L from a locally convex space to $\mathbb{C}$ is continuous if and only if there is a continuous semi-norm p in E such that for all $x \in E$ we have

$$|Lx| \leq p(x).$$

The set $\{\|\cdot\|_{k,i,j}\}_{k,i \in \mathbb{N}}$ forms a basis of continuous semi-norms in $\mathcal{S}(\mathbb{R} \times K_j)$, so the restriction $f|_{\mathcal{S}(\mathbb{R} \times U)}$ is continuous if and only if there exists $k, i \in \mathbb{N}$ such that for all $\varphi \in \mathcal{S}(\mathbb{R} \times K_j)$ we have

$$\left| f|_{\mathcal{S}(\mathbb{R} \times K_j)}(\varphi) \right| \leq C \|\varphi\|_{k,i,j}. \tag{C.1}$$

By [55, Proposition 8.5] a map L from a metrizable space E to a topological vector space F is continuous if and only if it is sequentially continuous. Hence we can rephrase the condition (C.1) for the continuity of $f|_{\mathcal{S}(\mathbb{R} \times K_j)}$ as follows: the map $f|_{\mathcal{S}(\mathbb{R} \times K_j)}$ is continuous if and only if for all sequences $(\varphi_n) \subset \mathcal{S}(\mathbb{R} \times K_j)$ such that $\|\varphi_n\|_{k,i,j} \xrightarrow{n \to \infty} 0$ for all $k, i \in \mathbb{N}$ we have

$$\lim_{n \to \infty} f|_{\mathcal{S}(\mathbb{R} \times K_j)}(\varphi_j) = 0. \tag{C.2}$$

Let $K \subset U$ be compact. Since for all $j \in \mathbb{N}$ the set K_j is a subset of the interior of K_{j+1}, we see that $U = \bigcup_{j \in \mathbb{N}} \operatorname{int}(K_j)$ and that there exists a $j \in \mathbb{N}$ such that $K \subset K_j$. Hence the condition (C.1) is equivalent to condition (i) of the Theorem and the condition (C.2) is equivalent to condition (ii) of the Theorem.

□

The previous theorem gives a practical condition for the continuity of a linear map $f : \mathcal{S}(\mathbb{R}\times U) \to \mathbb{C}$. In quite a few textbooks on PDEs, e.g. [49], [40] [48] and [26], this test is adopted as a starting point and the theory of Frechet spaces and LF spaces is skipped. In the same way we can agree on the following definitions, which represent the "operative end" of this theory.

Definition C.11 *A sequence $(\varphi_n) \subset \mathcal{S}(\mathbb{R} \times U)$ is said to converge to $\varphi \in \mathcal{S}(\mathbb{R} \times U)$ if there exists a compact $K \subset U$ such that for all n we have* $\mathrm{supp}(\varphi_n) \subset \mathbb{R} \times K$ *and for all $k, i \in \mathbb{N}$.*

$$\lim_{n\to\infty} \sup_{|\alpha|\leq k} \sup_{\substack{t\in\mathbb{R} \\ x\in K_j}} |\langle t\rangle^i \partial^\alpha (\varphi_n - \varphi)(t,x)| = 0,$$

where $\alpha \in \mathbb{N}^4$ is a multi-index.

Definition C.12 *A linear map $f : \mathcal{S}(\mathbb{R} \times U) \to \mathbb{C}$ is continuous if for all sequences $(\varphi_n) \subset \mathcal{S}(\mathbb{R}\times U)$ that converge to 0, in the sense of the Definition C.11, we have*

$$\lim_{n\to\infty} f(\varphi_n) = 0.$$

By Theorem C.10 Definition C.12 is equivalent to the following: $f : \mathcal{S}(\mathbb{R}\times U)$ is continuous as a topological map when $\mathcal{S}(\mathbb{R} \times U)$ is equipped with the LF topology and $\mathbb{C}$ is equipped with the Euclidean topology.

Appendix D

Legend

Sets

$\mathbb{N}$	Numbers $\{0, 1, 2, \ldots\}$
$\mathbb{Z}$	Numbers $\{\ldots, -1, 01, \ldots\}$
$\mathbb{Z}^-$	Numbers $\{-1, -2, -3, \ldots\}$
$\mathbb{Z}^+$	Numbers $\{1, 2, 3, \ldots\}$
$\mathbb{R}$	The set of real numbers
$\mathbb{R}_+$	The set of positive real numbers
$\mathbb{R}_-$	The set of negative real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{R}^n$	The n-tuples of real numbers
$B(0, r)$	Ball of radius r in $\mathbb{R}^n$
D	Open and bounded set with a C^2 boundary in $\mathbb{R}^3$
U	A set in $\mathbb{R}^n$
∂U	The boundary of the set U
$\overline{U}$	closure of the set U
D_R	Equals $B(0, r) \setminus \overline{D}$

Measures

$\mathcal{H}^s$	s-dimensional Hausdorff measure on $\mathbb{R}^n$
$\mathcal{L}^n$	n-dimensional Lesque measure on $\mathbb{R}^n$

Spaces

$C^k(U)$	k times differentiable functions on U. (U open)
$C^\infty(U)$	Infinitely many times differentiable functions on U
$C_0^k(U)$	k times differentiable functions on U with compact support
$\mathcal{D}(U)$	Same as $C_0^\infty(U)$
$C_0^\infty(U)$	Infinitely many times differentiable functions on U with compact support
$\mathcal{D}(U)$	Test fuctions, same as $C_0^\infty(U)$
$\mathcal{S}(\mathbb{R}^n)$	Rapidly vanishing functions
$\mathcal{S}(\mathbb{R}\times U)$	Rapidly decreasing functions on the set $\mathbb{R}\times U$
$\mathcal{S}_{m0}(\mathbb{R}\times\mathbb{S}^2)$	Tempered functions on $\mathbb{R}\times\mathbb{S}^2$ with zero mean integral over time slices
$\mathcal{E}(U)$	Same as $C^\infty(\mathbb{R}^n)$
$\mathcal{D}'(U)$	Distributions, act on testfunctions $\mathcal{D}(U)$
$\mathcal{S}'(\mathbb{R}^n)$	Distributions, act on testfunctions $\mathcal{S}(\mathbb{R}^n)$
$\mathcal{S}'(\mathbb{R}\times U)$	Distributions on $\mathcal{S}(\mathbb{R}\times U)$
$\mathcal{E}'(U)$	Distributios, act on testfunctions $\mathcal{E}(U)$
$L^p(U)$	Measurable classes on U with finite $\|\|\cdot\|\|_{L^p(U)}$ norm
ℓ^p	Sequences $\alpha=(\alpha_j)_{j\in\mathbb{N}}$ s.t. $\|\|\alpha\|\|_{\ell^p}^p := \sum_{j\in\mathbb{N}} \|\alpha_j\|^p < \infty$
$W_p^k(U)$	L^p-based Sobolev space on U
$H^s(\mathbb{R}^n)$	Sobolev Space on $\mathbb{R}^n$
$H^s(U)$	Space of restrictions of $H^s(\mathbb{R}^n)$ on U
$H_0^s(U)$	Closure of $\mathcal{D}(U)$ on $H^s(U)$
$H^s(\partial D)$	Sobolev space on ∂D

Bracets

$(\cdot,\cdot)_{\mathcal{D}'\times\mathcal{D}(U)}$	Distribution action, complex conjugate on the first element
$(\cdot,\cdot)_{L^2(U)}$	L^2 innerproduct
$(\cdot,\cdot)_{H^{-s}\times H^s(A)}$	Complex Sobolev dual action between $H^{-s}(A)$ and $H^s(A)$

Functions, Distributions

Φ(x,y)	Fundamental Solution of the Helmholtz Equation
$u\otimes v$	Tensorproduct of u and v
$\hat{u}$	For a function or distribution u; $\mathcal{F}\{u\}$
$u * v$	Convolution of u and v
$u *_y v$	Convolution of u and v with respect to the parameter(s) y

Operators

∂	Partial derivative
∂_ν	Normal derivative
∂_ν^-	Interior normal derivative
∂_ν^+	Exterior normal derivative
Δ	Laplace operator
$\Box$	Wave operator
$\mathrm{Ran}\,(T)$	Range of operator T
$\mathrm{Dom}\,(T)$	Domain of operator T
$\mathcal{F}\{\cdot\}$	Fourier transformation on $\mathbb{R}^n$
$\mathcal{F}^{-1}\{\cdot\}$	Inverse Fourier transformation
tr	Trace operator
tr^-	Trace operator from the inside of a domain
tr^+	Trace operator from the outside of a domain
tr^*	Dual of the trace operator tr
$\mathcal{F}_{k\to t}$	Fourier transformation with respect to one variable in a multidimensional space
$\mathcal{F}^{-1}_{t\to k}$	The inverse of the previous
$\mathcal{G}_k$	Volume potential operator
SL_k	Single layer operator
S_k	Single layer operator on the boundary
DL_k	Double layer operator
Grad	Surface gradient
∇	Gradient $\nabla = (\partial_{x_1}, \partial_{x_2} \ldots, \partial_{x_n})$
∇_y	Gradient with respect to the variable y
$\text{s-}\lim_{\lambda\to\mu} T(\lambda)$	Strong limit of operators
$\{P(\lambda)\}$	Resolution of identity
U_k	Solution operator to the single frequency Dirichlet problem
$U_{rob,k}$	Solution operator to the single frequency Robin Problem
Λ_k	The Diriclet to Neumann map
$\Lambda_{k,-}$	Interior Diriclet to Neumann map
$\Lambda_{k,+}$	Exterior Dirichlet to Neumann map
T_k	Double layer operator on the boundary in a Sobolev space setting
R_k	Single layer operator on the boundary in a Sobolev space setting
$\mathrm{Im}\,(T)$	Imaginary part of the operator T
$\mathrm{Re}\,(T)$	Real part of the operator T
F_k	The far field operator
G_k	Robin boundary values to the far field -map
$G_{D,k}$	Dirichlet boundary values to the far field -map

L_k	Equals $-k\mathrm{Im}(F_k)$
L	Combination of L_k into an $L^2(\mathbb{R} \times \mathbb{S}^2)$ operator
F	Combination of F_k into an $L^2(\mathbb{R} \times \mathbb{S}^2)$ operator
$\widehat{F}$	Partial Fourier transformation of F
F_{time}	Time side far field operator

Miscallenous notations

$\nu(U)$	Exterior surface normal of ∂U
$\langle k \rangle$	Equals $(1+k^2)^{\frac{1}{2}}$
$\|\cdot\|_{\mathrm{op}}$	Functional analytic operator norm
J_φ	The Jacobian of φ. Defined for $\varphi : \mathbb{R}^n \to \mathbb{R}^m$
$d(x, U)$	Distance of x from the set U
$\hat{x}$	Unit vector to the same direction as $x \neq 0$
∂_{x_i}	Partial derivative w.r.t. the x_i variable
$\|\cdot\|_D$	Dirichlet norm
$\mathcal{O}(f(x))$	Asymptotic term bounded by $Cf(x)$
Im	Imaginary part
Re	Real part
$[A]$	Martix corresponding to the operator A

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