Metric compactification of Banach spaces

Armando W. Gutiérrez

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Abstract

This dissertation contains a detailed examination of the metric compactification of some classical Banach spaces. The results are reported in three different publications.

In the first publication, we present explicit representation formulas for all the elements of the metric compactification of finite-dimensional ℓ_p spaces for $p \in [1, \infty]$. Also, we investigate the metric compactification for the variation norm in $\mathbb{R}^n$ and the Hilbert projective metric on the standard cone $\mathbb{R}_{>0}^n$.

In the second publication, we give a complete characterization of the metric compactification of infinite-dimensional ℓ_p spaces for $p \in [1, \infty[$.

In the third publication, we use random measures to give explicit representation formulas for all the elements of the metric compactification of $L_p(\Omega, \mathbf{P})$ for $p \in [1, \infty[$, where $(\Omega, \mathbf{P})$ is a non-atomic standard probability space.

Keywords metric compactification, metric functional, horofunction, Busemann function, isometry, semicontraction, metric geometry, Banach space

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Preface

The research work presented in this dissertation was carried out in the Department of Mathematics and Systems Analysis at Aalto University. This work is part of a research project on *Metric Functional Analysis* proposed by Professor Anders Karlsson. The preliminary ideas for the project were sketched out in the summer of 2015 after attending a conference on Boundaries and Ergodic Geometry at University of Notre Dame.

I would like to express my sincere gratitude to Professor Anders Karlsson for his valuable advice, enthusiastic guidance and constant encouragement. The work meetings held in Uppsala and Geneva were essential to complement my independent research.

I am very thankful to my supervisor Professor Kalle Kytölä for all the helpful technical discussions and comments.

I will always be grateful to my mentors Professor Timo Eirola and Professor Olavi Nevanlinna for their friendly and generous support during all these years working at Aalto University.

Otaniemi, October 14, 2019,

Armando W. Gutiérrez

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List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I Armando W. Gutiérrez. The horofunction boundary of finite-dimensional ℓ_p spaces.

Colloquium Mathematicum,
volume 155, issue 1, pp. 51–65,
doi: 10.4064/cm7320-3-2018,
published online: 19 October 2018.

II Armando W. Gutiérrez. On the metric compactification of infinite-dimensional ℓ_p spaces.

Canadian Mathematical Bulletin,
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doi:10.4153/S0008439518000681,
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III Armando W. Gutiérrez. Characterizing the metric compactification of L_p spaces by random measures.

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Author's Contribution

Publication I: “The horofunction boundary of finite-dimensional ℓ_p spaces”

This is a single-author article by Armando W. Gutiérrez.

Publication II: “On the metric compactification of infinite-dimensional ℓ_p spaces”

This is a single-author article by Armando W. Gutiérrez.

Publication III: “Characterizing the metric compactification of L_p spaces by random measures”

This is a single-author article by Armando W. Gutiérrez.

1. Introduction

The foundations of the metric compactification first appeared in Gromov's work [19] on isometry groups of hyperbolic metric spaces. The term *metric compactification* was proposed by Rieffel in a work on group C^* -algebras [38]. Rieffel identified the metric compactification of a complete locally compact metric space with the maximal ideal space of a unital commutative C^* -algebra. In Gromov's original construction of the metric compactification, the metric space is required to be proper and geodesic [5]. In this setting, the method produces an ideal boundary at infinity of the underlying metric space.

The difference between Gromov's original method and the one investigated in this dissertation is that the latter can be applied to metric spaces that are not locally compact. This is a matter of great importance as we are often interested in describing the metric compactification of infinite-dimensional Banach spaces. The generalization of Gromov's original ideas for the construction of the metric compactification was already discussed in [15, 35]. It is now well established that every metric space is continuously injected into its metric compactification; a compact Hausdorff space. It becomes metrizable if the underlying metric space is separable. Moreover, isometries between metric spaces can be extended continuously to homeomorphisms between their corresponding metric compactifications.

Banach spaces form an important class of metric spaces. Influential studies of the metric compactification in this context can be found in [22, 39, 43]. In this dissertation, we present explicit formulas for all the elements of the metric compactification of classical Banach spaces. In Publication I, we give a complete characterization of the metric compactification of finite-dimensional ℓ_p spaces for all $p \in [1, \infty]$. Also, we present complete characterizations of the metric compactification for the variation norm in $\mathbb{R}^n$ and the Hilbert projective metric on the standard cone $\mathbb{R}_{>0}^n$. In Publica-

tion II, we give a complete characterization of the metric compactification of infinite-dimensional ℓ_p spaces for all $p \in [1, \infty[$. Finally, in Publication III, we give a complete characterization of the metric compactification of $L_p(\Omega, \Sigma, \mathbf{P})$ for all $p \in [1, \infty[$, where $(\Omega, \Sigma, \mathbf{P})$ is a non-atomic standard probability space.

The metric compactification is a fundamental concept used by Karlsson to lay the foundations of a metric spectral theory [25]. Several applications of the metric compactification in different contexts can be found in the literature. For example, in the study of isometry groups [29, 34, 40, 41, 42, 31], random walks on hyperbolic groups [6, 17], random product of semicontractions [27, 26, 18, 16], Denjoy-Wolff theorems [23, 24, 8, 1, 32], Teichmüller spaces [2], limit graphs [11], random triangulations [10], and first-passage percolation [3].

2. Preliminaries

This chapter introduces some fundamental concepts in Metric Geometry that appear throughout this dissertation. Suggested references for this chapter include Ballmann [4], Bridson–Haefliger [7] and Papadopoulos [37]. For additional material on General Topology, the standard references include Kelley [28] and Munkres [36].

2.1 Pseudo-metric spaces

Let X be a non-empty set. A *pseudo-metric* on X is a function $d : X \times X \rightarrow \mathbb{R}$ satisfying the following properties:

- (i) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.
- (iii) $d(x, x) = 0$ for all $x \in X$.

Property (i) is known as the *triangle inequality*. A pseudo-metric d on X necessarily satisfies the property $d(x, y) \geq 0$ for all $x, y \in X$. A *pseudo-metric space* is a pair (X, d) , where X is a non-empty set and d is a pseudo-metric on X . For $x, y \in X$ the non-negative number $d(x, y)$ is called the *distance* from x to y .

Example 2.1 (variation pseudo-metric). Let F be a non-empty finite set. Let $\mathbb{R}^F$ denote the set of all real-valued functions on F . The function $d_{\text{var}} : \mathbb{R}^F \times \mathbb{R}^F \rightarrow \mathbb{R}$ given by

$$d_{\text{var}}(x, y) = \max_{s \in F} \{x(s) - y(s)\} - \min_{s \in F} \{x(s) - y(s)\} \quad (2.1)$$

defines a pseudo-metric on $\mathbb{R}^F$; see [33].

Example 2.2 (Hilbert’s projective distance). Let F be a non-empty finite set. Let $\mathbb{R}_{>0}^F$ denote the set of all $x \in \mathbb{R}^F$ such that $x(s) > 0$ for all $s \in F$.

The function $d_H : \mathbb{R}_{>0}^F \times \mathbb{R}_{>0}^F \rightarrow \mathbb{R}$ given by

$$d_H(x, y) = \log \max_{s \in F} \{x(s)/y(s)\} - \log \min_{s \in F} \{x(s)/y(s)\} \quad (2.2)$$

defines a pseudo-metric on $\mathbb{R}_{>0}^F$; see [33].

A pseudo-metric d on X is called a *metric* if $d(x, y) > 0$ for all $x, y \in X$ with $x \neq y$. A *metric space* is a pair (X, d) , where X is a non-empty set and d is a metric on X .

Example 2.3. Let $p \in [1, \infty[$ and let S be a non-empty set. Let $\ell_p(S)$ denote the set of all $x \in \mathbb{R}^S$ such that $\sum_{s \in S} |x(s)|^p$ is finite. The set $\ell_p(S)$ becomes a metric space with the metric

$$d_p(x, y) = \left(\sum_{s \in S} |x(s) - y(s)|^p \right)^{1/p}. \quad (2.3)$$

If $p = 2$ and S is the finite set $\{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$, the function (2.3) is known as the *Euclidean metric* on $\mathbb{R}^n$. In particular, if $n = 1$ then (2.3) becomes the *usual metric* $d(x, y) = |x - y|$ on the set of real numbers. It is sometimes convenient to denote by $\mathbb{E}^n$ the *Euclidean space* $\ell_2(\{1, \dots, n\})$.

Let (X, d) be a pseudo-metric space. For $x \in X$ and $r > 0$, an *open ball* of radius r around x is the set $B_r(x) = \{z \in X \mid d(x, z) < r\}$, and a *closed ball* of radius r around x is the set $\mathbf{B}_r(x) = \{z \in X \mid d(x, z) \leq r\}$. The collection of all open balls generates a topology on (X, d) , the so-called *pseudo-metric topology* [28, p. 119]. So a subset O of X is *open* if for every $x \in O$ there is $r_x > 0$ such that the open ball $B_{r_x}(x)$ is contained in O . A subset C of X is *closed* if $X \setminus C$ is open.

A sequence $(x_n)_{n \in \mathbb{N}}$ in a pseudo-metric space (X, d) *converges* to $x \in X$ whenever $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$, i.e., for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$. A sequence in a metric space cannot converge to more than one point.

The concept of *closure* is defined in general topological spaces [28, p. 42]. If Z is a subset of a topological space X , then the closure of Z in X is denoted by $\text{cl}(Z)$ and can be characterized as follows: $x \in \text{cl}(Z)$ if and only if there is a *net* in Z converging to x [28, p. 66]. Nets are widely used in topology, and essentially generalize the notion of sequences; see [28, pp. 62–83] or [36, pp. 187–188]. For pseudo-metric spaces, the word "net" can be replaced by "sequence" in the above characterization; see [28, p. 72] or [36, p. 190]. The following proposition is another useful characterization of the closure of a set in a pseudo-metric space.

Proposition 2.4. [28, p. 120] Let (X, d) be a pseudo-metric space and let Z be a non-empty subset of X . Then

$$\text{cl}(Z) = \left\{ x \in X \mid \inf_{z \in Z} d(x, z) = 0 \right\}.$$

The closed ball $B_r(x)$ may be strictly larger than the closure of the open ball $B_r(x)$. For example, let X be a set with at least two elements and let d be the *discrete metric* defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

The open ball $B_1(x)$ of radius 1 around $x \in X$ is the set $\{x\}$. Next, we observe that $\text{cl}(B_1(x)) = \{x\}$, however, the closed ball $B_1(x)$ is the whole set X .

A set Z is said to be *dense* in a topological space X if $\text{cl}(Z) = X$. A topological space X is said to be *separable* if there is a countable subset which is dense in X . For example, the set $\mathbb{Q}$ of rational numbers is dense in the set $\mathbb{R}$ of real numbers with the usual metric.

A *Cauchy sequence* in a metric space (X, d) is a sequence $(x_n)_{n \in \mathbb{N}}$ such that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ satisfying $d(x_m, x_n) < \epsilon$ for all $m, n \geq N$. A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges.

Example 2.5. The set $\mathbb{R}$ of real numbers with the usual metric becomes a complete metric space [36, p. 264]. The open interval $] -1, 1[$ in $\mathbb{R}$ with the same metric is not complete, e.g., the Cauchy sequence $(1 - 1/n)_{n \in \mathbb{N}}$ does not converge.

Example 2.6. Let S be a non-empty set. Let $\ell_\infty(S)$ denote the set of all $\varphi \in \mathbb{R}^S$ such that $\sup_{s \in S} |\varphi(s)|$ is finite. This set becomes a complete metric space [36, p. 266] with respect to the *uniform metric*

$$d_\infty(\varphi, \psi) = \sup_{s \in S} |\varphi(s) - \psi(s)|. \quad (2.4)$$

An *isometry* between pseudo-metric spaces (X, d) and (Y, ρ) is a mapping f from X onto Y such that $\rho(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. If such a mapping exists, the pseudo-metric spaces (X, d) and (Y, ρ) are said to be *isometric*. The set of all isometries from a metric space (X, d) to itself forms a group; the *isometry group* $\text{Isom}(X, d)$.

If a metric space (X, d) is not complete, we can always find a complete

metric space $(\widehat{X}, \widehat{d})$ and an isometry $f : X \rightarrow f(X) \subset \widehat{X}$ such that $f(X)$ is a dense subset of $\widehat{X}$. The set $\widehat{X}$ is known as the *completion* of X .

Proposition 2.7 (Fréchet [13], Kuratowski [30]). *Every metric space is isometric to a dense subset of a complete metric space.*

A tedious manipulation of Cauchy sequences in combination with a quotient space provides a typical proof of Proposition 2.7 that is taught in a basic course on metric spaces. The original ideas of Fréchet and Kuratowski to prove Proposition 2.7 reveal geometric aspects of the metric and are combined as follows. Fix $x_0 \in X$ and let $f : (X, d) \rightarrow (\ell_\infty(X), d_\infty)$ be the mapping $x \mapsto f(x)$ defined by

$$[f(x)](y) = d(x, y) - d(x_0, y) \text{ for all } y \in X. \quad (2.5)$$

By the triangle inequality, it follows that $|[f(x)](y)| \leq d(x, x_0)$ for all $y \in X$. Hence $f(x) \in \ell_\infty(X)$ for each $x \in X$. Furthermore, for $x, x' \in X$,

$$|[f(x)](y) - [f(x')](y)| \leq d(x, x') \text{ for all } y \in X,$$

and so $d_\infty(f(x), f(x')) \leq d(x, x')$. On the other hand, since

$$|[f(x)](x') - [f(x')](x')| = d(x, x'),$$

it follows that $d_\infty(f(x), f(x')) = d(x, x')$. Now, let $\widehat{X}$ be the closure of $f(X)$ in $\ell_\infty(X)$ with respect to the uniform metric $\widehat{d} = d_\infty$. Hence $(\widehat{X}, \widehat{d})$ becomes a complete metric space.

Compactness is also a notion defined in general topological spaces [28, p. 135]. The following proposition characterizes compactness in terms of nets.

Proposition 2.8. [28, p. 136] *A topological space X is compact if and only if each net in X has a subnet which converges to some point of X .*

Subnet is a generalized notion of *subsequence* [28, pp. 69–70]. Recall that $(x_{n_k})_{k \in \mathbb{N}}$ is a subsequence of $(x_n)_{n \in \mathbb{N}}$ if $(n_k)_{k \in \mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$. The following proposition characterizes compactness of a pseudo-metric space.

Proposition 2.9. [28, p. 138] *A pseudo-metric space X is compact if and only if each sequence in X has a subsequence which converges to some point of X .*

The completion of a metric space (X, d) is obtained via the mapping $x \mapsto f(x)$ given by (2.5) which depends explicitly on the metric d . One could naturally ask whether there is an analogous mapping that yields a compact space containing the metric space as a dense subset. A positive answer to this question is given in Chapter 3. To achieve this we need to re-interpret (2.5) and define a new mapping f whose image sits inside a different topological space.

2.2 Proper and geodesic spaces

Let (X, d) be a pseudo-metric space. A subset B of X is said to be *bounded* if $\sup\{d(x, y) \mid x, y \in B\}$ is finite. A *bounded sequence* $(x_n)_{n \in \mathbb{N}}$ in X is a sequence for which there is a bounded set B of X with $x_n \in B$ for all $n \in \mathbb{N}$.

A pseudo-metric space (X, d) is said to be *proper* if for every $x \in X$ and every $r > 0$ the closed ball $B_r(x)$ is compact. Another classical terminology for proper is *finitely compact*. A proper metric space satisfies an analogue of the Bolzano–Weierstrass property for sequences in the Euclidean space $\mathbb{E}^n$, i.e., each bounded sequence has a convergent subsequence.

Example 2.10. Let F be a non-empty finite set. The set $\mathbb{R}^F$ endowed with the pseudo-metric d_{var} given by (2.1) is proper. Also, the set $\mathbb{R}_{>0}^F$ with the pseudo-metric d_{H} given by (2.2) is proper.

Example 2.11. Let $p \in [1, \infty[$. The set $\ell_p(S)$ with the metric d_p given by (2.3) is proper if and only if S is a finite set. Similarly, the set $\ell_\infty(S)$ with the metric d_∞ given by (2.4) is proper if and only if S is a finite set.

Example 2.12. Let X be the subset of $\mathbb{E}^2$ containing $(0, 0)$ and all the points $(a, b) \in \mathbb{E}^2$ with $a > 0$. The metric space (X, d_2) is not proper.

Let (X, d) be a pseudo-metric space. A *path* in X is a continuous mapping $\gamma : I \rightarrow X$, where I is an interval of real numbers. For $x, y \in X$ with $d(x, y) > 0$, a *geodesic path* joining x to y is a mapping $\gamma : [0, d(x, y)] \rightarrow X$ satisfying the following properties:

- (i) $\gamma(0) = x$ and $\gamma(d(x, y)) = y$.
- (ii) $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [0, d(x, y)]$.

The image $\gamma([0, d(x, y)])$ is called a *geodesic segment* with endpoints x and y . A *geodesic ray* in X issuing from $x \in X$ is a mapping $\gamma : [0, \infty[\rightarrow X$ such that for each $t > 0$ the restriction $\gamma|_{[0, t]}$ is a geodesic path joining x

to $\gamma(t)$. A *geodesic line* is a mapping $\gamma : \mathbb{R} \rightarrow X$ such that for every $t_0 \in \mathbb{R}$ the mapping $\tilde{\gamma} : [0, \infty[\rightarrow X$ defined by $\tilde{\gamma}(\cdot) = \gamma(t_0 + \cdot)$ is a geodesic ray. A *straight line* in X is the image of a geodesic line.

A pseudo-metric space (X, d) is called *geodesic* if for every $x, y \in X$ with $d(x, y) > 0$ there exists a geodesic path joining x to y . We say that (X, d) is *uniquely geodesic* if for every $x, y \in X$ with $d(x, y) > 0$ there exists exactly one geodesic path joining x to y .

Example 2.13. Let X be the set $\mathbb{E}^n \setminus \{p\}$ with $p \in \mathbb{E}^n$. The metric space (X, d_2) is not geodesic. Recall that a subset K of a real linear space V is said to be *convex* if $(1 - \theta)x + \theta y \in K$ for all $x, y \in K$ and all $\theta \in [0, 1]$. The Euclidean space $\mathbb{E}^n$ has linear structure. A subset X of $\mathbb{E}^n$ equipped with the Euclidean metric d_2 is geodesic if and only if X is convex [7, p. 5].

Proposition 2.14. [7, p. 54] *The set $\ell_p(S)$ with the metric d_p given by (2.3) is uniquely geodesic for all $p \in]1, \infty[$.*

Example 2.15. Let S be the finite set $\{1, 2\}$. The points $(x(1), x(2)) = (-1, -1)$ and $(y(1), y(2)) = (1, 1)$ in the metric space $(\ell_1(S), d_1)$ can be joined by more than one geodesic path, e.g., $\gamma, \tilde{\gamma} : [0, 4] \rightarrow \ell_1(S)$

$$\gamma(t) = \begin{cases} (t-1, -1) & \text{if } t \in [0, 2], \\ (1, t-3) & \text{if } t \in [2, 4]. \end{cases} \quad \tilde{\gamma}(t) = \begin{cases} (-1, t-1) & \text{if } t \in [0, 1], \\ (t-2, 0) & \text{if } t \in [1, 3], \\ (1, t-3) & \text{if } t \in [3, 4]. \end{cases}$$

Proposition 2.16. [4, p. 12] *If for every pair of points x, y in a complete pseudo-metric space (X, d) there exists a point $m \in X$ such that*

$$d(x, m) = d(y, m) = \frac{d(x, y)}{2},$$

then (X, d) is geodesic.

Example 2.17 (The Hilbert metric). Let X be a non-empty, open, bounded, convex subset of the Euclidean space $\mathbb{E}^n$. For two distinct points $x, y \in X$, let x_∞ and y_∞ be the points in $\text{cl}(X) \setminus X$ such that x_∞, x, y, y_∞ are collinear and arranged in this order along the Euclidean straight line in which they lie. The cross-ratio of the four points x_∞, x, y, y_∞ is defined by

$$[x_\infty, x, y, y_\infty] = \frac{t}{s} \cdot \frac{1-s}{1-t},$$

where $x = (1-s)x_\infty + sy_\infty$ and $y = (1-t)x_\infty + ty_\infty$ with $0 < s < t < 1$. In

a letter to Klein, Hilbert [20] defined the distance from x to y on X to be

$$\rho_X(x, y) = \begin{cases} \log[x_\infty, x, y, y_\infty] & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

The metric space (X, ρ_X) is proper and geodesic. It is often referred to as the *Hilbert geometry* on X . In the particular case where X is the Euclidean open ball $B_1(0)$, the Hilbert geometry on X becomes Klein's model of hyperbolic geometry. For more details see [37, pp. 158–167] or [9, pp. 105–106].

2.3 Banach spaces

Let X be a real linear space, i.e., a vector space over the field $\mathbb{R}$ of real numbers. A *pseudo-norm* on X is a mapping $\|\cdot\| : X \rightarrow \mathbb{R}$ satisfying the following properties:

- (i) $d_{\|\cdot\|}(x, y) := \|x - y\|$ is a pseudo-metric on X .
- (ii) $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in X$ and all $\lambda \in \mathbb{R}$.

The triangle inequality for the pseudo-metric $d_{\|\cdot\|}$ induced by the pseudo-norm $\|\cdot\|$ is equivalent to the statement that

$$\|x + y\| \leq \|x\| + \|y\| \text{ for all } x, y \in X.$$

A *norm* on X is a pseudo-norm $\|\cdot\|$ that vanishes only at the zero vector $o \in X$. A real *Banach space* is a pair $(X, \|\cdot\|)$, where X is a real linear space and $\|\cdot\|$ is a norm on X such that $(X, d_{\|\cdot\|})$ is a complete metric space.

Example 2.18. Let S be any non-empty set. The set $\mathbb{R}^S$ of all mappings $x : S \rightarrow \mathbb{R}$ is a real linear space with the usual operations of addition and scalar multiplication:

$$(x + y)(s) = x(s) + y(s), \quad (\lambda x)(s) = \lambda x(s).$$

The zero vector $o \in \mathbb{R}^S$ is defined by $o(s) = 0$ for all $s \in S$. For $p \in [1, \infty[$ the real linear space $\ell_p(S) \subset \mathbb{R}^S$ becomes a Banach space with the norm

$$\|x\|_{\ell_p} := d_p(o, x) = \left(\sum_{s \in S} |x(s)|^p \right)^{1/p}.$$

Also, the real linear space $\ell_\infty(S) \subset \mathbb{R}^S$ is a Banach space with the norm

$$\|x\|_{\ell_\infty} := d_\infty(o, x) = \sup_{s \in S} |x(s)|.$$

If S is a non-empty finite set, the pseudo-metric d_{var} given by (2.1) induces a pseudo-norm on the real linear space $\mathbb{R}^S$ defined by

$$\|x\|_{\text{var}} := d_{\text{var}}(o, x) = \max_{s \in F} x(s) - \min_{s \in S} x(s).$$

If $\|\cdot\|$ is a pseudo-norm on a real linear space X , then the pseudo-metric space $(X, d_{\|\cdot\|})$ is geodesic. Indeed, for every $x, y \in X$ with $\|x - y\| > 0$ the mapping $\gamma_{x,y} : [0, \|x - y\|] \rightarrow X$ defined by

$$\gamma_{x,y}(t) = \left(1 - \frac{t}{\|x - y\|}\right)x + \frac{t}{\|x - y\|}y$$

is a geodesic path in X joining x to y .

A norm $\|\cdot\|$ on a real linear space X is said to be *strictly convex* if for every $x, y \in X$ with $\|x - y\| > 0$ and $\|x\| = \|y\| = 1$, the strict inequality $\|\gamma_{x,y}(t)\| < 1$ holds for all $0 < t < \|x - y\|$. We denote the image $\gamma_{x,y}([0, \|x - y\|])$ by $\llbracket x, y \rrbracket$.

Proposition 2.19. *Let $\|\cdot\|$ be a norm on a real linear space X . The metric space $(X, d_{\|\cdot\|})$ is uniquely geodesic if and only if the norm $\|\cdot\|$ is strictly convex.*

The proof of Proposition 2.19 follows from the observation that $\llbracket x, y \rrbracket$ is a geodesic segment for any pair of points $x, y \in X$. Hence X is uniquely geodesic if and only if for every $x, y, w \in X$,

$$d_{\|\cdot\|}(x, w) + d_{\|\cdot\|}(w, y) = d_{\|\cdot\|}(x, y) \implies w \in \llbracket x, y \rrbracket.$$

Let $a = x - w$ and $b = w - y$. Then X is uniquely geodesic if and only if $\|a + b\| < \|a\| + \|b\|$ whenever a and b are linearly independent. Now, by letting $u = \|a\|^{-1}a$ and $v = \|b\|^{-1}b$, it follows that the real number

$$t = \frac{\|b\|}{\|a\| + \|b\|} \|u - v\|$$

is such that

$$\begin{aligned} a + b &= (\|a\| + \|b\|) \left(\frac{\|a\|}{\|a\| + \|b\|} u + \frac{\|b\|}{\|a\| + \|b\|} v \right) \\ &= (\|a\| + \|b\|) \left(\left(1 - \frac{t}{\|u - v\|}\right) u + \frac{t}{\|u - v\|} v \right). \end{aligned}$$

Therefore $\|a + b\| < \|a\| + \|b\|$ if and only if $\|\gamma_{u,v}(t)\| < 1$.

Example 2.20. Let S be a set with at least two elements. The norm $\|\cdot\|_{\ell_1}$ on $\ell_1(S)$ is not strictly convex. For $s_1, s_2 \in S$ with $s_1 \neq s_2$ define $x, y \in \ell_1(S)$ by the formulas

$$x(s) = \begin{cases} 1 & \text{if } s = s_1, \\ 0 & \text{otherwise.} \end{cases} \quad y(s) = \begin{cases} 1 & \text{if } s = s_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|x\|_{\ell_1} = \|y\|_{\ell_1} = 1$ and $\|x - y\|_{\ell_1} = 2$. However, for every $0 < t < 2$,

$$\|\gamma_{x,y}(t)\|_{\ell_1} = \left\| \left(1 - \frac{t}{2}\right)x + \frac{t}{2}y \right\|_{\ell_1} = 1.$$

Let X be a real linear space. An *inner product* on X is a mapping $(\cdot | \cdot) : X \times X \rightarrow \mathbb{R}$ satisfying the following properties:

- (i) $(x | y) = (y | x)$ for all $x, y \in X$.
- (ii) $(x + y | z) = (x | z) + (y | z)$ for all $x, y, z \in X$.
- (iii) $(\lambda x | y) = \lambda(x | y)$ for all $x, y \in X$ and all $\lambda \in \mathbb{R}$.
- (iv) $(x | x) > 0$ for all $x \in X \setminus \{o\}$.

Every inner product $(\cdot | \cdot)$ on a real linear space X induces a norm on X given by the formula

$$\|x\|_i = (x | x)^{1/2} \quad \text{for all } x \in X,$$

and hence X becomes a metric space with the metric

$$d_i(x, y) = (x - y | x - y)^{1/2} \quad \text{for all } x, y \in X.$$

The pair $(X, (\cdot | \cdot))$ is called a real *Hilbert space* if the associated metric space (X, d_i) is complete.

Example 2.21. Let S be a non-empty set. The real linear space $\ell_2(S)$ endowed with the inner product

$$(x | y) = \sum_{s \in S} x(s)y(s)$$

is a real Hilbert space.

If $(\cdot | \cdot)$ is an inner product on a real linear space X , then the identity

$$\|x - y\|_i^2 + \|x + y\|_i^2 = 2\|x\|_i^2 + 2\|y\|_i^2 \quad (2.6)$$

holds for all $x, y \in X$. Geometric properties of the identity (2.6) with arbitrary norms were analyzed by Jordan and von Neumann [21], and by Fréchet [14] in 1935. The results of these works are summarized in the following proposition.

Proposition 2.22 (Fréchet-Jordan-von Neumann). *Let $\|\cdot\|$ be a norm on a real linear space X . If the identity*

$$\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

holds for all $x, y \in X$, then there exists an inner product $(\cdot | \cdot)$ on X such that $\|x\| = (x | x)^{1/2}$ for all $x \in X$. More precisely, the inner product $(\cdot | \cdot)$ is defined by the formula

$$(x | y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2).$$

Proposition 2.23. [7] *If $(\cdot | \cdot)$ is an inner product on a real linear space X , then the associated metric space (X, d_i) is uniquely geodesic.*

2.4 Length spaces

Let a, b be two real numbers with $a < b$. A *subdivision* of the interval $[a, b]$ is a finite subset $\sigma = \{t_0, t_1, \dots, t_k\}$ of $[a, b]$ with the property that $a = t_0 < t_1 < \dots < t_k = b$. Let (X, d) be a pseudo-metric space. The *length* of a path $\gamma : [a, b] \rightarrow X$ is defined to be

$$L(\gamma) = \sup_{\sigma} \sum_{j=0}^{k-1} d(\gamma(t_j), \gamma(t_{j+1})),$$

where the supremum is taken over the set of all subdivisions $\sigma = \{t_0, \dots, t_k\}$ of $[a, b]$. It is clear that $d(\gamma(a), \gamma(b)) \leq L(\gamma)$ and $0 \leq L(\gamma) \leq \infty$. A path is said to be *rectifiable* if its length is finite.

Example 2.24. The path $\gamma : [0, 1] \rightarrow \mathbb{R}$ given by

$$\gamma(t) = \begin{cases} t \sin(1/t) & \text{if } t > 0, \\ 0 & \text{if } t = 0, \end{cases}$$

is non-rectifiable; see [37, p. 18].

A pseudo-metric space (X, d) is said to be a *length space* if the identity

$$d(x, y) = \inf_{\gamma} L(\gamma)$$

holds for all $x, y \in X$, where the infimum is taken over the set of all rectifiable paths γ joining x to y .

Example 2.25. Euclidean spaces $\mathbb{E}^n$ with $n \geq 1$ and Banach spaces are examples of length spaces. However, the set $\mathbb{E}^n \setminus B_1(o)$ with the Euclidean metric is not a length space; see [37, p. 42].

Example 2.26. Let (G, e) be a group and let $S \subset G \setminus \{e\}$ be a set of generators of G . The *word length* $\mathcal{L}_S(g)$ of an element $g \in G \setminus \{e\}$ with respect to S is defined to be

$$\mathcal{L}_S(g) = \inf \left\{ r \mid g = s_1^{\varepsilon_1} \cdots s_r^{\varepsilon_r}, s_i \in S, \varepsilon_i \in \{+1, -1\} \text{ for } 1 \leq i \leq r \right\}.$$

The *word metric* on G associated to S is the mapping $d_S : G \times G \rightarrow \mathbb{R}$ defined by

$$d_S(g, h) = \begin{cases} \mathcal{L}_S(g^{-1}h) & \text{if } g \neq h, \\ 0 & \text{if } g = h. \end{cases}$$

The *Cayley graph* $C(G, S)$ of G with respect to S is defined to be a graph with the following properties:

1. The vertices of $C(G, S)$ are the elements of G .
2. Two vertices g_1, g_2 are connected by an edge if and only if there is a generator $s \in S$ such that $g_2 = g_1 s$.

The Cayley graph $C(G, S)$ has a natural metric ρ_S which makes it into a length space where each edge is isometric to the unit interval $[0, 1]$. The word metric d_S on the group G is the restriction of ρ_S to the set of vertices of the Cayley graph $C(G, S)$; see [37, p. 43].

Every geodesic space is a length space. However, a length space may not be a geodesic space. For example the set $X = \mathbb{E}^2 \setminus \{o\}$ endowed with the Euclidean metric becomes a length space, but there is no geodesic path joining $x \in X$ to $-x$. It is important to observe that X is not complete.

A metric space (X, d) is said to be *locally compact* if each point has at least one compact neighborhood (with respect to the topology generated by

the metric). Every proper metric space is locally compact. The converse is not always true; the real line $\mathbb{R}$ endowed with the metric

$$d(x, y) = \min\{1, |x - y|\}$$

becomes a locally compact metric space, but it is not proper.

The following result gives certain conditions under which a metric space becomes proper and geodesic. It is due to Cohn-Vossen and generalizes the so-called Hopf-Rinow theorem of Differential Geometry.

Theorem 2.27. *[4, p. 14][7, p. 35] Every complete locally compact length space is a proper geodesic metric space.*

3. The metric compactification

In this chapter we introduce the notion of metric compactification in the setting of arbitrary metric spaces. We include a review of the main facts and important observations about the metric compactification. Suggested references for this chapter include Gromov [19], Rieffel [38], Gaubert–Vigeral [15], Maher–Tiozzo [35], and Karlsson [25].

3.1 Metric functionals

Let (X, d) be a metric space. Consider the topological space $(\mathbb{R}^X, \tau_{pw})$ where $\mathbb{R}^X$ denotes the set of all mappings from X to $\mathbb{R}$ and τ_{pw} denotes the topology of pointwise convergence [28, p. 92, p. 217]. For an arbitrary point $x_0 \in X$ let $\text{Lip}_{x_0}^1(X)$ denote the subset of $\mathbb{R}^X$ containing all mappings $h : X \rightarrow \mathbb{R}$ with the following properties:

- (i) $h(x_0) = 0$.
- (ii) $|h(x) - h(y)| \leq d(x, y)$ for all $x, y \in X$. (1-Lipschitz)

Proposition 3.1. *$\text{Lip}_{x_0}^1(X)$ is a closed subset of $\mathbb{R}^X$ relative to the topology τ_{pw} of pointwise convergence.*

Proof. Let $(h_\alpha)_\alpha$ be a net in $\text{Lip}_{x_0}^1(X)$ converging to h . We want to show that $h \in \text{Lip}_{x_0}^1(X)$. For each $x \in X$ the net $(h_\alpha(x))_\alpha$ of real numbers converges to $h(x)$. In particular, $h(x_0) = 0$. The fact that h satisfies property (ii) follows immediately from the inequality

$$|h(x) - h(y)| \leq |h(x) - h_\alpha(x)| + d(x, y) + |h_\alpha(y) - h(y)|$$

for all $x, y \in X$ and all α . □

Proposition 3.2. *$\text{Lip}_{x_0}^1(X)$ is a compact subset of $\mathbb{R}^X$ relative to the topology τ_{pw} of pointwise convergence.*

Proof. If $\mathbf{h} \in \text{Lip}_{x_0}^1(X)$ then $|\mathbf{h}(x)| = |\mathbf{h}(x_0) - \mathbf{h}(x)| \leq d(x_0, x)$ for all $x \in X$, and hence

$$\text{Lip}_{x_0}^1(X) \subset \prod_{x \in X} [-d(x_0, x), d(x_0, x)]. \quad (3.1)$$

Due to Tychonoff's theorem [28, p. 143], the Cartesian product in (3.1) is compact relative to the topology τ_{pw} of pointwise convergence. Therefore, compactness of $\text{Lip}_{x_0}^1(X)$ follows readily from Proposition 3.1. $\square$

Consider the mapping $y \mapsto \mathbf{h}_y$ from the metric space (X, d) to the topological space $(\mathbb{R}^X, \tau_{pw})$ defined by

$$\mathbf{h}_y(x) := d(x, y) - d(x_0, y) \quad (3.2)$$

for all $x \in X$. We observe that for each $y \in X$ the element $\mathbf{h}_y \in \mathbb{R}^X$ satisfies $\mathbf{h}_y(x_0) = 0$ and $|\mathbf{h}_y(z) - \mathbf{h}_y(x)| \leq d(z, x)$ for all $x, z \in X$. Hence, the set $\{\mathbf{h}_y \mid y \in X\}$ is contained in the compact space $\text{Lip}_{x_0}^1(X)$.

Proposition 3.3. *The mapping $y \mapsto \mathbf{h}_y$ from (X, d) to $(\mathbb{R}^X, \tau_{pw})$ given by (3.2) is continuous and injective.*

Proof. For injectivity we assume that $\mathbf{h}_y = \mathbf{h}_z$ for $y, z \in X$. Hence we observe that

$$0 = \mathbf{h}_z(y) - \mathbf{h}_y(y) = d(y, z) - d(x_0, z) + d(x_0, y), \quad (3.3)$$

$$0 = \mathbf{h}_y(z) - \mathbf{h}_z(z) = d(z, y) - d(x_0, y) + d(x_0, z). \quad (3.4)$$

By adding the identities (3.3) and (3.4), we obtain $d(y, z) = 0$, and so $y = z$. For continuity it suffices to observe that for every $y, z \in X$,

$$\begin{aligned} |\mathbf{h}_y(\cdot) - \mathbf{h}_z(\cdot)| &= |d(\cdot, y) - d(x_0, y) - d(\cdot, z) + d(x_0, z)| \\ &\leq |d(\cdot, y) - d(\cdot, z)| + |-d(x_0, y) + d(x_0, z)| \\ &\leq 2d(y, z). \end{aligned}$$

$\square$

Definition 3.4. Let (X, d) be a metric space and let x_0 be an arbitrary point in X . The *metric compactification* of X is denoted by $\overline{X}^h$ and defined to be the closure of the set $\{\mathbf{h}_y \mid y \in X\}$ in the topology τ_{pw} of pointwise convergence. Each element $\mathbf{h} \in \overline{X}^h$ is called a *metric functional* on X . Metric functionals of the form (3.2) are called *internal*.

For every metric functional $\mathbf{h} \in \overline{X}^h$ there exists a net of points $(y_\alpha)_\alpha$ in

X such that the net of internal metric functionals $(h_{y_\alpha})_\alpha$ converges to h pointwise on X .

The choice of x_0 in Definition 3.4 is irrelevant; the metric compactification is unique up to a homeomorphism. Assume that the net of internal metric functionals $(h_{y_\alpha})_\alpha$ given by (3.2) converges to $h \in \overline{X}^h$. If z_0 is a point in X such that $z_0 \neq x_0$, then we observe that

$$d(\cdot, y_\alpha) - d(z_0, y_\alpha) = (d(\cdot, y_\alpha) - d(x_0, y_\alpha)) - (d(z_0, y_\alpha) - d(x_0, y_\alpha)).$$

Hence $\lim_\alpha (d(\cdot, y_\alpha) - d(z_0, y_\alpha)) = h(\cdot) - h(z_0)$, i.e., the metric functional h and the one obtained with respect to the reference point z_0 differ only by a constant.

Some authors endow $\text{Lip}_{x_0}^1(X)$ with the topology τ_{uc} of uniform convergence on compact subsets of X , or equivalently with the compact-open topology [28, p. 230]. Recall that a set E of mappings from a metric space (X, d) to another (Y, D) is said to be *equicontinuous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every element $f \in E$, if $d(x, y) < \delta$ then $D(f(x), f(y)) < \epsilon$. Since $\text{Lip}_{x_0}^1(X)$ is equicontinuous, the topology τ_{uc} for $\text{Lip}_{x_0}^1(X)$ coincides with the topology τ_{pw} of pointwise convergence [28, p. 232].

If the metric space X is separable, its metric compactification $\overline{X}^h$ becomes a second-countable compact Hausdorff space [35]. Hence due to Urysohn's metrization theorem [28, p. 125], it follows that $\overline{X}^h$ is metrizable. Therefore, sequences are sufficient to describe metric functionals on separable metric spaces.

We emphasize that the mapping $y \mapsto h_y$ from an arbitrary metric space (X, d) to the topological space $(\mathbb{R}^X, \tau_{pw})$ given by (3.2) may fail to be a homeomorphism onto its image. For example, let $X = \ell_1(\mathbb{N})$ endowed with the ℓ_1 -metric (2.3) and let $x_0 = o = (0, 0, 0, \dots) \in X$. Consider the sequence $(y_n)_{n \in \mathbb{N}}$ in X given by $y_n = (0, \dots, 0, n, 0, \dots)$, with n in the n -th coordinate. Then for every $x \in \ell_1(\mathbb{N})$,

$$\lim_{n \rightarrow \infty} h_{y_n}(x) = \lim_{n \rightarrow \infty} (d_1(x, y_n) - d_1(o, y_n)) = \sum_{s \in \mathbb{N}} |x(s)| = h_o(x).$$

However, $d_1(o, y_n) \rightarrow \infty$ as $n \rightarrow \infty$.

Proposition 3.5. *Let (X, d) be a proper and geodesic metric space. Then the mapping $y \mapsto h_y$ from (X, d) to the topological space $(\mathbb{R}^X, \tau_{pw})$ given by (3.2) is a homeomorphism onto its image.*

Proof. We need to show that the map $\mathbf{h}_y \mapsto y$ from $(\{\mathbf{h}_y \mid y \in X\}, \tau_{pw})$ to (X, d) is continuous. Since X is proper, it will be sufficient to use sequences rather than nets. Assume that $\mathbf{h}_{y_n}$ converges to $\mathbf{h}_z$ pointwise on X . We will show that $d(y_n, z) \rightarrow 0$ as $n \rightarrow \infty$.

First suppose that $(y_n)_{n \in \mathbb{N}}$ is a bounded sequence in X and let $(y_{n_k})_{k \in \mathbb{N}}$ be any subsequence of $(y_n)_{n \in \mathbb{N}}$. Since X is proper, we may assume that y_{n_k} converges to some $\tilde{z} \in X$ as $k \rightarrow \infty$. Hence $\mathbf{h}_{y_{n_k}}$ converges to $\mathbf{h}_{\tilde{z}}$ in $\overline{X}^h$, and so $\tilde{z} = z$. This shows that the sequence $(y_n)_{n \in \mathbb{N}}$ converges to z .

Now suppose that $(y_n)_{n \in \mathbb{N}}$ is unbounded. By taking a subsequence if necessary, we may assume that $d(x_0, y_n) \geq 2d(x_0, z) + 1$ for all $n \in \mathbb{N}$. Since X is geodesic, for each $n \in \mathbb{N}$ there is a geodesic path γ_n joining z to y_n . Let $[[z, y_n]]$ denote the geodesic segment $\gamma_n([0, d(z, y_n)])$. Since X is proper, the subset $S = \{x \in X \mid d(x, z) = 1 + d(x_0, z)\}$ is compact. By continuity of the path γ_n and the distance function $d(\cdot, z)$, there exists x_n in the intersection $[[z, y_n]] \cap S$. Then

$$\begin{aligned} \sup_{x \in S} |\mathbf{h}_z(x) - \mathbf{h}_{y_n}(x)| &\geq \mathbf{h}_z(x_n) - \mathbf{h}_{y_n}(x_n) \\ &= d(x_n, z) - d(x_0, z) - d(x_n, y_n) + d(x_0, y_n) \\ &= 1 - d(x_n, y_n) + d(x_0, y_n) \\ &\geq 1 - d(x_n, y_n) + d(y_n, z) - d(z, x_0) \\ &= 1 + d(z, x_n) - d(z, x_0) = 2. \end{aligned}$$

Since the topologies τ_{pw} and τ_{uc} coincide in $\overline{X}^h$, and $\mathbf{h}_{y_n}$ converges to $\mathbf{h}_z$ pointwise on X , we obtain the following contradiction

$$2 \leq \sup_{x \in S} |\mathbf{h}_z(x) - \mathbf{h}_{y_n}(x)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence, the sequence $(y_n)_{n \in \mathbb{N}}$ must be bounded in X , and so $d(y_n, z) \rightarrow 0$ as $n \rightarrow \infty$. $\square$

Example 3.6. The real line $\mathbb{R}$ with the metric induced by the absolute value is proper and geodesic. Its metric compactification $\overline{\mathbb{R}}^h$ contains the internal metric functionals

$$s \mapsto \mathbf{h}_r(s) = |s - r| - |r| \text{ with } r \in \mathbb{R},$$

and the two additional "points at infinity" given by

$$s \mapsto \mathbf{h}_{+\infty}(s) = -s, \quad s \mapsto \mathbf{h}_{-\infty}(s) = s.$$

3.2 Busemann functions

Let (X, d) be a metric space. Assume that there exists a geodesic ray $\gamma : [0, \infty[\rightarrow X$ issuing from $\gamma(0) = x_0$, i.e.,

$$d(\gamma(s), \gamma(t)) = |s - t| \text{ for all } s, t \geq 0.$$

Since $\gamma(t) \in X$ for all $t \geq 0$, the internal metric functional (3.2) becomes

$$\mathbf{h}_{\gamma(t)}(x) = d(x, \gamma(t)) - t \text{ for all } x \in X.$$

We observe that for every $t \geq 0$,

$$\mathbf{h}_{\gamma(t)}(\cdot) \geq d(x_0, \gamma(t)) - d(\cdot, x_0) - t = -d(\cdot, x_0).$$

Also, $\mathbf{h}_{\gamma(t)} \leq \mathbf{h}_{\gamma(s)}$ whenever $s \leq t$. Hence for every $x \in X$ the limit

$$\mathbf{h}_{\gamma(\infty)}(x) := \lim_{t \rightarrow \infty} \mathbf{h}_{\gamma(t)}(x)$$

exists. The mapping $x \mapsto \mathbf{h}_{\gamma(\infty)}(x)$ is said to be a *Busemann function* associated to the geodesic ray γ . A Busemann function is a special type of metric functional on X .

Example 3.7. Suppose that X is a real linear space endowed with a norm $\|\cdot\|$. Let $v \in X$ such that $\|v\| = 1$. The mapping $\gamma : [0, \infty[\rightarrow X$ given by $\gamma(t) = tv$ is a geodesic ray issuing from $\gamma(0) = o \in X$. Hence the Busemann function associated to γ is

$$\mathbf{h}_{\gamma(\infty)}(x) = \lim_{t \rightarrow \infty} (\|x - tv\| - t) \text{ for all } x \in X.$$

In particular, if X is a real Hilbert space with norm $\|x\| = \sqrt{(x|x)}$, then

$$\mathbf{h}_{\gamma(\infty)}(x) = -(x|v) \text{ for all } x \in X.$$

Example 3.8. Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$. The Poincaré metric in $\mathbb{D}$ is given by the infinitesimal form

$$ds = 2(1 - |z|^2)^{-1} |dz|.$$

The distance between any pair of points $z, w \in \mathbb{D}$ with respect to the

Poincaré metric is given by

$$d_{\mathbb{D}}(z, w) = \log \left(\frac{|1 - z\bar{w}| + |z - w|}{|1 - z\bar{w}| - |z - w|} \right).$$

Let ξ be a complex number with $|\xi| = 1$. Let γ be the geodesic ray in $(\mathbb{D}, d_{\mathbb{D}})$ issuing from $0 \in \mathbb{D}$ to ξ . The Busemann function associated to γ is

$$\mathbf{h}_{\gamma(\infty)}(z) = \log \left(\frac{|z - \xi|^2}{1 - |z|^2} \right) \text{ for all } z \in \mathbb{D}.$$

We emphasize that Busemann functions on an arbitrary metric space may fail to exist, however, metric functionals always exist. Rieffel [38] and Walsh [39] consider Busemann functions as limits of *almost-geodesic* rays, a notion that we do not use in this dissertation.

3.3 Horofunctions

In [19], Gromov introduced the notion of *horofunction* as follows. Let (X, d) be a complete metric space. Endow $\text{Lip}_{x_0}^1(X)$ with the topology τ_{ub} of uniform convergence on bounded subsets of X . The topology τ_{ub} is stronger than τ_{uc} , i.e., $\tau_{uc} \subseteq \tau_{ub}$. The mapping $y \mapsto \mathbf{h}_y$ given by (3.2) also injects (X, d) continuously into the topological space $(\text{Lip}_{x_0}^1(X), \tau_{ub})$. Let $\hat{X}^G$ denote the closure of $\{\mathbf{h}_y \mid y \in X\}$ in $(\text{Lip}_{x_0}^1(X), \tau_{ub})$. Every element of $\hat{X}^G \setminus \{\mathbf{h}_y \mid y \in X\}$ is said to be a *horofunction* on X . The set $\hat{X}^G$ is said to be the *horofunction bordification* of X .

We observe that $\hat{X}^G$ may not be compact. For example, let $X = \ell_2(\mathbb{N})$ with the ℓ_2 -metric given by (2.3) and consider the sequence $y_n = (0, \dots, 0, n, 0, \dots)$ with n in the n -th coordinate. Then for every $x \in X$,

$$\begin{aligned} \mathbf{h}_{y_n}(x) &= d_2(x, y_n) - d_2(o, y_n) \\ &= \left(\sum_{s \neq n} |x(s)|^2 + |x(n) - n|^2 \right)^{1/2} - n \\ &= \left(\sum_{s \in \mathbb{N}} |x(s)|^2 - |x(n)|^2 + |x(n) - n|^2 \right)^{1/2} - n \\ &= n \left(n^{-2} \sum_{s \in \mathbb{N}} |x(s)|^2 - 2n^{-1}x(n) + 1 \right)^{1/2} - n \\ &\xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

The closed unit ball $B_1(o)$ is bounded in X and contains the sequence

$e_n = (0, \dots, 0, 1, 0, \dots)$ with 1 in the n -th coordinate. Hence for every $n \in \mathbb{N}$,

$$\sup_{x \in B_1(o)} |\mathbf{h}_{y_n}(x)| \geq |\mathbf{h}_{y_n}(-e_n)| = 1.$$

This shows that the sequence $(h_{y_n})_{n \in \mathbb{N}}$ contains no convergent subsequences in $\hat{X}^G$. However, it is important to observe that the sequence of internal metric functionals $(h_{y_n})_{n \in \mathbb{N}}$ does converge in the metric compactification $\overline{X}^h$ to the zero functional 0 on X .

If (X, d) is a proper geodesic metric space, it follows readily from Proposition 3.5 that the set $\hat{X}^G$ coincides with the metric compactification $\overline{X}^h$. In this setting, X is identified with $\{\mathbf{h}_y \mid y \in X\}$, and so $\overline{X}^h$ is a strict compactification of X in the sense that it contains X as an open dense subset [15, 35, 25]. The set $\overline{X}^h \setminus X$ is said to be the *horofunction boundary* of X .

For an arbitrary metric space, Gromov's horofunction boundary may be empty. An example of a metric space with this property can be found in [25].

4. The search for metric functionals

Let (X, d) be a metric space and let $f : X \rightarrow X$ be an isometry. The mapping $\tilde{f} : \overline{X}^h \rightarrow \overline{X}^h$ defined by

$$[\tilde{f}(\mathbf{h})](x) = \mathbf{h}(f^{-1}(x)) - \mathbf{h}(f^{-1}(x_0)) \text{ for all } x \in X$$

is a homeomorphism. This property is a fundamental tool in the study of (group of) isometries; see [5, 7, 4]. In Applied Sciences one often encounters more general morphisms of metric spaces than isometries. A *semicontraction* is a mapping $f : X \rightarrow X$ such that

$$d(f(x), f(y)) \leq d(x, y) \text{ for all } x, y \in X.$$

A subadditive argument shows that the limit

$$L := \lim_{n \rightarrow \infty} \frac{1}{n} d(f^n(x), x)$$

exists and is independent of $x \in X$. Karlsson [23, 25] uses the quantity L in combination with the metric compactification to describe the dynamics of semicontractions as follows.

Proposition 4.1 (Metric spectral principle). *Let (X, d) be a metric space and let $f : X \rightarrow X$ be a semicontraction. Then there exists a metric functional $\mathbf{h} \in \overline{X}^h$ such that for each $k \geq 0$,*

$$\mathbf{h}(f^k(x_0)) \leq -Lk.$$

Moreover, for every $x \in X$,

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \mathbf{h}(f^n(x)) = L.$$

In [18], Gouëzel and Karlsson present a random version of Proposition 4.1

that gives a geometrical approach of random product of semicontractions.

Geometric analysis of iterations of semicontractions becomes effective once we know explicit characterizations of all the metric functionals on the metric space being considered.

An analogous reasoning appeared a century ago in the context of linear mappings between function spaces. In [12, pp. 121–128], Dieudonné presents a historical review of the search for continuous linear functionals on the space $(C([0, 1]), d_\infty)$. Following Volterra’s work on integral equations, Hadamard initiated the formal study of mappings $\varphi : C([0, 1]) \rightarrow \mathbb{R}$ that are linear and continuous. Soon after Hadamard presented a characterization as the limit of a certain sequence of functions, Fréchet and F. Riesz got involved in Hadamard’s project and independently presented characterizations that nowadays are known as measures. Characterizations of continuous linear functionals on L_p spaces are known as the Riesz representation theorem.

This dissertation contributes to the search for metric functionals in the setting of Banach spaces.

5. Summary of Results

Let S be a non-empty set. For each $p \in [1, \infty[$ let $\ell_p(S)$ denote the space of functions $x : S \rightarrow \mathbb{R}$ such that $\sum_{s \in S} |x(s)|^p$ is finite. The space $\ell_p(S)$ endowed with the norm

$$\|x\|_{\ell_p} = \left(\sum_{s \in S} |x(s)|^p \right)^{1/p}$$

is a Banach space. Let $\ell_\infty(S)$ denote the space of functions $x : S \rightarrow \mathbb{R}$ such that $\sup_{s \in S} |x(s)|$ is finite. The space $\ell_\infty(S)$ endowed with the norm

$$\|x\|_{\ell_\infty} = \sup_{s \in S} |x(s)|$$

is also a Banach space.

5.1 Summary of Publication I

In Publication I, we investigate the metric compactification of finite-dimensional ℓ_p spaces for all $p \in [1, \infty]$.

In Publication I, Sec. 3, we provide explicit formulas for all the metric functionals on the finite-dimensional ℓ_1 space.

Theorem 5.1. *Let F be a non-empty finite set. Every metric functional on $\ell_1(F)$ is precisely of the form*

$$x \mapsto \mathbf{h}(x) = \sum_{s \in I} \varepsilon(s)x(s) + \sum_{s \in F \setminus I} (|x(s) - z(s)| - |z(s)|), \quad (5.1)$$

where $\emptyset \subseteq I \subseteq F$, $\varepsilon \in \{-1, +1\}^I$, and $z \in \mathbb{R}^{F \setminus I}$.

Remark 5.2. The formula (5.1) represents the internal metric functional $x \mapsto \mathbf{h}(x) = \|x - z\|_{\ell_1} - \|z\|_{\ell_1}$ if and only if the set I is empty. The metric functional (5.1) is linear if and only if $I = F$.

In Publication I, Sec. 4, we provide explicit formulas for all the metric functionals on the finite-dimensional ℓ_p space for $p \in]1, \infty[$.

Theorem 5.3. *Let $p \in]1, \infty[$ and let F be a non-empty finite set. Every metric functional on $\ell_p(F)$ has exactly one of the following forms:*

$$A) \quad x \mapsto \mathbf{h}(x) = \|x - z\|_{\ell_p} - \|z\|_{\ell_p},$$

where z is an element of $\mathbb{R}^F$.

$$B) \quad x \mapsto \mathbf{h}(x) = - \sum_{s \in F} \zeta(s)x(s),$$

where ζ is an element of $\mathbb{R}^F$ such that $\|\zeta\|_{\ell_{p/(p-1)}} = 1$.

In Publication I, Sec. 5, we provide explicit formulas for all the metric functionals on the finite-dimensional ℓ_∞ space.

Theorem 5.4. *Let F be a non-empty finite set. Every metric functional on $\ell_\infty(F)$ has exactly one of the following forms:*

$$A) \quad x \mapsto \mathbf{h}(x) = \|x - z\|_{\ell_\infty} - \|z\|_{\ell_\infty},$$

where z is an element of $\mathbb{R}^F$.

$$B) \quad x \mapsto \mathbf{h}(x) = \max_{s \in F} \max\{x(s) - \kappa(s), -x(s) - \lambda(s)\},$$

where κ and λ are elements of $[0, \infty]^F$ such that $\kappa(s) + \lambda(s) = \infty$ for all $s \in F$ and $\min_{s \in F} \min\{\kappa(s), \lambda(s)\} = 0$.

In Publication I, Sec. 6, we investigate the metric compactification of the finite-dimensional Hilbert metric cone $(\mathbb{R}_{>0}^F, d_H)$. As an illustration, we present a new proof of Perron's theorem which states that there is a unique fixed point for every positive matrix acting on the projective space $(\mathbb{R}_{>0}^F/\mathbb{R}, d_H)$.

The Hilbert projective metric d_H is defined on $\mathbb{R}_{>0}^F$ by

$$d_H(x, y) = \log \max_{s \in F} \{x(s)/y(s)\} - \log \min_{s \in F} \{x(s)/y(s)\}$$

for all $x, y \in \mathbb{R}_{>0}^F$. The Hilbert metric cone $(\mathbb{R}_{>0}^F, d_H)$ is isometric to the pseudo-normed space $\ell_{\text{var}}(F) := (\mathbb{R}^F, \|\cdot\|_{\text{var}})$, where

$$\|x\|_{\text{var}} = \max_{s \in F} \{x(s)\} - \min_{s \in F} \{x(s)\}$$

for all $x \in \mathbb{R}^F$.

Theorem 5.5. *Let F be a finite set with at least two elements. Every metric functional on $\ell_{\text{var}}(F)$ has exactly one of the following forms:*

$$A) \quad x \mapsto \mathbf{h}(x) = \|x - z\|_{\text{var}} - \|z\|_{\text{var}}$$

where z is an element of $\mathbb{R}^F$.

$$B) \quad x \mapsto \mathbf{h}(x) = \max_{s \in U} \{x(s) - \mu(s)\} - \min_{s \in V} \{x(s) + \nu(s)\}$$

where $\emptyset \subsetneq U, V \subsetneq F$ such that $U \cap V = \emptyset$, μ is an element of $\mathbb{R}_{\geq 0}^U \setminus \mathbb{R}_{> 0}^U$ and ν is an element of $\mathbb{R}_{\geq 0}^V \setminus \mathbb{R}_{> 0}^V$.

Theorem 5.6. *Let F be a finite set with at least two elements and let $\mathbf{1}$ denote the element of $\mathbb{R}_{> 0}^F$ given by $\mathbf{1}(s) = 1$ for all $s \in F$. Every metric functional on the Hilbert metric cone $(\mathbb{R}_{> 0}^F, d_H)$ has exactly one of the following forms:*

$$A) \quad x \mapsto \mathbf{h}(x) = d_H(x, z) - d_H(\mathbf{1}, z)$$

where z is an element of $\mathbb{R}_{> 0}^F$.

$$B) \quad x \mapsto \mathbf{h}(x) = \log \max_{s \in F} \{u(s)x(s)\} + \log \max_{s \in F} \{v(s)/x(s)\}$$

where u and v are elements of $\mathbb{R}_{\geq 0}^F$ such that $u(s)v(s) = 0$ for all $s \in F$, $\max_{s \in F} \{u(s)\} = 1$ and $\max_{s \in F} \{v(s)\} = 1$.

5.2 Summary of Publication II

In Publication II, we give a complete description of the metric compactification of infinite-dimensional ℓ_p spaces for all $p \in [1, \infty[$.

In Publication II, Sec. 3, we provide explicit formulas for all the metric functionals on the infinite-dimensional ℓ_1 space.

Theorem 5.7. *Let S be an infinite (countably infinite or uncountable) set. Every metric functional on $\ell_1(S)$ is precisely of the form*

$$x \mapsto \mathbf{h}(x) = \sum_{s \in I} \varepsilon(s)x(s) + \sum_{s \in S \setminus I} (|x(s) - z(s)| - |z(s)|), \quad (5.2)$$

where $\emptyset \subseteq I \subseteq S$, $\varepsilon \in \{-1, +1\}^I$, and $z \in \mathbb{R}^{S \setminus I}$.

Remark 5.8. There is an important difference between (5.1) and (5.2). The metric functional (5.2) is internal if and only if $I = \emptyset$ and $z \in \ell_1(S)$. For example, let $S = \mathbb{N}$ and for each $n \geq 1$ let $y_n = e_1 + e_2 + \dots + e_n \in \ell_1(\mathbb{N})$ where $e_n = (0, \dots, 0, 1, 0, \dots)$ has 1 in the n -th coordinate. Hence $\|y_n\|_{\ell_1} = n \rightarrow \infty$, however, for every $x \in \ell_1(\mathbb{N})$,

$$\lim_{n \rightarrow \infty} \mathbf{h}_{y_n}(x) = \mathbf{h}(x) = \sum_{s \in \mathbb{N}} (|x(s) - 1| - 1).$$

Here $I = \emptyset$ but $z = (1, 1, \dots)$ is not in $\ell_1(\mathbb{N})$.

In Publication II, Sec. 4, we provide explicit formulas for all the metric functionals on the infinite-dimensional ℓ_p spaces for $p \in]1, \infty[$.

Theorem 5.9. *Let $p \in]1, \infty[$. Let S be an infinite (countably infinite or uncountable) set. Every metric functional on $\ell_p(S)$ has exactly one of the following forms:*

$$A) \quad x \mapsto \mathbf{h}(x) = (\|x - z\|_{\ell_p}^p - \|z\|_{\ell_p}^p + C^p)^{1/p} - C,$$

where z is an element of $\ell_p(S)$ and $C \geq \|z\|_{\ell_p}$.

$$B) \quad x \mapsto \mathbf{h}(x) = - \sum_{s \in S} \zeta(s)x(s),$$

where ζ is an element of $\ell_{p/(p-1)}(S)$ such that $\|\zeta\|_{\ell_{p/(p-1)}} \leq 1$.

Remark 5.10. The formula A) represents the internal metric functional $\mathbf{h}_z(\cdot) = \|\cdot - z\|_{\ell_p} - \|z\|_{\ell_p}$ if and only if $C = \|z\|_{\ell_p}$.

Remark 5.11. Metric functionals of the form B) are elements of the closed unit ball of the dual space $\ell_p^* \cong \ell_{p/(p-1)}$. In particular, the zero functional $\mathbf{h} \equiv 0$ is an element of the metric compactification of the infinite-dimensional ℓ_p space for all $p \in]1, \infty[$.

5.3 Summary of Publication III

For every $p \in [1, \infty[$ let $L_p(\Omega, \Sigma, \mathbf{P})$ denote the Banach space of (classes of) Σ -measurable functions $f : \Omega \rightarrow \mathbb{R}$ with finite L_p -norm

$$\|f\|_{L_p} = \left(\int_{\Omega} |f(\omega)|^p d\mathbf{P}(\omega) \right)^{1/p}.$$

In Publication III, we investigate the metric compactification of the Banach space $L_p(\Omega, \Sigma, \mathbf{P})$ for $p \in [1, \infty[$, where $(\Omega, \Sigma, \mathbf{P})$ is a non-atomic standard probability space.

The metric compactification of the real line $(\mathbb{R}, |\cdot|)$ is the set

$$\overline{\mathbb{R}}^h = \{\eta_r \mid r \in \mathbb{R}\} \cup \{\eta_{+\infty}, \eta_{-\infty}\},$$

where the internal points are given by $\eta_r(a) = |a - r| - |r|$, and the points at infinity are $\eta_{+\infty}(a) = -a$, $\eta_{-\infty}(a) = a$.

Theorem 5.12. *Let $(\Omega, \Sigma, \mathbf{P})$ be a non-atomic standard probability space. Every metric functional on $L_1(\Omega, \Sigma, \mathbf{P})$ is precisely of the form*

$$f \mapsto \mathbf{h}(f) = \int_{\Omega} \int_{\overline{\mathbb{R}}^h} \eta(f(\omega)) d\xi_{\omega}(\eta) d\mathbf{P}(\omega),$$

where ξ_{ω} is a probability measure on the compact space $\overline{\mathbb{R}}^h$ for $\mathbf{P}$ -almost every $\omega \in \Omega$.

Theorem 5.13. *Let $p \in]1, \infty[$ and let $(\Omega, \Sigma, \mathbf{P})$ be a non-atomic standard probability space. Every metric functional on $L_p(\Omega, \Sigma, \mathbf{P})$ has exactly one of the following forms:*

$$A) \quad f \mapsto \mathbf{h}(f) = \left(\int_{\Omega} \int_{\mathbb{R}} (|f(\omega) - r|^p - |r|^p) d\xi_{\omega}(r) d\mathbf{P}(\omega) + C^p \right)^{1/p} - C,$$

where ξ_{ω} is a probability measure on $\mathbb{R}$ for $\mathbf{P}$ -almost every $\omega \in \Omega$ and $C^p \geq \int_{\Omega} \int_{\mathbb{R}} |r|^p d\xi_{\omega}(r) d\mathbf{P}(\omega)$.

$$B) \quad f \mapsto \mathbf{h}(f) = - \int_{\Omega} f(\omega) \zeta(\omega) d\mathbf{P}(\omega),$$

where ζ is an element of $L_{p/(p-1)}(\Omega, \Sigma, \mathbf{P})$ such that $\|\zeta\|_{L_{p/(p-1)}} \leq 1$.

Remark 5.14. The metric functional A) becomes the internal metric functional $\mathbf{h}_g(\cdot) = \|\cdot - g\|_{L_p} - \|g\|_{L_p}$ if and only if there is $g \in L_p(\Omega, \Sigma, \mathbf{P})$ such that $\xi_{\omega} = \delta_{g(\omega)}$ for $\mathbf{P}$ -almost every $\omega \in \Omega$ and $C^p = \int_{\Omega} \int_{\mathbb{R}} |r|^p d\xi_{\omega}(r) d\mathbf{P}(\omega)$.

In Publication III, Sec. 5, we use the metric functionals on L_p for $p \in]1, \infty[$ to give a proof of the L_p -mean ergodic theorem. Also, we show how to apply the metric compactification of L_1 to analyze Alspach's example of an isometry defined on a subset of L_1 having no fixed points.

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