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## Problems around the Stanley-Wilf Limits of Permutation

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## Abstract

The Theory of Matrix Avoidance explores the problem of determining the maximum number of 1 -entries in an $n \times n$ binary matrix that avoids a fixed pattern $P$. For permutation matrices, the Furedi-Hajnal conjecture posits a linear relationship between this number, known as extremal function of $n$, and the matrix size $n$. This conjecture was initially proven by Marcus and Tardos, and subsequently, the linear constant was further improved.

Another class of matrices, known as light matrices, exhibits a quasi-linear extremal function. Although, the proof for this class relies on the connection between pattern avoidance and the theory of Davenport-Schinzel sequences.

This thesis presents a proof for light matrices in terms of matrices without applying known results from connected topics, followed by an alternative proof for the Furedi-Hajnal conjecture. By addressing these topics, this research contributes to the understanding of matrix avoidance and its implications for different matrix classes.

## Keywords Permutation avoidance, Forbidden matrix , Stanley-Wilf limit, Furedi-Hajnal limit, Light matrix

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## 1 Introduction

Permutation patterns have been studied in combinatorial theory due to their rich mathematical structure and wide range of applications. One intriguing aspect of permutation patterns is their avoidance. Understanding the properties and limitations of permutation avoidance has led to the development of various combinatorial techniques and has found applications in diverse areas such as linguistics ([15], [21]), genetics ([2]), and social networks. Moreover, forbidden matrices offer valuable insights into the realm of data structures [20]. To embark on our exploration, we begin by introducing the definitions and providing an insightful overview of the historical progression of this topic.

### 1.1 Definitions

We call $k$-permutation a permutation of $\{1,2, \ldots, k\}$. For each $k$-permutation $\pi$ there exist a corresponding permutation matrix $P$, such that $p_{i j}=1$ if and only if $\pi(i)=j$. Such matrix contain a single element in each row and column. It follows from the correspondence, that there is a bijection between $k$-permutations and $k \times k$ permutation matrices. Such property allows us to work in terms of matrices or permutations when it is more suitable.

A generalization of permutation matrices are binary matrices: these are matrices with entries only from $\{0,1\}$. A mass of a binary matrix is the number of 1 entries, denoted as $|M|$. In this setting we will refer to one entry of a matrix as entry or element. For simplicity, we will denote elements as dots and zeros as nothing. In the future, we will skip the word "binary".

Next we introduce the concept of "avoidance" in several contexts.
Definition 1. An $n \times m$ matrix $M$ contains $k \times l$ matrix $A$, if there exist indices $1 \leq a_{1}<\ldots<a_{k} \leq n$ and $1 \leq b_{1}<\ldots<b_{l} \leq m$, such that for all $i \in[1, k]$, $j \in[1, k]$ if $A_{i, j}=1$, then $M_{a_{i}, b_{j}}=1$.

It is easy to see that the definition is equivalent to saying that matrix $M$ contains $A$ if $M$ can by obtained from $A$ by removing rows, columns or turning 1-entries into 0 -entries.

Otherwise $M$ avoids $A$.
For example, let $A$ be a $2 \times 2$ matrix. Notice, that the following matrix $M_{1}$ contains $A$, but $M_{2}$ does not contain $A$.

$$
\begin{gathered}
\left.A=\left(\begin{array}{ll}
\bullet & \bullet \\
\bullet & \\
M_{1}=\left(\begin{array}{lll}
\bullet & & \\
& \bullet & \bullet \\
& & \bullet
\end{array}\right) ; M_{2}=\left(\begin{array}{lll}
\bullet & & \\
& & \bullet \\
& & \bullet \\
\bullet & \bullet &
\end{array}\right)
\end{array}\right) . \begin{array}{llll} 
& \bullet & &
\end{array}\right)
\end{gathered}
$$

Definition 2. For a matrix $A$ with positive mass and $n, m \in \mathbb{N}$, ex $(n, m, A)$ denotes the maximum mass of an $n \times m$ matrix avoiding $A$.

When working with square $n \times n$ matrices, the notation ex $(n, P)$ is used instead of ex $(n, n, P)$.

The concept of avoiding a substructure is defined in terms of permutations the following way.

Definition 3. A n-permutation $\alpha$ contains $k$-permutation $\pi$ if there exist indices $1 \leq a_{1}<\ldots<a_{k} \leq n$, such that

$$
\pi(i)<\pi(j) \Leftrightarrow \alpha\left(a_{i}\right)<\alpha\left(a_{j}\right)
$$

Otherwise, $\alpha$ avoids $\pi$. Let $S_{n}(\pi)$ be a number of $n$-permutations avoiding $\pi$.
Notice that the definitions are related. If we restrict the Definition 1 on only permutation matrices, the two are equivalent. Thus, $S_{n}(\pi)$ also denotes the number of permutation matrices avoiding permutation matrix of $\pi$. We will write $S_{n}(P)$ for the number of permutation matrices avoiding permutation matrix $P . S_{n}(P)=S_{n}(\pi)$, when $P$ is $\pi$ permutation matrix.

A lot of researches were interested in the behaviour of matrices avoiding permutation matrix. For permutations of length 3 it was thoroughly studied [22]. For example, it is known, that 3 -avoiding permutations are counted by the Catalan numbers. On the other hand, permutations of length 4 introduce a significantly heightened level of complexity [3], [4].

Matrix avoidance has deep connections with various other combinatorial structures, including the avoidance of subgraphs in graphs and the theory of Davenport-Schinzel sequences. Understanding these connections provides valuable insights into the underlying combinatorial principles.

Graphs can be represented as adjacency matrices, where the presence or absence of edges is encoded by entries in the matrix. This area of research proved to be closely related to matrix avoidance, as patterns in matrices can be translated into subgraphs in graphs. By studying the avoidance of subgraphs, researchers sought to understand the structure and properties of graphs [19].

Davenport-Schinzel sequences are sequences of symbols with certain restrictions on the arrangement of repeated symbols. The avoidance of Davenport-Schinzel sequences is intimately linked to both matrix and subgraph avoidance. Specifically, the length of the longest Davenport-Schinzel sequence of a given order corresponds to the maximum size of a matrix or subgraph that can be avoided.

The history of these connections dates back several decades. The connection between forbidden matrices and graphs was first noted in a special case by Füredi and Hajnal [11] and was developed later by Klazar [16]. Their contributions laid the foundation for subsequent research in this area, and their insights into permutation patterns paved the way for exploring connections with subgraph avoidance and Davenport-Schinzel sequences.

Techniques from one area have been adapted and applied to the others, leading to new breakthroughs and results. The interdisciplinary nature of these connections continues to inspire researchers in various fields, including combinatorics, graph theory, and algorithm design.

For example, the results from theory of Davenport-Schinzel sequences can be used as a tool in studying light matrices. Light matrix is a matrix with one element in each column. Thus, permutation matrices are light. The extremal function quasi-linear form. For example, for a light matrix of size 4,

$$
\operatorname{ex}(n,(\bullet \text { • • }))=n \alpha(n),
$$

where $\alpha(n)$ is inverse Ackermann function. In a general case, for any light matrix $A$, there exist a constant $c$, such that

$$
\operatorname{ex}(n, A) \leq n \cdot 2^{\alpha(n)^{c}}
$$

The result uses reduction to Davenport-Schinzel sequences [14]. There are also generalizations of the results in matrices of higher dimension [12].

### 1.2 Introducing the question

The intriguing question is to study the properties of avoidance. For example,

- How does ex $(n, A)$ and $S_{n}(P)$ grow when $n$ goes to infinity?
- How are $S_{n}(P)$ and ex $(n, P)$ related?
- How does ex $(n, A)$ differ for different classes of matrices, e.g. light matrices, permutation matrices, all-ones matrices.

In the late 1980s Richard P. Stanley and Herbert Wilf independently formulated the following conjecture:
Stanley-Wilf conjecture: For every permutation $P$ there exists a constant $c=c_{p}$, such that $S_{n}(\pi) \leq c^{n}$.

Later it was proven to be equivalent to the existence of a specific limit, as demonstrated by Arratia [1]. This limit is defined as follows:

$$
L(P)=\lim _{n \rightarrow \infty} \sqrt[n]{S_{n}(P)}
$$

In 1993, Zoltán Füredi and Péter Hajnal introduced the Füredi-Hajnal conjecture [11], which seeks to investigate the linearity of complexity in all permutation configurations. The conjecture poses the question:
Füredi-Hajnal conjecture: For every permutation $P, \operatorname{ex}(n, P)=O(n)$.

$$
c(P)=\lim _{n \rightarrow \infty} \frac{\operatorname{ex}(n, P)}{n}
$$

Subsequently, Klazar demonstrated [17] that the Füredi-Hajnal conjecture implies the Stanley-Wilf conjecture. The first partial results were proved in 1999, when Miklós Bóna showed that Stanley-Wilf conjecture holds for all layered patterns [5]. A pattern is layered if it consists of the disjoint union of substrings, so that the entries decrease
within each layer, and increase between the layers. Then in 2004, Marcus and Tardos provided a proof [18] of the Füredi-Hajnal conjecture, establishing double exponential upper bounds for the Stanley-Wilf limit:

$$
L(P) \leq 15^{2 k^{4}\binom{k^{2}}{k}},
$$

where $k$ represents the size of the permutation matrix $P$. It was mentioned in their paper that the constant factor is not optimized. Prior to this, Arratia had conjectured [1] that for all $k$-permutations, it holds that $L(P)=(k-1)^{2}$.

Subsequently, in 2013, Jacob Fox [9] significantly improved the bound and provided a tight lower bound, thereby refuting Arratia's conjecture. He showed that for every permutation matrix $L(P)=2^{O(k)}$ and, moreover, there are permutations, for which $L(P)=2^{\Omega\left(k^{1 / 4}\right)}$. The estimate was improved further by Josef Cibulka and Jan Kyncl [7] in 2019. In their work they showed that for every permutation $P, c(P) \leq 2^{(4+o(1)) k}$.

The mentioned works employ a concept involving the division of matrices into blocks and subsequent contraction. This operation allows to write recursive formula for extremal function, solution to which gives an upper bound.

### 1.3 Structure

This thesis will present a comprehensive review of the existing literature on permutation patterns. Building upon this foundation, we will propose a novel approach and explore the potential for improving the upper bounds on $c(P)$.

This thesis is structured as follows. In Sections 2,3 we cover all necessary preliminaries. Section 3 also includes the Marcus and Tardos' and Fox's proofs of Füredi-Hajnal conjecture. In Section 4 we present the upper bound for light matrices proved in terms of matrices without referencing the results in theory of DavenportSchinzel sequences. The proof also uses more intuitive definition of the inverse Ackermann function. The main result is presented in section 5 . We give a double exponential upper bound on Füredi-Hajnal limit.

## 2 Basic facts about forbidden 0-1 matrices

In this section we will see basic properties of extremal function, examples and reduction rules.

### 2.1 Properties

Proposition 2.1. If $P$ contains $Q$, then $\operatorname{ex}(n, m, Q) \leq \operatorname{ex}(n, m, P)$.
Proof. If a matrix avoids $Q$, then it must also avoid $P$, and the proposition follows.
Lemma 2.2. Let $P$ be a binary matrix. Then

$$
\operatorname{ex}(n, m, P)=\operatorname{ex}\left(m, n, P^{\top}\right)
$$

where $P^{\top}$ denotes the transpose of $P$.
Proof. If a matrix $M$ avoids $P, M^{\top}$ avoids $P^{\top}$, thus ex $\left(m, n, P^{\top}\right) \leq \operatorname{ex}(n, m, P)$. Using the same reasoning for matrix $P^{\top}$ we get the ex $\left(m, n, P^{\top}\right) \geq \operatorname{ex}(n, m, P)$.

Corollary 2.2.1. Let $P$ a binary matrix. Then, $\operatorname{ex}(n, P)=\operatorname{ex}\left(n, P^{\top}\right)$.
If we rotate of reflect a matrix $P$, the extremal function will not change, up to change of places of variables $n$ and $m$ (in case of $90^{\circ}$ rotating). For example, it is clear that

$$
\operatorname{ex}(n, m,(\bullet \bullet \quad \bullet))=\operatorname{ex}\left(m, n,\left(\begin{array}{lll}
\bullet & \\
& \bullet \\
& \bullet
\end{array}\right)\right)
$$

It follows from the definition, that $\operatorname{ex}(n, m, P) \leq n m$ for any $P$, since $n m$ is the maximum number of elements a matrix can have.

If matrix $P$ is a single any 1 -entry, then ex $(n, m, P)=0$, because matrix avoiding $P$ is not allowed to have any elements. Furedi-Hajnal conjecture states that for permutation matrix $P$ the extremal function is linear.

There are patterns for which extremal function has higher order. In Section 4 we show that light matrices have non-linear extremal functions. Also, the following example has order $O(n \log n)$. It was proved by Furedi in [10].

Theorem 2.3.

$$
\operatorname{ex}(n,(\bullet) \quad \bullet))=\Theta(n \log n)
$$

Proof. For simplicity, denote the matrix from the statement as $P$. To prove lower bound, consider matrix $A_{n}$, such that $\left(A_{n}\right)_{i j}=1$ if and only if $j-i=2^{k}$ for $k \geq 1$. Then,

$$
A_{n}=\sum_{k=1}^{\left\lfloor\log _{2} n\right\rfloor}\left(n-2^{k}\right) \geq n \log _{2} n-n .
$$

We prove the stronger statement: for every $i<i_{1} \leq i_{2}$ and $j>j_{1} \geq j_{2}$ we can not have $A_{i, j}=A_{i, j_{1}}=A_{i_{1}, j}=A_{i_{2}, j_{2}}$. Adding constraints $i_{1}=i_{2}$ and $j_{1}>j_{2}$ guarantees the case in the formulation of the theorem.

Suppose $A_{i, j}=A_{i, j_{1}}=A_{i_{1}, j}$. Then $j-i$ must be some power of $2, j-i=2^{k}$, and $j-i_{1} \leq 2^{k-1}$ and $j_{1}-i \leq 2^{k-1}$.

$$
j_{2}-i_{2} \leq j_{1}-i_{1} \leq\left(j_{1}-i\right)+\left(j-i_{1}\right)-(j-i) \leq 2^{k-1}+2^{k-1}-2^{k}=0
$$

Thus, $A_{i_{2}, j_{2}}$ can not be one.
Let $A$ be $n \times n$ matrix avoiding $P,|A|=\operatorname{ex}(n, P)$. For the row $i$ let $f(i)=\min \{j$ : $\left.A_{i j}=1\right\}$ denote the first entry. Consider an entry $A_{i j}$, and let $j^{\prime}$ be the column index of its left neighbor (closest entry in the same row with smaller column index). Suppose $j \neq f(i)$ and $j^{\prime} \neq f(i)$. Define $p=j^{\prime}-f(i), q=j-j^{\prime}$. We call an entry far, if $p \leq q$ and close, otherwise. In the row each element, except for the first two, is either far or close. Let $F$ and $C$ be matrices containing only far and close elements, respectively.

Consider $i$ th row of $F$. Suppose it has $k 1$-entries with indices $f(i)<j_{1}<j_{2}<$ $\ldots<j_{k}$. Let $j_{l}$ be the column index of left neighbor of $A_{i j l}, 1<l \leq k$. Then,

$$
j_{l-1}-f(i) \leq j_{l}^{\prime}-f(i) \leq j_{l}-j_{l}^{\prime} \leq j_{l}-j_{l-1} .
$$

Adding $j_{l-1}-f(i)$ to both sides of the resulting inequality,

$$
2\left(j_{l-1}-f(i)\right) \leq j_{l}-f(i)
$$

That means that $k \leq \log _{2} n$ and $|F| \leq n \log _{2} n$.
Similar argument works for the close elements. Consider $j$ th column of $C$. Let $i_{1}<i_{2}<\ldots<i_{k}$ denote row indices of 1-entries. Let $j_{l}^{\prime}$ be the column index of the left neighbor of $A_{i_{j} j}$ in $A$. Then,

$$
j_{l}^{\prime} \leq f\left(i_{l+1}\right), 1 \leq l<k
$$

Otherwise, we can find representation of $P$ in $A$. Thus,

$$
\begin{aligned}
j-f\left(i_{l+1}\right) & \leq j-j_{l}^{\prime} \leq j_{l}^{\prime}-f\left(i_{l}\right), \quad \text { and } \\
2(j-f(l+1)) & \leq j-j_{l}^{\prime}+j_{l}^{\prime}-f\left(i_{l}\right)=j-f\left(i_{l}\right) .
\end{aligned}
$$

It follows that $k \leq \log _{2} n$ and $|C| \leq n \log _{2} n$.
In total,

$$
|A| \leq|F|+|C|+2 n=2 n \log _{2} n+2 n .
$$

### 2.2 Super additivity

The next result is proved in [19] and shows that extremal function is super additive.
Theorem 2.4. $\operatorname{ex}\left(n_{1}+n_{2}, m_{1}+m_{2}, P\right) \geq \operatorname{ex}\left(n_{1}, m_{1}, P\right)+\operatorname{ex}\left(n_{2}, m_{2}, P\right)$.

Proof. Firstly, suppose $P$ that the first and the last rows of $P$ are not empty. Take the first non-zero element of the first row $p_{1 i}$ and the last nonzero element of the last row $p_{k j}$. Assume $j \leq i$ without loss of generality. Now let $A$ and $B$ be maximum mass matrices avoiding $P$ of sizes $n_{1} \times m_{1}$ and $n_{2} \times m_{2}$, respectively. Construct block diagonal matrix $D$ by putting $A$ and $B$ on the main diagonal. $D$ has size $\left(n_{1}+n_{2}\right) \times\left(m_{1}+m_{2}\right)$. Suppose for a contradiction that $D$ contains $P$. Since $A$ and $B$ avoid $P, D$ must contain elements corresponding to $p_{1 i}$ and $p_{k j}$ in the different blocks. But in that case the second element would be placed to the right from the first one. That gives a contradiction.

If $j>k$, construct $D$ by placing blocks $A$ and $B$ on the anti-diagonal. The same reasoning holds. Thus,

$$
\operatorname{ex}\left(n_{1}+n_{2}, m_{1}+m_{2}, P\right) \geq \operatorname{ex}\left(n_{1}, m_{1}, P\right)+\operatorname{ex}\left(n_{2}, m_{2}, P\right) .
$$

Consider the case when the first row of $P$ does not contain 1-entry. Denote as $P^{\prime}$ the matrix obtained by deleting the first row of $P$. If some matrix $A$ avoids $P$, then $A^{\prime}$, which is matrix by deleting the first row of $A$, avoids $P^{\prime}$. Thus,

$$
\begin{gathered}
\operatorname{ex}\left(n_{1}+n_{2}, m_{1}+m_{2}, P\right)=\operatorname{ex}\left(n_{1}+n_{2}-1, m_{1}+m_{2}, P^{\prime}\right)+m_{1}+m_{2} \leq \\
\leq \operatorname{ex}\left(n_{1}-1, m_{1}, P^{\prime}\right)+\operatorname{ex}\left(n_{2}, m_{2}, P^{\prime}\right)+m_{1}+m_{2} \leq \operatorname{ex}\left(n_{1}, m_{1}, P\right)+\operatorname{ex}\left(n_{2}, m_{2}, P\right) .
\end{gathered}
$$

If several rows are zero, repeat this process until matrix has a nonzero element in the first row. If the last row is zero, similar argument holds.

Corollary 2.4.1. ex $(n+m, P) \geq \operatorname{ex}(n, P)+\operatorname{ex}(m, P)$.
The construction mentioned in the previous theorem motivates the following definition.

Definition 4. Sum of matrices $A$ and $B$ of sizes $n \times m, k \times l$ respectively, is a block diagonal $n+k \times m+l$ matrix $A \oplus B$, constructed bu putting $A$ and $B$ on the main diagonal.
Skew sum $A \ominus B$ is a matrix constructed by putting $A$ and $B$ on anti-diagonal.

$$
\begin{gathered}
(A \oplus B)_{i j}=\left\{\begin{array}{l}
A_{i, j}, i \in[1, n], j \in[1, m] \\
B_{i-n, j-m}, i \in[n+1, n+k], j \in[m+1, m+l] \\
0, \text { otherwise }
\end{array}\right. \\
(A \ominus B)_{i j}=\left\{\begin{array}{l}
A_{i, j-l}, i \in[1, n], j \in[1+l, m+l] \\
B_{i-n, j}, i \in[n+1, n+k], j \in[1, l] \\
0, \text { otherwise }
\end{array}\right.
\end{gathered}
$$

Observe that a permutation can not be a sum and a skew sum of two permutations at the same time. It is also possible that a permutation is neither a sum or a skew sum.

For example, the following 4-permutation is neither.

$$
\left(\begin{array}{lll}
\bullet & & \\
\bullet & & \bullet \\
& & \bullet
\end{array}\right)
$$

A proposition proved in [1], shows that the limit $L(P)$ exists and is finite if Stanley-Wilf conjecture holds.

## Proposition 2.5.

(1) $S_{n+m}(P) \geq S_{n}(P) \cdot S_{m}(P)$,
(2) $c_{P}=\lim _{n \longrightarrow \infty} S_{n}(P)^{1 / n}$ exists.

Proof. The first statements follows from the same construction as in lemma 2.4. Fekete's lemma [8] states that if $a_{1}, a_{2}, \ldots \in \mathbb{R}$ and $a_{m}+a_{m} \leq a_{n+m}$, then there exists $\lim _{n \rightarrow \infty} a_{n} / n \in[-\infty, \infty)$. Apply Fekete's lemma with $a_{i}=-\log S_{n}(P)$.

### 2.3 Reduction rules

In this section we will see examples of how the extremal function changes with some matrix transformations mentioned in [13].

Theorem 2.6. (1) Let $P^{\prime}$ be a matrix obtained by adding a new column to $P$ and placing a single 1-entry there next to an existing one. Then

$$
\operatorname{ex}\left(n, m, P^{\prime}\right) \leq \operatorname{ex}(n, m, P)+n .
$$

(2) Let $P^{\prime}$ be a matrix obtained by adding a new row to $P$ and placing a single 1-entry there next to an existing one. Then

$$
\operatorname{ex}\left(n, m, P^{\prime}\right) \leq \operatorname{ex}(n, m, P)+m
$$

Proof. (1) Without loss of generality, suppose the column was added to the left side. Let $A^{\prime}$ be the maximum mass $n \times m$ matrix avoiding $P^{\prime}$. Let $A$ be a matrix constructed by deleting the rightmost element in each row. Then $A$ has at least ex $\left(n, m, P^{\prime}\right)-n$ elements. Notice that $A$ avoids $P$. Indeed, if $A$ contains $P$, the the representation together with deleted element of the appropriate row would give the representation of $P^{\prime}$.
(2) Similar argument, but with deleting elements in columns, works.

Lemma 2.7. (1) If $P^{\prime}$ is a matrix constructed from $P$ by adding two empty columns to the boundary,

$$
\operatorname{ex}\left(n, m, P^{\prime}\right) \leq \operatorname{ex}(n, m, P)+2 n .
$$

(2) If $P^{\prime}$ is a matrix constructed from $P$ by adding two empty rows to the boundary,

$$
\operatorname{ex}\left(n, m, P^{\prime}\right) \leq \operatorname{ex}(n, m, P)+2 m .
$$

Proof. Let $M^{\prime}$ be a $n \times m$ matrix avoiding $P^{\prime}$ and $\left|M^{\prime}\right|=\operatorname{ex}\left(n, m, P^{\prime}\right)$. Let $M$ be the matrix constructed by deleting all elements in the first and last columns. It has at least $\operatorname{ex}\left(n, m, P^{\prime}\right)-2 n$ entries. Notice that $M$ avoids $P$. Indeed, if $M$ contains $P$, then $M^{\prime}$ contains $P^{\prime}$. Thus, ex $\left(n, m, P^{\prime}\right) \leq \operatorname{ex}(n, m, P)+2 n$.

To prove the second statement repeat the reasoning above with deleting all elements of $M^{\prime}$ of the first and the last rows.

Lemma 2.8. (1) If $P^{\prime}$ is a matrix constructed from $P$ by adding $k$ consecutive empty rows into the matrix,

$$
\operatorname{ex}\left(n, m, P^{\prime}\right) \leq(k+1) \operatorname{ex}\left(\frac{n}{k+1}, m, P\right)+2 k m
$$

(2) If $P^{\prime}$ is a matrix constructed from $P$ by adding $k$ consecutive empty columns into the matrix,

$$
\operatorname{ex}\left(n, m, P^{\prime}\right) \leq(k+1) \operatorname{ex}\left(n, \frac{m}{k+1}, P\right)+2 k n
$$

Proof. Let $M^{\prime}$ be a $n \times m$ matrix avoiding $P^{\prime}$ and $\left|M^{\prime}\right|=\operatorname{ex}\left(n, m, P^{\prime}\right)$. Let $M$ be the matrix constructed by deleting all elements in the first and last $k$ rows. Let $M_{a}, 0 \leq a \leq k$ be a matrix:

$$
\begin{gathered}
M_{a}(i, j)=M(i, j), \text { if } i \bmod (k+1)=a \\
M_{a}(i, j)=0, \text { otherwise } .
\end{gathered}
$$

This way $M_{a}$ contains rows of modulo $a$ of $k$. For $a \neq a^{\prime}$ matrices $M_{a}$ and $M_{a^{\prime}}$ do not intersect and

$$
\sum_{a=0}^{k-1}\left|M_{a}\right|=|M|
$$

Each $M_{a}$ avoids $P$. Indeed, if $M_{a}$ contains $P$, then $M$ will contain $P^{\prime}$. Thus,

$$
\operatorname{ex}\left(n, m, P^{\prime}\right)=\left|M^{\prime}\right| \leq|M|+2 k m \leq k \operatorname{ex}\left(n, \frac{m}{k+1}, P\right)+2 k m
$$

The proof of the second statement is similar. Repeat the reasoning above but with deleting column elements instead of rows.

Here is an example of how these properties could be applied to estimate the extremal function for a particular matrix.

Lemma 2.9.

$$
\operatorname{ex}\left(n, m,\left(\begin{array}{lll}
\bullet & & \bullet \\
& \bullet & )
\end{array}\right) \leq 2 n+m\right.
$$

Proof. Let us first apply theorem 2.1, and then theorem 2.6.

$$
\left.\left.\begin{array}{rl}
\operatorname{ex}\left(n, m,\left(\begin{array}{lll}
\bullet & \bullet
\end{array}\right)\right) & \leq \operatorname{ex}\left(n, m,\left(\begin{array}{lll}
\bullet & \bullet & \bullet \\
& \bullet
\end{array}\right)\right) \\
& \leq \operatorname{ex}(n, m,(\bullet \bullet \\
\bullet & \bullet
\end{array}\right)+m\right)+\left(\begin{array}{ll} 
& \\
& \leq 2 n+m,
\end{array}\right.
$$

the last inequality holds, because a matrix avoiding $1 \times 3$ all-ones matrix can not have more than 2 elements in each row.

The following reduction rule was proved in [13].
Theorem 2.10. Let $A$ and $B$ be two matrices. Assume that $A$ has an entry in its lower right and B at its upper left corner. Let $C$ be a pattern consisting of $A$ at its upper left part and $B$ at its lower right part with exactly one common 1-entry ( $C$ is almost $A \oplus B$, but with overlapping in 1 element $)$. Then $\max (\operatorname{ex}(n, A), \operatorname{ex}(n, B)) \leq$ $\mathrm{ex}(n, C) \leq \operatorname{ex}(n, A)+\operatorname{ex}(n, B)$.

Proof. The first inequality is trivial. Now suppose for a contradiction that $M$ avoids $C$ and $|M|=\operatorname{ex}(n, A)+\operatorname{ex}(n, B)+1 . M$ must contain $A$, since $|M| \geq \operatorname{ex}(n, A)+1$. Fix a representation of $A$ in $M$ and delete the entry corresponding to the lower right element of $A$. The mass of $M$ decreases by one. Repeat this process ex $(B)+1$ times, until $|M|=\operatorname{ex}(n, A)$. Let $M^{\prime}$ be the matrix containing only deleted elements, $\left|M^{\prime}\right|=\operatorname{ex}(n, B)+1$. It follows that $M^{\prime}$ contains $B$. Take this representation of $B$ in $M$. The upper right element is from $M^{\prime}$ meaning that there is a representation of $A$, where this entry is a lower right element of $A$. That gives a representation of $C$.

## 3 Permutation matrices

In this section, we focus on the properties and examples of permutation matrices avoidance. Also, we review two proofs of Füredi-Hajnal conjecture.

### 3.1 Properties

The notion of contraction matrix will be used. Let $M$ be an $n \times m$ matrix. Divide $M$ into blocks of size $B_{1} \times B_{2}$, where $B_{1}$ divides $n$ and $B_{2}$ divides $m$ : let $S_{k l}$, $1 \leq k \leq n / B_{1}$ and $1 \leq l \leq m / B_{2}$, be $B_{1} \times B_{2}$ matrix containing all elements $M_{i j}$ for $i \in\left[(k-1) \cdot B_{1}+1, k B_{1}\right], j \in\left[(l-1) \cdot B_{2}+1, l B_{2}\right]$. A contraction matrix $M^{\prime}$ corresponding to this division of size $\frac{n}{B_{1}} \times \frac{m}{B_{2}}$ is defined the following way:

$$
\left(M^{\prime}\right)_{k l}=\left\{\begin{array}{l}
0,\left|S_{k l}\right|=0 \\
1, \text { otherwise }
\end{array} .\right.
$$

We refer as superrow to a row of blocks, and as a supercolumn to a columns of blocks.
Lemma 3.1. If $M$ avoids some permutation $P$, then $M^{\prime}$ avoids $P$.
Proof. Suppose for a contradiction that $M^{\prime}$ contains $P$. Fix a realization of $P$ in $M^{\prime}$. Each 1-entry of $M^{\prime}$ corresponds to a block in $M$, that has at least one element. Fix an entry in each of the blocks, giving realization of $P^{\prime}$. Notice that since $P$ is a permutation, all these entries are in different rows and columns. They represent $P$ in $M$, which gives a contradiction.

Notice that this reasoning only works with permutation matrices. The statement is not necessarily true if $P$ is a random pattern. Indeed, consider the following counterexample.

$$
P=\left(\begin{array}{ll}
\bullet & \bullet \\
\bullet &
\end{array}\right) .
$$

Let $A$ be $6 \times 6$ matrix, avoiding $P$. Let $A^{\prime}$ be the contraction, corresponding to division into blocks of size $2 \times 2$ :


Clearly, $A^{\prime}$ contains $P$.

### 3.2 Examples

Theorem 3.2. For every 2-permutation $P, \operatorname{ex}(n, m, P)=n+m-1$.
Proof. There are only 2 possible permutations:

$$
(1,2)=\left(\begin{array}{ll}
\bullet & \\
& \bullet
\end{array}\right) \text { and }(2,1)=(\bullet \quad \bullet) .
$$

They are symmetric, thus it is enough to show the estimation for the first one. Let $A$ be a matrix of maximum mass, $|A|=\operatorname{ex}(n, m,(1,2))$, avoiding (1,2). Suppose it has two 1 -entries on the same diagonal. Thus, these two entries give realization of permutation $(1,2)$. There are $(n+m-1)$ diagonals and each has at most 1 entry,

$$
\operatorname{ex}(n, m,(1,2)) \leq n+m-1 .
$$

To show that this bound is the best, consider an $n \times m$ matrix, that has 1 entries in the last row and in the first column. It has exactly $n+m-1$ entries and it avoids $(1,2)$.

Theorem 3.3. For every 3-permutation $P$, ex $(n, m, P) \leq 2(n+m)$.
Proof. There 2 essentially different classes of 3-permutations [22]. Permutations (123) and (321) are symmetric. As well as (132), (213), (231) and (312). Thus, we only need to show the estimation for one element of each class.
(123): Suppose there are 3 elements on one diagonal. Then they give a 3-permutation. Thus, each of $m+n-1$ diagonals have at most 2 elements. Also, there are 2 diagonals of size 1 (the bottom-left element and the top-right element),

$$
\operatorname{ex}(n, m,(123)) \leq 2(n+m-3)+2=2(n+m-2) .
$$

To show that this is the best bound, consider an $n \times m$ matrix, that has 1-entries in the last two rows and first two columns. It has exactly $2(n+m-2)$ elements and avoids (123).
(213): By theorem 2.1,

$$
\operatorname{ex}\left(n, m,\left(\bullet \begin{array}{lll}
\bullet & & \\
& & \bullet
\end{array}\right)\right) \leq \operatorname{ex}\left(n, m,\left(\bullet \bullet_{0}\right.\right.
$$

To estimate this extremal function, use theorem 2.6:

$$
\operatorname{ex}\left(n, m,\left(\bullet \begin{array}{lll}
\bullet & \bullet \\
& \bullet & \bullet
\end{array}\right)\right) \leq \operatorname{ex}\left(n, m,\left(\begin{array}{l}
\bullet \\
\bullet \\
\bullet
\end{array}\right)\right)+2 n \leq 2(n+m) .
$$

The last inequality is true, since matrix avoiding this permutation can not have more than 2 elements in each of $m$ columns.

Notice that the argument for identity permutations of size 2 and 3 works in a general case.

Theorem 3.4. If $P$ is an identity matrix of size $k$, then $\operatorname{ex}(n, m, P)=O((n+m) k)$.
Proof. Let $M$ be $n \times m$ matrix, avoiding $P$. Notice that it can not have more than $k$ elements on each diagonal, because otherwise these entries would represent $P$. There are $n+m-1$ diagonals. Thus, ex $(n, m, P) \leq(n+m-1) k$. Also, the exists a matrix with $(k-1) n+(k-1) m-(k-1)^{2}$ elements avoiding $P$ : fill first $(k-1)$ columns and $(k-1)$ rows with 1 entries.

### 3.3 Connection between $c(P)$ and $L(P)$

Theorem 3.5. Füredi-Hajnal conjecture implies Stanley-Wilf conjecture. [17]
Definition 5. Let $T_{n}(P)$ be the number of $n \times n$ matrices avoiding $P$.
It is clear that $S_{n}(P) \leq T_{n}(P)$.
Theorem 3.6. $T_{n}(P) \leq C_{P}^{n}$ for some constant $C_{P}$.
Proof. Suppose we have a $2 n \times 2 n$ matrix $M$, avoiding $P$. Consider a contraction matrix $M^{\prime}$ constructed the following way: partition $M$ into $2 \times 2$ blocks, an entry of $M^{\prime}$ corresponds to a block of $M$. If the block has at least 1 element, the corresponding entry of $M^{\prime}$ is 1 . Otherwise, it is 0 . Since $M^{\prime}$ is a contraction matrix, $M^{\prime}$ also avoids $P$. The matrix $M^{\prime}$ is also an image of $15^{\mid} M^{\prime} \mid$ matrices under the described contraction. Indeed, each 1-entry of $M^{\prime}$ corresponds to a $2 \times 2$ block in $M$ with at least 1 element, there are 15 options. Since $\left|M^{\prime}\right| \leq \operatorname{ex}(n, P)$, it allows us write the following recursion:

$$
T_{2 n}(P) \leq T_{n}(P) 15^{\operatorname{ex}(n, P)}
$$

Then, since ex $(n, P)=c(P) n$, choose $C_{P}=15^{c(P)}$ and the statement will follow from induction.

Since $\left(S_{n}(P)\right)^{\frac{1}{n}} \leq T_{n}(P)^{\frac{1}{n}}$,

$$
L(P) \leq\left(T_{n}(P)\right)^{\frac{1}{n}} \leq 15^{c(P)} .
$$

Later it was shown by Cibulka [6], that these two limits are polynomials of each other. [6]

Lemma 3.7. Let $P$ be a permutation matrix and $t, n$ be integer numbers. Let $N=t n$. Then,

$$
S_{N}(P) \leq T_{n}(P) t^{2 N} .
$$

Proof. Let $A$ be $N \times N$ permutation matrix avoiding $P$. Partition $A$ into $t \times t$ blocks and consider a contraction matrix $B$ of size $n \times n, B$ also avoid $P$. Thus, $B$ is one of $T_{n}(P)$ possible matrices. Let us see how many different matrices $A$ would give $B$ after contraction. Notice that since $A$ is a permutation matrix, $t$ rows of $A$ contain at most $t$ elements. Thus, each row of $B$ contains at most $t 1$-entries. Each row of $A$ has one element. It can be located in positions corresponding to nonzero elements of $B$. There
are $t$ of them, and the width of the block is $t$. Thus, given $B$, there are $t^{2}$ options where the element of each row is located. This gives an estimate

$$
S_{N}(P) \leq T_{n}(P) t^{2 N}
$$

Corollary 3.7.1. $L(P)=O\left(c(P)^{2}\right)$.
Proof. Substitute $t=c(P)$ and $n=N / c(P)$ into this theorem. Also notice, that by the proof of $3.6, T_{n}(P) \leq 2^{O(e x(n, P))} \leq 2^{O(c(P) n)}$.

$$
S_{N}(P) \leq T_{n}(P) c(P)^{2 N} \leq 2^{O(c(P) n)} c(P)^{2 N}=2^{O(N)} c(P)^{2 N}=\left(2^{O(1)} c(P)\right)^{2 N} .
$$

Take the $N$ th root of this inequality:

$$
L(P)=\lim _{N \rightarrow \infty} \sqrt[N]{S_{N}(P)} \leq 2^{O(1)} c(P)^{2} .
$$

Cibulka also shows a polynomial relation in the second direction in [6].
Theorem 3.8. Stanley-Wilf conjecture implies Furedi-Hajnal conjecture, and for every permutation matrix $P$,

$$
c(P)=O\left(L(P)^{4.5}\right)
$$

The proof requires a technical lemma.
Lemma 3.9. Let $B$ be $b \times c$ matrix, which has at least $b$ entries in each row and avoids some permutation $P$. Then,

$$
\left|S_{b}(P)\right| \geq\left(\frac{b^{2}}{e^{2} c}\right)^{b}
$$

Proof. Since each of $b$ rows contains at least $b$ elements, there are at leas $b$ ! occurrences of $b$-permutation in $B$. Each permutation avoids $P$ since $B$ avoids $P$. Some of these occurrences correspond to same permutations. Since the width of $B$ is $c$, a given permutation avoiding $P$ can occur at most $\binom{c}{b}$ times.

$$
S_{b}(P) \geq \frac{b!}{\binom{c}{b}} \geq \frac{\left(\frac{b}{e}\right)^{b}}{\left(\frac{c e}{b}\right)^{b}} \geq\left(\frac{b^{2}}{e^{2} c}\right)^{b}
$$

Theorem 3.10. Let $l$ be an integer, such that $\sqrt[7]{l}$ is also an integer. Let $P$ be a permutation matrix. If

$$
S_{l^{10 / 7}}(P)<\left(\frac{l^{6 / 7}}{2 e^{2}}\right)^{l^{10 / 7}}
$$

then

$$
\operatorname{ex}(n, P) \leq n\left(2 l^{27 / 7}+10 l^{24 / 7}+8 l^{2}\right) .
$$

Proof. Firstly, it follows from 2.4, that $S_{n k}(P) \geq S_{k}(P)^{n}$. We can apply this to the statement of the theorem and get

$$
\begin{aligned}
& S_{l}(P)<\left(\frac{l^{6 / 7}}{e^{2}}\right)^{l} \\
& S_{l^{8 / 7}}(P)<\left(\frac{l^{6 / 7}}{e^{2}}\right)^{l^{8 / 7}} .
\end{aligned}
$$

Notice that if $l=1$, theorem holds. Let $A$ be $n \times n$ permutation matrix avoiding $P$. Divide $A$ into blocks of size $2 l^{2} \times 2 l^{2}$ and delete incomplete not square blocks on the right and at the bottom. We will have $\rfloor^{n} / 2 l^{2}\lfloor$ blocks in a row and in a column. We call a block wide if it has at least $l$ nonzero columns, very wide if the number is at least $l^{8 / 7}$, and ultra wide if it is at least $l^{10 / 7}$. Similarly define tall, very tall and ultra tall blocks. Notice that number of nonzero blocks is at most ex $\left(n / 2 l^{2}, P\right)$, because contraction matrix of size $n / 2 l^{2} \times n / 2 l^{2}$ also avoids $P$.

We will count number of entries in these types of blocks separately.

- Deleted blocks have at most $2 \cdot 2 l^{2} n$ elements.
- Neither wide or tall blocks have at most $l^{2}$ elements. There are at most $\operatorname{ex}\left(\left\lfloor n / 2 l^{2}\right\rfloor, P\right)$ of them.
- If the block is ultra wide or ultra tall, it has at most $4 l^{4}$ elements. There are to most $l^{10 / 7}$ ultra wide blocks in each column of blocks. Suppose for a contradiction, that there are at least $l^{10 / 7}$ ultra wide blocks in a column. Contract each of these blocks into a row of length $2 l^{2}$. We will get a $l^{10 / 7} \times 2 l^{2}$ matrix, where each row has at least $l^{10 / 7}$ entries. By lemma 3.9,

$$
S_{l^{10 / 7}}(P) \geq\left(\frac{l^{10 / 7^{2}}}{2 e^{2} l^{2}}\right)^{1^{10 / 7}}=\left(\frac{l^{6 / 7}}{e^{2}}\right)^{l^{210 / 7}} .
$$

That gives a contradiction. The same holds for the number of ultra tall blocks in a row of blocks.

- If the block is very wide or very tall, but not ultra wide or ultra tall, it has at most $l^{20 / 7}$ elements. Now we need to find maximum possible number of very wide blocks in a columns. To do that, contract every wide block in a row with at least $l^{8 / 7}$ elements. Notice that if there are $l^{8 / 7}$ consecutive rows, such that all of their entries are located in $l^{10 / 7}$ columns, this would give an $l^{8 / 7} \times l^{10 / 7}$ matrix, and it would imply that

$$
S_{l^{8 / 7}} P \geq\left(\frac{l^{6 / 7}}{2 e^{2}}\right)^{8^{8 / 7}},
$$

which gives a contradiction. It follows, that in every $l^{8 / 7}$ consecutive rows, there are at least $l^{10 / 7}$ non-zero columns. If we contract each group of $l^{8 / 7}$ consecutive
rows, we will get rows with at least $l^{10 / 7}$. By the previous point, there can not be more than $l^{10 / 7}$ of such rows. Thus, the number of very wide blocks in a column of blocks is at most

$$
l^{8 / 7} \cdot l^{10 / 7}=l^{18 / 7}
$$

Same hold for the number of very tall blocks in a row.

- If the block is wide or tall, but not very wide or very tall, it has at most $l^{16 / 7}$ entries. To count the number of wide blocks in a column of blocks, contract each $l$ consecutive wide blocks into a single row. If all the elements of these $l$ rows are located in $l^{8 / 7}$ columns, that would give a contradiction, since

$$
S_{l}(P) \geq\left(\frac{l^{6 / 7}}{e^{2}}\right)^{l}
$$

Thus, we in the resulting rows, there would be at least $l^{8 / 7}$ elements. By the previous point, there are at most $l^{18 / 7}$ of such rows. Thus, the number of wide blocks in a column of blocks is at most

$$
l \cdot l^{18 / 7}=l^{25 / 7}
$$

Now we are ready to prove the bound by induction on $n$.

$$
\begin{aligned}
\operatorname{ex}(n, P) & \leq 2 \cdot 2 l^{2} n+l^{2} \operatorname{ex}\left(\left\lfloor n / 2 l^{2}\right\rfloor, P\right)+2\left(4 l^{4} \cdot l^{10 / 7}+l^{20 / 7} \cdot l^{18 / 7}+l^{16 / 7} \cdot l^{25 / 7}\right) \frac{n}{2 l^{2}} \\
& \leq 2 l^{2} n+\frac{n}{2}\left(2 l^{27 / 7}+10 l^{24 / 7}+8 l^{2}\right)+n\left(4 l^{24 / 7}+l^{24 / 7}+l^{27 / 7}\right) \\
& \leq n\left(2 l^{27 / 7}+10 l^{24 / 7}+8 l^{2}\right)
\end{aligned}
$$

Proof. (Theorem 3.7) Choose the smallest $l$, such that it is a seventh power of an integer and

$$
l>\left(2 e^{2} L(P)\right)^{7 / 6}
$$

Notice that $l \leq 2^{7}\left(2 e^{2} L(P)\right)^{7 / 6}$. Since $S_{n}(P) \leq(L(P))^{n}$, then

$$
S_{n}(P) \leq\left(\frac{l^{6 / 7}}{2 e^{2}}\right)^{n}
$$

By theorem 3.10, and since $l \leq 2^{7}\left(2 e^{2} L(P)\right)^{7 / 6}$,

$$
c(P) \leq 2 l^{27 / 7}+10 l^{24 / 7}+8 l^{2}=O\left(L(P)^{4.5}\right)
$$

This proof uses similar technique to the one appeared in [18]. In the next section we will review this result.

### 3.4 Marcus and Tardos' proof

In this subsection we give a proof of Füredi-Hajnal conjecture by Adam Marcus and Gábor Tardos [18].

Theorem 3.11. For any permutation matrix $P$, ex $(n, P)=O(n)$.
Proof. Let $P$ have size $k$. Then take $M$ to be a maximum mass $n \times n$ matrix avoiding $P,|M|=\operatorname{ex}(n, P)$. Suppose $k^{2}$ divides $n$.

Consider a contraction matrix $B$ of size $\frac{n}{k^{2}} \times \frac{n}{k^{2}}$ corresponding to partition into blocks $S_{i j}$ of size $k^{2} \times k^{2}$. By lemma 3.1, $B$ avoids $P$.

Consider a block $S_{i j}$. We will call a block wide, if there are at least $k$ out of $k^{2}$ columns with a non-zero entry. We will call a block tall, if there are at least $k$ rows with non-zero entry.

Lemma 3.12. Consider a set of blocks $C_{j}=\left\{S_{i j}, 1 \leq i \leq \frac{n}{k^{2}}\right\}$, jth supercolumn. The number of blocks in $C_{j}$ that are wide is at most $k\binom{k^{2}}{k}$.

Proof. Suppose for a contradiction, that there are more than $k\binom{k^{2}}{k}$ wide blocks. Since it is greater than $\binom{k^{2}}{k}$, there are at least two blocks having 1-entries in the same columns $c_{1}<\ldots<c_{k}$. Applying the pigeonhole principle $k$ times, we will find $k$ blocks having 1 -entries in the columns $c_{1}<\ldots<c_{k}$. This gives a representation of all-ones $k \times k$ matrix. We will find a representation of $P$. That gives a contradiction.

Lemma 3.13. Consider a set of blocks $R_{i}=\left\{S_{i j}, 1 \leq j \leq n\right\}$, ith superrow. The number of tall blocks in $R_{i}$ is at most $k\binom{k^{2}}{k}$.

Proof. Similar to the previous lemma.
Since $B$ avoids $P$, the number of neither wide or tall blocks can be estimated as ex $\left(\frac{n}{k^{2}}, P\right)$.

Now let us estimate the mass of $M$. We will do it by estimating the masses of blocks separately.

- Blocks that are neither wide or tall have at most $(k-1)^{2}$ elements,
- Wide blocks have at most $k^{4}$ elements, which is the total number of elements in $k^{2} \times k^{2}$ matrix,
- Tall blocks have at most $k^{4}$ elements.

In each of the $\frac{n}{k^{2}}$ supercolumns the number of wide blocks is bounded by lemma 3.12. Thus, number of elements which are in the wide block is at most

$$
\frac{n}{k^{2}} k^{4}\binom{k^{2}}{k}
$$

The same estimation works for the number of elements in tall blocks. Now, the total mass of $M$ is at most

$$
\begin{aligned}
\operatorname{ex}(n, m) & \leq \operatorname{ex}\left(\frac{n}{k^{2}}, P\right) \cdot(k-1)^{2}+k\binom{k^{2}}{k} \cdot \frac{n}{k^{2}} \cdot k^{4}+k\binom{k^{2}}{k} \cdot \frac{n}{k^{2}} \cdot k^{4} \\
& =(k-1)^{2} \operatorname{ex}\left(\frac{n}{k^{2}}, P\right)+2 k^{3}\binom{k^{2}}{k} n
\end{aligned}
$$

Solving the recursion above will give linear estimation.
Lemma 3.14. $\operatorname{ex}(n, P) \leq 2 k^{4}\binom{k^{2}}{k} n$.
Proof. Let us prove the statement using induction by $n$. Suppose it is true for all $n^{\prime}<n$. Take $n_{0}$ to be the largest integer, such that $n_{0} \leq n$ and $k^{2}$ divides $n_{0}$. Then apply the induction hypothesis

$$
\begin{aligned}
\operatorname{ex}(n, P) & \leq \operatorname{ex}\left(n_{0}, P\right)+2 k^{2} n \leq(k-1)^{2} e x\left(\frac{n_{0}}{k^{2}}, P\right)+2 k^{3}\binom{k^{2}}{k} n_{0}+2 k^{2} n \\
& \leq(k-1)^{2} \frac{n_{0}}{k^{2}} 2 k^{4}\binom{k^{2}}{k}+2 k^{3}\binom{k^{2}}{k} n_{0}+2 k^{2} n \\
& \leq 2 n\binom{k^{2}}{k} k^{2}\left(1+k+(k-1)^{2}\right) \\
& \leq 2 k^{4}\binom{k^{2}}{k} n .
\end{aligned}
$$

Here the last inequality is true for all $k \geq 2$.
That concludes the proof of the theorem.

### 3.5 Fox' proof

As it was mentioned in the Introduction, the constant factor from the previous proof was improved later by Jacob Fox [9]. We will present his proof in the rest of this section.

The new notations will be used.
Definition 6. The interval contraction of $k$ consecutive rows (columns) replaces them with a new row (column), s.t. the entry is 1 if in at least one row (column) has an element in the same place.

A matrix $P$ is an interval minor of $M$, if some interval contraction contains $P$. Otherwise, $M$ avoids $P$ as an interval minor.

For a matrix $P$, denote as $m(n, P)$ the maximum mass of $n \times n$ matrix avoiding $P$ as an interval minor. For a permutation matrix $P$ this definition is equivalent to avoidance, and ex $(n, P)=m(n, P)$. Denote as $f_{P}(t, s)$ the maximum number $N$, such that there exists an $N \times t$ matrix with at lest $s$ elements in each row, avoiding $P$ as an
interval minor. Also, denote as $g_{P}(t, s)$ the minimum number $N$, such that any $t \times N$ matrix with at least $s$ elements in each column avoids $P$ as an interval minor.

Firstly, we generalize the construction of Marcus and Tardos in the following lemma:

Lemma 3.15. For any positive integers $s, t, n, s \leq t$ and matrix $P$,

$$
m(t n, P) \leq m(s-1, P) m(n, P)+m(t, P) f_{P}(t, s) n++m(t, P) g_{P}(t, s) n .
$$

Proof. Let $A$ be $t n \times t n$ maximum mass matrix avoiding $P$ as an interval minor. Divide $A$ into blocks of size $t \times t$ and obtain a contraction matrix $B$ of size $n \times n$. $B$ also avoids $P$ as an interval minor. Call a block wide (tall), if it has at least $s$ non-zero rows (columns). The number of blocks that are neither wide or tall is at most $m(n, P)$ and they have at most $m(s-1, P)$ elements, since each block avoids $P$ as well.

Notice that each supercolumn has at most $f_{P}(t, s)$ wide blocks, because otherwise the contraction of rows in each block would give a matrix that contains $P$ as an interval minor by definition of $f_{P}(t, s)$. Same holds for the number of tall blocks in each superrow: it is at most $g_{P}(t, s)$. There are $n$ superrows and supercolumns. Each wide or tall block has at most $m(t, P)$ elements. Together it gives the following estimation:

$$
m(t n, P) \leq \underbrace{m(s-1, P) m(n, P)}_{\text {neither wide or tall }}+\underbrace{m(t, P) f_{P}(t, s) n}_{\text {wide }}+\underbrace{m(t, P) g_{P}(t, s) n}_{\text {tall }} .
$$

Notice that in the lemma below we did not require matrix $P$ to be a permutation matrix. This is because of the difference between the usual avoiding a matrix and avoiding as an interval minor.

The plan of the proof of Füredi-Hajnal conjecture is to upper bound the extremal function $m\left(n, J_{k}\right)$ for all-ones matrix of size $k$, denoted as $J_{k}$, with exponential bound. Then, since $J_{k}$ contains all possible $k$-permutations, the result will follow. Denote as $J_{r, k}$ all-ones matrix of size $r \times k$. The proof of next lemmas are rather technical.

Lemma 3.16. Let $s \leq t$ be positive integers and $t$ be even. Then,

$$
f_{r, k}(t, s) \leq 2 f_{r, k}\left(\frac{t}{2}, s\right)+2 f_{r, k-1}\left(\frac{t}{2}, \frac{s}{2}\right),
$$

where $f_{r, k}(t, s)=f_{J_{r, k}}(t, s)$.
Proof. Let $A$ be $N \times t$ matrix avoiding $J_{r, k}$ as an interval minor with at least $s$ ones in each row, where $N=f_{r, k}(t, s)$. Divide $A$ into two supercolumns of width $t / 2$. There are three types of rows in $A$ : (1) rows, that has elements only in the first supercolumn, (2) rows with elements only in the second supercolumn, (3) rows with elements in both supercolumns. There are at most $f_{r, k}(t / 2, s)$ elements in rows of type (1) and (2).

For the third type, consider only rows, that have at least $s / 2$ elements in the first supercolumn. Then contract the last $t / 2$ columns into one to get a matrix of width $t / 2+1$. There are at least $s / 2+1$ non-zero entries and the last element should be one.

That means, that there are at most $f_{r, k-1}\left(\frac{t}{2}, \frac{s}{2}\right)$ such rows. Same argument holds for rows, have at least $s / 2$ elements in the second supercolumn. Now all the rows are considered and that gives the desired estimation:

$$
f_{r, k}(t, s) \leq 2 f_{r, k}\left(\frac{t}{2}, s\right)+2 f_{r, k-1}\left(\frac{t}{2}, \frac{s}{2}\right) .
$$

Lemma 3.17. Let $2^{k-1} \leq s \leq t$ be integers and $t$ is a power of 2 . Then

$$
f_{r, k}(t, s) \leq r 2^{k-1} \frac{t^{2}}{s}
$$

Proof. The proof uses induction on $k, t$ and lemma 3.16.
Base: $f_{r, 1}(t, s)=r \leq r \frac{t^{2}}{s}$.
Ind. step: Suppose it holds for all $k^{\prime}<k$ or $t^{\prime}<t$. Then,

$$
f_{r, k}(t, s) \leq 2 f_{r, k}\left(\frac{t}{2}, s\right)+2 f_{r, k-1}\left(\frac{t}{2}, \frac{s}{2}\right) \leq r 2^{k} \frac{t^{2}}{4 s}+r 2^{k-1} \frac{t^{2}}{2 s}=r 2^{k-1} \frac{t^{2}}{s}
$$

Theorem 3.18. $m\left(n, J_{n}\right) \leq k 2^{8 k} n$.
Proof. To see how recursion works, apply lemma 3.15 with $P=J_{k}, t=2^{2 k}$ and $s=2^{k-1}$. We can use trivial bounds $m(s-1, P) \leq s^{2}$ and $m(t, P) \leq t^{2}$ in the formula. Also, since $J_{k}$ is symmetric, $f_{J_{k}}(t, s)=g_{J_{k}}(t, s)$. Together we get an estimate

$$
m\left(2^{2 k} n, J_{k}\right) \leq 2^{2 k-2} m\left(n, J_{k}\right)+2 t^{2} f_{J_{k}}\left(2^{2 k}, 2^{k-1}\right) n .
$$

Apply lemma 3.17 with $r=k$ :

$$
m\left(2^{2 k} n, J_{k}\right) \leq 2^{2 k-2} m\left(n, J_{k}\right)+2 k 2^{8 k} n .
$$

Now we can write this recursion for $m\left(n, J_{k}\right)$ and iterate this process for $s$ steps, where $s$ is such that $n 2^{-2 k s} \leq 2^{2 k}$ :

$$
\begin{aligned}
m\left(n, J_{k}\right) & \leq 2^{2 k-2} m\left(n 2^{-2 k}, J_{k}\right)+2 k 2^{8 k} n 2^{-2 k} \\
& \leq 2^{4 k-4} m\left(n 2^{-4 k}, J_{k}\right)+2 k 2^{6 k-2} n+2 k 2^{6 k} n \\
& \leq 2 k 2^{6 k} n \underbrace{\left(1+\frac{1}{4}+\frac{1}{4^{2}}+\ldots\right)}_{s \text { terms }}+\left(2^{2 k-2}\right)^{s} m\left(n 2^{-2 k s}, J_{k}\right) \\
& \leq \frac{4}{3} 2 k 2^{8 k} n+n m\left(2^{2 k}, J_{k}\right) \leq n\left(\frac{4}{3} 2 k 2^{8 k}+2^{4 k}\right) \\
& \leq 3 n k 2^{8 k} .
\end{aligned}
$$

The Füredi-Hajnal conjecture follows from the fact that for permutation matrix $P$, $e x(n, P)=m(n, P) \leq m\left(n, J_{k}\right)$, where $k$ is the size of $P$.
Corollary 3.18.1. For any permutation matrix $P, c(P) \leq O\left(k 2^{8 k}\right)$.

## 4 Light matrices

We will call an $k \times s$ matrix light if there is only one element in each column and no empty rows. Denote light matrix with $s$ elements as $L_{s}$. Permutation matrices are light in both rows and columns. In this section we show the bounds of extremal function for light matrices. First, we will prove the bound for matrices of size four and five. Then, we will prove the general case for matrices of size $s$. The deduction in these three cases will be similar and consist of the following two steps: we explain the construction giving the recursive formula and solve the recurrence.

To start, consider two trivial cases when the matrix has only two or three elements.
The are only three different light matrices of size two:

$$
\left(\begin{array}{lll}
\bullet & \bullet
\end{array}\right), \quad\left(\begin{array}{lll}
\bullet & \\
& \bullet
\end{array}\right), \quad(\bullet \bullet)
$$

Matrix of size $n \times m$ avoiding any of these three matrices can have at most $n+m$ elements. Indeed, apply lemma 3.2

Thus,

$$
\operatorname{ex}\left(n, m, L_{2}\right) \leq n+m .
$$

For light matrices of size three, there are more options. All permutations of size three are light. Besides them, there are three more different matrices:

$$
\left(\begin{array}{lll}
\bullet & & \bullet \\
& \bullet &
\end{array}\right), \quad\left(\begin{array}{llll}
\bullet & & \\
& \bullet & \bullet
\end{array}\right), \quad(\bullet \bullet \bullet)
$$

and their symmetric matrices. Thus, notice that

$$
\operatorname{ex}\left(n, m, L_{3}\right) \leq 2(n+m)
$$

It hold for permutation matrices by lemma 3.3. It also holds for matrices above by lemma 2.9 and reduction rules.

To work with light matrices of a bigger sizes, we need the definition and some properties of the inverse Ackermann function.

### 4.1 Inverse Ackermann function

Denote

$$
f^{[k]}(x)=\underbrace{f(f(\ldots f(x) \ldots))}_{k \text { times }} .
$$

Definition 7. $\log ^{*}(n)$ denotes the number of times we need to apply $\log _{2}(x)$ until the result is at most 2 ,

$$
\log ^{*}(n)=\min \left\{k: \log ^{[k]} n \leq 2\right\} .
$$

The next proposition follows from the definition.

## Proposition 4.1.

> (1) $\log ^{*} n \leq \log n$
> (2) $\log ^{[k] *} n \leq \log ^{[k-1] *} n$
> (3) $\log ^{*}(\log (n))=\log ^{*}(n)-1$.

Definition 8. We define inverse Ackermann function the following way:

$$
\alpha(n, m)=\min \left\{k: \log ^{[k] *} m \leq \frac{n}{m}+1\right\},
$$

where

$$
\log ^{[k] *} n=\log ^{* \ldots *} n .
$$

It differs from the standard definition by a small constant. Denote $\alpha(n)=\alpha(n, n)=$ $\min \left\{k: \log ^{[k] *} n \leq 2\right\}$.

It turns out that if we slightly change the definition of a star function it will not affect the inverse Ackermann function. In particular, consider the following definition.

## Definition 9.

$$
\alpha_{(s)}(n, m)=\min \left\{k:\left(\log ^{s} m\right)^{[k] *} \leq \frac{n}{m}+1\right\} .
$$

In the Definition 9 instead of $\log ^{[k] *} m$ we use $\left(\log ^{s} m\right)^{[k] *}$. Then, the next theorem holds.

Theorem 4.2. $\alpha_{(s)}(n) \leq \alpha(n)+O(1)$ for big enough $n$.
We might apply this result in the where it seems more relevant to work with power of logarithm. The rest of the subsection is dedicating to proving theorem 4.2.

Lemma 4.3. For any $s \geq 2$ and any $k \geq 1$,

$$
\left(\log ^{s}(n)\right)^{[k]} \leq\left(s^{3} \log ^{[k]}(n)\right)^{s} .
$$

Proof. Fix $s \geq 2$. We will prove the statement by induction on $k$. In the base case $k=1$ the inequality is trivial. Suppose it holds for all $k^{\prime}<k$. Then,

$$
\begin{aligned}
\left(\log ^{s}(n)\right)^{[k]} & \leq \log ^{s}\left(\left(\log ^{s}(n)\right)^{[k-1]}\right) \leq\left(\log \left(s^{3 s}\left(\log ^{[k-1]}(n)\right)^{s}\right)\right)^{s} \\
& =\left(3 s \log s+s \log ^{[k]}(n)\right)^{s} \\
& \leq(3 s \log s+s)^{s}\left(\log ^{[k]} n\right)^{s} \\
& \leq\left(s^{3}\right)^{s}\left(\log ^{[k]} n\right)^{s} .
\end{aligned}
$$

Corollary 4.3.1. If $s \geq 2,\left(\log ^{s} n\right)^{*} \leq 2 \log ^{*} n$ for big enough $n$.

Proof. Substitute $k=\log ^{*}(n)$ into the previous theorem.
By definition, $\log ^{\left[\log ^{*}(n)\right]}(n) \leq 2$. Then,

$$
\left(\log ^{s}(n)\right)^{\left[\log ^{*}(n)\right]} \leq\left(2 s^{3}\right)^{s} .
$$

It follows that $\log ^{*} n \leq\left(\log ^{s} n\right)^{*}$ since we still need to apply $\log ^{s}$ a number of times until the result is at most 2 .

$$
\left(\log ^{s}(n)\right)^{*} \leq \log ^{*}(n)+\left(\log ^{s}\left(2^{s} s^{3 s}\right)\right)^{*} \leq 2 \log ^{*} n,
$$

for big enough $n$, since $s$ is a constant.
We can show the similar inequality for any number of stars.
Lemma 4.4. For integer $k, s \geq 2$ and big enough $n$

$$
\left(\log ^{s} n\right)^{* \ldots *} \overbrace{}^{k} \leq 2 \log ^{* \ldots *} n
$$

Proof. Let us prove the statement by induction on $k$. The base case follows from corollary 4.3.1. Suppose it is true for $k^{\prime}<k$. Then

$$
\left(\log ^{s} n\right)^{[k] *} \leq\left(\left(\log ^{s} n\right)^{[k-1] *}\right)^{*} \leq\left(2 \log ^{[k-1] *} n\right)^{*} \leq 2 \log ^{[k] *} n .
$$

Proof. (Theorem 4.2) Substitute $k=\alpha(n)$ into theorem 4.4. Then $\log ^{[k] *} n \leq 2$ and

$$
\left(\log ^{s} n\right)^{[k] *} \leq 4 .
$$

Notice that the same holds for general definition $\alpha(n, m)$ for big enough $m$ and $n$. The proof repeats the reasoning above. We can choose $m$ to be such that $n / m \leq 1$.

### 4.2 Matrix of size 4

In this subsection we present the proof of quasi-linear bound for light patterns of size four. The proof will be divided into two main steps: derivation of recurrence, and solving the recurrence.

Let $L_{4}$ be light matrix of size 4.
Theorem 4.5. $\operatorname{ex}\left(n, m, L_{4}\right)=O(n \alpha(n, m))$.
Proof. The goal of the first step of the proof is to derive the recursive formula for ex $\left(n, m, L_{4}\right)$. Let $M$ be an $n \times m$ matrix of maximum mass avoiding $L_{4},|M|=$ ex $\left(n, m, L_{4}\right)$. Let $B>0$ and divide $M$ into $m / B$ supercolumns: $i$ th supercolumn is a block $M_{k l}, 1 \leq k \leq n$ and $\left.i-1\right) \cdot m / B+1 \leq l \leq i \cdot m / B$. We call an element $M_{k l}$ local, if there are no elements in the row $i$ outside of the supercolumn of column $j$. Notice
that if the element is local, all elements in its row are also local. We will call such rows local. The are at most $B$ elements in a local row. Denote the number of local rows, which element lie in $i$ th supercolumn as $n_{i}$.
The rest of the rows are called global, the number is denoted as

$$
n_{g}=n-\sum_{i=1}^{\frac{m}{B}} n_{i} .
$$

An element is called first, if it's in a global row, and there are no elements in the supercolumns before. It means it is in the first non-zero block in its row. An element is last, if the are no elements in the supercolumns after. The rest of 1 entries in global rows are called middle. See figure 1.


Figure 1: The local row contains elements in the second supercolumn. The global row has first, last and middle elements. There are no elements in the supercolumns before first or after last.

We will estimate $|M|$ by considering local first, last and middle elements separately. The number of local elements is estimated as

$$
\mid \text { Local } \left\lvert\, \leq \sum_{i=1}^{\frac{m}{B}} \operatorname{ex}\left(n_{i}, B, L_{4}\right) .\right.
$$

Indeed, there are $n_{i}$ local rows non-zero in $i$ th supercolumn. They form a submatrix of size $n_{i} \times B$, which must avoid $L_{4}$.

Now we can focus on the rest of the matrix. First, last and middle elements use $n_{g}$ rows. Notice that a matrix can not be sum decomposable and skew-sum decomposable at the same time. Without loss of generality, suppose $L_{4}$ is not sum decomposable.

To estimate the number of first entries, consider a block-diagonal matrix $D_{f}$ : the sum of supercolumns of $M$, containing only first elements. Delete all zero rows from $D_{f}$. Notice that $D_{f}$ has size $n_{g} \times m$. Indeed, it has the same number of columns as $M$.

The number of nonzero rows is the number of rows, containing first elements, which is $n_{g}$.

Let $P_{f}$ be the matrix constructed the following way: delete the last column from $P$ with a single element. In case the first of last row of $P_{f}$ has no entries, add an element next to the one from the second or second to last row. For example,

$$
P=\left(\begin{array}{llll} 
& & & \bullet \\
& \bullet & & \\
\bullet & & \bullet &
\end{array}\right), \quad P_{f}=\left(\begin{array}{lll} 
& \bullet & \\
& \bullet & \\
\bullet & & \bullet
\end{array}\right)
$$

Then, the following lemma holds.
Lemma 4.6. $D_{f}$ avoids $P_{f}$.
Proof. Suppose for a contradiction $D_{f}$ contains $P_{f}$. Since $L_{4}$ is not sum decomposable, $P_{f}$ is also not sum decomposable. Then, $D_{f}$ must contains $P_{f}$ in a single block (supercolumn). Consequently, the original matrix $M$ also contains $P_{f}$ in a single supercolumn. Consider the element of $D_{f}$ corresponding to the row of deleted right entry of $L_{4}$. In the original matrix $M$ there would be an element to the right in the same row (from the last block). Thus, that element together with realization of $P_{f}$ that would give a realization of $L_{4}$.

In case $P_{f}$ has no empty rows, it is a light matrix of size three and

$$
\mid \text { First } \mid \leq \operatorname{ex}\left(n_{g}, m, L_{3}\right) \leq 2\left(n_{g}+m\right)
$$

In case $P_{f}$ has an empty row in the middle, we can apply lemma 2.8 with $k=1$ and get

$$
\mid \text { First } \left\lvert\, \leq 2 \mathrm{ex}\left(\frac{n_{g}}{2}, m, L_{3}\right)+2 m \leq 2 n_{g}+6 m\right.
$$

In case a new element was added into the first or the last row of $P_{f}$, use lemma 2.6 and get

$$
\mid \text { First } \mid \leq \operatorname{ex}\left(n_{g}, m, L_{3}\right)+m \leq 2 n_{g}+3 m
$$

The second estimate is more general. Thus, we will use it.
Analogously, by considering block diagonal matrix $D_{l}$ containing only last elements, we can get estimate

$$
\mid \text { Last } \mid \leq 2 n_{g}+6 m
$$

This follows from the similar reasoning, since the $D_{l}$ will avoid a pattern constructed by deleting the first column of $L_{4}$ and adding an element in the first or last row if needed.

To estimate the number of middle elements, consider a block diagonal matrix $D_{m}$ : the sum of supercolumns of $M$, containing only middle elements. Notice that $D_{m}$ must avoid the matrix $P_{m}$, which is constructed by deleting the first and the last columns of $L_{4}$. If $P_{m}$ has zero top or bottom rows, add elements next to the existing ones.

Lemma 4.7. $D_{m}$ avoids $P_{m}$.

Proof. Indeed, if $D$ contains it, it must contain it in a single block. Thus, $M$ would also contain the same permutation. Since $D$ consists of only middle elements, there is at least 1 element to the right and to the left for each entry of $D$. Thus, $M$ would contain $L_{4}$.

Lemma 4.8. $D_{m}$ has size $n^{\prime} \times m$, where $n^{\prime} \leq \operatorname{ex}\left(n, m / B, L_{4}\right)$.
Proof. Suppose $D_{m}$ has more than ex $\left(n_{g}, m / B, L_{4}\right)$ rows. Let $M^{\prime}$ be contraction matrix of size $n / B \times m$, corresponding to the division of into supercolumns. Each element of $M^{\prime}$ corresponds to a row of matrix $D_{m}$. As $M^{\prime}$ is a contraction matrix, it avoids $L_{4}$. This holds, since we did not contract any rows, and $L_{4}$ is light. Thus, if $D_{m}$ has more than $\operatorname{ex}\left(n_{g}, m / B, L_{4}\right)$ rows, matrix $M^{\prime}$ has mass greater than $\operatorname{ex}\left(n_{g}, m / B, A\right)$, which gives a contradiction.

To get the most general bound on the mass of $D_{m}$ we need to apply lemma 2.8 with $k=1$ two times. Then,

$$
\mid \text { Middle } \left\lvert\, \leq 4 \mathrm{ex}\left(\frac{n^{\prime}}{4}, m, L_{2}\right)+6 m=n^{\prime}+10 m .\right.
$$

For simplicity, denote $T(n, m)=\operatorname{ex}(n, m, P)$. The resulting recurrence formula is

$$
\begin{align*}
T(n, m) & \leq \sum_{i=1}^{\frac{m}{B}} T\left(n_{i}, B\right)+4 n_{g}+12 m+T\left(n_{g}, \frac{m}{B}\right)+10 m \\
& =\sum_{i=1}^{\frac{m}{B}} T\left(n_{i}, B\right)+4 n_{g}+22 m+T\left(n_{g}, \frac{m}{B}\right) \tag{1}
\end{align*}
$$

This concludes the first step of the proof. The rest of the subsection is dedicated to solving recurrence 1 . Lemmas 4.9 and 4.10 are intermediate results, which are base cases for the main result, lemma 4.11.

Lemma 4.9. $T(n, m) \leq 12(n+m \log m)$.
Proof. We prove using induction by $n$ and $m$. The inequality is trivial in base cases, when $n, m \leq 4$. Now suppose it is true for all $n^{\prime}<n$ and for $m^{\prime}<m$ if $n^{\prime}=n$. Choose $B=m / 2$. Now we only have two blocks and there are no middle elements. Then, by induction hypothesis,

$$
\begin{aligned}
T(n, m) & \leq T\left(n_{1}, B\right)+T\left(n_{2}, B\right)+4 n_{g}+12 m \\
& \leq 12\left(n_{1}+B \log B\right)+12\left(n_{2}+B \log B\right)+4 n_{g}+12 m .
\end{aligned}
$$

Notice that $n_{1}+n_{2}+n_{g}=n$, and thus

$$
\begin{aligned}
T(n, m) & \leq 12\left(n+m+m \log \left(\frac{m}{B}\right)\right) \\
& =12(n+m(1+(\log m-\log 2)))=12(n+m \log m)
\end{aligned}
$$

Lemma 4.10. $T(n, m) \leq 34\left(n+m \log ^{*} m\right)$.
Proof. Use induction by $n$ and $m$. The inequality is trivial in base cases, when $n, m \leq 4$. Now suppose it is true for all $n^{\prime}<n$ and for $m^{\prime}<m$ if $n^{\prime}=n$. Here let us substitute $B=\log m$.

$$
T(n, m) \leq \underbrace{\sum_{i=1}^{m / B} T\left(n_{i}, B\right)}_{\text {local }}+4 n_{g}+22 m+\underbrace{T\left(n_{g}, \frac{m}{B}\right)}_{\text {middle }}
$$

Apply induction hypothesis to the terms corresponding to the local elements, and lemma 4.9 to the last term.

$$
\begin{aligned}
T(n, m) \leq & \sum_{i=1}^{m / B} 34\left(n_{i}+B \log ^{*} B\right)+4 n_{g}+22 m+12\left(n_{g}+\frac{m}{B} \log \frac{m}{B}\right) \\
\leq & 34\left(n-n_{g}+m \log ^{*}(\log m)\right)+16 n_{g}+22 m \\
& \quad+\frac{12 m}{\log m}(\log m-\log \log m) \\
\leq & 34\left(n+m\left(1+\log ^{*}(\log m)\right)\right) \leq 34\left(n+m \log ^{*} m\right),
\end{aligned}
$$

where the last inequality is true by the properties of $\log ^{*}$ function.
We can apply the same reasoning to get the following estimation.
Lemma 4.11. For every $k \in \mathbb{N}, T(n, m) \leq 22(k+1)\left(n+m \log ^{[k] *} m\right)$, where $[k] *$ denotes $k$ stars.

Proof. We will prove that by induction on $k$, where base case is covered by lemma 4.10. Suppose the statement is true for all $k^{\prime}<k$. We will show it for $k$ using the same induction by $n, m$.

Choose $B=\log ^{[k-1] *} m$. Now

$$
T(n, m) \leq \sum_{i=1}^{\frac{m}{B}} T\left(n_{i}, B\right)+4 n_{g}+22 m+T\left(n_{g}, \frac{m}{B}\right)
$$

By induction hypothesis,

$$
\begin{aligned}
T\left(n_{i}, B\right) & \leq 22(k+1)\left(n_{i}+B \log ^{[k] *} B\right), \\
T\left(n_{g}, \frac{m}{B}\right) & \leq 22 k\left(n_{g}+\frac{m}{B} \log ^{(k-1) *} \frac{m}{B}\right) .
\end{aligned}
$$

Substitute that into estimation:

$$
\begin{aligned}
T(n, m) \leq & \sum_{i=1}^{\frac{m}{B}} 22(k+1)\left(n_{i}+B \log ^{[k] *} B\right)+4 n_{g}+22 m \\
& +22 k\left(n_{g}+\frac{m}{B} \log ^{(k-1) *} \frac{m}{B}\right) \\
\leq & 22(k+1)\left(n-n_{g}\right)+22(k+1) m \log ^{[k] *} B+4 n_{g}+22 m \\
& \quad+22 k n_{g}+22 k m \\
\leq & 22(k+1) n+22(k+1) m\left(1+\log ^{[k] *}\left(\log ^{[k-1] *} m\right)\right) \\
\leq & 22(k+1)\left(n+m\left(1+\log ^{[k] *} m-1\right)\right)=22(k+1)\left(n+m \log ^{[k] *} m\right) .
\end{aligned}
$$

Now substitute $k=\alpha(n, m)$ into lemma 4.11. Then

$$
\operatorname{ex}\left(n, m, L_{4}\right)=O(n \alpha(n, m)) .
$$

That finishes the proof of theorem 4.5.

### 4.3 Matrix of size 5

Let $L_{5}$ be a light matrix of size 5 .
Theorem 4.12. ex $\left(n, m, L_{5}\right)=O\left(2^{\alpha(n, m)} n\right)$.
Proof. Let $M$ be a maximum mass $n \times m$ matrix avoiding $L_{5}$. The proof is similar the theorem 4.5. First we divide matrix $M$ into supercolumns of width $B$. Define local, first, last and middle elements with respect to this division in a similar manner. Let $n_{i}$ denote the number of local rows with elements in supercolumn $i$, and $n_{g}$ denote the number of global rows. Recall that

$$
\sum_{i=1}^{n / B} n_{i}+n_{g}=n
$$

We will estimate $|M|$ by considering local first, last and middle elements separately. The number of local entries is at most

$$
\mid \text { Local } \mid \leq \sum_{i=1}^{m / B} \operatorname{ex}\left(n_{i}, B, L_{2}\right)
$$

because each supercolumn has $n_{i}$ local elements, they must avoid $P$. The number of first entries is at most

$$
\mid \text { First } \left\lvert\, \leq 2 \mathrm{ex}\left(\frac{n_{g}}{2}, m, L_{4}\right)+2 m .\right.
$$

For the last,

$$
\mid \text { Last } \left\lvert\, \leq 2 \operatorname{ex}\left(\frac{n_{g}}{2}, m, L_{4}\right)+2 m .\right.
$$

The first and last elements avoid a light matrix of size 4 with an empty row in the middle in the worst case.

To estimate the number of middle entries, again consider a block diagonal matrix $D_{m}$ and use similar argument. The size of $D_{m}$ is $n^{\prime} \times m$, where $n^{\prime} \leq \operatorname{ex}\left(n_{g}, m / B, L_{5}\right)$. In the worst case there will be two empty rows in the middle. Then, apply lemma 2.8 two times. The number of middle elements is at most

$$
\mid \text { Middle } \left\lvert\, \leq 4 \mathrm{ex}\left(\frac{n^{\prime}}{4}, m, L_{3}\right)+6 m \leq 8\left(\frac{n^{\prime}}{4}+m\right)+6 m \leq 2 n^{\prime}+14 m .\right.
$$

For simplicity, denote $T(n, m)=\operatorname{ex}\left(n, m, L_{5}\right)$. In total it gives:

$$
T(n, m) \leq \sum_{i=1}^{m / B} T\left(n_{i}, B\right)+4 \operatorname{ex}\left(\frac{n_{g}}{2}, m, L_{4}\right)+4 m+2 T\left(n_{g}, \frac{m}{B}\right)+14 m
$$

Lemma 4.13. $T(n, m) \leq 52\left(n+m \log ^{2} m\right)$.
Proof. Use induction on $n, m$. The base cases $n, m \leq 4$ are trivial. Substitute $B=\frac{m}{2}$. In that case the first sum has only two terms and there are no middle elements. Use induction hypothesis for the first terms.

$$
\begin{aligned}
T(n, m) & \leq T\left(n_{1}, B\right)+T\left(n_{2}, B\right)+4 \operatorname{ex}\left(\frac{n_{g}}{2}, m, L_{4}\right)+4 m \\
& \leq 52\left(n-n_{g}+m \log ^{2} \frac{m}{2}\right)+4 m+48\left(\frac{n_{g}}{2}+m \log m\right) \\
& \leq 52 n+52 m\left(\log ^{2} \frac{m}{2}+\log m\right) \\
& \leq 52\left(n+m \log ^{2} m\right) .
\end{aligned}
$$

Now let us consider the next step.
Lemma 4.14. $T(n, m) \leq 258\left(n+m\left(\left(\log ^{2} m\right)^{*}\right)^{2}\right)$.
Proof. Use induction on $n, m$. The base cases $n, m \leq 4$ are trivial. Substitute
$B=\log ^{2} m$.

$$
\begin{array}{rlr}
T(n, m) \leq & \sum_{i=1}^{m / B} T\left(n_{i}, B\right)+4 \operatorname{ex}\left(\frac{n_{g}}{2}, m, L_{4}\right)+4 m+2 T\left(n_{g}, m / B\right)+14 m & \\
\leq & 258\left(n-n_{g}+m\left(\left(\log ^{2} B\right)^{*}\right)^{2}\right) & \text { \{local\} } \\
& +4 m+34 \cdot 4\left(\frac{n_{g}}{2}+m \log ^{*} m\right) & \text { \{first and last\} } \\
& +52 \cdot 2\left(n_{g}+\frac{m}{B} \log ^{2} \frac{m}{B}\right)+14 m & \text { \{middle\} } \\
\leq & 258 n+122 m+258\left(m\left(\left(\log ^{2} B\right)^{*}\right)^{2}\right)+136\left(m \log ^{*} m\right) & \\
\leq & 258\left(n+m\left(\left(\log ^{2}\left(\log ^{2} m\right)\right)^{*}\right)^{2}+\log ^{*} m\right) & \\
\leq & 258\left(n+m\left(\left(\left(\log ^{2} m\right)^{*}-1\right)^{2}+\log ^{*} m\right)\right) \\
\leq & 258\left(n+m\left(\left(\log ^{2} m\right)^{*}\right)^{2}\right) .
\end{array}
$$

Lemma 4.15. $T(n, m) \leq 980 \cdot 2^{k}\left(n+m\left(\left(\log ^{2} m\right)^{[k] *}\right)^{2}\right)$.
Proof. We will prove the stronger inequality. Define the sequence

$$
A_{1}=258, \ldots A_{k}=2 A_{k-1}+88(k+1)+18
$$

We will show that $T(n, m) \leq A_{k}\left(n+m\left(\log ^{[k] *} m\right)^{2}\right)$. The solution to the recursion above is

$$
A_{k}=980 \cdot 2^{k}-176 k-56 \leq 980 \cdot 2^{k}
$$

which gives the estimate from the statement.
Prove the statement using induction on $k$. The base case is described in the previous lemma. Use induction on $n, m$. Substitute $B=\left(\left(\log ^{2} m\right)^{[k-1] *}\right)^{2}$. For the local terms, use induction on $n, m$. For the first and last terms, use estimate from lemma 4.11, since first and last elements avoid $L_{4}$. For the $T\left(n_{g}, m / B\right)$ use induction on $k$. Thus, by induction hypothesis,

$$
\begin{aligned}
T\left(n_{i}, B\right) & \leq A_{k}\left(n_{i}+B\left(\left(\log ^{2} B\right)^{[k] *}\right)^{2}\right) \\
\operatorname{ex}\left(\frac{n_{g}}{2}, m, L_{4}\right) & \leq 22(k+1)\left(\frac{n_{g}}{2}+m \log ^{[k] *} m\right), \\
T\left(n_{g}, m / B\right) & \leq A_{k-1}\left(n_{g}+\frac{m}{B}\left(\left(\log ^{2} \frac{m}{B}\right)^{[k-1] *}\right)^{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
T(n, m) \leq & \sum_{i=1}^{m / B} T\left(n_{i}, B\right)+4 \operatorname{ex}\left(\frac{n_{g}}{2}, m, L_{4}\right)+4 m+2 T\left(n_{g}, m / B\right)+14 m \\
\leq & A_{k}\left(n-n_{g}+m\left(\left(\log ^{2} B\right)^{[k] *}\right)^{2}\right) \\
& +88(k+1)\left(\frac{n_{g}}{2}+m \log ^{[k] *} m\right)+4 m \\
& +2 A_{k-1}\left(n_{g}+\frac{m}{B}\left(\left(\log ^{2} \frac{m}{B}\right)^{[k-1] *}\right)^{2}\right)+14 m \\
\leq & A_{k}\left(n-n_{g}\right)+44(k+1) n_{g}+2 A_{k-1} n_{g} \\
& +A_{k} m\left(\left(\log ^{2} m\right)^{[k] *}-1\right)^{2} \\
& +88(k+1) m \log ^{[k] *} m+2 A_{k-1} m+18 m \\
\leq & A_{k} n+A_{k} m\left(\left(\log ^{2} m\right)^{[k] *}-1\right)^{2} \\
& +\left(88(k+1)+2 A_{k-1}+18\right) m \log ^{[k] *} m \\
= & A_{k} n+A_{k} m\left(\left(\left(\log ^{2} m\right)^{[k] *}-1\right)^{2}+\log ^{[k] *} m\right) \\
\leq & \left.A_{k}\left(n+m\left(\left(\log ^{2} m\right)^{[k] *}\right)\right)^{2}\right)
\end{aligned}
$$

Now substitute $k=\alpha_{(2)}(n, m)$ into the previous lemma. We will get

$$
\operatorname{ex}\left(n, m, L_{2}\right) \leq O\left(2^{\alpha_{(2)}(n, m)} n\right)=O\left(2^{\alpha(n, m)} n\right)
$$

That concludes the proof.

### 4.4 General case

Let $L_{s}$ be light matrix of size $s$.
Theorem 4.16. ex $\left(n, m, L_{s}\right) \leq O\left(2^{(5 \alpha(n, m))^{(s-3)}} n\right)$.
Proof. For simplicity denote $T_{s}(n, m)=\operatorname{ex}\left(n, m, L_{s}\right)$. Using the same construction
as in theorems 4.5 and 4.12 we get the recursive formula.

$$
\begin{aligned}
& \mid \text { Local } \mid \leq \sum_{i=1}^{m / B} T_{s}\left(n_{i}, B\right), \\
& \mid \text { First } \left\lvert\, \leq 2 T_{s-1}\left(\frac{n_{g}}{2}, m\right)+2 m\right., \\
& \mid \text { Last } \left\lvert\, \leq 2 T_{s-1}\left(\frac{n_{g}}{2}, m\right)+2 m\right., \\
& \mid \text { Middle } \left\lvert\, \leq 4 T_{s-2}\left(\frac{n^{\prime}}{4}, m\right)+6 m\right., \text { where } n^{\prime} \leq \operatorname{ex}\left(n_{g}, \frac{m}{B}, L_{s}\right) . \\
& T_{s}(n, m) \leq \sum_{i=1}^{m / B} T_{s}\left(n_{i}, B\right)+4 T_{s-1}\left(\frac{n_{g}}{2}, m\right)+4 m \\
& \quad+4 T_{s-2}\left(\frac{n^{\prime}}{4}, m\right)+6 m .
\end{aligned}
$$

We will prove 3 analogous lemmas by induction on $s$.
Lemma 4.17. $T(n, m) \leq 3 \cdot 5^{s-3}\left(n+m \log ^{s-3} m\right)$.
Proof. The base case when $s=5$ follows from lemma 4.13. Use induction on $n, m$. Substitute $B=m / 2$. In that case there are no middle elements.

$$
\begin{aligned}
T_{s}(n, m) & \leq T_{s}\left(n_{1}, B\right)+T_{s}\left(n_{2}, B\right)+4 T_{s-1}\left(\frac{n_{g}}{2}, m, L_{s-1}\right)+4 m \\
& \leq 3 \cdot 5^{s-3}\left(n-n_{g}+m \log ^{s-3} B\right)+12 \cdot 5^{s-4}\left(\frac{n_{g}}{2}+m \log ^{s-4} m\right)+4 m \\
& \leq 3 \cdot 5^{s-3}\left(n+m\left(\log ^{s-3} \frac{m}{2}+\log ^{s-4} m\right)\right) \\
& \leq 3 \cdot 5^{s-3}\left(n+m \log ^{s-3} m\right) .
\end{aligned}
$$

Lemma 4.18. $T_{s}(n, m) \leq 2^{5^{s-3}}\left(n+m\left(\left(\log ^{s-3} m\right)^{*}\right)^{s-3}\right)$.
Proof. The base case is $s=5$, it follows from lemma 4.14. We will use induction on $n, m$ to estimate local terms and induction on $s$ to estimate first and last. For the
middle terms we will use induction on $s$ and lemma 4.17. Substitute $B=\log ^{s-3} m$.

$$
\begin{array}{rlr}
T_{s}(n, m) \leq & \sum_{i=1}^{m / B} T_{s}\left(n_{i}, B\right)+4 T_{s-1}\left(\frac{n_{g}}{2}, m\right)+4 m & \\
& +4 T_{s-2}\left(\frac{1}{4} T_{s}\left(n_{g}, \frac{m}{B}\right), m\right)+6 m \\
\leq & 2^{5^{s-3}}\left(n-n_{g}+m\left(\left(\log ^{s-3} B\right)^{*}\right)^{s-3}\right) & \text { \{local\} } \\
& +4 \cdot 2^{5^{s-4}}\left(\frac{n_{g}}{2}+m\left(\left(\log ^{s-4} m\right)^{*}\right)^{s-4}\right)+4 m & \text { \{first and last\} } \\
& +2^{5^{s-5} 5^{s-3}\left(n_{g}+\frac{m}{B} \log ^{s-3} \frac{m}{B}\right)} \\
& +4 \cdot 2^{5^{s-5} m} m\left(\left(\log ^{s-5} m\right)^{*}\right)^{s-5}+6 m .
\end{array}
$$

For simplicity, consider terms with $n$ and with $m$ separately.

$$
2^{5^{s-3}}\left(n-n_{g}\right)+2 \cdot 2^{5^{s-4}} n_{g}+2^{5^{s-5}} 5^{s-3} n_{g} \leq 2^{5^{s-3}} n_{g} .
$$

Now consider all the terms with $m$.

$$
\begin{aligned}
& 2^{5^{s-3}} m\left(\left(\log ^{s-3} B\right)^{*}\right)^{s-3}+4 \cdot 2^{2^{s-4}} m\left(\left(\log ^{s-4} m\right)^{*}\right)^{s-4} \\
& \quad+2^{s^{s-5}} 5^{s-3} \frac{m}{B} \log ^{s-3} \frac{m}{B}+2^{5^{s-5}} m\left(\left(\log ^{s-5} m\right)^{*}\right)^{s-5}+10 m \\
& \leq 2^{5^{s-3}} m\left(\left(\log ^{s-3} B\right)^{*}\right)^{s-3} \\
& \quad+\left(4 \cdot 2^{5^{s-4}}+2^{5^{s-5}} 5^{s-3}+2^{2^{s-5}}+10\right) m\left(\left(\log ^{s-3} m\right)^{*}\right)^{s-4} \\
& \leq 2^{5^{s-3}} m\left(\left(\left(\log ^{s-3} m\right)^{*}-1\right)^{s-3}+\left(\left(\log ^{s-3} m\right)^{*}\right)^{s-4}\right) \\
& \left.\leq 2^{5^{s-3}} m\left(\left(\log ^{s-3} m\right)^{*}\right)^{s-3}\right)
\end{aligned}
$$

The last inequality is true since

$$
(a-1)^{k}+a^{k-1} \leq a^{k},
$$

for $a \geq 1$ and integer $k$. We estimated separately all the terms with $n$ and with $m$. In total it gives:

$$
T(n, m) \leq 2^{5^{s-3}}\left(n+m\left(\left(\log ^{s-3} m\right)^{*}\right)^{s-3}\right)
$$

Lemma 4.19. For every $k, T_{s}(n, m) \leq 2^{(5 k)^{s-3}}\left(n+m\left(\left(\log ^{s-3} m\right)^{[k] *}\right)^{s-3}\right)$.

Proof. Induction by $k$. Base case $k=1$ is lemma 4.18. Again, use induction on $n, m$ to estimate local terms and induction on $s$ to estimate first and last. For the middle terms, use induction on $k$ and $s$. Substitute $B=\left(\log ^{s-3} m\right)^{[k-1] *}$.

$$
\begin{aligned}
& T_{s}(n, m) \leq \sum_{i=1}^{m / B} T_{s}\left(n_{i}, B\right)+4 T_{s-1}\left(\frac{n_{g}}{2}, m\right)+4 m \\
&+4 T_{s-2}\left(\frac{n^{\prime}}{4}, m\right)+6 m \\
& \leq 2^{(5 k)^{s-3}\left(n-n_{g}+m\left(\left(\log ^{s-3} B\right)^{[k] *}\right)^{s-3}\right)} \quad\{\text { local\} } \\
&+4 \cdot 2^{(5 k)^{s-4}\left(\frac{n_{g}}{2}+m\left(\left(\log ^{s-4}\right)^{[k] *}\right)^{s-4}\right)+4 m} \quad \\
& \quad+2^{\left.(5 k)^{s-5} 2^{(5(k-1))^{s-3}}\left(n_{g}+\frac{m}{B}\left(\left(\log \frac{m}{B}\right)^{s-3}\right)^{[k-1] *}\right)^{s-3}\right)} \quad\{\text { first and last\}} \\
&+4 \cdot 2^{(5 k)^{s-5}} m\left(\left(\log ^{s-5}\right)^{[k]^{*}}\right)^{s-5}+6 m .
\end{aligned}
$$

Terms with $n$ and $n_{g}$ :

$$
2^{(5 k)^{s-3}}\left(n-n_{g}\right)+2 \cdot 2^{(5 k)^{s-4}}+2^{(5 k)^{s-5}} 2^{(5(k-1))^{s-3}} n_{g} \leq 2^{(5 k)^{s-3}} n
$$

Terms with $m$ :

$$
\begin{aligned}
& 2^{(5 k)^{s-3}} m\left(\left(\log ^{s-3} B\right)^{[k] *}\right)^{s-3}+4 \cdot 2^{(5 k)^{s-4} m\left(\left(\log ^{s-4}\right)^{[k] *}\right)^{s-4}} \begin{array}{l}
\left.\quad+2^{(5 k)^{s-5}} 2^{(5(k-1))^{s-3} \frac{m}{B}}\left(\left(\log \frac{m}{B}\right)^{s-3}\right)^{[k-1] *}\right)^{s-3} \\
\quad+4 \cdot 2^{(5 k)^{s-5}} m\left(\left(\log ^{s-5}\right)^{[k]^{*}}\right)^{s-5}+10 m \\
\leq 2^{(5 k)^{s-3}} m\left(\left(\log ^{s-3} m\right)^{[k] *}-1\right)^{s-3} \\
\quad+\left(4 \cdot 2^{(5 k)^{s-4}}+2^{(5 k)^{s-5}} 2^{(5(k-1))^{s-3}}+4 \cdot 2^{(5 k)^{s-5}}+10\right) m\left(\left(\log ^{s-3} m\right)^{[k] *}\right)^{s-4} \\
\leq 2^{(5 k)^{s-3}} m\left(\left(\log ^{s-3} m\right)^{[k] *}\right)^{s-3} \cdot
\end{array} .
\end{aligned}
$$

Now substitute $k=\alpha_{(s-3)}(n, m)$ into the previous lemma. We will get

$$
\operatorname{ex}\left(n, m, L_{s}\right) \leq O\left(2^{\left(5 \alpha_{(s-3)}(n, m)+5\right)^{s-3}} n\right) \leq O\left(2^{(5 \alpha(n, m))^{s-3}} n\right)
$$

## 5 Main results

In this section the main results of the thesis are presented. We will prove Füredi-Hajnal conjecture using a new approach based on construction from theorem 4.5. The construction similar to one from light matrices will provide a recurrence relation. Solving it will give us a double exponential upper bound.

Theorem 5.1. For any $k$-permutation $P$,

$$
c(P)=2^{o\left(2^{\frac{k}{2}}\right)} .
$$

Notice that this upper bound is bigger than the best known mentioned in the Introduction. Still, the proof shows how the different method can be applied.

In the proof we will work with matrices of size $n \times m$, which is a more general case. For this form, the limit is defined as

$$
c(P)=\lim _{n, m \rightarrow \infty} \frac{\operatorname{ex}(n, m, P)}{n+m}
$$

which differs from the original definition on case $n=m$. But the difference is in constant factor, which does not affect the theorem.

Consider the following recurrence relation:

$$
\begin{align*}
\operatorname{ex}(n, m, P) \leq & n\left(2 B+4 c_{k-2}^{2}+14 c_{k-2}+12\right) \\
& +m\left(6 c_{k-2}+10\right)+\left(c_{k-2}+2\right)^{2} \operatorname{ex}\left(\frac{n}{B}, \frac{m}{B}, P\right) \tag{2}
\end{align*}
$$

Suppose it holds and let us show how the proof of theorem 5.1 will follow.

### 5.1 The proof

Lemma 5.2. $\operatorname{ex}(n, m, P) \leq\left(20 c_{k-2}^{2}+76 c_{k-2}+70\right)(n+m)$.
Proof. We will prove it using induction by $n, m$. Suppose the inequality holds for every ( $n^{\prime}, m^{\prime}$ ) such that $n^{\prime}<n$ or $n^{\prime}=n$ and $m^{\prime}<m$. Then for given $n$ and $m$ let $n_{0}$ and $m_{0}$ be largest integers, such that $n_{0} \leq n, m_{0} \leq m$ and $2\left(c_{k-2}+2\right)^{2}$ divides both of them. By induction hypothesis,

$$
\operatorname{ex}\left(\frac{n_{0}}{B}, \frac{m_{0}}{B}, P\right) \leq\left(20 c_{k-2}^{2}+76 c_{k-2}+70\right) \frac{n_{0}+m_{0}}{B} .
$$

Set $B=2\left(c_{k-2}+2\right)^{2}$. Then

$$
\begin{aligned}
\operatorname{ex}(n, m, P) \leq & \operatorname{ex}\left(n_{0}, m_{0}, P\right)+(B-1)(n+m) \\
\leq & n\left(2 B+4 c_{k-2}^{2}+14 c_{k-2}+12+B-1\right)+m\left(6 c_{k-2}+10+B-1\right) \\
& \quad+\left(c_{k-2}+2\right)^{2}\left(20 c_{k-2}^{2}+76 c_{k-2}+70\right) \frac{n+m}{2\left(c_{k-2}+2\right)^{2}} \\
= & n\left(6\left(c_{k-2}+2\right)^{2}+4 c_{k-2}^{2}+14 c_{k-2}+11\right) \\
& \quad+m\left(2\left(c_{k-2}+2\right)^{2}+6 c_{k-2}+9\right) \\
& \quad+\left(10 c_{k-2}^{2}+38 c_{k-2}+35\right)(n+m) \\
= & n\left(20 c_{k-2}^{2}+76 c_{k-2}+70\right)+m\left(2 c_{k-2}^{2}+14 c_{k-2}+9\right) \\
\leq & \left(20 c_{k-2}^{2}+76 c_{k-2}+70\right)(n+m) .
\end{aligned}
$$

Proof. (Theorem 5.1) We will prove the statement for even $k$. If $k$ is odd, notice that

$$
c_{k} \leq c_{k+1}=2^{o\left(2^{\frac{k+1}{2}}\right)}=2^{o\left(2^{\frac{k}{2}}\right)} .
$$

By the lemma 5.2,

$$
c_{k} \leq 20 c_{k-2}^{2}+76 c_{k-2}+70 \leq 166 c_{k-2}^{2} .
$$

By taking the logarithm, we get

$$
\log c_{k} \leq \log 166+2 \log c_{k-2}
$$

Define the sequence

$$
d_{1}=\log c_{2}, d_{2}=\log c_{4}, d_{k / 2}=\log c_{k} .
$$

Then,

$$
d_{m} \leq 2 d_{m-1}+f, \text { where } f=\log 166
$$

The solution to this recurrence is

$$
d_{m} \leq 2^{m-1} d_{1}+\left(2^{m-1}-1\right) f
$$

Then,

$$
\begin{gathered}
\log c_{k} \leq 2^{\frac{k}{2}-1} d_{1}+\left(2^{\frac{k}{2}}-1\right) \log 166, \\
c_{k} \leq 2^{O\left(2^{\frac{k}{2}}\right)} .
\end{gathered}
$$

The rest of the section is devoted to proving the formula 2 .

### 5.2 Construction: Step 1

We will use similar argument to the one used in the proof for light matrices. Consider an $n \times m$ matrix $A$, which avoids $k$-permutation $P$ and $|A|=\operatorname{ex}(n, m, P)$. We assume an integer number $B$ divides $n$ and $m$. Divide $A$ into $m / B$ supercolumns of width $B$ : for $1 \leq i \leq m / b$, denote the submatrix formed by consecutive columns $(i-1) B+1,(i-1) B+2 \ldots, i B$ as $i$ th supercolumn.

Next we introduce the notion of local, first, last and middle elements similar to the way it was used in the theorem 4.5 .

We will call an entry $(i, j)$ local, the row $i$ has entries only in one supercolumn. Let $A_{l o c}$ denote $n \times m$ matrix containing only local elements of $A$. If a row has a local entry, then all of its entries are local. We will call such rows local. Notice that local row has at most $B$ elements. Otherwise, a row is global. Let $0 \leq n_{l} \leq n$ be the number of local rows. Denote as $n_{g}$ the number of global rows. Then $n_{l}+n_{g}=n$.

In a global row, we call an entry $(i, j)$ first if there are no 1 -entries in a supercolumn before. Analogously, the entry is last, if there are no entries in the supercolumns after. Let $A_{f}$ and $A_{l}$ denote matrices containing first and last elements, respectively.

The rest of entries are middle. Let $A_{m}$ denote the matrix containing all middle elements. See figure 1 for illustration. Notice that

$$
|A|=\left|A_{l o c}\right|+\left|A_{f}\right|+\left|A_{l}\right|+\left|A_{m}\right| .
$$

We will study each matrix separately.
Local $A_{l o c}$ : Notice that in each local row entries may occur only in one supercolumn of width $B$. In other words, each local row contains at most $B$ elements. Thus, $\left|A_{l o c}\right| \leq B n_{l}=B\left(n-n_{g}\right)$.

First and last $A_{f}, A_{l}$ : There are at most $B$ first elements in each global row, because they may occur only in one supercolumn. Thus, there are no more than $B n_{g}$ first elements. Similarly, there are $B n_{g}$ last elements. In total, last and first sum to

$$
\left|A_{f}\right|+\left|A_{l}\right| \leq 2 B n_{g} .
$$

Middle $A_{m}$ : Define a $k \times k$ matrix $P^{\prime}$ in the following way.

- Let $\left(p_{r}, k\right)$ and $\left(p_{l}, 1\right), 1 \leq p_{r}, p_{l} \leq l$ be the rightmost and the leftmost entries of $P$, respectively.
- Delete these two elements from $P$.
- If $p_{r}$ or $p_{l}$ is 1 , then add element in the first row such that it is next to the element in the second row.
- If $p_{r}$ or $p_{l}$ is $k$, then add element in the last row such that it is next to the element in the $(k-1)$ th row.

See figure 2 for illustration. Notice that $P^{\prime}$ has 2 empty columns and at most 2 empty rows. Matrix $P^{\prime}$ is an altered ( $k-2$ )-permutation matrix.


Figure 2: The first matrix is a permutation matrix $P$. The second pattern is $P^{\prime}$ constructed by deleting elements in the first and the last columns and adding an element in the first row.

Matrix $P^{\prime}$ can not be sum decomposable and skew-sum decomposable at the same time, since the first and the last rows are not empty. Without loss of generality, suppose $P^{\prime}$ is not sum decomposable. Then let $D$ be a block-diagonal matrix, which is skew-sum of supercolumns of $A_{m}$. This construction of block diagonal matrix $D$ allows us to write different avoidance restrictions.

## Lemma 5.3. $D$ avoids $P^{\prime}$.

Proof. Suppose for a contradiction that $D$ contains $P^{\prime}$. Let us fix the realization of $P^{\prime}$ in $D$. Since $P^{\prime}$ cannot be written as a skew-sum, the realization must occur in one block of $D$. Indeed, if at least 2 blocks contain it, it gives skew decomposition of $P^{\prime}$. Consequently, matrix $A_{m}$ contains $P^{\prime}$. Now let us consider the row of $A_{m}$ that contains row $p_{r}$ of $P^{\prime}$. This row might not have any entries, but in the original matrix $A$ this row has the element in the last block, and it would realize the rightest entry of $P$. Analogously, in the row $p_{l}$ of $A$ we will find the leftmost element in the first block. In total, the realization of $P^{\prime}$ together with the rightmost and the leftmost elements will give us the realization of $P$ in $A$. That gives a contradiction.

The goal is to estimate the mass of $D$ and, consequently, mass of $A_{m}$. Firstly, we need to determine size of $D$.

Lemma 5.4. $D$ has size $n^{\prime} \times m$, where $n^{\prime} \leq \operatorname{ex}\left(n_{g}, \frac{m}{B}, P\right)$.
Proof. It has the same number of columns as $A$, thus this number is $m$.
Suppose $D$ has more than $\operatorname{ex}\left(n_{g}, m / B, P\right)$ rows. Let $A^{\prime}$ be contraction matrix of size $n / B \times m$, where element $(i, j)$ is 0 if and only if elements of $i$ th row of supercolumn $j$ are all zeros. Each element of $A^{\prime}$ corresponds to each row of matrix $D$. As $A^{\prime}$ is a contraction matrix, it avoids $P$. Thus, if $D$ has more than ex $\left(n_{g}, m / B, P\right)$ rows, matrix $A^{\prime}$ has mass greater than $\operatorname{ex}\left(n_{g}, m / B, A\right)$, which gives a contradiction.

Lemma 5.5. $|D| \leq\left(c_{k-2}+2\right) \operatorname{ex}\left(n_{g}, \frac{m}{B}, P\right)+6 m+4 c_{k-2} m$.
Proof. For simplicity, let us denote $n^{\prime}=\operatorname{ex}\left(n_{g}, m / B, P\right)$. Then, matrix $D$ has size at most $n^{\prime} \times m$. Matrix $D$ avoids $P^{\prime}$, which is a matrix of $(k-2)$-permutation with 2 empty columns on the boundary and 2 modified rows. Let $P_{1}^{\prime}$ be a matrix constructed by deleting 2 empty columns on the boundary. Then, by lemma 2.7,

$$
|D| \leq \operatorname{ex}\left(n^{\prime}, m, P^{\prime}\right) \leq \operatorname{ex}\left(n^{\prime}, m, P_{1}^{\prime}\right)+2 n^{\prime}
$$

In case $P_{1}^{\prime}$ has the first or last row altered, with an entry added next to an existing one, we can apply lemma 2.6. In case $P_{1}^{\prime}$ has empty rows, lemma 2.8 is needed. The case when $P_{1}^{\prime}$ has two empty rows of $P_{1}^{\prime}$ which are not consecutive is the worst case, because the bound is bigger then. Thus, we need to apply lemma 2.8 two times with $k=1$. Let $P_{2}^{\prime}$ be the matrix constructed by deleting the first empty row, and $P_{3}^{\prime}$ be the matrix after deleting both empty rows. Matrix $P_{3}^{\prime}$ is a $(k-2)$-permutation matrix.

$$
\begin{aligned}
\operatorname{ex}\left(n^{\prime}, m, P_{1}^{\prime}\right) & \leq 2 \operatorname{ex}\left(\frac{n^{\prime}}{2}, m, P_{2}^{\prime}\right)+2 m \\
\operatorname{ex}\left(\frac{n^{\prime}}{2}, m, P_{2}^{\prime}\right) & \leq 2 \operatorname{ex}\left(\frac{n^{\prime}}{4}, m, P_{3}^{\prime}\right)+2 m \\
& \leq 2 c_{k-2}\left(\frac{n^{\prime}}{4}+m\right)+2 m=c_{k-2} \frac{n^{\prime}}{2}+2 m+2 c_{k-2} m
\end{aligned}
$$

In total it would give

$$
\begin{aligned}
\operatorname{ex}\left(n^{\prime}, m, P_{1}^{\prime}\right) & \leq c_{k-2} n^{\prime}+6 m+4 c_{k-2} m, \text { and } \\
|D| & \leq c_{k-2} n^{\prime}+2 n^{\prime}+6 m+4 c_{k-2} m \\
& =\left(c_{k-2}+2\right) \operatorname{ex}\left(n_{g}, \frac{m}{B}, P\right)+6 m+4 c_{k-2} m .
\end{aligned}
$$

Now we are ready to estimate the whole matrix $A$ :

$$
\begin{aligned}
|A| & =\left|A_{l o c}\right|+\left|A_{f}\right|+\left|A_{l}\right|+\left|A_{m}\right| \\
& \leq B\left(n-n_{g}\right)+2 B n_{g}+\left(c_{k-2}+2\right) \operatorname{ex}\left(n_{g}, \frac{m}{B}, P\right)+6 m+4 c_{k-2} m \\
& \leq B n+B n_{g}+4 c_{k-2} m+6 m+\left(c_{k-2}+2\right) \operatorname{ex}\left(n_{g}, \frac{m}{B}, P\right) .
\end{aligned}
$$

The formula above describes recursion for $m$, but there is no recursion for $n$. In the next step we estimate the extremal function ex $\left(n_{g}, m / B, P\right)$ further the same way, but with respect to the first coordinate $n$.

### 5.3 Construction: Step 2

To get recursion in $n$, we can repeat step 1 but with respect to horizontal division into blocks.

Let $C$ be a $n_{g} \times m / B$ matrix avoiding $P$, such that $|C|=\operatorname{ex}\left(n_{g}, m / B, A\right)$. Assume that $B$ divides $n_{g}$. We will split matrix $C$ horizontally into $n_{g} / B$ superrows of size $B$. Define local, first, last and middle elements with respect to this division.

We will call an element of $C(i, j)$ local, if its column $j$ has no 1-entries outside the superrow of $(i, j)$. Then, the column with local elements is called local. The rest of the columns are called global. Let $m_{g} \leq m / B$ denote the number of global columns and $m_{l}=\frac{m}{B}-m_{g}$ denote the number of local columns.

The element in the global column is called first, if there are no elements in the superrows above. Similarly, the element is last, if there are no entries in the suprerows below. The rest of the entries are called middle. Define matrices $C_{l o c}, C_{f}, C_{l}$ and $C_{m}$, in a similar manner, as matrices containing local, first, last and middle elements, respectively.

We will estimate the mass by counting elements of $C_{l o c}, C_{f}, C_{l}$ and $C_{m}$.
Local $C_{l o c}$ : Each of $n_{l}$ columns has at most $B$ elements,

$$
\left|C_{l o c}\right| \leq B\left(\frac{m}{B}-m_{g}\right)
$$

First and last $C_{f}, C_{l}$ : each global column has at most $B$ first and $B$ last entries,

$$
\left|C_{f}\right|+\left|C_{l}\right| \leq 2 B m_{g}
$$

Middle $C_{m}$ : Let $P^{\prime}$ be $k \times k$ matrix constructed by deleting the top and the bottom elements of matrix $P$. Thus, $P^{\prime}$ is matrix corresponding to $(k-2)$ permutation with 2 empty columns and to empty rows on the boundary. Suppose it can not be written as a skew sum of 2 matrices. Let $F$ be a block matrix which is a skew sum of superrows of $C$. Then, the following lemmas hold.

Lemma 5.6. $F$ avoids $P^{\prime}$.
Proof. The proof is similar to 5.3.
Lemma 5.7. $F$ has size at most $n_{g} \times \operatorname{ex}\left(\frac{n_{g}}{B}, m_{g}, P\right)$.
Proof. The proof is similar to lemma 5.4.
Lemma 5.8. $|F| \leq\left(c_{k-2}+2\right) \operatorname{ex}\left(\frac{n_{g}}{B}, m_{g}, P\right)+6 n_{g}+4 c_{k-2} n_{g}$.
Proof. The proof is similar to lemma 5.5.
Then,

$$
\begin{aligned}
|C| & \leq B\left(\frac{m}{B}-m_{g}\right)+2 B m_{g}+\left(c_{k-2}+2\right) \operatorname{ex}\left(\frac{n_{g}}{B}, m_{g}, P\right)+6 n_{g}+4 c_{k-2} n_{g} \\
& =m+B m_{g}+\left(c_{k-2}+2\right) \operatorname{ex}\left(\frac{n_{g}}{B}, m_{g}, P\right)+6 n_{g}+4 c_{k-2} n_{g}
\end{aligned}
$$

Let us substitute the estimate above in. Notice that $n_{g} \leq n$ and $m_{g} \leq m / B$.

$$
\begin{aligned}
\mathrm{ex}(n, m, P) \leq & B n+B n_{g}+4 c_{k-2} m+6 m+\left(c_{k-2}+2\right) \operatorname{ex}\left(n_{g}, \frac{m}{B}, P\right) \\
\leq & B n+B n_{g}+4 c_{k-2} m+6 m+ \\
& \left(c_{k-2}+2\right)\left(m+B m_{g}+\left(c_{k-2}+2\right) \operatorname{ex}\left(\frac{n_{g}}{B}, m_{g}, P\right)+6 n_{g}+4 c_{k-2} n_{g}\right) \\
\leq & 2 B n+\left(4 c_{k-2}+6\right) m \\
& \quad+\left(c_{k-2}+2\right)\left(2 m+\left(c_{k-2}+2\right) \operatorname{ex}\left(\frac{n}{B}, \frac{m}{B}, P\right)+\left(4 c_{k-2}+6\right) n\right) \\
\leq & 2 B n+\left(6 c_{k-2}+10\right) m \\
& \quad+4 c_{k-2}^{2} n+14 c_{k-2} n+12 n+\left(c_{k-2}+2\right)^{2} \operatorname{ex}\left(\frac{n}{B}, \frac{m}{B}, P\right) \\
\leq & n\left(2 B+4 c_{k-2}^{2}+14 c_{k-2}+12\right) \\
& \quad+m\left(6 c_{k-2}+10\right)+\left(c_{k-2}+2\right)^{2} \operatorname{ex}\left(\frac{n}{B}, \frac{m}{B}, P\right) .
\end{aligned}
$$

### 5.4 Remarks

In conclusion, we applied the technique from the light matrices into a different setting, using the fact that permutation matrices are light in both directions. It made it possible to use contraction of superrows and supercolumns.

In lemma 5.2 we substitute $B=2\left(c_{k-2}+2\right)^{2}$. Notice that this choice is optimal. It follows from the resulting recursive formula. Suppose we want to choose optimal $B$ to find the best estimation for $c_{k}$ by induction.

$$
\begin{aligned}
\operatorname{ex}(n, m, P) \leq n & \left(2 B+4 c_{k-2}^{2}+14 c_{k-2}+12+B-1\right)+m\left(6 c_{k-2}+10+B-1\right) \\
& +\left(c_{k-2}+2\right)^{2} c_{k} \frac{n+m}{B} .
\end{aligned}
$$

Thus, we want to choose $B$ to minimize the solution $c_{k}$ to the following inequality

$$
\begin{aligned}
& n\left(3 B+4 c_{k-2}^{2}+14 c_{k-2}+11\right)+m\left(6 c_{k-2}+9+B\right) \\
& \quad+\left(c_{k-2}+2\right)^{2} c_{k} \frac{n+m}{B} \leq c_{k}(n+m) .
\end{aligned}
$$

Notice that if $B \leq\left(c_{k-2}+2\right)^{2}$, there are no solutions for $c_{k}$. Let $B=A \cdot\left(c_{k-2}+2\right)^{2}$. Thus we have

$$
\begin{aligned}
& n\left(3 A \cdot\left(c_{k-2}+2\right)^{2}+4 c_{k-2}^{2}+14 c_{k-2}+11\right)+m\left(6 c_{k-2}+9+A \cdot\left(c_{k-2}+2\right)^{2}\right) \\
& \quad+c_{k} \frac{n+m}{A} \leq c_{k}(n+m)
\end{aligned}
$$

Let us find the solution to the stronger inequality:

$$
\begin{aligned}
& 3 A \cdot\left(c_{k-2}+2\right)^{2}+4 c_{k-2}^{2}+14 c_{k-2}+11 \leq \frac{(A-1) c_{k}}{A}, \\
& c_{k} \geq \frac{A}{A-1}\left(3 A \cdot\left(c_{k-2}+2\right)^{2}+4 c_{k-2}^{2}+14 c_{k-2}+11\right)
\end{aligned}
$$

The right part is minimal when $A=2$. Thus, $B=2\left(c_{k-1}+2\right)^{2}$ is optimal.
In concept, the result has a double exponential bound because we have the square term in recurrence $c_{k}=O\left(c_{k-2}^{2}\right)$. It appears because the construction has two steps: each of them includes multiplication of the main term by coefficient $O\left(c_{k-2}\right)$.

Compared to the proof for light matrices, notice that we used a different bound for the number of local, first and last elements. Here we used trivial bound, saying that in each row the number of local, first and last is at most $B$. But there are different ways to estimate the number of these elements. $P$ or a smaller pattern. For example,

$$
\mid \text { Local } \mid \leq \sum_{i=1}^{m / B} \operatorname{ex}\left(n_{i}, B, P\right)
$$

For the first and last elements, we can consider the block diagonal matrix $D$, defined in the proof, but consisting only first or last elements. By the same reasoning, the size of $D$ is at most $n^{\prime} \times m$, where $n^{\prime}=\operatorname{ex}\left(n_{g}, m / B, P\right)$. Then

$$
\begin{aligned}
& \mid \text { First } \mid \leq \operatorname{ex}\left(n^{\prime}, m, P_{1}^{\prime}\right) \\
& \mid \text { Last } \mid \leq \operatorname{ex}\left(n^{\prime}, m, P_{2}^{\prime}\right),
\end{aligned}
$$

where $P_{1}^{\prime}$ is matrix constructed by deleting the rightmost element of $P$, and $P_{2}^{\prime}$, by deleting the leftmost element. Even though, these bounds seem more tight, the solution will still be of double exponential order, since we will have a quadratic term.

Possible future research might be focused on continuing working with the construction described in section 5 . There are different options for choosing $B$. Suppose that in the beginning we choose $B$ to be large, for example, $B=m / 10$. Define local, first, last and middle elements as before. For the middle entries use the same construction with block diagonal matrix, avoiding a smaller pattern. The estimation will be

$$
\mid \text { Middle } \left\lvert\, \leq\left(c_{k-2}+2\right) \operatorname{ex}\left(n_{g}, \frac{m}{B}, P\right)+6 m+4 c_{k-2} m\right.
$$

Here $m / B=10$ and then the term $\operatorname{ex}\left(n_{g}, m / B, P\right)$ can be trivially bounded as

$$
\operatorname{ex}\left(n_{g}, \frac{m}{B}\right) \leq 10 n_{g} .
$$

Then the number of middle elements is at most

$$
\mid \text { Middle } \mid \leq 10\left(c_{k-2}+2\right) n_{g}+6 m+4 c_{k-2} m
$$

This estimation is linear and does not use recurrence at all. Although, for the local, first and last elements trivial bound will not work anymore, since

$$
\begin{aligned}
B\left(n-n_{g}\right) & =\frac{m}{10}\left(n-n_{g}\right), \\
B n_{g} & =\frac{m}{10} n_{g}
\end{aligned}
$$

The problem with this approach is that we can not now estimate $n_{g}$ and we always upper bound it by $n$. In that case we will have quadratic estimate, which is too weak. The possible future work may include analyzing $n_{g}$ variable and trying to find more tight upper bound depending on $B$ in a random matrix.

## References

[1] Arratia, R. On the Stanley-Wilf conjecture for the number of permutations avoiding a given pattern. The Electronic Journal of Combinatorics 6 (1999), 4.
[2] Bouvel, M., and Rossin, D. A variant of the tandem duplication random loss model of genome rearrangement. Theoretical Computer Science 410 (2009), 847-858.
[3] Bóna, M. Exact enumeration of 1342 -avoiding permutations: A close link with labeled trees and planar maps. Journal of Combinatorial Theory, Series A 80, 2 (1997), 257-272.
[4] Bóna, M. Permutations avoiding certain patterns: The case of length 4 and some generalizations. Discrete Mathematics 175 (1997), 55-67.
[5] Bóna, M. The solution of a conjecture of stanley and wilf for all layered patterns. Journal of Combinatorial Theory, Series A 85, 1 (1999), 96-104.
[6] Cibulka, J. On constants in the Füredi-Hajnal and the Stanley-Wilf conjecture. Journal of Combinatorial Theory, Series A 116, 1 (2009), 290-302.
[7] Cibulka, J., and Kyncl, J. Better upper bounds on the Füredi-Hajnal limits of permutations. In ACM-SIAM Symposium on Discrete Algorithms (2016).
[8] Fekete, M. Uber die verteilung der wurzeln bei gewissen algebraischen gleichungen mit ganzzähligen koeffizienten. Math. Z. 17 (1923), 228-249.
[9] Fox, J. Stanley-Wilf limits are typically exponential. arXiv:1310.8378 (2013).
[10] Füredi, Z. The maximum number of unit distances in a convex n-gon. Journal of Combinatorial Theory, Series A 55 (1990), 316-320.
[11] Füredi, Z., and Hajnal, P. Davenport-Schinzel theory of matrices. Discrete Mathematics 153(1-3) (1996), 185-196.
[12] Geneson, J. Forbidden formations in multidimensional 0-1 matrices. European Journal of Combinatorics 78 (2019), 147-154.
[13] Keszegh, B. Forbidden submatrices in 0-1 matrices. Master's thesis, Eötvös Loránd University, (2005).
[14] Keszeg h, B. On linear forbidden submatrices. Journal of Combinatorial Theory, Series A 107, 1 (2009), 232-241.
[15] Kitaev, S. Patterns in permutations and words. Monographs in Theoretical Computer Science. An EATCS Series. Springer Verlag, (2011).
[16] Klazar, M. A general upper bound in extremal theory of sequences, comment. Commentationes Mathematicae Universitatis Carolinae 33, 4 (1992), 737-747.
[17] Klazar, M. The Füredi-Hajnal conjecture implies the Stanley-Wilf conjecture. Formal Power Series and Algebraic Combinatorics (2000), 250-255.
[18] Marcus, A., and Tardos, G. Excluded permutation matrices and the Stanley-Wilf conjecture. Journal of Combinatorial Theory, Series A 107, 1 (2004), 153-160.
[19] Pach, J., and Tardos, G. Forbidden paths and cycles in ordered graphs and matrices. Israel Journal of Mathematics 155 (2006), 359-380.
[20] Pettie, S. Applications of forbidden 0-1 matrices to search tree and path compression-based data structures. Proceedings of the Annual ACM-SIAM Symposium on Discrete Algorithms (2010), 1457-1457.
[21] Stanojevic, M., and Steedman, M. Formal basis of a language universal. Computational Linguistics 47, 1 (2021), 9-42.
[22] Vatter, V. Permutation Classes. Handbook of Enumerative Combinatorics (2015), 754-833.

