Department of Mathematics and Systems Analysis

Nonlinear variational problems on metric measure spaces

Cintia Pacchiano Camacho

Aalto University

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Abstract

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Abstract

This dissertation studies existence and regularity properties of functions related to the calculus of variations on metric measure spaces that support a weak Poincaré inequality and doubling measure. The work consists of four articles in which we study the total variation flow and quasiminimizers of a (p,q)-Dirichlet integral.

More specifically, we define variational solutions to the total variation flow in metric measure spaces. We establish existence and, using energy estimates and the properties of the underlying metric, we give necessary and sufficient conditions for a variational solution to be continuous at a given point.

We then take a purely variational approach to a (p,q)-Dirichlet integral, define its quasiminimizers, and using the concept of upper gradients together with Newtonian spaces, we develop interior regularity, as well as regularity up to the boundary, in the context of general metric measure spaces. For the interior regularity, we use De Giorgi type conditions to show that quasiminimizers are locally Hölder continuous and they satisfy Harnack inequality. Furthermore, we give a pointwise estimate near a boundary point, as well as a sufficient condition for Hölder continuity and a Wiener type regularity condition for continuity up to the boundary. Lastly, we prove higher integrability and stability results in metric measure spaces, for quasiminimizers related to the (p,q)-Dirichlet integral.

The results and the methods used in the proofs are discussed in detail, and some related open questions are presented.

Keywords partial differential equations, parabolic, nonlinear analysis, total variation flow, Dirichlet integral, regularity theory, calculus of variations, energy estimates, quasiminimizers, metric spaces, doubling measure, Poincaré inequality, upper gradients, Newtonian spaces, Harnack estimate, existence theory, higher integrability, stability, comparison principle.

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Preface

First and foremost, I would like to thank Prof. Dr. Juha Kinnuen who made this work possible. Thanks for introducing me to the world of metric measure spaces. I have been extremely fortunate over the last four years to have had a supervisor who has fostered not only research collaborations, but also friendships that will hopefully last beyond my time at Aalto University.

I would like to acknowledge Prof. Dr. Mónica Alicia Clapp Jiménez Labora for her constant support and the excellent example she has provided as a successful woman mathematician and professor. A huge thank you to my co-authors. To Antonella Nastasi, for her patient support, constructive discussions, and for all of the opportunities I was given to further my research. To Michael Collins and Vito Buffa for guiding me through these important publications, and for the stimulating questions that expanded my viewpoints. Thank you all for the good collaborations. I would also like to thank Professors Anna Zatorska-Goldstein and Simone Di Marino for carrying out the preliminary examination of the thesis. Moreover, I am grateful to Professor Cristiana De Filippis for agreeing to act as my opponent in the public examination.

On a different note, during these last two years I have had the chance to fully concentrate on my research thanks to the funding I received from the Finnish Academy of Science and Letters, the Vilho, Yrjö and Kalle Väisälä Foundation.

My time in Finland was made more enjoyable in large part due to the friends that became a part of my life. Thanks to Julian for being a true friend and for saving me on more than one occasion. Thanks to Florian for being someone I can talk to, thanks for not only listening to me, but also for always making me smile, thanks for all the cooking and for watching movies with me. Thanks to Estuardo for showing me how pretty Helsinki really is, thanks for lifting me up whenever I needed it. Thanks to Matias for all the interesting discussions, piano sessions, and delicious coffee. Special thanks go to Valentina, Bjørn, Silvia, and Sorrachai, thank you all for being with me, and for helping me in difficult moments. I wish to thank my other co-workers in the Nonlinear PDE group, Emma, Kim, and Kristian, thanks for all the laughs, and for those quiet afternoons we spent working together. I would also like to acknowledge my all-time friend Vivi, we grew up together and, even though we do not live in the same country anymore, I know I can always count on you, thank you for being my friend all these years. To Joaquín, thanks for all the mathematical and non-mathematical discussions we have had, I appreciate that we can talk about anything, and that I always get an honest response from you. Thanks to Jorge for being a constant friend in this ever-changing world.

Last but definitely not least, I want to thank my family for always supporting me. I dedicate this work to you all. Special thanks goes to my brother Aldo, you have always been the best teacher, and I am extremely proud of you. I love you. Mamá, papá, no sé cómo agradecerles todo lo que han hecho por nosotros, es por ustedes que me encuentro en donde estoy y es por ustedes que seguiré adelante. Los quiero muchísimo.

Helsinki, August 17, 2022,

Cintia Pacchiano Camacho

Contents

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I Antonella Nastasi and Cintia Pacchiano Camacho. Regularity properties for quasiminimizers of a (p,q)-Dirichlet integral. *Calculus of Variations and Partial Differential Equations*, 60(6):Paper No. 227, 37, DOI 10.1007/s00526-021-02099-y, September 2021.

II Vito Buffa, Michael Collins and Cintia Pacchiano Camacho. Existence of parabolic minimizers to the total variation flow on metric measure spaces. *Manuscripta Mathematica*, DOI 10.1007/s00229-021-01350-2, January 2022.

III Vito Buffa, Juha Kinnunen and Cintia Pacchiano Camacho. Variational solutions to the total variation flow on metric measure spaces. *Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal*, 220:Paper No. 112859, DOI 10.1016/j.na.2022.112859, February 2022.

IV Antonella Nastasi and Cintia Pacchiano Camacho. Higher integrability and stability of (p,q)-quasiminimizers. Submitted to *a journal*, March 2022.

Author's Contribution

Publication I: "Regularity properties for quasiminimizers of a (p,q)-Dirichlet integral"

Both co-authors contributed equally to all parts of the paper.

Publication II: "Existence of parabolic minimizers to the total variation flow on metric measure spaces"

The author has played a central role in preparing the article.

Publication III: "Variational solutions to the total variation flow on metric measure spaces"

The author has made significant contributions to all parts of the paper. She played a central role in preparing the article.

Publication IV: "Higher integrability and stability of (p,q)-quasiminimizers"

The author contributed to proving the main theoretical results as an equal member of the research team. She also participated in the writing process.

Abbreviations

TVF total variation flow

BV bounded variation

a.e. almost every

q.e. quasi everywhere

1. Introduction

Variational methods appeared as an answer to the problem of finding minima of functionals. It is about giving a necessary and sufficient condition for the existence of the minimum, as well as conditions that allow its calculation and algorithms that let us to compute it. Variational calculus is intimately linked with the theory of partial differential equations, since the conditions for existence of a solution to the minimization problem normally depend on the fact that said solution satisfies a certain differential equation.

The main interest in this dissertation is to extend some classical results of the calculus of variations to metric measure spaces (X,d,μ). We focus on some methods which, in Euclidean spaces, are related to existence and regularity of nonlinear parabolic and elliptic partial differential equations. Some typical nonlinear variational problems are the evolution (parabolic) p-Laplace equation

$$\frac{\partial u}{\partial t} - \operatorname{div}(|Du|^{p-2}Du) = 0, \qquad 1 < p < \infty,$$

and the p-Laplace equation

$$\operatorname{div}(|Du|^{p-2}Du) = 0, \qquad 1 < p < \infty.$$

Since, in a general metric measure space, we do not necessarily have access to directions, it is not clear how to define the partial derivatives of a function and thus, it is not clear what the counterparts of the above equations are. However, in Euclidean spaces, these equations can be formulated into equivalent problems. Meaning that, a function is a weak solution of the equation if and only if it is a minimizer to the corresponding *variational* problem. In this new approach, the modulus of the gradient plays an important role. This is a major advantage when working on a variational level, because first-order Sobolev spaces on a metric measure space can be defined in terms of the modulus of the gradient without the notion of distributional derivatives, see [59, 60]. Hence, the theory of nonlinear parabolic and elliptic partial differential equations can be developed and studied in the metric space setting, thus combining analysis of nonlinear partial differential equations with analysis in metric spaces.

The subject of first-order calculus in metric measure spaces provides an integrating structure for ideas and questions from many different areas of mathematics. Analysis on metric spaces is nowadays an active and independent field, bringing together researchers from different parts of the mathematical spectrum, see [29, 30, 14, 31, 39, 40]. It has applications to disciplines as diverse as geometric group theory, nonlinear PDEs, and even theoretical computer science. This can offer us a better understanding of the phenomena and also lead to new results, even in the classical Euclidean case.

This dissertation is about various classes of functions related to a Dirichlet type integral. We concentrate especially on parabolic minimizers, variational solutions and quasiminimizers. We study existence and regularity, including the integrability and stability properties of the solutions and their upper gradients.

The text is organized in the following way. In Chapter 2 we present the preliminary concepts needed in the proofs of our existence and all of our different regularity results. In Chapter 3 we present the main theorems of articles **[II]** and **[III]**, and discuss in detail the methods and ideas behind them. In Chapter 4 we discuss articles **[I]** and **[IV]**, present the main results and techniques used in the proofs. The last part of this thesis contains the four original articles.

2. Basic concepts and preliminary results

This chapter is devoted to collecting the basic calculus rules and properties of metric measure spaces, as well as laying the preliminaries for what has been achieved in the attached papers. For further details, see [3, 29, 32].

2.1 Sobolev spaces on metric spaces

Let X be a set with at least two elements, and $d : X \times X \to [0,\infty]$ be a metric on X. We assume the metric space (X,d) is *complete, separable and connected*. Let μ be a Borel measure on the metric space (X,d). Then, the triple (X,d,μ) is called a *metric measure space*.

Sobolev spaces can be defined in a metric setting by introducing the notion of an *upper gradient*. These spaces are called *Newtonian spaces*.

Let u be a function on X. A non-negative Borel measurable function g on X is said to be an *upper gradient* of u if

$$|u(x) - u(y)| \leq \int_\gamma g \ \mathrm{d}s, \tag{2.1}$$

holds for all rectifiable paths γ joining points x and y in X, whenever $u(x)$ and $u(y)$ are both finite; otherwise, the path integral is defined as being equal to infinity. Upper gradients have been studied, for example in [14, 31, 44, 60, 59].

Unfortunately, inequality (2.1) implies that upper gradients are not unique. Indeed, if g is an upper gradient of u then $g + C$ remains an upper gradient for any positive constant C. In $\mathbb{R}^n$, we have that $g = |Du|$ is an upper gradient of a smooth function u. Therefore, we can think of upper gradients as a generalization of $|Du|$. Another drawback of upper gradients is that they are not preserved by L^p-convergence. More specifically, the set of upper gradients of a function u is not necessarily a closed subset of $L^p(X)$. This makes it necessary for us to work with the so-called *p-weak upper gradients*.

Let Γ be a family of paths in X and $1 \leq p < \infty$. The p-modulus of Γ is defined as

$$\operatorname{Mod}_p(\Gamma) = \inf \int_X g^p \, d\mu,$$

where the infimum is taken over all Borel functions $g : X \to [0,\infty]$ satisfying

$$\int_\gamma g \, ds \geq 1,$$

for all locally rectifiable $\gamma \in \Gamma$. For more information regarding these definitions, see [31, 59].

If (2.1) fails only for a set of paths that is of zero p-modulus (i.e. holds for p-almost all paths), then g is said to be a *p-weak upper gradient* of u.

Moreover, there exists a *minimal weak upper gradient*. For any u that has a p-integrable weak upper gradient, there exists a p-weak upper gradient denoted g_u, such that for all p-weak upper gradients g of u, there holds

$$g_u \leq g \, \mu\text{-a.e. on } \Omega, \text{ and } \|g_u\|_{L^p(X)} = \inf_g \|g\|_{L^p(X)}.$$

We refer to g_u as the *minimal p-weak upper gradient* of u. It is unique up to sets of measure zero, see [3]. One major drawback about upper gradients is that they do not have some of the good qualities of the Euclidean gradient, such as linearity. To be precise, consider the sum of two functions, u and v. The sum of their individual weak upper gradients is indeed a weak upper gradient of $u+v$. On the other hand, if g and h are weak upper gradients of u and v, respectively, the difference $g-h$ may not be a weak upper gradient of $u-v$. Nevertheless, some properties of a weak upper gradient are very useful. Indeed, it has good local properties. For example, a weak upper gradient of a function can be chosen to be zero almost everywhere the function is constant, see [3, 59].

2.1.1 Newtonian spaces

We define, for $1 \leq p < \infty$, the space $\widetilde{N}^{1,p}(X)$ to be the set of all p-integrable functions u on X that have a p-integrable p-weak upper gradient g on X. The space is equipped with the seminorm

$$\|u\|_{\widetilde{N}^{1,p}(X)} = \|u\|_{L^p(X)} + \inf \|g\|_{L^p(X)},$$

where the infimum is taken over all p–weak upper gradients of u. Note that the norm in $\widetilde{N}^{1,p}(X)$ is precisely the sum of the L^p–norm of the function and of the L^p–norm of the minimal weak upper gradient.

We define an equivalence relation in $\widetilde{N}^{1,p}(X)$ by saying that $u \sim v$ if

$$\|u - v\|_{\widetilde{N}^{1,p}(X)} = 0.$$

The *Newtonian space* $N^{1,p}(X)$ is then defined as the quotient space $\widetilde{N}^{1,p}(X)/\sim$ with the norm

$$\|u\|_{N^{1,p}(X)} = \|u\|_{\widetilde{N}^{1,p}(X)}.$$

The normed space $\left(N^{1,p}(X), \|\cdot\|_{N^{1,p}(X)}\right)$ is a Banach space.

Let $\Omega \subset X$ open. We can naturally consider Ω as a metric space in its own right (with the restrictions of d and μ). Therefore, we can define the Newtonian space $N^{1,p}(\Omega)$ by substituting X with Ω in $N^{1,p}(X)$.

The corresponding local Newtonian space is defined by $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ if $u \in N^{1,p}(\Omega')$ for all $\Omega' \Subset \Omega$, see [3]. Here $\Omega' \Subset \Omega$ means that Ω' is a compact subset of Ω.

As already anticipated, in $\mathbb{R}^n$, equipped with the n–dimensional Lebesgue measure and the Euclidean metric, this definition coincides with the classical definition of Sobolev spaces. We note that, the concept of an upper gradient and thus of Newtonian spaces can be defined in any metric space. For more information regarding Newtonian spaces, we refer the reader to [59, 36, 3, 29].

2.2 Parabolic function spaces

Since the papers **[II]** and **[III]** deal with a *time dependent (parabolic) problem*, we need suitable function spaces in which such objects can be properly defined. More specifically, the spaces we consider rely on Bochner's theory for Banach space-valued functions.

Let $T > 0$ and B be a Banach space equipped with the norm $\|\cdot\|_B$. A function $\rho : (0,T) \to B$ is called a *simple function* if there exist vectors $v_1, \dots, v_n \in B$ and a partition $E_1, \dots, E_n$ of measurable and pairwise disjoint subsets of $(0,T)$ such that

$$\rho = \sum_{i=1}^{n} v_i \mathbb{1}_{E_i}. \tag{2.2}$$

A function $u : (0,T) \to B$ is called *strongly measurable* (in the *sense of Bochner*) if there is a sequence $(\rho_k)_{k\in\mathbb{N}}$ of simple functions, such that $u(t)$ is the limit of $\rho_k(t)$ as $k \to \infty$ for a.e. $t \in (0,T)$.

Let $\langle \cdot, \cdot \rangle$ denote the pairing between B and its dual space B^*. A function $u : (0,T) \to B$ is called *weakly measurable* (in the sense of *Pettis*) if the mapping $t \mapsto \langle v^*, u(t) \rangle$ is Lebesgue measurable for all $v^* \in B^*$. Furthermore, u is called *essentially separably valued* if there exists a subset $N \subset (0,T)$ such that $u((0,T) \setminus N)$ is a separable subset of B and $\mu(N) = 0$. By Pettis' measurability theorem [32], a function $u : (0,T) \to B$ is *strongly measurable* if and only if it is *essentially separably valued* and *weakly measurable*. Therefore, we see that if B is a separable Banach space, then the strong and weak measurability are equivalent.

For a simple function $\rho : (0,T) \to B$ as in (2.2), we define the *Bochner integral* by

$$\int_0^T \rho(t)\ \mathrm{d}t = \sum_{i=1}^{n} v_i \mu(E_i).$$

Notice that, the Bochner integral of a simple function is well defined in B since we have

$$\left\|\int_0^T \rho(t)\ \mathrm{d}t\right\| \le \sum_{i=1}^n \|v_i\|\mu(E_i) < \infty.$$

A strongly measurable function $u:(0,T)\to B$ is called *Bochner integrable* if for the sequence $(\rho_k)_{k\in\mathbb{N}}$ in the definition of the strong measurability of u there holds

$$\lim_{k\to\infty}\int_0^T \|\rho_k(t)-u(t)\|_B\ \mathrm{d}t = 0.$$

The *Bochner integral* of a Bochner integrable function u is then defined as

$$\int_0^T u(t)\ \mathrm{d}t = \lim_{k\to\infty}\int_0^T \rho_k(t)\ \mathrm{d}t \in B.$$

In [32], it is shown that a function $u:(0,T)\to B$ is Bochner integrable if and only if its norm $\|u\|_B:(0,T)\to[0,\infty)$ is Lebesgue integrable.

The next step is to define function spaces for Banach space-valued functions. A Bochner-measurable function $u:(0,T)\to B$ belongs to the space $L^p(0,T;B)$, for $1\le p<\infty$, if

$$\|u\|_{L^p(0,T;B)} = \left(\int_0^T \|u(t)\|_B^p\ \mathrm{d}t\right)^{\frac{1}{p}} < \infty.$$

If $p=\infty$, we say the function u belongs to the space $L^\infty(0,T;B)$ if

$$\|u\|_{L^p(0,T;B)} = \operatorname*{ess\,sup}_{t\in(0,T)}\|u(t)\|_B < \infty.$$

The space $C^0([0,T];B)$ consists of continuous functions $u:[0,T]\to B$ for which there holds

$$\|u\|_{C^0([0,T];B)} = \max_{t\in[0,T]}\|u(t)\|_B < \infty.$$

For $\alpha\in(0,1)$, the space $C^{0,\alpha}([0,T];B)$ consists of functions u in $C^0([0,T];B)$ for which there additionally holds

$$\sup_{s,t\in[0,T]}\frac{\|u(s)-u(t)\|_B}{|s-t|^\alpha} < \infty.$$

In our work, we mainly choose $B=N^{1,p}(\Omega)$ and similar spaces for a bounded and open subset $\Omega\subset X$. We study functions u that depend on the spatial variable x but also on a time variable t. In this Bochner's theory, we have that the functions depend only on the time variable and their values are functions that depend on the spatial variable. Therefore, we need to make sure that these different perspectives are consistent.

With this in mind, let $1\le p<\infty$ and consider a function $u\in L^p(\Omega_T)$, where $\Omega_T=\Omega\times(0,T)$ is a space-time cylinder with $\Omega\subset X$ open and bounded. By Fubini's theorem, for a.e. $t\in(0,T)$ the mapping

$$u(t):\Omega\to\mathbb{R},\quad x\mapsto u(x,t),$$

belongs to $L^p(\Omega)$ and we have

$$\|u\|^p_{L^p(\Omega_T)} = \int_{\Omega_T} |u|^p \ \mathrm{d}(\mu \otimes \mathfrak{L}^1) = \int_0^T \int_\Omega |u(x,t)|^p \ \mathrm{d}\mu \mathrm{d}t$$
$$= \int_0^T \|u(t)\|^p_{L^p(\Omega)} \ \mathrm{d}t = \|u\|^p_{L^p(0,T;L^p(\Omega))},$$

where $\mathfrak{L}^1$ denotes the Lebesgue measure on $\mathbb{R}$. To be able to show that $u \in L^p(0,T;L^p(\Omega))$, we need to establish strong measurability (in the sense of Bochner). Since $L^p(\Omega)$ is a separable Banach space, by Pettis' theorem it is enough to prove that the mapping $t \mapsto \langle u(t), v\rangle$ is Lebesgue measurable for any $v \in L^q(\Omega)$, where q is the Hölder-conjugate of p. Therefore, consider $v \in L^q(\Omega)$ and extend it to $v \in L^q(\Omega_T)$ independently of the time variable. Then, by Hölder's inequality, $uv \in L^1(\Omega_T)$ and by Fubini's theorem we get

$$\infty > \int_{\Omega_T} |uv| \ \mathrm{d}(\mu \otimes \mathfrak{L}^1) = \int_0^T \int_\Omega |u(t)\cdot v| \ \mathrm{d}\mu \mathrm{d}t$$
$$\geq \int_0^T \left| \int_\Omega u(t)\cdot v \ \mathrm{d}\mu \right| \ \mathrm{d}t = \int_0^T |\langle u(t), v\rangle| \ \mathrm{d}t.$$

Thus, the mapping $t \mapsto \langle u(t), v\rangle$ is Lebesgue integrable, and therefore, Lebesgue measurable. Since $v \in L^q(\Omega)$ was arbitrary, this means that u is weakly measurable, as wanted. Lastly, Pettis' theorem implies that u is strongly measurable. Hence, $u \in L^p(0,T;L^p(\Omega))$.

We now show the opposite direction. Let $u \in L^p(0,T;L^p(\Omega))$. Then there is a sequence $(\rho_k)_{k\in\mathbb{N}}$ of simple functions

$$\rho_k(t) = \sum_{i=1}^{n_k} v_i^{(k)} \mathbb{1}_{E_i^{(k)}}(t),$$

with some $n_k \in \mathbb{N}$, $v_i^{(k)} \in L^p(\Omega)$ for $i = 1,\dots,n_k$ and a partition $E_1^{(k)},\dots,E_{n_k}^{(k)}$ of measurable subsets of $(0,T)$, such that $\rho_k \to u$ pointwise a.e. on $(0,T)$ and in $L^p(0,T;L^p(\Omega))$, respectively, as $k \to \infty$. The functions ρ_k are $(\mu \otimes \mathfrak{L}^1)$-measurable and belong to $L^p(\Omega_T)$. Because of the equality of the norms in $L^p(\Omega_T)$ and $L^p(0,T;L^p(\Omega))$ and the convergence of ρ_k to u in $L^p(0,T;L^p(\Omega))$, the sequence $(\rho_k)_{k\in\mathbb{N}}$ is a Cauchy-sequence in $L^p(\Omega_T)$. Since $L^p(\Omega_T)$ is a complete space, there is a $\widetilde{u} \in L^p(\Omega_T)$ such that $\rho_k \to \widetilde{u} \in L^p(\Omega_T)$ as $k \to \infty$. Since $L^p(\Omega_T) \subset L^p(0,T;L^p(\Omega))$, we have that u and $\widetilde{u}$ coincide, which gives us, $L^p(0,T;L^p(\Omega)) \subset L^p(\Omega_T)$. This concludes the proof that these two function spaces and the different concepts of measurability are equivalent. This in turn, allows us to consider functions in $L^p(0,T;L^p(\Omega))$ as functions in space and time, depending on two variables.

2.2.1 Parabolic upper gradients and parabolic Newtonian spaces

Let us now consider the space $L^p(0,T;N^{1,p}(\Omega))$, for $1 \leq p < \infty$. It is easy to see that $L^p(0,T;N^{1,p}(\Omega))$ is contained in the space $L^p(0,T;L^p(\Omega))$. Functions

u in this so-called *parabolic Newtonian space* take values in $N^{1,p}(\Omega)$ for a.e. $t \in (0,T)$, so on these time slices we can find the minimal upper gradient $g_{u(t)}$. We now know that u can be seen as a function of two variables, therefore is it natural to ask if the same applies to the mapping $t \mapsto g_{u(t)}$.

Since $u \in L^p(0,T;N^{1,p}(\Omega))$, there is a sequence $(\rho_k)_{k\in\mathbb{N}}$ of simple functions

$$\rho_k(t) = \sum_{i=1}^{n_k} v_i^{(k)} \mathbb{1}_{E_i^{(k)}}(t),$$

with some $n_k \in \mathbb{N}$, $v_i^{(k)} \in N^{1,p}(\Omega)$ for $i = 1,\dots,n_k$ and a partition $E_1^{(k)},\dots,E_{n_k}^{(k)}$ of measurable subsets of $(0,T)$, such that $\rho_k \to u$ pointwise a.e. on $(0,T)$ and in $L^p(0,T;N^{1,p}(\Omega))$, respectively, as $k \to \infty$. Since the $E_i^{(k)}$ are pairwise disjoint, we find that

$$g_{\rho_k(t)} = \sum_{i=1}^{n_k} g_{v_i^{(k)}} \mathbb{1}_{E_i^{(k)}}(t),$$

The mappings $(0,T) \ni t \mapsto g_{\rho_k(t)}$ are simple functions with values in $L^p(\Omega)$. Furthermore, we have

$$\|g_{\rho_k(t)} - g_{u(t)}\|_{L^p(\Omega)} \le \|g_{(\rho_k(t)-u(t))}\|_{L^p(\Omega)} \le \|\rho_k(t) - u(t)\|_{N^{1,p}(\Omega)} \to 0,$$

consequently,

$$\begin{aligned}\int_0^T \|g_{\rho_k(t)} - g_{u(t)}\|^p_{L^p(\Omega)}\,\mathrm{d}t &\le \int_0^T \|g_{(\rho_k(t)-u(t))}\|^p_{L^p(\Omega)}\,\mathrm{d}t\\ &\le \int_0^T \|\rho_k(t) - u(t)\|^p_{N^{1,p}(\Omega)}\,\mathrm{d}t \to 0,\end{aligned}$$

as $k \to \infty$. Thus, we have that the mapping $t \mapsto g_{u(t)}$ is in the space $L^p(0,T;L^p(\Omega))$, as expected.

Thanks to the equivalence between $L^p(0,T;L^p(\Omega))$ and $L^p(\Omega_T)$, we can give a well posed definition of the *parabolic upper gradient* g_u of functions $u \in L^p(0,T;N^{1,p}(\Omega))$, that is

$$g_u(x,t) = g_{u(\cdot,t)}(x),$$

for $(\mu \otimes \mathfrak{L}^1)$-a.e. $(x,t) \in \Omega_T$.

2.3 Poincaré inequalities and Sobolev embeddings

Throughout our work, we make two rather standard, yet nontrivial, assumptions. We require that the metric space X supports a *doubling measure* μ and a *weak* $(1,p)$*–Poincaré inequality*. In this section we discuss with detail these different notions.

2.3.1 Doubling measures

A positive Borel regular measure is said to be *doubling* if there exists a constant $C_d > 0$, called the *doubling constant*, such that

$$\mu(B(x,2r)) \le C_d \mu(B(x,r)),$$

for every $x \in X$ and for all $r > 0$. The doubling condition implies the following growth condition. For all $x, y \in X$ and $R \ge r$ we have

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \ge C\left(\frac{r}{R}\right)^Q, \tag{2.3}$$

where $Q = \log_2 C_d$ and $C = C_d^{-2}$. The Euclidean space $\mathbb{R}^N$ is doubling with the doubling constant 2^N and the best exponent in (2.3) is $Q = N$. Therefore, the constant Q is sometimes called the *doubling dimension*. It serves as a counterpart of dimension for our space, it implies that a metric space with a doubling measure is in some sense finite-dimensional.

A metric space equipped with a doubling measure has many useful properties. For example, such a space is always *locally compact*. In addition, if the space is complete, then it is *proper*, meaning that its closed and bounded subsets are compact. Furthermore, in a space with a doubling measure we have Vitali-type covering theorems, see [29]. Vitali's theorem then implies Lebesgue's differentiation theorem, i.e. for every non-negative locally integrable function u of X and μ-almost every $x \in X$ there holds

$$\lim_{r \searrow 0} \fint_{B(x,r)} u \, \mathrm{d}\mu = u(x).$$

2.3.2 Weak Poincaré inequality

Since we need a connection between the upper gradient and the function itself, we also assume that (X, d, μ) is a metric measure space supporting a *weak* $(1,p)$*-Poincaré inequality*. A metric measure space is said to support a *weak* $(1,p)$*-Poincaré inequality* if there exist a constant $C_P > 0$ and a dilation factor $\lambda > 1$, such that

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, \mathrm{d}\mu \le C_P r \left(\fint_{B(x,\lambda r)} g_u^p \, \mathrm{d}\mu\right)^{\frac{1}{p}},$$

for every $u \in N^{1,p}(X)$ and $B(x,r) \subset X$. Here,

$$u_{B(x,r)} = \fint_{B(x,r)} u \, \mathrm{d}\mu = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, \mathrm{d}\mu.$$

In the literature, the case when $\lambda = 1$ is only called *Poincaré inequality*. For our results, $\lambda > 1$ is sufficient and therefore, we sometimes call it a

Poincaré inequality and omit "weak". A point worth mentioning is that, the property of supporting a weak Poincaré inequality is preserved under biLipschitz transformations of X. On the contrary, Poincaré inequalities need not be preserved, see [3].

Supporting a Poincaré inequality entails many geometric properties for a metric space. Some of these implications are fundamental in our work. An immediate consequence is that a space supporting a Poincaré inequality has to be *connected*, [3]. Moreover, with the Poincaré inequality, we get a relation between the oscillation of u and its minimal p-weak upper gradient, via the underlying measure μ. As we know, when working in the usual Euclidean space $\mathbb{R}^n$, a $(1,p)$-Poincaré inequality is always available for any $1 < p < \infty$.

2.3.3 Sobolev embeddings

One essential implication of working with a doubling measure μ and a weak $(1,p)$-Poincaré inequality, is that together they imply a *Sobolev embedding*. More specifically, if X is a metric measure space equipped with a doubling measure μ and supporting a weak $(1,p)$-Poincaré inequality, then there exist positive constants $C > 0$ and $\lambda \geq 1$ such that

$$\left(\fint_{B(x,r)} |u - u_{B(x,r)}|^t \, \mathrm{d}\mu\right)^{\frac{1}{t}} \leq Cr \left(\fint_{B(x,\lambda r)} g_u^p \, \mathrm{d}\mu\right)^{\frac{1}{p}}, \tag{2.4}$$

where $t > p$. For more details, see [27].

Let $1 < p < \infty$. The smaller the exponent p, the stronger the $(1,p)$-Poincaré inequality. Indeed, if X supports a weak $(1,p)$-Poincaré, then, by Hölder inequality, it supports a weak $(1,p')$-Poincaré for all $p' > p$. Another important result we use, is a self-improving property for the Poincaré-inequality, established by Keith and Zhong in 2008 [35]. This principle says that if a complete metric space X is equipped with a doubling measure μ and supports a weak $(1,p)$-Poincaré inequality, then there exists $\varepsilon > 0$ for which X supports a $(1,s)$-Poincaré inequality for all $p - \varepsilon < s < p$. This plays an important role in the higher integrability results in **[IV]**.

2.3.4 Newtonian spaces with zero boundary values

The *p-capacity* of a set $E \subset X$ is the number

$$\mathrm{C}_p(E) = \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$, such that $u \geq 1$ on E. This capacity is sometimes referred to as the *Sobolev capacity*. We say that a property holds p-quasieverywhere (p-q.e.) if the set of points for which it does not hold has p-capacity zero.

In articles **[I]** and **[IV]** we work with *boundary value problems*. To be able to compare boundary values of Newtonian functions and solve the associated Dirichlet problem we need the concept of *Newtonian spaces with zero boundary values*.

Let Ω be an open and bounded subset of X. We define $N_0^{1,p}(\Omega)$ to be the set of functions $u \in N^{1,p}(X)$ that are zero on $X \setminus \Omega$ p-q.e. The space $N_0^{1,p}(\Omega)$ is equipped with the norm $\|\cdot\|_{N^{1,p}}$. Note also that if $C_p(X \setminus \Omega) = 0$, then $N_0^{1,p}(\Omega) = N^{1,p}(X)$. We shall therefore always assume that $C_p(X \setminus \Omega) > 0$. We state relevant results regarding these spaces with more detail in Section 2.5 of **[IV]**.

2.3.5 Variational capacity

For a measurable set $E \subset \Omega$, the *variational capacity* is defined as

$$\mathrm{cap}_p(E,\Omega) = \inf_u \int_\Omega g_u^p \, d\mu,$$

where the infimum is taken over all $u \in N_0^{1,p}(\Omega)$ such that $u \geq 1$ on E. The variational capacity is sometimes referred to as the *relative capacity*. One can show that if the space X is equipped with a doubling measure and supports a weak p-Poincaré inequality, then there exists a positive constant C such that

$$\frac{\mu(E)}{Cr^p} \leq \mathrm{cap}_p(E,B(x,2r)) \leq \frac{C\mu(B(x,r))}{r^p}, \tag{2.5}$$

when $E \subset B(x,r)$. The last inequality allows us to estimate the variational capacity of a set, see [4]. Furthermore, it is possible to compare the capacities, cap_p and C_p, and show that they are in many cases equivalent, see [6].

The variational capacity can be used to give some sort of regularity condition for the boundary of a set without actually having to define the boundary as a curve. This is done by defining a so called thickness condition, we say that a set E is *uniformly p-fat* if there exist positive constants C_0 and r_0 such that for all x in E and $0 < r < r_0$ we have

$$\mathrm{cap}_p(E \cap B(x.r), B(x,2r)) \geq C_0 \mathrm{cap}_p(B(x,r),B(x,2r)).$$

As with the weak Poincaré inequality, the uniform p-fatness also satisfies a *self-improving property*. Namely, let X be *proper, linearly locally convex* and equipped with a *doubling measure*. If a set $E \subset X$ is uniformly p-fat with $p > 1$, and X supports a weak $(1,p)$-Poincaré inequality, then E is also uniformly p_0-fat for some $1 < p_0 < p$, see [6].

A space X is called *linearly locally convex* if there exists constants $C_1 > 0$ and $r_1 > 0$ such that for all balls $B(x,r)$ in X with radius at most r_1, every pair of distinct points in the annulus $B(x,2r) \setminus B(x,r)$ can be connected by a curve lying in the annulus $B(x,2C_1 r) \setminus B(x,C_1^{-1} r)$.

The following proposition is a capacity version of the Sobolev-Poincaré inequality (2.4), also referred as *Maz'ya type estimate*. The proof is a straightforward generalization of the Euclidean case and it can be found in [4].

Let X be a doubling metric measure space supporting a weak $(1,p)$-Poincaré inequality. Then there exists C and $\lambda \geq 1$ such that for all balls B in X, $u \in N^{1,p}(X)$ and $S = \{x \in \frac{1}{2}B : u(x) = 0\}$, then

$$\left(\fint_B |u|^t \, \mathrm{d}\mu \right)^{\frac{1}{t}} \leq \left(\frac{C}{\operatorname{cap}_p(S,B)} \int_{\lambda B} g^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}, \tag{2.6}$$

for C depending on C_{PI} and $t > p$.

The concepts presented in this subsection are mainly used in article **[IV]**, where we prove higher integrability up to the boundary. For more details we refer the reader to [3, 6] and the references therein.

Next we move on to discuss in more detail the results established in articles **[I]-[IV]**.

3. Total variation flow on metric measure spaces

In the Euclidean case, the *total variation flow* (TVF) corresponds to the partial differential equation

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 \quad \text{on} \quad \Omega_T = \Omega \times (0,T),$$

where $\Omega \subset \mathbb{R}^N$ is an open set and $T > 0$. This can be seen as the limiting case of the evolution (parabolic) p-Laplace equation

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(Du|^{p-2}Du\right) = 0 \qquad 1 < p < \infty,$$

as $p \to 1$. The investigation of parabolic problems on metric measure spaces started not long ago with the work of Kinnunen, Marola, Miranda and Paronetto [38], concerning regularity problems. Since then, most contributions in this field of research have been made to stability theory [23], and regularity problems [54].

The motivation for the TVF arises from the image processing problems, [2]. Therefore, it is natural that the main interest has been directed towards questions such as the asymptotic behavior of solutions to the TVF or numerical aspects. As far as we know, this is the first time when existence and regularity questions are discussed for parabolic problems with linear growth on metric measure spaces. Our intention is to prove *existence and regularity* of the solutions. To this end, we take a purely variational approach to the study of the TVF. The advantages of this approach include better convergence and stability properties. This is an essential advantage as the solutions naturally lie in the space of *bounded variation* (BV functions), [1].

Before we discuss the results and preliminaries of papers **[II]** and **[III]**, let us first have a look at the space BV and its associated parabolic spaces.

3.1 Functions of bounded variation and parabolic spaces

The total variation flow does not have any regularizing effects, therefore it is natural to expect that the existence of solutions is related to the class of *functions of bounded variation* (BV functions), see [1]. We are mainly interested in local properties such as existence and regularity of the solutions. In order to reach these goals, there is a need to develop a new theory for the study of TVF based on a variational definition for the solution.

Functions of bounded variation, abbreviated as *BV functions*, are a somewhat more general class than Sobolev functions, in the sense that they may have discontinuities and even "jumps", but are nontheless differentiable in a very weak sense. The class has many applications, for example, as generalized solutions to partial differential equations with linear growth conditions, which often arise in calculus of variations, physics and image processing.

For an open and bounded set $\Omega \subset \mathbb{R}^n$ and $u \in L^1_{\mathrm{loc}}(\Omega)$ the *total variation* is defined as

$$\|Du\|(\Omega) = \sup\left\{ \int_\Omega u \operatorname{div}(\varphi)\, \mathrm{d}x : \ \varphi \in C^1_c(\Omega;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\Omega)} \le 1 \right\}. \tag{3.1}$$

We have as well that, in the Euclidean setting, functions of bounded variation are defined as integrable functions whose weak partial derivatives are Radon measures of finite mass. In the metric setting, we again need a somewhat different definition, which at the end turns out equivalent to the one we already have in Euclidean spaces. Therefore, for an open subset Ω of the metric measure space (X,d,μ), the *total variation* of $u \in L^1_{\mathrm{loc}}(\Omega)$ is defined as

$$\|Du\|(\Omega) = \inf\left\{ \liminf_{i\to\infty} \int_\Omega g_{u_i}\, \mathrm{d}\mu \right\},$$

where the infimum is taken over all sequences $(u_i)_{i\in\mathbb{N}}$ with $u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ for every $i \in \mathbb{N}$ and $u_i \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ as $i \to \infty$. Here g_{u_i} is a 1-weak upper gradient of u_i and $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ denotes the class of functions that are Lipschitz continuous on compact subsets of Ω. We say that a function $u \in L^1(\Omega)$ is of *bounded variation*, and denote $u \in BV(\Omega)$, if $\|Du\|(\Omega) < \infty$. We refer to this definition of the total variation as the *relaxation approach*. For more details, see [10, 57] and the references therein.

3.1.1 BV space via derivations

When working with the *total variation flow*, one cannot simply set $p = 1$ in the parabolic function space $L^p(0,T;N^{1,p}(\Omega))$, since the Newtonian space $N^{1,1}$ lacks important properties such as *reflexivity*. Therefore, it is replaced by BV, the space of functions with *bounded variation*. Unfortunately, choosing the space $L^1(0,T;BV(\Omega))$ still is not appropriate for our tasks, since Bochner measurability is too restrictive to ask, see Section 2.2. To this end, we consider a *weak version* of this parabolic space, denoted by $L^1_w(0,T;BV(\Omega))$. The Bochner measurability for Banach space valued functions is replaced by a weaker measurability condition that makes use of the so called *derivation* approach for BV. In this short section we see what the concept of *derivation* entails.

In paper **[II]**, we focus on how to overcome certain difficulties given by working in the setting of metric measure spaces. The main difficulty is given by the fact that the standard definition (*relaxation* approach) of the space BV on a metric measure space does not rely on an integration by parts formula, like (3.1), and therefore it is unsure if BV can be characterized as the dual space of a separable Banach space, as suggested by [1]. This complicates finding a weak measurability condition similar to the one posed in [8], and other works concerning the total variation flow and functionals with linear growth. To overcome this obstacle, and to be able to give a suitable definition of a *parabolic function space*, we make use of an alternative approach.

In [19], Di Marino introduced a concept that allows a characterization of BV via an integration by parts formula. It is based on so-called *derivations*, i.e. mappings on the space of Lipschitz functions with bounded support. Since the space X is proper, we consider Lip_c (set of Lipschitz functions with compact support) instead. Let $L^0(X)$ denote the space of measurable functions on X. By a (Lipschitz) *derivation* we denote a linear map $\eth : \mathrm{Lip}_c(X) \to L^0(X)$ such that the Leibniz rule

$$\eth(fg) = f\eth(g) + g\eth(f)$$

holds true for all $f,g \in \mathrm{Lip}_c(X)$, and for which there exists a function $h \in L^0(X)$ such that for μ-a.e. $x \in X$ and all $f \in \mathrm{Lip}_c(X)$ there holds

$$|\eth(f)|(x) \leq h(x) \cdot \mathrm{Lip}_a(f)(x), \tag{3.2}$$

where $\mathrm{Lip}_a(f)(x)$ denotes the asymptotic Lipschitz constant of f at x. The set of all such derivations is be denoted by $\mathrm{Der}(X)$. The smallest function h satisfying (3.2) is denoted by $|\eth|$, and we write $\eth \in L^p(X)$ when we mean to say $|\eth| \in L^p(X)$.

For given $\eth \in \mathrm{Der}(X)$ with $\eth \in L^1_{\mathrm{loc}}(X)$ we define the *divergence operator* as

$$\begin{aligned} \mathrm{div}(\eth) : \mathrm{Lip}_c(X) &\to \mathbb{R} \\ f &\mapsto -\int_X \eth(f)\ \mathrm{d}\mu. \end{aligned}$$

We say that $\mathrm{div}(\eth) \in L^p(X)$, if this operator admits an integral representation via a unique L^p-function $\widetilde{h}$, i.e.

$$\int_X \eth(f)\,\mathrm{d}\mu = -\int_X \widetilde{h} f\,\mathrm{d}\mu.$$

For all $p, q \in [1, \infty]$, we set

$$\mathrm{Der}_p(X) = \left\{ \eth \in \mathrm{Der}(X) : \ \eth \in L^p(X) \right\},$$

and

$$\mathrm{Der}_{p,q}(X) = \left\{ \eth \in \mathrm{Der}(X) : \ \eth \in L^p(X), \ \mathrm{div}(\eth) \in L^q(X) \right\}.$$

When $p = \infty = q$ we write $\mathrm{Der}_b(X)$ instead of $\mathrm{Der}_{\infty,\infty}(X)$. For $u \in L^1(X)$ we say that u is of *bounded variation (in the sense of derivations)* in X , denoted $u \in BV_\eth(X)$, if there is a linear and continuous map $L_u : \mathrm{Der}_b(X) \to M(X)$ such that

$$\int_X \mathrm{d}L_u(\eth) = -\int_X u \,\mathrm{div}(\eth)\ \mathrm{d}\mu, \tag{3.3}$$

for all $\eth \in \mathrm{Der}_b(X)$ and satisfying $L_u(h\eth) = hL_u(\eth)$ for any bounded $h \in \mathrm{Lip}(X)$, where $M(X)$ denotes the space of finite signed Radon measures on X. This characterization of BV, in the sense of derivations, is well-posed, see [19]. If we take any two maps $L_u, \widetilde{L}_u$ as in (3.3), the Lipschitz-linearity of derivations ensures that $L_u(\eth) = \widetilde{L}_u(\eth)$ μ-a.e. for all $\eth \in \mathrm{Der}_b(X)$. The common value is then denoted by $Du(\eth)$.

An important result is that for a complete and separable metric measure space (X, d, μ) with a locally finite measure μ (as in our case) the spaces $BV(X)$ and $BV_\eth(X)$ are equivalent. Hence, we obtain the following representation formula

$$\|Du\|(\Omega) = \sup\left\{ \int_\Omega u \,\mathrm{div}(\eth)\ \mathrm{d}\mu : \ \eth \in \mathrm{Der}_b(X), \ \mathrm{supp}(\eth) \Subset \Omega, \ |\eth| \le 1 \right\},$$

as a generalization of (3.1). For further details, see [12, 28, 57].

The derivation approach allows us now to give a suitable definition of our weak parabolic function space. More specifically, the space $L^1_w(0,T;BV(\Omega))$ consists of those v in $L^1(\Omega_T)$, for which there holds:

1. $v(\cdot, t) \in BV(\Omega)$ for a.e. $t \in (0, T)$,

2. $\int_0^t \|Dv(t)\|(\Omega)\,\mathrm{d}t < \infty$,

3. the mapping $t \mapsto v(\cdot, t)$ is weakly measurable, i.e. the mapping

$$(0, T) \ni t \mapsto \int_\Omega v(t)\mathrm{div}(\eth)\,\mathrm{d}\mu, \tag{3.4}$$

   is measurable for all $\eth \in \mathrm{Der}_b(\Omega)$ with $\mathrm{supp}(\eth) \Subset \Omega$.

3.2 Existence of parabolic minimizers to the total variation flow

This section is devoted to a joint work **[II]** with Buffa and Collins on the *existence* of parabolic minimizers to the *total variation flow* on *metric* *measure spaces*, see [12]. More precisely, we consider minimizers of integral functionals that are related to scalar functions $u:\Omega\times(0,T)\to\mathbb{R}$ which satisfy the inequality

$$\iint_{\Omega_T} u\partial_t\varphi \, \mathrm{d}\mu\mathrm{d}t + \int_0^T \|Du(t)\|(\Omega)\, \mathrm{d}t \le \int_0^T \|D(u+\varphi)(t)\|(\Omega)\, \mathrm{d}t, \tag{3.5}$$

for all test functions $\varphi\in \mathrm{Lip}_c(\Omega_T)$ where $\|Du(t)\|(\Omega)$ denotes the total variation of $u(\cdot,t)$ on Ω. Here, $\Omega\subset X$ is a bounded domain, where (X,d,μ) is a *complete, separable* and *connected* metric space with a metric d and endowed with a Borel *doubling* measure μ. In addition to the doubling property, we demand that the metric measure space (X,d,μ) supports a *weak* $(1,1)$-*Poincaré inequality*.

This project generalizes the results given in [8], while considering the case where the functional depends only on the total variation.

Existence for parabolic problems on metric measure spaces has already been dealt with in [15], this paper treats time-dependent boundary data, the author considered integral functionals with p-growth for $p>1$. We recall that the total variation flow corresponds to the case $p=1$. In the elliptic case, existence for functions of least gradient has been considered by Korte, Lahti, Li and Shanmugalingam in [43].

Once having a reasonable definition of the parabolic function space where our concept of solution lies, see Section 3.1, our method of proof is aligned to the one proposed in the work of Bögelein, Duzaar and Marcellini [8].

3.2.1 Assumptions and definitions

Since traces of BV-functions are a delicate issue, we have to be careful on how to formulate the Cauchy-Dirichlet problem. We follow an approach that is well known in the Euclidean case and can also be applied in the metric setting.

We consider an open and bounded domain Ω^* that is slightly larger than Ω, i.e. $\Omega\Subset\Omega^*$. For given $u_0\in BV(\Omega^*)$, we denote $u\in BV_{u_0}(\Omega)$ if and only if $u\in BV(\Omega^*)$ and $u=u_0$ μ-a.e. in $\Omega^*\setminus\Omega$. The condition on the lateral boundary can then be understood in the sense that $u(\cdot,t)\in BV_{u_0}(\Omega)$ for a.e. $t\in(0,T)$. Furthermore, we consider an initial datum

$$u_0\in BV(\Omega^*)\cap L^2(\Omega^*). \tag{3.6}$$

Based on this, a map $u:\Omega_T^*\to\mathbb{R}$, $T\in(0,\infty)$ in the class $L^1_w(0,T;BV_{u_0}(\Omega))\cap C^0\left([0,T];L^2(\Omega^*)\right)$ is a *variational solution* on Ω_T to the Cauchy-Dirichlet problem for the total variation flow if and only if the *variational inequality*

$$\begin{aligned}\int_0^T \|Du(t)\|(\Omega^*)\, \mathrm{d}t \le& \int_0^T \left[\int_{\Omega^*} \partial_t v(v-u)\mathrm{d}\mu + \|Dv(t)\|(\Omega^*)\right] \mathrm{d}t\\ &- \frac{1}{2}\|(v-u)(T)\|^2_{L^2(\Omega^*)} + \frac{1}{2}\|v(0)-u_0\|^2_{L^2(\Omega^*)}\end{aligned} \tag{3.7}$$

holds for any $v \in L^1_w(0,T;BV_{u_0}(\Omega))$ with $\partial_t v \in L^2(\Omega_T^*)$ and $v(0) \in L^2(\Omega^*)$. Moreover, a map $u : \Omega_\infty^* \to \mathbb{R}$ is termed a *global variational solution* if $u \in L^1_w(0,T;BV_{u_0}(\Omega)) \cap C^0\left([0,T];L^2(\Omega^*)\right)$ for any $T > 0$, and u is a variational solution on Ω_T for any $T \in (0,\infty)$.

On the other hand, we say that a measurable function $u : \Omega_\infty^* \to \mathbb{R}$ is a *parabolic minimizer* to the total variation flow if and only if for any $T > 0$ one has $u \in L^1_w(0,T;BV_{u_0}(\Omega))$ together with the following condition

$$\int_0^T \left(\int_{\Omega^*} u \cdot \partial_t \varphi \, \mathrm{d}\mu + \|Du(t)\|(\Omega^*) \right) \mathrm{d}t \le \int_0^T \|D(u+\varphi)(t)\|(\Omega^*) \, \mathrm{d}t \tag{3.8}$$

for all $\varphi \in \mathrm{Lip}_c(\Omega_T^*)$.

3.2.2 Main results

The main results in paper **[II]** concern the *existence, uniqueness* and *regularity* of variational solutions, see Theorem 2.8 and Theorem 2.9 in [12]. More specifically, we show the existence of a unique global variational solution. In addition, we prove that any variational solution on Ω_T with $T \in (0,\infty]$ satisfies

$$\partial_t u \in L^2(\Omega^*) \text{ and } u \in C^{0,\frac{1}{2}}\left([0,\tau];L^2(\Omega^*)\right) \text{ for all } \tau \in \mathbb{R} \cap (0,T]. \tag{3.9}$$

Furthermore, for the time derivative $\partial_t u$ there holds the quantitative bound

$$\int_0^T \int_{\Omega^*} |\partial_t u|^2 \, \mathrm{d}\mu \, \mathrm{d}t \le \|Du_0\|(\Omega^*). \tag{3.10}$$

Finally, for any $t_1, t_2 \in \mathbb{R}$ with $0 \le t_1 < t_2 \le T$ one has the energy estimate

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \|Du(t)\|(\Omega^*) \, \mathrm{d}t \le \|Du_0\|(\Omega^*). \tag{3.11}$$

Moreover, it is shown that variational solutions are also parabolic minimizers, which implies the existence of the latter, see Prop. 6.2 in [12]. In the overall context, this last result is of high importance for the author of this dissertation, since it ties the *existence theory* of paper **[II]** with the *regularity theory* of paper **[III]**, as we will see ahead.

3.2.3 Remarks on the method of proof

As already mentioned, our method of proof is based on the approach in [8]. Before we explain the actual existence proof, we talk about a few preliminary results and tools that helped us achieve it.

Smoothing procedures in time
As variational solutions lack the appropriate time-regularity, they are in general not admissible as comparison maps in (3.7). This is why a

mollification procedure with respect to time (also known as *time-smoothing*) has to be performed. Let X be a Banach space, and $v_0 \in X$. Consider some $v \in L^r(0,T;X)$ for some $1 \le r \le \infty$, and define for $h \in (0,T]$ and $t \in [0,T]$ the *mollification in time* by

$$[v]_h^{v_0}(t) = e^{-\frac{t}{h}} v_0 + \frac{1}{h} \int_0^t e^{-\frac{s-t}{h}} v(s)\ \mathrm{d}s.$$

Regarding the basic properties of the mollification $[\cdot]_h^{v_0}$, we refer the reader to [7, 16].

In [12], we first show that the space $L^1_w(0,T;BV(\Omega))$ is closed under time-smoothing. This *mollification* technique makes it possible to show that the time derivative of a variational solution exists and belongs to L^2.

Localization on subcylinders

Next, we show that a variational solution on Ω_T^* is also a variational solution on any subcylinder $\Omega^*_{t_1,t_2}$, with $0 \le t_1 < t_2 \le T$.

To obtain this, we consider a comparison function for the subcylinder, i.e. $v \in L^1_w(t_1,t_2;BV_{u_0}(\Omega))$ with $\partial_t v \in L^2(\Omega^*_{t_1,t_2})$ and $v(t_1) \in L^2(\Omega^*)$. We consider a specific cutoff function $\zeta_\theta(t)$, for a fixed θ in the interval $(0, \frac{1}{2}(t_2-t_1))$. We then use $\widetilde{v} = \zeta_\theta v + (1-\zeta_\theta)[u]_h^{u_0}$ as a comparison function in (3.7). By rewriting terms, using the convexity of the total variation and exploiting properties of the mollification in time, we then find that the variational inequality holds on the subcylinder.

The initial condition

We now make use of the localization described above to show that variational solutions attend the initial datum u_0 in the L^2-sense, meaning

$$\lim_{t \searrow 0} \|u(t) - u_0\|^2_{L^2(\Omega^*)} = 0.$$

Since u_0, as in (3.6), is admissible as a comparison function in the variational inequality on any subcylinder Ω^*_τ for $\tau \in (0,T)$. We have

$$\int_0^\tau \|Du(t)(\Omega)\|\ \mathrm{d}t + \frac{1}{2}\|u(\tau) - u_0\|^2_{L^2(\Omega^*)} \le \tau \|Du_0\|(\Omega) < \infty.$$

By discarding the first term on the left-hand side and then letting $\tau \searrow 0$ we obtain the assertion.

Regularity of variational solutions

Our next aim is to prove the regularity properties (3.9), (3.10) and (3.11) for a variational solution. To this end, let $\widetilde{u}(s) = u(s+t_1)$ for $0 \le t_1 < t_2 \le T$ and $s \in (0, t_2 - t_1)$. Then, $\widetilde{u}$ fulfills the variational inequality (3.7) on the subcylinder $\Omega^*_{t_2-t_1}$ with initial datum $u(t_1)$. We test this variational inequality with $v = [\widetilde{u}]_h^{u(t_1)}$. We then arrive at

$$\int_0^{t_2-t_1} \int_{\Omega^*} |\partial_t [\widetilde{u}]_h^{u(t_1)}|^2 \,\mathrm{d}\mu\,\mathrm{d}t \leq \|D\widetilde{u}(0)\|(\Omega^*) - \left[\|D\widetilde{u}(t)\|(\Omega^*)\right]_h^{\|D\widetilde{u}(0)\|(\Omega^*)} (t_2 - t_1)$$

$$\leq \|D\widetilde{u}(0)\|(\Omega^*).$$

By letting $h \to 0$, we infer the existence of $\partial_t u \in L^2(\Omega^*_{t_2-t_1})$. Furthermore, the previous inequality yields

$$\int_0^{t_2-t_1} \int_{\Omega^*} |\partial_t \widetilde{u}|^2 \,\mathrm{d}\mu\,\mathrm{d}t \leq \|D\widetilde{u}(0)\|(\Omega^*) - \|D\widetilde{u}(t_2 - t_1)\|(\Omega^*),$$

and

$$\int_{t_1}^{t_2} \int_{\Omega^*} |\partial_t u|^2 \,\mathrm{d}\mu\,\mathrm{d}t \leq \|Du(t_1)\|(\Omega^*) - \|Du(t_2)\|(\Omega^*),$$

respectively. The latter estimate holds true for a.e. $0 \leq t_1 < t_2 \leq T$ and in particular for $t_1 = 0$ and $t_2 = T$. In the case $T = \infty$, we simply let $t_2 \to \infty$. This estimate also implies

$$\|u(t_2) - u(t_1)\|^2_{L^2(\Omega^*)} \leq |t_1 - t_2| \|Du_0\|(\Omega^*),$$

and

$$\int_{\Omega^*} |u(T)|^2 \,\mathrm{d}\mu \leq 2t \|Du_0\|(\Omega^*),$$

from which we infer $u \in C^{0,\frac{1}{2}}([0,\tau]; L^2(\Omega^*))$ for any $\tau \in \mathbb{R} \cap (0,T]$.

To establish the estimate (3.11), we use the just established fact that $\partial_t u \in L^2(\Omega^*_T)$, to apply an integration by parts to the variational inequality (3.7) which yields

$$\int_0^{\tau} \|Du(t)\|(\Omega^*) \ \mathrm{d}t \leq \int_0^{\tau} \left[\int_{\Omega^*} \partial_t u(v-u)\,\mathrm{d}\mu + \|Dv(t)\|(\Omega^*) \right] \mathrm{d}t,$$

for any $\tau \in \mathbb{R} \cap (0,T]$. Now, for $t_1, t_2 \in \mathbb{R}$ with $0 \leq t_1 < t_2 \leq \tau$ we define a specific $\zeta_{t_1,t_2}(t)$ and let $v = u + \zeta_{t_1,t_2}([u]_h^{u_0} - u)$. We see that v is indeed admissible as a comparison function in the minimality condition on Ω^*_τ. Using the convexity of the total variation and rearranging terms, then leads to

$$\begin{aligned} 0 \leq & -h \int_0^{t_2} \int_{\Omega^*} \zeta_{t_1,t_2} \partial_t u \partial_t [u]_h^{u_0} \,\mathrm{d}\mu\,\mathrm{d}t \\ & + h \int_0^{t_2} \zeta'_{t_1,t_2} [\|Du(t)\|(\Omega^*)]_h^{\|Du(0)\|(\Omega^*)} \mathrm{d}t + h \|Du_0\|(\Omega^*). \end{aligned}$$

Dividing both sides by $h > 0$ and letting $h \to 0$ finally lets us obtain (3.11).

Uniqueness

To show the uniqueness of the variational solution, we use a *comparison principle*, see Lemma 4.1 in [12]. The precise statement of the lemma is, if one considers two variational solutions u and $\widetilde{u}$, with initial data u_0 and $\widetilde{u}_0$, respectively, then $u_0 \leq \widetilde{u}_0$ μ-a.e on Ω^* implies $u \leq \widetilde{u}$ $(\mu \otimes \mathscr{L}^1)$-a.e. on Ω_T, see [12] for the complete proof. Applying this to two variational solutions $u, \widetilde{u}$ with the same initial datum, we obtain *uniqueness*.

The existence proof

Finally, we briefly explain the proof of the existence of variational solutions. We consider the relaxed convex functionals

$$\mathscr{F}_\varepsilon[v] = \int_0^T e^{-\frac{t}{\varepsilon}} \left[\frac{1}{2} \int_{\Omega^*} |\partial_t v|^2 \, \mathrm{d}\mu + \frac{1}{\varepsilon} \|Dv(t)\|(\Omega^*) \right] \mathrm{d}t,$$

for $\varepsilon \in (0,1]$. The properties of the total variation allow the application of standard methods in the *calculus of variations* to ensure the existence of minimizers u_ε of $\mathscr{F}_\varepsilon$. To prove the existence of these minimizers in our setting, we apply a compactness result by Simon, see Theorem 1 in [61].

Energy estimates

Now, the sequence of minimizers (u_ε) are expected to converge to a variational solution (an therefore to a parabolic minimizer), based on an idea in the Euclidean case. For this, we first establish the following energy estimate

$$\begin{aligned}\int_0^T \zeta(t) \|Du_\varepsilon(t)\|(\Omega^*) \, \mathrm{d}t \le \int_0^T \zeta(t) \|D(u_\varepsilon + \varphi)(t)\|(\Omega^*) \, \mathrm{d}t + \int_0^T \int_{\Omega^*} \zeta \partial_t u_\varepsilon \varphi \, \mathrm{d}\mu \, \mathrm{d}t \\ + \varepsilon \int_0^T \int_{\Omega^*} \left[\zeta' \partial_t u_\varepsilon \varphi + \zeta \partial_t u_\varepsilon \partial_t \varphi \right] \mathrm{d}\mu \, \mathrm{d}t,\end{aligned}$$

for any Lipschitz map $\zeta : (0,T) \to [0,1]$ and any test function $\varphi \in L^1_w(0,T;BV_0(\Omega))$ with $\partial_t \varphi \in L^2(\Omega_T^*)$ satisfying

$$\int_0^T \|D(u_\varepsilon + \varphi)(t)\|(\Omega^*) \, \mathrm{d}t < \infty,$$

and such that either $\zeta(0) = 0$ and $\varphi(0) \in L^2(\Omega^*)$ or $\varphi(0) = 0$.

With this energy estimate we are able to get the next inequalities

$$\int_0^T \int_{\Omega^*} |\partial_t u_\varepsilon|^2 \, \mathrm{d}\mu \, \mathrm{d}t \le \|Du_0\|(\Omega^*),$$

and

$$\int_{t_1}^{t_2} \|Du_\varepsilon(t)\|(\Omega^*) \, \mathrm{d}t \le \left(t_2 - t_1 + \frac{\varepsilon}{2} \right) \|Du_0\|(\Omega^*).$$

These last estimates allow us to obtain uniform bounds for u_ε in $L^2(\Omega_T^*)$ and $C^{0,\frac{1}{2}}([0,T];L^2(\Omega^*))$, as well as establish uniform bounds for ∂u_ε.

The limit procedure

Again, using Simon's compactness result [61], the previous uniform bounds for (u_ε) allow us to extract a subsequence, that we shall still denote by (u_ε), and find a measurable function $u : \Omega_T^* \to \mathbb{R}$ such that

$$\begin{cases} u_\varepsilon \to u & \text{strongly in } L^1(\Omega_T^*), \\ u_\varepsilon \to u & \text{a.e on } \Omega_T^*, \\ u_\varepsilon \rightharpoonup u & \text{weakly in } L^2(\Omega_T^*), \\ \partial_t u_\varepsilon \rightharpoonup \partial_t u & \text{weakly in } L^2(\Omega_T^*). \end{cases}$$

We then show that $u \in L^1_w(0,t:BV_{u_0}(\Omega))$.

Finally, all that is left to prove is that u is indeed a variational solution. Meaning, we prove that u satisfies the variational inequality (3.7).

3.3 Regularity of variational solutions

Once having existence of solutions to the total variation flow, our interest turned into studying regularity theory to the TVF in general metric measure spaces.

The regularity theory of nonlinear parabolic problems in the metric space context has been developed and studied in [33, 34, 38, 51, 52, 53, 54]. All of these results consider variational inequalities with p-growth for $p > 1$. In paper **[III]**, a joint work with Buffa and Kinnunen [13], our main goal is to extend the results of DiBenedetto, Gianazza and Klaus [21] to a metric measure space. Throughout the work, we assume that (X,d,μ) is a *complete* metric measure space endowed with a Borel measure μ. Furthermore, we ask for the measure μ to be *doubling* and that the metric measure space (X,d,μ) supports a *weak* $(1,1)$-*Poincaré inequality*. Our assumption on the time regularity of a variational solution is initially weaker than in [21] and thus our results may be interesting also in the Euclidean case.

3.3.1 Variational solutions in metric measure spaces

As previously mentioned, the last important result in article **[II]** is that any *variational solution* to the TVF is a *parabolic minimizer*. This is why, in the context of paper **[III]**, we say that a function $u \in L^1_{\mathrm{loc}}(0,T;BV_{\mathrm{loc}}(\Omega))$ is a *variational solution* to the total variation flow in Ω_T, with with $\Omega \subset X$ and $0 < T < \infty$, if

$$\int_0^T \left(\int_\Omega -u(t)\frac{\partial\varphi}{\partial t}(t)\,\mathrm{d}\mu + \|Du(t)\|(\Omega) \right) \mathrm{d}t \le \int_0^T \|D(u+\varphi)(t)\|(\Omega)\,\mathrm{d}t, \qquad (3.12)$$

for every $\varphi \in \mathrm{Lip}(\Omega_T)$ with $\mathrm{supp}\varphi \Subset \Omega_T$.

It is worth mention that, if we take any test function $\tilde{\varphi} \in \mathrm{Lip}(\Omega_T)$ with $\mathrm{supp}\tilde{\varphi} \Subset \Omega_T$ and define $\Phi(x,t) = \tilde{\varphi}(x, T-t)$, then testing (3.8) gives us (3.12), but with $\tilde{\varphi}$ evaluated in $(T-t)$. Since the resulting inequality holds for any test function $\tilde{\varphi}$, we can infer that (3.8) implies (3.12). This is why, for paper **[III]**, we kept and worked with the definition given by DiBenedetto, Gianazza and Klaus in [21].

Next we move on to discuss in more detail the results established in article **[III]**.

3.3.2 Main results

In paper **[III]**, we discuss a purely variational approach to the total variation flow on metric measure spaces with a doubling measure and supporting a Poincaré inequality. We apply the concept of parabolic De Giorgi classes, to prove a *necessary and sufficient* condition for a variational solution to be *continuous* at a given point, in terms of sufficient fast decay of the total variation of the function about the point. More specifically, if $u \in L^1_{\mathrm{loc}}(0,T;BV_{\mathrm{loc}}(\Omega))$ is a variational solution to the total variation flow in Ω_T. Then u is continuous at some $(x_0,t_0) \in \Omega_T$ if and only if

$$\limsup_{\rho\to 0+} \frac{\rho}{(\mu\otimes\mathscr{L}^1)(Q^-_{\rho,1}(x_0,t_0))} \int_{t_0-\rho}^{t_0} \|Du(t)\|(B_\rho(x_0))\,\mathrm{d}t = 0.$$

3.3.3 Remarks on the method of proof

Before we explain the actual continuity proof, we talk about a few preliminary results and tools that helped us achieve it.

Parabolic De Giorgi Class.
The first and, in many ways, most important step in the proofs of the regularity results presented in this section, is to establish an energy estimate for the variational solution. The regularity results (namely, continuity at a given point), are then proved based only on this estimate, and on the assumptions made on the underlying metric measure space.

In our case, the energy estimate comes in terms of belonging to a *De Giorgi class*. Indeed, we say that a function $u \in L^1_{\mathrm{loc}}(0,T;BV_{\mathrm{loc}}(\Omega))$ belongs to the *parabolic De Giorgi class* $DG^{\pm}(\Omega_T;\gamma)$, with $\gamma>0$, if

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t_0-\theta\rho\le t\le t_0} \int_{B_\rho(x_0)} \varphi(t)(u(t)-k)^2_\pm\,\mathrm{d}\mu + \int_{t_0-\theta\rho}^{t_0} \|D(\varphi(u-k)_+)(t)\|(B_\rho(x_0))\,\mathrm{d}t \\
&\le \gamma \iint_{Q^-_{\rho,\theta}(x_0,t_0)} \left|\frac{\partial\varphi}{\partial t}(t)\right| (u(t)-k)^2_\pm\,\mathrm{d}\mu\,\mathrm{d}t + \gamma \iint_{Q^-_{\rho,\theta}(x_0,t_0)} g_\varphi(t)(u(t)-k)_+\,\mathrm{d}\mu\,\mathrm{d}t \\
&\quad - \left[\int_{B_\rho(x_0)} \varphi(t)(u(t)-k)^2_\pm\,\mathrm{d}\mu\right]_{t=t_0-\theta\rho}^{t_0},
\end{aligned} \tag{3.13}$$

for every $Q^-_{\rho,\theta}(x_0,t_0) = B_\rho(x_0)\times(t_0-\theta\rho,t_0] \Subset \Omega_T$, $k\in\mathbb{R}$ and $\varphi \in \mathrm{Lip}(\Omega_T)$ with $\operatorname{supp}\varphi \Subset B_\rho(x_0)\times(0,T)$ and $0\le\varphi\le 1$.

The *parabolic De Giorgi class* $DG(\Omega_T;\gamma)$ is then defined as

$$DG(\Omega_T;\gamma) = DG^+(\Omega_T;\gamma)\cap DG^-(\Omega_T;\gamma).$$

The motive behind defining the De Giorgi class is that, it enables us to extend the study of parabolic partial differential equations to metric measure spaces. This in turn helps us to better understand those aspects of the theory which are independent of the geometry of the space, where the partial differential equation is originally defined.

Establishing the energy estimate
As already anticipated, the proof of the necessary and sufficient conditions for continuity of a variational solution to the TVF, only uses the local integral inequalities in (3.13) and the assumptions made on the underlying metric measure space. Therefore, we need to establish the energy estimate for the variational solution, in other words we show that a variational solution u to the total variation flow in Ω_T belongs to the parabolic De Giorgi class $DG(\Omega_T;\gamma)$.

There is a technical difficulty present when establishing these energy estimates, it is not clear that the time regularity of a variational solution is a priori sufficient for placing it as the test function and performing the usual techniques used for obtaining an energy estimate. We treat this issue by using a *mollification technique*. For $u \in L^p_{\mathrm{loc}}(0,T;N^{1,p}_{\mathrm{loc}}(\Omega))$, $1 \le p < \infty$, we consider the *time mollification*

$$u_\varepsilon(t) = \int_{-\varepsilon}^{\varepsilon} \eta_\varepsilon(s) u(t-s)\,\mathrm{d}s,$$

where $\eta_\varepsilon(s) = \frac{1}{s}\eta(\frac{s}{\varepsilon})$, $\varepsilon > 0$, is a standard mollifier. The idea of this technique is to deduce the energy estimate for the mollification and finally to establish the same estimate at the limit.

In the metric space setting, one runs into unexpected difficulties when taking the limit. To establish convergence of the estimate, one needs that the weak upper gradient of the difference also tends to zero. In the Euclidean case this poses no difficulties as we can use the linearity of the gradients. In the general metric setting the situation is not that simple, as taking an upper gradient does not preserve linearity. Nonetheless, it turns out to be problematic, but not impossible, to establish the convergence by using only the theory of upper gradients.

Indeed, assume that $u \in L^p_{\mathrm{loc}}(0,T;N^{1,p}_{\mathrm{loc}}(\Omega))$, $1 \le p < \infty$. Then $u_\varepsilon \to u$ in $L^p_{\mathrm{loc}}(0,T;N^{1,p}_{\mathrm{loc}}(\Omega))$ as $\varepsilon \to 0$. In particular, we have $g_{u_\varepsilon - u} \to 0$ in $L^p_{\mathrm{loc}}(\Omega_T)$ as $\varepsilon \to 0$. Moreover, as $s \to 0$, we have $g_{u(\cdot,t-s)-u(\cdot,t)} \to 0$ in $L^p_{\mathrm{loc}}(\Omega_T)$ uniformly in t. This important approximation is proved in more generality in [11].

De Giorgi Lemma
After establishing the energy estimates, we focus on proving that functions in a parabolic De Giorgi class are bounded from below. More specifically, let $\rho,\theta > 0$ be such that $Q^-_{\rho,\theta}(x_0,t_0) \subset \Omega_T$ and let

$$m_+ \ge \operatorname*{ess\,sup}_{Q^-_{\rho,\theta}(x_0,t_0)} u, \quad m_- \le \operatorname*{ess\,inf}_{Q^-_{\rho,\theta}(x_0,t_0)} u \quad \text{and} \quad \omega \ge m_+ - m_-.$$

Assume that $u \in DG^-(\Omega_T;\gamma)$, for $\gamma > 0$. Then

(i) For $a,\xi \in (0,1)$ and $\overline{\theta} \in (0,\theta)$, there exists a constant $\nu_- > 0$ such that if

$$(\mu \otimes \mathscr{L}^1)(Q^-_{\rho,\theta}(x_0,t_0) \cap \{u \le m_- + \xi\omega\}) \le \nu_-(\mu \otimes \mathscr{L}^1)(Q^-_{\rho,\theta}(x_0,t_0)),$$

then $u \ge m_- + a\xi\omega$ $(\mu \otimes \mathscr{L}^1)$-almost everywhere in $B_{\frac{\rho}{2}}(x_0) \times (t_0 - \overline{\theta}\rho, t_0]$.

(ii) For $a,\xi \in (0,1)$ and $\overline{\theta} \in (0,\theta)$, there exists a constant $\nu_+ > 0$ such that if

$$(\mu \otimes \mathscr{L}^1)(Q^-_{\rho,\theta}(x_0,t_0) \cap \{u \ge m_+ - \xi\omega\}) \le \nu_+(\mu \otimes \mathscr{L}^1)(Q^-_{\rho,\theta}(x_0,t_0)),$$

then $u \le m_+ - a\xi\omega$ $(\mu \otimes \mathscr{L}^1)$-almost everywhere in $B_{\frac{\rho}{2}}(x_0) \times (t_0 - \overline{\theta}\rho, t_0]$.

The proof of these bounds is relatively straightforward, it uses the estimates from the De Giorgi class definition, an standard iteration lemma, see Lemma 5.1 in [22], as well as an isoperimetric inequality for functions of bounded variation, see Lemma 2.6 in [13].

Time expansion of positivity

The last key ingredient for the proof of the continuity result is the proof of the *expansion of positivity*, see Lemma 7.1 in [21]. Roughly speaking, it asserts that information on the measure of the positivity set of u, at time level t_0, over the ball $B_\rho(x_0)$, translates into an expansion of positivity set in time (from t_0 to $t_0 + \theta\rho$, for some suitable θ). For a cylinder $Q^+_{2\rho,\theta}(x_0,t_0) = B_{2\rho}(x_0) \times (t_0, t_0 + \theta\rho) \subset \Omega_T$, let

$$m_+ \ge \operatorname*{ess\,sup}_{Q^+_{2\rho,\theta}(x_0,t_0)} u, \quad m_- \le \operatorname*{ess\,inf}_{Q^+_{2\rho,\theta}(x_0,t_0)} u \quad \text{and} \quad \omega \ge m_+ - m_-.$$

The parameter θ is actually determined in the proof. Let $\xi \in (0,1)$ be a fixed parameter. For $u \in DG^-(\Omega_T;\gamma)$ with $\gamma > 0$, assume that

$$\mu(\{x \in B_\rho(x_0) : u(x,t_0) \ge m_- + \xi\omega\}) \ge \frac{1}{2}\mu(B_\rho(x_0)),$$

for some $(x_0,t_0) \in \Omega_T$ and some $\rho > 0$. Then there exist $\delta \in (0,1)$ and $\varepsilon \in (0,1)$ such that

$$\mu(\{x \in B_\rho(x_0) : u(x,t) \ge m_- + \varepsilon\xi\omega\}) \ge \frac{1}{4}\mu(B_\rho(x_0)),$$

for every $t \in (t_0, t_0 + \delta\xi\omega\rho)$.

As in the previous section, the proof of the time expansion of positivity is straighforward, most of the arguments are based on the energy estimates and the boundedness from below of functions in the De Giorgi class.

Characterization of continuity

Finally, we are ready to prove the main result of paper **[III]**, the characterization of continuity.

We begin with the necessary part, assuming that u is continuous at $(x_0,t_0) \in \Omega_T$. We define ζ, a Lipschitz cutoff function with $0 \le \zeta \le 1$, $\zeta = 0$ on

$(X\times\mathbb{R})\setminus Q^-_{2\rho,1}(x_0,t_0)$, $\zeta=1$ on $Q^-_{\frac{3}{2}\rho,1}(x_0,t_0)$, $\zeta(\cdot,t_0-2\rho)=0$, $\zeta_t\ge 0$ and $g_\zeta+\zeta_t\le\frac{3}{\rho}$. Applying (3.13) with $\theta=1$, $k=0$, neglecting the supremum term of the left-hand side and using the definition of ζ, we obtain

$$\frac{\rho}{(\mu\otimes\mathscr{L}^1)(Q^-_{\rho,1}(x_0,t_0))}\int_{t_0-2\rho}^{t_0}\|Du(t)\|(B_\rho(x_0))\,\mathrm{d}t \le 6C_\mu\gamma\iint_{Q^-_{2\rho,1}(x_0,t_0)}(|u(t)|+u(t)^2)\,\mathrm{d}\mu\,\mathrm{d}t.$$

The right-hand side tends to zero as $\rho\to 0$, implying the necessary condition.

For the sufficient part, we proceed by contradiction. Namely, we assume that u is not continuous at (x_0,t_0). For $\rho>0$ small enough, so that $Q^-_{\rho,1}(x_0,t_0)=B_\rho(x_0)\times(t_0-\rho,t_0]\subset\Omega_T$, we set

$$m_+=\operatorname*{ess\,sup}_{Q^-_{\rho,1}(x_0,t_0)}u,\quad m_-=\operatorname*{ess\,inf}_{Q^-_{\rho,1}(x_0,t_0)}u\quad\text{and}\quad \omega=m_+-m_-=\operatorname*{ess\,osc}_{Q^-_{\rho,1}(x_0,t_0)}u.$$

Without loss of generality, we assume that $\omega\le 1$ so that

$$Q^-_{\rho,\omega}(x_0,t_0)=B_\rho(x_0)\times(t_0-\omega\rho,t_0]\subset Q^-_{\rho,1}(x_0,t_0)\subset\Omega_T.$$

Therefore,

$$\operatorname*{ess\,inf}_{Q^-_{\rho,\omega}(x_0,t_0)}u\ge m_-,\quad \operatorname*{ess\,sup}_{Q^-_{\rho,\omega}(x_0,t_0)}u\le m_+\quad\text{and}\quad \omega\ge\operatorname*{ess\,osc}_{Q^-_{\rho,\omega}(x_0,t_0)}u.$$

Since, we are assuming that u is not continuous at (x_0,t_0), there exists $\rho_0>0$ and $\omega_0>0$ such that

$$\omega_{\tilde\rho}=\operatorname*{ess\,osc}_{Q^-_{\tilde\rho,1}(x_0,t_0)}u\ge\omega_0>0,$$

for all $0<\tilde\rho\le\rho_0$. By the time expansion of positivity, there is a $\delta>0$, and $\varepsilon=\frac{1}{32}>0$ such that

$$\mu\left(\left\{x\in B_\rho(x_0):u(x,t)\ge m_-+\frac{\omega}{64}\right\}\right)\ge\frac{1}{4}\mu(B_\rho(x_0)),$$

for every $t\in(t_0-\frac{\tilde\delta\omega\rho}{2},t_0]$, where $\tilde\delta=\frac{1}{2^8\gamma Q}$. By using again the isoperimetric inequality, see Lemma 2.6 in [13], we obtain

$$\tilde\xi\omega\mu(\{x\in B_\rho(x_0):u(x,t)<m_-+\tilde\xi\omega\})\le C\rho\|Du(t)\|(\{x\in B_\rho(x_0):u(x,t)>m_-+\tilde\xi\omega\},$$

with $2\tilde\xi=\frac{1}{64}\tilde\delta$.

Integrating over the time interval $(t_0-\tilde\xi\omega\rho,t_0]$ gives

$$\frac{(\mu\otimes\mathscr{L}^1)(Q^-_{\rho,\tilde\xi\omega}(x_0,t_0)\cap\{u<m_-+\tilde\xi\omega\})}{(\mu\otimes\mathscr{L}^1)(Q^-_{\rho,\tilde\xi\omega}(x_0,t_0))}\le\frac{\rho}{(\mu\otimes\mathscr{L}^1)(Q^-_{\rho,1}(x_0,t_0))}\cdot\frac{C}{(\tilde\xi\omega_0)^2}\int_{t_0-\tilde\xi\omega\rho}^{t_0}\|Du(\cdot,t)\|(B_\rho(x_0))\,\mathrm{d}t.$$

By assumption, the right-hand side tends to zero as $\rho \to 0+$. Hence, there exits $\rho > 0$ small enough such that

$$\frac{(\mu \otimes \mathscr{L}^1)(Q^-_{\rho,\tilde{\xi}\omega}(x_0,t_0) \cap \{u < m_- + \tilde{\xi}\omega\})}{(\mu \otimes \mathscr{L}^1)(Q^-_{\rho,\tilde{\xi}\omega}(x_0,t_0))} \le \nu_-,$$

where, ν_- is given by the De Giorgi lemma. Furtheremore, this lemma implies $u \ge m_- + \frac{1}{2}\tilde{\xi}\omega$ $(\mu \otimes \mathscr{L}^1)$-almost everywhere in $Q^-_{\frac{1}{2}\rho,\tilde{\xi}\omega}(x_0,t_0)$ and consequently

$$\operatorname*{ess\,inf}_{Q^-_{\frac{1}{2}\rho,\tilde{\xi}\omega}(x_0,t_0)} u \ge m_- + \frac{\tilde{\xi}\omega}{2}.$$

This, in turn, implies

$$\begin{aligned}
\omega_{\rho_1} = \operatorname*{ess\,osc}_{Q^-_{\rho_1,1}(x_0,t_0)} u &= \operatorname*{ess\,sup}_{Q^-_{\rho_1,1}(x_0,t_0)} u - \operatorname*{ess\,inf}_{Q^-_{\rho_1,1}(x_0,t_0)} u \\
&\le \operatorname*{ess\,sup}_{Q^-_{\frac{1}{2}\rho,\tilde{\xi}\omega}(x_0,t_0)} u - \operatorname*{ess\,inf}_{Q^-_{\frac{1}{2}\rho,\tilde{\xi}\omega}(x_0,t_0)} u = \operatorname*{ess\,osc}_{Q^-_{\frac{1}{2}\rho,\tilde{\xi}\omega}(x_0,t_0)} \le \eta\omega.
\end{aligned}$$

By repeating the same argument, starting from the cylinder $Q^-_{\rho_1,1}(x_0,t_0)$ and proceeding recursively, we generate a decreasing sequence of radii $\rho_n \to 0$ such that

$$\omega_0 \le \operatorname*{ess\,osc}_{Q^-_{\rho_n,1}(x_0,t_0)} u \le \eta^n\omega,$$

for every $n \in \mathbb{N}$. This is a contradiction to the assumption u is not continuous at (x_0,t_0).

4. (p,q)-Dirichlet integral on metric measure spaces

This chapter is dedicated to papers **[I]** and **[IV]**. Here, we focus on the study of quasiminimizers of the following anisotropic energy (p,q)-Dirichlet integral

$$\int_\Omega (a g_u^p + b g_u^q)\,\mathrm{d}\mu, \tag{4.1}$$

in metric measure spaces, with g_u the minimal q-weak upper gradient of u. Where, $\Omega \subset X$ is an open bounded set and $1 < p < q$. Throughout this chapter, we consider a *complete, metric measure space* (X,d,μ) with metric d and a *doubling Borel regular measure* μ. Moreover, we assume that X supports a *weak* $(1,p)$*-Poincaré inequality* and we ask for the coefficient functions a and b to be *measurable* and to satisfy $0 < \alpha \le a, b \le \beta$, for some positive constants α, β.

4.1 Quasiminimizers on metric measure spaces

In [24], Giaquinta and Giusti introduced the notion of quasiminimizers in $\mathbb{R}^N$. They proved several of their fundamental properties, such as local Hölder continuity and the strong maximum principle. Since, in a metric measure space, it is possible to define (quasi)minimizers of Dirichlet integrals, this approach is particularly useful.

We say that a function $u \in N^{1,q}(\Omega)$ is a (p,q)*-quasiminimizer* on Ω if there exists $K > 0$, called *quasiminimizing constant*, such that for every open $\Omega' \Subset \Omega$ and every test function $v \in N^{1,q}(\Omega')$ with $u - v \in N_0^{1,q}(\Omega')$ the inequality

$$\int_{\Omega'} (a g_u^p + b g_u^q)\,\mathrm{d}\mu \le K \int_{\Omega'} (a g_v^p + b g_v^q)\,\mathrm{d}\mu \tag{4.2}$$

holds, where g_u, g_v are the minimal q-weak upper gradients of u and v in Ω, respectively. Furthermore, a function $u \in N^{1,q}(\Omega)$ is a *global* (p,q)*-quasiminimizer* on Ω if (4.2) is satisfied with Ω instead of Ω', for all $v \in N^{1,q}(\Omega)$, with $u - v \in N_0^{1,q}(\Omega)$.

Since their introduction, elliptic quasiminimizers have been extensively studied, both in the Euclidean case and more recently in metric measure spaces. Moreover, many authors have explored different generalizations of classical elliptic and parabolic partial differential equations, such as the nonlinear p and (p,q)-Laplace equations, see [7, 17, 55]. However, there are still new and interesting open mathematical questions in the setting of anisotropic nonlinear elliptic and parabolic partial differential equations driven by (p,q)-Laplace operators.

4.2 Regularity for (p,q)-quasiminimizers

This section is devoted to paper **[I]**, a joint work with Nastasi [58]. Our first goal is to answer questions regarding regularity for quasiminimizers of the anisotropic energy (p,q)-Dirichlet integral (4.1).

Local properties of quasiminimizers of the p-energy integral on metric spaces were studied by Kinnunen and Shanmugalingam in [42]. More precisely, they used the De Giorgi method [18], to prove that, if the metric measure space is equipped with a doubling measure and it supports a Poincaré inequality, quasiminimizers of the p-energy functional are locally Hölder continuous, they satisfy the Harnack inequality and the maximum principle. Paper **[I]** generalizes the results in [42, 4, 6], since, one of the new features is that we include both p-Laplace and q-Laplace operators.

There exists a rich literature concerning regularity results for solutions to partial differential equations, both elliptic and parabolic, under p and (p,q)-growth conditions in the Euclidean setting. In this study, we focus on the anisotropic energy integral as presented by Marcellini in [49, 50], but the new feature is that, we worked in metric measure spaces. We have considered this setting to prove that the (p,q)-growth condition (4.1) can be treated also in a general context, thus obtaining several relevant properties for (quasi)minimizers even in a metric framework.

Furthermore, there are also some regularity results concerning the boundary behaviour for (quasi)minimizers, both in the Euclidean, see [20, 62, 64], and in the metric setting see [4, 5]. More specifically, Ziemer [64] proved a Wiener type condition for the continuity of a quasiminimum at a boundary point of a bounded open subset of $\mathbb{R}^n$. On the other hand, Björn [4], extended these results to the general metric setting and also gave sufficient condition for Hölder continuity. Another important contribution concerning boundary behaviour was by Björn, MacManus and Shanmugalingam in [6]. They obtained an estimate for the oscillation of p-harmonic functions and p-energy minimizers near a boundary point. However, the study of boundary behavior for the (p,q)-problems can be considered mostly still open, at least in its full generality.

Throughout paper **[I]**, we consider a *complete* metric measure space

(X,d,μ) with metric d and a *doubling Borel regular* measure μ. Moreover, we assume that X supports a *weak* $(1,p)$*-Poincaré inequality* with $1<p<q<p^*$. We also fix $1<s<p$ for which X also admits a weak $(1,s)$-Poincaré inequality. Such s is given by the theorem of Keith and Zhong in [35], and is used in various of our results. As before, we consider a non empty open subset $\Omega\subset X$ such that $\mu(X\setminus\Omega)>0$.

4.2.1 Main results

The first important results in **[I]** concern interior regularity for (p,q)-quasiminizers. By adapting the approach in [42], we establish the local boundedness for quasiminimizers of the convex integral (4.1). We do this by proving a *weak Harnack inequality*. More specifically, we prove that if a function u is a (p,q)-quasiminimizer, then there exists a constant $C>0$ such that

$$\operatorname*{ess\,sup}_{B(y,\frac{R}{2})} u \le k_0 + C\left(\fint_{B(y,R)} (u-k_0)_+^q \,\mathrm{d}\mu\right)^{\frac{1}{q}}, \tag{4.3}$$

for any $k_0\in\mathbb{R}$.

After proving local boundedness for quasiminimizers, we move to proving *local Hölder continuity*. If u is a (p,q)-quasiminimizer, and $0<\rho<R$ with $B(y,2\lambda' R)\subset\Omega$. Then, there exists $0<\eta<1$ such that

$$\operatorname{osc}(\tilde u, B(y,\lambda'\rho)) \le 4^\eta\left(\frac{\rho}{R}\right)^\eta \operatorname{osc}(\tilde u, B(y,\lambda' R)),$$

where $\operatorname{osc}(\tilde u, B(y,\cdot)) = \sup_{B(y,\cdot)}\tilde u - \inf_{B(y,\cdot)}\tilde u$ is the oscillation of $\tilde u$, and

$$\tilde u(x) = \limsup_{\rho\to 0}\fint_{B(x,\rho)} u\,\mathrm{d}\mu, \tag{4.4}$$

is given by Lebesgue's differentiation theorem, see Section 2.3.1. In particular, $\tilde u$ is locally Hölder continuous on Ω, and therefore u can be modified on a set of capacity zero so that it becomes locally Hölder continuous on Ω.

The last important result, regarding interior regularity, is *Harnack's inequality*. Assume that $u>0$, u and $-u$ are (p,q)-quasiminimizers. Then there exists a constant $C\ge 1$ so that

$$\operatorname*{ess\,sup}_{B(y,R)} u \le C \operatorname*{ess\,inf}_{B(y,R)} u$$

for every ball $B(y,R)$ for which $B(y,6R)\subset\Omega$ and $R>0$.

Furthermore, we study regularity results up to the boundary. We first give a *pointwise estimate near a boundary point*. For $u\in N^{1,q}(X)$ a (p,q)-quasiminimizer on Ω and $w\in N^{1,q}(X)$ with $u-w\in N_0^{1,q}(\Omega)$. We prove that there exist constants $C_0, C_1>0$ such that

$$M(\rho,r_0)\le C_1 M(r_0,r_0)\exp\left(-\frac{1}{4}\int_\rho^{r_0}\exp\left(-C_0\gamma(s,r)^{\frac{p}{p-s}}\right)\frac{\mathrm{d}r}{r}\right), \tag{4.5}$$

where

$$M(r,r_0)=\Big(\operatorname*{ess\,sup}_{B(x_0,r)} u-\operatorname*{ess\,sup}_{B(x_0,r_0)} w\Big)_+,$$

with $0<r\le r_0$.

This result implies a sufficient condition for the *Hölder continuity of (p,q)-quasiminimizers at a boundary point*. More specifically, if $u\in N^{1,q}(X)$ is a (p,q)-quasiminimizer on Ω and $w\in N^{1,q}(X)$ is a Hölder continuous function at $x_0\in\partial\Omega$, with $u-w\in N_0^{1,q}(\Omega)$. Then there exists a constant $C_0>0$ such that

$$\liminf_{\rho\to 0}\frac{1}{|\log\rho|}\int_\rho^1 \exp\left(-C_0\gamma(s,r)^{\frac{p}{p-s}}\right)\frac{\mathrm{d}r}{r}>0.$$

Thus u is Hölder continuous at x_0.

Moreover, if we assume that $w\in N^{1,q}(X)$ is continuous at $x_0\in\partial\Omega$, we get a *Wiener type regularity condition*. Namely, there exists $t>0$ such that

$$\int_0^1\left(\frac{\operatorname{cap}_s(B(x_0,r)\setminus\Omega,B(x_0,2r))}{r^{-\frac{qs}{p}}\mu(B(x_0,r))}\right)^t\frac{\mathrm{d}r}{r}=+\infty,$$

and therefore, u is continuous at x_0.

Lastly, we consider (p,q)-minimizers and we give an *estimate for their oscillation at boundary points*. More clearly, let $w\in N^{1,q}(X)\cap C(\overline{\Omega})$. Consider a (p,q)-minimizer u on Ω such that $w-u\in N_0^{1,q}(\Omega)$. If $x_0\in\partial\Omega$ and $0<\rho\le r$, then

$$\begin{aligned}\operatorname{osc}(u,\overline{\Omega}\cap B(x_0,\rho))\le&\operatorname{osc}(w,\partial\Omega\cap B(x_0,5r))\\&+\operatorname{osc}(w,\partial\Omega)\exp\left(-C\int_\rho^r\varphi(x_0,X\setminus\Omega,t)^{\frac{1}{q-1}}\frac{\mathrm{d}t}{t}\right),\end{aligned}\tag{4.6}$$

for some constant $C>0$ and $\varphi(x,E,r)=\dfrac{\operatorname{cap}_q(B(x,r)\cap E,B(x,2r))}{\operatorname{cap}_q(B(x,r),B(x,2r))}$.

4.2.2 Remarks on the method of proof

In this section, we talk in detail about the methods that helped us achieve the results previously described.

Energy estimates

As in paper **[III]**, the first and possibly most important step in the proofs of the regularity results presented in paper **[I]**, is to establish an energy estimate for the (p,q)-quasiminimizers. Most of the regularity results are then proved using this energy estimate, which comes in the form of the following *De Giorgi type inequality*

$$\begin{aligned}\int_{B(y,\rho)}(ag_u^p+bg_u^q)\,\mathrm{d}\mu\le C\bigg(&\frac{1}{(R-\rho)^p}\int_{B(y,R)}a(u-k)_+^p\,\mathrm{d}\mu\\&+\frac{1}{(R-\rho)^q}\int_{B(y,R)}b(u-k)_+^q\,\mathrm{d}\mu\bigg),\end{aligned}\tag{4.7}$$

where $k \in \mathbb{R}$ and $a_+ = \max\{a, 0\}$.

Therefore, we first show that if $u \in N^{1,q}_{\mathrm{loc}}(\Omega)$ is a (p,q)-quasiminimizer and $0 < \rho < R$, then there exists a constant $C > 0$ such that u satisfies (4.7).

Note that, (4.7) can be rewritten as

$$\int_{S_{k,\rho}} (ag_u^p + bg_u^q)d\mu \le C\left(\frac{1}{(R-\rho)^p}\int_{S_{k,R}} a(u-k)^p d\mu + \frac{1}{(R-\rho)^q}\int_{S_{k,R}} b(u-k)^q d\mu\right), \qquad (4.8)$$

where $S_{k,r} = \{x \in B(y,r) \cap \Omega : u(x) > k\}$, with $k \in \mathbb{R}$ and $r > 0$.

To prove (4.8), we use the definition of a (p,q)-quasiminimizer, define a specific test function w, and use Lemma 6.1 in [25].

After obtaining (4.8), we use the $(1,p)$-Poincaré inequality, together with Hölder inequality and the Sobolev-Poincaré inequality, see Section 2.3.3, to obtain

$$\left(\fint_{B(y,\rho)} (u-k)_+^l \,\mathrm{d}\mu\right)^{\frac{1}{l}} \le \frac{CR}{R-\rho}(k-h)^{-\theta}\left(\fint_{B(y,R)} (u-h)_+^l \,\mathrm{d}\mu\right)^{\frac{1+\theta}{l}}, \qquad (4.9)$$

for either $l = p$ or $l = q$, with $0 < \theta < 1$ and $0 < h < k$.

Local boundedness

The key step into proving the *weak Harnack inequality*, is by proving

$$\left(\fint_{B(y,\frac{R}{2})} (u-(k_0+d))_+^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} = 0,$$

for all $k_0 \in \mathbb{R}$ and for a specific $d > 0$ (defined later). The proof of this equality is by induction.

We first define two sequences, one of radii ρ_n, and the other one of levels k_n. Using inequality (4.9), independently of the case, we are able to obtain

$$\begin{aligned}
0 \le \left(\fint_{B(y,\frac{R}{2})} (u-(k_0+d))_+^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} &\le \left(\fint_{B(y,\frac{R}{2})} (u-(k_0+d))_+^l \,\mathrm{d}\mu\right)^{\frac{1}{l}} \\
&\le \left(\fint_{B(y,\rho_n)} (u-k_n)_+^l \,\mathrm{d}\mu\right)^{\frac{1}{l}} \\
&\le 2^{-\tau n}\left(\fint_{B(y,R)} (u-k_0)_+^q \,\mathrm{d}\mu\right)^{\frac{1}{q}} \to 0,
\end{aligned}$$

as $n \to +\infty$, where $\tau > 1$. Therefore,

$$\left(\fint_{B(y,\frac{R}{2})} (u-(k_0+d))_+^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} = 0,$$

as wanted.

The proof of the local boundedness for (p,q)-quasiminimizers can be easily deduced from the previous result. Indeed, this equality implies that $u \le k_0 + d$ almost everywhere in $B(y,\frac{R}{2})$, meaning

$$\operatorname*{ess\,sup}_{B(y,\frac{R}{2})} u \le k_0 + d = k_0 + C\left(\fint_{B(y,R)} (u-k_0)_+^q \, d\mu\right)^{\frac{1}{q}}.$$

Local Hölder continuity

In order to prove the local Hölder continuity, we first prove that for u, a (p,q)-quasiminimizer, there exists a constant $C>0$ such that

$$(k-h)\mu(S_{k,R}) \le C\mu(B(y,R))^{1-\frac{1}{s}}\left(\mu(S_{h,\lambda' R}) - \mu(S_{k,\lambda' R})\right)^{\frac{1}{s}-\frac{1}{q}} \cdot \left(\int_{S_{h,2\lambda' R}} (u-h)^q \, d\mu\right)^{\frac{1}{q}}, \tag{4.10}$$

is satisfied for either $l=p$ or $l=q$, and $\lambda' \ge 1$ is given by the weak Poincaré inequality. We recall, s is such that $1<s<p<q$ and our space X also supports a $(1,s)$-Poincaré inequality. The proof of (4.10) relies heavily on the De Giorgi type inequality (4.8) and the Poincaré inequality.

Now, for $B(y,\rho) \subset \Omega$, let $m(\rho) = \operatorname{ess\,inf}_{B(y,\rho)} u$ and $M(\rho) = \operatorname{ess\,sup}_{B(y,\rho)} u$. We define $M = M(2\lambda' R)$, $m = m(2\lambda' R)$ and $k_0 = \frac{M+m}{2}$. Since u is a (p,q)-quasiminimizer, we know that it satisfies (4.8). Assume, in addition, that $\mu(S_{k_0,R}) \le \gamma\mu(B(y,R))$ for some $0<\gamma<1$.

Let $k_j = M - 2^{-(j+1)}(M-m)$, $j \in \mathbb{N}\cup\{0\}$. By (4.10), for either $l=p$ or $l=q$, we deduce that

$$2^{-(j+1)}(M-m)\mu(S_{k_j,R}) \le C\mu(B(y,R))^{1-\frac{1}{s}+\frac{1}{l}}\left(\mu(S_{k_{j-1},\lambda' R}) - \mu(S_{k_j,\lambda' R})\right)^{\frac{1}{s}-\frac{1}{l}} \cdot 2^{-j}(M-m).$$

If $n>j$, then $\mu(S_{k_n,R}) \le \mu(S_{k_j,R})$, and so

$$\mu(S_{k_n,R}) \le C\mu(B(y,R))^{1-\frac{1}{s}+\frac{1}{l}}\left(\mu(S_{k_{j-1},\lambda' R}) - \mu(S_{k_j,\lambda' R})\right)^{\frac{1}{s}-\frac{1}{l}}.$$

By summing the above inequality over j, up until n, we get (independently if $l=p$ or $l=q$) $\lim_{n\to+\infty}\mu(S_{k_n R}) = 0$. Since $\mu(S_{k,R})$ is a monotonic decreasing function of k, we conclude that $\lim_{k\to M}\mu(S_{k,R}) = 0$.

We can finally discuss the proof of the *local Hölder continuity*. Firstly, we observe that u and $-u$ satisfy (4.8) (if u is a (p,q)-quasiminimizer, then $-u$ is a (p,q)-quasiminimizer). We consider $k_0 = \frac{M+m}{2}$, where M and m are as before. Furthermore, we can assume that

$$\mu(S_{k_0,R}) \le \frac{\mu(B(y,R))}{2}. \tag{4.11}$$

By the weak Harnack inequality (4.3), replacing k_0 with $k_n = M - 2^{-n-1}(M - m)$, $n \in \mathbb{N} \cup \{0\}$, we have

$$M\left(\frac{\lambda' R}{2}\right) \le k_n + C(M(2\lambda' R) - k_n)\left(\frac{\mu(S_{k_n,R})}{\mu(B(y,R))}\right)^{\frac{1}{q}}. \tag{4.12}$$

Inequality (4.11) ensures that it is possible to choose an integer n, independent from $B(y,R)$ and u, large enough such that

$$C\left(\frac{\mu(S_{k_n,R})}{\mu(B(y,R))}\right)^{\frac{1}{q}} < \frac{1}{2}.$$

Thanks to this choice, from (4.12) we deduce

$$\begin{aligned} M\left(\frac{\lambda' R}{2}\right) - m\left(\frac{\lambda' R}{2}\right) &\le M\left(\frac{\lambda' R}{2}\right) - m(2\lambda' R) \\ &\le (M(2\lambda' R) - m(2\lambda' R))(1 - 2^{-(n+2)}). \end{aligned}$$

As a consequence of the previous inequality, we can write

$$\operatorname{osc}\left(\tilde{u}, B\left(y, \frac{\lambda' R}{2}\right)\right) < \tau \operatorname{osc}(\tilde{u}, B(y, 2\lambda' R)), \tag{4.13}$$

where $\tau = 1 - 2^{-(n+2)} < 1$. Now, we consider an index $j \ge 1$ such that

$$4^{j-1} \le \frac{R}{\rho} < 4^j.$$

Then, from inequality (4.13), at the end, we obtain

$$\operatorname{osc}(\tilde{u}, B(y, \lambda' \rho)) \le 4^{\eta}\left(\frac{R}{\rho}\right)^{-\eta} \operatorname{osc}(\tilde{u}, B(y, \lambda' R)),$$

which completes the proof.

Harnack inequality

We define $D_{\tau,R} = \{x \in B(y,R) : u(x) < \tau\}$. Let $R > 0$ be such that $B(y,R) \subset \Omega$. Using the weak Harnack inequality (4.3) and inequality (4.10), we first prove that for $\tau > 0$, if $u \ge 0$ and $-u$ satisfy (4.8), then there exists $\gamma_0 \in (0,1)$, independent of the ball $B(y,R)$, such that if

$$\mu(D_{\tau,R}) \le \gamma_0 \mu(B(y,R)), \tag{4.14}$$

then

$$\operatorname*{ess\,inf}_{B(y,\frac{R}{2})} u \ge \frac{\tau}{2}.$$

After getting this last inequality, we then prove that for every γ with $0 < \gamma < 1$ there is a constant $\lambda > 0$ such that if $\mu(D_{\tau,R}) \le \gamma \mu(B(y,R))$, then

$$\operatorname*{ess\,inf}_{B(y,\frac{R}{2})} u \ge \lambda \tau.$$

By Krylov-Safonov covering Theorem [42], we show the existence of two constants $C > 0$ and $\sigma > 0$ such that

$$\operatorname*{ess\,inf}_{B(y,3R)} u \geq C \left(\fint_{B(y,R)} u^{\sigma} \, d\mu \right)^{\frac{1}{\sigma}}, \tag{4.15}$$

for every $B(y,R)$ with $B(y,6R) \subset \Omega$ and $R > 0$.

The proof of Harnack's inequality follows from the previous results.

One thing worth mentioning is that, we have two important consequences of Harnack's inequality. These involve the so-called (p,q)-harmonic functions. Let $u \in N^{1,q}_{loc}(\Omega)$ be a (p,q)-minimizer on Ω. If u is a continuous function, then we say that u is a *(p,q)-harmonic function*.

With this in mind, we first obtain the *strong maximum principle*. If Ω is connected, u is a *(p,q)-harmonic function* in Ω and u attains its maximum in Ω, then u is constant in Ω. The second corollary of Harnack's inequality is the *Liouville's Theorem*. If u is a (p,q)-harmonic function bounded from below in Ω, then u is constant.

Continuity up to the boundary

Until now, we have been interested in local properties, now we focus in boundary value problems.

We first prove that for $u \in N^{1,q}(X)$, a (p,q)-quasiminimizer on Ω, and $w \in N^{1,q}(X)$ with $u - w \in N^{1,q}_0(\Omega)$, there exist $C > 0$ and $\lambda \geq 1$ such that, for all $x_0 \in \partial\Omega$ and $0 < 2\lambda r \leq r_0$, the next inequality holds

$$M\left(\frac{r}{2}, r_0\right) \leq (1 - 2^{-n(r)-1}) M(2\lambda r, r_0),$$

where

$$n(r) = C\gamma\left(s, \frac{r}{2}\right)^{\frac{p}{p-s}}.$$

and

$$\gamma(s,r) = \frac{r^{-s}\mu(B(x_0,r))}{\operatorname{cap}_s(B(x_0,r) \setminus \Omega, B(x_0,2r))}$$

with x_0 fixed.

This result is used in the proof of the *pointwise estimate near a boundary point* (4.5). Afterwards, both *Hölder continuity at the boundary* and the *Wiener type regularity condition* follow from (4.5).

Boundary regularity for (p,q)-minimizers

In this section, we focus on (p,q)-minimizers and give control over the oscillation of (p,q)-minimizer functions at boundary points (4.6). The key role in the proof is played by the *comparison principle*, which unfortunately fails for quasiminimizers. However, we are able to prove a comparison principle for (p,q)-minimizers.

More specifically, we consider $V \subset \Omega' \subset \overline{\Omega'} \subset \Omega$ and $u_1, u_2 \in N^{1,q}(\Omega)$ (p,q)-minimizers. We show that, if $v \leq u_1$ q-q.e. in $\overline{\Omega'} \setminus V$, then $u_2 \leq u_1$ q-q.e. in V.

4.3 Higher integrability and stability

Article **[IV]** is another joint work with Nastasi. It is motivated by the work of Maasalo and Zatorska-Goldstein [48] and is a continuation of [58]. The novelty is that, we include both p-Laplace and q-Laplace operators, involving also some measurable coefficient functions a and b, assuming only they are bounded away from zero and infinity.

Throughout this paper, we assume that (X,d,μ) is a *complete, locally linearly connected* (LLC) metric measure space with metric d and a *doubling* Borel regular measure μ. We work on $\Omega \subset X$, an open and bounded subset such that $X \setminus \Omega$ is of *positive q-capacity and uniformly p-fat*, with $1 < p < q$, see Section 2.3.5. Moreover, we assume that X supports a *weak $(1,p)$-Poincaré inequality*. As in paper **[I]**, we fix $1 < s < p < q < s^*$ for which X also admits a weak $(1,s)$-Poincaré inequality. We recall that, such s is given by a result in [35].

Using purely variational methods, we prove *local* and *global higher integrability* results for upper gradients of (p,q)-quasiminimizers of (4.1), with fixed boundary data w, assuming the latter belongs to a slightly better Newtonian space. We also obtain a *stability property* with respect to the varying exponents p and q.

Quasiminimizers have been an active research topic for various years in the setting of a doubling metric measure space with a Poincaré inequality. In the elliptic setting, one of the first higher integrability results were by Bojarski [9]. Later, Elcrat and Meyers proved local higher integrability for nonlinear elliptic systems [56]. In [26], Granlund showed that an elliptic minimizer has the higher integrability property if the complement of the domain satisfies a certain measure density condition. Kilpeläinen and Koskela [37] generalized this result to a uniform capacity density condition.

Regarding stability results, Li and Martio [45] examined a quasilinear elliptic operator and proved a convergence result for solutions of an obstacle problem in a bounded subset of $\mathbb{R}^n$. In [41], Kinnunen and Parviainen showed that, if the complement of a cylindrical domain satisfies a uniform capacity density condition, then an initial and boundary value problem related to the parabolic p-Laplace equation is stable with respect to p. Another important reference concerning stability results in the Euclidean setting is [46].

4.3.1 Main results

As already anticipated, we first show a global higher integrability for upper gradients of (p,q)-quasiminimizers of the Dirichlet integral (4.1) with fixed boundary data. To be precise, we prove that for $w \in N^{1,\bar{q}}(\Omega)$, with $\bar{q} > q$. If $u \in N^{1,q}(\Omega)$ is a (p,q)-quasiminimizer with boundary data w

(meaning $w-u \in N_0^{1,q}(\Omega)$), then there exists $\delta_0 > 0$ such that $g_u \in L^{q+\delta}(\Omega)$ for all $0 < \delta < \delta_0$ and

$$\left(\fint_\Omega g_u^{q+\delta}\,\mathrm{d}\mu\right)^{\frac{1}{q+\delta}} \le C\left(\left(\fint_\Omega g_u^q\,\mathrm{d}\mu\right)^{\frac{1}{q}} + \left(\fint_\Omega g_w^{q+\delta}\,\mathrm{d}\mu\right)^{\frac{1}{q+\delta}} + 1\right).$$

Our second main theorem is a *stability result*. We consider a sequence (u_i) where $u_i \in N^{1,q_i}(\Omega)$ is a (p_i,q_i)-quasiminimizer in an open bounded subset Ω of X. We assume that all functions u_i have the same boundary data w and same quasiminimizing constant K. We prove that if $p = \lim_{i\to\infty} p_i$, $q = \lim_{i\to\infty} q_i$, with $1 < s < p_i < q_i < s^*$ and $u_i \to u$ μ-a.e. in Ω, then $u \in N^{1,q}(\Omega)$ is a (p,q)-quasiminimizer with boundary data w.

As previously stated, we require a regularity condition for the complement of the domain. We recall that, $X \setminus \Omega$ is assumed to be uniformly p-fat.

4.3.2 Remarks on the method of proof

In this section, we talk about some of the details on the proofs of the higher integrability and stability results.

Higher integrability

We first note that, global higher integrability of the upper gradient is essential when proving the stability theorem. With this in mind, the general idea of our higher integrability proof is to show that the minimal upper gradients satisfy a weak *reverse Hölder inequality*, apply *Gehring lemma* [47, 63], and generalize the resulting local higher integrability to the whole Ω. To this end, we need a suitable covering argument.

Since we are considering quasiminimizers with boundary data, we are able to work near and on the boundary. This gives us the opportunity to cover Ω by balls B that are inside the set, together with those that intersect the complement.

Inside Ω, the De Giorgi inequality (4.8), implies immediately that the minimal upper gradient satisfies a reverse Hölder inequality.

Near the boundary, we have to be careful. Here the p-fatness of the complement $X \setminus \Omega$ plays a big role. Furthermore, we use two self-improving properties, that of the weak Poincaré inequality and that of the p-fatness condition, see Section 2.3.3 and Section 2.3.5, respectively. In addition to the De Giorgi type inequality (4.8), we use the capacity version of a Sobolev-Poincaré-type inequality (Maz'ya estimate), see [4], and obtain as well a reverse Hölder inequality in this case.

Overall, we get that if $4\lambda B \subset 2B_0$, with B_0 a ball in X such that $\overline{\Omega} \subset B_0$, then

$$\fint_B g^\sigma\,\mathrm{d}\mu \le C\left(\left(\fint_{4\lambda B} g\,\mathrm{d}\mu\right)^\sigma + \fint_{4\lambda B} f^\sigma\,\mathrm{d}\mu\right),$$

where $\sigma = \frac{q}{p_0} > 1$,

$$g = \begin{cases} g_u^{p_0} & \text{in } \Omega, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f = \begin{cases} g_w^{p_0} \chi_{4\lambda B \cap \Omega} + 1 & \text{in } \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $1 < s \leq p_0 < p < q$ is given by the self-improving property of the p-fatness condition for $X \setminus \Omega$, and λ comes from the dilation factor from the Poincaré inequalities.

By applying Gehring lemma, we obtain the existence of a δ_0>0, such that the following inequality

$$\left(\fint_B g_u^{q+\delta} \,\mathrm{d}\mu \right)^{\frac{1}{q+\delta}} \leq C \left(\left(\fint_{4\lambda B} g_u^q \,\mathrm{d}\mu \right)^{\frac{1}{q}} + \left(\fint_{4\lambda B} g_w^{q+\delta} \,\mathrm{d}\mu \right)^{\frac{1}{q+\delta}} + 1 \right),$$

holds for all $\delta \in [0, \delta_0]$.

Finally, since Ω is bounded, we find a finite number balls covering it. Therefore, obtaining the desired inequality in the whole Ω.

Stability result

As a first step into proving the stability property, we use the metric version of Rellich-Kondrachov Theorem, see [48], and prove that there exists $\epsilon_0 > 0$ such that

$$\begin{cases} u, u_i \in L^{q+\epsilon_0}(\Omega), \\ g_{u_i}, g \in L^{q+\epsilon_0}(\Omega), \end{cases}$$

and there is a subsequence (u_i) such that

$$\begin{cases} u_i \to u & \text{in } L^{q+\epsilon_0}(\Omega), \\ g_{u_i} \to g & \text{in } L^{q+\epsilon_0}(\Omega), \end{cases}$$

where g is a q-weak upper gradient of u.

Notice that in the previous result, we have convergence to some q-weak upper gradient of u and not necessarily to the minimal q-weak upper gradient g_u.

Now, we consider a compact set $D \subset \Omega$ and define $D(t) = \{x \in \Omega : \operatorname{dist}(x, D) < t\}$, for every $t > 0$. We then obtain a *local uniform integrability estimate* for the minimal upper gradients. To be precise, we get

$$\limsup_{i \to \infty} \int_{D(t)} (g_{u_i}^{p_i} + g_{u_i}^{q_i}) \,\mathrm{d}\mu \leq C \int_{D(t)} (g_u^p + g_u^q) \,\mathrm{d}\mu,$$

for almost every $0 < t < t_0$.

At this point, we are left to show that u is indeed a (p,q)-quasiminimizer with boundary data w. In order to do that, we begin by proving that $u - w \in N_0^{1,q}(\Omega)$. Afterwards, we move on to showing a *lower semicontinuity* result in the varying exponent case, meaning that we prove the following two inequalities

$$\int_E g_u^q \, \mathrm{d}\mu \le \liminf_{i\to\infty} \int_E g_{u_i}^{q_i} \, \mathrm{d}\mu, \tag{4.16}$$

and

$$\int_E g_u^p \, \mathrm{d}\mu \le \liminf_{i\to\infty} \int_E g_{u_i}^{p_i} \, \mathrm{d}\mu, \tag{4.17}$$

for every μ-measurable subset E of Ω.

Finally, we show that u satisfies the quasiminimizing inequality

$$\int_{\Omega'} (a g_u^p + b g_u^q) \, \mathrm{d}\mu \le C \int_{\Omega'} (a g_{u+\varphi}^p + b g_{u+\varphi}^q) \, \mathrm{d}\mu.$$

for every bounded open subset Ω' of Ω with $\Omega' \Subset \Omega$ and for all functions $\varphi \in N_0^{1,q}(\Omega')$, where $C > 0$.

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This dissertation studies existence and regularity properties of functions related to the calculus of variations on metric measure spaces that support a weak Poincaré inequality and doubling measure. The work consists of four articles in which we study the total variation flow and quasiminimizers of a (p,q)-Dirichlet integral.

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