Regularity theory for nonlinear parabolic PDEs: gradient estimates, stability and the obstacle problem

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Abstract

This thesis concerns different aspects of regularity theory for weak solutions of nonlinear parabolic partial differential equations. We focus on such equations with porous medium type and p-growth structure. In particular, we consider solutions which are defined either via a weak formulation of the equation with test functions under the integral sign, or as functions that obey a parabolic comparison principle. Our research concerns the regularity of both the solution and its gradient. For the gradient of a weak solution of porous medium type systems we prove a higher integrability result up to the boundary of the domain. We derive reverse Hölder inequalities in intrinsic cylinders near the boundary, for which we prove a Vitali type covering property that is applied to obtain the higher integrability result. We also show that under suitable assumptions, weak solutions as well as their gradients are stable with respect to small fluctuations of the parameter characterizing the equation. In particular, we prove that solutions to the approximating problems converge to the corresponding solution of the limit problem in the natural parabolic Sobolev space. For the parabolic p-Laplace equation we study supersolutions, which are defined via a parabolic comparison principle. We show that in the fast diffusion case these functions can be divided into two mutually exclusive classes, for which we give several characterizations. An important tool in regularity theory is the obstacle problem, which is also interesting in its own right. In the case of signed obstacles we study Hölder continuity for solutions to the porous medium type equations defined via a variational inequality. We use a De Giorgi type iteration argument to show that solutions to obstacle problems are locally Hölder continuous, provided that the obstacle is Hölder continuous.

Tiivistelmä

Väitöskirjassa tutkitaan säännöllisyysteoriaa eri näkökulmista epälineaaristen parabolisten osittaisdifferentiaaliyhtälöiden heikoille ratkaisuille. Tutkimus keskittyy yhtälöihin, joilla on huokoisen aineen tyyppiset tai p-kasvuehdot. Työssä tarkastellaan erityisesti ratkaisuja, jotka on joko määritelty yhtälön heikon muodon avulla testifunktioita vastaan integroiden, tai jotka kytketään yhtälöön käyttäen parabolista vertailuperiaatetta. Tutkimus kohdistuu sekä itse ratkaisun että ratkaisun gradientin säännöllisyyteen liittyviin kysymyksiin. Huokoisen aineen systeemien heikon ratkaisun gradientille todistetaan korkeampi integroituvuus alueen reunalle asti. Gradientille johdetaan käänteisen Hölderin epäyhtälöt lähellä alueen reunaa sellaisissa sylintereissä, joiden geometria on yhtälön rakenteelle ominainen. Kyseisille sylintereille osoitetaan Vitali-tyyppinen peitelause, jota sovelletaan korkeamman integroituvuuden todistuksessa. Väitöskirjassa näytetään myös, että asianmukaisilla oletuksilla sekä heikot ratkaisut että ratkaisujen gradientit ovat stabiileja ongelmaa karakterisoivan parametrin heilahtelujen suhteen. Työssä todistetaan, että approksimoivien ongelmien ratkaisut suppenevat rajaongelman ratkaisuun luonnollisessa parabolisessa Sobolevin avaruudessa. Parabolisen p-Laplacen yhtälölle tutkitaan superratkaisuja, jotka määritellään parabolisen vertailuperiaatteen avulla. Työssä osoitetaan, että nopean diffuusion tapauksessa kyseiset superratkaisut voidaan jakaa kahteen erilliseen luokkaan, joille esitetään useita eri karakterisointeja. Esteongelma on tärkeä työkalu säännöllisyysteoriassa mutta myös kiinnostava tutkimuskohde itsessään. Variaatioepäyhtälön avulla määritellyn huokoisen aineen yhtälön esteongelman ratkaisun Hölder-jatkuvuutta tutkitaan merkkiä vaihtavan esteen tapauksessa. De Giorgi-tyyppistä iteraatioargumenttia soveltaen työssä todistetaan ratkaisun lokaali Hölder-jatkuvuus esteen ollessa Hölder-jatkuva.

Preface

First and foremost, I would like to express my deepest gratitude to my supervisor Professor Juha Kinnunen for his patient guidance throughout my doctoral studies. He has been an excellent supervisor, not only for the valuable advice and constructive feedback he has given, but also for providing an opportunity to work on several interesting research problems during the process of writing this thesis.

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Espoo, June 7, 2022,

Kristian Moring

Contents

Contents

Publications 49

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I Kristian Moring, Christoph Scheven, Sebastian Schwarzacher and Thomas Singer. Global higher integrability of weak solutions of porous medium systems. *Communications on Pure and Applied Analysis*, Volume 19, Issue 3, pages 1697–1745, March 2020.

II Ratan Kr. Giri, Juha Kinnunen and Kristian Moring. Supercaloric functions for the parabolic p-Laplace equation in the fast diffusion case. *Nonlinear Differential Equations and Applications NoDEA*, Volume 28, Issue 3, Article number 33, 21 pages, April 2021.

III Kristian Moring and Rudolf Rainer. Stability for systems of porous medium type. *Journal of Mathematical Analysis and Applications*, Volume 506, Issue 1, Article number 125532, 36 pages, February 2022.

IV Kristian Moring and Leah Schätzler. On the Hölder regularity for obstacle problems to porous medium type equations. Submitted to *a journal*, 30 pages, Available at arXiv:2202.11565, June 2022.

Author's contribution

Publication I: "Global higher integrability of weak solutions of porous medium systems"

The author has made significant contributions to all parts of the paper.

Publication II: "Supercaloric functions for the parabolic p-Laplace equation in the fast diffusion case"

All co-authors contributed equally to all parts of the paper.

Publication III: "Stability for systems of porous medium type"

The author has made significant contributions to all parts of the paper.

Publication IV: "On the Hölder regularity for obstacle problems to porous medium type equations"

The author has made significant contributions to all parts of the paper. The author proposed the problem.

1. Introduction

This thesis concerns different aspects of regularity theory for two types of nonlinear parabolic PDEs that can be written in the general form

$$\partial_t u - \operatorname{div} \mathbf{A}\left(x, t, u, D\left(|u|^{m-1}u\right)\right) = 0, \tag{1.1}$$

for $m > 0$. Here $u = u(x, t)$ is a function depending on spatial and temporal variables that can be scalar or vector valued, whereas the parameter m depends on the specific problem. It is also assumed that the vector field $\mathbf{A}$ fulfills suitable measurability and continuity properties together with the structure conditions

$$\mathbf{A}(x, t, u, \xi) \cdot \xi \geq C_o |\xi|^p, \tag{1.2}$$

$$|\mathbf{A}(x, t, u, \xi)| \leq C_1 |\xi|^{p-1}, \tag{1.3}$$

for $p > 1$ and some positive constants C_o and C_1. These type of equations can be viewed as generalizations of the linear heat equation (for which $m = 1$ and $p = 2$). However, besides mathematical interest many real life phenomena are nonlinear in nature, which makes it important to develop theory and techniques that are not based on the linear structure. It is typical that there exists no general theory for nonlinear PDEs, which is why different equations require different tools. Depending on a specific problem in question, the structural conditions (1.2) and (1.3) will be chosen differently.

We restrict ourselves to two special cases of (1.1). These are equations of p-growth structure, when $m = 1$, and of porous medium type structure, when $p = 2$. A prototype of the former is the parabolic p-Laplace equation that can be written in the form

$$\partial_t u - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = 0. \tag{1.4}$$

For the standard theory of this equation we refer to the monographs by DiBenedetto [24], DiBenedetto et al. [26] and Wu et al. [66]. Despite the inhomogeneous scaling behavior of (1.4), one notable advantage of the

equation is that adding a constant to the solution will produce another solution.

A prototype of (1.1) with $p = 2$ is the porous medium equation (PME for short), that can be written as

$$\partial_t u - \Delta \left(|u|^{m-1} u \right) = 0. \tag{1.5}$$

For the standard theory of this equation we refer to the monographs by Vázquez [64], [63], and Daskalopoulos & Kenig [22] in addition to the references above. In contrast to the equation (1.4), there is an additional technical difficulty with the porous medium equation. This is due to the fact that together with the anisotropic scaling behavior, one cannot add constants to a solution to produce another solution.

The physics of these equations depends on the different parameter ranges. We call the regimes $p > 2$ for (1.4) and $m > 1$ for (1.5) the slow diffusion or degenerate case. In this case, disturbances propagate with finite speed, and free boundaries occur. Conversely, we call the regimes $p < 2$ in (1.4) and $m < 1$ in (1.5), the fast diffusion or singular case. In this case disturbances propagate with infinite speed, and solutions may become extinct in finite time. Moreover, at the points where $|\nabla u| = 0$, the equation (1.4) degenerates when $p > 2$ and becomes singular for $p < 2$. Analogously for (1.5), the equation degenerates at the points $u = 0$ when $m > 1$ and becomes singular when $m < 1$.

One may define and study different notions of solutions to the problems above. The usual starting point for equations in the divergence form is to study weak solutions, which are defined with test functions under the integral sign. Then, one does not need to assume any differentiability for the solution with respect to time, and only sufficiently integrable first derivatives in space. Depending on this level of integrability of the gradient, one may consider weak solutions (with p-integrable gradient) or very weak solutions (with less than p-integrable gradient). In this thesis we will focus on the former definition. It is often useful to relax the equation by requiring only inequality instead of equality in (1.4) or (1.5), which leads to the concept of (weak) super- or subsolutions. However, in some cases even these definitions fail to include all interesting solutions in the theory. This motivates us to also study the so-called supercaloric functions, which are defined via a parabolic comparison principle.

In Publication II we consider supercaloric functions for the parabolic p-Laplace equation in the fast diffusion case, and we prove a dichotomy and several characterizations for the two classes of these functions. The theory is discussed in Chapter 2. In Publication I we prove higher integrability of the gradient for weak solutions to porous medium type systems, which is described in Chapter 3. The stability of solutions to the systems with same structure is studied in Publication III, which is discussed in Chapter 4. In case an obstacle is restricting the behavior of the solution, we show

in Publication IV, which corresponds to Chapter 5, that the solution is locally Hölder continuous provided that the obstacle is Hölder continuous.

1.1 Weak solutions

We begin with some notation. From now on we consider space-time cylinders $\Omega_T := \Omega \times (0, T) \subset \mathbb{R}^{n+1}$, in which Ω is a bounded open set. The parabolic boundary of the space time cylinder Ω_T is defined as $\partial_p \Omega_T := \overline{\Omega} \times \{0\} \cup \partial\Omega \times (0, T)$. In the case of vector-valued functions i.e., for $u = (u_1, u_2, ..., u_N) : \Omega_T \to \mathbb{R}^N$, for some $N \geq 1$, we will also use shorthand notation for the power by setting $\boldsymbol{u}^m := |u|^{m-1}u$. In the scalar case $N = 1$, we will denote the (weak) gradient of u by $\nabla u = (\partial_{x_1} u, \partial_{x_2} u, ..., \partial_{x_n} u) : \Omega_T \to \mathbb{R}^n$. Here ∂_{x_i} is the weak derivative in the direction of the x_i-axis defined via integration by parts. In the vectorial case the gradient is taken component-wise producing another vector of length Nn, and in this case we use the notation $Du = (\nabla u_1, \nabla u_2, ..., \nabla u_N) : \Omega_T \to \mathbb{R}^{Nn}$.

We will also use the framework of parabolic Sobolev spaces. Especially we consider functions u in $L^p(0, T; W^{1,p}(\Omega, \mathbb{R}^N))$, which we interpret as measurable maps $u : \Omega_T \to \mathbb{R}^N$, such that $x \mapsto u(x, t) \in W^{1,p}(\Omega, \mathbb{R}^N)$ for almost every $t \in (0, T)$ and

$$\iint_{\Omega_T} (|u|^p + |Du|^p)\, \mathrm{d}x\mathrm{d}t < \infty.$$

The space $L^p_{\text{loc}}(0, T; W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^N))$ is defined analogously on compact subsets of Ω_T. Moreover, we consider a vector field $\mathbf{A} : \Omega_T \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ which is a Carathéodory function with the structure (1.2) and (1.3). We will always assume that either $m = 1$ or $p = 2$ in this thesis.

A standard starting point in regularity theory is to define weak solutions for (1.1) as maps $u : \Omega_T \to \mathbb{R}^N$ with $\boldsymbol{u}^m \in L^p(0, T; W^{1,p}(\Omega, \mathbb{R}^N))$ satisfying

$$\iint_{\Omega_T} \left(-u \cdot \partial_t \varphi + \mathbf{A}(x, t, u, D\boldsymbol{u}^m) \cdot D\varphi\right) \mathrm{d}x\mathrm{d}t = 0 \qquad (1.6)$$

for all test functions $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N)$. This can be heuristically motivated by multiplying each equation in (1.1) by a test function, and integrating by parts. Especially when considering equations ($N = 1$), it can be useful to study solutions for which (1.6) is relaxed. We call function u a weak supersolution if the integral in (1.6) is nonnegative for all nonnegative test functions φ. Similarly, we call u a weak subsolution if the integral is nonpositive for all such φ. When deriving estimates for weak solutions some additional integrability assumptions are sometimes adequate. This becomes apparent when deriving energy estimates as in Section 1.2, and in particular when p is close to 1 or m is close to 0.

An advantageous property with weak solutions is that for the prototype cases we consider the existence is relatively well-known. We do not focus

on these questions in this thesis, but as classical references we mention [3] and monographs by Lions [50] and Showalter [60]. Furthermore, regularity theory is attainable via energy estimates. Locally bounded weak solutions are Hölder continuous, for which standard references for the parabolic p-Laplace equation are [23] and [17], and for nonnegative solutions to porous medium equations [25] and the monograph [26] by DiBenedetto, Gianazza & Vespri. Hölder continuity of signed solutions were treated in a unified way for both equations in [48]. Also, for (1.4) locally bounded gradients are Hölder continuous by [24]. Moreover, weak supersolutions possess lower semicontinuous representatives, which was proven in [44] for the parabolic p-Laplace equation, and in [49] for doubly nonlinear equations. We also mention that for the porous medium equation (1.5) in the degenerate case $m > 1$, an alternative definition of the solution is used for example in [26] and [28] by requiring that $|u|^{\frac{m+1}{2}} \in L^2(0, T; W^{1,2}(\Omega))$ instead of $\boldsymbol{u}^m \in L^2(0, T; W^{1,2}(\Omega))$. In this thesis we do not consider the former definition, but for a study between the connection of these two definitions we refer to [12].

The definition of a weak solution above is local in nature. In some cases, we will also consider solutions to the porous medium type system ($p = 2$), for which some specific boundary values are prescribed. These we will call Cauchy-Dirichlet problems, which can formally be introduced as a problem

$$
\begin{cases}
\partial_t u - \operatorname{div} \mathbf{A}(x, t, u, D\boldsymbol{u}^m) = 0 & \text{in } \Omega_T, \\
u = g & \text{on } \partial_p \Omega_T,
\end{cases}
\tag{1.7}
$$

for a given boundary function g. The standard assumptions we make for g are similar as for the solution u, and at least that $\boldsymbol{g}^m \in L^2(0, T; W^{1,2}(\Omega, \mathbb{R}^N))$. The formulation in (1.7) is formal in a sense that it is not specified in detail in what sense the solution u attains the boundary values g. In more precise terms, we assume that u is a solution to the corresponding Cauchy-Dirichlet problem if u satisfies (1.6) with $p = 2$, and in addition

$$
(\boldsymbol{u}^m - \boldsymbol{g}^m)(\cdot, t) \in W_0^{1,2}(\Omega, \mathbb{R}^N) \quad \text{for a.e. } t \in (0, T), \text{ and}
$$

$$
\frac{1}{h} \int_0^h \int_\Omega \left| \boldsymbol{u}^{\frac{m+1}{2}} - \boldsymbol{g}^{\frac{m+1}{2}} \right|^2 \, \mathrm{d}x\mathrm{d}t \longrightarrow 0,
\tag{1.8}
$$

as $h \to 0$. In particular, we consider boundary value problems in Publication I and Publication III, which are discussed in Chapters 3 and 4.

We also consider weak solutions to obstacle problems. An obstacle problem can be interpreted as a constraint problem, when there is an obstacle function ψ restricting the behavior of the weak solution u, namely $u \geq \psi$ pointwise. Solutions to such problems can be defined via a variational inequality, or as minimal supersolutions lying above the given obstacle ψ. In Publication IV, we take the former as a starting point to show

Hölder regularity for the solution. In connection with supercaloric functions in Publication II, we exploit an existence result proved in [42] based on the latter definition. These are discussed more in Chapters 5 and 2.

1.2 Energy estimates

The first step to obtain regularity results for weak solutions is typically by deriving energy estimates, which are also called Caccioppoli inequalities, starting from (1.6). These can be obtained in different forms, but a standard approach is to choose suitable test function in (1.6), which typically depends on the solution itself. For weak solutions, by testing heuristically with $\varphi = \boldsymbol{u}^m \eta^p$ with a regular enough nonnegative cut-off function η vanishing in compact subset of Ω_T, one can derive an estimate of the form

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_{\Omega\times\{t\}} \eta^p |u|^{m+1}\,\mathrm{d}x + \iint_{\Omega_T} \eta^p \left|D\boldsymbol{u}^m\right|^p\,\mathrm{d}x\mathrm{d}t$$

$$\leq c \iint_{\Omega_T} |\partial_t \eta^p|\,|u|^{m+1}\,\mathrm{d}x\mathrm{d}t + c \iint_{\Omega_T} |D\eta|^p\,|\boldsymbol{u}^m|^p\,\mathrm{d}x\mathrm{d}t, \qquad (1.9)$$

for some positive constant c depending on m, p, C_o and C_1. In this estimate $m = 1$ in the case of p-growth and $p = 2$ in the case of porous medium structure. First terms on both sides result from estimates for the parabolic part of the equation, whereas second terms are due to estimates for the divergence part. In contrast to the Sobolev inequality in Section 1.3, the energy estimate gives an integral bound for the gradient of the solution in terms of the solution itself. In order to extract information from (1.9) the right hand side should be finite. In particular, the integrability of the first term should be ensured. Also, already when testing the equation with given φ and deriving the estimate, the first integral in (1.6) should be integrable. A requirement $u \in L^{m+1}_{\mathrm{loc}}(\Omega_T, \mathbb{R}^N)$ is adequate, and this can be included in the definition of weak solution especially in the singular case ($p < 2$ in case of p-growth, $m < 1$ with porous medium structure). The estimate presented in (1.9) is local, and in the case of global problem as in (1.7), one may derive the estimate up to the parabolic boundary. In this case, testing heuristically with $\varphi = \xi(t)(\boldsymbol{u}^m - \boldsymbol{g}^m)$ results in an analogous estimate to (1.9). For example, one may derive an estimate in which the integrands $|u|^{m+1}$ are replaced on both sides of (1.9) with $|\boldsymbol{u}^{\frac{m+1}{2}} - \boldsymbol{g}^{\frac{m+1}{2}}|^2$. In this case, the initial condition in the form $(1.8)_2$ can be used. In addition, this type of estimate typically produces integrals involving the gradient $D\boldsymbol{g}^m$ and time derivative $\partial_t \boldsymbol{g}^m$ on the right hand side of (1.9).

A notable difficulty in deriving the energy estimate of the type (1.9) becomes apparent when testing (1.6) with test function depending on $\boldsymbol{u}^m$. The problem is due to the fact that no regularity requirements are imposed on solution in the time direction in the first place, whereas the

test function should be at least weakly differentiable with respect to the temporal variable. One can overcome this difficulty by approximating solution u^m in a suitable sense, so that approximants possess sufficient regularity in time. In practical terms this can be achieved by mollifying u^m in time, so that suitable convergence properties hold true and estimates are recovered for the solution u^m itself after passing to the limit. There exist different options for this procedure, e.g. standard mollification or Steklov averages (as in [24]) but especially in case of porous medium type structure so called exponential mollification has turned out to be particularly useful. This type of mollification has been introduced in [54] and useful properties have been established in [37] and [9]. For $v \in L^1(\Omega_T, \mathbb{R}^N)$ and $h > 0$, we define the mollification by

$$[\![v]\!]_h(x,t) := \frac{1}{h} \int_0^t e^{\frac{s-t}{h}} v(x,s) \, \mathrm{d}s.$$

In some cases it is useful to use the reverse time mollification, which is defined analogously as a weighted integral from t to T. The standard properties for mollifications include convergence $[\![v]\!]_h \to v$ in $L^p(\Omega_T, \mathbb{R}^N)$ as $h \to 0$, provided that $v \in L^p(\Omega_T, \mathbb{R}^N)$. Similarly, the L^p-convergence holds true for the spatial gradient of the function v as well. Feature that stands out with the exponential mollification is that one has a formula for the time derivative, namely

$$\partial_t [\![v]\!]_h = \frac{1}{h} \left(v - [\![v]\!]_h \right).$$

This will be particularly useful when deriving energy estimate of type (1.9), since when testing the equation (1.6) with function depending on $[\![u^m]\!]_h$, the parabolic part produces a term with a specific sign, which allows to conclude the estimate (1.9).

1.3 Sobolev inequality

Another fundamental tool used in regularity theory is the parabolic Sobolev embedding, which implies inclusions for certain function spaces. Indeed, many results in regularity theory follow the idea of combining energy estimates and Sobolev type inequalities. The order of which these inequalities are used depends for example whether one wishes to prove estimates for the solution itself or for the gradient. There exist different variants for the latter, and one way is to write it in the following form (see [24]): If $v \in L^p(0, T; W_0^{1,p}(\Omega, \mathbb{R}^N))$, $1 \le p < \infty$ and $0 < r < \infty$, there exists a constant $c = c(p, r, n, N)$ such that

$$\iint_{\Omega_T} |v|^q \, \mathrm{d}x\mathrm{d}t \le c \iint_{\Omega_T} |Dv|^p \, \mathrm{d}x\mathrm{d}t \left(\operatorname*{ess\,sup}_{t \in (0,T)} \int_{\Omega \times \{t\}} |v|^r \, \mathrm{d}x \right)^{\frac{p}{n}}, \qquad (1.10)$$

in which $q = p\left(1 + \frac{r}{n}\right)$. Observe that there is product of two terms on the right hand side, and the second one is not present at all in the elliptic setting. Essentially, the inequality (1.10) implies that the function v is integrable to a higher power than p, provided that the function belongs in addition to $L^\infty(0, T; L^r(\Omega, \mathbb{R}^N))$. For a weak solution it is natural to assume $\boldsymbol{u}^m \in L^p(0, T; W^{1,p}(\Omega, \mathbb{R}^N))$ a priori as presented in Section 1.1. In order to conclude that the second term is finite, one may exploit an energy estimate with suitable choice of r.

2. Theory for supercaloric functions

In order to include some fundamental solutions in the theory, the definition of weak solutions as in (1.6) turns out to be too restrictive. A well-known example of a fundamental solution is the Barenblatt solution (discovered in [5]), which can be written for the p-Laplace equation in the case $p > \frac{2n}{n+1}$ as

$$U(x,t) = (\lambda t)^{-\frac{n}{\lambda}} \left(c - \frac{p-2}{p} (\lambda t)^{-\frac{p}{\lambda(p-1)}} |x|^{\frac{p}{p-1}} \right)_{+}^{\frac{p-1}{p-2}} \chi_{\{t>0\}}, \qquad (2.1)$$

for $(x,t) \in \mathbb{R}^n \times (-\infty, \infty)$, in which $\lambda = n(p-2) + p$ and $c > 0$. In the upper half-space, the Barenblatt solution is a solution to the parabolic p-Laplace equation. However, in any domain containing the origin, this function fails to be even a weak supersolution. This is due to the fact that the a priori integrability property fails, i.e., $|\nabla U|^p$ is not locally integrable in any such domain. This suggests that we define a more general class of solutions. We will consider a class called supercaloric functions, which has been studied in the elliptic case e.g. in [31], and in the linear case for the heat equation in [65]. We say that $u : \Omega_T \to (-\infty, \infty]$ is a supercaloric function for the parabolic p-Laplace equation if the following conditions are satisfied:

(i) u is lower semicontinuous,

(ii) u is finite in a dense subset of Ω_T,

(iii) u satisfies comparison principle in the following sense: let $\Omega'_{t_1,t_2} \Subset \Omega_T$ be a space-time cylinder and $h \in C(\overline{\Omega'_{t_1,t_2}})$ be a weak solution to (1.4). If $u \geq h$ on the parabolic boundary $\partial_p \Omega'_{t_1,t_2} = \overline{\Omega'} \times \{t_1\} \bigcup \partial \Omega' \times (t_1, t_2)$, then $u \geq h$ in Ω'_{t_1,t_2}.

Weak supersolutions are required to be integrable, which implies finiteness condition (ii). They also possess lower semicontinuous representatives that has been proven in [44], [49]. The third condition is a comparison principle, which is shown for weak supersolutions in [33]. This implies that weak supersolutions are supercaloric functions when restricted to lower

semicontinuous representatives, so the class of supercaloric functions is a generalization in that sense. Observe that for supercaloric functions the only connection to the equation is via the comparison principle (iii).

We consider local properties of supercaloric functions for the parabolic p-Laplace equation in this chapter. In order to emphasize this particular equation in question, we may also denote these functions by p-supercaloric functions. A technical advantage that is present with the parabolic p-Laplace equation is the property that allows one to add constants to solutions. Further, since any supercaloric function u is lower semicontinuous and $u > -\infty$ in Ω_T by definition, it follows that u is locally bounded from below. Due to these two properties it is sufficient to consider nonnegative u when studying for example local integrability of u. Indeed, in a compact subset $K \Subset \Omega_T$ instead of u one may consider $u + |\inf_K u|$ which is nonnegative with essentially same integrability properties as u. It is also immediate from the definition that pointwise minimum of finite number of supercaloric functions is supercaloric. Further, supercaloric functions are closed under increasing convergence, provided that the limit is finite in a dense set, which has been shown in [41].

2.1   Slow diffusion case

In Publication II we consider classification theory for supercaloric functions in the singular supercritical case. In the degenerate case the theory is well developed, see e.g. [37], [45], and for the corresponding theory for the porous medium equation see [34], [38]. Already in the degenerate case, it is well-known that the classification theory is dictated by two leading examples: the Barenblatt solution (2.1) and the friendly giant (defined below in (2.2)). First one is a representative of a relatively well behaving class of supercaloric functions (Barenblatt class), and latter of a class with strong singularities. These classes can be characterized by integrability properties. In the degenerate case, the integrability exponent $p - 2$ is decisive; if a supercaloric function is integrable to this particular exponent, it belongs to the Barenblatt class and if not, it belongs to the complementary class. Furthermore, it can be shown that any function in the Barenblatt class is integrable up to any power less than $p - 1 + \frac{p}{n}$. For a supercaloric function this means that the function is either integrable up to power $p - 1 + \frac{p}{n}$, or not even to $p - 2$. Another feature that stands out in the complementary class is the rate of blow-up. In the degenerate case, there exists a time slice t_o such that the supercaloric function blows up at least with rate $(t - t_o)^{-\frac{1}{p-2}}$, when t approaches t_o from the future. Furthermore, this blow-up takes place in the whole time slice. Friendly giant gives this mildest possible rate for blow-up for the functions in the complementary class.

In the degenerate case the Barenblatt class can be characterized by the following theorem (see [45]).

Theorem 1. *Let $p > 2$ and Ω be an open set in $\mathbb{R}^n$. For a p-supercaloric function u the following conditions are equivalent:*

(i) $u \in L^{p-2}_{\mathrm{loc}}(\Omega_T)$,

(ii) ∇u *exists and* $|\nabla u| \in L^q_{\mathrm{loc}}(\Omega_T)$ *whenever* $q < p - 1 + \frac{1}{n+1}$,

(iii)

$$\operatorname*{ess\,sup}_{t \in (\delta, T-\delta)} \int_{\Omega'} |u(x,t)| \, \mathrm{d}x < \infty$$

whenever $\Omega' \Subset \Omega$ *and* $\delta \in (0, \frac{T}{2})$.

There is also a connection between measure data problems and supercaloric functions in the Barenblatt class. Especially, the Barenblatt solution U satisfies the equation

$$\partial_t U - \operatorname{div}\left(|\nabla U|^{p-2}\nabla U\right) = M\delta$$

in the weak sense, where $M > 0$ is constant depending on c in (2.1) and δ is Dirac's delta concentrated in the origin. Furthermore, any supercaloric function in the Barenblatt class (characterized in Theorem 1 in the slow diffusion case and in Theorem 3 below in the fast diffusion case) is a solution to the measure data problem with a Radon measure μ on the right hand side as above. In the degenerate case the converse has also been studied in [39].

As mentioned above, the leading example of a class of supercaloric functions that do not satisfy properties in Theorem 1 is so-called friendly giant. This function has the following explicit form

$$V(x,t) = U(x)(t - t_o)^{-\frac{1}{p-2}}\chi_{\{t > t_o\}}, \tag{2.2}$$

where $x \in \Omega$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain. Furthermore, $U > 0$ satisfies an elliptic eigenvalue problem

$$\operatorname{div}\left(|\nabla U|^{p-2}\nabla U\right) + \frac{1}{p-2}U = 0$$

with zero boundary values on $\partial\Omega$. Observe that the function V blows up for every spatial point $x \in \Omega$ when approaching the time instant t_o from the future. The function V in (2.2) fails to satisfy items in the Theorem 1. This is an example from the complementary class, for which there exists the following characterization (see [45]).

Theorem 2. *Let $p > 2$ and Ω be an open set in $\mathbb{R}^n$. For a p-supercaloric function u the following conditions are equivalent:*

(i) $u \notin L^{p-2}_{\mathrm{loc}}(\Omega_T)$,

(ii) for some $\delta \in (0, \frac{T}{2})$,

$$\operatorname*{ess\,sup}_{t \in (\delta, T-\delta)} \int_{\Omega'} |u(x,t)| \,\mathrm{d}x = \infty$$

whenever $\Omega' \Subset \Omega$ *and* $|\Omega'| > 0$.

(iii) there exists $(x_o, t_o) \in \Omega_T$ *such that*

$$\liminf_{\substack{(x,t) \to (x_o, t_o) \\ t > t_o}} u(x,t)\,(t - t_o)^{\frac{1}{p-2}} > 0.$$

2.2 Fast diffusion case

Publication II extends the classification theory for supercaloric functions for the parabolic p-Laplace equation from range $p > 2$ to $\frac{2n}{n+1} < p < 2$. In a similar fashion as in the degenerate case, supercaloric functions can be divided into two mutually exclusive classes also in the singular case. In contrast to the degenerate case, the decisive integrability exponent is $\frac{n}{p}(2-p)$, which is less than one. The Barenblatt solution is still the leading example in the good class with some regularity properties in terms of integrability (up to the power $p - 1 + \frac{p}{n} > 1$) representing the worst possible behavior. However, there is a remarkable difference in the complementary class. The friendly giant is apparent only in the degenerate case, but ceases to exist in the singular case. Instead, a prime example representing the behavior in the complementary class is given by the infinite point source solution (IPSS, see [16]). A qualitative difference compared to the friendly giant is that space and time change roles in the blow-up. Whereas the friendly giant is unbounded on the whole time slice but at a single moment of time, the blow-up for IPSS happens at a single point x_o in space but for all large enough instances of time. This is a standing singularity, that does not decay as time passes. A minorant for the blow-up in the complementary class is given by the IPSS, which is $|x - x_o|^{-\frac{p}{2-p}}$ near point x_o and large enough instances of time.

The first step in proving characterizations for the two classes of supercaloric functions (in both slow and fast diffusion cases) is by showing that supercaloric functions can be approximated pointwise by truncations of weak supersolutions. Indeed, we can show that locally bounded supercaloric functions are weak supersolutions. The proof of this especially for the more general form of the equation with coefficients relies on the obstacle problem, see [41] and [42]. Idea is first to approximate the lower

semicontinuous supercaloric function pointwise from below by more regular (e.g. smooth) functions. Then, in each space time box one solves a boundary value problem with these approximants acting as obstacles inducing the boundary values as well. In this way, one obtains a weak supersolution above the obstacle, and by using appropriate comparison principle, one can deduce that these supersolutions are bounded pointwise from above by u. This implies a pointwise converging sequence of weak supersolutions to the original supercaloric function u from below. Since u is locally bounded, it ends up being a weak supersolution itself.

When boundedness is not required for supercaloric functions a priori, these functions are divided into two mutually exclusive classes as mentioned above. When $\frac{2n}{n+1} < p < 2$, Barenblatt solution (2.1) can be written as

$$U(x,t) = (\lambda t)^{-\frac{n}{\lambda}} \left(c + \frac{2-p}{p} (\lambda t)^{-\frac{p}{\lambda(p-1)}} |x|^{\frac{p}{p-1}} \right)^{-\frac{p-1}{2-p}} \chi_{\{t>0\}},$$

where $\lambda = n(p-2) + p$ and the constant c is a positive number. In Publication II we obtain the following characterization for supercaloric functions in the Barenblatt class in the singular case.

Theorem 3. *Let $\frac{2n}{n+1} < p < 2$ and Ω be an open set in $\mathbb{R}^n$. Assume that u is a p-supercaloric function in Ω_T. Then the following assertions are equivalent:*

(i) $u \in L^q_{\mathrm{loc}}(\Omega_T)$ for some $q > \frac{n}{p}(2-p)$,

(ii) $u \in L^{\frac{n}{p}(2-p)}_{\mathrm{loc}}(\Omega_T)$,

(iii)

$$\operatorname*{ess\,sup}_{\delta < t < T-\delta} \int_{\Omega'} |u(x,t)| \, \mathrm{d}x < \infty,$$

whenever $\Omega' \Subset \Omega$ and $\delta \in (0, \frac{T}{2})$.

In particular, we can show that functions beloning to this class are actually integrable up to any power $q < p - 1 + \frac{p}{n}$. This improvement in integrability can be shown by using Moser type iteration scheme. Essentially this is a result of combining the Sobolev embedding (1.10) and an energy estimate for weak supersolutions. A suitable form of energy estimate for a nonnegative weak supersolution u can be written as

$$\iint_{\Omega_T} |\nabla u|^p u^{-\varepsilon-1} \varphi^p \, \mathrm{d}x\mathrm{d}t + \operatorname*{ess\,sup}_{0<t<T} \int_\Omega u^{1-\varepsilon} \varphi^p \, \mathrm{d}x$$

$$\leq c(p,\varepsilon) \iint_{\Omega_T} u^{p-1-\varepsilon} |\nabla \varphi|^p \, \mathrm{d}x\mathrm{d}t + c(p,\varepsilon) \iint_{\Omega_T} u^{1-\varepsilon} |\partial_t(\varphi^p)| \, \mathrm{d}x\mathrm{d}t,$$

for any nonnegative test function $\varphi \in C_0^\infty(\Omega_T)$. The parameter $\varepsilon \in (0,1)$ can be chosen freely, and the constant $c(p,\varepsilon) \to \infty$ if $\varepsilon \to 0$ or $\varepsilon \to 1$. One can employ the energy estimate above via approximation by truncating the supercaloric function as $\min\{u,k\}$ for $k = 1,2,...$ which is a weak supersolution. In order to combine the Sobolev embedding (1.10) and the energy estimate above, one chooses $r > 0$ and $\varepsilon \in (0,1)$ in these estimates suitably so that the right hand side of the Sobolev embedding can be estimated by the energy estimate. In order to increase the integrability exponent for u in this manner, it is crucial to assume that the parameter p is above the critical exponent $\frac{2n}{n+1}$. In the end, the improvement in integrability for u is obtained by passing to the limit $k \to \infty$, where k is the truncation level. This procedure can be repeated until the integrability exponent exceeds $p - 1 + \frac{p}{n}$.

As the Theorem 4 shows below, if a supercaloric function does not belong to the Barenblatt class, it is not integrable even to a power $\frac{n}{p}(2-p)$. This indicates that there exists a gap

$$\left[\frac{n}{p}(2-p), p-1+\frac{p}{n}\right)$$

of integrability exponents, which is nonempty in the case $\frac{2n}{n+1} < p < 2$, the lower bound being less than 1 and the upper bound greater than 1.

In the fast diffusion case, instead of the friendly giant (2.2) a prime example from the complementary class is the IPSS. This one can write as

$$U(x,t) = \left(\frac{ct}{|x|^p}\right)^{\frac{1}{2-p}}, \quad (x,t) \in \mathbb{R}^n \times (0,\infty),$$

for a specific positive constant c depending on n and p. One can verify that any of the items in Theorem 3 fails to hold in a domain Ω_T for which $0 \in \Omega$. In fact, the IPSS can be modified into a supercaloric function with even worse singularity in space (Example 4.2 in Publication II).

The next theorem that has been proven in Publication II gives a characterization for the complementary class of Barenblatt type supercaloric functions in the singular case.

Theorem 4. *Let $\frac{2n}{n+1} < p < 2$ and Ω be an open set in $\mathbb{R}^n$. Assume that u is a p-supercaloric function in Ω_T. Then the following properties are equivalent:*

(i) $u \notin L_{\text{loc}}^q(\Omega_T)$ for any $q > \frac{n}{p}(2-p)$,

(ii) $u \notin L_{\text{loc}}^{\frac{n}{p}(2-p)}(\Omega_T)$,

(iii) there exists $\Omega' \Subset \Omega$ and $\delta \in (0, \frac{T}{2})$ such that

$$\operatorname*{ess\,sup}_{\delta < t < T-\delta} \int_{\Omega'} |u(x,t)|\, \mathrm{d}x = \infty,$$

(iv) there exists $(x_o, t_o) \in \Omega_T$ such that

$$\liminf_{\substack{(x,t) \to (x_o, s) \\ t > s}} u(x,t)|x - x_o|^{\frac{p}{2-p}} > 0$$

for every $s > t_o$.

The last item in the Theorem 4 represents a very interesting behavior of supercaloric functions in the complementary class. It states that functions in that class possess a singularity in space. Even more, the statement gives a lower bound for the blow-up rate and the singularity does not decay as time passes. An important tool in obtaining this blow-up rate is a weak Harnack inequality, which asserts that on time slices pointwise infimum of a nonnegative supersolution can be bounded from below by the integral average of the supersolution over the same set. In the case $\frac{2n}{n+1} < p < 2$, for a supersolution u this can be written as

$$\inf_{B_{2\varrho}(x_o)} u(\cdot, t) \geq c_1 \fint_{B_{2\varrho}(x_o)} u(x, s) \, \mathrm{d}x \tag{2.3}$$

for any $t \in \left[s + \frac{3}{4}\theta\varrho^p, s + \theta\varrho^p \right]$, in which

$$\theta = c_2 \left(\fint_{B_{2\varrho}(x_o)} u(x, s) \, \mathrm{d}x \right)^{2-p}.$$

Here c_1 and c_2 are positive constants depending only on n and p. For the proof of this form of the weak Harnack inequality, see [27]. Observe that in this result there is a waiting time depending on the integral average, which increases as the integral average itself increases. One way of showing the pointwise blow-up property is first to assume for example item (iii) in Theorem 4. This implies that there exists a sequence (t_i) of instants of time, for which integrals over space blow up in the limit, i.e.

$$\lim_{i \to \infty} \fint_{B_{2\varrho}(x_o)} u(x, t_i) \, \mathrm{d}x = \infty,$$

for some ball $B_{2\varrho}(x_o) \subset \Omega'$. Then, one may roughly deduce that as the integral average blows up on the right hand side of (2.3), it also forces the pointwise infimum on the left hand side to blow up. In order to keep the time interval in the domain, i.e. $\theta\varrho^p$ in control, one shrinks the radius ϱ along the sequence, which gives an idea how the desired blow-up rate can be deduced.

3.   Higher integrability

For functions that satisfy a reverse Hölder inequality, an application of Gehring's lemma can be used to obtain a higher integrability result. In the case of parabolic PDEs, higher integrability of the weak solution itself is typically a direct consequence of Sobolev and Caccioppoli inequalities together with the standard integrability assumptions for the solution and its gradient. However, a similar property on the gradient level is usually a more subtle issue, in which technique based on self-improving property of suitable reverse Hölder inequalities can be exploited. In Publication I we show that the gradient of a weak solution to porous medium type systems is integrable to a higher power than assumed a priori, and up to the boundary of the domain. More precisely, we prove this result for solutions to a Cauchy-Dirichlet type problem (1.7) in the case $m > 1$. Prior to our result there was a breakthrough for the local problem in [28] by Gianazza & Schwarzacher, and the result was generalized in [8] by Bögelein et al. The overall strategy of our proof follows the ideas presented in the local case, although the boundary imposes additional difficulties one needs to solve.

There is a long history of higher integrability results for the gradients of solutions that goes back to the paper by Elcrat & Meyers, [53], in which the result was proven for elliptic systems of p-Laplace type. This result was extended up to the boundary by Kilpeläinen & Koskela in [32]. In the parabolic setting the first higher integrability result was proven for quasilinear systems by Giaquinta & Struwe in [30]. This corresponds to (1.1) with $m = 1$ and $p = 2$. However, this technique was not applicable to more general nonlinear PDEs. Almost 20 years later, higher integrability of the gradient was proven for parabolic p-Laplace type systems by Kinnunen & Lewis in [35], and similar result up to the boundary was shown later by Parviainen in [55], [56]. Higher order systems has been studied by Bögelein & Parviainen in [15]. Again over a decade later of the result for parabolic problems with p-growth, the self-improving property was shown to hold for gradients of nonnegative solutions to degenerate porous medium type equations by Gianazza & Schwarzacher in [28], which was

extended for systems in [8] and to singular case in [29] and [11]. This type of technique has been applied to doubly nonlinear systems in [7], and also for obstacle problem for the PME in [18] and [19]. The higher integrability property has also been studied in the context of very weak solutions for parabolic p-Laplace type systems in [36] and [1]. The case of variable exponents has been investigated e.g. in [2].

3.1 Main result

Our result extends the higher integrability for porous medium type systems up to the boundary. We consider a Cauchy-Dirichlet problem written in the form

$$
\begin{cases}
\partial_t u - \operatorname{div} \mathbf{A}(x, t, u, D\boldsymbol{u}^m) = \operatorname{div} F & \text{in } \Omega_T, \\
u = g & \text{on } \partial_p \Omega_T,
\end{cases}
\tag{3.1}
$$

where the vector field $\mathbf{A}$ satisfies porous medium type growth conditions and F is a source term. Boundary values g are assigned in the sense of (1.8). For a sufficiently regular boundary function g, source term F and domain Ω there exists $\varepsilon > 0$, such that for a weak solution u to (3.1) there holds

$$
|D\boldsymbol{u}^m| \in L^{2+\varepsilon}(\Omega_T),
\tag{3.2}
$$

together with an estimate as stated in Theorem 1.4 in Publication I. For a weak solution u it is assumed a priori only that $|D\boldsymbol{u}^m| \in L^2(\Omega_T)$, so that the result implies an improvement. Parameter ε in (3.2) depends on n, N, m, C_o, C_1 and parameters connected to the properties of the domain Ω, the source term F, and the boundary function g, which are discussed in the next section.

Key challenges in deriving gradient estimates in the case of porous medium type equations is due to the structure of the equation. Especially, a difficulty arises from the fact that the degeneracy of the equation depends on the solution rather than its gradient, while aim is to prove the estimates for the latter. To be able to handle this issue, in Publication I we will work with intrinsic cylinders of the form

$$
Q_\varrho^{(\theta)}(x_o, t_o) = B_\varrho(x_o) \times \left(t_o - \theta^{1-m}\varrho^{\frac{m+1}{m}}, t_o + \theta^{1-m}\varrho^{\frac{m+1}{m}}\right),
$$

with a scaling parameter θ, which roughly corresponds to $\frac{1}{\varrho}|u|^m$. Furthermore, as the cylinders are intersecting the parabolic boundary of the domain Ω_T, the boundary function g will be coupled with the scaling parameter θ as well. We will prove most of the estimates separately near the lateral and near the initial boundary of the domain.

The first step in the proof is to derive suitable Caccioppoli and Sobolev-Poincaré inequalities. By combining these estimates we are able to prove

reverse Hölder inequalities for the gradient in cylinders satisfying specific intrinsic couplings. We construct a collection of intrinsic cylinders in which the derived reverse Hölder inequalities can be applied, and in addition, other favorable properties such as a Vitali type covering result can be established. This allows us to extend the higher integrability result to the whole domain.

3.2 Domain and extensions

To begin with, in addition to the standard assumptions on the boundary function g in (1.7), we assume continuity in time for g (from the interval $[0, T]$ to the space $L^{m+1}(\Omega, \mathbb{R}^N)$) together with some higher integrability properties for g^m. In particular, we suppose $g^m \in L^{2+\varepsilon}(0, T; W^{1,2+\varepsilon}(\Omega, \mathbb{R}^N))$ and $\partial_t g^m \in L^{\frac{m(2+\varepsilon)}{2m-1}}(\Omega_T, \mathbb{R}^N)$. Furthermore, the boundary of the spatial domain Ω plays a role in boundary regularity. A standard assumption we make is the capacity density condition on the complement, which is already necessary in the case of elliptic equations with p-growth as shown in [32]. More precisely, we assume that the complement of Ω is uniformly 2-thick, which means that for some positive constants μ and ϱ_o,

$$\operatorname{cap}_2 \big((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_\varrho(x_o), B_{2\varrho}(x_o) \big) \geq \mu \operatorname{cap}_2 \big(\overline{B}_\varrho(x_o), B_{2\varrho}(x_o) \big)$$

holds true for all $x_o \in \mathbb{R}^n \setminus \Omega$ and $0 < \varrho < \varrho_o$. Heuristically, this means that there is a substantial amount of complement around every point on the lateral boundary $\partial\Omega$ in the sense of capacity. As mentioned before, this type of assumption is already present in the proof of equations with p-growth, see [32], [55], [56].

In order to prove the result for porous medium systems, we assume further that the domain Ω is Sobolev $W^{1,2+\varepsilon}$-extension domain. In more accurate terms this means that there exists a linear operator $E : W^{1,2+\varepsilon}(\Omega) \to W^{1,2+\varepsilon}(\mathbb{R}^n)$ such that $Eu(x) = u(x)$ for a.e. $x \in \Omega$ with the bound

$$\|Eu\|_{W^{1,2+\varepsilon}(\mathbb{R}^n)} \leq c_E \|u\|_{W^{1,2+\varepsilon}(\Omega)}$$

for any $u \in W^{1,2+\varepsilon}(\Omega)$ and some constant $c_E \geq 0$.

To be able to work in cylinders that intersect complement of Ω_T, we use extensions $\hat{u}$ and $\hat{g}$ coinciding with u and g in Ω_T. Essentially the idea is that g is extended slice-wise as a Sobolev function outside Ω, which is possible since Ω is a Sobolev extension domain. Further, this extension (still denoted by g) in the strip $\mathbb{R}^n \times [0, T]$ is reflected to the negative times such that $\hat{g}(\cdot, t) = g(\cdot, -t)$ for $t \in (-T, 0)$ in $\mathbb{R}^n$. Inside Ω, we define $\hat{u}(\cdot, t) = g(\cdot, -t)$ for $t \in (-T, 0)$. Moreover, we exploit the property $(1.8)_1$, which allows us to extend the function $u^m - g^m$ by zero slice-wise outside Ω. We may define $\hat{u}$ outside Ω such that $\hat{u}^m - \hat{g}^m = 0$ holds in $\mathbb{R}^{n+1} \setminus \Omega_T$.

3.3 Towards reverse Hölder inequalities

In this section we will focus mostly on the case where the cylinder $Q_\varrho^\theta(z_o)$ intersects at least the lateral boundary, and is allowed to touch the initial boundary as well. We call this case near the lateral boundary, while the term near the initial boundary represents the case when the cylinder intersects only the initial boundary. By deriving a Caccioppoli inequality near the lateral boundary, we essentially establish a bound for the integral of $|D\boldsymbol{u}^m|^2$ in terms of

$$\iint_{Q_\varrho^{(\theta)}(z_o)\cap\Omega_T} \frac{|\boldsymbol{u}^m - \boldsymbol{g}^m|^2}{\varrho^2}\,\mathrm{d}x\mathrm{d}t \tag{3.3}$$

and

$$\iint_{Q_\varrho^{(\theta)}(z_o)\cap\Omega_T} \theta^{m-1} \frac{\left|\boldsymbol{u}^{\frac{m+1}{2}} - \boldsymbol{g}^{\frac{m+1}{2}}\right|^2}{\varrho^{\frac{m+1}{m}}}\,\mathrm{d}x\mathrm{d}t \tag{3.4}$$

on the right hand side. The term (3.3) is connected to the diffusion part and (3.4) to the parabolic part of the equation.

The Sobolev-Poincaré inequality is another fundamental building block when deriving reverse Hölder inequalities. The idea of the Sobolev-Poincaré inequality is to bound the term (3.3) by an integral involving the gradient $|D\boldsymbol{u}^m|^q$, where the power satisfies $q < 2$. To establish this, we work in cylinders satisfying sub-intrinsic scaling, that can be written as

$$\iint_{Q_\varrho^{(\theta)}(z_o)} 2\frac{|\hat{\boldsymbol{u}}^m - \hat{\boldsymbol{g}}^m|^2 + |\hat{g}|^{2m}}{\varrho^2}\,\mathrm{d}x\mathrm{d}t \le 2^{d+2}\theta^{2m} \tag{3.5}$$

near the lateral boundary, and

$$\iint_{Q_\varrho^{(\theta)}(z_o)} \frac{|\hat{u}|^{2m}}{\varrho^2}\,\mathrm{d}x\mathrm{d}t \le 2^{d+2}\theta^{2m} \tag{3.6}$$

near the initial boundary (when not touching the lateral boundary).

Then, in order to deduce reverse Hölder inequalities, the task is to combine Caccioppoli and Sobolev-Poincaré inequalities. Roughly the idea is to bound (3.4) by (3.3). For this we wish to bound θ in the integrand of (3.4) from above. In particular, near the lateral boundary we consider cases where either the coupling

$$\theta^{2m} \le K \iint_{Q_\varrho^{(\theta)}(z_o)} 2\frac{|\hat{\boldsymbol{u}}^m - \hat{\boldsymbol{g}}^m| + |\hat{g}|^{2m}}{\varrho^2}\,\mathrm{d}x\mathrm{d}t, \tag{3.7}$$

or

$$\theta^{2m} \le K \iint_{Q_\varrho^{(\theta)}(z_o)\cap\Omega_T} \left(|D\boldsymbol{u}^m|^2 + |F|^2 + |\partial_t\boldsymbol{g}^m|^{\frac{2m}{2m-1}}\right)\,\mathrm{d}x\mathrm{d}t \tag{3.8}$$

$$+ K \iint_{Q_\varrho^{(\theta)}(z_o)\cap\{t>0\}} |D\boldsymbol{g}^m|^2\,\mathrm{d}x\mathrm{d}t$$

holds true for some constant $K \geq 1$, in addition to (3.5). We call the former the non-degenerate case and the latter the degenerate case. The proofs for reverse Hölder inequalities differ depending on these cases. The non-degenerate case is more involved, and there we use (3.7) together with the Poincaré inequality for $\hat{g}^m$ (Lemma 4.3 in Publication I) in order to estimate the term (3.4) by (3.3) in the Caccioppoli inequality, after which the Sobolev-Poincaré inequality is applicable. In the degenerate case we exploit Young's inequality to split the integral in (3.4) into two parts, so that one term is exactly $\delta\theta^{2m}$ with a small δ, and the other term can be estimated from above by (3.3). By using the bound (3.8) for $\delta\theta^{2m}$, the resulting integral of $|D\boldsymbol{u}^m|^2$ can be absorbed to the left hand side, and the Sobolev-Poincaré inequality is again applicable.

By following the steps described above we may deduce a reverse Hölder inequality, that can be written in the form

$$
\frac{1}{|Q_\varrho^{(\theta)}(z_o)|} \iint_{Q_\varrho^{(\theta)}(z_o) \cap \Omega_T} |D\boldsymbol{u}^m|^2 \, \mathrm{d}x\mathrm{d}t
$$

$$
\leq c \left(\frac{1}{|Q_{8\varrho}^{(\theta)}(z_o)|} \iint_{Q_{8\varrho}^{(\theta)}(z_o) \cap \Omega_T} |D\boldsymbol{u}^m|^q \, \mathrm{d}x\mathrm{d}t \right)^{\frac{2}{q}} \tag{3.9}
$$

$$
+ c \iint_{Q_{8\varrho}^{(\theta)}(z_o) \cap \{t>0\}} \left[\left(|F|^2 + |\partial_t \boldsymbol{g}^m|^{\frac{2m}{2m-1}} \right) \chi_{\Omega_T} + |D\boldsymbol{g}^m|^2 \right] \, \mathrm{d}x\mathrm{d}t
$$

near the lateral boundary, where the exponent $q < 2$ is connected to the self-improving property of uniformly 2-thick sets (in this case $\mathbb{R}^n \setminus \Omega$), see [47]. Near the initial boundary similar type of bound can be deduced, where the exponent on the right hand side has explicit form $q = \frac{2n}{d}$, in which $d = n + 1 + \frac{1}{m}$ is the parabolic dimension associated to our cylinders. Here we have also used the fact that Ω is a $W^{1,2+\varepsilon}$-extension domain, which allows us to extend the gradient of $\boldsymbol{g}^m$ on the right hand side slice-wise outside Ω.

3.4 Final arguments

The strategy of the final proof is to construct a collection of sub-intrinsic cylinders, in which the reverse Hölder inequalities hold true, and for which one can show a Vitali type covering property. Then we are able to derive a gradient estimate in the super-level sets of the gradient $|D\boldsymbol{u}^m|$ leading to the higher integrability result by a Fubini type argument.

The construction of these cylinders follows the idea of [8] and [28] in the interior case. The scaling parameter θ will depend on a considered reference point in space z_o and radius ϱ. So-called rising sun construction is exploited, since in particular it is not clear a priori if the constructed system of cylinders is nested with respect to the radius ϱ. This challenge

is already present in the interior case. However, there are additional complications related to the boundary of the domain. Especially, the sub-intrinsic scaling will depend on the boundary values g if the particular cylinder is close to the lateral boundary. In more technical terms this means that sub-intrinsicness is understood in the sense of (3.6) if $\varrho < \frac{1}{2}\operatorname{dist}(x_o, \partial\Omega)$ and (3.5) if $\varrho \geq \frac{1}{2}\operatorname{dist}(x_o, \partial\Omega)$. By rising sun type construction, we can show that the resulted parameter $\theta_{z_o} = \theta_{z_o;\varrho}$ is decreasing and continuous with respect to ϱ, so that the cylinders $Q_\varrho^{(\theta_{z_o})}(z_o)$ are nested w.r.t. ϱ. Furthermore, we are able to show a Vitali type covering for cylinders constructed like this (Lemma 6.3 in Publication I). Moreover, by considering a point z_o in super-level set $\{|D\boldsymbol{u}^m| > \lambda^m\}$ for large enough λ, we can find a maximal radius $\varrho_{z_o} > 0$ such that

$$\iint_{Q_{\varrho_{z_o}}^{(\theta_{z_o})}(z_o)} \left[|D\boldsymbol{u}^m|^2 \chi_{\Omega_T} + G^2\right]\,\mathrm{d}x\mathrm{d}t = \lambda^{2m} \tag{3.10}$$

holds true, so that the integral average is smaller than λ^{2m} for all $\varrho > \varrho_{z_o}$. Here G will depend on spatial and temporal derivatives of $\boldsymbol{g}^m$, as well as the source term F. For points z_o in the super-level set $\{|D\boldsymbol{u}^m| > \lambda^m\}$ we are able to show that the reverse Hölder inequality holds in the form

$$\frac{1}{\left|Q_{\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)\right|} \iint_{Q_{\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)\cap\Omega_T} |D\boldsymbol{u}^m|^2 \,\mathrm{d}x\mathrm{d}t$$

$$\leq \left(\frac{c}{\left|Q_{32\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)\right|} \iint_{Q_{32\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)\cap\Omega_T} |D\boldsymbol{u}^m|^q \,\mathrm{d}x\mathrm{d}t\right)^{\frac{2}{q}} \tag{3.11}$$

$$+ c\iint_{Q_{32\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)\cap\{t>0\}} G^2 \,\mathrm{d}x\mathrm{d}t,$$

for some exponent $q < 2$. This is a result by extensive case-by-case analysis by deducing that either one of the bounds (3.7) or (3.8) hold true so that (3.9) is applicable. Additional complications are induced by the fact that it is not clear a priori how close to the lateral boundary a particular cylinder in question is. Especially in the degenerate case, coupling (3.10) is crucial.

The conclusion of higher integrability is obtained by covering the super-level sets $\{|D\boldsymbol{u}^m| > \lambda^m\}$ by enlarged cylinders of disjoint and countable subcollection of $\{Q_{32\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)\}$, which is possible by Vitali type covering in Lemma 6.3 in Publication I. Then, a reverse Hölder type inequality can be deduced in these super-level sets, and the integrability exponents on both sides of the equation can be increased by using Fubini type argument. This implies the final result given in Theorem 1.4 in Publication I.

4. Stability

Publication III concerns stability for porous medium systems. Stability here answers the question "do weak solutions to a sequence of problems converge to the solution of the limit problem, if the parameter characterizing the problems converges?". This type of result can also be considered to be practically motivated, as in the applications the parameter can be measured only approximately. Then, it is important to know that the corresponding solutions are stable with respect to small fluctuations of that parameter. In Publication III we considered stability in both local and global, in the sense of Cauchy-Dirichlet problem (1.7), setting.

In the case of the parabolic p-Laplace equation, stability of the solutions to Cauchy-Dirichlet problems was proven by Kinnunen and Parviainen in [40]. In particular, they showed that weak solutions of problems with varying p converge to the solution of the limit problem in a suitable parabolic Sobolev space. There the result was shown for solutions to boundary value problems, which are known to be unique. In particular, to show convergence of the gradients in suitable sense, the authors in [40] used the higher integrability result from [55]. This requires an additional condition for the lateral boundary, namely that the complement of the domain satisfies a uniform capacity density condition. Furthermore, in the case of parabolic p-Laplace equation stability has been studied for Cauchy problems in the upper half-space in [52] up to generality of Radon measure as the initial data. In case of the porous medium equation, we also mention that there exists a manuscript [51] concerning stability of nonnegative solutions.

One remarkable feature in our result for porous medium systems is that in contrast to the parabolic p-Laplace equation treated in [40], we do not need the higher integrability result or additional regularity requirements for the lateral boundary. This is connected to the fact that in the case of equations with p-growth, the function spaces of the solutions vary if p varies, while for porous medium equation the corresponding function space stays fixed even if m varies. Essentially, our proof for stability relies on suitable energy estimates and a compactness result from [61] together

with appropriate interpolation lemma from [6].

4.1   Main results and assumptions

The result we obtain roughly tells that if (m_i) is a sequence of real numbers in the interval $\left(\frac{(n-2)_+}{n+2}, \infty \right)$ converging to some m inside the same interval, then weak solutions u_i to the corresponding system (1.1) (where $p = 2$) with m_i converge to the solution u of the limit problem in a suitable norm. Essentially we show that the solutions converge in the sense

$$\boldsymbol{u}_i^{m_i} \to \boldsymbol{u}^m \quad \text{in } L^2(0, T; W^{1,2}(\Omega, \mathbb{R}^N)), \qquad (4.1)$$

as $i \to \infty$, provided that appropriate assumptions are made. For a local problem, the convergences are naturally understood on compact subsets of Ω_T. The results for local and global problems are stated precisely in Theorems 2.3 and 2.7 in Publication III.

In addition to the structure conditions in (1.2) and (1.3) (with $p = 2$), we further assume monotonicity for the vector field $\mathbf{A}$. This means that for some positive constant μ,

$$(\mathbf{A}(x, t, u, \xi) - \mathbf{A}(x, t, v, \eta)) \cdot (\xi - \eta) \geq \mu \, |\xi - \eta|^2, \qquad (4.2)$$

holds true for a.e. $(x, t) \in \Omega_T$ and all pairs $(u, \xi), (v, \eta) \in \mathbb{R}^N \times \mathbb{R}^{Nn}$.

In Publication III we consider both local and global (Cauchy-Dirichlet) problem. Especially in the local case, some additional assumptions are in order. Namely, we suppose that

$$\boldsymbol{u}_i^{m_i} \rightharpoonup \boldsymbol{u}^m \quad \text{weakly in } L_{\text{loc}}^2(\Omega_T, \mathbb{R}^N), \qquad (4.3)$$

when $i \to \infty$ for some function $\boldsymbol{u}^m \in L^2(\Omega_T, \mathbb{R}^N)$. In addition, if $\frac{(n-2)_+}{n+2} < m < 1$, we suppose that

$$\boldsymbol{u}_i^{m_i+1} \quad \text{is bounded in } L_{\text{loc}}^1(\Omega_T, \mathbb{R}^N). \qquad (4.4)$$

The assumption (4.3) allows us to identify the limit $\boldsymbol{u}^m$ such that it is unique for the whole sequence. The boundedness assumption in (4.4) ensures that the Caccioppoli inequality gives a uniform (with respect to i) bound, also in the singular case. In the degenerate case this is already implied by (4.3).

4.2   Local case

The first step is in the proof is to derive a suitable Caccioppoli inequality for weak solutions u_i to the porous medium system corresponding the

parameter m_i. This can be obtained in a form

$$\sup_{t \in (t_1, t_2)} \int_\Omega \eta^2 |u_i|^{m_i+1} \, \mathrm{d}x + \iint_{\Omega_T} \eta^2 |D\boldsymbol{u}_i^{m_i}|^2 \, \mathrm{d}x\mathrm{d}t$$

$$\leq \frac{c_m}{\delta} \iint_{\Omega \times (t_1-\delta, t_2)} \eta^2 |u_i|^{m_i+1} \, \mathrm{d}x\mathrm{d}t + c_m \iint_{\Omega_T} |D\eta|^2 |\boldsymbol{u}_i^{m_i}|^2 \, \mathrm{d}x\mathrm{d}t,$$

where $\eta \in C_0^\infty(\Omega)$ and $c = c(m, C_o, C_1) > 0$ with the constant c being uniformly bounded with respect to m. Now assumptions (4.3) and (4.4) imply that the right hand side is uniformly bounded with respect to i, which implies

$$\boldsymbol{u}_i^{m_i} \rightharpoonup \boldsymbol{u}^m \quad \text{weakly in } L_{\mathrm{loc}}^2(0, T; W_{\mathrm{loc}}^{1,2}(\Omega, \mathbb{R}^N))$$

as $i \to \infty$ along a subsequence. In particular, the weak convergence in (4.3) identifies the limit also for the gradients. Then, the strategy is to improve this convergence from weak to strong. At first, the strong convergence is shown for the functions $\boldsymbol{u}_i^{m_i}$ in L_{loc}^2. The main difficulty is to show that a compactness result in [61, Theorem 3] is applicable. Especially, in order to apply the result we have to establish the condition

$$\|\tau_h u_i - u_i\|_{L^1(t_1, t_2-h; L^1(K, \mathbb{R}^N))} \to 0, \tag{4.5}$$

uniformly in i, as $h \to 0$ for any compact subset $K \Subset \Omega$ and $0 < t_1 < t_2 < T$. This can be verified by using a suitable application of interpolation lemma [6, Lemma 4.4], which implies (4.5) once we ensure that (u_i) is uniformly equicontinuous in $C([t_1, t_2]; (W_0^{1,2}(\Omega, \mathbb{R}^N))')$. This can be shown by using the weak formulation with test function $\xi_\delta w$, in which ξ_δ is a cut-off function in time and $w \in W_0^{1,2}(K, \mathbb{R}^N)$. Then, by an application of Gagliardo-Nirenberg inequality and standard arguments, the convergence can be improved from $L_{\mathrm{loc}}^1(\Omega_T, \mathbb{R}^N)$ to $L_{\mathrm{loc}}^2(\Omega_T, \mathbb{R}^N)$.

Finally, to show the convergence of the gradients we use monotonicity property (4.2) of the vector field $\mathbf{A}$, which implies

$$\iint_{\Omega_T} \eta\xi |D\boldsymbol{u}_i^{m_i} - D\boldsymbol{u}^m|^2 \, \mathrm{d}x\mathrm{d}t$$

$$\leq c \iint_{\Omega_T} \eta\xi \mathbf{A}(x, t, u_i, D\boldsymbol{u}_i^{m_i}) \cdot (D\boldsymbol{u}_i^{m_i} - D\boldsymbol{u}^m) \, \mathrm{d}x\mathrm{d}t \tag{4.6}$$

$$- c \iint_{\Omega_T} \eta\xi \mathbf{A}(x, t, u, D\boldsymbol{u}^m) \cdot (D\boldsymbol{u}_i^{m_i} - D\boldsymbol{u}^m) \, \mathrm{d}x\mathrm{d}t$$

for a cut-off function η in space and ξ in time. The second term on the right hand side vanishes as $i \to \infty$ due to structural conditions of $\mathbf{A}$ together with weak convergence of the gradients. For the first term we use (mollified) equation with a test function $\varphi = \eta\xi(\boldsymbol{u}_i^{m_i} - [\![\boldsymbol{u}^m]\!]_h)$. In this way, we are able to show that also the first term on the right hand side

of (4.6) will vanish when $i \to \infty$, which implies the desired convergence for the gradients in $L^2_{\text{loc}}(\Omega_T, \mathbb{R}^{Nn})$. Furthermore, the limit function u is indeed a solution to the limit problem with exponent m. Since $u_i \to u$ in $L^1_{\text{loc}}(\Omega_T, \mathbb{R}^N)$, and $(u_i, D\boldsymbol{u}_i^{m_i}) \to (u, D\boldsymbol{u}^m)$ pointwise, one can deduce

$$\lim_{i \to \infty} \left(\iint_{\Omega_T} [-u_i \partial_t \varphi + \mathbf{A}(x, t, u_i, D\boldsymbol{u}_i^{m_i})] \, \mathrm{d}x\mathrm{d}t \right)$$

$$= \iint_{\Omega_T} [-u \partial_t \varphi + \mathbf{A}(x, t, u, D\boldsymbol{u}^m)] \, \mathrm{d}x\mathrm{d}t,$$

along a subsequence, since $\mathbf{A}(x, t, u_i, D\boldsymbol{u}_i^{m_i})$ is bounded in $L^2_{\text{loc}}(\Omega_T, \mathbb{R}^{Nn})$ and continuous in the last two variables.

Observe that a passage to a subsequence occurs several times in our arguments. However, any subsequence of the original sequence $(\boldsymbol{u}_i^{m_i})$ has a converging subsequence which converges to the same limit $\boldsymbol{u}^m$ in $L^2_{\text{loc}}(0, T; W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^N))$ by the proof. Identification of the limit is essentially due to (4.3). Indeed, if the original sequence would not converge to this limit, then given a compact subset $K \subset \Omega$ and $0 < t_1 < t_2 < T$, one could extract a subsequence for which there exists $\varepsilon > 0$ such that $\|\boldsymbol{u}_i^{m_i} - \boldsymbol{u}^m\|_{L^2(t_1, t_2; W^{1,2}(K, \mathbb{R}^N))} \geq \varepsilon$ for all i. For this particular subsequence, the proof above still gives a convergent subsequence in the same norm. This is a contradiction, which implies that the original sequence must converge.

4.3   From local to global result

The proof of the global result applies the local result and uses similar arguments. However, in the global case we do not assume the conditions (4.3) and (4.4). In contrast, we need some assumptions for the boundary function g, which are specified in (2.10) in Publication III. For example, the assumption $\boldsymbol{g}^{\widetilde{m}} \in C^1(\overline{\Omega}_T)$ for some $\widetilde{m} < m$ will be sufficient (but it does not need to be this strong). This will imply convergence of $\boldsymbol{g}^{m_i}$, as well as its temporal and spatial derivatives to corresponding of $\boldsymbol{g}^m$ in appropriate spaces.

Again, the Caccioppoli inequality (Lemma 6.2 in Publication III) together with the assumptions on g imply that $(\boldsymbol{u}_i^{m_i})$ and $(D\boldsymbol{u}_i^{m_i})$ are bounded sequences in L^2. This further implies that along a subsequence, $\boldsymbol{u}_i^{m_i}$ converges to some $\boldsymbol{u}^m$ weakly in $L^2(0, T; W^{1,2}(\Omega, \mathbb{R}^N))$. By applying the local result we can immediately conclude that this convergence holds locally in the strong sense for a subsequence. This implies pointwise convergence almost everywhere (by passing to a subsequence again), which can be exploited to obtain a strong convergence in $L^q(0, T; W^{1,q}(\Omega, \mathbb{R}^N))$ for any $q < 2$ by similar arguments as in the local case. By application of Gagliardo-Nirenberg inequality, this convergence can be improved for the functions $\boldsymbol{u}_i^{m_i}$ to hold in $L^2(\Omega_T, \mathbb{R}^N)$. With the results at this stage, we are already

able to show that the limit function u (along a specific subsequence) attains the boundary values g in the sense of (1.8).

It is left to show the strong convergence of the gradients in $L^2(\Omega_T, \mathbb{R}^{Nn})$, which is obtained in a similar fashion as in the local case. In this case the proof becomes a bit more involved, since the test function in the weak form of the equation should have zero boundary values on the lateral boundary in the Sobolev sense. We are able to overcome this problem splitting the first term on the right hand side of (4.6) into two terms by adding and subtracting integral of $\xi \mathbf{A}\left(x, t, u_i, D\boldsymbol{u}_i^{m_i}\right) \cdot D[\![\boldsymbol{u}^m]\!]_h$. Then by using $\varphi = \xi(\boldsymbol{u}_i^{m_i} - \boldsymbol{g}^{m_i} + [\![\boldsymbol{g}^m]\!]_h - [\![\boldsymbol{u}^m]\!]_h)$ as a test function in the weak formulation of the equation with a cut-off function ξ in time, we are able to show the convergence of the gradients in $L^2(\Omega_T, \mathbb{R}^{Nn})$.

As in the local case, the convergences hold up to now for a subsequence and some limit function u solving the equation in the weak sense with exponent m. If we restrict ourselves to the model system $\mathbf{A}(x, t, u, D\boldsymbol{u}^m) = D\boldsymbol{u}^m$, we can show that the solutions are unique as in the monograph [63] by Vázquez, which implies convergence of the whole sequence with similar argument as in the local case.

5. Hölder regularity for the obstacle problem

The obstacle problem can be interpreted as a constraint problem, when there are other constraints affecting the behavior of the solution than only the data on the boundary of the domain. In Publication IV we study local Hölder continuity for solutions to obstacle problems to porous medium type equations, for which we introduced a prototype in (1.5). In this case, we use an alternative starting point for the equation, namely

$$\partial_t \boldsymbol{u}^q - \Delta u = 0,$$

for $q \in (0, \infty)$. Here the nonlinearity is shifted from the divergence part to the parabolic part. We define the solution to the obstacle problem via a variational inequality. This can be written as follows: for a given obstacle ψ, a solution u to the obstacle problem satisfies $u \geq \psi$ and

$$\langle\langle \partial_t \boldsymbol{u}^q, \varphi(v - u) \rangle\rangle + \iint_{\Omega_T} \mathbf{A}(x, t, u, \nabla u) \cdot \nabla\left(\varphi(v - u)\right) \, \mathrm{d}x\mathrm{d}t \geq 0,$$

for any nonnegative cut-off function φ and comparison map v that also stays above the given obstacle, i.e. $v \geq \psi$. Here the vector field $\mathbf{A}$ satisfies structure conditions (1.2) and (1.3) with $p = 2$. The term $\langle\langle \cdot, \cdot \rangle\rangle$ is defined via integration by parts in the time direction. In order to prove the Hölder continuity of the solution u, we assume that the obstacle ψ is Hölder continuous as well. This is a natural assumption to make in the sense that the solution may touch the obstacle in some parts of the domain.

The obstacle problem has been studied in different contexts. In the case of variational solutions, problems related to parabolic p-Laplace type equations has been studied for example in [59], [10] and [46]. One can also define the solution to the obstacle problem as the smallest supersolution staying above the given obstacle. In the case of parabolic p-Laplace type equations this approach has been considered in [42] and for the porous medium equation in [4] and [43].

In the obstacle-free case, the proof of Hölder continuity for weak solutions to degenerate porous medium equations goes back to the paper by DiBenedetto and Friedman [25]. The detailed argument in the singular

case can be extracted from [26]. Hölder continuity was proven for the solutions of obstacle problem to quasilinear equations in [62] and [21]. In the case of porous medium type equations, Hölder continuity was established with nonnegative obstacles in [14] in the degenerate case and in [20] in the singular case. Novelty in the Publication IV is that we obtain the result with an obstacle that may take any real values, and prove the Hölder regularity in a unified way for both singular and degenerate cases. Publication IV focuses purely on this regularity result. A question of existence for variational solutions to the obstacle problem to porous medium type equations has been addressed e.g. in [3], [13], [57] and [58].

Our proof relies on a De Giorgi type iteration argument. In the heart of this approach are energy estimates for truncations of solutions. Idea is to show Hölder continuity near each point in the domain by constructing a sequence of cylinders shrinking to a common vertex. When passing from one cylinder to the subsequent one in the sequence, we show that the oscillation of the solution is reduced by a fixed amount. Essentially, information from the level of measure is transferred to pointwise information by using applications of energy and logarithmic estimates with suitable truncation levels.

The main difficulty when treating the obstacle problem is already present in the energy estimates. The estimate for truncations from above take the same form as in the obstacle-free case, but for truncations from below a restriction appears. Namely, the truncation level in the latter case must be above the obstacle ψ. In the De Giorgi type iteration argument, this is taken into account in the upper bound for the oscillation of the solution u. More precisely, the upper bound for the oscillation of u must be large enough compared to the oscillation of the obstacle ψ. Moreover, when the solution is negative and substantially below its level of oscillation, a technical argument is needed to show that the De Giorgi iteration can be carried out in this case as well.

For a given reference point $z_o = (x_o, t_o) \in \Omega_T$, in the degenerate case in [14] the authors used cylinders of the form

$$Q_{\varrho,\theta\varrho^2}(z_o) = B_\varrho(x_o) \times (t_o - \theta\varrho^2, t_o) \Subset \Omega_T, \qquad (5.1)$$

in the proof of Hölder continuity. These cylinders are intrinsic, and the scaling parameter θ is comparable to u^{1-m} for nonnegative u. In contrast, in the singular case in [20], the authors exploit a slightly different form of cylinders, namely

$$Q_{\theta^{1/2}\varrho,\varrho^2}(z_o) = B_{\theta^{1/2}\varrho}(x_o) \times (t_o - \varrho^2, t_o) \Subset \Omega_T, \qquad (5.2)$$

in which the scaling parameter θ is comparable to u^{m-1}. In the proof for both of these regimes, the cases when u is close to zero (degenerate/singular), and when it is away from zero (non-degenerate/non-singular) were treated separately.

In the Publication IV, we treat signed solutions in cylinders of the form (5.1), in which θ is comparable to $|u|^{q-1}$. In particular, we are able to use this same form of intrinsic cylinders in the whole parameter range $0 < q < \infty$ without switching to the form (5.2) in the singular case. We can avoid this by exploiting the scaling invariance property of the PME. More precisely, it can be verified that for given obstacle ψ and corresponding variational solution u, for any $M > 0$ the function

$$\tilde{u}(x,t) := \frac{1}{M} u(x, M^{q-1} t)$$

is a solution to the obstacle problem with obstacle $\tilde{\psi}(x,t) = \frac{1}{M}\psi(x, M^{q-1}t)$ and vector field $\widetilde{\mathbf{A}}(x,t,v,\xi) = \frac{1}{M}\mathbf{A}(x, M^{q-1}t, Mv, M\xi)$ in the domain $\Omega_{\widetilde{T}}$ with $\widetilde{T} = M^{1-q}T$. Furthermore, the property of Hölder continuity with this transformation is preserved with the same Hölder exponent.

In the following, $\boldsymbol{\mu}^+$ and $\boldsymbol{\mu}^-$ will represent pointwise bounds from above and below, respectively, for the solution u in suitable parabolic cylinders. Furthermore, $\boldsymbol{\omega} = \boldsymbol{\mu}^+ - \boldsymbol{\mu}^-$ will then naturally be an upper bound for the oscillation of u. We will use slightly different factors for $\theta \approx |u|^{q-1}$ in different cases that are described in Section 5.3.

5.1 The obstacle

The obstacle ψ plays obviously a role in our argument. To show Hölder continuity for the solution u, we require this condition for the obstacle as well. More precisely, we will assume that

$$\psi \in C^{0;\beta,\frac{\beta}{2}}(\Omega_T) \quad \text{for some } \beta \in (0,1).$$

The obstacle will also be present already in the energy estimates. In cylinders $Q_{\varrho,s}(z_o) = B_\varrho(x_o) \times (t_o - s, t_o)$, the energy estimates for the truncations of a solution u can be written as

$$\max\left\{ \operatorname*{ess\,sup}_{t_o-s<t<t_o} \int_{B_\varrho(x_o)\times\{t\}} \varphi^2 \mathfrak{g}_\pm(u,k)\,\mathrm{d}x, \iint_{Q_{\varrho,s}(z_o)} \varphi^2 |\nabla(u-k)_\pm|^2\,\mathrm{d}x\mathrm{d}t \right\}$$

$$\leq c \iint_{Q_{\varrho,s}(z_o)} [(u-k)_\pm^2 |\nabla\varphi|^2 + \mathfrak{g}_\pm(u,k)|\partial_t\varphi^2|]\,\mathrm{d}x\mathrm{d}t \qquad (5.3)$$

$$+ \int_{B_\varrho(x_o)\times\{t_o-s\}} \varphi^2 \mathfrak{g}_\pm(u,k)\,\mathrm{d}x$$

for level k and regular enough cut-off function $\varphi \geq 0$. Constant c in the estimate depends on C_o and C_1 (the quantity $\mathfrak{g}_\pm$ depends on q). Here we may interpret $\mathfrak{g}_\pm(u,k) \approx (|u| + |k|)^{q-1}(u-k)_\pm^2$, up to a constant depending on q. The upper signs $(+)$ in the inequality correspond estimate for the truncations from below, and lower signs $(-)$ for the truncations from above.

There is a notable difference compared to the obstacle-free case. While $k \in \mathbb{R}$ can be chosen arbitrarily in that case, in presence of an obstacle we need an additional requirement for the truncations from below $(+)$. More precisely, we require that in that case

$$k \geq \sup_{Q_{\varrho,s}(z_o)} \psi,$$

so that the level of truncation must be above the obstacle in the whole cylinder. When applying the De Giorgi type iteration argument, this will result in a condition of type

$$\sup_{Q_{\varrho,\theta\varrho^2}(z_o)} \psi \leq \tfrac{1}{2}\left(\boldsymbol{\mu}^+ + \boldsymbol{\mu}^-\right). \tag{5.4}$$

When proving the reduction of oscillation in shrinking cylinders, this condition is ensured by choosing the the upper bound $\boldsymbol{\omega}$ for the oscillation of u to be large enough compared to the oscillation of the obstacle ψ.

5.2 The alternatives

Two different types of alternatives are used in our proof; either

$$\begin{cases} \left|Q_{\varrho,\theta\varrho^2}(z_o) \cap \left\{u \leq \boldsymbol{\mu}^- + \tfrac{1}{2}\boldsymbol{\omega}\right\}\right| \leq \nu_o \left|Q_{\varrho,\theta\varrho^2}(z_o)\right|, \\ \left|Q_{\varrho,\theta\varrho^2}(z_o) \cap \left\{u \leq \boldsymbol{\mu}^- + \tfrac{1}{2}\boldsymbol{\omega}\right\}\right| > \nu_o \left|Q_{\varrho,\theta\varrho^2}(z_o)\right|, \end{cases} \tag{5.5}$$

or

$$\begin{cases} \left|Q_{\varrho,\theta\varrho^2}(z_o) \cap \left\{u \geq \boldsymbol{\mu}^+ - \tfrac{1}{2}\boldsymbol{\omega}\right\}\right| \leq \nu_o \left|Q_{\varrho,\theta\varrho^2}(z_o)\right|, \\ \left|Q_{\varrho,\theta\varrho^2}(z_o) \cap \left\{u \geq \boldsymbol{\mu}^+ - \tfrac{1}{2}\boldsymbol{\omega}\right\}\right| > \nu_o \left|Q_{\varrho,\theta\varrho^2}(z_o)\right|, \end{cases} \tag{5.6}$$

in which $\nu_o \in (0,1)$. Here $\boldsymbol{\mu}^+$ can be viewed as an upper bound and $\boldsymbol{\mu}^-$ as a lower bound for u in the cylinder $Q_{\varrho,\theta\varrho^2}(z_o)$, so that $\boldsymbol{\omega}$ represents an upper bound for the oscillation of u in the same cylinder. The choice whether considering (5.5) or (5.6) is connected to different cases described in Section 5.3. The upper inequalities, $(5.5)_1$ and $(5.6)_1$ are called first alternatives, and the lower ones, $(5.5)_2$ and $(5.6)_2$, second alternatives. Next we will present a heuristic idea on the reduction of oscillation of u when passing from the intrinsic cylinder $Q_{\varrho,\theta\varrho^2}(z_o)$ to a smaller one by using alternatives in (5.5). The scaling parameter in intrinsic cylinders here could be considered to be either $\theta = \boldsymbol{\omega}^{q-1}$ (case (5.9)) or $\theta \approx (\boldsymbol{\mu}^+)^{q-1}$ (case (5.10)). We will omit the vertex z_o from the notation of the cylinders. By using a De Giorgi type lemma relying on the energy estimate (5.3) with $(-)$, the alternative $(5.5)_1$ will essentially result in a reduction of oscillation in a subcylinder of $Q_{\varrho,\theta\varrho^2}$ as

$$\operatorname*{ess\,inf}_{Q_{\frac{\varrho}{2},\theta(\frac{\varrho}{2})^2}} u \geq \boldsymbol{\mu}^- + \tfrac{1}{4}\boldsymbol{\omega}. \tag{5.7}$$

The parameter ν_o in (5.5) will be determined by this De Giorgi type lemma, and it will be a small number depending only on the data C_o, C_1, n and q. After this, one is left to deal with the second alternative $(5.5)_2$. By using the definition $\boldsymbol{\omega} = \boldsymbol{\mu}^+ - \boldsymbol{\mu}^-$, one can write this condition equivalently in the form

$$\left| Q_{\varrho,\theta\varrho^2} \cap \left\{ u > \boldsymbol{\mu}^+ - \tfrac{1}{2}\boldsymbol{\omega} \right\} \right| < (1 - \nu_o) \left| Q_{\varrho,\theta\varrho^2} \right|.$$

In order to conclude pointwise information for u and reduce the oscillation also in this case, aim is to use an application of De Giorgi type lemma again. A problem that appears now is that the parameter $1 - \nu_o$ is close to 1, while De Giorgi type lemmas are applicable only when this parameter is small. By applying a logarithmic estimate, De Giorgi's isoperimetric inequality and the energy estimate (5.3) with $(+)$, we are able to show that

$$\left| Q_{\varrho,\frac{1}{2}\nu_o\theta\varrho^2} \cap \left\{ u > \boldsymbol{\mu}^+ - \eta\boldsymbol{\omega} \right\} \right| < \nu_1 \left| Q_{\varrho,\frac{1}{2}\nu_o\theta\varrho^2} \right|,$$

holds true for any small parameter $\nu_1 > 0$ and some small constant $\eta = \eta(n, q, C_o, C_1, \nu_1)$. Observe that here the restriction (5.4) needs to be taken into account. By choosing ν_1 sufficiently small, an application of De Giorgi type lemma can again be exploited. In this case ν_1 will depend only on the data n, q, C_o and C_1, and so will η. In this way we obtain a reduction of oscillation in a subcylinder of $Q_{\varrho,\theta\varrho^2}$ again as

$$\operatorname*{ess\,sup}_{Q_{\frac{\varrho}{2},\frac{1}{2}\nu_o\theta\left(\frac{\varrho}{2}\right)^2}} u \leq \boldsymbol{\mu}^+ - \tfrac{1}{2}\eta\boldsymbol{\omega}. \tag{5.8}$$

The cylinder $Q_{\frac{\varrho}{2},\frac{1}{2}\nu_o\theta\left(\frac{\varrho}{2}\right)^2}$ is a subcylinder of $Q_{\frac{\varrho}{2},\theta\left(\frac{\varrho}{2}\right)^2}$ in (5.7) and the coefficient η will be smaller than $\frac{1}{2}$ resulting in a coefficient smaller than $\frac{1}{4}$ in front of $\boldsymbol{\omega}$ in (5.8). Now (5.7) and (5.8) together imply that

$$\operatorname*{ess\,osc}_{Q_{\frac{\varrho}{2},\frac{1}{2}\nu_o\theta\left(\frac{\varrho}{2}\right)^2}} u \leq \delta\boldsymbol{\omega},$$

in which $\delta = 1 - \frac{1}{2}\eta$ which only depends on the data n, q, C_o and C_1. By passing to this smaller cylinder $Q_{\frac{\varrho}{2},\frac{1}{2}\nu_o\theta\left(\frac{\varrho}{2}\right)^2}$, the upper bound for the oscillation of u is reduced by a fixed amount δ. Similar ideas can also be used when considering the alternatives in (5.6).

5.3   Cases near zero and away from zero

In the construction of cylinders, we will distinguish between three different cases. We refer to near zero case, if

$$\boldsymbol{\mu}^+ \geq \tfrac{1}{4}\boldsymbol{\omega} \quad \text{and} \quad \boldsymbol{\mu}^- \leq \tfrac{1}{4}\boldsymbol{\omega}. \tag{5.9}$$

If this fails, we refer to away from zero case, which splits into two subcases. We call the subcase above (and away from) zero if

$$\boldsymbol{\mu}^- \geq \tfrac{1}{4}\boldsymbol{\omega}, \tag{5.10}$$

and below (and away from) zero if

$$\mu^+ \leq -\tfrac{1}{4}\omega. \tag{5.11}$$

In each of the cases, we are able to use the machinery described in section 5.2 and reduce the oscillation of u when passing to a smaller cylinder. Significant feature in the construction of shrinking cylinders is that it can be done in such a way that there is a pattern how different cases (5.9), (5.10) and (5.11) can occur when passing from one cylinder to the next one. Assume that we have defined first cylinder Q_o in the sequence and consider Q_i for some $i \in \mathbb{N}$. If (5.9) holds true, then the oscillation can be reduced by the scheme described Section 5.2 when passing to the cylinder Q_{i+1}. In this cylinder, any of the three cases can again occur. However, if we reach some index j for which either (5.10) or (5.11) holds, the oscillation can be again reduced when passing to Q_{j+1} but in the cylinder Q_{j+1} the same case holds as in Q_j. This means that when passing from a cylinder in the near zero case to the subsequent one, any of the three cases can occur, but once above or below zero case holds in some cylinder in the sequence, in every subsequent cylinder the same case holds as well. This pattern is exploited in the final argument to prove Hölder continuity of u.

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