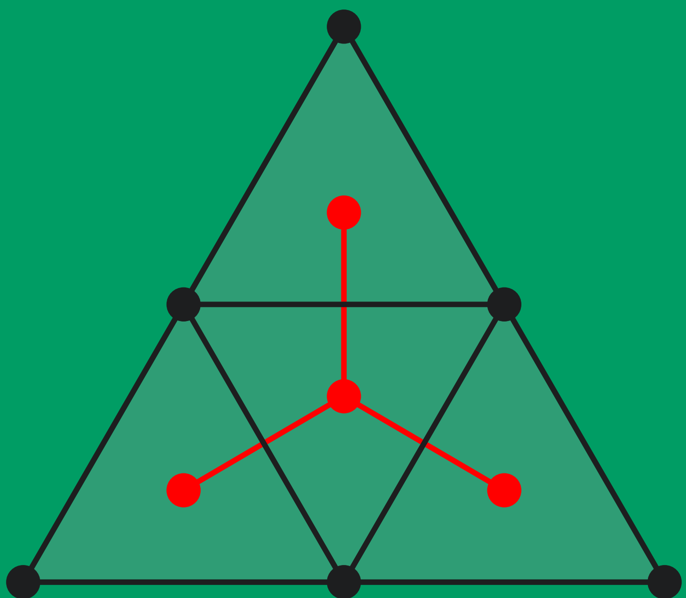
Mathematics and Systems Analysis

Homology and Combinatorics of Monomial Ideals

Milo Orlich



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Homology and Combinatorics of Monomial Ideals

Milo Orlich

A doctoral thesis completed for the degree of Doctor of Science (Technology) to be defended, with the permission of the Aalto University School of Science, at a public examination held at the lecture hall M1 of the school on 25 February 2022 at 1 pm.

Aalto University
School of Science
Mathematics and Systems Analysis
Engström group

Supervising professor
Associate Professor Alexander Engström, Aalto University, Finland

Preliminary examiners
Associate Professor Venorica Crispin Quiñonez, Uppsala University, Sweden
Associate Professor Russ Woodroofe, University of Primorska, Slovenia

Opponent
Doctor Emil Sköldberg, National University of Ireland, Galway, Ireland

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Abstract

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Abstract

This thesis is in combinatorial commutative algebra. It contains four papers, the first three of which concern homological properties and invariants of monomial ideals.

In Publication I we examine a construction originally defined in complexity theory to reduce the isomorphism problem for arbitrary graphs to so-called Booth-Lueker graphs. The map associating to a graph G its Booth-Lueker graph $BL(G)$ can be interpreted from an algebraic point of view as a construction that associates to a squarefree quadratic monomial ideal a squarefree quadratic monomial ideal with a 2-linear resolution. We study numerical invariants coming from the minimal resolutions of the edge ideals of Booth-Lueker graphs, in particular their Betti numbers and Boij-Söderberg coefficients. We provide very explicit formulas for these invariants.

Publication II concerns a generalization of the construction in Publication I: starting from an arbitrary monomial ideal I we define its linearization $\mathrm{Lin}(I)$, which is an equigenerated monomial ideal with linear quotients, and hence in particular with a linear resolution. We moreover introduce another construction, called equification, that to an arbitrary monomial ideal associates an equigenerated monomial ideal. We study several properties of both constructions, with particular attention to their homological invariants.

In Publication III we address the central open problem in the theory of edge ideals of describing their regularity. We prove new results in this direction by employing the methods of critical graphs. We introduce the concept of parabolic Betti number and provide structural descriptions for almost all graphs whose edge ideal has some parabolic Betti numbers equal to zero. For a parabolic Betti number in row r of the Betti table, we show that, for almost all graphs whose edge ideal has that Betti number equal to zero, the regularity of the edge ideal is r-1.

Publication IV deals with separations (i.e., a generalization of the classical concept of polarization) of the Stanley-Reisner ideals of stacked simplicial complexes. We study combinatorial and algebraic properties of the Stanley-Reisner ideals of triangulated balls, and in particular those of triangulated polygons

Keywords monomial ideals, free resolutions, graphs, simplicial complexes

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Preface

This dissertation is the result of the research conducted as a doctoral student at Aalto University from September 2017 to August 2021, under the supervision of Alexander Engström.

Without Alex the dissertation could not have been written—in fact not even the title in its current form. He is the person who proposed the research project to begin with and the core ideas behind three of the papers contained in this thesis. He has always been supportive, even when I wrongly claimed that I had proved some long-standing conjectures, or when I came up with some preposterous ones of my own. And he has always trusted me, even when I did not feel up to my tasks. He is smart, knowledgeable and perceptive. I am not able to do justice here to all the insightful advice I have received from Alex, but I warmly thank him for that.

Right before the pandemic started affecting international travels in 2020, I had the chance to work with Gunnar Fløystad in Bergen for a couple of months, and one of the papers constituting this thesis is the upshot of that visit. I am very grateful to Gunnar for giving me the chance to collaborate on that project, for inviting me to Bergen and for being very friendly and hospitable. Related to this, but also more in general, I greatly appreciate the financial support received from Aalto for many of my trips to conferences, workshops and summer schools. I was also supported financially by the Finnish Academy of Science and Letters, in 2020 and 2021, with the Vilho, Yrjö and Kalle Väisälä Fund.

The third mathematical guiding figure to whom I owe the opportunity of writing this dissertation is Aldo Conca, my supervisor for my Bachelor and Master's degrees, who still now often helps and supports me. In particular, Aldo and Alex were the people who motivated me to write the solo paper in the thesis, and who raised most of the questions studied in it.

I thank the other members of the Engström group: Laura, for all the fun time spent together and for all the help she gave to me, essentially being like a big sister since the very beginning; Florian, who always has plenty of clever advice for me in math, language and general life matters; Tuomas, whom I have only known for few months but who has already been very friendly to me. Alex and Florian deserve special thanks for helping me proof-read the thesis and giving lots of useful suggestions about it.

I thank the pre-examiners Veronica Crispin Quiñonez and Russ Woodroofe for their helpful comments on the dissertation, and I thank Emil Sköldberg for agreeing to act as opponent in the defense.

Many more people in the Aalto math department, outside of the Engström group, gifted me with lots of delightful time over the past four years: in particular I am grateful to Valentina, for all the supportive conversations, for sharing her insights and for studying Finnish together, and to Vesa, for his modest, dry and rich wisdom, for essentially acting as a Finnish teacher to me and for spending many afternoons doing geometry and category theory together, partly with Laura, too. Out of all the time I passed in the department during these years, those afternoons are probably my favorite experience. I also cherish fondly all the time that went into teaching activities in these past years. I have greatly enjoyed being an assistant for each course I got, thanks to all the excellent teachers I met and all the committed students who made the courses so fun. I believe these and many more people have contributed to some extent in shaping my dissertation.

I am very lucky regarding the friends that I met in Finland also outside of the math department, especially Emanuele, Fausto, Rodolfo and Teemu. I thank them for countless conversations, walks, meals and for watching so many movies and series together over these years. With these friends around me I never felt isolated, even during the pandemic.

I am blessed with luck and thankful also for the love from my family. I might not have developed interest for math at all, without following in my parents Daniela and Franco's footsteps, and possibly also without my brother Alessandro: I believe I owe a great deal of my liking for puzzles to a childhood mostly spent playing with him.

Lastly, I am deeply grateful to my girlfriend Yinggy for all the love and support she has given to me, in so many ways, and for all the time that she has dedicated to me.

Helsinki, December 31, 2021,

Milo Orlich

Contents

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I Alexander Engström, Laura Jakobsson and Milo Orlich. Explicit Boij–Söderberg theory of ideals from a graph isomorphism reduction. *J. Pure Appl. Algebra*, 224(11), 17, 2020.

II Milo Orlich. Linearization of monomial ideals. *Submitted to a journal*, 36pp., submission date: July 2021.

III Alexander Engström and Milo Orlich. The regularity of almost all edge ideals. *Submitted to a journal*, 24pp., submission date: August 2021.

IV Gunnar Fløystad and Milo Orlich. Triangulations of polygons and stacked simplicial complexes: separating their Stanley–Reisner ideals. *Submitted to a journal*, 27pp., submission date: August 2021.

Author’s Contribution

Publication I: “Explicit Boij–Söderberg theory of ideals from a graph isomorphism reduction”

The last two authors contributed equally to the article, largely wrote the article, and established the results. The first author proposed the problem and computed examples.

Publication II: “Linearization of monomial ideals”

The author did everything.

Publication III: “The regularity of almost all edge ideals”

All co-authors contributed equally to all parts of the paper.

Publication IV: “Triangulations of polygons and stacked simplicial complexes: separating their Stanley–Reisner ideals”

All co-authors contributed equally to all parts of the paper.

1. Introduction

This thesis is in combinatorial commutative algebra. It contains four publications, preceeded by a four-chapter overview. In Chapters 2 and 3 we assemble the necessary background, respectively graph-theoretic and simplicial in the former chapter and algebraic in the latter. In Chapter 4 we summarize the publications and showcase the main results.

In this introductory chapter we provide context for the publications: in Section 1.1 we condense some of the notions discussed in Chapters 2 and 3, and in Section 1.2 we outline briefly the content of the publications.

1.1 Interplay between combinatorics and commutative algebra

The leitmotif in combinatorial commutative algebra is that there are bijections between sets of combinatorial objects (simplicial complexes, graphs, trees, ...) and sets of algebraic objects (squarefree monomial ideals, quadratic squarefree monomial ideals, polarizations, ...), and often the combinatorial properties of a combinatorial object are equivalent to algebraic properties of the corresponding algebraic object. The algebraic objects recurring in all four publications are squarefree monomial ideals of some polynomial ring $\mathbb{K}[x_1,\dots,x_n]$ over a field $\mathbb{K}$. The main bridge between combinatorics and commutative algebra exploited in this thesis is the well-known bijection

$$\left\{ \begin{array}{c} \text{simplicial complexes} \\ \text{on } \{1,\dots,n\} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{squarefree monomial} \\ \text{ideals in } \mathbb{K}[x_1,\dots,x_n] \end{array} \right\},$$

called the *Stanley–Reisner correspondence*, associating to a simplicial complex Δ its so-called *Stanley–Reisner ideal*

$$I_\Delta := (x_\sigma \mid \sigma \notin \Delta),$$

where $x_\sigma := \prod_{i\in\sigma} x_i$. Among squarefree monomial ideals, those generated by quadratic monomials play a central role in Publications I, III and IV. These arise as *edge ideals* of graphs or alternatively as Stanely–Reisner ideals of *flag* simplicial complexes; in the first case one may write the ideal as

$$I_G = (x_i x_j \mid \{i,j\} \text{ is an edge of } G).$$

We adopt the first point of view in Publications I and III, and the second in Publication IV. The theory concerning Stanley–Reisner ideals has been around since the 1970's, and the systematic study of edge ideals was initiated by Villarreal in the early 1990's.

Free resolutions and Betti numbers, recalled below, are one of the main topics of Publications I, II and III. For a finitely generated graded $\mathbb{K}[x_1,\dots,x_n]$-module M, so in particular for any monomial ideal of $\mathbb{K}[x_1,\dots,x_n]$, a *free resolution* of M consists of a sequence of module maps $(d_i\colon F_i \to F_{i-1})_{i=1}^{+\infty}$ between free modules, plus an additional surjective module map $\varepsilon\colon F_0 \to M$, such that the image of every map in

$$\dots \longrightarrow F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} M$$

is equal to the kernel of the map to its right. Up to isomorphism, there is a unique free resolution of M satisfying certain "gradedness" and "minimality" conditions, and from such a resolution one may define numerical invariants of M called its *graded Betti numbers*, denoted by $\beta_{i,j}(M)$ or just $\beta_{i,j}$. These numbers are arranged as entries in the *Betti table* $\beta(M)$ of M:

$$\begin{array}{c|ccccc}
 & 0 & 1 & 2 & 3 & \cdots \\ \hline
0 & \beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \beta_{3,3} & \cdots \\
1 & \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \beta_{3,4} & \\
2 & \beta_{0,2} & \beta_{1,3} & \beta_{2,4} & \beta_{3,5} & \\
3 & \beta_{0,3} & \beta_{1,4} & \beta_{2,5} & \beta_{3,6} & \\
\vdots & \vdots & & & & \ddots
\end{array}$$

For any finitely generated graded module M, there is only a finite number of non-zero entries in the Betti table $\beta(M)$.

The issue with free resolutions and Betti numbers is that, although there are algorithms to compute them, a priori one does not know what to expect in general. So the main problem in this subject is to find an explicit and canonical description of the resolutions for large classes of modules, or, when those are out of reach, explicit formulas for the Betti numbers. The latter problem may also be very challenging, in which case one may at least be satisfied with bounding the non-zero region of the Betti table. A well-studied invariant is the *regularity* of M

$$\operatorname{reg}(M) := \max\{j \mid \beta_{i,i+j}(M) \neq 0 \text{ for some } i\},$$

equal to the highest index of a non-zero row in the Betti table of M.

The fruitful interplay between commutative algebra and combinatorics mentioned above is exemplified by the following celebrated theorem, which is of essential use in Publication I.

Theorem (Fröberg, [42]). *Let G be a graph. The following are equivalent:*

- *the edge ideal I_G has a 2-linear resolution (which means that the only non-zero Betti numbers of I_G are in row 2 of the Betti table $\beta(I_G)$, which one could also write in short as $\operatorname{reg}(I_G) = 2$);*

- *the complement $\overline{G}$ of G is chordal, that is, the only induced cycles in $\overline{G}$ are triangles.*

Aside from describing the resolutions or the numerical invariants of some class of modules or giving combinatorial interpretations of them, another focal problem is that of classifying the possible Betti numbers attainable by some module. This has been wide open for more than a century, since Hilbert first introduced the idea of a free resolution in the 1890's. A major breakthrough occurred, starting in 2006, when Boij and Söderberg (correctly) conjectured that any Betti table sits inside a cone whose extremal rays are generated by the so-called *pure tables*. That is, the Betti table $\beta(M)$ of a module M can be written as a weighted finite sum

$$\beta(M) = \sum_j c_j \pi_j,$$

where the π_j are such pure tables and the numbers $c_j \in \mathbb{Q}_{\geq 0}$ are called *Boij–Söderberg coefficients* of M. Just as in the case of resolutions, it is possible to algorithmically find these Boij–Söderberg coefficients, but there are only few (and partial) results providing explicit formulas for them, or combinatorial interpretations of them.

1.2 Outline of the publications

Publications I, II and III concern the resolutions or the numerical homological invariants (i.e., the Betti numbers, Boij–Söderberg coefficients and regularity) of some monomial ideals. The most closely related works are Publications I and II, the latter being a generalization of the former but more algebraic in flavor. Publications III and IV are both about squarefree monomial ideals generated in degree 2; in the former we are interested in Betti numbers and regularity, and we view these ideals as edge ideals of graphs, relating their homological properties to graph-theoretic properties of the associated graph, whereas in the latter paper we view these objects as Stanley–Reisner ideals, defined in terms of directed trees, and we focus on the associated simplicial complexes and trees.

Below follow more details about each publication:

- In Publication I we consider a construction from early complexity theory that takes any graph G and returns a graph $BL(G)$ that in particular has chordal complement, so that the edge ideal $I_{BL(G)}$ has a 2-linear resolution by Fröberg's theorem above. We prove very explicit formulas for the Boij–Söderberg coefficients of $I_{BL(G)}$ and the Betti numbers of $I_{BL(G)}$, in terms of the degrees of the vertices of G.

- In Publication II we generalize the construction in Publication I, seen from a purely algebraic point of view. This more general construction, called *linearization*, takes as input any monomial ideal and returns (injectively) a monomial ideal with a d-linear resolution. We introduce another, auxiliary construction that gives an equigenerated monomial ideal starting from an arbitrary one. We study several (in particular, homological) properties of both constructions.

- There are still no explicit formulas that express the regularity of an edge ideal I_G in terms of "easy" invariants of G such as the degrees of the vertices. Publication III is about the regularity of edge ideals. There we say that $\beta_{i,j}$ is a *parabolic Betti number* if i and j satisfy certain conditions. This defines a region of the Betti table that grows wider and wider as the row index grows. Our main result is that if $\beta_{i,j}$ is a parabolic Betti number on row r of the Betti table, then almost all graphs G with $\beta_{i,j}(I_G) = 0$ are such that $\operatorname{reg}(I_G) = r - 1$. This follows partially from a structural result we prove in Publication III: almost all the graphs G with $\beta_{i,j}(I_G) = 0$ can be covered with $r - 2$ cliques and one independent set.

- In Publication IV we consider monomial ideals $I(T) \subseteq S$ that are defined starting from directed trees T. We study the quotients of $S/I(T)$ by regular sequences consisting of variable differences (that is, we do the "opposite of polarizing" the ideals $I(T)$) and study the ideals associated to these quotients. Among the ideals defining the quotients we classify in particular the squarefree ones and their associated simplicial complexes, and among these ideals we find in particular the Stanley–Reisner ideals of triangulated polygons.

To conclude this first chapter, we remark that in this thesis the set $\mathbb{N}$ of *natural numbers* consists of the non-negative integers, including 0.

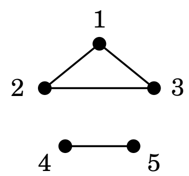
2. Graphs and Simplicial Complexes

The notions in Sections 2.1 and 2.3 are all very standard and quite commonly used in combinatorial commutative algebra. Section 2.2 contains more involved concepts.

2.1 Basic definitions in graph theory

Denote by $\binom{V}{2} := \{W \subseteq V \mid \#W = 2\}$ the set of two-element subsets of a set V.

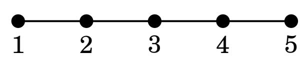
A *finite simple graph* is an ordered pair (V,E) where V is a finite set and $E \subseteq \binom{V}{2}$. The elements of V and E are called respectively the *vertices* and *edges* of G, and the sets V and E are called respectively the *vertex set* and *edge set* of G. Two vertices v and w are *adjacent* if $\{v,w\} \in E$. If $e = \{v,w\}$ is an edge of G, the vertices v and w are called the *ends* of e. One may visualize a graph pictorially: for instance the drawing



represents the finite simple graph $G = (V,E)$ with vertex set $V = \{1,2,3,4,5\}$ and edge set $E = \{\{1,2\},\{1,3\},\{2,3\},\{4,5\}\}$.

Definition 2.1.1. Let G be a finite simple graph with vertex set V. The *degree* of a vertex $v \in V$, written $\deg_G(v)$ or simply $\deg(v)$, is the number of vertices that are adjacent to v. Denoting by n the cardinality of V, the *degree vector* of G is defined as $\mathbf{d}_G := (d_0,\ldots,d_{n-1})$, where d_i is the number of vertices in G of degree i.

In the finite simple graph G drawn above, the vertices 1, 2 and 3 have degree 2, whereas the vertices 4 and 5 have degree 1. The degree vector in this case is $\mathbf{d}_G = (0,2,3,0,0)$. Notice that graphs that “look very different” might have the same degree vector. For instance



has the same degree vector $(0,2,3,0,0)$ as the other finite simple graph above.

Intuitively the "combinatorial structure" of a finite simple graph doesn't change if we rename the labels of its vertices, and this idea is made precise by the concept of graph isomorphism.

Definition 2.1.2. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two finite simple graphs. An *isomorphism from G to H* is a bijection $\varphi\colon V_G \to V_H$ such that

$$\{v, w\} \in E_G \quad \Leftrightarrow \quad \{\varphi(v), \varphi(w)\} \in E_H.$$

If there exists an isomorphism from G to H, we say that G and H are *isomorphic*. An *unlabeled graph* is the isomorphism class of a finite simple graph.

In Publications I, II and IV we refer to a finite simple graph simply by *graph*, whereas in Publication III a *graph* is usually an unlabeled graph. For this reason, in Chapters 2, 3 and 4 the distinction will be as explicit as possible, in order to avoid confusion. Sometimes we refer to a finite simple graph as a *labeled* graph, as opposed to *unlabeled*.

Definition 2.1.3. Given a finite simple graph $G = (V, E)$, the *complement of* G is the finite simple graph $\overline{G} := (V, E')$, with vertex set V and edge set $E' := \binom{V}{2} \setminus E$. That is, $\overline{G}$ has the same vertices as G and exactly the edges that G does not have.

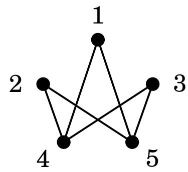
For an unlabeled graph G, one defines the complement $\overline{G}$ as the isomorphism class of the complement of any labeled graph in the class G. In fact, many of the terms defined in this section are usually introduced just for labeled graphs but are used also in the unlabeled context in an intuitive way.

Definition 2.1.4. Given a set V, the *clique* (or *complete graph*) *on* V is the finite simple graph with edge set $\binom{V}{2}$. That is, all the vertices are adjacent to each other. We denote by K^n the isomorphism class of a clique on a vertex set with n elements, and call it the *clique* (or *complete graph*) *on n vertices*. The disjoint union of r cliques is called an *r-cluster*. The complement of a (labeled or unlabeled) clique, that is, a graph where no two vertices are adjacent, is called an *independent set*.

Definition 2.1.5. A finite simple graph G is *r-partite* if the vertex set V of G can be written as a disjoint union $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_r$ so that, for all $i \in \{1, \ldots, r\}$, no two vertices in V_i are adjacent. We say that G is *complete r-partite* if additionally, for all $i \neq j$, every vertex in V_i is adjacent to every vertex in V_j. The 2-partite graphs are called *bipartite graphs*. We denote by $K_{a_1, a_2, \ldots, a_r}$ the isomorphism class of a finite simple graph whose vertex set consists of r disjoint sets $V_1, \ldots, V_r$ where $\#V_i = a_i$, and where, for all $i \neq j$, every vertex of V_i is adjacent to every vertex of V_j.

Notice that the complete r-partite graphs are exactly the complements of the r-clusters.

Example 2.1.6. The first finite simple graph G drawn on the previous page is a 2-cluster. The isomorphism class of G can be written as the disjoint union $K^3 \sqcup K^2$. The complement of G is the following bipartite graph, whose isomorphism class is $K_{3,2}$ according to the definition above:



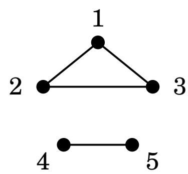
Notice that this graph is complete bipartite.

2.1.1 Subgraphs

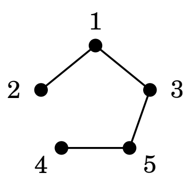
Definition 2.1.7. Let $G = (V, E)$ be a finite simple graph.

- We say that $G' = (V', E')$ is a *subgraph of* G if $V' \subseteq V$ and $E' \subseteq E \cap \binom{V'}{2}$.
- A subgraph $G' = (V', E')$ of G is an *induced subgraph of* G if $E' = E \cap \binom{V'}{2}$, that is, if E' consists of all the edges in E with both ends in V'.
- Given a subset $W \subseteq V$, we denote by $G[W]$ the finite simple graph with vertex set W and edge set $E \cap \binom{W}{2}$, that is, with edge set consisting of all the edges in E with both ends in W. We call $G[W]$ the *subgraph of* G *induced by* W.

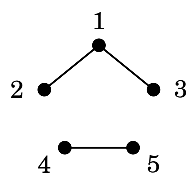
Example 2.1.8. Consider the finite simple graph G



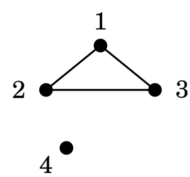
and the following:



not a subgraph of G



a subgraph of G



an induced subgraph of G

The subgraph in the middle is not induced because it has both the vertices 2 and 3 but lacks the edge $\{2, 3\}$. The subgraph on the right can be written as $G[W]$, for $W = \{1, 2, 3, 4\}$.

Definition 2.1.9. Given two unlabeled graphs G and H, we say that H is an *induced subgraph of* G if there are labeled graphs G' and H', respectively in the classes G and H, such that H' is an induced subgraph of G' in the "labeled sense".

Definition 2.1.10. Given a finite simple graph $G = (V, E)$, a *matching in* G is a subset $M \subseteq E$ such that no two distinct elements of M share an end.

Example 2.1.11. For instance, $M = \big\{\{1,2\},\{4,5\}\big\}$ is a matching in the graph G of Example 2.1.8. In fact, M is a *maximal* matching, in the sense that one cannot add any more edges of E to M and obtain a matching in G.

In Publication III, matchings will occur in the following unlabeled version, and without an “ambient” graph G:

Definition 2.1.12. We call *unlabeled matching on n edges* the unlabeled graph that is the isomorphism class of the finite simple graph $G = (V, E)$ with

$$V = \{v_1, \dots, v_n, w_1, \dots, w_n\} \quad \text{and} \quad E = \big\{\{v_i, w_i\} \mid i = 1, \dots, n\big\},$$

where the elements $v_1, \dots, v_n, w_1, \dots, w_n$ are pairwise distinct. Alternatively, we call this the *unlabeled matching on $2n$ vertices*.

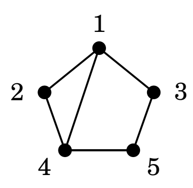
Notice that matchings are a special case of clusters, in which each clique is of cardinality 2.

2.1.2 Chordal and split graphs

Definition 2.1.13. For $n \geq 3$, a finite simple graph is called a *cycle on n vertices* (or *cycle of length n*) if it is isomorphic to (V, E), where the elements of $V = \{v_1, \dots, v_n\}$ are pairwise distinct and $E = \big\{\{v_i, v_{i+1}\} \mid i = 1, \dots, n-1\big\} \cup \big\{\{v_1, v_n\}\big\}$. We denote the isomorphism class of a cycle on n vertices by C_n. A cycle on three vertices is called a *triangle*.

Definition 2.1.14. A finite simple graph G is *chordal* if the only cycles that are induced subgraphs of G are triangles. That is, if G' is a subgraph of G and G' is a cycle of length greater than 3, then there is an edge e in G and not in G' connecting two vertices of G'; such an edge is called a *chord* of G'.

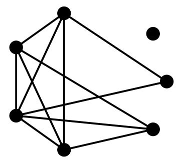
Example 2.1.15. The finite simple graph



is not chordal because it contains an induced cycle of length 4.

Definition 2.1.16. A finite simple graph $G = (V, E)$ is *split* if V can be written as a disjoint union $V = V_1 \sqcup V_2$ so that $G[V_1]$ is a clique and $G[V_2]$ is an independent set.

Example 2.1.17. The unlabeled graph



is (the isomorphism class of) a split graph. The subgraph induced by the four vertices on the left is a clique, and the three vertices on the right form an independent set.

Remark 2.1.18. Notice that all split graphs are chordal, and that the complement of a split graph is also split. This is a key observation, with Publication I in mind.

2.1.3 Trees

Definition 2.1.19. For $n \geq 2$, a finite simple graph is called a *path on n vertices* if it is isomoprhic to (V,E), where the elements of $V = \{v_1,\dots,v_n\}$ are pairwise distinct and

$$E = \Big\{\{v_1,v_2\},\{v_2,v_3\},\dots,\{v_{n-2},v_{n-1}\},\{v_{n-1},v_n\}\Big\}.$$

More precisely, the graph (V,E) above is called a *path from v_1 to v_n*. A finite simple graph G is *connected* if, for any distinct vertices a and b of G, G contains a path from a to b as subgraph.

The first finite simple graph drawn on page 5 is not connected. The second one is, and in fact the second one is a path on five vertices.

Definition 2.1.20. A finite simple graph G is called a *forest* if G does not contain any cycle as a subgraph. A forest that is also connected is called a *tree*.

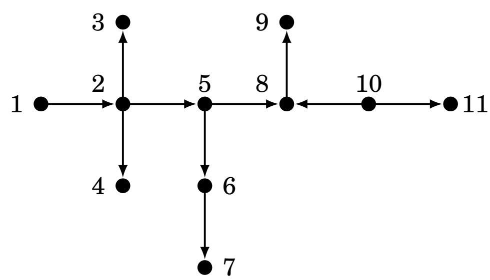
Trees are characterized by the following property (part of Theorem 1.5.1 in [20]):

Proposition 2.1.21. *For a finite simple graph G, the following are equivalent:*

- *G is a tree;*
- *for any two vertices v and w in G, there is a unique path in G linking v and w.*

Definition 2.1.22. Let T be a tree and let v and w be vertices of T. We denote by vTw the unique path in T linking v and w.

In Publication IV we consider *directed* (or *oriented*) *trees*. The *edges* in a directed graph are defined as ordered pairs. For instance, the picture



represents the directed tree $T = (V, E)$ with $V = \{1, 2, \ldots, 11\}$ and

$$E = \big\{(1,2),(2,3),(2,4),(2,5),(5,6),(5,8),(6,7),(8,9),(10,8),(10,11)\big\}.$$

From a directed graph one gets an "underlying" undirected one by regarding the edges as sets, forgetting the order.

Notice that for any edge e and any vertex v in a directed tree, by Proposition 2.1.21, one has that either e points towards v or not. In particular, for any edge e of a directed tree T, one gets a partition of the vertex set V of T into two non-empty sets $\{v \in V \mid e \text{ points towards } v\}$ and $\{v \in V \mid e \text{ does not point towards } v\}$.

2.2 Extremal graph theory: critical graphs

Extremal graph theory is a branch of graph theory that deals in particular with counting and describing the structure of the graphs that do not contain some given subgraphs. We refer to Chapter 7 of [20] for a general introduction.

Two of the first major results in extremal graph theory, proven in the 1940's, are the well-known theorem by Turán [72], bounding the number of edges of a graph that does not contain a clique of a given size, and the generalization of it known as the Erdős–Stone theorem [35]. The results of the kind we are interested in, from the point of view of Publication III, state that the graphs (or at least *almost all* graphs) that do not contain some given subgraph can be covered with a determined number of cliques and independent sets. An example of such a result, by Erdős, Kleitman and Rothschild [34], states that almost all graphs that do not contain triangles are bipartite. (See the details in the subsections below.)

The rather technical concept of "critical graph" discussed in this section was introduced relatively recently (compared to the rest of the topics discussed in this chapter) by Balogh and Butterfield [3]. (See also Butterfield's thesis [13] for additional information and references, although with a different notation.) They prove a characterization of critical graphs in terms of the graphs that do not contain them as induced subgraphs (see Theorem 2.2.17 below), and this result is a key tool for us in Publication III. Whereas the other sections in this chapter contain material that is very commonly used in commutative algebra, we believe that in Publication III we are the first to make use of critical graphs to prove results about Betti numbers of edge ideals.

One of the motivating prior results for [3] was the following (for the details involved, see the sections below):

Theorem 2.2.1 (Prömel and Steger, [64]). *Let H be an unlabeled graph with chromatic number $\chi(H) = k+1$. The following are equivalent:*

- *almost every graph G that does not contain H as a (not necessarily induced) subgraph is k-colorable;*
- *there exists an edge e such that $\chi(H - e) = k$.*

The point of view of Balogh and Butterfield [3] is that H in the theorem above is "critical", in the sense that removing an edge changes the chromatic number, and their goal is to generalize the theorem to the induced setting, for a suitable definition of "critical". Notice that Prömel and Steger [65] previously defined a different concept of "critical graph" (see Definition 2.10 of [65]). The authors of [3] write that both their own definition and that of [65] are so technical that they do not know whether one is a generalization of the other.

We collect all the concepts involved in the definition of critical graph in Sections 2.2.1 and 2.2.2. But first of all we recall the notion of "almost every graph".

Definition 2.2.2. Let $\mathscr{A}$ and $\mathscr{B}$ be two families of unlabeled graphs, with $\mathscr{B} \subseteq \mathscr{A}$. For each positive integer n, denote

$$\mathscr{A}_n := \{\text{unlabeled graphs on } n \text{ vertices belonging to } \mathscr{A}\},$$

$$\mathscr{B}_n := \{\text{unlabeled graphs on } n \text{ vertices belonging to } \mathscr{B}\},$$

and assume that $\mathscr{B}_n \neq \emptyset$ (at least for all large n). We say that *almost every graph in $\mathscr{A}$ is in $\mathscr{B}$* if

$$\lim_{n\to\infty} \frac{|\mathscr{A}_n|}{|\mathscr{B}_n|} = 1.$$

We also use expressions like "almost all graphs" or "for almost all graphs", etc., all with the same connotation as above. This concept is used in Publication III with a slightly different notation.

2.2.1 Covering pairs and (s,t)-templates

Let $G = (V, E)$ be a finite simple graph and let X be a finite set, whose elements are called *colors*. A function $f : V \to X$ is called a *coloring of* G if, for all v and w in V,

$$\{v,w\} \in E \quad \Rightarrow \quad f(v) \neq f(w),$$

that is, if any two adjacent vertices do not share the same color. If such a coloring exists for a set X such that $\#X = k$, one says that G is *k-colorable*. The *chromatic number of* G, denoted $\chi(G)$, is the smallest k such that G is k-colorable. The concepts of (s,t)-template and coloring number defined below generalize this situation.

Remark 2.2.3. Another way to phrase the definition of coloring, aiming for a generalization of this concept, is as follows: if for all $i \in X$ we denote $V_i := \{v \in V \mid f(v) = i\}$, then f is a coloring of G if and only if each induced subgraph $G[V_i]$ is an independent set.

In Publication III we consider a more general version of graph coloring, which is also well studied in graph theory but perhaps less popular in other branches of math. Instead of partitioning the vertex set into independent sets, we partition it into *cliques and independent sets*. Recall that we denote $[n] := \{1, 2, \ldots, n\}$.

Definition 2.2.4. Let $G=(V,E)$ be a finite simple graph. We say that (s,t) is a *covering pair* for G if there is a function $f\colon V\to[s+t]$ such that the following holds: if for all $i\in[s+t]$ we denote $V_i:=\{v\in V \mid f(v)=i\}$, then $G[V_i]$ is a clique for $1\le i\le s$ and $G[V_i]$ is an independent set for $s+1\le i\le s+t$. If (s,t) is a covering pair for G, we call G an *(s,t)-template*.

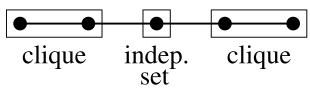
Remark 2.2.5. Observe the following:

- The $(0,k)$-templates are exactly the k-colorable graphs.
- Any of the cliques or independent sets in the definition above may be empty. So in particular if G is an (s,t)-template, then G is also an (s',t')-template for any $s'\ge s$ and $t'\ge t$.
- The $(1,1)$-templates are exactly the split graphs. (See Definition 2.1.16.)
- The complement of an (s,t)-template is a (t,s)-template.

The definitions above can be used for unlabeled graphs in an intuitive way, since the existence of colorings does not depend on the labeling. For instance, one may give the following:

Definition 2.2.6. For an unlabeled graph G, we say that G is an *(s,t)-template* if one (or equivalently all) of the labeled graphs in the isomorphism class G is an (s,t)-template according to the definition above.

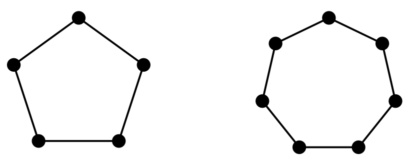
Example 2.2.7. Consider P_5, the path on five vertices. The pair $(2,0)$ is not a covering pair for P_5 because the largest cliques in P_5 are edges, and two edges are not enough to cover all of P_5. On the other hand, $(2,1)$ is a covering pair for P_5: one may pick as independent set the middle point of P_5 and as cliques the first and last edge:



Notice that this is not the only way to cover P_5 with two cliques and one independent set. For instance, the independent set could alternatively consist of the first, third and fifth vertex, and the two cliques would then consist of the two remaining vertices, one vertex each.

Definition 2.2.8. Let G be a finite simple graph. The *coloring number of* G, denoted $\chi_c(G)$, is the smallest number k such that every pair (s,t) of non-negative integers such that $s+t=k$ is a covering pair for G. If a pair (s,t) with $s+t=\chi_c(G)-1$ is not a covering pair for G, then (s,t) is called a *witnessing pair* for G.

Example 2.2.9. Consider the five-cycle C_5 and the seven-cycle C_7:



C_5 C_7

One may check that $\chi_c(C_5) = 3$ and that $(2,0)$, $(1,1)$ and $(0,2)$ all are witnessing pairs for C_5. That is, C_5 cannot be covered by two cliques, nor one clique and one independent set, nor two independent sets. One may also check that $\chi_c(C_7) = 4$ and that the witnessing pairs for C_7 are $(3,0)$ and $(2,1)$, whereas $(1,2)$ and $(0,3)$ are covering pairs for C_7. Observe that $\chi_c(C_5) = \chi(C_5)$ and $\chi_c(C_7) > \chi(C_7) = 3$.

2.2.2 Critical graphs

The presence of an induced subgraph in G up to isomorphism clearly does not depend on whether we are considering G or any other labeled graph isomorphic to G. So in the following definition by "graph" one may consider both a labeled or an unlabeled graph.

Definition 2.2.10. Given graphs G and H, we say that G is *H-free* if G does not contain an induced subgraph that is isomorphic to H (or "an induced subgraph that is a member of the isomorphism class H" in the unlabeled case). More generally, given a family of graphs $\mathscr{F}$, we say that G is *$\mathscr{F}$-free* if G is H-free for all $H \in \mathscr{F}$.

Example 2.2.11. For instance, one has the following:

- Any bipartite graph is triangle-free. The converse is not true: cycles of odd length greater than 3 are triangle-free but not bipartite. However, Erdős, Kleitman and Rothschild [34] proved that *almost all* (in the sense of Definition 2.2.2) triangle-free graphs are bipartite.
- The definition of chordal graph may be shortened by saying that a graph G is chordal if G is $\mathscr{F}$-free, where $\mathscr{F} := \{C_n \mid n \geq 4\}$ consists of all cycles of length $n \geq 4$.

Definition 2.2.12. For a family $\mathscr{F}$ of unlabeled graphs, define

$$\mathscr{P}(n,\mathscr{F}) := \{\text{unlabeled graphs } G \text{ on } n \text{ vertices} \mid \text{for all } H \in \mathscr{F}, G \text{ is } H\text{-free}\}.$$

In the case of $\mathscr{F} = \{H\}$, we simply write $\mathscr{P}(n,H)$ for $\mathscr{P}(n,\{H\})$.

We need to measure how much of an unlabeled graph H is left when we cover as much as possible of H with s cliques and t independent sets:

Definition 2.2.13. For an unlabeled graph H and non-negative integers s and t, denote by $\mathscr{F}(H,s,t)$ the set of minimal (by induced containment) unlabeled graphs F such that H can be covered by s cliques, t independent sets, and a copy of F. In other words, $\mathscr{F}(H,s,t)$ consists of the unlabeled graphs in the set

$$\left\{H - U \mid U \subseteq V_H \text{ and } H[U] \text{ is an } (s,t)\text{-template}\right\}$$

which are minimal with respect to induced containment.

In particular, one gets $\mathscr{F}(H,s,t) = \{\emptyset\}$ if H itself is an (s,t)-template. Notice moreover that in practice when trying to determine what a specific set $\mathscr{F}(H,s,t)$ is, we consider *maximal* cliques and independent sets, because if they were not maximal then we would just end up with a graph F that is not minimal with respect to induced containment, that is, a graph F that strictly contains some other F', obtained with maximal cliques and independent sets, as an induced subgraph.

Example 2.2.14. Consider the five-path $H = P_5$. The set $\mathscr{F}(P_5,1,0)$ consists of exactly two graphs F_1 and F_2: F_1 is the path on three vertices, obtained when we choose to cover one of the external edges of P_5, and F_2 is the the disjoint union of an edge and a vertex, obtained when we choose to cover one of the internal edges of P_5. In the following picture the covered clique is in gray and dashed, whereas the remaining "uncovered" graph (in black) is F_1 and F_2, respectively:

Indeed, neither F_1 nor F_2 is an induced subgraph of the other.

Definition 2.2.15. An unlabeled graph H is *critical* if, for all non-negative integers s and t with $s+t = \chi_c(H)-2$ and for all large enough n, there are at most two graphs in $\mathscr{P}(n,\mathscr{F}(H,s,t))$.

Example 2.2.16. Consider the five-cycle C_5 and the seven-cycle C_7. From Example 2.2.9 we know that $\chi_c(C_5) = 3$ and $\chi_c(C_7) = 4$. In order to determine whether C_5 is critical, one needs to consider the pairs (s,t) such that $s+t = 3-2$, namely the pairs $(1,0)$ and $(0,1)$. We have

$\mathscr{F}(C_5,1,0) = \{$$\}$, $\mathscr{F}(C_5,0,1) = \{$$\}$.

Then C_5 is not critical, because for large n the set $\mathscr{P}(n,\mathscr{F}(C_5,1,0))$ consists of more than two elements: in particular it always contains at least the graph on n vertices with no edges, the graph on n vertices with one edge, and the complete graph K^n. On the other hand, C_7 is critical: as $\chi_c(C_7) = 4$, we need to inspect values of s and t such that $s+t = 4-2$, which give

$\mathscr{F}(C_7,2,0) = \{$, $\}$, $\mathscr{F}(C_7,1,1) = \{$$\}$, $\mathscr{F}(C_7,0,2) = \{$$\}$.

Thus, for large n we get the sets

$$\mathscr{P}(n,\mathscr{F}(C_7,2,0)) = \{K^n,\overline{K^n}\}, \quad \mathscr{P}(n,\mathscr{F}(C_7,1,1)) = \{K^n\}, \quad \mathscr{P}(n,\mathscr{F}(C_7,0,2)) = \emptyset,$$

which all have cardinality at most 2, and therefore C_7 is critical.

The following is the main result of [3] and one of the key tools that we used in Publication III.

Theorem 2.2.17 (Theorem 1.9 of [3]). *Let H be an unlabeled graph with $\chi_c(H) \geq 3$. The following are equivalent:*

- *almost every H-free graph is an (s,t)-template, for some s and t such that (s,t) is a witnessing pair for H (that is, $s+t=\chi_c(H)-1$ and (s,t) is not a covering pair for H);*

- *H is critical.*

Example 2.2.18. Consider the seven-cycle $H=C_7$. By Example 2.2.9 we know that $\chi_c(C_7)=4$ and that the witnessing pairs are $(3,0)$ and $(2,1)$. By Example 2.2.16 we also know that C_7 is critical. Hence, by the theorem above, almost every C_7-free graph is a $(3,0)$-template or a $(2,1)$-template.

2.3 Simplicial complexes

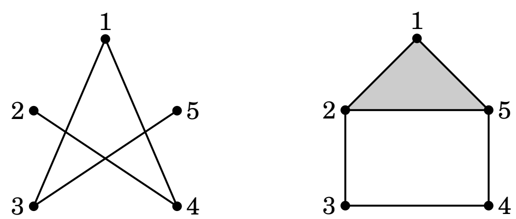
Given a set V, a *simplicial complex* Δ *on* V is a family of subsets of V such that, whenever $\sigma\in\Delta$ and $\sigma'\subseteq\sigma$, we have $\sigma'\in\Delta$. (Very often in the literature this is called an *abstract* simplicial complex.) The elements of V are called the *vertices* of Δ. We call the elements of Δ its *faces*, and the faces that are maximal with respect to inclusion are called the *facets* of Δ. For a face $\sigma\in\Delta$ we say that the *dimension* of σ is $\dim\sigma := |\sigma|-1$. The dimension of Δ is defined as the largest dimension of any of its faces. If all the facets of Δ have the same dimension, Δ is *pure*. Lastly, given some subsets $F_1,\dots,F_s$ of V, denote by $\langle F_1,\dots,F_s\rangle$ the simplicial complex on V consisting of all the subsets of F_i, for all i.

The most important simplicial complexes for us, used in Publication III, are independence complexes of graphs:

Definition 2.3.1. Let G be a finite simple graph. The simplicial complex whose faces are the independent sets of G is called the *independence complex of* G and denoted $\mathrm{Ind}(G)$. (See also Remark 2.3.15 below.)

Indeed $\mathrm{Ind}(G)$ is a simplicial complex: if I is an independent set of vertices, any subset $I'\subseteq I$ still consists of independent vertices. Another famous complex in the literature is the *clique complex* of G, having as faces the cliques of G. The independence complex of G is equal to the clique complex of the complement $\overline{G}$.

Example 2.3.2. Consider the finite simple graph G with vertex set $[5]=\{1,\dots,5\}$ drawn below, along with its independence complex:

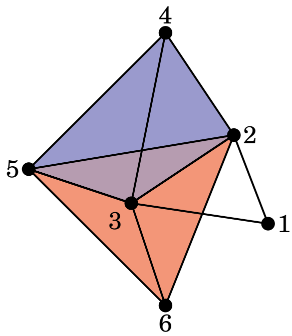


G $\qquad$ $\mathrm{Ind}(G)$

Consider now the following simplicial complex Δ on vertex set $[6]$, where we write $i_1 i_2 \dots i_s$ for the set $\{i_1, i_2, \dots, i_s\}$, in order to make the notation less heavy:

$$\Delta := \langle 12,\ 13,\ 23,\ 234,\ 235,\ 236,\ 245,\ 256,\ 345,\ 356\rangle.$$

That is, Δ consists of two hollow tetrahedra (one in blue and one in red) glued along the (full) triangle 235, plus an additional triangle (without the interior) glued to the tetrahedra along the edge 23:



Observe that Δ cannot be the independence complex of a graph, because 12, 13 and 23 are all faces of Δ but 123 is not.

Definition 2.3.3. Let Δ be a simplicial complex. We say that Δ is *flag* if all the minimal non-faces of Δ have cardinality 2.

The second simplicial complex Δ in Example 2.3.2 is not flag because $\{1,2,3\}$ is a minimal non-face of Δ.

Remark 2.3.4. Notice that $\mathrm{Ind}(G)$ is a flag simplicial complex for any G. Conversely, every flag simplicial complex is the independence complex of some graph. (More precisely: The 1-skeleton of a simplicial complex is its underlying graph. A flag complex Δ is the independence complex of the complement of the 1-skeleton of Δ. See for instance Lemma 9.1.3 of [50].)

Definition 2.3.5. Let Δ be a simplicial complex. We say that Δ is *(non-pure) shellable* if the facets of Δ can be ordered as $F_1, F_2, \dots, F_m$ so that, for all $2 \le j \le m$, the subcomplex

$$\langle F_1, \dots, F_{j-1}\rangle \cap \langle F_j\rangle$$

is pure and has dimension $\dim F_j - 1$. An order of the facets as above is called a *shelling order*.

Equivalently, $F_1, F_2, \dots, F_m$ is a shelling order of Δ if and only if, for all $j < i$, there exist $\ell \in F_i \setminus F_j$ and $k < i$ such that $F_i \setminus F_k = \{\ell\}$.

Example 2.3.6. Consider once more the simplicial complex Δ in Example 2.3.2, and again write $i_1 i_2 \dots i_s$ for the set $\{i_1, i_2, \dots, i_s\}$ if $s \ge 2$, so that for instance $\{24, 25\}$

stands for $\{\{2,4\},\{2,5\}\}$. The complex Δ is shellable, and the ordering of the facets as in Example 2.3.2 is a shelling order. Indeed, one has

$$\langle 12\rangle \cap \langle 13\rangle = \{\{1\}\}$$

$$\langle 12,\ 13\rangle \cap \langle 23\rangle = \{\{2\},\{3\}\}$$

$$\langle 12,\ 13,\ 23\rangle \cap \langle 234\rangle = \{23\}$$

$$\langle 12,\ 13,\ 23,\ 234\rangle \cap \langle 235\rangle = \{23\}$$

$$\langle 12,\ 13,\ 23,\ 234,\ 235\rangle \cap \langle 236\rangle = \{23\}$$

$$\langle 12,\ 13,\ 23,\ 234,\ 235,\ 236\rangle \cap \langle 245\rangle = \{24,25\}$$

$$\langle 12,\ 13,\ 23,\ 234,\ 235,\ 236,\ 245\rangle \cap \langle 256\rangle = \{25,26\}$$

$$\langle 12,\ 13,\ 23,\ 234,\ 235,\ 236,\ 245,\ 256\rangle \cap \langle 345\rangle = \{34,35,45\}$$

$$\langle 12,\ 13,\ 23,\ 234,\ 235,\ 236,\ 245,\ 256,\ 345\rangle \cap \langle 356\rangle = \{35,36,56\},$$

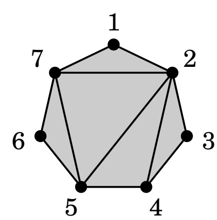
and all the complexes above are pure and of the correct dimension.

Definition 2.3.7. Let Δ be a pure simplicial complex. We say that Δ is *stacked* if the facets of Δ can be ordered as $F_1, F_1, \ldots, F_m$ so that, for all $2 \le j \le m$, the facet F_j is attached to $X_{j-1} := \langle F_1, \ldots, F_{j-1}\rangle$ along a single codimension-one face of X_{j-1}. An order of the facets as above is called a *stacking*.

Remark 2.3.8. Stacked simplicial complexes are flag complexes, and they are also shellable.

The most relevant example of stacked simplicial complexes for us (in Publication IV) are triangulations of balls (see [19]).

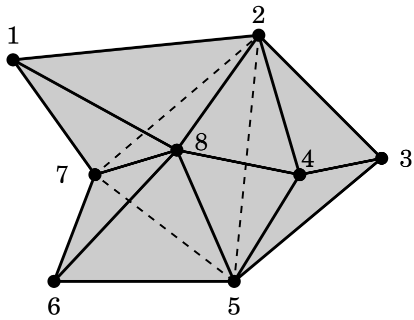
Example 2.3.9. Consider the following simplicial complex



which is a triangulation of a ball (specifically, of a heptagon). This is a stacked simplicial complex, and for instance

$$127,\ 257,\ 567,\ 245,\ 324.$$

is a stacking of its facets. A triangulation of a ball in one dimension higher is for instance the simplicial complex



and

$$1278,\ 2578,\ 5678,\ 2458,\ 2345.$$

is a stacking of its facets.

Remark 2.3.10. Stacked complexes are also known in the literature as *facet constructible complexes*. They are related to stacked polytopes. For an explanation of this, and an interesting source of references about these complexes, we refer to Section 4.5 of Goeckner's thesis [43].

We recall a couple more notions that will be useful later.

Definition 2.3.11. Let Δ be a simplicial complex with vertex set V. For any $W \subseteq V$, the *restriction of* Δ *to* W is the simplicial complex

$$\Delta[W] := \{\sigma \in \Delta \mid \sigma \subseteq W\}.$$

Definition 2.3.12. Let Δ be a simplicial complex with vertex set V. For any $\sigma \subseteq V$, denote $\overline{\sigma} := V \setminus \sigma$. The *Alexander dual of* Δ is the simplicial complex

$$\Delta^\vee := \{\overline{\sigma} \mid \sigma \subseteq V \text{ and } \sigma \notin \Delta\}.$$

2.3.1 Simplicial homology

One may define the reduced homology of a simplicial complex Δ over a field $\mathbb{K}$ as follows. For additional details see for instance Chapter 1 of [60].

Definition 2.3.13. Let Δ be a simplicial complex on $[n] = \{1, \dots, n\}$. For any face $\sigma \in \Delta$, write $\sigma = \{j_1, \dots, j_d\}$, assuming that $j_1 < \dots < j_d$, and define $\operatorname{sgn}(j_r, \sigma) := (-1)^{r+1}$. Denote by F_i the set of faces of Δ of dimension i, and consider the $\mathbb{K}$-vector spaces $\mathbb{K}^{F_i}$ with bases indexed by the faces of dimension i. Define

$$\partial_i : \mathbb{K}^{F_i} \longrightarrow \mathbb{K}^{F_{i-1}}, \qquad \partial_i(e_\sigma) := \sum_{j \in \sigma} \operatorname{sgn}(j, \sigma) e_{\sigma \setminus \{j\}},$$

where $\sigma \in F_i$. This way one gets the chain complex

$$0 \longrightarrow \mathbb{K}^{F_{n-1}} \xrightarrow{\partial_{n-1}} \dots \longrightarrow \mathbb{K}^{F_i} \xrightarrow{\partial_i} \mathbb{K}^{F_{i-1}} \xrightarrow{\partial_{i-1}} \dots \longrightarrow \mathbb{K}^{F_0} \xrightarrow{\partial_0} \mathbb{K}^{F_{-1}} \longrightarrow 0.$$

The *i-th reduced homology of* Δ *over* $\mathbb{K}$ is defined as

$$\widetilde{H}_i(\Delta;\mathbb{K}) := \ker(\partial_i)/\mathrm{im}(\partial_{i+1}),$$

and if $\mathbb{K}$ is understood from the context we simply write $\widetilde{H}_i(\Delta) = \widetilde{H}_i(\Delta;\mathbb{K})$.

The intuitive idea is that the dimension (as a $\mathbb{K}$-vector space) of $\widetilde{H}_i(\Delta,\mathbb{K})$ is the number of i-dimensional holes of Δ. Notice, as a straightforward consequence of the definition, that

$$\widetilde{H}_i(\Delta) = 0 \quad \text{for all} \quad i > \dim(\Delta).$$

Example 2.3.14. Consider the simplicial complex Δ in Example 2.3.2, and again write $i_1 i_2 \ldots i_s$ for the set $\{i_1, i_2, \ldots, i_s\}$, for $s \geq 2$. Then one has

$$\begin{aligned}
F_{-1} &= \{\emptyset\},\\
F_0 &= \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\},\\
F_1 &= \{12,13,23,24,25,26,34,35,36,45,56\},\\
F_2 &= \{234,235,236,245,256,345,356\},
\end{aligned}$$

and $F_i = \emptyset$ for $i > 2$, so that the complex in the definition above is

$$0 \longrightarrow \mathbb{K}^7 \xrightarrow{\partial_2} \mathbb{K}^{11} \xrightarrow{\partial_1} \mathbb{K}^6 \xrightarrow{\partial_0} \mathbb{K} \longrightarrow 0.$$

One may check that $\dim(\ker\partial_0) = \dim(\mathrm{im}\partial_1) = 0$, so that $\widetilde{H}_0(\Delta) = 0$. In general the dimension of $\widetilde{H}_0$ is equal to the number of connected components minus 1, and indeed Δ is connected. One may also check that

$$\dim \widetilde{H}_1(\Delta) = 1 \qquad \text{and} \qquad \dim \widetilde{H}_2(\Delta) = 2,$$

where $\widetilde{H}_1(\Delta)$ accounts for the one-dimensional hole in the triangle with edges 12, 13 and 23, and the $\widetilde{H}_2(\Delta)$ for the two two-dimensional holes inside the tetrahedra.

Remark 2.3.15. Notice that the concept of independence complex above is defined for labeled graphs, but in Publication III we consider independence complexes of *unlabeled* graphs. In fact we are only interested in the homology of $\mathrm{Ind}(G)$, and that does not change if we relabel the vertices of G.

3. Tools from Commutative Algebra

We refer to [59] for the basics on rings and modules. The rings considered in this thesis are commutative and with a multiplicative identiy: they will usually be polynomial rings $\mathbb{K}[x_1,\dots,x_n]$ over a field $\mathbb{K}$.

Given a commutative ring R with identity, a *chain complex* $\mathscr{C}$ is a sequence of maps of R-modules

$$\dots \longrightarrow M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \longrightarrow \dots$$

such that $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$, that is, such that $\mathrm{im}(d_{i+1}) \subseteq \ker(d_i)$ for all $i \in \mathbb{Z}$. The chain complexes considered in this thesis, called *resolutions*, are such that $M_i = 0$ for all $i < 0$. We say that the complex $\mathscr{C}$ is *exact in M_i* if $\mathrm{im}(d_{i+1}) = \ker(d_i)$. One may measure how far the complex $\mathscr{C}$ is from being exact in M_i by considering the *i-th homology* of $\mathscr{C}$

$$H_i(\mathscr{C}) := \ker(d_i)/\mathrm{im}(d_{i+1}).$$

Then $\mathscr{C}$ is exact in M_i if and only if $H_i(\mathscr{C}) = 0$. We call $\mathscr{C}$ an *exact complex* if $H_i(\mathscr{C}) = 0$ for all $i \in \mathbb{Z}$.

3.1 Free resolutions

For a general commutative ring R with identity, the free R-modules are very rare: they are exactly the direct sums of (finitely or infinitely many) copies of the ring R. In a nutshell, what one does with free resolutions is "approximating" an arbitrary R-module M with a sequence of free R-modules.

Definition 3.1.1. A *free resolution* of an R-module M is a chain complex

$$\dots \longrightarrow F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{d_0} F_0,$$

where each F_i is a free R-module, together with an additional module homomorphism $\varepsilon\colon F_0 \to M$, called *augmentation map*, such that the chain complex

$$\dots \longrightarrow F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is exact. That is, $\mathrm{im}(d_i) = \ker(d_{i-1})$ for all $i \in \mathbb{N}$, and ε is surjective.

In the literature, people sometimes refer to the second complex in the definition above as a resolution of M, and sometimes as an *augmented resolution*, possibly without the zero-module on the right.

Assuming that a set $\{m_i \mid i \in \Lambda_0\}$ of generators of M is known, and that one knows how to compute kernels of R-module maps, it is intuitively possible to construct a free resolution of M as follows. Let $F_0 := \bigoplus_{i\in\Lambda_0} R$ be the the direct sum of as many copies of the ring R as the generators of M, and let $\{e_i \mid i \in \Lambda_0\}$ be a basis of F_0. The map

$$\begin{aligned} \varepsilon\colon F_0 &\longrightarrow M \\ e_i &\longmapsto m_i \end{aligned}$$

is a surjection, and by the first isomorphism theorem one has

$$F_0/\ker(\varepsilon) \cong M.$$

Now, the map ε might also be injective, which happens if and only if $\ker(\varepsilon) = 0$. In this case the module M is actually isomorphic to F_0 via ε, which means that M is free and there are no non-trivial relations among the generators of M. If otherwise this is not the case, then $\ker(\varepsilon)$ is a non-zero module, with a system of generators $\{g_i \mid i \in \Lambda_1\}$, describing the relations among the generators of M. Define then the next module in the resolution as $F_1 := \bigoplus_{i\in\Lambda_1} R$ and the map

$$\begin{aligned} d_1\colon F_1 &\longrightarrow F_0 \\ \eta_i &\longmapsto g_i, \end{aligned}$$

where $\{\eta_i \mid i \in \Lambda_1\}$ is a basis of the free module F_1. Then by construction one has $\mathrm{im}(d_1) = \ker(\varepsilon)$, and again by the first isomorphism theorem

$$F_1/\ker(d_1) \cong \ker(\varepsilon).$$

There are two cases again: if $\ker(d_1) = 0$ then the module $\ker(\varepsilon)$ is free, isomorphic to F_1, and this is a satisfactory description of it; if otherwise $\ker(d_1) \neq 0$, then we keep going, constructing a free module F_2 with as many generators as $\ker(d_1)$ and a map $d_2\colon F_2 \to F_1$. And so on... This way one constructs a resolution of M, possibly going on endlessly.

In the setting of $\mathbb{K}[x_1,\ldots,x_n]$-modules there are indeed algorithms to compute kernels, implemented in several computer algebra systems. If M is a finitely generated such module, the free modules F_i constructed at each step are themselves finitely generated, and if one considers systems of generators that are not redundant, then the procedure described above to construct a free resolution of M ends after a finite number of steps—that is, $F_i = 0$ for $i \gg 0$. This will be made more precise in the next section.

3.2 Minimal graded free resolutions

In this thesis the ring R will almost always be a polynomial ring $\mathbb{K}[x_1,\dots,x_n]$, where $\mathbb{K}$ is a field, and the R-modules M that we consider will always be finitely generated. In fact we fix once and for all the notation

$$S := \mathbb{K}[x_1,\dots,x_n],$$

where the letter S stands for "standard grading".

3.2.1 Graded resolutions

We recall a few basic definitions concerning graded rings and modules, and refer to Sections 1 and 2 of [63] for additional content and fundamental results such as the graded version of Nakayama's lemma.

Definition 3.2.1. A commutative ring R with identity is a *graded ring* if there is a direct sum decomposition $R = \bigoplus_{i\in\mathbb{Z}} R_i$ into abelian groups such that $R_iR_j \subseteq R_{i+j}$ for all i and j. The elements of R_i are called *homogeneous elements of degree* i. An ideal I of a graded ring R is called a *homogeneous* (or *graded*) *ideal* if I is generated by homogeneous elements.

The graded ring we will consider is always $S = \mathbb{K}[x_1,\dots,x_n]$, for some n, where the decomposition $S = \bigoplus_{i\in\mathbb{Z}} S_i$ is obtained by taking the abelian groups

$$\begin{aligned} S_i &:= \langle x_1^{a_1}x_2^{a_2}\dots x_n^{a_n} \mid a_j \in \mathbb{N} \text{ for all } j, \text{ and } a_1+\dots+a_n = i\rangle \\ &= \{f \in S \mid \text{every monomial of } f \text{ has degree } i\}, \end{aligned}$$

so that in particular $S_i = 0$ for all $i < 0$. This agrees with the "usual" notion of *homogeneous polynomial*.

Definition 3.2.2. An S-module M is called a *graded module* if there is a direct sum decomposition $M = \bigoplus_{i\in\mathbb{Z}} M_i$ into abelian groups such that $R_jM_i \subseteq M_{j+i}$ for all i and j. The elements of M_i are called *homogeneous elements of degree* i. A submodule $N \subseteq M$ is called a *graded submodule* if it is generated by homogeneous elements.

A module map $\varphi\colon M \to N$ between graded modules is called a *graded map* if

$$\varphi(M_i) \subseteq N_i \qquad \text{for all } i \in \mathbb{Z},$$

that is, if φ preserves the degree. In most cases the maps we will consider can be represented as multiplications by some homogeneous element, possibly of high degree, and clearly such multiplication maps cannot preserve the degrees of the elements. A way to fix this issue is by "shifting" the grading in the module M.

Definition 3.2.3. Let M be a graded M-module. For $j \in \mathbb{Z}$, one defines the *module M shifted by j degrees* as

$$M(-j) := \bigoplus_{i\in\mathbb{Z}} M(-j)_i, \quad \text{where} \quad M(-j)_i := M_{i-j}.$$

The integer j is called the *shift*.

Observe that the elements and the algebraic structure of M and $M(-j)$ are exactly the same. The only difference is that we consider a different grading. The ring S itself is a graded S-module and it can be shifted: for instance, if $S = \mathbb{K}[x,y,z]$, then the homogeneous polynomial $x^2y^3 + z^5$ has degree 2 in $S(-3)$ and degree 4 in $S(-1)$. The multiplication map

$$S(-4) \xrightarrow{\cdot x^3} S(-1)$$

is a graded map. More generally, one can take a direct sum of shifted copies of S. For instance, $(x^2 + yz,\ x^4)$ is a homogeneous element of degree 1 in $S(-1) \oplus S(-3)$.

Definition 3.2.4. A free resolution is called *graded* if all the maps are graded.

One can construct a graded resolution by refining the discussion in Section 3.1. If M is generated by homogeneous elements $m_1, \ldots, m_t$ with degrees $d_i = \deg(m_i)$, then the module F_0 can be defined by shifting suitably the direct summands as

$$F_0 := \bigoplus_{i=1}^{t} S(-d_i)$$

and the augmentation map $\varepsilon \colon e_i \mapsto m_i$ is graded. Then it turns out that $\ker(\varepsilon)$ is a graded submodule of F_0, and one may proceed by defining F_1 with suitable shifts. We refer to Construction 4.2 of [63] for additional details.

Example 3.2.5. Let $S = \mathbb{K}[x,y,z]$ and $I = (x^2, xy, y^3)$. One may check that the complex

$$0 \longrightarrow S(-5) \xrightarrow{\begin{bmatrix} y^2 \\ -x \\ -1 \end{bmatrix}} S(-3) \oplus S(-4) \oplus S(-5) \xrightarrow{\begin{bmatrix} y & 0 & y^3 \\ -x & -y^2 & 0 \\ 0 & x & -x^2 \end{bmatrix}}$$

$$S(-2)^2 \oplus S(-3) \xrightarrow{\begin{bmatrix} x^2 & xy & y^3 \end{bmatrix}} S$$

is a graded resolution of S/I.

3.2.2 Minimal resolutions

For a $\mathbb{K}$-vector space, all minimal systems of generators have the same cardinality, namely the dimension of the space. For general R-modules there might be different systems of generators that are minimal but have different cardinality: for instance the $\mathbb{Z}$-module—that is, abelian group—$\mathbb{Z}$ is generated by $\{1\}$ or by $\{2,3\}$ and they are both minimal systems of generators. However, as a consequence of Nakayama's lemma, for a graded S-module M the situation is as beautiful as it can be: not only all minimal systems of *homogeneous* generators of M have the same cardinality, but the number of generators of a fixed degree d is the same in all miminal systems of homogeneous generators of M. (See Theorem 2.12 of [63].) In what is called a *minimal* resolution of M, one pushes this even further.

Theorem 3.2.6. *For a graded free resolution, the following conditions are equivalent:*

- *at each step, the generators for each module considered in the resolution form a minimal system of generators;*
- *we have* $\mathrm{im}(d_i) \subseteq (x_1, \ldots, x_n)F_{i-1}$ *for all* i. *That is, if we choose bases for the free modules* F_i, *and we represent the maps in the resolution as multiplications by matrices, then the entries of these matrices are elements of the ideal* $(x_1, \ldots, x_n)$, *namely the entries of the matrices cannot be non-zero constants.*

For a proof, see for instance Theorem 7.3 of [63].

Definition 3.2.7. If a graded free resolution satisfies the equivalent conditions above, then it is called a *minimal graded free resolution*. We will just call such a resolution a *minimal resolution* for short.

Example 3.2.8. The resolution in Example 3.2.5 is not minimal, because there is an entry $-1 \notin (x_1, \ldots, x_n)$ in one of the matrices. More explicitly, the reason is that there is a redundant generator

$$\begin{pmatrix} y^3 \\ 0 \\ -x^2 \end{pmatrix} = y^2 \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} - x \begin{pmatrix} 0 \\ -y^2 \\ x \end{pmatrix}$$

for the kernel of the map $S(-2)^2 \oplus S(-3) \to S$. On the other hand,

$$0 \longrightarrow S(-5) \xrightarrow{\begin{bmatrix} y^2 \\ -x \end{bmatrix}} S(-3) \oplus S(-4) \xrightarrow{\begin{bmatrix} y & 0 \\ -x & -y^2 \\ 0 & x \end{bmatrix}} S(-2)^2 \oplus S(-3) \xrightarrow{\begin{bmatrix} x^2 & xy & y^3 \end{bmatrix}} S,$$

where we got rid of that redundant generator, *is* a minimal resolution for S/I.

A *homomorphism of complexes* between

$$\ldots \to M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \to \ldots \quad \text{and} \quad \ldots \to N_{i+1} \xrightarrow{d'_{i+1}} N_i \xrightarrow{d'_i} N_{i-1} \to \ldots$$

is a sequence of module maps $(\varphi_i : M_i \to N_i)_{i \in \mathbb{Z}}$ (with $\varphi_i = 0$ for all $i < 0$ in the case of resolutions) such that all the squares in the diagram

$$\begin{tikzcd}
\ldots \arrow[r] & M_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "\varphi_{i+1}"'] & M_i \arrow[r, "d_i"] \arrow[d, "\varphi_i"'] & M_{i-1} \arrow[r] \arrow[d, "\varphi_{i-1}"'] & \ldots \\
\ldots \arrow[r] & N_{i+1} \arrow[r, "d'_{i+1}"'] & N_i \arrow[r, "d'_i"'] & N_{i-1} \arrow[r] & \ldots
\end{tikzcd}$$

commute. See Section 3 of [63] for additional details about homomorphism of complexes, and in particular isomorphisms of resolutions. For the following fundamental result, we refer to Theorem 7.8 of [63].

Theorem 3.2.9. *Let M be a graded finitely generated S-module. There exists a minimal graded free resolution for M, and it is unique up to isomorphism.*

In virtue of this theorem one often speaks of “the” minimal resolution of M.

Some good news is that minimal resolutions are “finite”. Hilbert already showed in the 1890’s (albeit in primordial form) some of the properties that would later become fundamental cornerstones in the theory of polynomial rings with coefficients in a field:

Theorem 3.2.10 (Hilbert [53])**.** *Let $S = \mathbb{K}[x_1,\ldots,x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$ and let M be a finitely generated graded S-module. Then,*

- *all the modules in a minimal resolution of M are finitely generated;*
- *denoting by $F_0,\ldots,F_p$ the non-zero modules in a minimal resolution of M, one has $p \le n$.*

The second part of the theorem above is referred to as *Hilbert’s syzygy theorem*. See for instance Theorem 15.2 of [63] for a proof. For a reference in English about Hilbert’s work, see for instance the translation [54] of some notes taken during one of his courses on the topic.

Another piece of good news, as anticipated in Section 3.1, is that there are algorithms, implemented in computer algebra systems such as CoCoA [1] or Macaulay2 [44], that produce minimal resolutions. See for instance Section 4.8.B of [58] for the explicit technical details about such algorithms. The bad news is that, although algorithms are known, one doesn’t in general know a priori what to expect. There are very few classes of S-modules for which an explicit description of the minimal resolution is known, with a canonical construction and closed formulas for the maps in the resolution. Quoting Peeva’s words in [63], the general trend in the last decades has been to *“introduce new ideas and constructions which either have strong applications or/and are beautiful”*. More explicitly, the following are some of the main ways in which the problem has been tackled:

- One considers a special class of modules (or in particular ideals), and gives an explicit description for a minimal resolution of the elements of that class. (See some of the items in the list in Section 3.3.1).
- One gives constructions that work in great generality but that are usually not minimal resolutions. For instance the Koszul complex (see Section 14 of [63]) is in general a complex, not a resolution, and it is a resolution exactly when the elements defining it form a regular sequence. Other such constructions are mentioned in Section 3.3.1.
- A well-known result (see for instance Theorem 7.5 of [63]) states that the minimal resolution is contained as a subcomplex in any resolution. One may start from a given non-minimal resolution and essentially “eliminate” the redundant parts in order to get a minimal resolution. See for instance Theorem 4.8.6 of [58].

Even for monomial ideals (which one would guess to be a preposterously simple case) the problem of producing an explicit canonical description of minimal resolutions has been elusive for the last half a century. For the case of minimal resolutions of monomial ideals, see Section 3.3.1.

3.2.3 Betti numbers, Betti tables and regularity

When we write the free modules in a minimal graded free resolution as

$$F_i = \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{i,j}},$$

the natural numbers $\beta_{i,j}$ are uniquely determined.

Definition 3.2.11. The numbers $\beta_{i,j}$ are called the *(coarsely graded) Betti numbers of M*. We shall write $\beta^S_{i,j}(M)$ or $\beta_{i,j}(M)$ if there is ambiguity about the module or even the ring. The *i-th total Betti number of M* is $\beta_i := \sum_{j\in\mathbb{Z}} \beta_{i,j}$.

There are well-known strict inequalities for the smallest shifts (see Definition 3.2.3), following straightforwardly from the fact that the Betti numbers come from a resolution that is graded and minimal:

Proposition 3.2.12. *For all i, denote by $t_i := \min\{j \mid \beta_{i,j} \neq 0\}$ the smallest shift in the module F_i. Then $t_i < t_{i+1}$.*

So, if one were to arrange the Betti numbers in a table in the "intuitive way", namely by putting the number $\beta_{i,j}$ in column i and row j, there would be a part of the table that is automatically always zero, above the diagonal:

	0	1	2	3	4	$\cdots$
0	$\beta_{0,0}$	0	0	0	0	$\cdots$
1	$\beta_{0,1}$	$\beta_{1,1}$	0	0	0	
2	$\beta_{0,2}$	$\beta_{1,2}$	$\beta_{2,2}$	0	0	
3	$\beta_{0,3}$	$\beta_{1,3}$	$\beta_{2,3}$	$\beta_{3,3}$	0	
4	$\beta_{0,4}$	$\beta_{1,4}$	$\beta_{2,4}$	$\beta_{3,4}$	$\beta_{4,4}$	
$\vdots$	$\vdots$					$\ddots$

For this reason, in order to save space, one usually arranges the Betti numbers by "lifting" column i by i positions, as follows.

Definition 3.2.13. The *Betti table of M*, denoted $\beta(M)$, is a table where in colum i and row j one places the Betti number $\beta_{i,i+j}$ of M.

That is, the Betti numbers are arranged like this:

$$
\begin{array}{c|cccccc}
 & 0 & 1 & 2 & 3 & 4 & \cdots \\
\hline
0 & \beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \beta_{3,3} & \beta_{4,4} & \cdots \\
1 & \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \beta_{3,4} & \beta_{4,5} & \\
2 & \beta_{0,2} & \beta_{1,3} & \beta_{2,4} & \beta_{3,5} & \beta_{4,6} & \\
3 & \beta_{0,3} & \beta_{1,4} & \beta_{2,5} & \beta_{3,6} & \beta_{4,7} & \\
4 & \beta_{0,4} & \beta_{1,5} & \beta_{2,6} & \beta_{3,7} & \beta_{4,8} & \\
\vdots & \vdots & & & & & \ddots
\end{array}
$$

Notice that often in the literature the name “Betti diagram” is used instead of “Betti table”. This also applies to “pure diagram” in place of “pure table” as in Definition 3.6.2, most notably for us in the literature about Boij–Söderberg theory.

One has the following immediate consequence of Theorem 3.2.10:

Corollary 3.2.14. *For any finitely generated graded S-module M, there is only a finite number of non-zero Betti numbers.*

In practice one usually just writes the smallest possible sub-rectangle of the Betti table containing all the non-zero Betti numbers, and one refers to that as the Betti table. Moreover, often in the literature dashes or dots are used in place of zeros.

Example 3.2.15. Consider for instance

$$I = (x_1x_2x_4,\ x_1^2x_2^2x_3,\ x_3^3x_4^3) \quad \subset \quad \mathbb{Q}[x_1,x_2,x_3,x_4].$$

A minimal resolution of I is for instance

$$0 \longrightarrow S(-6)\oplus S(-8) \xrightarrow{\begin{bmatrix} x_1x_2x_3 & x_3^3x_4^2 \\ -x_4 & 0 \\ 0 & -x_1x_2 \end{bmatrix}}$$

$$S(-3)\oplus S(-5)\oplus S(-6) \xrightarrow{\begin{bmatrix} x_1x_2x_4 & x_1^2x_2^2x_3 & x_3^3x_4^3 \end{bmatrix}} I \longrightarrow 0.$$

Therefore, the Betti table $\beta(I)$ is

$$
\begin{array}{c|cc}
 & 0 & 1 \\
\hline
3 & 1 & - \\
4 & - & - \\
5 & 1 & 1 \\
6 & 1 & - \\
7 & - & 1.
\end{array}
$$

A first measure of the complexity of the graded minimal resolution of M is given by how “wide” and “tall” the (non-zero region of the) Betti table of M is. The width is measured by the so-called projective dimension of M and is easier to understand.

Definition 3.2.16. The *projective dimension of M* is the number

$$\begin{aligned}\operatorname{proj\,dim}(M) &:= \max\{i \mid F_i \neq 0\}\\ &= \max\{i \mid \beta_{i,i+j}(M) \neq 0 \text{ for some } j\},\end{aligned}$$

where F_i denotes the i-th module in a minimal resolution of M.

The following is a restatement of the second part of Theorem 3.2.10, with the notation just introduced.

Theorem 3.2.17 (Hilbert’s syzygy theorem)**.** *If $S = \mathbb{K}[x_1,\dots,x_n]$ is a polynomial ring in n variables over a field $\mathbb{K}$ and M is a finitely generated graded S-module, then* $\operatorname{proj\,dim}(M) \leq n$.

That is, the largest index of a non-zero column of the Betti table can be at most n, the number of variables. On the other hand, although the number of non-zero rows is also finite, there is a priori no bound to the highest index of a non-zero row. One may consider for instance the ideal (x_1^d), generated by a single element of degree d: the highest—actually, the only—index of a non-zero row of the Betti table of this ideal is d, and one may do this for arbitrary d.

The (Castelnuovo–Mumford) regularity is the invariant that measures how “tall” the Betti table of M is:

Definition 3.2.18. The highest index of a non-zero row of the Betti table of M is the *(Castelnuovo–Mumford) regularity of M*, denoted $\operatorname{reg}(M)$. That is,

$$\operatorname{reg}(M) := \max\{j \mid \beta_{i,i+j}(M) \neq 0 \text{ for some } i\}.$$

This numerical invariant is intimately related to linear resolutions (see Section 3.5).

3.3 Monomial ideals

Denote as usual $S = \mathbb{K}[x_1,\dots,x_n]$, where $\mathbb{K}$ is a field. Recall that a *monomial* is a product of (possibly repeated) variables.

Definition 3.3.1. An ideal I of S is called a *monomial ideal* if I is generated by monomials.

The S-modules considered most often in this thesis will be monomial ideals I or quotients S/I by monomial ideals. Our main reference for most results about monomial ideals is the namesake monography [50] by Herzog and Hibi.

Recall that a polynomial $f \in S$ can be written in a unique way as a finite linear combination of monomials, with coefficients in $\mathbb{K} \setminus \{0\}$. The set formed by these monomials is usually called the *support of f*. A very useful characterization of the monomial ideals is the following (see Corollary 1.1.3 of [50]):

Proposition 3.3.2. *For an ideal I of S, the following are equivalent:*

- *I is a monomial ideal;*
- *for all $f \in S$, f belongs to I if and only if every monomial in the support of f belongs to I.*

The word "support" has a different meaning in this thesis when it is applied to a monomial (as in Definition 3.3.6 below), so we refrain from using it any longer in the sense of summands of f with a non-zero coefficient, as in the proposition above.

In general an ideal has infinitely many *minimal* systems of generators. For an arbitrary homogeneous ideal, even if one asks for "monic" and homogeneous generators, usually one still needs to fix an ordering of the monomials in order to get a uniquely determined system of generators. Consider for instance $I = (x_1 - x_2,\ x_2 - x_3) = (x_1 - x_2,\ x_1 - x_3)$. However, for monomial ideals this matter is very simple (see Proposition 1.1.6 of [50]):

Proposition 3.3.3. *A monomial ideal I has a unique minimal system of monomial generators, consisting of the monomials in I which are minimal with respect to divisibility.*

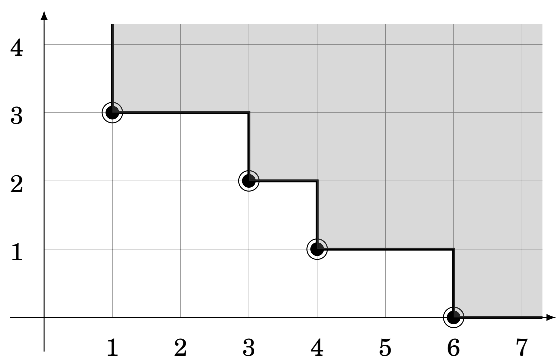
Definition 3.3.4. We denote by $G(I)$ the unique minimal system of monomial generators of a monomial ideal I.

It is clear at this point that a monomial ideal I is a very combinatorial object: it is determined by a finite number of vectors in $\mathbb{N}^n$, namely the vectors of exponents of the monomials in $G(I)$.

Example 3.3.5. Consider $S = \mathbb{K}[x, y]$ and the ideal $I = (xy^3,\ x^3y^2,\ x^4y,\ x^6)$. The exponent vectors in $\mathbb{N}^2$ of the minimal monomial generators of I are respectively

$$(1,3), \quad (3,2), \quad (4,1), \quad (6,0).$$

It is very common to visualize these as points in the plane, as in the following picture:



A monomial $x^i y^j$ is divisible by xy^3 if and only if $i \geq 1$ and $j \geq 3$, which defines a cone with vertex $(1,3)$. The monomials that belong to I are then easily readable from the picture above: they are exactly the monomials with exponent vector lying in the union of the four cones associated to the minimal generators of I. This union of cones is the gray region in the picture above.

One may visualize monomial ideals in three variables in a very similar fashion as in the example above, drawing cones in the three-dimensional space instead of the plane.

Definition 3.3.6. Let m be a monomial in S. We call the *support of* m, denoted by $\operatorname{supp}(m)$, the set consisting of the variables of S that divide m.

For instance, $\operatorname{supp}(x_1^3 x_2 x_5^2) = \{x_1, x_2, x_5\}$.

Monomial ideals arise mainly as initial ideals of arbitrary ideals in S. The theory of Gröbner bases (see Chapter 15 of [25] or Section 39 of [63]) offers a variety of algorithms to solve problems about arbitrary ideals, and many of these problems become trivial for monomial ideals. To name one, the ideal membership problem has a very easy answer in the monomial case, implicit in the results recalled above. Another example of such a situation, which we recall next and which is useful in what follows, concerns colon ideals.

Definition 3.3.7. Let I and J be ideals of S. The *quotient ideal* (or *colon ideal*) *of* I *by* J is defined as

$$I : J := \{g \in S \mid \text{for all } f \in J,\, fg \in I\}.$$

When $J = (f)$ is a principal ideal, we shall write simply $I : f = I : (f)$.

Lemma 3.3.8. *Let $I \subset S$ be a monomial ideal and let $m \in S$ be a monomial. Then*

$$I : m = \left(\frac{u}{\gcd(u,m)} \mid u \in G(I) \right).$$

For a proof, see Proposition 1.2.2 of [50].

Example 3.3.9. Notice that the generators of $I : m$ on the right-hand side in the lemma above are not necessarily minimal. For instance, in $\mathbb{K}[x_1, x_2, x_3]$ one has

$$(x_1x_2,\, x_1x_3) : x_2 = (x_1,\, x_1x_3) = (x_1).$$

3.3.1 Resolutions of monomial ideals

In the 1960's Kaplansky raised the problem of systematically studying resolutions of monomial ideals. Despite the apparently easy structure of monomial ideals, the problem of describing their *minimal* resolutions explicitly (with closed formulas, in a universal and canonical way) has been quite elusive for more than half a century. Here follows a first indication that the problem is not as harmless as it might seem.

Remark 3.3.10. *In general the characteristic of the ground field* $\mathbb{K}$ *matters.* Consider, inside the ring $\mathbb{K}[x_1, \ldots, x_6]$, the monomial ideal

$$\begin{aligned} I = (&x_1x_2x_3,\ x_1x_2x_6,\ x_1x_3x_4,\ x_1x_4x_6,\ x_1x_5x_6,\\ &x_2x_3x_6,\ x_2x_4x_5,\ x_2x_4x_6,\ x_3x_4x_5,\ x_3x_5x_6). \end{aligned}$$

Depending on whether $\mathrm{char}(\mathbb{K}) \neq 2$ or $\mathrm{char}(\mathbb{K}) = 2$, the Betti table of I is respectively

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 3 & 10 & 15 & 6 \end{array} \qquad \text{or} \qquad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 3 & 10 & 15 & 6 & 1 \\ 4 & - & - & 1 & -. \end{array}$$

This is however only a mild complication: in practice, for almost all finite characteristics one gets the same result as in characteristic 0. Moreover, there are some Betti numbers that do not depend on the characteristic, and in some situations *all* Betti numbers are independent of the field $\mathbb{K}$. Almost all the Betti numbers in this thesis (save a couple of explicit exceptions) have been computed for $\mathrm{char}(\mathbb{K}) = 0$, so we shall assume this from now on.

A great deal of research on resolutions of monomial ideals has been done after Kaplansky first posed the problem, starting with Taylor's PhD thesis [70], where she defined a canonical construction that works for every monomial ideal but gives a highly non-minimal resolution in general (see Section 26 of [63]). Since then, many more constructions have been introduced, and they either produce minimal resolutions only for certain classes of monomial ideals, or complexes that work in great generality but are not always minimal resolutions, or not even resolutions to begin with. Some of these constructions, however "partial", provide fruitful combinatorial or topological interpretations of what happens on the algebraic side. We mention some of the most well-known such constructions:

- The Koszul complex (see Section 14 of [63]) constitutes a famous complex that works in great generality. It is usually not a resolution but provides anyway useful information. In the case of a monomial ideal I, the Koszul complex is a resolution if and only if no two elements of $G(I)$ have a non-trivial common divisor.

- The Eliahou–Kervaire resolution [31] is a minimal resolution defined for strongly stable monomial ideals. (See also Section 28 of [63].)

- Cellular resolutions [4, 5] have an interesting underlying topological structure. (See also [56, 57], where these objects are categorified.)

- The Scarf complex [4] is always *contained* in the minimal resolution, and sometimes constitutes a minimal resolution. (See also Section 59 of [63].)

- In 2019, Eagon, Miller and Ordog [23] described a canonical minimal resolution for every monomial ideal, for characteristic zero and most positive characteristics. In the positive-characteristic case, the constructions still work but are non-canonical. After the paper [23] was made public, Tchernev [71] produced minimal resolutions for all monomial ideals; these resolutions work in every characteristic, and they are canonical but not in closed form.

- The most notable case of interest to us is the following: Hochster, Stanley and Reisner introduced a machinery where one associates bijectively squarefree monomial ideals to simplicial complexes, via the so-called *Stanley–Reisner correspondence*, which allows to understand the Betti numbers of squarefree monomial ideals in terms of simplicial homology (see [67], [68], Chapter 5 of [12], Section 62 of [63] and Section 3.4.1 below).

We refer to [23] and Ordog's thesis [62] for additional information and references.

To conclude this subsection we consider the very first map in a resolution of a monomial ideal, the augmentation map, and its kernel. Let I be a monomial ideal with $G(I) = \{f_1, \dots, f_m\}$, and write $d_j := \deg(f_j)$ for all j. One defines

$$\varepsilon \colon S^m = \bigoplus_{j=1}^{m} S(-d_j) \longrightarrow I, \qquad e_j \longmapsto f_j,$$

where $e_1, \dots, e_m$ is a basis for S^m. Of course one may get a different map just by rearranging the order of the generators, but this does not affect significantly what $\ker(\varepsilon)$ is. With a very mild abuse of notation, one introduces the following.

Definition 3.3.11. We call $\mathrm{Syz}(I) := \ker(\varepsilon)$ the *(first) sygyzy module of I* and we call the elements of $\mathrm{Syz}(I)$ the *syzygies of I*.

That is, a syzygy of I is a tuple $(p_1, p_2, \dots, p_m) \in S^m$ such that

$$p_1 f_1 + p_2 f_2 + \cdots + p_m f_m = 0.$$

Determining syzygies for arbitrary homogeneous ideals is not a completely trivial matter, but the situation for monomial ideals is much simpler: observe that for any i and j we have

$$\varepsilon(f_j e_i - f_i e_j) = f_j f_i - f_i f_j = 0.$$

The elements $f_j e_i - f_i e_j$ can be "refined" to

$$\begin{aligned} \sigma_{ij} &:= \frac{f_j}{\gcd(f_i, f_j)} e_i - \frac{f_i}{\gcd(f_i, f_j)} e_j \\ &= \frac{\mathrm{lcm}(f_i, f_j)}{f_i} e_i - \frac{\mathrm{lcm}(f_i, f_j)}{f_j} e_j, \end{aligned}$$

and these still map to zero.

Definition 3.3.12. The σ_{ij}'s above are called the *reduced trivial syzygies of I*.

For the following well-known result we refer to Theorem 15.10 of [25].

Theorem 3.3.13 (Schreyer). *The reduced trivial syzygies of I generate* $\mathrm{Syz}(I)$.

Example 3.3.14. The reduced trivial sygygies are not necessarily a *minimal* system of generators for $\mathrm{Syz}(I)$. Consider for instance $I = (x_1x_2x_4,\ x_1^2x_2^2x_3,\ x_3^3x_4^3) \subset \mathbb{K}[x_1, x_2, x_3, x_4]$. Then

$$\sigma_{12} = (x_1x_2x_3,\ -x_4,\ 0), \qquad \sigma_{13} = (x_3^3x_4^2,\ 0,\ -x_1x_2),$$

and σ_{23} is redundant, since $\sigma_{23} = -x_3^2x_4^2\sigma_{12} + x_1x_2\sigma_{13}$.

3.4 Squarefree monomial ideals

Among all monomial ideals, those generated by squarefree monomials play a special role. Their particularly nice structure allows one to use topological tools. In particular, one can compute their Betti numbers by means of simplicial homology. Moreover, as we shall see in Section 3.7, one can always transform an arbitrary monomial ideal into a squarefree one, preserving in particular the Betti numbers.

3.4.1 The Stanley–Reisner correspondence and Hochster's formula

Definition 3.4.1. A monomial ideal $I \subset S = \mathbb{K}[x_1, \dots, x_n]$ is said to be *squarefree* if none of the minimal monomial generators of I is divided by the square of some variable. That is, the entries of the exponent vectors of the monomials in $G(I)$ can only be 0 or 1.

For a subset $\sigma \subseteq [n] = \{1, \dots, n\}$, we denote

$$x_\sigma := \prod_{i \in \sigma} x_i.$$

(Notice that for instance in [60] the same monomial is denoted by x^σ.)

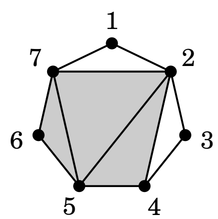
Definition 3.4.2. Given a simplicial complex Δ on $[n] = \{1, \dots, n\}$, the *Stanley–Reisner ideal of* Δ is

$$I_\Delta := (x_\sigma \mid \sigma \notin \Delta),$$

that is, the ideal generated by the non-faces of Δ, in the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$.

In practice, I_Δ is generated by the *minimal* non-faces of Δ.

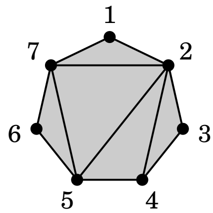
Example 3.4.3. Consider the following simplicial complex Δ



which has three facets of dimension 2 and the hollow triangles 127 and 234. The Stanley–Reisner ideal I_Δ lives in $\mathbb{K}[x_1, \dots, x_7]$ and it is the ideal

$$I_\Delta = (x_1x_2x_7,\ x_1x_3,\ x_1x_4,\ x_1x_5,\ x_1x_6,\ x_2x_3x_4,\ x_2x_6,\ x_3x_5,\ x_3x_6,\ x_3x_7,\ x_4x_6,\ x_4x_7).$$

Notice that this ideal has two generators of degree 3. Instead, the Stanley–Reisner ideal of the simplicial complex



is the ideal

$$(x_1x_3,\ x_1x_4,\ x_1x_5,\ x_1x_6,\ x_2x_6,\ x_3x_5,\ x_3x_6,\ x_3x_7,\ x_4x_6,\ x_4x_7),$$

and all of its minimal generators are of degree 2.

Remark 3.4.4. Flag simplicial complexes (see Definition 2.3.3) are exactly the complexes whose Stanley–Reisner ideal is generated in degree 2.

The map

$$\left\{\begin{array}{c}\text{simplicial complexes}\\ \text{on } \{1,\dots,n\}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{squarefree monomial}\\ \text{ideals of } \mathbb{K}[x_1,\dots,x_n]\end{array}\right\}$$

$$\Delta \longmapsto I_\Delta$$

is a bijection, and it is called the *Stanley–Reisner correspondence*. The following result and variations of it are commonly known as *Hochster's formula*.

Theorem 3.4.5 (Hochster)**.** *Let Δ be a simplicial complex on $[n]$. For any $i,j \geq 0$,*

$$\beta_{i,j}(I_\Delta) = \sum_{W \in \binom{[n]}{j}} \dim_{\mathbb{K}} \widetilde{H}_{j-i-2}(\Delta[W];\mathbb{K}).$$

The original formulation of the result above is in [55]. For additional (finer) versions of it see for instance Corollaries 1.40 and 5.12 of [60], or Theorem 62.15 of [63].

3.4.2 Edge ideals of graphs

Definition 3.4.6. Given a finite simple graph $G = (V,E)$, the *edge ideal of* G is the ideal

$$I_G := (x_v x_w \mid \{v,w\} \in E)$$

in the polynomial ring $\mathbb{K}[x_v \mid v \in V]$.

In Publication III we talk about "edge ideals of *unlabeled* graphs", a concept that does not seem to be well-defined. However, what we actually investigate in that paper are the Betti numbers of such ideals, and those do not depend on the labelings. Namely, if $G \cong G'$ is an isomorphism of labeled graphs, then $\beta_{i,j}(I_G) = \beta_{i,j}(I_{G'})$ for all i and j.

Edge ideals, whose systematic study was initiated by Villarreal [73] about thirty years ago, have a very rich literature. The main goal is to relate algebraic properties of the ideal I_G to combinatorial or topological properties of G. We refer to Chapter 9 of [50] for a general introduction to such results, in particular relating properties of G to the Cohen–Macaulayness of I_G or homological invariants of I_G. A few results of the latter kind are discussed below.

Remark 3.4.7. Recall that the independence complex Ind(G) is the simplicial complex with the the independence sets of G as faces. Notice that I_G is equal to the Stanley–Reisner ideal $I_{\mathrm{Ind}(G)}$, and recall that the independence complexes of graphs are precisely the flag simplicial complexes (Remark 2.3.4).

Clearly the only non-zero Betti number in column 0 of the Betti table is on row 2, and it is the number of edges in G. By Proposition 3.2.12, this implies that all of row 0 and row 1 are zero. Moreover, it is a well-known fact that for all $i \geq 0$ and $j > 2(i+1)$, one has $\beta_{i,j}(I_G) = 0$. (See for instance [15] or [36].) That is, for edge ideals we always have zero entries under the diagonal consisting of the Betti numbers $\beta_{i,2(i+1)}(I_G)$:

	0	1	2	3	4	$\cdots$
2	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	$\beta_{3,5}$	$\beta_{4,6}$	$\cdots$
3	–	$\beta_{1,4}$	$\beta_{2,5}$	$\beta_{3,6}$	$\beta_{4,7}$	
4	–	–	$\beta_{2,6}$	$\beta_{3,7}$	$\beta_{4,8}$	
5	–	–	–	$\beta_{3,8}$	$\beta_{4,9}$	
6	–	–	–	–	$\beta_{4,10}$	
$\vdots$	$\vdots$					$\ddots$

Definition 3.4.8. We refer to the diagonal consisting of the Betti numbers $\beta_{i,2(i+1)}$, for $i \geq 0$, as the *main diagonal of the Betti table*.

Hochster's formula 3.4.5 reduced to the case of edge ideals gives the following:

Theorem 3.4.9 (Hochster's formula for edge ideals). *Let G be a finite simple graph with vertex set V. For any $i, j \geq 0$,*

$$\beta_{i,j}(I_G) = \sum_{W \in \binom{V}{j}} \dim_{\mathbb{K}} \widetilde{H}_{j-i-2}\big(\mathrm{Ind}(G)[W]; \mathbb{K}\big).$$

For the concept of linear resolution in the following, celebrated result by Fröberg, see Definition 3.5.1 below. For a proof, see for instance Theorem 9.2.3 of [50].

Theorem 3.4.10 (Fröberg, [42]). *Let G be a finite simple graph. The following are equivalent:*

- *I_G has a 2-linear resolution, that is, all the non-zero Betti numbers of I_G are on row 2 of the Betti table;*
- *the complement $\overline{G}$ of G is chordal.*

The following is a refinement by Dochtermann and Engström [22] of Hochster's formula, in the case of edge ideals of graphs with chordal complement.

Theorem 3.4.11. *For a finite simple graph G with vertex set V and with chordal complement, one has*

$$\beta_{i,i+2}(I_G) = \sum_{W \in \binom{V}{i+2}} \left(-1 + \text{the number of connected components of } \overline{G}[W]\right).$$

In spite of the apparently easy combinatorial structure of edge ideals and the fact that Hochster's result and variations of it have been around for more than forty years, to the best of our knowledge it is still not known explicitly what the exact value of $\operatorname{reg}(I_G)$ is for an arbitrary graph G, in terms of easy numerical invariants of G, such as the degrees of the vertices. Very often, only *bounds* for the regularity are known, or alternatively exact values only for edge ideals of graphs in some special class. We recall a few such results:

- Let $m(G)$ be the *matching number* of a graph G, that is, the largest size of a maximal matching in G. Hà and Van Tuyl [45] showed that $\operatorname{reg}(I_G) \leq m(G)+1$, a result later generalized for arbitrary squarefree monomial ideals in [46] (see also [47]).
- An exact result, also proven in [45], states that for a chordal graph G one has $\operatorname{reg}(I_G) = c+1$, where c is the largest number of pairwise 3-disjoint edges of G.
- Dirac's theorem [21] characterizing chordal graphs was imported to commutative algebra in [51], thereby "expanding" Fröberg's Theorem 3.4.10. (See a comprehensive summary of this in Theorem 9.2.12 of [50].) In particular, $\overline{G}$ is chordal if and only if $\operatorname{proj\,dim}(I_{\operatorname{Ind}(G)^\vee}) = 1$ (if and only if $\operatorname{reg}(I_G) = 2$).
- If exact data about the Betti numbers or other numerical invariants of an edge ideal are out of reach, one is at least interested in knowing some asymptotic behavior. The work [36], somewhat related to our Publication III, is an example of a paper containing such asymptotic results.

Among the plethora of papers concerning Betti numbers of edge ideals, of particular importance to us is [22].

3.5 Linear resolutions

Definition 3.5.1. A finitely generated graded S-module M has a *d-linear resolution* if $\beta_{i,j}(M) = 0$ for $j \neq i+d$, that is, if all the non-zero entries of the Betti table of M are in the d-th row.

The following list contains a few reasons why linear resolutions are interesting. Items 4 and 5 are the most relevant to us, for Publications I and II.

1. The concept of linear resolution is intimately related to that of regularity (see Definition 3.2.18): M has a linear resolution iff all the generators of M are in the same degree d and $\operatorname{reg}(M)$ is equal to d, the smallest possible value.

2. Since the Hilbert series is additive with an alternating sign on exact sequences, one can compute the Hilbert series of M from its graded Betti numbers: if

$$0 \longrightarrow \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{p,j}} \longrightarrow \ldots \longrightarrow \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{0,j}} \longrightarrow M \longrightarrow 0$$

   is an exact complex (not necessarily a minimal resolution, namely the numbers $\beta_{i,j}$ do not actually need to be the Betti numbers of M), then

$$\mathrm{HS}_M(t) = \frac{\sum_{i=0}^{p}(-1)^i \sum_{j\in\mathbb{Z}} \beta_{i,j} t^j}{(1-t)^n}.$$

   One cannot in general go the other way around, because there might occur some cancellations: for instance, $S/(x^2, xy, y^3)$ and $S/(x^2, y^2)$ have the same Hilbert series but different Betti numbers. However, when M has a linear resolution one *can* get the Betti numbers from the Hilbert series.

3. Any minimal resolution has some linear complexes as its building blocks. In particular, the *linear strand* is the linear part of a minimal resolution. We refer to Chapter 7 of [26], which contains several results and applications of this topic to algebraic geometry. Observe that a minimal linear resolution corresponds to its own linear strand.

4. The well-known Eagon–Reiner theorem (see Theorem 3.5.5 below) says that I has a linear resolution if and only if the Alexander dual $I^\vee$ is Cohen–Macaulay.

5. Graph theory is related to 2-linear resolutions by the famous result by Fröberg (Theorem 3.4.10) stating that the edge ideal of a graph G has a linear resolution if and only if the complement of G is chordal (see Definition 2.1.14).

6. Expressing a polynomial $f \in \mathbb{R}[x_1,\ldots,x_n]$ as a sum of squares provides a certificate of non-negativity for f. Sums of squares are a hot topic of interest in real algebraic geometry and optimization (among others), with a long and rich history. In [7] it was shown that if X is a non-degenerate totally-real projective variety, then every non-negative quadratic form on X is a sum of squares, modulo the defining ideal of X, if and only if X is a 2-regular variety. See also the survey [6] for a gentle introduction and a more general discussion, relating also to the fifth item of this list.

Among the many influential papers concerning linear resolutions, we mention that of Eisenbud and Goto [28] and the classical work by Steurich [69]. More recent directions of research involve families of ideals such that every product of elements in the family has a linear resolution (see for instance [11] or [14]). We refer to Chapter 7 of [26] and Section 17 of [63] for additional information and references.

Having a linear resolution implies in particular that there is only one non-zero Betti number in column 0, that is, all the elements in a minimal system of homogeneous generators of M have the same degree. This motivates the following definition.

Definition 3.5.2. A monomial ideal I is *equigenerated* if all the elements of $G(I)$ have the same degree.

In order to state the next famous result we recall what the Alexander dual of a squarefree monomial ideal is.

Definition 3.5.3. For any subset $\sigma \subseteq [n] := \{1,\dots,n\}$, denote $x_\sigma := \prod_{i\in\sigma} x_i$ and $\mathfrak{m}_\sigma := (x_i \mid i \in \sigma)$. Given a squarefree monomial ideal $I = (x_{\sigma_1},\dots,x_{\sigma_s}) \subseteq S = \mathbb{K}[x_1,\dots,x_n]$, the *Alexander dual of* I is the ideal

$$I^\vee := \mathfrak{m}_{\sigma_1} \cap \cdots \cap \mathfrak{m}_{\sigma_s}.$$

Equivalently, $I^\vee$ is the ideal generated by the monomials with non-trivial common divisor with every monomial in I (equivalently, in $G(I)$). In terms on Stanley–Reisner ideals, one can relate the Alexander dual of an ideal to that of a simplicial complex (see Definition 2.3.12) by noticing that $I_{\Delta^\vee} = (I_\Delta)^\vee$. For other equivalent descriptions of $I^\vee$ and additional information, see for instance Section 62 of [63] or Section 1.5.2 of [50].

We also recall the notion of "Cohen–Macaulay ring", ubiquitous in commutative algebra, without specifying all the details involved because anyway it is just instrumental for our purposes.

Definition 3.5.4. For a Noetherian local ring R with maximal ideal $\mathfrak{m}$, a finitely generated R-module $M \neq 0$ is a *Cohen–Macaulay module* if $\operatorname{depth}(M) = \dim(M)$. If R is any Notherian ring, M is called a *Cohen–Macaualy module* if the localization $M_\mathfrak{m}$ is Cohen–Macaulay as defined in the local case, for any maximal ideal $\mathfrak{m}$ in the support of M. If R is Cohen–Macaualy as an R-module, then we say that R is a *Cohen–Macaulay ring*.

We only use results about Cohen–Macaulay rings in Publication II, and even there it is actually in an indirect way, thanks to the following well-known result:

Theorem 3.5.5 (Eagon–Reiner, [24]). *For a squarefree monomial ideal $I \subseteq S$, the following are equivalent:*

- *I has a linear resolution;*
- *$S/I^\vee$ is Cohen–Macaulay.*

For a proof, see for instance Corollary 62.9 of [63].

3.5.1 Ideals with linear quotients

Recall that, given an ideal $I \subseteq S$ and a polynomial $f \in S$, one defines the colon ideal

$$I : f := \{g \in S \mid fg \in I\}.$$

Definition 3.5.6. Let $I \subseteq S = \mathbb{K}[x_1,\dots,x_n]$ be a homogeneous ideal. We say that I has *linear quotients* if there exists a system of homogeneous generators $f_1, f_2, \dots, f_m$ of I such that the colon ideal $(f_1,\dots,f_{i-1}) : f_i$ is generated by linear forms for all i.

Example 3.5.7. *The order in which we take the generators matters.* In the ring $\mathbb{K}[x_1,\dots,x_5]$, take $I = (x_1x_2x_3, x_3x_4x_5, x_2x_3x_4)$. If we took the generators in the given order, we would get in particular the colon ideal $(x_1x_2x_3) : x_3x_4x_5 = (x_1x_2)$, whose only generator is quadratic. On the other hand, if we order the generators as

$$f_1 = x_1x_2x_3, \quad f_2 = x_2x_3x_4, \quad f_3 = x_3x_4x_5,$$

then we get $(f_1) : f_2 = (x_1)$ and $(f_1, f_2) : f_3 = (x_2)$. So indeed I has linear quotients.

The following two results are respectively Proposition 8.2.1 and Corollary 8.2.2 in [50]:

Proposition 3.5.8. *Let $I \subseteq S = \mathbb{K}[x_1,\dots,x_n]$ be a homogeneous ideal equigenerated in degree d and with linear quotients. Then I has a d-linear resolution.*

Corollary 3.5.9. *Let $I \subseteq S$ be an equigenerated homogeneous ideal with linear quotients. For $k = 1,\dots,m$, let r_k be the number of generators of $(f_1,\dots,f_{k-1}) : f_k$. Then*

$$\beta_i(I) = \sum_{k=1}^{m} \binom{r_k}{i}.$$

In particular it follows that $\operatorname{proj\,dim}(I) = \max\{r_1, r_2, \dots, r_m\}$.

With the notation above, notice in particular that $r_1 = 0$, and this only affects the Betti number β_0, with a summand $\binom{r_1}{0} = 1$.

Example 3.5.10. Continuing Example 3.5.7, with the notation of Corollary 3.5.9 we get $r_1 = 0$, $r_2 = 1$ and $r_3 = 1$. So the projective dimension of I is 1 and we have

$$\beta_0(I) = \sum_{k=1}^{3} \binom{r_k}{0} = 1 + 1 + 1 = 3, \qquad \beta_1(I) = \sum_{k=1}^{3} \binom{r_k}{1} = 0 + 1 + 1 = 2.$$

The following is a combination of Lemma 8.2.3 and Corollary 8.2.4 in [50]:

Lemma 3.5.11. *Let I be a monomial ideal and write $G(I) = \{u_1, u_2, \dots, u_m\}$. The following are equivalent:*

- *I has linear quotients with respect to $u_1, u_2, \dots, u_m$;*
- *for all $j < i$, there exist an integer $k < i$ and an integer ℓ such that*

$$\frac{u_k}{\gcd(u_k, u_i)} = x_\ell \quad \text{and} \quad x_\ell \quad \text{divides} \quad \frac{u_j}{\gcd(u_j, u_i)}.$$

Assume now that I is squarefree and let $F_i := \operatorname{supp}(u_i)$ for all $i \in \{1,\dots,m\}$. The equivalent conditions above hold if and only if

- *for all i and all $j < i$, there exist an integer $\ell \in F_j \setminus F_i$ and an integer $k < i$ such that $F_k \setminus F_i = \{\ell\}$.*

Ideals with linear quotients are intimately related to the concept of shellability for simplicial complexes (see Definition 2.3.5) by the following well-known equivalence, which is Proposition 8.2.5 in [50]:

Proposition 3.5.12. *Let Δ be a simplicial complex. The following conditions are equivalent:*

- *I_Δ has linear quotients with respect to a monomial system of generators;*
- *the Alexander dual $\Delta^\vee$ of Δ is shellable.*

More precisely, if $G(I_\Delta) = \{u_1, u_2, \dots, u_m\}$ and $F_i := \operatorname{supp}(u_i)$ for $i \in \{1, \dots, m\}$, then I_Δ has linear quotients with respect to $u_1, \dots, u_m$ if and only if $\overline{F_1}, \overline{F_2}, \dots, \overline{F_m}$ is a shelling order of $\Delta^\vee$, where $\overline{F}$ denotes $[n] \setminus F$.

3.6 Boij–Söderberg theory

Denote by $\beta(M)$ the Betti table of a finitely generated graded S-module M. The problem of classifying the possible Betti tables attainable by the modules of the form S/I, for some homogeneous ideal I, has been open for many decades, and it is still apparently out of reach. Boij–Söderberg theory, born in 2006, constitutes the main major breakthrough since the problem was first raised. The main merits of Boij and Söderberg were:

- considering the Betti tables of *all* finitely generated graded S-modules M and not only those of the form S/I, observing that there is a semigroup structure on the set of all Betti tables:

$$\beta(M_1) + \beta(M_2) = \beta(M_1 \oplus M_2);$$

- considering Betti tables *up to rational multiples*, viewing them as elements in a cone inside an infinite-dimensional $\mathbb{Q}$-vector space.

The first preprint version of [8] appeared on the arXiv in 2006. There Boij and Söderberg first proposed their conjectures, describing a cone whose extremal rays are generated by "pure tables" (called "pure diagrams" in the literature) and stating that (1) there are indeed modules with Betti tables on those rays, and (2) any Betti table of a Cohen–Macaulay graded S-modules lies inside this cone. A good deal of research spanning a few months resulted in proofs by Eisenbud, Fløystad and Weyman [27] and Eisenbud and Schreyer [29] of the original conjectures. Inspired by these, Boij and Söderberg went on and were able to extend the results to the case of graded S-modules that are not necessarily Cohen–Macaulay, in [9]. For a detailed early survey of the theory and a general introduction, we refer to [37].

Definition 3.6.1. The minimal resolution of a finitely generated graded S-module M is called a *pure resolution of type* $\mathbf{s} = (s_0, s_1, \ldots, s_p)$ if it has the form

$$0 \to S(-s_p)^{\beta_{p,s_p}} \to \cdots \to S(-s_0)^{\beta_{0,s_0}},$$

that is, if there is at most one non-zero entry in each column of the Betti table.

By Proposition 3.2.12, one necessarily has $s_0 < s_1 < \cdots < s_p$: now the smallest shift in each homological position is the *only* shift. Notice that linear resolutions are a special case of pure resolutions, attained when $s_i = s_0 + i$ for all $i \in \{1, \ldots, p\}$.

Definition 3.6.2. A *degree sequence* is a finite sequence of increasing non-negative integers. Given a degree sequence $\mathbf{s} = (s_0, \ldots, s_p)$, we define the *pure table associated to* $\mathbf{s}$, denoted $\pi(\mathbf{s})$, as the table with entry

$$\pi(\mathbf{s})_{i,j} := \begin{cases} \prod_{k \neq 0,i} \left| \frac{s_k - s_0}{s_k - s_i} \right| & \text{if } i \in \{0, \ldots, p\} \text{ and } j = s_i, \\ 0 & \text{otherwise} \end{cases}$$

in column i and row j. We define a partial order on the set of degree sequences by setting

$$(s_0, \ldots, s_p) \geq (m_0, \ldots, m_t)$$

if $p \leq t$ and $s_i \geq m_i$ for all $i \in \{0, \ldots, p\}$.

Notice that the definition of the partial order is very natural: one could extend $\mathbf{s} = (s_0, \ldots, s_p)$ to the longer sequence $(s_0, \ldots, s_n)$, with $s_i = +\infty$ for $i \in \{p+1, \ldots, n\}$.

Remark 3.6.3. It is more customary to use the letter d instead of $\mathbf{s}$ for degree sequences. However, Boij–Söderberg theory appears in this thesis (in Publication I) in relation to the degrees of the vertices of some graphs, so the notational choice is meant to avoid any possible confusion. In fact, in Publication I we use the letter $\mathbf{n}$ to denote a degree sequence, but here that would result in a notationally hideous n_n, where the subscript is the number of variables in $S = \mathbb{K}[x_1, \ldots, x_n]$. Notice moreover that "degree sequence" has a different meaning in graph theory, but we never make use of that concept in this thesis.

The fundamental result in Boij–Söderberg theory of interest to us is the following. (See for instance [37], Theorem 5.1.)

Theorem 3.6.4. *Let M be a finitely generated graded S-module. There is a strictly increasing chain $\mathbf{s}_1 < \cdots < \mathbf{s}_t$ of degree sequences, and there are numbers $c_{\mathbf{s}_1}, \ldots, c_{\mathbf{s}_t} \in \mathbb{Q}_{\geq 0}$ such that the Betti table $\beta(M)$ of M can be written as*

$$\beta(M) = c_{\mathbf{s}_1} \pi(\mathbf{s}_1) + \cdots + c_{\mathbf{s}_t} \pi(\mathbf{s}_t).$$

For Cohen–Macaulay modules the decomposition above is unique. With the appropriate notation and some assumption on the "windows" where one can select the degree sequences, the decomposition above is unique (see [9]) also for more general modules. Of course as it is written above one could have some redundant degree sequence with coefficient zero. This is relevant to us for Publication I.

Definition 3.6.5. We call the decomposition in the theorem above a *Boij–Söderberg decomposition of* M, and we refer to the non-negative rational numbers $c_{\mathbf{s}_1},\dots,c_{\mathbf{s}_p}$ as *Boij–Söderberg coefficients of* M.

There is an easy algorithm to compute Boij–Söderberg decompositions:

1. Let M be a module with Betti table $\beta := \beta(M)$.

2. For each non-zero column i of β one picks the position s_i of the first non-zero entry and constructs a degree sequence $\mathbf{s} = (s_0, s_1, \dots, s_n)$. Let c be the largest non-negative rational number such that all the entries of $\beta - c\pi(\mathbf{s})$ are non-negative.

3. If $\beta - c\pi(\mathbf{s})$ is non-zero, replace β by $\beta - c\pi(\mathbf{s})$ and repeat the step above. Otherwise the algorithm has ended.

See [30] or [37] for additional details. Next we illustrate the algorithm with an example.

Example 3.6.6. Consider in $S = \mathbb{K}[x,y,z]$ the ideal $I = (x^2, xy, xz^2)$. The Betti table $\beta = \beta(S/I)$ of the S-module S/I is

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 2 & 1. \end{array}$$

The first degree sequence to consider is $(0,2,3,5)$, corresponding to the boxed entries:

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \boxed{1} & - & - & - \\ 1 & - & \boxed{2} & \boxed{1} & - \\ 2 & - & 1 & 2 & \boxed{1.} \end{array}$$

The largest rational number c such that all the entries of the table

$$\beta - c\pi(0,2,3,5) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix} - c \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

are non-negative is $c = \frac{1}{5}$. So next we need to consider

$$\beta' := \beta - \frac{1}{5}\pi(0,2,3,5) = \begin{pmatrix} \boxed{\frac{4}{5}} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 1 & \boxed{2} & \boxed{\frac{4}{5}} \end{pmatrix},$$

and the degree sequence corresponding to the boxed entries is $(0,2,4,5)$. The largest rational number c' such that all the entries of the table

$$\beta' - c'\pi(0,2,4,5) = \begin{pmatrix} \frac{4}{5} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{4}{5} \end{pmatrix} - c' \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 15 & 8 \end{pmatrix}$$

are non-negative is $c' = \frac{1}{10}$. The next table is then

$$\beta'' := \beta' - \frac{1}{10}\pi(0,2,4,5) = \begin{pmatrix} \boxed{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & \boxed{\frac{1}{2}} & 0 \end{pmatrix},$$

and the degree sequence corresponding to the boxed entries is $(0,3,4)$. The largest rational number c'' such that all the entries of the table

$$\beta'' - c''\pi(0,3,4) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \end{pmatrix} - c'' \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 3 & 0 \end{pmatrix}$$

are non-negative is $c'' = \frac{1}{6}$. The last table is then

$$\beta'' - \frac{1}{6}\pi(0,3,4) = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \end{pmatrix} = \frac{1}{3}\pi(0,3).$$

Putting all of this together one gets the Boij–Söderberg decomposition

$$\beta(S/I) = \frac{1}{5}\pi(0,2,3,5) + \frac{1}{10}\pi(0,2,4,5) + \frac{1}{6}\pi(0,3,4) + \frac{1}{3}\pi(0,3).$$

Similarly to what happens with minimal resolutions, *even though there is an algorithm to compute Boij–Söderberg decompositions, one does not know in general what outcome to expect*. To the best of our knowledge, there is no universal understanding of the Boij–Söderberg coefficients in terms of other invariants of the module. Over the last decade, several papers have been focusing on the problem of shedding light on the combinatorial meaning of Boij–Söderberg decompositions and coefficients (see for instance [16, 17, 32, 41, 61]), sometimes providing explicit formulas for the Boij–Söderberg coefficients of modules in some special class, like edge ideals of graphs with chordal complement, or more general modules with linear resolutions. Publication I is one such paper. Some useful results in [32], which we recall in the following section, are of particular interest to us.

3.6.1 Boij–Söderberg theory for 2-linear resolutions

Engström and Stamps proved very explicit formulas in [32] showing how Betti numbers and Boij–Söderberg coefficients are related for modules with 2-linear resolutions.

Definition 3.6.7. If a finitely generated graded S-module M has Betti table of the form

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & \dots & p \\ \hline 0 & m & 0 & 0 & \cdots & 0 \\ 1 & 0 & \beta_{1,2} & \beta_{2,3} & \cdots & \beta_{p,p+1} \end{array}$$

then we call $(\beta_{1,2},\dots,\beta_{p,p+1},0,\dots,0)$, by adding as many 0's as necessary to have n entries, the *Betti vector* of M and denote it by $\omega(M)$.

Notice that we are allowed to add 0's so that $\omega(M)$ has exactly n entries by Hilbert's Syzygy Theorem 3.2.17. Moreover, if an ideal $I \subseteq S$ has a 2-linear resolution, indeed the Betti table $\beta(S/I)$ of the quotient is of the form

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & \dots & p \\ \hline 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & \beta_{1,2} & \beta_{2,3} & \cdots & \beta_{p,p+1} \end{array}$$

Because in this section and in Publication I all the Betti tables can have non-zero entries only in this unambiguous way, we denote Betti tables like the one above as

$$\beta(S/I) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \beta_{1,2} & \beta_{2,3} & \cdots & \beta_{p,p+1} \end{pmatrix},$$

omitting the row and column indices. By Theorem 3.6.4, any Betti table is the weighted average of certain pure tables $\pi(\mathbf{s})$. For Betti tables such as the above, the only possible pure tables are those of the form $\pi(0,2,3,4,\dots,j,j+1)$, namely such that $\beta_{0,0} \neq 0$ and the other non-zero Betti numbers are all located after the first entry in the second row (i.e., the row with index 1) of the Betti table. For instance,

$$\pi(0,2) = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \end{pmatrix} \qquad \text{and} \qquad \pi(0,2,3) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 2 & 0 & \cdots \end{pmatrix}.$$

Using the same notation as in [32], we denote by π_j the Betti vector of the pure table $\pi(0,2,3,\dots,j,j+1)$. For the tables $\pi(0,2)$ and $\pi(0,2,3)$ above, we have

$$\pi_1 = (1,0,\dots) \qquad \text{and} \qquad \pi_2 = (3,2,0\dots).$$

Slightly larger examples are

$$\pi_7 = (28,112,210,224,140,48,7,0,0,\dots),$$

$$\pi_8 = (36,168,378,504,420,216,63,8,0,0,\dots),$$

$$\pi_9 = (45,240,630,1008,1050,720,315,80,9,0,0,\dots).$$

By Theorem 3.6.4, specified to the case of modules with a Betti table such as that in Definition 3.6.7, we can write

$$\beta(M) = \sum_{j=1}^{n} c_j \pi_j$$

for some unique $c_j \in \mathbb{Q}_{\geq 0}$.

Definition 3.6.8. With the same assumptions and notation as above, we call the coefficient c_j of π_j the *j-th Boij–Söderberg coefficient of M*.

Lemma 3.6.9 (Lemma 3.1 and Theorem 3.2 of [32])**.** *Let M be an S-module with a Betti table as in the definition above, and let $c = (c_1, \dots, c_n)$ be the vector with the Boij–Söderberg coefficients of M. Let Ω be the square matrix of order n whose (i,j)-entry is $j\binom{i+1}{j+1}$. Then Ω is invertible and the inverse Ω^{-1} has (i,j)-entry equal to $(-1)^{i-j}\frac{1}{i}\binom{i+1}{j+1}$. Moreover, we have $c = \omega(M)\Omega^{-1}$.*

Definition 3.6.10. A sequence of integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ such that

$$t \geq \frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \cdots \geq \frac{\lambda_n}{n} \geq 0$$

is called an *anti-lecture hall composition of length n bounded above by t*.

For general information about anti-lecture hall compositions, see [18], where they were introduced, and [66]. Engström and Stamps ([32], Section 4) show how to associate a unique anti-lecture hall composition with $t = 1$ and $\lambda_1 = 1$ to an ideal with a 2-linear resolution.

Lemma 3.6.11 (Proposition 4.11 of [32])**.** *Let $I \subseteq S$ be an ideal with a 2-linear resolution, and denote by $\lambda = (\lambda_1, \dots, \lambda_n)$ the anti-lecture hall composition associated to I. Let Ψ be the invertible $n \times n$ matrix with (i,j)-entry equal to $\Psi_{ij} = \binom{i-1}{j-1}$. Then we have $\lambda = \omega(S/I)\Psi^{-1}$.*

3.7 Polarizations

Polarization is a technique that allows one to get a squarefree monomial ideal starting from an arbitrary monomial ideal, while preserving some properties. In particular the Betti numbers stay the same (Proposition 3.7.6).

The classical notion of polarization was originally used by Hartshorne in his proof of the connectedness of the Hilbert scheme (see Chapter 4 of Hartshorne's paper [48]). Later it became a standard tool in commutative algebra thanks to the work of Hochster. More recently, new versions of "polarization" have been introduced, such as the b-polarization for strongly stable ideals (see [74]). Letterplace ideals (see [38, 39]) constitute polarizations for Artinian monomial ideals, and shifting (see Chapter 11 of [50]) can also be seen as a form of polarization.

In this context one often starts from a monomial ideal $I \subset S$ and gets a squarefree monomial ideal $\widetilde{I}$ in a larger polynomial ring $\widetilde{S}$, and one can go back from $\widetilde{S}/\widetilde{I}$ to S/I by taking a quotient modulo a regular sequence (see Section 3.7.1 below) consisting of variable differences. After a surprisingly long time, a systematic study of this kind of situation has begun in the recent paper [2], in the case of powers of graded maximal ideals. In the remaining sections of this chapter we introduce the notions involved in this, necessary for Publication IV and part of Publication II.

3.7.1 Regular sequences

Let R be a commutative ring with identity and let M be an R-module.

Definition 3.7.1. An element $a \in R$ is *M-regular* if the only element $m \in M$ such that $am = 0$ is $m = 0$. A sequence of elements $a_1, \dots, a_s \in R$ is an *M-regular sequence* if the following conditions hold:

1. a_i is $M/(a_1, \dots, a_{i-1})M$-regular for all $i \in \{1, \dots, s\}$;
2. $M/(a_1, \dots, a_s)M \neq 0$.

The second condition is often automatically satisfied, for instance in the context of graded rings and modules, if all the elements $a_1, \dots, a_s$ are of positive degree, as it will be the case for most sequences considered in this theses, in Publications II and IV: the regular sequences there will consist of differences of variables.

In a very "explicit" context, say R is a polynomial ring and $M = R/I$ is the quotient by some ideal of R, it is possible to check whether a given sequence of polynomials is M-regular: several computer algebra systems have algorithms to do that. It is however quite difficult in general to construct regular sequences. The most famous example in the literature of a regular sequence, consisting of specific variable differences, is described in the next section.

The following result is very well known. See for instance Theorem 2.1.3 of [12] for a proof.

Theorem 3.7.2. *Let R be a Notherian ring and let $a_1, \dots, a_s$ be an R-regular sequence. If R is a Cohen–Macaulay ring, then $R/(a_1, \dots, a_s)$ is a Cohen–Macaualy ring.*

The following result is also folklore, see Theorem 20.3 of [63] for a proof.

Proposition 3.7.3. *Let R be a positively graded ring and let M be a finitely generated graded R-module. Let $a \in R$ be homogeneous and of positive degree, and assume that a is both R-regular and M-regular. Let $\mathbf{F}$ be a minimal graded free resolution of M over R. Then $\mathbf{F} \otimes_R R/(a)$ is a minimal graded free resolution of M/aM over $R/(a)$. In particular, the graded Betti numbers of M over R are the same as those of M/aM over $R/(a)$; that is, for each i and j, we have*

$$\beta_{i,j}^{R}(M) = \beta_{i,j}^{R/(a)}(M/aM).$$

Example 3.7.4. Let $S = \mathbb{K}[x_1, x_2, x_3, x_4, x_5, x_6]$ and let I be the ideal generated by the three 2×2 minors of the matrix

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix},$$

that is, $I = (f_1 := x_1x_5 - x_2x_4,\ f_2 := x_1x_6 - x_3x_4,\ f_3 := x_2x_6 - x_3x_5)$. Then S/I has minimal resolution

$$0 \longrightarrow S(-3)^2 \xrightarrow{\begin{bmatrix} x_3 & x_6 \\ -x_2 & -x_5 \\ x_1 & x_4 \end{bmatrix}} S(-2)^3 \xrightarrow{\begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}} S \longrightarrow S/I \longrightarrow 0.$$

If we set $a := x_4 - x_2$ and $M := S/I$, then a is both S-regular and M-regular. The minimal resolution of M/aM as an $S/(a)$-module is

$$0 \longrightarrow \big(S/(a)\big)(-3)^2 \xrightarrow{\begin{bmatrix} \overline{x_3} & \overline{x_6} \\ \overline{-x_2} & \overline{-x_5} \\ \overline{x_1} & \overline{x_4} \end{bmatrix}} \big(S/(a)\big)(-2)^3 \xrightarrow{\begin{bmatrix} \overline{f_1} & \overline{f_2} & \overline{f_3} \end{bmatrix}} S/(a) \longrightarrow M/aM \longrightarrow 0,$$

where the bar denotes the equivalence class in the quotient. The graded Betti numbers are indeed the same.

By iterating Proposition 3.7.3, one gets in particular the following:

Corollary 3.7.5. *Let I be a homogeneous ideal of $S = \mathbb{K}[x_1,\dots,x_n]$ and denote $M := S/I$. Let $a_1,\dots,a_s$ be an M-regular sequence of homogeneous elements of S of positive degree and define the ideal $J := (a_1,\dots,a_s)$. Then, for all i and j,*

$$\beta^S_{i,j}(M) = \beta^{S/J}_{i,j}(M/JM).$$

3.7.2 Separations and polarizations

In the classical polarization one starts from an arbitrary monomial ideal and gets a squarefree one, replacing the monomial $x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$, in the ring $\mathbb{K}[x_i \mid i = 1,\dots,n]$, by the monomial

$$(x_{1,1}x_{1,2}\dots x_{1,a_1})(x_{2,1}x_{2,2}\dots x_{2,a_2})\dots(x_{n,1}x_{n,2}\dots x_{n,a_n}),$$

in the larger ring $\mathbb{K}[x_{i,1},\dots,x_{i,a_i} \mid i = 1,\dots,n]$. Using x, y and z instead of x_1, x_2 and x_3, consider for instance the ideal

$$I := (x^4y^2, x^2y^3z, xz^2) \quad \subset \quad S := \mathbb{K}[x,y,z].$$

With the classical polarization one would get

$$\begin{aligned} I^{\mathrm{pol}} &:= (x_1x_2x_3x_4y_1y_2,\ x_1x_2y_1y_2y_3z_1,\ x_1z_1z_2) \\ &\subset S^{\mathrm{pol}} := \mathbb{K}[x_1,x_2,x_3,x_4,\ y_1,y_2,y_3,\ z_1,z_2]. \end{aligned}$$

One can then recover S/I by taking the quotient of $S^{\mathrm{pol}}/I^{\mathrm{pol}}$ modulo the variable differences

$$x_1 - x_2, \quad x_2 - x_3, \quad x_3 - x_4, \quad y_1 - y_2, \quad y_2 - y_3, \quad z_1 - z_2$$

so that in the quotient one has $x_1 = x_2 = x_3 = x_4$, $y_1 = y_2 = y_3$ and $z_1 = z_2$. The variable differences above constitute an $S^{\mathrm{pol}}/I^{\mathrm{pol}}$-regular sequence. So in particular the homological invariants of $S^{\mathrm{pol}}/I^{\mathrm{pol}}$ and S/I are the same, as a special case of Proposition 3.7.3:

Proposition 3.7.6. *The graded Betti numbers of S/I and $S^{\mathrm{pol}}/I^{\mathrm{pol}}$ are equal.*

And moreover, I^{pol} being squarefree, one has the advantage that the Stanley–Reisner machinery can be applied to I^{pol}. We refer to Section 21 of [63] for additional information on the classical polarization.

This classical construction is not the only way to turn an arbitrary monomial ideal into a squarefree one. A small example where one has different ways to polarize an ideal is given by

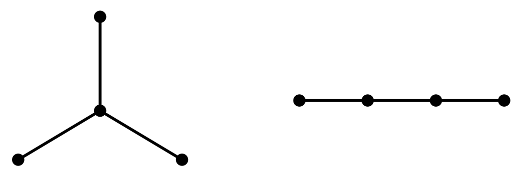
$$I := (x,y,z)^2 = (x^2, xy, xz, y^2, yz, z^2)$$

inside $S := \mathbb{K}[x,y,z]$. One may polarize I in exactly two non-isomorphic ways (up to permutation of the variables):

$$I_1 := (x_1x_2,\ x_1y_1,\ x_1z_1,\ y_1y_2,\ y_1z_1,\ z_1z_2),$$

$$I_2 := (x_1x_2,\ x_1y_2,\ x_1z_2,\ y_1y_2,\ y_1z_2,\ z_1z_2)$$

in $\widetilde{S} := \mathbb{K}[x_i, y_i, z_i \mid i = 1,2]$. Notice that the first is the classical polarization. In both cases the differences $x_1 - x_2$, $y_1 - y_2$ and $z_1 - z_3$ form a regular sequence, and taking the quotient modulo that regular sequence yields S/I. The reason behind the existence of these two distinct polarizations is that the polarizations of $(x_1,\dots,x_n)^2$ are in bijection with the unlabeled trees on $n+1$ vertices, as shown in [2] and explained in the following section. For $n = 3$, there are two unlabeled trees on $n+1$ vertices, and it turns out that I_1 above is associated to the tree on the left and I_2 to the one on the right, with respect to the bijection in [2]:



The goal with the setup introduced next, first appeared in [39], is to generalize the classical polarization.

Definition 3.7.7. Let $p\colon B' \to B$ be a surjection of finite sets with $|B'| = |B|+1$. Let b_1 and b_2 be two distinct elements of B' such that $p(b_1) = p(b_2)$. Denote for short $\mathbb{K}[x_B] := \mathbb{K}[x_i \mid i \in B]$ and $\mathbb{K}[x'_B] := \mathbb{K}[x_i \mid i \in B']$. Let I be a monomial ideal in $\mathbb{K}[x_B]$ and J a monomial ideal in $\mathbb{K}[x_{B'}]$. We say that J is a *separation of* I if the following conditions hold:

1. The ideal I is the image of J by the map $\mathbb{K}[x_{B'}] \to \mathbb{K}[x_B]$ induced by p.

2. Both the variables x_{b_1} and x_{b_2} occur in some minimal generators of J (usually in distinct generators).

3. The variable difference $x_{b_1} - x_{b_2}$ is a $\mathbb{K}[x_{B'}]/J$-regular element.

In general, if $p\colon B' \to B$ is a surjection of finite sets and $I \subseteq \mathbb{K}[x_B]$ and $J \subseteq \mathbb{K}[x_{B'}]$ are monomial ideals such that J is obtained by a succession of separations of I, we also call J a *separation of* I. If J is squarefree and a separation of I, then we say that J is a *polarization of* I.

The recent work [2] initiates the systematic study of polarizations in the connotation above, considering the polarizations of $(x_1,\dots,x_n)^m$, focusing in particular on the cases $n=3$ (with arbitrary m) and $m=2$ (with arbitrary n).

3.7.3 Polarizations of $(x_1,\dots,x_n)^2$ and tree ideals

The results presented here are taken from Section 6 of the fourth version of [2].

Definition 3.7.8. Let $I \subseteq S$ and $J \subseteq R$ be monomial ideals inside the polynomial rings S and R. We say that I and J are *isomorphic as monomial ideals* if there is a bijection from the set of variables of S to that of R that induces a bijection from $G(I)$ to $G(J)$.

Let T be a directed tree. Denote by V the set of vertices and by E the set of edges of T. For $e \in E$ and $v \in V$, we define

$$e_{\mathrm{to}}(v) := \begin{cases} 1 & \text{if } e \text{ points to } v, \\ 0 & \text{otherwise.} \end{cases}$$

For a field $\mathbb{K}$, consider the polynomial ring $S = \mathbb{K}[x_{e,0}, x_{e,1} \mid e \in E]$ that has two variables for each edge of T. Recall that, given two vertices v and w in T, we denote by vTw the only path in T linking v and w.

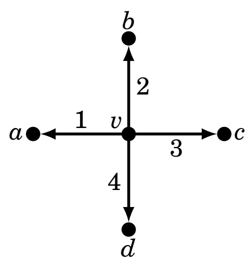
Definition 3.7.9. For a pair of vertices v,w in V, consider the path vTw. Let e and f be the edges on vTw incident to v and to w, respectively. Define the monomial

$$m_{v,w} := x_{e,e_{\mathrm{to}}(v)} x_{f,f_{\mathrm{to}}(w)}.$$

The *tree ideal of* T is the ideal of S

$$I(T) := (m_{v,w} \mid v,w \in V).$$

Example 3.7.10. Consider the tree T as in the following picture, the star with four edges pointing outwards:



For instance the path aTv consists only of the edge 1, so that both e and f in the definition of $m_{a,v}$ above are equal to the edge 1, and then $m_{a,v} = x_{1,1}x_{1,0}$. The tree ideal $I(T)$ is generated by the monomials

$$\begin{array}{llll} m_{a,b} = x_{1,1}x_{2,1} & m_{a,c} = x_{1,1}x_{3,1} & m_{a,d} = x_{1,1}x_{4,1} & m_{a,v} = x_{1,1}x_{1,0} \\ m_{b,c} = x_{2,1}x_{3,1} & m_{b,d} = x_{2,1}x_{4,1} & m_{b,v} = x_{2,1}x_{2,0} & m_{c,d} = x_{3,1}x_{4,1} \\ & m_{c,v} = x_{3,1}x_{3,0} & m_{d,v} = x_{4,1}x_{4,0}. & \end{array}$$

Consider the polynomial rings

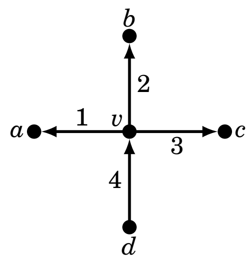
$$S = \mathbb{K}[x_{e,i} \mid e \in \{1,\dots,n\}, i \in \{0,1\}] \quad \text{and} \quad S' = \mathbb{K}[x_e \mid e \in \{1,\dots,n\}].$$

Theorem 3.7.11 ([2], Theorem 6.1). *The polarizations of $(x_1,\dots,x_n)^2 \subset S'$ are in bijection with the undirected trees on $n+1$ vertices. More precisely,*

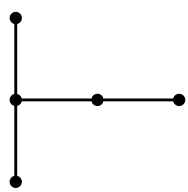
- *for any directed tree T on $n+1$ vertices, the ideal $I(T)$ is a polarization of $(x_1,\dots,x_n)^2 \subset S'$, and every polarization of $(x_1,\dots,x_n)^2$ is isomorphic as a monomial ideal to some $I(T)$;*
- *two polarizations $I(T)$ and $I(T')$ are isomorphic as monomial ideals if and only if the underlying undirected trees of T and T' are isomorphic.*

Remark 3.7.12. The star on $n+1$ vertices corresponds to (the monomial ideals isomorphic to) the classical polarization. Similarly, the path on $n+1$ vertices corresponds to the letterplace ideal $L(2,n)$ (see [39]).

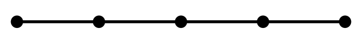
Example 3.7.13. Consider the following directed tree T':



The underlying undirected tree of T' is the same as the one of the directed tree T in Example 3.7.10. The only difference is in the direction of the edge 4. Indeed, $I(T)$ and $I(T')$ are isomorphic as monomial ideals, and $I(T')$ is obtained from $I(T)$ by simply swapping $x_{4,1}$ and $x_{4,0}$. On the other hand, any directed tree that has as underlying undirected tree one of



or



gives rise to a tree ideal that is not isomorphic to $I(T)$.

4. Summary of the Publications

This chapter contains summaries of Publications I, II, III and IV, respectively in Sections 4.1, 4.2, 4.3 and 4.4.

4.1 Explicit Boij–Söderberg theory of ideals from a graph isomorphism reduction

Question 4.1.1 (Graph Isomorphism Problem)**.** Are two given finite simple graphs G and H isomorphic? (See Definition 2.1.2.)

Of course this problem can be solved in a naïve way by checking all possible bijections $V_G \to V_H$ of the vertex sets of G and H until one finds an isomorphism or runs out of bijections. But this algorithm is *slow*. A more meaningful question is the following:

Question 4.1.2. Is there an algorithm that, given two graphs G and H, determines in polynomial time—that is, polynomial in the number of vertices of G and H—whether G and H are isomorphic?

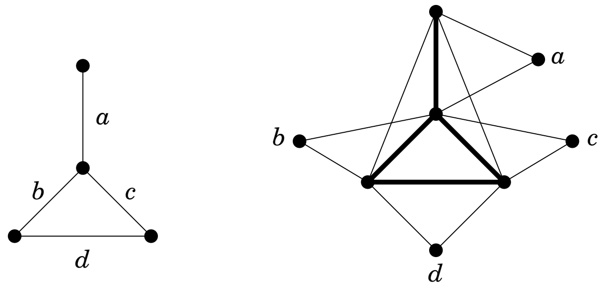
This is still an open problem, and it was the main motivation behind Publication I. In the origins of complexity theory, Booth and Lueker [10] introduced a construction that takes an arbitrary finite simple graph G and returns a *split* graph (see Definition 2.1.16):

Definition 4.1.3. Let $G = (V, E)$ be a finite simple graph. Let $BL(G)$ be the graph with vertex set $V \cup E$ and edge set

$$\Big\{\{u,v\} \mid u,v \in V\Big\} \cup \bigcup_{e\in E} \Big\{\{u,e\},\{v,e\} \mid e = \{u,v\}\Big\}.$$

That is, $BL(G)$ contains the complete graph on V and "new" vertices corresponding to the edges of G, with each $e \in E$ being adjacent to both ends of e. We call $BL(G)$ the *Booth–Lueker graph of* G.

Example 4.1.4. In the picture below a graph G is on the left and its Booth–Lueker graph on the right:



The elements of V are not labeled to keep the notation less heavy, and the copy of G inside $BL(G)$ is drawn with thicker lines for the sake of clarity.

Remark 4.1.5. Booth–Lueker graphs are split graphs: with the notation of the definition above, $BL(G)[V]$ is a clique and $BL(G)[E]$ is an independent set. Notice however that not all split graphs are Booth–Lueker graphs: for instance the one drawn in Example 2.1.17 cannot be a Booth–Lueker graph because it has a vertex of degree 0 (and one of degree 3). In Publication I it seemed psychologically helpful to refer to the clique $BL(G)[V]$ as the *left part of* $BL(G)$ and the independent set $BL(G)[E]$ as the *right part of* $BL(G)$. This is illustrated inside and immediately after Example 2.11 of Publication I.

Booth and Lueker proved the following:

Proposition 4.1.6 (Booth–Lueker, [10]). *For two finite simple graphs G and G', one has $G \cong G'$ if and only if $BL(G) \cong BL(G')$.*

The train of thought behind Publication I was:

1. The mapping $G \mapsto BL(G)$ is a polynomial-time reduction of the isomorphism problem for general finite simple graphs to that of split graphs, which are in particular chordal and with a complement that is also chordal.

2. Graphs G with a chordal complement are exactly those whose edge ideal I_G has a 2-linear resolution, by Theorem 3.4.10. Such resolutions and their Boij–Söderberg decompositions were studied by the first author of Publication I and Matthew Stamps in [32].

3. How much do the homological invariants (i.e., the Betti numbers, or equivalently the Boij–Söderberg coefficients) of the Booth–Lueker graph $BL(G)$ help in understanding the isomorphism class of G?

Unfortunately the answer to the last question is, not that much. We show that the Betti numbers (or equivalently, the Boij–Söderberg coefficients) of $BL(G)$ are equal to those of $BL(G')$ if and only if the degree vectors $\mathbf{d}_G$ and $\mathbf{d}_{G'}$ are equal (see Definition 2.1.1), and there are in general many non-isomorphic graphs sharing the same degree vector. However, we find our results particularly explicit, unlike others expressing the Boij–Söderberg coefficient of some module in terms of combinatorial data. To state the results precisely, recall that the Betti table of $S/I_{BL(G)}$ has the form

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & \dots & p \\ \hline 0 & \beta_{0,0} & 0 & 0 & \cdots & 0 \\ 1 & 0 & \beta_{1,2} & \beta_{2,3} & \cdots & \beta_{p,p+1} \end{array}$$

and that we denote by

$$\omega(BL(G)) := \omega(S/I_{BL(G)}) = (\beta_{1,2}, \beta_{2,3}, \dots, \beta_{p,p+1}, 0, \dots, 0)$$

the *Betti vector* of $S/I_{BL(G)}$. As explained in Section 3.6.1, it is equivalent to know

- the Betti vector of $S/I_{BL(G)}$,
- the Boij–Söderberg coefficients of $S/I_{BL(G)}$, or
- the anti-lecture hall composition associated to $BL(G)$.

Our main results in Publication I show that knowing any of the objects above is equivalent to knowing

- the degree vector of G.

We find explicit formulas to compute the Betti vector of $S/I_{BL(G)}$ knowing the degree vector of G, and vice versa we can compute the degree vector in terms of the (last non-zero) Betti numbers of $S/I_{BL(G)}$. The following is a combination of Propositions 3.1 and 3.4 of Publication I:

Proposition 4.1.7. *Let G be a finite simple graph with n vertices and m edges, and let $\mathbf{d}_G = (d_0, d_1, \dots, d_{n-1})^T$ be the degree vector of G. Let A be the $(n+m-1)\times n$ matrix defined by $A_{ij} = \binom{j+n-2}{i}$, and let v be the column $(n+m-1)$-vector defined by $v_i = \binom{n}{i+1}$. Then*

$$\omega(BL(G)) = A\mathbf{d}_G - v.$$

Let $\Delta(G)$ be the largest vertex degree in G. Let B be the square submatrix of A obtained by taking the first $\Delta(G)+1$ columns and the rows from $n-1$ to $n+\Delta(G)-1$. Then B is invertibe, with $(B^{-1})_{ij} = (-1)^{i+j}B_{ij}$, and

$$\mathbf{d}_G = B^{-1}(\beta_{n-1,n}+1,\ \beta_{n,n+1},\ \beta_{n+1,n+2},\ \dots,\ \beta_{n+\Delta(G)-1,n+\Delta(G)}).$$

(The "+1" in the first entry is not a typo.)

More interestingly, we give a very explicit description of the Boij–Söderberg coefficients of $S/I_{BL(G)}$, by applying Lemma 3.6.9 to the first part of Proposition 4.1.7, after some manipulations. The following is Theorem 3.5 of Publication I:

Theorem 4.1.8. *Let G be a graph with n vertices and $m \geq n$ edges. Denote by d_k the number of vertices of degree k in G. Then the j-th Boij–Söderberg coefficient of $S/I_{BL(G)}$ is*

$$c_j = \begin{cases} 0 & \text{if } j \leq n-2, \\ \frac{d_0}{n} & \text{if } j = n-1, \\ \frac{d_{j-n+1}}{j} + \frac{\sum_{i=j-n}^{n-1} d_i}{j(j+1)} & \text{if } n-1 < j \leq 2n-2, \\ 0 & \text{if } j > 2n-2. \end{cases}$$

As for the anti-lecture hall composition associated to $BL(G)$, the following is Proposition 3.7 of Publication I:

Proposition 4.1.9. *Let G be a finite simple graph with n vertices and $m \geq n-1$ edges. Denote by d_k the number of vertices of degree k in G, and denote by λ the anti-lecture hall composition associated to $BL(G)$. Then we have*

$$\lambda_j = \begin{cases} j & \text{for } j = 1, \dots, n, \\ d_{n-1} + d_{n-2} + \cdots + d_{j-n+1} & \text{for } j = n, \dots, 2n-2, \\ 0 & \text{for } j > 2n-2. \end{cases}$$

In particular, note that for $j = n$ we get $\lambda_n = d_{n-1} + d_{n-2} + \cdots + d_0 = n$.

The complement of a split graph is also split. The complement $\overline{BL(G)}$ of the Booth–Lueker graph of G also has edge ideal $I_{\overline{BL(G)}}$ with 2-linear resolution, to which one may apply the machinery of Section 3.6.1. And of course the complement is also such that $G \cong G'$ if and only if $\overline{BL(G)} \cong \overline{BL(G')}$. Therefore one may wonder how much the homological invariants of $\overline{BL(G)}$ help in distinguishing isomorphism class of G. It turns out that the answer is even worse than in the case of $BL(G)$. Knowing those invariants of $\overline{BL(G)}$ is equivalent to knowing the number of vertices and number of edges of G. We provide explicit formulas, analogously to the case of $BL(G)$, in Propositions 4.1 and 4.3 and Theorem 4.2 of Publication I.

4.2 Linearization of monomial ideals

The Booth–Lueker construction in Publication I can be interpreted as a map

$$\left\{ \begin{array}{c} \text{quadratic squarefree} \\ \text{monomial ideals} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{quadratic squarefree monomial} \\ \text{ideals with a linear resolution} \end{array} \right\}.$$

The initial goal of Publication II was to generalize as much as possible the Booth–Lueker construction seen from this viewpoint. As it turns out, a natural way to define such a generalization is as follows.

Definition 4.2.1. Let $I \subset \mathbb{K}[x_1, \dots, x_n]$ be a monomial ideal with minimal set of monomial generators $G(I) = \{f_1, \dots, f_m\}$, such that $f_1, \dots, f_m$ all have the same degree d. For all $i \in \{1, \dots, n\}$, denote by M_i the largest exponent with which x_i occurs in $G(I)$.

Let $M := \max\{M_1,\dots,M_n\}$. In the polynomial ring $R := \mathbb{K}[x_1,\dots,x_n,y_1,\dots,y_m]$ we define the *linearization of* I

$$\begin{aligned}\operatorname{Lin}(I) := &\left(x_1^{a_1}\cdots x_n^{a_n} \mid a_1+\cdots+a_n = d \text{ and } a_i \le M_i \text{ for all } i\right)\\ &+\left(f_j y_j/x_k \mid x_k \text{ divides } f_j,\ k=1,\dots,n,\ j=1,\dots,m\right)\end{aligned}$$

and the $*$*-linearization of* I

$$\begin{aligned}\operatorname{Lin}^*(I) := &\left(x_1^{a_1}\cdots x_n^{a_n} \mid a_1+\cdots+a_n = d \text{ and } a_i \le M \text{ for all } i\right)\\ &+\left(f_j y_j/x_k \mid x_k \text{ divides } f_j,\ k=1,\dots,n,\ j=1,\dots,m\right).\end{aligned}$$

We call the first summand the *complete part of* $\operatorname{Lin}(I)$ (respectively, of $\operatorname{Lin}^*(I)$) and the second summand the *last part of* $\operatorname{Lin}(I)$ (respectively, of $\operatorname{Lin}^*(I)$).

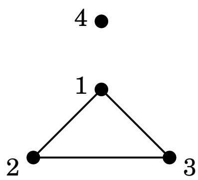
Remark 4.2.2. Notice that the last parts of $\operatorname{Lin}(I)$ and $\operatorname{Lin}^*(I)$ are always equal. The only difference is in the complete part: since $M_i \le M$ for each $i \in \{1,\dots,n\}$, we always have

$$\operatorname{Lin}(I) \subseteq \operatorname{Lin}^*(I),$$

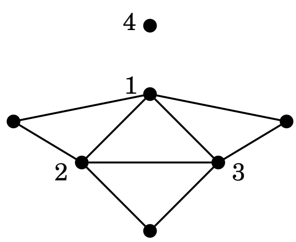
with equality only when $M_1 = M_2 = \cdots = M_n = M$. The reason for introducing $\operatorname{Lin}^*(I)$, which is "coarser" than $\operatorname{Lin}(I)$, is that given its symmetry it's easier to understand than $\operatorname{Lin}(I)$ in general. Moreover $\operatorname{Lin}^*(I)$ is a more direct generalization of the Booth–Lueker construction of Publication I, in the sense that

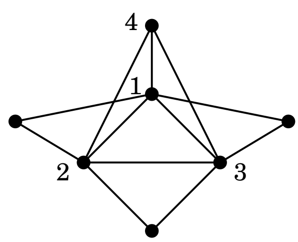
$$\operatorname{Lin}^*(I_G) = I_{BL(G)},$$

whereas with $\operatorname{Lin}(I_G)$ one would not take into account the vertices of G that do not have any neighbors: consider for instance the finite simple graph G



with edge ideal $I_G = \left(f_1 = x_1x_2,\ f_2 = x_1x_3,\ f_3 = x_2x_3\right)$. Then $\operatorname{Lin}(I_G)$ and $\operatorname{Lin}^*(I_G)$ are the edge ideals of the following graphs, respectively,





where we omit the labels of some vertices to lighten the notation.

4.2.1 Main results

Let $J \subseteq \mathbb{K}[z_1,\dots,z_r]$ be a monomial ideal with minimal system of monomial generators $G(J) = \{g_1,\dots,g_s\}$. Write $g_i = z_1^{a_{1i}} \cdots z_r^{a_{ri}}$ for all i. Fix a vector of non-negative integers $v = (v_1,\dots,v_r) \in \mathbb{N}^r$. We use v to "crop from above" the ideal J by keeping only the generators of J whose vector of exponents is componentwise at most as large as the vector v: that is, we define

$$J_{\le v} := (g_p \mid a_{pi} \le v_i \text{ for all } i = 1,\dots,r),$$

and we say that $J_{\le v}$ is obtained by *cropping J by v*. The following result (Proposition 2.9 of Publication II) is instrumental in the proof of the main property of $\mathrm{Lin}(I)$ and $\mathrm{Lin}^*(I)$, and it seemed interesting in its own right, especially as an analog for linear quotients of a general folklore result on resolutions (see Proposition 56.1 of [63]).

Proposition 4.2.3. *Let $J = (g_1,\dots,g_s) \subset \mathbb{K}[z_1,\dots,z_r]$ be a monomial ideal with linear quotients with respect to the given ordering of the generators. Fix $v \in \mathbb{N}^r$ and write $J_{\le v} = (g_{b_1},\dots,g_{b_t})$, where $b_1 < \cdots < b_t$ are the indices of the generators that survived the cropping. Then $J_{\le v}$ has linear quotients with respect to $g_{b_1},\dots,g_{b_t}$.*

Using the same notation as in the definition of linearization above, the main property is the following (Theorem 3.7 of II).

Theorem 4.2.4. *Assume that $f_1,\dots,f_m$ are in decreasing lexicographic order. List the generators of the complete part of $\mathrm{Lin}(I)$, and respectively of $\mathrm{Lin}^*(I)$, in decreasing lexicographic order. List the generators $\frac{f_j}{x_k} y_j$ of the last part first by increasing j, and secondly by increasing k. The ideals $\mathrm{Lin}(I)$ and $\mathrm{Lin}^*(I)$ have linear quotients with respect to the given ordering of the generators.*

This has an immediate consequence, by Proposition 3.5.8:

Corollary 4.2.5. *The ideals $\mathrm{Lin}(I)$ and $\mathrm{Lin}^*(I)$ have d-linear resolutions.*

We prove that also the radical $\sqrt{\mathrm{Lin}^*(I)}$ has linear quotients and compute its Betti numbers (see Theorem 3.18 and Corollary 3.19), where we consider the $*$-linearization to lighten the notation (which is still a bit heavy anyway).

Particular attention was given to the case when I is squarefree, which happens if and only if $\mathrm{Lin}(I)$ is squarefree, if and only if $\mathrm{Lin}^*(I)$ is squarefree. In this case we are able to give a (cumbersome but) explicit description of the Betti numbers of $\mathrm{Lin}^*(I)$. In particular, we answer one of the motivating questions: In Publication I we showed that the Betti numbers of $I_{BL(G)}$ only depend on the degrees of the vertices of G, so is there an analog to this in general for $\mathrm{Lin}^*(I)$? The answer is yes, and for $\mathrm{Lin}^*(I)$ it turns out that the Betti numbers depend only on how many generators of I (all of the same degree d) are divided by a common monomial of degree $d-1$. We introduce the following terminology:

Definition 4.2.6. Let $u = x_{i_1} \cdots x_{i_{d-1}}$, with $i_1 < \cdots < i_{d-1}$, be a squarefree monomial. We call u a $(d-1)$-*edge of* I if u divides some generator of I. The *multiplicity* of a $(d-1)$-edge u of I is the number

$$\operatorname{mult}(u) := \#\{f_i \in G(I) \mid u \text{ divides } f_i\},$$

where $G(I) = \{f_1, \dots, f_m\}$. We call a j-*cluster* a set of cardinality j consisting of generators of I that are divided by a same $(d-1)$-edge u.

Notice that a 1-edge, obtained for $d = 2$, is just a variable, and its multiplicity in the sense above corresponds to the graph-theoretic notion of degree. Notice moreover that the word "cluster" here does not have the same meaning as in Definition 2.1.4 and Publication III. The following is Corollary 4.10 in Publication II:

Corollary 4.2.7. *Let I be generated by m monomials of degree d as in the results above. Let $C_j \in \mathbb{N}$ be the number of maximal j-clusters, that is, j-clusters that are not contained in a $(j+1)$-cluster. Then*

$$\begin{aligned}\beta_i\left(\operatorname{Lin}^*(I)\right) = &\binom{i+d-1}{d-1}\binom{n}{i+d} \\ &+ \binom{n-d+1}{i}\left(md - \sum_{j \geq 2}(j-1)C_j\right) \\ &+ \sum_{j \geq 2} C_j \sum_{k=2}^{j} \binom{n-d+k}{i}.\end{aligned}$$

Example 4.2.8. Consider the ideal $I = (x_1x_2x_3,\ x_1x_2x_4,\ x_1x_2x_5) \subset S = \mathbb{K}[x_1, \dots, x_5]$. The hypergraph corresponding to this ideal consists of three triangles that share the common edge x_1x_2 (which has codimension 1). The $*$-linearization of I lives in $R = \mathbb{K}[x_1, \dots, x_5, y_1, y_2, y_3]$, and it's the ideal

$$\begin{aligned}\operatorname{Lin}^*(I) = &(x_1x_2x_3,\ x_1x_2x_4, \dots, x_2x_4x_5,\ x_3x_4x_5) \\ &+ (y_1x_2x_3,\ x_1y_1x_3,\ x_1x_2y_1, \dots, y_3x_2x_5,\ x_1y_3x_5,\ x_1x_2y_3),\end{aligned}$$

with $\binom{n}{d} + md = \binom{5}{3} + 3 \times 3 = 19$ generators. In fact, in this example one has $\operatorname{Lin}(I) = \operatorname{Lin}^*(I)$. One may check that the Betti table $\beta(\operatorname{Lin}^*(I))$ is

	0	1	2	3	4	5
3	19	45	43	21	6	1.

Next we compute the colon ideals $J_k := (g_1, \dots, g_{k-1}) : g_k$. For g_k in the complete part of $\operatorname{Lin}^*(I)$ we get the ideals written below on the left column, and after that for

g_k in the last part we get the ideals on the right:

$$
\begin{aligned}
J_1 &= (0) & J_{11} &= (x_1,\ x_4,\ x_5)\\
J_2 &= (x_3) & J_{12} &= (x_2,\ x_4,\ x_5)\\
J_3 &= (x_3,\ x_4) & J_{13} &= (x_3,\ x_4,\ x_5)\\
J_4 &= (x_2) & J_{14} &= (x_1,\ x_3,\ x_5)\\
J_5 &= (x_2,\ x_4) & J_{15} &= (x_2,\ x_3,\ x_5)\\
J_6 &= (x_2,\ x_3) & J_{16} &= (x_3,\ x_4,\ x_5,\ y_1)\\
J_7 &= (x_1) & J_{17} &= (x_1,\ x_3,\ x_4)\\
J_8 &= (x_1,\ x_4) & J_{18} &= (x_2,\ x_3,\ x_4)\\
J_9 &= (x_1,\ x_3) & J_{19} &= (x_3,\ x_4,\ x_5,\ y_1,\ y_2).\\
J_{10} &= (x_1,\ x_2)
\end{aligned}
$$

In this small example we can observe some facts that hold in general: the colon ideals J_k with g_k in the complete part are very regular, they clearly don't depend on the generators of I, and they have nothing to do with the variables y_j. They have at most $n-d$ generators, in this case $5-3=2$. On the other hand, the ideals J_k with g_k in the last part of $\mathrm{Lin}^*(I)$ have more generators, at least $n-d+1$, and some of these ideals happen to have additional generators, consisting of some variables y_j.

The squarefree case is discussed at length in Section 4 of Publication II, both with a direct argument and with a more conceptual proof suggested by Gunnar Fløystad, using his "polarization techniques" (see Section 3.7.2), based on the repeated use of the following well-known results:

- the quotient of a Cohen–Macaulay ring by a regular sequence is Cohen–Macaulay (see Theorem 3.7.2);
- a squarefree monomial ideal I has a linear resolution iff $S/I^\vee$ is Cohen–Macaulay (see Theorem 3.5.5).

Among the other results in Publication II, we only mention the following: One may wonder whether $\mathrm{Lin}(I)$ belongs to some famous class of ideals with linear quotients. Polymatroidal ideals, defined below, constitute one such large class (see also Section 12.6 of [50]). For a variable y and a monomial u, we denote by $\deg_y(u)$ the highest exponent $e \in \mathbb{N}$ such that y^e divides u.

Definition 4.2.9. An equigenerated monomial ideal $J \subset \mathbb{K}[z_1,\dots,z_r]$ is *polymatroidal* if the following holds: for any u and v in $G(J)$ and for any i such that $\deg_{z_i}(u) > \deg_{z_i}(v)$, there exists j such that $\deg_{z_j}(u) < \deg_{z_j}(v)$ and $\frac{u}{z_i}z_j \in G(J)$.

It was natural to ask how often $\mathrm{Lin}(I)$ is polymatroidal, and it turns out that it almost never is (Theorem 3.14 of Publication II):

Theorem 4.2.10. *For a monomial ideal $I \subset S = \mathbb{K}[x_1,\dots,x_n]$ equigenerated in degree d, in the following cases* $\operatorname{Lin}(I)$ *is polymatroidal:*

- $d = 1$, *that is, I is generated by some variables;*
- d *is arbitrary and I is principal.*

In all other cases $\operatorname{Lin}(I)$ *is not polymatroidal.*

4.2.2 Equification of monomial ideals

The linearization construction defined above works in a sensible way only when applied to *equigenerated* monomial ideals. So finding a way to extend it to arbitrary monomial ideals was a natural problem. This is done in Publication II by making a preliminary step, which amounts to defining an injective map

$$\left\{ \begin{array}{c} \text{monomial ideals} \\ \text{in } \mathbb{K}[x_1,\dots,x_n] \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{equigenerated monomial} \\ \text{ideals in } \mathbb{K}[x_1,\dots,x_n,z] \end{array} \right\}.$$

Definition 4.2.11. Let I be a monomial ideal in $S = \mathbb{K}[x_1,\dots,x_n]$, with minimal system of monomial generators $G(I) = \{f_1,\dots,f_m\}$. Denote $d_j := \deg(f_j)$ for all j, and let $d := \max\{d_j \mid j = 1,\dots,m\}$. We define the *equification of I* as

$$I^{\mathrm{eq}} := (f_1 z^{d-d_1},\ f_2 z^{d-d_2},\dots,\ f_m z^{d-d_m})$$

in the polynomial ring $S[z] = \mathbb{K}[x_1,\dots,x_n,z]$ with one extra variable z.

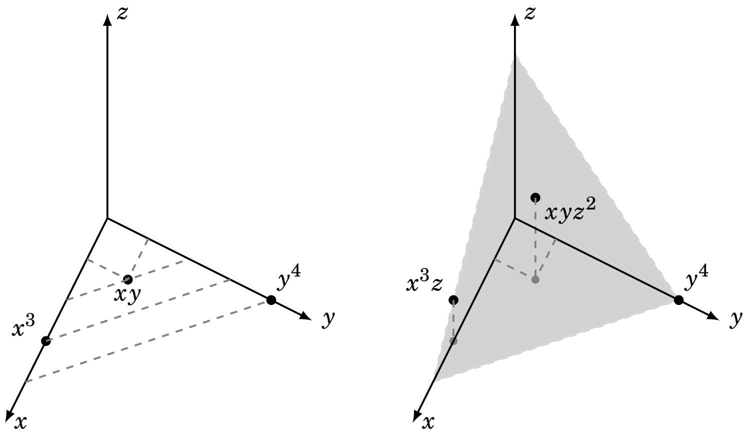
Remark 4.2.12. Observe that the words "equification" and "equify" already exist in English, as technical terms in trading and economics. In Publication II there is of course no relation at all to that connotation of equification. The word "equification" was suggested to me in analogy to "sheafification", which is a well-known process in algebraic geometry to make a presheaf into a sheaf.

In Section 5 of Publication II we start by investigating some basic properties of the equification, such as the following:

Lemma 4.2.13. *The generators of I^{eq} in Definition 4.2.11 are minimal.*

A way to illustrate pictorially what happens with I^{eq} for $n = 2$ is as follows. Think of the monomials in $\mathbb{K}[x,y]$ as lattice points in the plane with axes x and y. For all d, consider the line $x + y = d$, which goes through all monomials of degree d in x and y. Then, what $(\cdot)^{\mathrm{eq}}$ does is that we add a new axis z that comes out of the plane, we take the generators of I of degree d' (which are the ones lying on the line $x + y = d'$) and we "lift" them up to height $d - d'$. So, in particular, the generators of degree d stay on the plane.

Example 4.2.14. Consider for instance the ideal $I = (x^3, xy, y^4) \subset \mathbb{K}[x,y]$ and its equification $I^{\mathrm{eq}} = (x^3 z, xyz^2, y^4) \subset \mathbb{K}[x,y,z]$. We draw these in the following pictures, respectively on the left and on the right:



With a slight abuse of notation, the generator x^3 lies on the line $x+y=3$, xy on the line $x+y=2$ and y^4 on the line $x+y=4$, all of them on the plane $z=0$. Those three parallel lines are all dashed, in the left picture. The generators of I^{eq} all lie on the plane $x+y+z=4$, colored in gray on the right.

We then focus our attention on the homological properties of the equification: the main question is, what is the relation between the minimal resolutions of I and of I^{eq}? In particular, what is the relation between their Betti numbers? We prove some partial results in this direction, the first being Proposition 5.8 of Publication II, on total Betti numbers:

Proposition 4.2.15. *Let $I \subset S$ be a monomial ideal and consider its equification $I^{\mathrm{eq}} \subset T = S[z]$. We have $\beta_0^S(I) = \beta_0^T(I^{\mathrm{eq}})$ and $\beta_i^S(I) \le \beta_i^T(I^{\mathrm{eq}})$ for all $i > 0$.*

(Notice that the equality for β_0 is Lemma 4.2.13.) Later we focus on the first syzygy module. Recall Definition 3.3.12 and Theorem 3.3.13. Denote

$$\begin{aligned}\sigma_{ij}^{\mathrm{eq}} &:= \frac{\operatorname{lcm}(g_i, g_j)}{g_i} e_i - \frac{\operatorname{lcm}(g_i, g_j)}{g_j} e_j \\ &= \frac{\operatorname{lcm}(f_i, f_j) z^{\max\{0, d_i - d_j\}}}{f_i} e_i - \frac{\operatorname{lcm}(f_i, f_j) z^{\max\{0, d_j - d_i\}}}{f_j} e_j\end{aligned}$$

the reduced trivial syzygies for I^{eq}, where $g_i = f_i z^{d-d_i}$. The following is Proposition 5.11 of Publication II.

Proposition 4.2.16. *The reduced trivial syzygy $\sigma_{ij}^{\mathrm{eq}}$ is a redundant generator of* $\operatorname{Syz}(I^{\mathrm{eq}})$ *if and only if there exists $k \notin \{i,j\}$ such that* $\operatorname{lcm}(f_k, f_i)$ *and* $\operatorname{lcm}(f_k, f_j)$ *divide* $\operatorname{lcm}(f_i, f_j)$ *and* $\min\{d_i, d_j\} \le d_k$.

Example 4.2.17. We show an example where the equality for all Betti numbers is attained and an example where the inequalities are strict. First consider, inside $S := \mathbb{K}[x_1, x_2, x_3]$ and $T := S[z]$, respectively, the ideals

$$I = (x_1^2,\ x_1x_2^2x_3^2,\ x_2^3x_3^2) \qquad \text{and} \qquad I^{\mathrm{eq}} = (x_1^2z^3,\ x_1x_2^2x_3^2,\ x_2^3x_3^2).$$

Then the Betti tables $\beta^S(I)$ and $\beta^T(I^{\mathrm{eq}})$ are respectively

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 2 & 1 & - \\ 3 & - & - \\ 4 & - & - \\ 5 & 2 & 2 \\ \hline & 3 & 2 \end{array} \qquad \text{and} \qquad \begin{array}{c|cc} & 0 & 1 \\ \hline 5 & 3 & 1 \\ 6 & - & - \\ 7 & - & - \\ 8 & - & 1 \\ \hline & 3 & 2, \end{array}$$

and the minimal resolutions are

$$0 \longrightarrow S^2 \xrightarrow{\begin{bmatrix} 0 & x_2^2x_3^2 \\ x_2 & -x_1 \\ -x_1 & 0 \end{bmatrix}} S^3 \xrightarrow{\begin{bmatrix} x_1^2 & x_1x_2^2x_3^2 & x_2^3x_3^2 \end{bmatrix}} I \longrightarrow 0$$

and

$$0 \longrightarrow T^2 \xrightarrow{\begin{bmatrix} 0 & x_2^2x_3^2 \\ x_2 & -x_1z^3 \\ -x_1 & 0 \end{bmatrix}} T^3 \xrightarrow{\begin{bmatrix} x_1^2z^3 & x_1x_2^2x_3^2 & x_2^3x_3^2 \end{bmatrix}} I^{\mathrm{eq}} \longrightarrow 0.$$

In this case the total Betti numbers of I and I^{eq} are equal. Consider now, inside $S = \mathbb{K}[x_1,\ldots,x_4]$ and $T = S[z]$, respectively, the ideals

$$I = (x_1x_2x_4,\ x_1^2x_2^2x_3,\ x_3^3x_4^3) \qquad \text{and} \qquad I^{\mathrm{eq}} = (x_1x_2x_4z^3,\ x_1^2x_2^2x_3z,\ x_3^3x_4^3).$$

The Betti tables $\beta^S(I)$ and $\beta^T(I^{\mathrm{eq}})$ are respectively

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 3 & 1 & - \\ 4 & - & - \\ 5 & 1 & 1 \\ 6 & 1 & - \\ 7 & - & 1 \\ \hline & 3 & 2 \end{array} \qquad \text{and} \qquad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 6 & 3 & - & - \\ 7 & - & - & - \\ 8 & - & 1 & - \\ 9 & - & - & - \\ 10 & - & 2 & - \\ 11 & - & - & 1 \\ \hline & 3 & 3 & 1. \end{array}$$

In this case we have strict inequalities $\beta_i^S(I) < \beta_i^T(I^{\mathrm{eq}})$ for $i = 1,2$. The matrices corresponding to the first syzygies of I and I^{eq} are respectively

$$\begin{bmatrix} x_1x_2x_3 & x_3^3x_4^2 \\ -x_4 & 0 \\ 0 & -x_1x_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1x_2x_3 & x_3^3x_4^2 & 0 \\ -x_4z^2 & 0 & x_3^2x_4^3 \\ 0 & -x_1x_2z^3 & -x_1^2x_2^2z \end{bmatrix}.$$

For I the reduced trivial syzygy σ_{23} is redundant, because $\sigma_{23} = -x_3^2x_4^2\sigma_{12} + x_1x_2\sigma_{13}$. But for I^{eq} the corresponding syzygy, appearing as the third column in the matrix, is not redundant.

4.3 The regularity of almost all edge ideals

As remarked in Section 3.4.2, the Betti table of an edge ideal has some entries that are always zero, under (or equivalently, to the left of) the *main diagonal* consisting of the numbers

$$\beta_{i,2(i+1)}, \qquad i \geq 0.$$

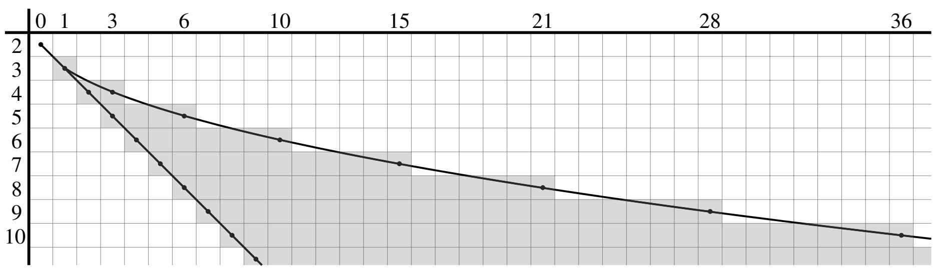
In Publication III we consider a region of the Betti table that extends from the main diagonal to the right, bounded on the right by a parabola.

Definition 4.3.1. Let $r \geq 3$. A Betti number $\beta_{i,j}$ on the r-th row of the Betti table is called a *parabolic Betti number* if the following hold:

$$r-2 \leq i \leq r-2+\binom{r-1}{2},$$

$$2(r-1) \leq j \leq 2(r-1)+\binom{r-1}{2}.$$

The following picture represents the top-left part of the Betti table, omitting rows 0 and 1 because they only contain zeros for edge ideals. The parabolic Betti numbers on row r, for $3 \leq r \leq 10$, are marked by gray squares. The main diagonal and the parabola of numbers $\beta_{i,j}$ attained for $i = r-2+\binom{r-1}{2}$ and $j = 2(r-1)+\binom{r-1}{2}$ are also drawn, for the sake of completeness and clarity.



Recall from Section 2.1 that a k-*cluster* is the disjoint union of k cliques, or equivalently the complement of a complete k-partite graph. We introduce the following notion.

Definition 4.3.2. We denote a k-cluster by

$$\overline{K_{a_1,\ldots,a_k}} = K^{a_1} \sqcup K^{a_2} \sqcup \cdots \sqcup K^{a_k},$$

where $a_1, a_2, \ldots, a_k$, that we assume being ordered as $a_1 \leq a_2 \leq \cdots \leq a_k$, are the number of vertices in the k cliques. Let $k \geq 2$. If $a_1 = 2$ and $2 \leq a_i \leq i$ for all $i \in \{2,3,\ldots,k\}$, then we say that $\overline{K_{a_1,\ldots,a_k}}$ is a *parabolic k-cluster*.

Example 4.3.3. We list the parabolic k-clusters for small values of k:

k	2	3	4	5
parabolic k-clusters	$\overline{K_{2,2}}$	$\overline{K_{2,2,2}}$ $\overline{K_{2,2,3}}$	$\overline{K_{2,2,2,2}}$ $\overline{K_{2,2,2,3}}$ $\overline{K_{2,2,2,4}}$ $\overline{K_{2,2,3,3}}$ $\overline{K_{2,2,3,4}}$	$\overline{K_{2,2,2,2,2}}$ $\overline{K_{2,2,2,2,3}}$ $\overline{K_{2,2,2,2,4}}$ $\overline{K_{2,2,2,2,5}}$ $\overline{K_{2,2,2,3,3}}$ $\overline{K_{2,2,2,3,4}}$ $\overline{K_{2,2,2,3,5}}$ $\overline{K_{2,2,2,4,4}}$ $\overline{K_{2,2,2,4,5}}$ $\overline{K_{2,2,3,3,3}}$ $\overline{K_{2,2,3,3,4}}$ $\overline{K_{2,2,3,3,5}}$ $\overline{K_{2,2,3,4,4}}$ $\overline{K_{2,2,3,4,5}}$

Parabolic Betti numbers and parabolic clusters are linked by the following special case of Lemma 21 of Publication III:

Lemma 4.3.4. *Let $\beta_{i,j}$ be a parabolic Betti number on the r-th row of the Betti table, for some $r \geq 3$. Let G be an unlabeled graph such that $\beta_{i,j}(I_G) = 0$. Then there is a parabolic $(r-1)$-cluster C such that G is C-free (see Definition 2.2.10).*

We show that the parabolic clusters are critical, and this, together with the lemma above, allows us to employ the critical graph machinery by Balogh and Butterfield [3] described in Section 2.2 of the thesis. The following are the main results of Publication III, appearing there as Theorems 32 and 33, respectively.

Theorem 4.3.5. *Let $\beta_{i,j}$ be a parabolic Betti number on the r-th row of the Betti table, for some $r \geq 3$. Almost every graph G with $\beta_{i,j}(I_G) = 0$ is an $(r-2,1)$-template.*

Theorem 4.3.6. *Let $\beta_{i,j}$ be a parabolic Betti number on the r-th row of the Betti table, for some $r \geq 3$. For almost every graph G with $\beta_{i,j}(I_G) = 0$, one has that*

1. $\operatorname{reg}(I_G) = r-1$, *and*
2. *every parabolic Betti number of I_G above row r is positive.*

We show that our results are “sharp”. More precisely, we were guided by to the following questions:

- Is the “almost” in the statements above necessary? How many graphs G are there with a vanishing parabolic Betti number $\beta_{i,j}(I_G)$ on row r and with regularity different from $r-1$?
- What about non-parabolic Betti numbers? Can one (easily) prove more general results?

We provide some answers to the first questions in Section 6 of Publication III, by proving the following (Theorem 36 in Publication III):

Theorem 4.3.7. *Let $r \geq 3$ and $i \geq 2r-4$ be integers. For large n, there are at least 2.99^n graphs G on n vertices such that*

(1) on row r, one has $\beta_{j,r+j}(I_G) > 0$ for $i < j \leq n-r$;

(2) on row r, one has $\beta_{j,r+j}(I_G) = 0$ for $j \leq i$;

(3) below row r, all Betti numbers are zero.

The second problem above is addressed in Section 7, considering the first non-parabolic Betti numbers and showing that things seem to fall apart quickly, as soon as we step out of the region of the Betti table consisting of the parabolic Betti numbers.

We relate our results to the following, famous conjecture in extremal graph theory.

Conjecture 4.3.8 (Erdős–Hajnal, [33]). *For every graph H there is a constant $\tau > 0$ (depending only on H) such that any H-free graph G contains a clique or an independent set of order at least $|G|^\tau$.*

We prove the following (Proposition 48 in Publication III).

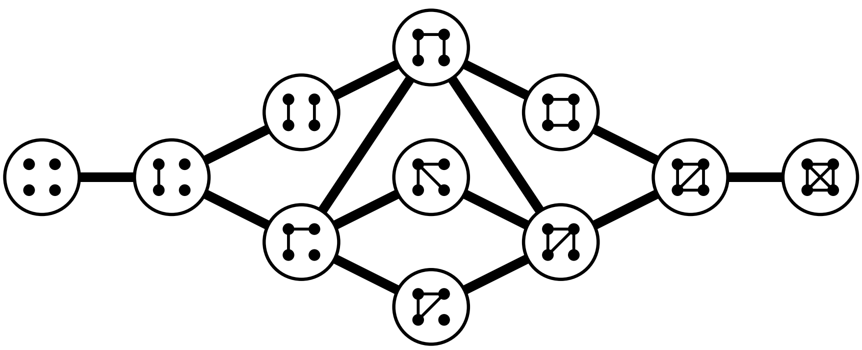
Proposition 4.3.9. *Let $i \geq 0$ and $2 \leq j \leq 2i+2$. There is a $\tau > 0$ such that if $\beta_{i,j}(I_G) = 0$ then there is a homogenous set of order $|G|^\tau$ in G.*

Notice that the condition on i and j in the proposition above means that $\beta_{i,j}$ is in the region of the Betti table bounded above by row 2 (including that row) and below by the main diagonal of numbers $\beta_{i,2i+2}$ (including that diagonal).

Lastly, we consider a "space" (actually, a graph) of graphs, defined as follows.

Definition 4.3.10. Let n be a non-negative integer. Let V be the set of unlabeled graphs on n vertices. We define the labeled graph $\mathscr{G}_n = (V, E)$, where $\{G_1, G_2\}$ is in E if G_1 can be obtained by adding exactly one edge to G_2, or vice versa.

In Publication III this appears simply as $\mathscr{G}$, suppressing the index n. For $n = 4$, $\mathscr{G}_4$ looks like this:



One may observe in this example a few easy facts that hold in general:

- One may partition the vertices of $\mathscr{G}_n$ in $\binom{n}{2} + 1$ independent sets, based on the cardinality of the edge set: the i-th independent set I_i consists of all unlabeled graphs on n vertices with exactly i edges.
- The edges of $\mathscr{G}_n$ only occur between independent sets of adjacent cardinalities (that is, between I_i and I_{i+1}).
- The map $G \mapsto \overline{G}$ that swaps G with its complement is an automorphism of $\mathscr{G}_n$.

The main results of Publication III revolve around (s,t)-templates, so it was natural to wonder how these graphs behave inside $\mathscr{G}_n$. We were in particular interested in the connectedness of the set of (s,t)-templates, and we proved the following (Proposition 50 in Publication III).

Proposition 4.3.11. *For any non-negative integers s, t and n, the set of the (s,t)-templates on n vertices is connected in $\mathscr{G}_n$.*

4.4 Triangulations of polygons and stacked simplicial complexes: separating their Stanley–Reisner ideals

In Publication IV we consider tree ideals $I(T) \subset S$ arising from directed trees T (see Section 3.7.3). A quotient of $S/I(T)$ modulo a sequence of variable differences can still be written as $\tilde{S}/\tilde{I}$, a polynomial ring modulo a monomial ideal. We characterize

- all the $S/I(T)$-regular sequences consisting of variable differences,

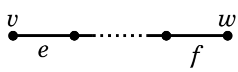
and among these, in particular,

- those such that the corresponding monomial ideal $\tilde{I}$ is squarefree;
- those such that $\tilde{I}$ is squarefree and $\tilde{I}$ is the Stanley–Reisner ideal of a simplicial ball. In particular, we find triangulations of polygons.

Fix a directed tree T with vertex set V and edge set E. Recall that for an edge e and a vertex v one defines

$$e_{\text{to}}(v) := \begin{cases} 1 & \text{if } e \text{ points to } v, \\ 0 & \text{otherwise.} \end{cases}$$

For two vertices v and w consider the unique path vTw linking v and w in T, and let e and f be the edges on vTw that are incident in v and w, respectively, so that (forgetting the direction of the edges) the path vTw looks like



Recall that one defines $m_{v,w} := x_{e,e_{\text{to}}(v)} x_{f,f_{\text{to}}(w)}$, and that the tree ideal of T is

$$I(T) := (m_{v,w} \mid v \neq w \in V),$$

in the polynomial ring $S := k[x_{e,0}, x_{e,1} \mid e \in E]$. Denote $\overline{1} = 0$ and $\overline{0} = 1$, and

$$h_{v,w} := x_{e,\overline{e_{\text{to}}(v)}} - x_{f,\overline{f_{\text{to}}(w)}}.$$

Before giving the necessary definitions to state the main results of the paper, notice that we prove in passing that there is a bijection between:

- the set of partitions of the edge set E into r sets, and

- the set of partitions of the vertex set V into $r+1$ independent sets,

constructing an explicit bijection before the statement of Theorem A.1 of Publication IV. The fact that such a bijection exists is not hard to prove and is implicitly stated for instance as Corollary 6 of [49], but to the best of our knowledge we are the first to explicitly show a bijection and state this result. More on this will appear in [40].

The first step in determining the $S/I(T)$-regular sequence consisting of variable differences is the following (Lemma 5.3 in Publication IV):

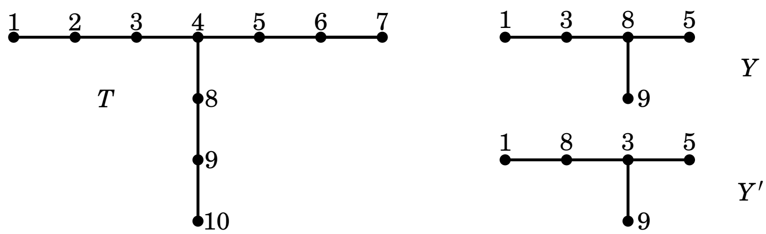
Lemma 4.4.1. *The differences of variables in S which are $S/I(T)$-regular are exactly those of the form $h_{v,w}$ above.*

We use the following terminology.

Definition 4.4.2. For any tree Y (regardless of the possible direction of the edges), we say that a sequence of vertices v,u,w is *Y-aligned* if u lies inside the unique path vYw in Y linking v and w.

Definition 4.4.3. Let $U \subseteq V$ be a subset of the vertex set of T, and let Y be a tree (unrelated to T) on U. We say that Y *flows with* T if whenever v,u,w is a T-aligned sequence of vertices in U, then v,u,w is a Y-aligned sequence.

Example 4.4.4. Consider the following trees T, Y and Y':



with $V = \{1,2,\dots,10\}$ and $U = \{1,3,5,8,9\}$. The T-aligned sequences of vertices in U are

$$1,3,5, \quad 1,3,8, \quad 1,3,9, \quad 1,8,9, \quad 3,8,9, \quad 5,8,9.$$

The tree Y flows with T, and the tree Y' does not flow with T, for instance because the sequence $5,8,9$ is not Y'-aligned.

Definition 4.4.5. Let Y be a tree with vertex set $U \subseteq V$. Let $L(Y)$ be the linear subspace of S generated by the variable differences $h_{v,w}$, for all the edges $\{v,w\}$ of Y. If $L(Y)$ has a basis which is a regular sequence of variable differences for $S/I(T)$, we say that $L(Y)$ is a *regular linear space for $S/I(T)$*.

The following are respectively a combination of Propositions 5.12 and 5.15, and then below Theorem 5.16, in Publication IV:

Proposition 4.4.6. *Let Y be a tree on $U \subseteq V$. The following are equivalent:*

- *Y flows with T;*

- *$L(Y)$ is a regular linear space for $S/I(T)$.*

Theorem 4.4.7. *There is a bijection between the set of regular linear spaces for $S/I(T)$ and the set of partitions of V.*

Notice that for instance the variable differences $x_{e,0} - x_{e,1}$, for all $e \in E$, form a regular sequence (corresponding to the trivial partition of V with only one part). The quotient of $S/I(T)$ modulo these differences yields $k[x_e \mid e \in E]/(x_e \mid e \in E)^2$. We are interested, for combinatorial reasons, in finding the regular quotients of $S/I(T)$ that can be written as a polynomial ring modulo a *squarefree ideal*. This is described in Lemma 6.1 and Theorem 6.2 of Publication IV:

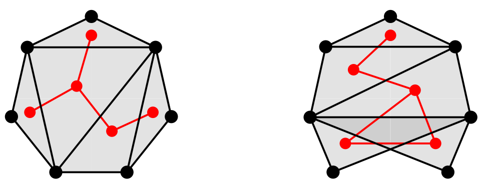
Lemma 4.4.8. *Let Y be a tree on $U \subseteq V$ flowing with T. The following are equivalent:*

- *the quotient of $S/I(T)$ by $L(Y)$ yields a squarefree ideal;*
- *the set U consists of independent vertices in T.*

Theorem 4.4.9. *There is a bijection between the set of regular linear spaces that yield a squarefree ideal and the set of partitions of V into independent sets of vertices.*

Lastly, among the regular quotients that yield a squarefree ideal, we are interested in particular in those whose associated simplicial complex is a triangulated ball. To state this more precisely, define the dual graph $G(\Delta)$ of a pure simplicial complex Δ as the following graph with vertex set equal to the set of facets of Δ: for two facets F and G, $G(\Delta)$ has the edge $\{F,G\}$ if and only if F and G share a codimension-one face. Then we call Δ a *triangulated ball* if $G(\Delta)$ is a tree. This is done with a different terminology, using hypergraphs and hypertrees, in Section 3 of Publication IV.

Example 4.4.10. Consider the following pure simplicial complexes of dimension 2, with their dual graphs drawn in red:



The complex on the left is a triangulated ball, whereas the one on the right is not. Indeed, the one on the right has three facets sharing the same codimension-one face.

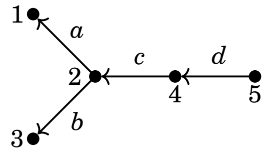
We describe in Section 8 of Publication IV how to get squarefree quotients of $S/I(T)$ whose associated simplicial complex is a triangulated ball, and in particular triangulations of polygons, whose dual tree has largest vertex degree at most 3. We prove the following (Theorem 8.7 in Publication IV):

Theorem 4.4.11. *There is a bijection between:*

- *the set of regular linear spaces for $S/I(T)$ yielding a squarefree ideal whose associated simplicial complex is a triangulated ball, and*

- *the set of partitions of the edge set E of T into sets of independent edges.*

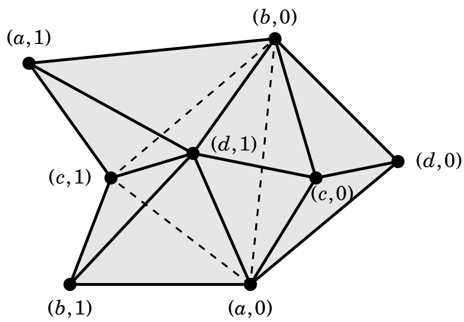
Example 4.4.12. Consider the directed tree T



whose tree ideal $I(T)$ is generated by the monomials

$$
\begin{array}{llll}
m_{12} = x_{a,1}x_{a,0} & m_{13} = x_{a,1}x_{b,1} & m_{14} = x_{a,1}x_{c,0} & m_{15} = x_{a,1}x_{d,0} \\
m_{23} = x_{b,0}x_{b,1} & m_{24} = x_{c,1}x_{c,0} & m_{25} = x_{c,1}x_{d,0} & m_{34} = x_{b,1}x_{c,0} \\
& m_{35} = x_{b,1}x_{d,0} & m_{45} = x_{d,1}x_{d,0}. &
\end{array}
$$

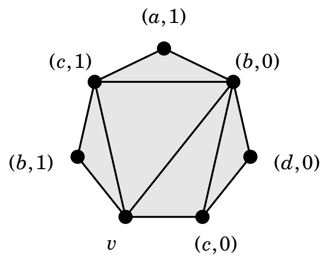
The ideal $I(T)$ is the Stanley–Reisner ideal of the following simplicial complex:



The difference of variables $h_{1,5} = x_{a,0} - x_{d,1}$ constitutes a regular sequence. The quotient of $S/I(T)$ by $x_{a,0} - x_{d,1}$ can be written as S'/I', where S' is almost the same polynomial ring as S, except that the variables $x_{a,0}$ and $x_{d,1}$ are identified as a single variable y, and I' is the monomial ideal obtained by replacing in the generators of $I(T)$ all occurrences of $x_{a,0}$ and $x_{d,1}$ by y. This ideal I' is squarefree and the associated simplicial complex is a triangulated ball. By Theorems 4.4.9 and 4.4.11, the sequence $x_{a,0} - x_{d,1}$ is associated to a partition of V into independent sets of vertices and to a partition of E into independent sets of edges. These partitions are respectively

$$V = \{1,5\} \cup \{2\} \cup \{3\} \cup \{4\} \qquad \text{and} \qquad E = \{a,d\} \cup \{b\} \cup \{c\}.$$

If the new variable y corresponds to a vertex v, the simplicial complex of the quotient by $x_{a,0} - x_{d,1}$ is



Notice that both the simplicial complexes drawn in this example have dual graphs that are trees isomorphic to the underlying undirected tree of T. This is not a coincidence: the dual graph of the simplicial complex of $I(T)$ is always isomorphic to T, a fact that follows from Lemma 4.2, and for the dual graph of the triangulated heptagon above this holds by Proposition 8.6 of Publication IV.

Remark 4.4.13 (The graph whose *edge ideal* is $I(T)$)**.** The ideal $I(T)$ is not defined starting from an undirected graph, but it is nonetheless the edge ideal of such a graph, call it G_T, on the set of vertices $\{x_{1,i},\dots,x_{n,i} \mid i = 0,1\}$. The graph G_T fits into the framework of the paper [52] by Herzog and Moradi, where they define the concept of a König graph as follows. For an arbitrary finite simple graph, the maximum cardinality of a matching is always at most as large as the minimum cardinality of a vertex cover; if for a graph G these two numbers are equal, then G is called a *König graph*. Corollary 2.4 of [52] states that G is König if and only if S/I_G (where I_G is the edge ideal of G) has a system of parameters consisting of variable differences. In the case of $I(T) = I_{G_T}$ one just gets the "trivial" regular sequence consisting of all the variable differences $x_{e,1} - x_{e,0}$.

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This thesis is in combinatorial commutative algebra. The leitmotif in combinatorial commutative algebra is that there are bijections between sets of combinatorial objects (simplicial complexes, graphs, trees, . . .) and sets of algebraic objects (squarefree monomial ideals, quadratic squarefree monomial ideals, polarizations, . . .), and often the combinatorial properties of a combinatorial object are equivalent to algebraic properties of the corresponding algebraic object.

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