Geometric properties of harmonic and polyharmonic mappings

Stavros Evdoridis

Geometric properties of harmonic and polyharmonic mappings

Stavros Evdoridis

A doctoral dissertation completed for the degree of Doctor of Science (Technology) to be defended, with the permission of the Aalto University School of Science, at a public examination held at the lecture hall M1 of the school on 14 April 2021 at 12 noon.

Aalto University
School of Science
Department of Mathematics and Systems Analysis

Supervising professor

Professor Juha Kinnunen, Aalto University, Finland

Thesis advisor

Professor Antti Rasila, GTIT, China

Preliminary examiners

Professor David Kalaj, University of Montenegro, Montenegro

Professor Karl-Joachim Wirths, Technical University of Braunschweig, Germany

Opponent

Professor María-José Martín-Gómez, Universidad de La Laguna, Spain

Aalto University publication series

DOCTORAL DISSERTATIONS 31/2021

© 2021 Stavros Evdoridis

ISBN 978-952-64-0294-9 (printed)

ISBN 978-952-64-0295-6 (pdf)

ISSN 1799-4934 (printed)

ISSN 1799-4942 (pdf)

<http://urn.fi/URN:ISBN:978-952-64-0295-6>

Unigrafia Oy

Helsinki 2021

Finland

Author

Stavros Evdoridis

Name of the doctoral dissertation

Geometric properties of harmonic and polyharmonic mappings

Publisher School of Science

Unit Department of Mathematics and Systems Analysis

Series Aalto University publication series DOCTORAL DISSERTATIONS 31/2021

Field of research Mathematics

Manuscript submitted 12 January 2021

Date of the defence 14 April 2021

Permission for public defence granted (date) 22 February 2021

Language English

☐ **Monograph**

☒ **Article dissertation**

☐ **Essay dissertation**

Abstract

This dissertation lies in the areas of Complex Analysis and Geometric Function Theory. Its subject concerns mainly the classes of harmonic and polyharmonic mappings as well as their relation to analytic functions. Harmonic mappings are defined as the solutions to the complex Laplacian equation. Since every analytic function solves the Laplacian equation, the harmonic mappings have been widely studied as generalisations of the analytic functions. Polyharmonic mappings form the wider class of functions which solve the p -Laplacian equation, for a certain positive p .

In the first paper, we improve the harmonic analogue of the classical Bohr inequality, proved by Kayumov, Ponnusamy, and Shakirov. We show that a non-negative term can be added and still obtain the same Bohr radius.

In the second paper, we examine how we can improve known results for analytic functions, given by R. Fournier and St. Ruscheweyh, for Bohr's phenomenon in more general domains. We also prove a sharp Bohr's inequality for harmonic mappings defined in disks larger than the unit disk.

In the third paper, we study the behaviour of harmonic mappings, defined either in the unit disk or in the upper half-plane, near the boundary. We also find sufficient conditions in order for the boundary function of such a mapping to be continuous, based on a result of Bshouty, Lyzzaik, and Weitsman.

In the fourth paper, we show that Radó's theorem remains true for the wider class of polyharmonic mappings. We also obtain results about the continuity of the boundary function of a polyharmonic mapping. At the end, we present a method for constructing a fully close-to-convex biharmonic mapping by using a convex harmonic one.

Keywords Complex Analysis, harmonic mappings, polyharmonic mappings, Bohr's inequality, boundary behavior, Radó's theorem

ISBN (printed) 978-952-64-0294-9

ISBN (pdf) 978-952-64-0295-6

ISSN (printed) 1799-4934

ISSN (pdf) 1799-4942

Location of publisher Helsinki

Location of printing Helsinki **Year** 2021

Pages 87

urn <http://urn.fi/URN:ISBN:978-952-64-0295-6>

Preface

The research work for this dissertation was carried out at the Department of Mathematics and Systems Analysis at Aalto University.

First of all, I would like to express my deepest gratitude to my thesis advisor, prof Antti Rasila, for his continuous support and guidance during all these years. I am also grateful to my thesis supervisor, prof Juha Kinnunen, for all the helpful discussions we have had since my first visit to the department.

Furthermore, I would like to express my sincere appreciation to prof Daoud Bshouty and prof Saminathan Ponnusamy for their motivation and the ideas they shared with me.

Without my friends, this journey would have never happened. I want to thank all of them in Finland, in Greece, in the UK, in Ireland, in Slovakia, in the USA, in Canada and in Hong Kong.

Finally, my gratitude goes to my family for being supportive for my studies and especially to my sister, Vasso, who helped me to overcome the obstacles I met all these years. A special thank you to Maria-Eirini for the wonderful moments we have shared.

Espoo, March 4, 2021,

Stavros Evdoridis

Contents

Preface	1
Contents	3
List of Publications	5
Author’s Contribution	7
1. Introduction	9
2. Preliminaries	11
2.1 Harmonic Mappings	11
2.2 Polyharmonic Mappings	13
2.3 Boundary Behaviour	14
2.4 Modulus of Families of Paths	17
2.5 Bohr’s inequality	19
References	21
Publications	23

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I S. Evdoridis, S. Ponnusamy, and A. Rasila. **Improved Bohr's inequality for locally univalent harmonic mappings.** *Indagationes Mathematicae*, 30, 201-213, DOI:/10.1016/j.indag.2018.09.008, January 2019.

II S. Evdoridis, S. Ponnusamy, and A. Rasila. **Improved Bohr's inequality for shifted disks.** *Results in Mathematics*, published online, DOI:10.1007/s00025-020-01325-x, 9 January 2021.

III D. Bshouty, J. Chen, S. Evdoridis, and A. Rasila. **Koebe and Carathéodory type boundary behavior results for harmonic mappings.** Accepted for publication in *Complex Variables and Elliptic Equations*, Published online, DOI:10.1080/17476933.2020.1851212, 16 December 2020.

IV D. Bshouty, S. Evdoridis, and A. Rasila. **A note on polyharmonic mappings.** Submitted to *Computational Methods and Function Theory*, 8 December 2020.

Author's Contribution

Publication I: “Improved Bohr’s inequality for locally univalent harmonic mappings”

SE can be considered as the first/principal author of this manuscript. SP and AR have provided ideas and guidance on all stages of the research process and checked the final version of the manuscript.

Publication II: “Improved Bohr’s inequality for shifted disks”

SE can be considered as the first/principal author of this manuscript. SP and AR have provided ideas and guidance on all stages of the research process and checked the final version of the manuscript.

Publication III: “Koebe and Carathéodory type boundary behavior results for harmonic mappings”

All authors had equal contribution

Publication IV: “A note on polyharmonic mappings”

All authors had equal contribution

1. Introduction

The main topic of the present PhD thesis is Complex Analysis. In particular, the class of harmonic mappings is being investigated as well as its connections with its subclass of analytic functions. The primary goal is to examine under which conditions it is possible to generalise well-known results about analytic functions to harmonic mappings.

The development of the theory of harmonic mappings was achieved in two different stages. At the beginning of the twentieth century, differential geometers were studying harmonic mappings due to their connection with minimal surfaces. Significant results were obtained during this period by important mathematicians such as Tibor Radó, Erhard Heinz and Johannes Nitsche. The second stage begins with the work of Mocanu [23] in 1981, where he obtains univalence criteria for certain classes of complex functions, including harmonic mappings. Next was the paper, which is considered to be the cornerstone of the area of harmonic mappings, by James Clunie and Terry Sheil-Small [9], in 1984. The authors proved harmonic analogues of known results for conformal mappings as well as a method for constructing a harmonic mapping by a given conformal one, when both image domains are convex in one direction. Motivated by this paper, many complex analysts worked on establishing the whole theory and answering the open questions posed by Clunie and Sheil-Small. Right after that, Hengartner and Schober published a collection of papers [16, 17, 18] with significant contribution to the investigation of the behaviour of a harmonic mapping near the boundary. In 2005, Peter Duren published his monograph “Harmonic mappings in the plane” [12], containing most of the known results on the topic. Other references could be the survey articles by D. Bshouty and W. Hengartner [7] and by S. Ponnusamy and A. Rasila [25].

2. Preliminaries

2.1 Harmonic Mappings

This chapter contains basic definitions and important results about the theory of planar harmonic mappings, that are necessary for the last part of the dissertation.

Let $f = u + iv$ be a complex-valued function defined in a domain Ω , where u and v denote its real and imaginary parts respectively. The function f is said to be a *harmonic mapping* if both u, v are real harmonic functions, i.e. $\Delta u = \Delta v = 0$. By writing $z = x + iy$, f is harmonic in Ω if and only if $\Delta f = 4f_{z\bar{z}} = 0$.

The above definition makes it clear that the Cauchy-Riemann equations need not be satisfied in order for a complex function to be harmonic. On the other hand, every analytic function is a harmonic mapping. An immediate consequence of the Cauchy-Riemann equations is that the analytic functions are independent of $\bar{z}$. For an analytic function h , this applies also to h_z (which is the derivative of h) and finally, $\Delta h = 0$.

A harmonic mapping f , defined in a simply connected domain Ω , admits the representation $f = h + \bar{g}$, where h, g are analytic in Ω . This is unique up to an additive constant and the special case where $g(0) = 0$ is called the *canonical representation* of f . The function h is said to be the analytic part of f and g is the co-analytic part.

Example 1. Let $f = z + \frac{1}{2}\bar{z}^2$ defined in the unit disk. Then, $f_{z\bar{z}} = 0$ implies that f is a harmonic mapping of the form $f = h + \bar{g}$, with $h(z) = z$ and $g(z) = \frac{1}{2}z$.

The Jacobian of a harmonic mapping f is given by the formula

$$J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2.$$

As h and g are analytic, it can also be expressed in the form

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2.$$

In 1936, Hans Lewy showed the direct connection between the Jacobian of a harmonic mapping and its local univalence by proving the following.

Theorem 2.1.1. *A harmonic mapping f is locally univalent in a neighbourhood of a point z_0 if and only if its Jacobian $J_f(z)$ does not vanish at z_0 .*

This implies that f is locally univalent in a domain Ω if and only if $J_f(z) \neq 0$ for all $z \in \Omega$. Then, f is called *sense-preserving* if $J_f(z) > 0$ in Ω or *sense-reversing* if $J_f(z) < 0$ in Ω .

Now, let f be a sense-preserving harmonic mapping in the simply connected domain Ω . The quantity

$$a_f(z) = \frac{\overline{f_z(z)}}{f_z(z)} = \frac{g'(z)}{h'(z)}, \quad z \in \Omega$$

is called the *second complex dilatation* of f in Ω . Since a_f has been defined for a sense-preserving function, it can be seen that it is an analytic function, bounded by the unity. The second complex dilatation is closely related with the quantity

$$D_f = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|},$$

known as the *dilatation* of f . A simple calculation shows that $|a_f(z)| \leq k < 1$ if and only if $D_f(z) \leq K < \infty$. A sense-preserving homeomorphism f with $D_f \leq K$ in Ω is called a *K-quasiconformal mapping* in Ω . Throughout the thesis, both terms “dilatation” and “second complex dilatation” will be used for describing the function $a_f(z)$ (also denoted by $\omega_f(z)$ or $a(z)$). When we refer to $D_f(z)$, this will be clearly stated.

For an analytic function ϕ with $\phi(z_0) = 0$, we say that ϕ has a zero of *multiplicity (or order) n* and we write $\mu(z_0, \phi)$ if

$$\phi(z_0) = \phi'(z_0) = \dots = \phi^{(n-1)}(z_0) = 0$$

and $\phi^{(n)}(z_0) \neq 0$. Consider a locally univalent sense-preserving harmonic mapping f , with the canonical representation $f = h + \bar{g}$. If $\mu(z_0, h) = n$ and $\mu(z_0, g) = m$, then $n \geq m$ and we say that f has multiplicity n at z_0 .

The composition of two analytic functions is always analytic. This is not true for the harmonic mappings. In addition, the square f^2 , the reciprocal $1/f$ and the inverse f^{-1} of a harmonic mapping are not necessarily harmonic. The same is also true for the composition of a harmonic with an analytic function. More precisely,

- If $f : \Omega \rightarrow \mathbb{C}$ is analytic and $g : f(\Omega) \rightarrow \mathbb{C}$ is harmonic, then $g \circ f$ is harmonic.
- If $f : \Omega \rightarrow \mathbb{C}$ is harmonic and $g : f(\Omega) \rightarrow \mathbb{C}$ is analytic, then $g \circ f$ is not necessarily harmonic.

The next two inequalities are due to Erhard Heinz [14], [15]. Both results, proved in the middle of the twentieth century, have played a significant role in the theory of harmonic mappings. The first one is known as *Heinz's Lemma* and states that

Theorem 2.1.2. (*Heinz's Lemma*) *For any univalent harmonic mapping f of the unit disk onto itself, with $f(0) = 0$, there is an absolute constant $c > 0$ such that*

$$|f_z(0)|^2 + |f_{\bar{z}}(0)|^2 \geq c.$$

The second theorem acts as a harmonic analogue of Schwarz's Lemma. In particular,

Theorem 2.1.3. (*Harmonic Schwarz's Lemma*) *Let f be a harmonic mapping of the unit disk into itself, with $f(0) = 0$. Then, the inequality*

$$|f(z)| \leq \tan^{-1} |z|$$

is sharp for every $z \in \mathbb{D}$. This upper bound remains sharp for univalent harmonic mappings of $\mathbb{D}$ onto itself under the same normalisation as above.

In the analytic case, the celebrated Riemann Mapping Theorem guarantees the existence of a univalent function from the unit disk (or more generally from a simply connected $\Omega \subsetneq \mathbb{C}$) onto any simply connected domain which is not equal to $\mathbb{C}$. On the other hand, as a result of Liouville's theorem, it is known that such a function does not exist when one of the domains is the whole complex plane. In the harmonic case, Riemann's Mapping Theorem still holds and Liouville's theorem can be used for proving the non-existence of a univalent harmonic mapping of $\mathbb{C}$ onto $\mathbb{D}$. The next theorem was firstly stated by Tibor Radó and shows that such a mapping from $\mathbb{D}$ onto $\mathbb{C}$ does not exist either. More precisely

Theorem 2.1.4. (*Radó's Theorem*) *There is no univalent harmonic mapping of the unit disk onto the whole complex plane.*

In 1926, Radó [26] gave a partial proof of it. Later L. Bers [3] and J. C. C. Nitsche [24] proved independently a more general version. Nitsche's proof is based on the Heinz lemma.

2.2 Polyharmonic Mappings

A complex-valued function f is called *polyharmonic mapping* (or p -harmonic mapping) if it satisfies the equation $\Delta^p f = 0$, for $p \geq 1$. When $p > 1$, Δ^p

is given by $\Delta^p f = \Delta(\Delta^{p-1} f)$ and $\Delta^1 f = \Delta f$. In the particular case where $p = 2$, the mapping is called *biharmonic*. These mappings have been studied extensively as they have many applications in different fields such as Physics and Engineering.

Restricting our attention to polyharmonic mappings defined in the unit disk, we see that they have a representation of the form

$$f(z) = \sum_{k=1}^p |z|^{2(k-1)} G_k(z),$$

where G_k are harmonic mappings in $\mathbb{D}$, for $p = 1, \dots, k$.

Example 2. Consider the function $f(z) = z - |z|^2 \frac{\bar{z}^2}{6}$. Clearly $\Delta^2 f = 0$ and hence f is a biharmonic mapping, with $G_1(z) = z$ and $G_2(z) = \frac{\bar{z}^2}{6}$.

2.3 Boundary Behaviour

The Dirichlet problem on a domain D is about finding a harmonic function in D , which is continuous in $\overline{D}$ and coincides with a given function on the boundary. The problem can be explicitly solved when the domain is the unit disk. Then, for an arbitrary continuous function ϕ in $[0, 2\pi]$, with $\phi(0) = \phi(2\pi)$, the solution is given by the integral

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - t) \phi(t) dt \quad 0 \leq r < 1,$$

where

$$P(r, t) = \frac{1 - r^2}{1 - 2r \cos t + r^2}$$

is the *Poisson kernel*. This is also a method to define a harmonic mapping in $\mathbb{D}$, for a given continuous function ϕ on $\mathbb{T}$. However, it can be used whenever ϕ has a finite number of jump discontinuities, as well. In this case, the sided limits both exist, but they are not equal. Then, $u(z)$ is harmonic in the unit disk, with

$$u(re^{i\theta}) = \frac{1}{2} (\phi(\theta^+) + \phi(\theta^-)),$$

where $\phi(\theta^-)$ and $\phi(\theta^+)$ denote the left-hand and right-hand limits respectively.

It is known that conformal mappings behave nicely near the boundary of the domain they are defined in. For example, Carathéodory's boundary extension theorem states that if f is a conformal mapping of the unit disk onto a Jordan domain Ω , then it can be extended to a homeomorphism

from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$. A similar result is true also for quasiconformal mappings ([21], pp 42-44). On the other hand, an analogue of Carathéodory's result cannot be proved for harmonic mappings. The following example shows that whenever the dilatation of a mapping is not bounded away from one, irregular behaviour may occur.

Before stating the example, we give the definition of the cluster set. For a function f defined in the unit disk, we say that the *cluster set* of f at the point e^{it} , is the quantity

$$C(f, e^{it}) = \bigcap_U \overline{f(U \cap \mathbb{D})},$$

where the intersection is taken over all neighborhoods U of e^{it} .

Example 3. Let f be the function given by the Poisson formula

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} e^{i\theta(t)} dt, \quad z \in \mathbb{D},$$

where

$$\theta(t) = \begin{cases} 0, & 0 \leq t \leq 2\pi/3, \\ 2\pi/3, & 2\pi/3 \leq t \leq 4\pi/3, \\ 4\pi/3, & 4\pi/3 \leq t < 2\pi. \end{cases}$$

Then f maps univalently $\mathbb{D}$ onto the triangle with vertices at the points $1, e^{2\pi i/3}, e^{4\pi i/3}$. Concerning its boundary function, the cluster sets of f at the points $1, e^{2\pi i/3}$ and $e^{4\pi i/3}$ are the three sides of the triangle and the three circular arcs that they define are mapped to the three vertices. This shows that the boundary function of f is not a homeomorphism on the unit circle.

In 1926, T. Radó conjectured that if ϕ is a homeomorphism from the boundary of the unit disk onto a closed curve, bounding a convex domain, then the solution f of the Dirichlet problem is univalent in $\mathbb{D}$. In the same year, H. Kneser proved the following stronger form

Theorem 2.3.1. Let ϕ be a homeomorphism from the unit circle $\mathbb{T}$ onto the boundary Γ of a bounded Jordan domain Ω . Then, the solution f of the Dirichlet problem, given by

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} \phi(e^{it}) dt,$$

is univalent if and only if $f(\mathbb{D}) = \Omega$.

Note that the quantity $P(z, t) = \frac{1 - |z|^2}{|e^{it} - z|^2}$ is equal to the Poisson kernel defined earlier.

Not having knowledge about Kneser's result, Gustave Choquet gave a second proof on Radó's conjecture in 1945 ([10]). Today, the theorem is

known as the Radó-Kneser-Choquet Theorem.

Sense-preserving harmonic mappings can be seen as the solutions of linear elliptic partial differential equations of the form

$$\overline{f_z(z)} = a(z)f_z(z), \quad z \in \mathbb{D}, \quad (2.1)$$

where a is an analytic mapping of the unit disk into itself. A famous conjecture was that a harmonic counterpart of the Riemann mapping theorem also exists. More precisely, for a simply connected domain Ω containing a point w_0 and a given analytic function a , defined as above, there is a unique harmonic mapping f from the unit disk onto Ω , fulfilling the equation (2.1), with $f(0) = w_0$ and $f_z(0) > 0$. The existence part of the conjecture appears to be true under the restriction $|a(z)| \leq k < 1$ in $\mathbb{D}$. The uniqueness part has not been proved for arbitrary simply connected domains yet. The closest result to Riemann's theorem was given by W. Hengartner and G. Schober in [16].

Theorem 2.3.2. *Let Ω be a bounded, simply connected Jordan domain whose boundary is locally connected. Suppose that $a(\mathbb{D}) \subset \mathbb{D}$ and w_0 is a fixed point in Ω . Then there exists a univalent function f such that $\overline{f_z(z)} = a(z)f_z(z)$ with the following properties*

- (a) $f(0) = w_0$, $f_z(0) > 0$ and $f(\mathbb{D}) \subset \Omega$.
- (b) *There is a countable set $E \subset \partial\mathbb{D}$ such that the unrestricted limits $f^*(e^{it}) = \lim_{z \rightarrow e^{it}} f(z)$ exist on $\partial\mathbb{D} \setminus E$ and they are on $\partial\Omega$.*
- (c) *The functions*

$$f^*(e^{it-}) = \operatorname{ess\,lim}_{s \uparrow t} f^*(e^{is}) \text{ and } f^*(e^{it+}) = \operatorname{ess\,lim}_{s \downarrow t} f^*(e^{is})$$
exist on $\partial\mathbb{D}$ and are equal on $\partial\mathbb{D} \setminus E$.
- (d) *The cluster set of f at $e^{it} \in E$ is the straight line segment joining $f^*(e^{it-})$ with $f^*(e^{it+})$.*

The boundary function f^* is said to have a *jump discontinuity* at every point of the set E . Otherwise it is continuous. The function f is called a *Generalised Riemann Mapping* from the unit disk onto Ω . Here, the term "onto" is used in a weaker sense than the usual one, as there could exist points in Ω which are not images of any point in $\mathbb{D}$ under the mapping f .

In [3], the authors obtained necessary and sufficient conditions for a Generalised Riemann Mapping to have continuous boundary function at a point of $\partial\mathbb{D}$.

Theorem 2.3.3. *Let $f = h + \bar{g}$ be a Generalised Riemann Mapping from the unit disk onto a bounded Jordan domain Ω with a rectifiable boundary. Then, the limits $\lim_{r \rightarrow 1-} (1-r)h'(re^{i\theta_0}) = c$ and $\lim_{r \rightarrow 1-} (1-r)g'(re^{i\theta_0}) = d$ exist, are finite and satisfy $ce^{i\theta_0} = \overline{de^{i\theta_0}}$. Moreover, either limit is zero if and only if f^* is continuous at $e^{i\theta_0}$.*

A second result from [4] about the continuity of the boundary function when the dilatation is a Blaschke product in $\mathbb{D}$ is the following

Theorem 2.3.4. *Let $f = h + \bar{g}$ be a Generalised Riemann Mapping from the unit disk onto a bounded convex domain Ω , whose dilatation a_f is a Blaschke product with zeros ζ_n , $n = 1, 2, \dots$. If*

$$\sum_{n=1}^{\infty} \frac{1 - |\zeta_n|}{|e^{i\theta_0} - \zeta_n|} = \infty,$$

then f^ is continuous at $e^{i\theta_0}$.*

2.4 Modulus of Families of Paths

The *modulus* of a family of paths Γ in a domain Ω , is a conformally invariant quantity, which characterises Ω with respect to Γ . A continuous mapping $\gamma : I \rightarrow \mathbb{C}$, where I is an interval on the real line, is called a *path* on the complex plane. Consider a domain Ω and a family of paths Γ in Ω . We denote by $\mathcal{F}(\Gamma)$ the set of all Borel functions $\rho : \Omega \rightarrow [0, \infty]$ with

$$\int_{\gamma} \rho(z) |dz| \geq 1,$$

for all paths $\gamma \in \Gamma$. Then, the modulus of Γ in Ω is given by the formula

$$m(\Omega, \Gamma) = \inf_{\rho \in \mathcal{F}(\Gamma)} \iint_{\Omega} \rho^2(z) dx dy.$$

Let Ω be a domain in $\mathbb{C}$. Then, the following properties hold for the modulus of the families Γ_j , $j = 1, 2, \dots$ in Ω .

- $m(\Omega, \emptyset) = 0$.
- If $\Gamma_1 \subset \Gamma_2$ in Ω , then $m(\Omega, \Gamma_1) \leq m(\Omega, \Gamma_2)$.
- $m(\Omega, \cup_j \Gamma_j) \leq \sum_j m(\Omega, \Gamma_j)$.
- Let $f : \Omega \rightarrow \Omega'$ be a conformal mapping, with $\Omega' = f(\Omega)$ and $\Gamma'_1 = f(\Gamma_1)$. Then $m(\Omega, \Gamma_1) = m(\Omega', \Gamma'_1)$.

The last property shows that the modulus of a family of paths remains invariant under conformal mappings.

Example 4. We consider the annulus $D = R(\mathbb{D}(R), \mathbb{D}(R')) = \{z = re^{i\theta} : R < r < R', 0 < \theta \leq 2\pi\}$. Let Γ be the family of those curves that connect the opposite boundary components of D . For the function

$$\rho^*(z) = \frac{1}{|z| \log \frac{R'}{R}}, \quad z \in D,$$

we have that

$$\int_{\gamma} \rho^*(z) |dz| \geq \frac{1}{\log \frac{R'}{R}} \int_R^{R'} \frac{1}{t} dt = 1,$$

for every curve $\gamma \in \Gamma$ and hence, $\rho^* \in \mathcal{F}(\Gamma)$.

In addition,

$$\iint_D \rho^*(z) dx dy = \left(\log \frac{R'}{R} \right)^2 \int_R^{R'} \int_0^{2\pi} \frac{1}{|re^{i\theta}|^2} r d\theta dr = \frac{2\pi}{\log \frac{R'}{R}},$$

which implies that $m(D, \Gamma) \leq \frac{2\pi}{\log \frac{R'}{R}}$.

For an arbitrary function $\rho \in \mathcal{F}(\Gamma)$ it holds that $\int_R^{R'} \rho(re^{i\theta}) dr \geq 1$ and thus,

$$\iint_D \frac{\rho(z)}{|z|} dx dy = \int_0^{2\pi} \int_R^{R'} \rho(re^{i\theta}) dr d\theta \geq 2\pi.$$

Finally,

$$\begin{aligned} 0 &\leq \iint_D \left(\frac{1}{|z| \log \frac{R'}{R}} - \rho(z) \right)^2 dx dy \\ &= \iint_D \rho^2(z) dx dy - \frac{2}{\log \frac{R'}{R}} \iint_D \frac{\rho(z)}{|z|} dx dy + \iint_D \left(\frac{1}{|z| \log \frac{R'}{R}} \right)^2 dx dy \\ &\leq \iint_D \rho^2(z) dx dy - \frac{2\pi}{\log \frac{R'}{R}}, \end{aligned}$$

or $\iint_D \rho^2(z) dx dy \geq \frac{2\pi}{\log \frac{R'}{R}}$, for any $\rho \in \mathcal{F}(\Gamma)$. therefore, we deduce that

$$m(D, \Gamma) = \frac{2\pi}{\log \frac{R'}{R}}.$$

A domain Ω in $\hat{\mathbb{C}}$ is called a *ring* if $\mathbb{C} \setminus \Omega$ consists of two components. Suppose that E and F are the boundary components of Ω . Then $\Delta(E, F; \Omega)$ denotes the family of paths in Ω , connecting E with F .

In certain cases that the domain Ω , where the family of paths Γ is defined, is either obvious or specified by the definition of the Borel functions ρ , the simplified notation $\mathcal{M}(\Gamma)$ is preferred, instead of $m(\Omega, \Gamma)$. Therefore, the modulus of the above-mentioned family of paths, joining the two boundary components of a ring domain, will be denoted by $\mathcal{M}(\Delta(E, F; \Omega))$.

2.5 Bohr's inequality

Harald Bohr was born in 1887 in Denmark. He was the younger brother of the nobelist Niels Bohr, who is known for his work on the atomic structure and the quantum theory. Harald studied mathematics at the University of Copenhagen and he became a professor at the Copenhagen's Polytechnic Institute, in 1915. He mainly worked on the field of almost periodic functions, while he is mostly known for the Bohr-Landau theorem, about the zeros of the zeta function. Bohr died on January 22, 1951, near Copenhagen.

In 1914, Bohr wrote a paper about the upper bound of the majorant series of a bounded analytic function in the unit disk (see [5]). More precisely, the simplest version of the so-called Bohr theorem (or Bohr inequality) states that

Theorem 2.5.1. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic function in $\mathbb{D}$, bounded by 1, then*

$$\sum_{n=0}^{\infty} |a_n| r^n \leq 1, \text{ for } |z| = r \leq 1/3,$$

and the inequality is sharp.

The series $\sum_{n=0}^{\infty} |a_n| r^n$ is called the *majorant series* of f .

H. Bohr originally proved the above result for $r \leq 1/6$ and then, M. Riesz, I. Schur and F. Wiener showed independently that it holds for all $r \leq 1/3$, which cannot be further improved. The constant $1/3$ is the *Bohr radius* for the analytic self-maps of the unit disk. The term is also used for any radius giving a sharp Bohr-type inequality.

Bohr realised that this type of inequalities are related to the convergence of the ordinary Dirichlet series. Although he proved the above theorem as a part of his study on the Dirichlet series and his attempt to prove the Riemann hypothesis, he noticed that the result could be of independent interest. However, for almost 80 years, only few researchers worked in this area. In 1995, Dixon [11] used Bohr's inequality to disprove an open question concerning Banach algebras satisfying the von Neumann inequality. Since then, Bohr's theorem has attracted the attention of several mathematicians and many Bohr-type results have been proved in different settings (see for example [2, 4, 6, 13]). In 2010, Y. Abu-Muhanna [1] was the first to prove a Bohr-type inequality for harmonic mappings. In this harmonic version of the classical theorem, Abu-Muhanna considered a harmonic mapping of the unit disk into itself:

Theorem 2.5.2. *Let $f(z) = h(z) + \overline{g(z)} = \sum_{k=0}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}$ be a harmonic mapping in $\mathbb{D}$, with $|f(z)| < 1$ for all $z \in \mathbb{D}$. Then,*

$$\sum_{k=1}^{\infty} (|a_k| + |b_k|) r^k \leq \frac{2}{\pi}, \text{ for } r \leq 1/3.$$

The main problem with considering Abu-Muhanna's theorem as a generalisation of Bohr's inequality to harmonic mappings is the absence of the constant term from the last inequality. However, the problem cannot be solved for harmonic mappings in the classical setting, where f is a self-map of the unit disk with majorant series bounded by one. A counterexample, given in [1], makes it clear.

Example 5. Consider a real-valued harmonic mapping f in the unit disk, such that $f(0) = 0$ and $|f(z)| < 1$ in $\mathbb{D}$. Then, $f(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{a_n z^n}$.

The mapping

$$F_{\beta}(z) = f(z) \sin \beta + i \cos \beta, \quad \beta \in \mathbb{R}$$

is still harmonic, with $|F_{\beta}(z)| < 1$ in $\mathbb{D}$. If $a_n \neq 0$, then

$$\frac{|\sin \beta| |a_n + a_n|}{1 - |\cos \beta|} \rightarrow \infty \quad \text{as } \beta \rightarrow 0.$$

Hence, there is not any $r > 0$ such that

$$|i \cos \beta| + \sum_{n=1}^{\infty} |\sin \beta (a_n + a_n)| r^n \leq 1.$$

Another harmonic analogue of Bohr's inequality was proved by Kayumov, Ponnusamy, and Shakirov in [19]. There, the authors take the constant term into account and change the starting assumption into sense-preserving mappings, with bounded analytic part. In particular, they prove that

Theorem 2.5.3. If $f(z) = h(z) + \overline{g(z)} = \sum_{k=0}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}$ is a harmonic mapping in $\mathbb{D}$, with $|h(z)| \leq 1$ and $|g'(z)| \leq |h'(z)|$ for all $z \in \mathbb{D}$, then

$$|a_0| + \sum_{k=1}^{\infty} (|a_k| + |b_k|) r^k \leq 1, \quad \text{for } r \leq 1/5$$

and the inequality is sharp.

References

- [1] Y. Abu-Muhanna, Bohr's phenomenon in subordination and bounded harmonic classes, *Complex Var. Elliptic Equ.* **55**(11) (2010), 1071-1078.
- [2] C. Bénéteau, A. Dahlner, and D. Khavinson, Remarks on the Bohr phenomenon, *Comput. Methods Funct. Theory* **4**(1) (2004), 1-19.
- [3] L. Bers, Isolated singularities of minimal surfaces, *Ann of Math.* **53** (1951), 364-386.
- [4] H. P. Boas and D. Khavinson, Bohr's power series theorem in several variables, *Proc. Am. Math. Soc.* **125**(10) (1997), 2975-2979.
- [5] H. Bohr, A theorem concerning power series, *Proc. London Math. Soc.* **13**(2) (1914), 1-5.
- [6] E. Bombieri and J. Bourgain, A remark on Bohr's inequality, *Int. Math. Res. Not.* **80** (2004), 4307-4330.
- [7] D. Bshouty and W. Hengartner, Univalent harmonic mappings in the plane, *Handbook of complex analysis: Geometric function theory* **2** (2005), 479-506.
- [8] D. Bshouty, A. Lyzzaik, and A. Weitsman, On the boundary behavior of univalent harmonic mappings, *Ann. Acad. Sci. Fenn. Math.* **37** (2012), 135-147.
- [9] J. G. Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **9** (1984), 3-25.
- [10] G. Choquet, Sur un type de transformation analytique généralisant la représentation conforme et définie au moyen de fonctions harmoniques, *Bull. Sci. Math.* **69** (1945), 156-165.
- [11] P. G. Dixon, Banach algebras satisfying the non-unital von Neumann inequality, *Bull. London Math. Soc.* **27**(4) (1995), 359-362.

- [12] P. L. Duren, *Harmonic mappings in the plane*, Cambridge University Press, New York, NY, USA (2004).
- [13] R. Fournier and St. Ruscheweyh, On the Bohr radius for simply connected plane domains, *Centre de Recherches Mathématiques CRM Proceedings and Lecture Notes* **51** (2010), 165-171.
- [14] E. Heinz, Über die Lösungen der Minimalflächengleichung, *Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl.* (1952), 51-56.
- [15] E. Heinz, On one-to-one harmonic mappings, *Pacific J. Math.* **9** (1959), 101-105.
- [16] W. Hengartner and G. Schober, Harmonic mappings with given dilatation, *J. London Math. Soc.* (2) **33** (1986), 473-483.
- [17] W. Hengartner and G. Schober, On the boundary behavior of orientation-preserving harmonic mappings, *Complex Variables Theory Appl.* **5** (1986), 197-208.
- [18] W. Hengartner and G. Schober, Univalent harmonic functions, *Trans. Amer. Math. Soc.* **299** (1987), 1-31.
- [19] I. R. Kayumov, S. Ponnusamy, and N. Shakirov, Bohr radius for locally univalent harmonic mappings, *Math. Nachr.* **11-12** (2018), 1757-1768.
- [20] H. Kneser, Lösung der Aufgabe 41, *Jahresber. Deutsch. Math.-Verein.* **35** (1926), 123-124.
- [21] O. Lehto and K. Virtanen, *Quasiconformal mappings in the plane*, Springer-Verlag, New York-Heidelberg, 1973.
- [22] H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings *Bull. Amer. Math. Soc.* **42** (1936), 689-692.
- [23] P. T. Mocanu, Sufficient conditions of univalence for complex functions in the class C^1 , *Rev. Anal. Numér. Théor. Approx.* **10** (1981), 75-79.
- [24] J. C. C. Nitsche, On Harmonic Mappings, *Proc. Amer. Math. Soc.* **9** (1958), 268-271.
- [25] S. Ponnusamy, A. Rasila, Planar harmonic and quasiregular mappings, *Topics in Modern Function Theory: Chapter in CMFT, RMS-Lecture Notes Series* No. **19** (2013).
- [26] T. Radó, Über den analytischen Charakter der Minimalflächen, *Math. Z.* **24** (1926), 321-327.
- [27] A. Vasil'ev, *Moduli of Families of Curves for Conformal and Quasiconformal Mappings*, Springer-Verlag (2002).

ISBN 978-952-64-0294-9 (printed)

ISBN 978-952-64-0295-6 (pdf)

ISSN 1799-4934 (printed)

ISSN 1799-4942 (pdf)

Aalto University
School of Science
Department of Mathematics and Systems Analysis
www.aalto.fi

**BUSINESS +
ECONOMY**

**ART +
DESIGN +
ARCHITECTURE**

**SCIENCE +
TECHNOLOGY**

CROSSOVER

**DOCTORAL
DISSERTATIONS**