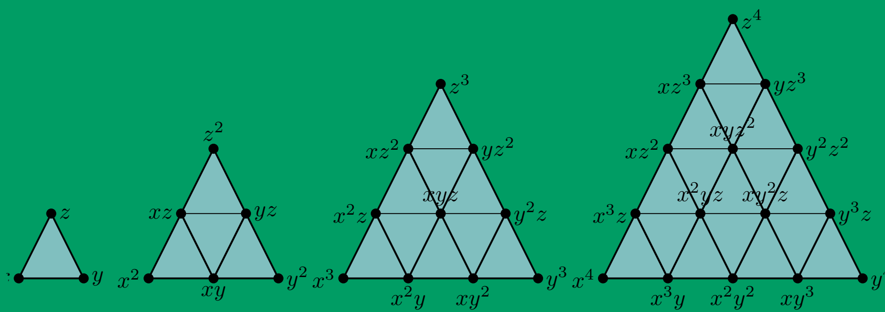
Department of Mathematics and Systems Analysis

Representation Stability for Cellular Resolutions

Laura Jakobsson



Aalto University

DOCTORAL DISSERTATIONS

Aalto University publication series
DOCTORAL DISSERTATIONS 17/2021

Representation Stability for Cellular Resolutions

Laura Jakobsson

A doctoral dissertation completed for the degree of Doctor of Science (Technology) to be defended, with the permission of the Aalto University School of Science, at a public examination held at the lecture hall M1 of the school and also via remote technology, on 25 March 2021 at 18:00.

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Aalto University publication series
DOCTORAL DISSERTATIONS 17/2021

ISBN 978-952-64-0263-5 (printed)
ISBN 978-952-64-0264-2 (pdf)
ISSN 1799-4934 (printed)
ISSN 1799-4942 (pdf)
http://urn.fi/URN:ISBN:978-952-64-0264-2

Unigrafia Oy
Helsinki 2021

Finland

Printed matter
4041-0619

Aalto University

Aalto University, P.O. Box 11000, FI-00076 Aalto www.aalto.fi

Abstract

Author
Laura Jakobsson

Name of the doctoral dissertation
Representation Stability for Cellular Resolutions

Publisher School of Science

Unit Department of Mathematics and Systems Analysis

Series Aalto University publication series DOCTORAL DISSERTATIONS 17/2021

Field of research Mathematics

Manuscript submitted 14 April 2020 **Date of the defence** 25 March 2021

Permission for public defence granted (date) 14 September 2020 **Language** English

☐ **Monograph** ☒ **Article dissertation** ☐ **Essay dissertation**

Abstract

This thesis is on cellular resolutions and the invariants of resolutions of monomial ideals.
The general area of these topics is combinatorial commutative algebra, and as much of pure mathematics, the studied questions in the thesis are motivated mainly by fascination towards these combinatorial mathematical objects and applying new tools to study them.
The questions on resolutions and their invariants have been around for a long time, and over the years they have become a rich topic, with a variety of directions including cellular resolutions.
We look at cellular resolutions from a category-theoretic point of view and apply tools from representation stability to study them.
In Publication I, we define the category of cellular resolutions and establish the basic properties for it. Among these results are showing that homotopy colimit lifts from topology and that discrete and algebraic Morse maps are morphisms in this category.
Having the category of cellular resolutions opens up cellular resolutions for applying tools of representations of categories, and we use these in Publication II to show that that specific families of cellular resolutions have finitely generated syzygies. The main tools used are defining a linear family that satisfies noetherianity properties of representation stability and viewing syzygies as a representation of the category of cellular resolutions. In particular, we show that the powers of maximal monomial ideals of a polynomial ring have finitely generated syzygies. We also touch upon the case of families with cellular resolutions over different rings and show that the finite generation of syzygies applies in this setting in special cases.
The last publication covers combinatorial formulas for algebraic invariants of edge ideals of Booth-Lueker graphs.

Keywords Cellular resolutions, representation stability

ISBN (printed) 978-952-64-0263-5 **ISBN (pdf)** 978-952-64-0264-2

ISSN (printed) 1799-4934 **ISSN (pdf)** 1799-4942

Location of publisher Helsinki **Location of printing** Helsinki **Year** 2021

Pages 195 **urn** http://urn.fi/URN:ISBN:978-952-64-0264-2

Preface

Firstly, I would like to express my gratitude to my advisor Alexander Engström. I am thankful for the guidance and help I have received, and I am glad to have been introduced to the topic in this thesis by him. I appreciate that I have been allowed to explore research directions as suits my interest, but receiving help and guidance when needed.

I want to extend my thanks to rest of the research group as well: I want to thank Milo Orlich for being a joy to collaborate with and having many helpful, but perhaps one-sided, conversations over the years; I thank Florian Kohl for the guidance, in math and other academic matters, and also for proofreading parts of my thesis.

Also, I want to thank Oleksandra Gasanova for proofreading the entire thesis and giving many useful comments during the thesis writing. I want to thank both Fatemeh Mohammadi and Veronica Crispin Quinonez for inviting me for research visits to Bristol University and Uppsala University, respectively, and to give talks during my thesis project, and thank them for the enlightening discussions we have had.

Next, I want to thank my pre-examiners Anton Dochtermann and Bruno Benedetti, and also I want to thank Anton Dochtermann for agreeing to be my opponent.

I would like to thank everyone at the Aalto math community as well.

Finally, I thank my family and friends for the support and encouragement I have received from them. Particular thanks goes to my beloved Taoufiq.

Helsinki, February 3, 2021,

Laura Jakobsson

Contents

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I Laura Jakobsson. The category of cellular resolutions. Submitted to *Journal of Commutative Algebra*, submission date April 2019.

II Laura Jakobsson. Families of cellular resolution, their syzygies, and stability. Submitted to *Algebraic Combinatorics*, submission date March 2020.

III Alexander Engström, Laura Jakobsson, and Milo Orlich. Explicit Boij-Söderberg theory of ideals from a graph isomorphism reduction. Accepted for publication in *Journal of Pure and Applied Algebra*, 25pp., February 2020.

Author's Contribution

Publication I: "The category of cellular resolutions"

The author did everything.

Publication II: "Families of cellular resolution, their syzygies, and stability"

The author did everything.

Publication III: "Explicit Boij-Söderberg theory of ideals from a graph isomorphism reduction"

The last two authors of the article contributed equally to the article, largely wrote the article, and established the results. The first author of the article proposed the problem and computed examples.

1. Introduction

This thesis is on representation stability of cellular resolutions and invariants of resolutions of monomial ideals. The general area of these topics is combinatorial commutative algebra, and as much of pure mathematics, the questions studied in the thesis are motivated mainly by fascination towards these mathematical objects and applying new tools for studying them.

The main algebraic objects that appear in this thesis are all related to resolutions. A resolution of a finitely generated graded module is a way to interpret the structure of the module, and every module has a resolution. We will be using free resolutions, where this information on the module will be stored in free modules and maps between those. Historically, resolutions first appeared in the work of Hilbert [35] and have been an active research topic ever since. From the early days of studying resolutions, we have known they can be algorithmically computed, but the results may not be unique since a module has multiple resolutions. The primary way to combat this issue is to require minimal resolutions that solve this issue up to isomorphism. A central piece of information that a resolution can give is the knowledge of the relations between the generators of a module, and the relations between those relations, and going on with all the relations of relations that follow. These are called the syzygies of the module. Syzygies have also given the name to a central theorem in studying resolutions, namely the Hilbert syzygy theorem, that states a resolution will always be finite. In terms of the relations, this means eventually they stop.

Over the years since Hilbert, many exciting results have been published on resolutions, for example, a very recent result of Eagon, Miller, and Ordog showing how to construct a canonical minimal resolution for a monomial ideal [21], to name one. Resolutions play a central role in studying invariants for modules and interact well with many functorial constructions, all of which contributes to their importance as mathematical objects. There have been different directions taken with the study of resolutions, and the two main ones appearing in this thesis are cellular resolutions for monomial modules and Boij–Söderberg theory.

A cellular resolution is a resolution of a monomial module that also contains the homological structure of a cell complex. It connects an algebraic construction to a combinatorial and topological object. Cellular resolutions were first defined and studied by Bayer, Peeva, and Sturmfels [4, 5]. With cellular resolutions, minimality is often a desired condition, too, and many results are on showing that minimal cellular resolutions exist for a specific class of ideals, for instance, the cointerval ideals by Dochtermann and Engström [19]. However, a significant result on cellular resolutions is that by Velasco [55] that shows that not every monomial ideal has a minimal cellular resolution. Another significant result on cellular resolutions is that they are compatible with Morse theory as shown by Batzies and Welker [3], and the examples of cellular resolutions in their work play a significant role in the results of this thesis. The interest in minimality has also motivated the application of discrete Morse theory to cellular resolutions, for example, in [41] it is shown how to make a resolution closer to a minimal one and Engström and Norén use it in [26] to show the existence of minimal resolutions for powers of certain ideals.

The other direction that our results take is the Boij–Söderberg theory. To each finitely generated graded module M one can associate numerical invariants called the graded betti numbers of M, which can be arranged in the so-called betti table of M. These betti numbers can be found in the minimal free resolution of a given module. The problem of characterising the possible betti tables of modules has been open for many decades, and only relatively recently a breakthrough was obtained by Boij and Söderberg [6]. They thought of betti tables as elements of a vector space and provided conjectures describing the cone in which all possible betti tables live. These conjectures were proved soon afterwards, and as a consequence, it turns out that the extremal rays of this cone are what we call pure betti tables. The cone interpretation can then give the central theorem of Boij–Söderberg theory: the betti table of a finitely generated graded module can be expressed as a weighted sum of pure tables with positive rational coefficients. This provides an excellent method of studying the betti numbers of modules and ideals.

Despite all the known facts about cellular resolutions, they have not been studied as a class of objects. There has been discussions on the general structure of cellular resolutions, see for example [20] for an open question on “moduli spaces” for a family of cellular resolutions, and even these cases often focus on the structure of a particular family of cellular resolutions. A natural question would be to ask how cellular resolutions behave in a more category-theoretic setting. This approach is supported by the existing conversation on higher structures on cellular resolutions, and that category theory is a fundamental tool in studying these in other fields like algebraic geometry and representation stability. These ideas give the fundamental areas of this thesis outside of commutative algebra: category theory and representation stability. Category theory was pioneered by Eilenberg and MacLane in the 1940s [22], and has since become a standard language to discuss mathematics. Representation stability is, on the other hand, much more recent. The term was first coined in the current use by Church and Farb in 2012 [15]; however, it does have connections to older ideas of homological stability, in particular in topology on cohomology and configuration spaces. The main idea behind representation stability, especially in its infancy, is to apply homological stability to an infinite sequence of representations. This has then been expanded by Ellenberg, Farb, Church and Nagpal [13, 14] to study what is called FI-modules, which form one of the prevailing directions of representation stability with very active research. Roughly at the same time as much of the expansion of representation stability from the first paper was growing, Sam and Snowden released their work on representations of combinatorial categories [48]. Their work can be seen as a more algebraic generalisation of the ideas of representation stability, and the results presented in [48] recover many of the previous ones. We are inspired by the paper of Sam and Snowden [48], which generalises ideas from representation stability, and a part of the thesis applies their tools to cellular resolutions. Many preliminary versions of the theorems appeared in earlier works on representation stability like those of Church and Farb [15], and Church, Ellenberg, and Farb [13].

The first results in this thesis are the generalisation of the definition of cellular resolutions to cases where the cell complex may not be connected, followed by a definition of what a map between two cellular resolutions is. The main idea behind the chosen morphisms is the concept of compatible cellular and chain maps, which says that "they both do the same thing". These lay the basis for defining a category of cellular resolutions and all the further category-theoretic results. Our main result from Publication I is the definition of the category of cellular resolutions, **CellRes**, and that it does indeed form a category.

Further results in Publication I focus on the study of the typical constructions in **CellRes** and note other worthwhile observations. These include mapping cones and cylinders, (co)products and (co)limits. Throughout these sections, we see the repeating pattern of well-behaved constructions if topological and algebraic constructions are essentially the same. Otherwise, they may not even exist in the category **CellRes** in general. An example of the non-existence is the product, which we do not have in the category. Among the significant results that appear is the homotopy colimits of cellular resolutions. They are a well-known construction in topology, and we show that the explicit construction lifts to **CellRes**. The final topic touched upon in this setting is discrete Morse theory on cellular resolutions. We show that the algebraic Morse theory and the discrete Morse theory for cellular resolutions work well together, that is, a pair of a cellular map and a chain map that come from the same Morse matching forms a morphism of cellular resolutions. This result shows that Morse maps are well behaved with respect to algebra and topology on cellular resolutions. Furthermore, the results on Morse theory give a basis for simple homotopy theory for cellular resolutions.

The computational results on cellular resolutions have long suggested that in some families we have finitely generated syzygies given by some finite number of resolutions in the family; however, there has not been a proof of this. In Publication II, we study the families of cellular resolutions from a categorical perspective and through using representations of categories that is motivated by computational results on syzygies. Using the tools from representation stability, we establish conditions for families of cellular resolutions to have finitely generated syzygies. The main idea is to define the syzygies as a representation of the family and then show finite generation for this representation using noetherianity and covering of the cell complexes. This method allows us to use families with non-minimal cellular resolutions to study the syzygies. Our main results from this publication are showing that a family of cellular resolutions satisfying linearity conditions will have a noetherian representation category and that a family of cellular resolutions with a noetherian representation category will have finitely generated syzygies if the cell complexes are covered.

Other than being able to show that certain families have finitely generated syzygies, using categorical representations to study cellular resolutions can give new insights into them and structures of families, in particular, it seems to be suited to studying cellular resolutions of powers of ideals. This is highlighted in the specific examples of the families we study. The powers of maximal ideals and edge ideals of paths are among the examples we prove to have finitely generated syzygies. This method does have limitations, and in particular, having to fix a polynomial ring for families that do not consist of powers of ideals turns out to be not such an interesting situation with edge ideals. Our last results propose, perhaps naive, approach to this situation by removing the requirement for a constant polynomial ring through defining a new category.

The results of Publication III change in theme towards the Boij–Söderberg theory. In general, there are algorithms to compute the Boij–Söderberg decomposition, but to the best of our knowledge, there are no general formulas that give it for any given module or even ideal. We show that the class of ideals considered in Publication III has a Boij–Söderberg decomposition that can be explicitly written. These ideals are edge ideals of graphs that come from a specific construction, the Booth-Lueker graph, linked to the graph isomorphism question. In one of the first results of complexity theory, Booth and Lueker [7] introduced the following construction: given a finite simple graph G with n vertices and m edges, define a new graph on $n+m$ vertices, which we call $BL(G)$. Booth and Lueker proved that two graphs G and G_0 are isomorphic if and only if their corresponding Booth–Lueker graphs $BL(G)$ and $BL(G_0)$ are isomorphic. This reduced the problem of graph isomorphism to the special class of Booth–Lueker graphs, which have several attractive properties. The main result of Publication III regards the edge ideal of the Booth–Lueker graph of a graph on n vertices and d_i vertices of degree i, for $i = 0,\dots,n-1$. We show that the weight of the pure table c_j with j non-zero entries on the second row, can be described explicitly in terms of n and d_i, and the other Boij–Söderberg coefficients vanish. Further on, we also describe the explicit Betti numbers and anti-lecture hall compositions of these ideals. Moreover, we study the dual graph situation, constructing the ideal of the complement of the Booth–Lueker graph. We show that if the original graph has n vertices and m edges, then the coefficients c_j only depend on the values of n and m, and the other coefficients vanish. We also give explicit results on Betti numbers and anti-lecture hall compositions for these ideals.

Lastly, we detail the outline of this thesis. Chapter 2 covers the relevant background material for the results in this thesis and defines in detail the concepts mentioned in this introduction. Chapter 3 gives an overview of the results of Publications I and II, and finally, Chapter 4 summarises the results of Publication III.

2. Mathematical preliminaries

This chapter covers the relevant background material for the thesis. We begin by going over the notions from category theory, commutative algebra and topology that are required for defining cellular resolutions. This is followed by a section on cellular resolutions, the definition and known results. The remaining sections are used to cover the background necessary for the results in this thesis.

2.1 Category theory

The language of much of modern mathematics is category theory. One of the goals of this thesis is to use categories to study cellular resolutions, and thus the definitions and concepts from category theory that are frequently used in this thesis are covered in this section. There are many good references for introductory category theory, like [8], [9], and [38], and for our primary reference on category theory, we will use the Chapter 1 from the book by Vakil [54].

One of the most important definitions that is required from category theory is the definition of a locally small category itself. Informally this can be thought of as a collection of objects that are linked by arrows. The collection of objects does not necessarily have to be a set, and thus one defines categories with classes, that are collections of sets.

Definition 2.1.1. *A locally small* category $\mathscr{C}$ *consists of a collection of objects* $\mathrm{obj}(\mathscr{C})$ *and a set of morphisms* $\mathscr{C}(a,b)$ *for each pair of objects* a,b. *For any triple* a,b,c *there is a composition map of the morphisms* $\mathscr{C}(b,c)\times\mathscr{C}(a,b)\to\mathscr{C}(a,c)$, *with the image of the pair* (ϕ,ψ) *denoted by* $\phi\circ\psi$. *The category* $\mathscr{C}$ *must satisfy the following two conditions:*

1. *For any object* $a\in\mathrm{obj}(\mathscr{C})$ *there exists an identity morphism* $\mathrm{id}_a\in\mathscr{C}(a,a)$ *such that* $\mathrm{id}_a\circ\phi=\phi$ *and* $\psi\circ\mathrm{id}_a=\psi$ *for any* $\phi:a\to b$ *and* $\psi:b\to a$.

2. *Composition of morphisms is associative, that is,* $(\phi\circ\psi)\circ\chi=\phi\circ(\psi\circ\chi)$ *for all* ψ,χ, *and* ϕ.

A further requirement is that the morphism sets $\mathscr{C}(a,b)$ and $\mathscr{C}(c,d)$ are disjoint unless $a = c$ and $b = d$.

We say that a category $\mathscr{C}$ is small *if the objects and morphisms form a set.*

Example 2.1.2. Common examples of categories include the category of sets **Set**, where the objects are sets and the morphisms are set maps, the category of topological spaces **Top** with objects being topological spaces and morphisms are continuous maps, and the category of S-modules Mod_S, where objects are modules over the ring S and morphisms are module homomorphisms.

A *subcategory* $\mathscr{C}'$ of $\mathscr{C}$ is a category where $\text{obj}(\mathscr{C}') \subseteq \text{obj}(\mathscr{C})$ and morphisms of $\mathscr{C}$ such that the source, target, and composition are the same as in $\mathscr{C}$. A subcategory $\mathscr{C}'$ of $\mathscr{C}$ is *full* if $\mathscr{C}'(a,b) = \mathscr{C}(a,b)$ for any pair $a,b \in \text{obj}(\mathscr{C}')$.

There are two categories in particular that we want to focus on, the *category of topological spaces* and *the category of chain complexes of S-modules*. The properties of these two categories influence the category of cellular resolutions and thus we will look at them in more detail in this section. we note that any of the topology books listed in Section 2.3 is a suitable reference for the category of topological spaces and [56] is a good introduction to the category of chain complexes.

The *category of topological spaces*, denoted by **Top**, is a category that has topological spaces as the objects and for any two spaces X, Y the set of morphisms $\textbf{Top}(X,Y)$ consists of all continuous maps between X and Y. CW-complexes and cellular maps form a subcategory of **Top**.

The *category of chain complexes* $C_\bullet(\text{Mod}_S)$ is the category with the objects being chain complexes of objects of the category Mod_S

$$\mathbf{C}: \ \ldots \leftarrow C_0 \leftarrow C_1 \leftarrow \ldots \leftarrow C_n \leftarrow \ldots$$

where C_i is in Mod_S and the maps, sometimes called differentials, $\partial_k : C_k \to C_{k-1}$ such that $\partial_i \circ \partial_{i+1} = 0$. The morphisms are given by *chain maps*, that is, a collection of module homomorphisms $\mathbf{f} = \{f_i\}$ with $f_i : C_i \to D_i$, such that the squares

$$\begin{array}{ccccc} \cdots \leftarrow & C_i & \leftarrow & C_{i+1} & \leftarrow \cdots \\ & \downarrow & & \downarrow & \\ \cdots \leftarrow & D_i & \leftarrow & D_{i+1} & \leftarrow \cdots \end{array} .$$

commute. The commuting square means the map is the same whether the first map is taken to be the differential map in the chain and then the module homomorphism, or vice versa, the resulting composition map is the same both ways. The commuting square property can also be expressed as

$$f_i \circ \partial_{i+1} = \partial_{i+1} \circ f_{i+1}.$$

Remark 2.1.3. We have stated the definition for the category of chain complexes of S-modules; however, chain complexes can be defined for any additive category [56, Chapter 1].

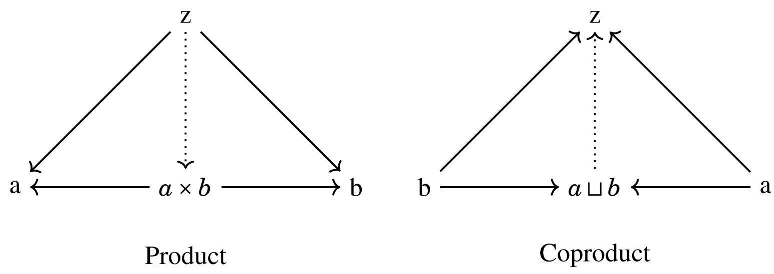


Figure 2.1. The diagrams for product and coproduct in a category $\mathscr{C}$.

The definitions in category theory are often unique up to an isomorphism. What it means is to say that the objects are essentially unique, and isomorphic objects are close enough to be considered the same in these circumstances. This uniqueness often comes from defining concepts in category theory via the universal property. A universal property provides a way to define properties and objects with the assurance that the resulting object has no more than the desired properties; that is, it is the universal object for that definition.

An object $a \in \mathscr{C}$ is said to be an *initial object* if for all objects $b \in \text{obj}(\mathscr{C})$ there is a single morphism $a \to b$. Similarly a is a *final object* if there is a unique morphism $b \to a$ for all $b \in \text{obj}(\mathscr{C})$. If the initial and final objects exists, they are unique up to an isomorphism.

The product and coproduct constructions play a significant role for the properties that the category of cellular resolutions has.

Definition 2.1.4. *A* product *of two objects a, b in the category $\mathscr{C}$ is an object $a \times b$ such that there exist morphisms $f : a \times b \to a$ and $g : a \times b \to b$, such that for any object z mapping both to a and b there exists a unique morphism $z \to a \times b$ that makes the product diagram in Figure 2.1 commute.*

Definition 2.1.5. *A* coproduct *of two objects a, b in the category $\mathscr{C}$ is an object $a \sqcup b$ in $\mathscr{C}$ such that there exist morphisms $f : a \to a \sqcup b$ and $g : b \to a \sqcup b$, such that for any object z where both a and b map to, there exists a unique morphism $a \sqcup b \to z$ that makes the product diagram in Figure 2.1 commute.*

If a product or a coproduct exists, then they are unique up to unique isomorphism. Comparing the diagrams for product and coproduct one notices that they are essentially the same, only the arrows have been reversed. Such reversing of arrows is a common phenomenon in category theory and the definitions related by reversing the arrows often are separated by the co- prefix.

The next few definitions cover the limits and colimits in the category setting. First, a definition of a diagram in a category is required, and then one can proceed to state the definitions of a limit and a colimit.

A *diagram D* in a category $\mathscr{C}$ is a covariant functor $F : I \to \mathscr{C}$ where I is a small category and F_i denotes the image of $i \in \text{obj}(I)$, and for any $\phi : i \to i'$ there is a map $F(\phi) : F_i \to F_{i'}$.

Definition 2.1.6. *A* limit *of the diagram D is an object* $\lim D$ *with maps* $f_i : \lim D \to F_i$*, satisfying* $f_i = F(\phi) \circ f_j$ *for all* $\phi : i \to j$ *in* $\mathscr{I}$*, and for any* $W \in \mathrm{obj}(\mathscr{C})$ *and any family of maps* $t_i : W \to F_i$ *such that* $t_i = F(\phi) \circ t_j$ *for all* $\phi : i \to j$ *in* $\mathscr{I}$*, there exists a morphism* $t : W \to \lim D$ *such that* $t_i = f_i \circ t$ *for any object* $i \in \mathscr{I}$*.*

Definition 2.1.7. *A* colimit *of a diagram D in $\mathscr{C}$ is an object* $\mathrm{colim}\ D$ *in $\mathscr{C}$ with a map* $\iota_i : F_i \to \mathrm{colim}\ D$*. The colimit must satisfy* $\iota_i = \iota_j \circ F(\phi)$ *for all* $\phi : i \to j$ *in* $\mathscr{I}$*, and for any* $W \in \mathrm{obj}(\mathscr{C})$ *and any family of maps* $t_i : F_i \to W$ *such that* $t_i = t_j \circ F(\phi)$ *for all* $\phi : i \to j$ *in* $\mathscr{I}$*, there exists a morphism* $t : \mathrm{colim}\ D \to W$ *such that* $t_i = t \circ \iota_i$ *for any object* $i \in \mathscr{I}$*.*

If limits and colimits exist, they are unique up to a unique isomorphism.

Definition 2.1.8. *A map F between two categories $\mathscr{C}$ and $\mathscr{D}$ is called a* (covariant) functor *and consists of a map* $F : \mathrm{obj}(\mathscr{C}) \to \mathrm{obj}(\mathscr{D})$*. For all pairs* $a, b \in \mathrm{obj}(\mathscr{C})$ *there is a map* $F : \mathscr{C}(a,b) \to \mathscr{C}(F(a), F(b))$*. The functor F must also satisfy* $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$ *and* $F(id_a) = id_{F(a)}$*. A* contravariant functor *is a functor that is a map* $F : \mathscr{C}(a,b) \to \mathscr{C}(F(b), F(a))$ *for all pairs* $a, b \in \mathrm{obj}(\mathscr{C})$ *.*

One can also construct a category where the objects are functors. The morphisms between functors are given by natural transformations.

Definition 2.1.9. *A* natural transformation η *between two functors* $F, G : \mathscr{C} \to \mathscr{D}$ *is a collection of maps* $\{\eta_a : F(a) \to G(a)\}_{a \in obj(\mathscr{C})}$ *in $\mathscr{D}$ such that the diagram*

$$\begin{array}{ccc} F(a) & \xrightarrow{\eta_a} & G(a) \\ \downarrow & & \downarrow \\ F(b) & \xrightarrow{\eta_b} & G(b) \end{array}$$

commutes for any morphisms $\phi : a \to b$ *in* $\mathscr{C}$*. The functors F and G are said to be isomorphic if* η_a *is an isomorphism for all a, and η is called a* natural isomorphism.

Many of the known constructions in algebra and topology can be seen as functors, like the fundamental group of a space. Functors are not only useful to generalise the concept of functions but also for defining things in category theory. Let $F : \mathscr{C} \to \mathscr{D}$ and $F' : \mathscr{D} \to \mathscr{C}$ be functors between categories $\mathscr{C}$ and $\mathscr{D}$. Then $\mathscr{C}$ is *equivalent* to $\mathscr{D}$ if we have that $F \circ F' \cong \mathrm{id}_{\mathscr{C}}$ and $F' \circ F \cong \mathrm{id}_{\mathscr{D}}$ are natural isomorphisms where id denotes the identity functor. A category $\mathscr{C}$ is *essentially small* if it is equivalent to a small category.

As with subcategories, a *subfunctor* can be defined for a given functor $F : \mathscr{C} \to \mathscr{D}$ as the functor $F' : \mathscr{C} \to \mathscr{D}$ such that there is a natural transformation $i : F' \to F$ with components $i_x : F'(x) \to F(x)$ being monomorphisms. A monomorphism is a map $f : x \to y$ such that $h \circ f = g \circ f$ implies $h = g$ for maps $h, g : z \to x$.

Definition 2.1.10. *The category $\mathscr{C}$ is a* monoidal category *if it has a bifunctor* $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$*, an object e, a natural isomorphism* $\alpha : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$*, and natural isomorphisms* $\lambda : (e \otimes -) \to -$ *and* $\rho : (- \otimes e) \to -$*, that satisfy the triangle equality*

$$\rho_x \otimes 1_y(x,e,y) = (1_x \otimes \lambda_y) \circ \alpha(x,e,y)$$

and the pentagon identity

$$\alpha \otimes 1 \circ \alpha \circ 1 \otimes \alpha(x, y, z, w) = \alpha \circ \alpha(x, y, z, w).$$

Returning to our categories of interest, in the category of chain complexes of S-modules the product is given by the direct sum of two complexes. The direct sum of chain complexes $\mathbf{C}$ and $\mathbf{D}$ is $\mathbf{C} \oplus \mathbf{D}$ with $(\mathbf{C} \oplus \mathbf{D})_k = C_k \oplus D_k$ in the finite case. In the case of finite coproducts, they are also given by the direct sum. Limits and colimits can be computed degree wise in the category of chain complexes.

The category **Top** has an initial object, the empty space, as there is a continuous map from the empty space to any other topological space. The products in the category **Top** are just the usual products of topological spaces, where the underlying space is the Cartesian product, and it has the product topology. The coproducts in **Top** are disjoint unions of topological spaces.

Limits and colimits in **Top** are lifted from the category of sets, that is, the limit of the diagram D in **Top** is the limit of the diagram in the in the category of sets with initial topology, and final topology in the case of colimit. All finite limits and colimits exist in the category **Top**.

2.2 Commutative Algebra

The main references for this section are the books by Eisenbud [23, 24] for general commutative algebra and resolutions, and the one by Weibel [56] for chain complexes.

A *graded ring* is a ring S such that it has a direct sum decomposition

$$S = S_0 \oplus S_1 \oplus S_2 \oplus \dots$$

of abelian groups satisfying $S_i S_j \subseteq S_{i+j}$ for $i, j \geq 0$.

In particular, we are working over *polynomial rings* $S = k[x_1, x_2, \dots, x_n]$ where k is a field and n is some positive integer. These have a natural grading by taking elements in k to be the grade-0 elements and polynomials of degree d to be the elements grade-d in the graded ring. Unless otherwise specified, S will denote a graded polynomial ring.

A ring R is *noetherian* if it satisfies the ascending chain condition on ideals, that is, every strictly ascending chain of ideals stabilises. Equivalently, R is noetherian if every ideal is finitely generated.

A *graded S-module* is an module M with a decomposition $M = \bigoplus_{-\infty}^{\infty} M_i$ into abelian groups such that $S_i M_j \subseteq M_{i+j}$ for all i and j. A module is *finitely generated* if it has a finite generating set. A *free module* is a module that is isomorphic to a direct sum of copies of S, and in the finitely generated case it is denoted by S^k for direct sum of k copies. The ring S can be viewed as a free graded module over itself. Then one can define the shifted free module $S(-a)$ by $S(-a)_i = S_{i+a}$, and it is a free module with a generator in degree a.

A chain complex is *exact* if it satisfies $\operatorname{im} \partial_{i+1} = \ker \partial_i$ for all $i \geq 1$.

Let M be a S-module. A *free resolution* of M is a chain complex of free modules

$$\mathbf{F} : F_0 \xleftarrow{\partial_1} F_1 \leftarrow F_2 \leftarrow \ldots \xleftarrow{\partial_n} F_n \leftarrow \ldots$$

such that $\operatorname{coker} \partial_1 = M$ and $\mathbf{F}$ is exact. If M is a finitely generated S-module and $m_1, m_2, \ldots, m_k$ are the generators of M with degrees $d_1, d_2, \ldots, d_k$, then a free resolution can be constructed as follows: Take a free module $F_0 = \bigoplus_{i=1}^{k} S(-d_i)$ and define a map $\partial_0 : F_0 \to M$ by sending the generator e_i of $S(-d_i)$ to the generator m_i of M. The kernel M_1 of ∂_0 is also a finitely generated module, with some generators $g_1, g_2, \ldots, g_r$. Then we can define a free module $F_1 = \bigoplus_{i=1}^{r} S(-d'_i)$, where d'_i is the degree of g_i, and a map $\partial_1 : F_1 \to M_1$ by sending the generator of $S(-d'_i)$ to g_i in M_1. Again the kernel of this map ∂_1 is finitely generated S-module, and the process of defining a free module can be repeated and same for a map from that free module to the kernel. Iterating this process gives the following chain complex

$$M \xleftarrow{\partial_0} F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} \ldots \xleftarrow{\partial_i} F_i \leftarrow \ldots$$

The kernel M_1 is called the first *syzygy module* of M and the elements in M_1 are called syzygies. Similarly the kernel of ∂_i is called the i-th syzygy module of M.

Example 2.2.1. For this example let $S = k[x,y,z,w]$ and let $I = (xy, xz, xw, yzw)$ be a monomial ideal. Then the module S/I is finitely generated and has a resolution

$$S/I \leftarrow S(-0) \xleftarrow{\partial_1} S(-2)^3 \oplus S(-3) \xleftarrow{\partial_2} S(-3)^3 \oplus S(-4) \xleftarrow{\partial_3} S(-4) \leftarrow 0$$

with maps

$$\partial_1 = \begin{bmatrix} xy & xz & xw & yzw \end{bmatrix}, \partial_2 = \begin{bmatrix} -z & -w & 0 & 0 \\ y & 0 & -w & -yw \\ 0 & y & z & 0 \\ 0 & 0 & 0 & x \end{bmatrix}, \text{ and } \partial_3 = \begin{bmatrix} w \\ -z \\ y \\ 0 \end{bmatrix}.$$

A natural question to ask about the free resolutions is if they are finite. The answer is yes and is given by the Hilbert Syzygy theorem.

Theorem 2.2.2 (Hilbert Syzygy Theorem)**.** *If $S = k[x_1, x_2, \ldots, x_n]$, then every finitely generated graded S-module has a finite graded free resolution of length $\leq n$, by finitely generated free modules.*

There exist both a constructive and a non-constructive proof of the above theorem, and both are found in [23, p. 340 and 478].

The given definition of a free resolution allows multiple resolutions for the same module. For instance, we might not have chosen a minimal generating set for one of the kernels. This brings us to one of the most desired properties of a free resolution, minimality.

Definition 2.2.3. *A free resolution*

$$\mathbf{F} : F_0 \xleftarrow{\partial_1} F_1 \leftarrow F_2 \leftarrow \ldots \xleftarrow{\partial_n} F_n \leftarrow \ldots$$

is minimal *if for each i the image of ∂_i is contained in $\mathfrak{m}F_{i-1}$ where $\mathfrak{m}$ is the graded maximal ideal of S.*

The resolution of Example 2.2.1 is a minimal free resolution. In the explicit construction, this means choosing a minimal generating set for all of the kernels. All minimal resolutions for a module M are isomorphic, and a useful fact is that a minimal resolution of some finitely generated module M is contained in any non-minimal resolution of M as a direct summand [23, Thm 20.2].

Resolutions, in general, are central in commutative algebra as one can think of them as a general presentation of the module, and importantly every module has a resolution. They offer a way to compute many invariants for the module and also provide a good way to compute functorial invariants. A functorial invariant refers to an invariant that is obtained by applying a functor to something. In the case of free resolutions, the free modules in them are special cases of projective modules, and thus free resolutions are well suited for particular functors, the main one being Tor [23, Section A3.10]. Moreover, Tor is an example of a derived functor, that is, a functor derived from another one, in this case, the tensor product of modules. Derived functors could also be viewed as a canonical way for making an almost exact sequence, like a resolution that is not augmented with the module, to an exact sequence [23, Section A3.9]. Derived functors do not play a central role in the results or proofs of Publications I, II, or III, but it is good to be aware of them as they are central to much of modern mathematics, and in particular of Tor for their usefulness in computing invariants for the modules.

Many times minimal resolutions are the topic of interest in commutative algebra as the uniqueness of them allows for definitions of invariants for the module without having to consider multiple resolutions. One of the properties that a minimal resolution has is the fixed length that is known to be the shortest possible resolution for a given finitely generated module. The length of a minimal resolution is the projective dimension of the module. Another invariant that can be obtained from the minimal resolution is the Castelnuovo-Mumford regularity; it can be thought of as a measure for how complex it is to resolve the given module, and to define it, the existence of unique resolution is required. In a minimal resolution any minimal set of homogeneous generators of F_i contains exactly $\dim_k(\mathrm{Tor}_i^S(k,M)_j)$ generators of degree j, where Tor is the left derived functor of the tensor product of modules. The degrees of the generators of the minimal resolution give the Hilbert polynomial of the module, $H_M(d) = \dim_k M_d$, as $H_M(d) = \sum_i (-1)^i H_{F_i}(d)$ [24, p.3].

For a finitely generated graded S-module M, the (i,j)-th *graded betti number* is defined as

$$\beta_{i,j}(M) := \dim_k(\mathrm{Tor}_i^S(k,M)_j).$$

The betti numbers can also be found from the minimal resolution of the given module as the number of generators of F_i of degree j. It is customary to arrange the betti numbers of M in the *betti table* of M, which has as ji-th entry the number $\beta_{i,i+j}(M)$.

Free resolutions are special cases of chain complexes of S-modules and thus the operations of chain complexes can be applied to them.

Definition 2.2.4. *Let* $\mathbf{f},\mathbf{g} : \mathbf{C} \to \mathbf{D}$ *be two chain maps. A* homotopy *between* f *and* g

is a collection of maps $h_i : C_i \to D_{i+1}$ *such that*

$$f_i - g_i = \partial_{i+1} \circ h_i + h_{i-1} \circ \partial_i.$$

If a collection of the maps h_i *exists, then we write* $\mathbf{f} \sim \mathbf{g}$. *Two complexes* $\mathbf{C}$ *and* $\mathbf{D}$ *are said to be* homotopy equivalent, *denoted by* $\mathbf{C} \simeq \mathbf{D}$, *if there are chain maps* $\mathbf{f} : \mathbf{C} \to \mathbf{D}$ *and* $\mathbf{g} : \mathbf{D} \to \mathbf{C}$ *such that* $\mathbf{f} \circ \mathbf{g} \sim \mathrm{id}_{\mathbf{C}}$ *and* $\mathbf{g} \circ \mathbf{f} \sim \mathrm{id}_{\mathbf{D}}$ *where* id *denotes the identity map.*

Example 2.2.5. Let $S = k[x,y,z]$. Let $\mathbf{C}$ be the resolution of the module S/I where $I = (x,y,z)$ given by

$$S \xleftarrow{\partial_1} S^3 \xleftarrow{\partial_2} S^3 \xleftarrow{\partial_3} S \leftarrow 0$$

with maps

$$\partial_1 = \begin{bmatrix} x & y & z \end{bmatrix}, \partial_2 = \begin{bmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{bmatrix}, \text{ and } \partial_3 = \begin{bmatrix} z \\ -y \\ x \end{bmatrix}.$$

and let $\mathbf{D}$ be a resolution of the same module given by

$$S \xleftarrow{\partial'_1} S^4 \xleftarrow{\partial'_2} S^5 \xleftarrow{\partial'_3} S^2 \leftarrow 0$$

with maps

$$\partial'_1 = \begin{bmatrix} x & y & z & z \end{bmatrix}, \partial'_2 = \begin{bmatrix} y & z & 0 & z & 0 \\ -x & 0 & z & 0 & 0 \\ 0 & -x & -y & 0 & -1 \\ 0 & 0 & 0 & x & 1 \end{bmatrix}, \text{ and } \partial'_3 = \begin{bmatrix} z & 0 \\ -y & -1 \\ x & 0 \\ 0 & 1 \\ 0 & x \end{bmatrix}.$$

Let $\mathbf{f}, \mathbf{g} : \mathbf{C} \to \mathbf{D}$ be two chain maps given by

$$f_0 = [1], f_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, f_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } f_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and

$$g_0 = [1], g_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, g_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & x & 0 \end{bmatrix}, \text{ and } g_3 = \begin{bmatrix} 1 \\ -y \end{bmatrix}.$$

Let $h_i : C_i \to D_{i+1}$, $i = 0,1,2$, be a collection maps given by

$$h_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, h_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & y & z \\ -x & -y & -z \end{bmatrix}, \text{ and } h_2 = \begin{bmatrix} 0 & 0 & 0 \\ y & 1+z & 0 \end{bmatrix},$$

and $h_i = 0$ for all other i. Then the maps h_i give the homotopy between **f** and **g**.

There are many ways to construct new chain complexes from existing ones, and the main ones used in this thesis are tensor product, mapping cones and mapping cylinders.

The tensor product for two S-modules M_1 and M_2 constructs a new module $M_1 \otimes_S M_1$ that is the universal object making bilinear maps from the product $M_1 \times M_2$ linear, that is, the tensor product makes the following diagram commute

$$\begin{array}{ccc} M_1 \times M_2 & \longrightarrow & M_1 \otimes M_2 \\ & \searrow & \downarrow \\ & & P \end{array}$$

where $M_1 \times M_2 \to P$ is any bilinear map. This gives the category of S-modules a monoidal structure [23, Section A2.2].

Keeping the above in mind, we want a construction for chain complexes that is in the same spirit. Let **C** and **D** be two chain complexes. The *tensor product* of **C** and **D**, $\mathbf{C} \otimes \mathbf{D}$, is a chain complex given by

$$(\mathbf{C} \otimes \mathbf{D})_k = \bigoplus_{i+j=k} C_i \otimes D_j$$

with a differential

$$\partial_k(c \otimes d) = \partial_i^{\mathbf{C}}(c) \otimes d + (-1)^i c \otimes \partial_j^{\mathbf{D}}(d) \text{ where } c \in C_i, d \in D_j.$$

This definition of the tensor product provides a way to combine two chain complexes, satisfying the property that it is still a chain complex of S-modules and that the differentials respect the homological grading. The given formula is also known as the Künneth formula for complexes [56, Thm 3.6.3]. One way to look at the differentials is that since the differentials of the individual cell complexes $\partial^{\mathbf{C}}$ and $\partial^{\mathbf{D}}$ increase the grading by 1, then the map in the tensor product also must only increase the grading by 1. For an element of the form $c \otimes d$ this can be done by applying the differential to either c or d but not both at once; hence we get the sum, and the -1 component is there to assure that the composition of maps behaves well. This tensor product also equips the category of chain complexes $C_\bullet(\mathrm{Mod}_S)$ with a monoidal structure.

The mapping cone and cylinder are again constructions that originated in topology, like generally all of homological algebra, and the definitions for chain complexes

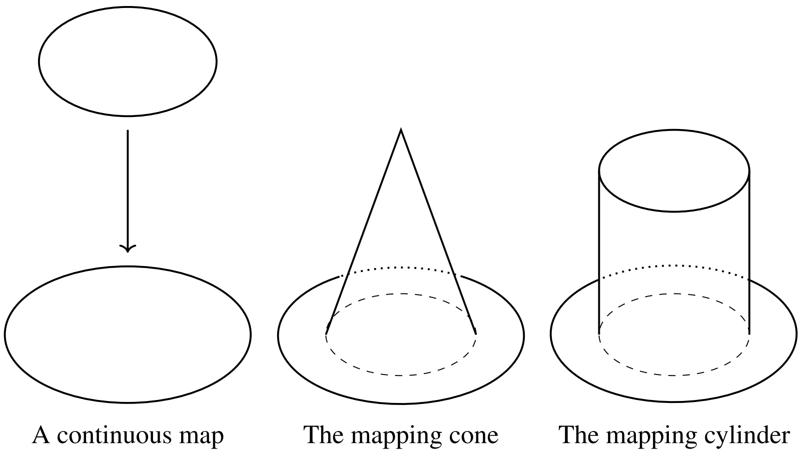


Figure 2.2. A simplified example of topological mapping cones and cylinders.

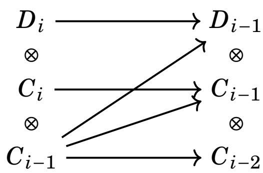
are strongly motivated by them. For this, it may be helpful to have an elementary topological picture in mind presented in Figure 2.2. Let $\mathbf{f}:\mathbf{C}\to\mathbf{D}$ be a map of chain complexes. The mapping cylinder in Figure 2.2 consists of copies of the spaces involved and an extra part that comes from the map and appears to be more dependent on starting space. The *mapping cylinder* of **f**, **Cy(f)** is the chain complex

$$\mathbf{Cy(f)}_i = D_i \oplus C_i \oplus C_{i-1}$$

with a differential map

$$\partial_i(d,c,c') = (-\mathbf{f}(c)+\partial(d), \partial(c)+\mathrm{id}(c'), -\partial(c')).$$

This consists of modules made of copies of the involved chain complexes and an extra piece of the chain complex **C** to mimic the added part in the cylinder. The differential here can be presented in the following diagram



which shows that the copies of **C** and **D** are only mapped by their respective differentials, and only the shifted copy is mapped to all components.

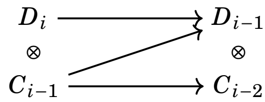
In Figure 2.2, the mapping cone is obtained from the cylinder by pinching the copy of the starting space into a single point. This pinching is mimicked in the chain complexes by leaving out the copy of the chain complex **C**. The *mapping cone* of **f**, **Co(f)**, is the chain complex

$$\mathbf{Co(f)}_i = D_i \oplus C_{i-1}$$

with differential map

$$\partial_i(d,c) = (-\mathbf{f}(c)+\partial(d), -\partial(c)).$$

Here again the diagram of the differential is presented to make the maps clearer. The arrow from C_{i-1} to D_{i-1} corresponds to the differential component of $-\mathbf{f}(c)$, and the other to arrows are the differentials coming from the original chain complexes.



2.3 Simplicial and CW-complexes

This section introduces the "second half" of background for defining cellular resolutions. Many standard topology books cover simplicial and CW-complexes, our primary references are the books by Spanier [52] and Munkres [42]. We want to note that due to the nature of the objects we study in this thesis, we will only focus on finite cell complexes.

Let $a_0, a_1, \dots, a_n$ be an independent set of points in $\mathbb{R}^N$, where N is some positive integer. A geometric *n-simplex* $\mathbf{s}$ is the space spanned by $a_0, a_1, \dots, a_n$, that is, the set of points $\mathbf{x} \in \mathbb{R}^N$ such that $\mathbf{x} = \sum_{i=0}^{n} t_i a_i$ where $\sum_{i=0}^{n} t_i = 1$ and $t_i \geq 0$ for all i. Any subset of $a_0, a_1, \dots, a_n$ spans a *face* of $\mathbf{s}$, and it is called a proper face if it does not contain $\mathbf{s}$. All the proper faces of $\mathbf{s}$ form its *boundary*. The dimension of the simplex $\mathbf{s}$ is n and the empty simplex has dimension -1 by definition. A (geometric) *simplicial complex* Δ is a collection of simplices satisfying the following conditions:

(i) If f is a face of a simplex $\mathbf{s}$ in Δ, then f is in Δ,

(ii) The non-empty intersection of two simplices $\mathbf{s}_1$ and $\mathbf{s}_2$ in Δ is a face in both $\mathbf{s}_1$ and $\mathbf{s}_2$.

One can define a simplicial complex abstractly as well. An *abstract simplicial complex* Δ is a set of vertices $V = \{1, \dots, n\}$ with collection of subsets of V such that if $A \subseteq \Delta$ and $B \subseteq A$ then $B \in \Delta$. The subsets are called simplices and they satisfy $\dim A = |A| - 1$. The dimension of a simplicial complex Δ is the maximum dimension of its simplices. A face of A in Δ is a nonempty subset $B \subseteq A$.

There exists a geometric realization of an abstract simplicial complex that makes it a geometric simplicial complex. Commonly this is represented as a functor, and we will return to it in Section 2.3.1. Simplicial complexes have many nice properties, but they are many times not general enough to be used. Often the restrictions of simplicial complexes can be overcome with regular CW-complexes that share many of the nice properties of simplicial complexes.

Let B^d denote the d-dimensional unit ball and $\mathrm{Int} B^d$ denote the interior of the ball. We will use the standard notations ∂X and $\overline{X}$ to denote boundary and closure respectively for some topological space X. A *cell* e of dimension d is a topological space that is homeomorphic to B^d. The open cell e is a space that is homeomorphic to $\mathrm{Int}B^d$.

Definition 2.3.1. *A* CW-complex *is a topological space X and a collection of disjoint open cells e_α whose union X satisfies*

(i) X is Hausdorff,

(ii) for each open d-cell e_α of the collection, there exists a continuous map $f_\alpha : B^d \to X$ that maps $\mathrm{Int}B^d$ homeomorphically onto e_α and ∂B^d is mapped into a finite union of open cells, each of dimension less that d, and

(iii) a set A is closed in X if $A \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for each α.

A finite CW-complex *is a CW-complex with a finite collection of disjoint open cells.*

In the case of a finite CW-complex the Hausdorff condition is implied by the finiteness. The maps f_α are called characteristic maps.

Example 2.3.2. A geometric realization of a d-simplex is homeomorphic to a unit ball of dimension d; thus, a simplex is a cell. The open cells are the interiors of the simplices and they satisfy conditions of Definition 2.3.1. Therefore, simplicial complexes are examples of CW-complexes.

Definition 2.3.3. *A CW-complex is* regular *if all of the characteristic maps are homeomorphisms.*

Regular CW-complexes have geometric properties that a general CW-complex does not always satisfy. These are central to the definition of cellular resolutions to get well-behaving chain complexes, and these are collected into a proposition below.

Proposition 2.3.4 ([17], Chapter 2)**.** *Let X be a regular CW-complex and e_n an n-cell of X, then*

(i) If $m < n$ and e_m and e_n are cells such that their intersection is non-empty, then $e_m \subset e_n$.

(ii) For $n \geq 0$, $\overline{e_n}$ and ∂e_n are subcomplexes, and furthermore ∂e_n is the union of closures of (n-1)-cells.

(iii) If e_n and e_{n+2} are cells such that e_n is a face of e_{n+2}, then there are exactly two (n+1)-cells between them.

One example of regular cell complex is a polyhedral cell complex. A *polyhedral complex P* is a finite collection of convex polytopes in $\mathbb{R}^N$ such that if f is a face of polytope P' in P then f is in P, and if P_1 and P_2 are polytopes in P then the intersection $P_1 \cap P_2$ is a face of both. A geometric simplicial complex is also a polyhedral complex.

Example 2.3.5. Figure 2.3 shows three different cell complexes. The cell complex (a) is a simplicial complex and (b) is not a simplicial but still regular. The cell complex (c) is an example of a non-regular CW-complex as it fails the condition (iii) of Proposition 2.3.4. Another example of a non-regular CW-complex is the space given by taking the projective space with one cell in each dimension.

After defining the cell complexes, we want to look at the maps between them. Classically in topology, the maps are continuous maps. However, just continuity is not enough to preserve the cell structure.

Definition 2.3.6. *Let X and Y be CW-complexes. A continuous map $f : X \to Y$ is* cellular *if $f(X_n) \subseteq Y_n$, where X_n and Y_n are the unions of the cells of dimension $\leq n$ in X and Y, respectively.*

If both of the complexes are simplicial, then the map is called a simplicial map and can be described by a function on the vertex set which has the requirement that any simplex in X is mapped to another simplex in Y.

A fundamental concept in topology is homotopy of maps. We have already encountered homotopy for chain complexes, which copies the topological definition. Let f and f' be two continuous maps from X to Y, then f is *homotopic* to f' if there exists a function $h : X \times [0,1] \to Y$ such that $h(x,0) = f(x)$ and $h(x,1) = f'(x)$. We write $f \sim f'$ for the homotopy. Two spaces X and Y are homotopic if there are morphisms $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim \mathrm{id}_X$ and $g \circ f \sim \mathrm{id}_Y$.

Example 2.3.7. Let X be the cell complex consisting of a triangle and let Y be the cell complex consisting of a square split into two triangles. These are shown in Figure 2.4. The cell complexes have been labelled with numbers to clarify the example, and we will refer to the cells by the vertices they contain. For example, an edge will be denoted by 13 if it goes between the vertices 1 and 3. We want to look at two cellular maps, f and g. The map f maps the vertices of X to the vertices of Y with the same labels and the edges between them accordingly, in other words, f maps X to the upper triangle of Y. The map g maps the vertices in the same way but the edge 13 is mapped to the edges 14 and 43, and the 2-cell in X gets mapped to the whole square. These images of the edge 13 under these maps are drawn in Figure 2.5 with blue describing the map f and red marking g.

The maps f and g are homotopic. To show this we want to construct the map $h : X \times [0,1] \to Y$. Note that to define how the cell complex X maps, it is enough to define the continuous map for the edges and map the 2-cell into the cell bounded by this. Let us parametrise the line in Y from the midpoint of the edge 13, denoted by p_1, to the vertex 4, denoted p_2, by $p_1 t + (1-t)p_2$ for $t \in [0,1]$. Here we assume an embedding to $\mathbb{R}^n$ for a suitable n. Then define the map h such that $h(x,t)$ is the cell complex where the edges 12 and 23 get mapped to the ones with the same label and the edge 14 gets mapped such that the midpoint goes to $p_1 t + (1-t)p_2$ and the halves form edges from there to 1 and 3. Figure 2.5 illustrates the middle stages in the homotopy with the purple edges.

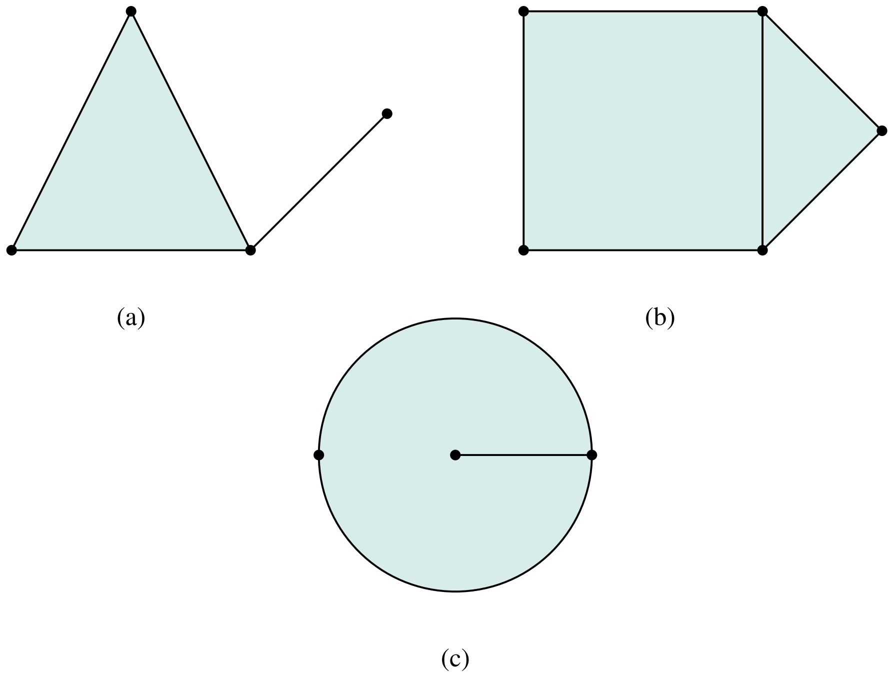


Figure 2.3. Cell complexes of Example 2.3.5.

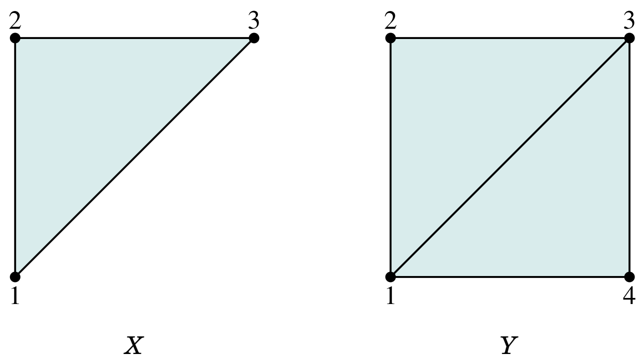


Figure 2.4. The cell complexes of Example 2.3.7.

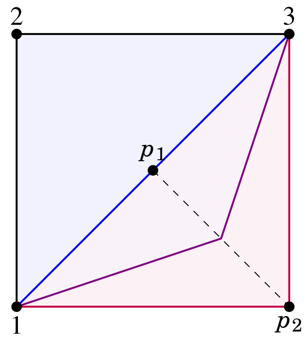


Figure 2.5. The images of maps f and g and the intermediate stage in homotopy on the cell complex Y of Example 2.3.7.

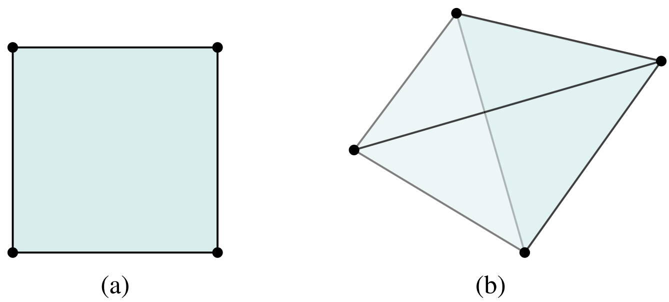


Figure 2.6. Cell complexes of Example 2.3.9.

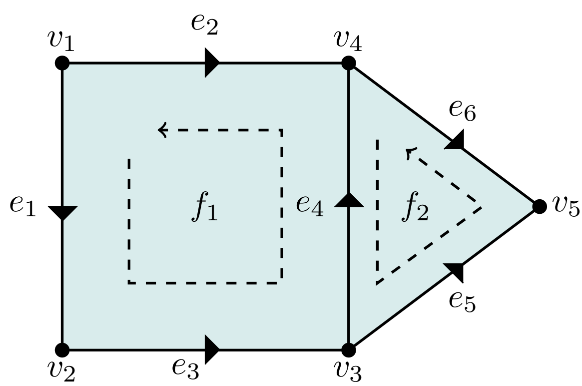


Figure 2.7. The oriented cell complex of Example 2.3.11.

Continuous maps between CW-complexes are cellular up to homotopy and this is captured in the Cellular approximation theorem.

Theorem 2.3.8 (Cellular approximation theorem)**.** *Let X and Y be any CW-complexes with a continuous map $f : X \to Y$, then f is homotopic to a cellular map.*

See [52, Thm 17 p. 404] for a proof on CW-pairs. One can get the version with a single CW-complex by choosing the empty complex as the subcomplex in the pair.

One can also construct new CW-complexes from existing ones. The two main ways to do this are a join and a product. A *product* of X and Y is the CW-complex $X \times Y$ given by each cell in $X \times Y$ being a product of a cell in X and a cell in Y with the weak topology. The underlying set of the product is the Cartesian product. Since we have finite CW-complexes, the product will also be a finite CW-complex. A *join* of topological spaces X and Y, $X * Y$, is the quotient space of the product $X \times Y \times [0,1]$. The elements in the quotient space are sets $\{x\} \times Y \times \{0\}$ where $x \in X$, $X \times \{y\} \times \{1\}$ where $y \in Y$ and points from the set $X \times Y \times [0,1] \setminus (X \times Y \times \{0\} \cup X \times Y \times \{1\})$. The topology on this space is the quotient topology [52, pp. 25; 437–444]. Let X and Y be regular CW-complexes and assume that we have an embedding into $\mathbb{R}^n$. Then the join of X and Y is the complex we get by connecting every vertex of X to all vertices of Y with an edge, and filling in the higher degrees accordingly.

Example 2.3.9. Let us consider the cell complexes consisting of a single edge and two vertices. Figure 2.6 shows the product and the join of two copies of the described cell complex. The cell complex (a) is the product, and the cell complex (b) is the join. Note that the product can be obtained from the join by cutting with a suitable hyperplane.

Mapping cones and cylinders are typical in topology, and they can be used to build new cell complexes from the existing ones.

Definition 2.3.10. *Let $f : X \to Y$ be a continuous map. Then the* mapping cone *of f, denoted with C_f, is the space $(X \times [0,1]) \sqcup_f Y$ with the identification of $X \times \{0\}$ with a single point and $(x,1) \sim f(x)$.*

The mapping cylinder *is constructed in the same way, but instead identifying $X \times \{0\}$ with a single point, every point in X is identified with itself.*

As a final part for theory on the cell complexes, we want to bring up the homology of these cell complexes. Homology assigns a sequence of groups to a space X and morphisms between homology groups for any map $f : X \to Y$. Often in practice these homology groups are computed using a chain complex such that the n-th homology group is given by $\ker \partial_n / \operatorname{im} \partial_{n+1}$. For CW-complexes, the n-th homology group $C_n(X)$ is a free abelian group with generators in one-to-one correspondence with the n-cells of X. However, finding the maps in the cell complex is not always easy, but luckily for regular CW-complexes and simplicial complexes, this is doable.

Firstly it requires the notion of orientation. On a simplex $\mathbf{s}$ the orientation can be defined through an ordering on the vertices. Ordering on the vertex set is equivalent to another one if one can be reached from the other by an even permutation. If $\mathbf{s}$ has dimension greater than zero, these permutations will give two equivalence classes, each being an orientation.

More generally, let X be a regular CW-complex. Then X comes equipped with an orientation of the faces, and a function $\text{sign}(e',e)$ on pairs of faces e,e'. The functions take values in $\{0,1,-1\}$, with $\text{sign}(e',e)$ non-zero if and only if e' is a facet of e, and $\text{sign}(e',e)=1$ if the orientation of e' induces the orientation for e.

The $\text{sign}(e',e)$ can also be thought of as giving the sign of e' in the boundary map of e.

Example 2.3.11. Let X be the cell complex consisting of a square and a triangle in Figure 2.7. The cells have been labelled with v_i for the vertices, e_i for the edges, and f_i for the faces for clarity. The labelling is also showing an ordering on the cells by starting with subscript 1, and Figure 2.7 shows the orientation of the cells with arrows. The vertices have canonical orientation 1.

First look at the pairs of cells coming from the vertices and edges. The value of $\text{sign}(v_1,e_1)$ is 1 since our chosen edge orientation goes from v_1 to v_2, and the value of $\text{sign}(v_2,e_1)$ is -1 following the same orientation. For any other vertex we have that $\text{sign}(v_i,e_1)=0$ for $i\neq 1,2$ as the edge has no other facets. Similarly for the other edges one vertex gives 1 and the other gives -1.

The sign-function for the faces behaves similarly. For the square face f_1, the orientation chosen is counter-clockwise. The edges that are oriented in this directon, e_1, e_3, and e_4, have $\text{sign}(e_i,f_1)=1$, and the remaining edge e_2 has $\text{sign}(e_2,f_1)=-1$, since they do not match the orientation of f_1, that is, it does not induce it. Using the same arguments one can compute $\text{sign}(e_4,f_2)=1$, $\text{sign}(e_5,f_2)=1$, and $\text{sign}(e_6,f_2)=-1$.

Observing the values from the sign-function, they match up with the coefficients in the boundary maps of the cells.

Proposition 2.3.12 ([39], Lemma 7.1)**.** *The* sign *function given above exists for regular CW-complexes and satisfies the described properties.*

In the case of regular CW-complexes, their gluing structure and desirable properties of the sign function allows one to compute the maps in the chain complex. We will present the case of X being a simplicial complex and how to compute the reduced chain complex for it.

Let X be a simplicial complex defined on the set $\{1,2,\dots,n\}$. A *reduced chain complex* $\tilde{C}(X;k)$ for X is a chain complex

$$0 \leftarrow k^{F_{-1}(X)} \xleftarrow{\partial_0} \dots \xleftarrow{\partial_i} k^{F_i(X)} \xleftarrow{\partial_{i+1}} \dots$$

where $F_i(X)$ is the set of i-dimensional cells in X and k^{F_i} is the vector space over a field k with a basis elements e_s corresponding to $s\in F_i(X)$. In the simplicial case the sign function can defined as $\text{sign}(j,s)=(-1)^r$ if j is the r-th element of simplex $\mathbf{s}$ with the points arranged in an increasing order. Then the differential is given by

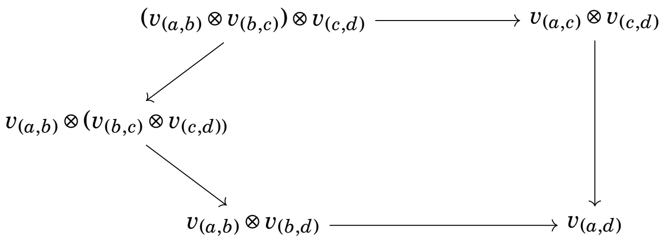
$$\partial(e_s)=\sum_{j\in s}\text{sign}(j,s)e_{s\setminus j}.$$

2.3.1 Homotopy colimits

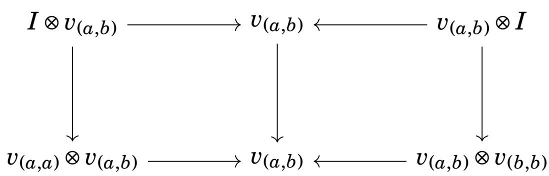
In the category **Top** the colimits are not invariant under homotopy of the diagram. However, this is a desirable property so one can define the homotopy colimit in **Top**. The remaining definitions in this section are required to define homotopy colimits and simplicial set enrichment. We will mostly follow the notation and construction of [57], and in particular, use the definition of homotopy colimit that is given in it.

Let $\mathscr{C}$ be a category. Then the *opposite category* $\mathscr{C}^{op}$ is the category with the objects of $\mathscr{C}$ and morphism $b \to a$ for every morphism $a \to b$ in $\mathscr{C}$.

Definition 2.3.13. *Let $\mathcal{V}$ be a monoidal category. Then a category $\mathscr{C}$* enriched with *$\mathcal{V}$ is the category with objects* obj($\mathscr{C}$)*, and for every pair of objects we have an object $v_{(a,b)} \in \mathcal{V}$. For any triple $a, b, c \in \mathscr{C}$, we have the composition $v_{(a,b)} \otimes v_{(b,c)} \to v_{(a,c)}$. Finally, the following diagrams must commute for the given data:*



and

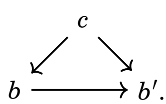


An example of the enriched category is **Top** with simplicial sets that are defined below.

The *category OF* is the category of ordered finite sets, denoted by $[n] = \{0, 1, \dots, n\}$, as the objects and order preserving functions as morphisms. A *simplicial set* is defined to be a contravariant functor $X : OF \to$ **Set**. The category of simplicial sets is denoted by **sSet**.

The nerve of the under category appears in the definition of the homotopy colimit, and these two concepts are defined below.

Definition 2.3.14. *Let $\mathscr{C}$ be a category and $c \in \mathscr{C}$ an object. Then the* under category*, or category of objects of $\mathscr{C}$ under c, $\mathscr{C}_{\downarrow c}$, is a category with objects (b, f) where $b \in \mathscr{C}$ and $f : c \to b$, and the morphisms $(b, f) \to (b', f')$ is a map $g : b \to b'$ that makes the triangle below commute*



The under category is sometimes called the category of arrows and in particular one can find this name in [57].

Definition 2.3.15. *Let $\mathscr{C}$ be a small category. The* nerve *of $\mathscr{C}$ is the simplicial set $N\mathscr{C}$ where the n-simplex σ is a diagram in $\mathscr{C}$ of the form $c_0 \to c_1 \to \ldots \to c_n$ with maps $d_i : N\mathscr{C}_n \to N\mathscr{C}_{n-1}$ by composing at i-th object, and $s_i : N\mathscr{C}_n \to N\mathscr{C}_{n+1}$ by adding an identity morphisms at i, where $N\mathscr{C}_n$ is the collection of all n-simplices.*

Homotopy colimits in Top are defined using the category **Ord** as follows, see [57] for more details. The category **Ord** consists of finite sets $[n] = \{0, 1, \ldots, n\}$ as the objects and non-decreasing maps, that is, $f : [n] \to [m]$ then $f(i) \leq f(i+1)$ as the morphisms. The morphisms in **Ord** are generated by two maps, namely $\delta_n^i : [n] \to [n-1]$ and $\sigma_n^i : [n] \to [n+1]$. These maps are often called face and degeneracy in the literature.

Definition 2.3.16. *A* simplicial space *is a contravariant functor F from* **Ord** *to* **Top**. *The functors form a category of simplicial spaces with the morphisms being the natural transformations between the functors.*

A particular case of the simplicial space is the simple geometric realization functor $R : \mathbf{Ord} \to \mathbf{Top}$ taking the set $[n]$ to the standard n-dimensional simplicial complex Δ_n.

Definition 2.3.17. *The* geometric realization *of a simplicial space F is the direct sum $\bigsqcup F_n \times \Delta_n$ quotiented out by the relations $(d^i(x), p) \sim (x, R(\delta^i)(p))$ and $(s^i(x), p) \sim (x, R(\sigma^i)(p))$ where d^i and s^i are the images of δ^i and σ^i under F.*

Definition 2.3.18. *The* classifying space *of a category $\mathscr{A}$ is the geometric realization of the simplicial space $F_{\mathscr{A}}$ associated to $\mathscr{A}$, which is the functor $F_{\mathscr{A}} : \mathbf{Ord} \to \mathbf{Set}$ taking the set $[n]$ to the sequence $\alpha_n \leftarrow \ldots \leftarrow \alpha_0$.*

Remark 2.3.19. The classifying space is a special case of the nerve of a category.

For some small category A and objects, let $A_{\downarrow a}$ be the category of all arrows $a \to b$ with commutative triangles as the morphisms. Let $B(A_{\downarrow a})$ be the classifying space of $A_{\downarrow}$.

Definition 2.3.20. *The* homotopy colimit *of the diagram $D : A \to$* **Top**, *denoted by* hocolim$D$ *is the quotient of the coproduct $\sqcup_{a \in A} B(A_{\downarrow a}) \times D_a$. The equivalence relation $\sim$ for the quotient is the transitive closure of $\alpha(p, x) \sim \beta(p, x)$, where α and β are the following maps*

$$\alpha : B(A_{\downarrow b}) \times D_a \longrightarrow B(A_{\downarrow b}) \times D_b, \ \alpha(p, x) = (p, d_f(x)),$$

$$\beta : B(A_{\downarrow b}) \times D_a \longrightarrow B(A_{\downarrow a}) \times D_a, \ \alpha(p, x) = (p, d_f(x))$$

for all morphisms $f : a \to b$.

One can also approach the homotopy colimit from a more concrete view and take it as "gluing in mapping cylinders" to the diagram.

Definition 2.3.21. *The homotopy category of* **Top** *is the category where the objects are the same as in* **Top***, but the morphisms are homotopy classes of the morphisms in* **Top***.*

Informally the colimits in **Top** can be viewed as gluing the diagram together along the images of the maps and homotopy colimits by gluing in mapping cylinders in the diagram. Homotopy colimits are of course not limited to topological spaces, and one way to construct them is to use derived functors. The more categorical approach to homotopy colimits and the more explicit construction we presented above are, in fact, equivalent. Shulman provides both proof of this and definitions of the different constructions and expansion to enriched categories in [49]. The approach through derived functors often requires a Quillen model category [18] and the category of cellular resolutions studied later do not have this property, which is why we have chosen the more explicit approach for homotopy colimits.

2.4 Cellular resolutions

In this section we define cellular resolutions. This is the most important definition in this chapter and central to the results of Publications I and II. Cellular resolutions were introduced initially to study resolutions of monomial modules, and the first definition only covered simplicial complexes that were subsequently expanded to regular CW-complexes [4, 5]. The book of Miller and Sturmfels [40] presents a good introduction to cellular resolutions and is our main reference.

A *labelled cell complex* X is a regular CW-complex with monomial labels on the faces. The vertices of X have labels $m_1, m_2, \dots, m_r$ with exponent vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_r \in \mathbb{N}^n$, respectively. If $\mathbf{a}_i = (a_{i1}, a_{i2}, \dots, a_{in})$, then the monomial m_i is $x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_n^{a_{in}}$. The faces F of X have the least common multiple of the monomial labels of the vertices it contains, $m_F = \operatorname{lcm}\{m_i : i \in F\}$. The label on the empty face is 1, that is, $x_1^0 x_2^0 \cdots x_n^0$. The *degree* of a face F is the exponent vector $\mathbf{a}_F$ of the monomial label.

Recall that for a non-labelled cell complex, we can construct the reduced chain complex of free modules. In the case of a labelled cell complex, there is also the algebraic data of the monomial labels, which one would like to see included in the data of the chain complex.

Definition 2.4.1. *Let $S(-\mathbf{a}_F)$ be the free S-module with a generator e_F in degree $\mathbf{a}_F$. Given a labelled and oriented cell complex X, the* cellular complex $\mathbf{F}_X$ *is given by*

$$(\mathbf{F}_X)_i = \bigoplus_{\substack{F \in X \\ \dim F = i-1}} S(-\mathbf{a}_F)$$

with a differential

$$\partial(F) = \sum_{G \subset F} \operatorname{sign}(G, F) m_{F-G} e_G,$$

where e_G is the generator corresponding to the face G and m_{F-G} is the monomial with the exponent vector $\mathbf{a}_F - \mathbf{a}_G$

Note that the above definition uses fine grading on the modules, so the monomials x_1x_2 and x_1x_3 would have a different gradings, for example. It is common not to write all of the grading in examples, and indeed, if one has the differential maps, the fine graded degrees can be deduced from those.

Definition 2.4.2. *The chain complex* $\mathbf{F}_X$ *is a* cellular resolution *if it is acyclic, that is,* $\mathbf{F}_X$ *has non-zero homology only at degree 0.*

Example 2.4.3. Let us consider the simplicial complex of Example 2.3 and give it two labellings shown in Figure 2.8. For the cell complex (i), the cellular chain complex is the following:

$$S \xleftarrow{\begin{bmatrix} xy & xz & xw & yzw \end{bmatrix}} S^4 \xleftarrow{\begin{bmatrix} -z & -w & 0 & 0 \\ y & 0 & -w & -yw \\ 0 & y & z & 0 \\ 0 & 0 & 0 & x \end{bmatrix}} S^4 \xleftarrow{\begin{bmatrix} w \\ -z \\ y \\ 0 \end{bmatrix}} S \leftarrow 0$$

Now looking at the maps we can see that $\ker\partial_i = \operatorname{im}\partial_{i+1}$ for all $i > 0$, thus it is acyclic and is a cellular resolution. Moreover, this resolution is the same minimal resolution as Example 2.2.1.

The labelled cell complex in (ii) gives the cellular chain complex

$$S \xleftarrow{\begin{bmatrix} xy & xz & xw & yzw \end{bmatrix}} S^4 \xleftarrow{\begin{bmatrix} -z & -zw & 0 & 0 \\ y & 0 & -yw & 0 \\ 0 & 0 & 0 & -yz \\ 0 & x & x & x \end{bmatrix}} S^4 \xleftarrow{\begin{bmatrix} w \\ -1 \\ 1 \\ 0 \end{bmatrix}} S \leftarrow 0$$

If we look at the maps, we note that $\ker\partial_1$ contains the element

$$e = \begin{pmatrix} -w \\ 0 \\ y \\ 0 \end{pmatrix}$$

and $\operatorname{im}\partial_2$ does not contain e. Thus $\ker\partial_1 \neq \operatorname{im}\partial_2$ and this cellular chain complex is not a cellular resolution, even if the chosen labels are the same as in (i).

The differentials in the cellular complex can also be described by monomial matrices, with the columns and rows having the corresponding faces as labels and the scalar entries coming from the usual differential for reduced chain complexes. The free S-modules of $\mathbf{F}_X$ are then the ones represented by the matrices.

Another useful result for cellular resolutions makes use of order of vectors. If $\mathbf{a}$ and $\mathbf{b}$ are two vectors in $\mathbb{N}^n$, then set $\mathbf{a} \preceq \mathbf{b}$ if $\mathbf{b} - \mathbf{a} \in \mathbb{N}^n$. Let X be a labelled cell complex and define the subcomplex $X_{\preceq \mathbf{b}}$ to be the complex consisting of all the faces with labels $\preceq \mathbf{b}$.

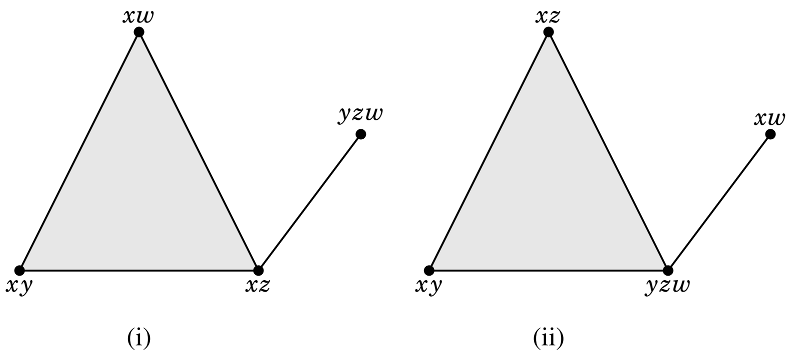


Figure 2.8. The labelled cell complexes of Example 2.4.3.

Proposition 2.4.4 ([40], Prop 4.5)**.** *The cellular free complex $\mathbf{F}_X$ supported on X is a cellular resolution if and only if $X_{\leq \mathbf{b}}$ is acyclic over $\mathbf{k}$ for all $\mathbf{b} \in \mathbb{N}^n$. When $\mathbf{F}_X$ is acyclic, it is a free resolution of S/I, where $I = (m_v \mid v$ is a vertex in $X)$ is generated by the monomial labels on vertices.*

Building cellular resolutions is not as simple as the definition may make it look. Given a labelled cell complex one may not get a resolution from the cellular chain complex, take, for example, a cell complex that has a hole in it. A hole will cause the cellular complex not to be acyclic. Thus one can infer that a necessary condition for the cell complex is to be contractible. Still, a lot depends on the chosen labels as can be seen in Example 2.4.3. There are some particular cases where the knowledge on the cell complex is enough to deduce cellular resolutions. One of these is utilising standard labelling on specific cell complexes that we know give cellular resolutions and then use those. Examples of this can be seen with the subdivision of Minkowski sums by Norén [43].

Another operation that can be performed on a cell complex is discrete Morse theory. Batzies and Welker [3] established that one can do discrete Morse theory on cellular resolutions as well, with a few restrictions, and this has also motivated one approach to algebraic Morse theory. See Section 2.5 for more details.

Going the other way, given a resolution and asking if it is supported on a complex, is not necessarily any easier question. In particular, finding a minimal resolution supported on a cell complex is a problem that we do not have general solutions to. Velasco showed that there exist monomial ideals that do not have minimal cellular resolutions [55]. Trying to answer these questions has provided some well-known ways to build cell complexes like the Taylor resolution and the Hull resolution. Taylor resolutions for the module S/I, with I having m generators, is built by taking the m-simplex and labelling the vertices with the generators of I. The benefit of the Taylor resolution is that given any monomial ideal I it will have a cellular resolution that is even simplicial. However, this resolution is very far from the minimal one for almost all ideals. The quest for minimality in the resolutions has provided many results about specific classes of ideals that have a minimal cellular resolution, like particular edge ideals [1, 26], co-interval ideals [19], and a construction for ideals with linear quotients to build the minimal resolutions [20]. Building or finding the minimal resolution can be demanding, and there has also been work to prune the resolutions closer to a minimal one [41], and we too make use of cellular resolutions that are not minimal but close enough in Publications I and II.

2.5 Discrete and algebraic Morse theory

Both the algebraic and topological objects bring us to the homological setting, and this section introduces a handy tool for working with these, namely discrete and algebraic Morse theory. Traditionally, Morse theory has studied the topology of smooth manifolds and could be thought of as an extension of the connection between critical points on a manifold and critical points of a smooth function on it. Morse theory has been a strong and useful theory, and naturally many variations of it have been born. We are interested in the combinatorial versions of Morse theory, namely the discrete Morse theory of Forman [30] and the algebraic Morse theory of Jöllenbeck and Welker [36], and of Sköldberg [50]. These give us useful tools to remove unnecessary pieces of a resolution or a cell complex while still preserving the homotopy type and in the latter two cases the structure of the resolution is preserved as well. The discrete Morse theory motivated Jöllenbeck and Welker, and Sköldberg as well to develop algebraic Morse theory but with different goals. The theory of Jöllenbeck and Welker was developed with the intention to be applied to cellular resolutions, and in this direction, there is also an earlier paper by Batzies and Welker [3]. Sköldberg's work is directed towards having a purely algebraic version of the discrete Morse theory.

2.5.1 Discrete Morse theory

The main reference used for this discrete Morse theory section is [30]. We will again focus only on the case where all the cell complexes are at least regular CW-complexes or even simplicial complexes at the starting point. The cell complexes may deform a little to be non-regular with discrete Morse theory, but they are all what can be considered "not weird". A *cell complex* will mean a regular CW-complex.

Firstly, we require a few preliminary definitions. A *graph* G is a set of vertices $V(G)$ and set of edges $E(G)$ consisting of pairs of vertices. A *directed graph* has a direction for the edges.

A *poset*, or a partially ordered set, is a set P together with a relation $\leq$ that satisfies:

(i) $a \leq a$,

(ii) $a \leq b$ and $b \leq a$ implies $a = b$,

(iii) and $a \leq b$ and $b \leq c$ implies $a \leq c$ for all $a, b, c \in P$.

The cells in a regular CW-complex form a poset with the inclusion relation.

Let X be a cell complex. A *face poset* diagram P_X for X is a directed graph with vertices corresponding to n-cells of the cell complex. There is an edge from β to α if and only if α is a codimension 1 face of β.

Definition 2.5.1. *A* matching *on a graph is a set of pairwise non-adjacent edges. Let X be a cell complex with face poset P_X. Then a* Morse matching *on P_X is a matching M such that P_X has no directed cycles when the edges in M are reversed. A vertex is* critical *if it is not in the Morse matching.*

The defined Morse matching is the central tool in poset-based discrete Morse theory. The main theorem of discrete Morse theory is given in the form from [26, Thm 5.1] since it will be convenient for the results for cellular resolutions in Publication I.

Theorem 2.5.2 (The main theorem of discrete Morse theory)**.** *If X is a regular CW-complex with a Morse matching (giving at least one critical vertex and not matching the empty cell to anything), then there exists a CW complex $\tilde{X}$ that is homotopy equivalent to X, where the number of d-dimensional cells of $\tilde{X}$ equals the number of d-dimensional critical cells of X for every d.*

Example 2.5.3. Let us consider the cell complexes X and Y in Figure 2.9. The face poset for each complex is drawn in Figure 2.10. The cell complexes and face posets have been labelled by numbers to clarify the example. Consider the following matchings, on P_X we choose a single edge between the vertex 4 and 34; on P_Y we take the edges $14 \leftarrow 134$, $24 \leftarrow 234$ and $124 \leftarrow 1234$. The edges in the matchings have been coloured in Figure 2.10. It is not hard to check on these small examples that the chosen matchings do not create cycles. Thus they are Morse matchings. Now using the chosen Morse matchings, the cell complex X is homotopy equivalent to a cell complex that has three vertices, three edges and a single 2-cell, in other words, a triangle. The cell complex Y is homotopy equivalent to X since the critical cells form the face poset of X.

We have chosen to use the face poset version of the theorems in [30]. It is possible to also define discrete Morse theory in terms of discrete Morse functions and finding critical cells with that method [30].

A Morse matching with a single edge gives an elementary collapse in the cell complex. This can be explicitly described on the CW-complex by the following definition, see [16, Chapter 2] for more details.

Definition 2.5.4. *Let X be a finite CW-complex and let Y be a subcomplex of X. Then there is an* elementary collapse *of X to Y, $X \searrow^{e} Y$ if there exists a ball B^n, where $\partial B^n = B_+^{n-1} \cup B_-^{n-1}$ with $\partial B_+^n = \partial B_-^n$, and a map $\varphi : B^n \to X$ such that*

(i) φ is a characteristic map for e^n,

(ii) $\varphi|_{B_+^{n-1}}$ is a characteristic map for e^{n-1}, and

(iii) $\varphi(\partial B_-^{n-1}) \subset Y$.

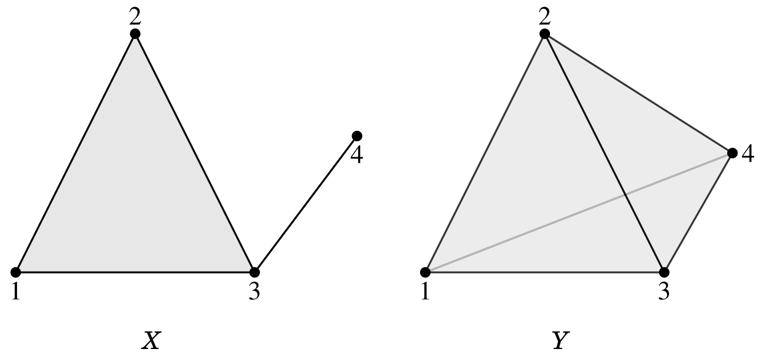


Figure 2.9. The cell complexes of the Example 2.5.3.

Example 2.5.5. Returning to the cell complexes and matchings of Example 2.5.3, the change from the complexes X and Y to the homotopy equivalent ones X' and Y' can be seen through elementary collapses. In the case of X, it is straightforward. There was only one edge chosen; thus, it gives an elementary collapse of the edge in the cell complex. For the cell complex Y, the chosen matching consists of three edges. Hence it can be viewed as a process of three elementary collapses. First one can collapse the 3-cell and the matched the 2-cell, and then the two following elementary collapses are those of a face and an edge.

The elementary collapses form the basis of simple homotopy theory defined by Whitehead which is covered for cellular resolutions in Publication I.

2.5.2 Algebraic Morse theory

The focus of this section is the algebraic Morse theory and aims to cover the definitions required for the results in Publication I. For a more complete and detailed overview of algebraic Morse theory, the reader may look up the original works by Sköldberg [50] and Jöllenbeck and Welker [36]. The notation used in this section follows that of [50], and the ring is assumed to be a polynomial ring S even if the theory permits a more general setting.

A *based chain complex* is a chain complex of R-modules such that the modules in the complex $\mathbf{N}$ have a direct sum decomposition $N_i = \bigoplus_{j \in I_i} N_j$ with $\{\mathscr{I}_i\}$ being a collection of index sets. A free resolution is an example of a based chain complex.

Let $\mathbf{N}$ be a based chain complex of S-modules

$$N_0 \xleftarrow{\partial_1} N_1 \xleftarrow{\partial_2} N_2 \longleftarrow \cdots$$

with $N_i = \bigoplus_j N_{i,j}$, where $N_{i,j}$ is an S-module and ∂ is the differential in the chain complex. The double indexing for $N_{i,j}$ has the first component to denote the homological degree of the module that it is the summand of, and the second subscript corresponds to the direct sum decomposition.

The *directed graph associated to* $\mathbf{N}$, denoted by Γ_N, is defined to be the graph where vertices are given by the summands in each homological degree and the directed edges

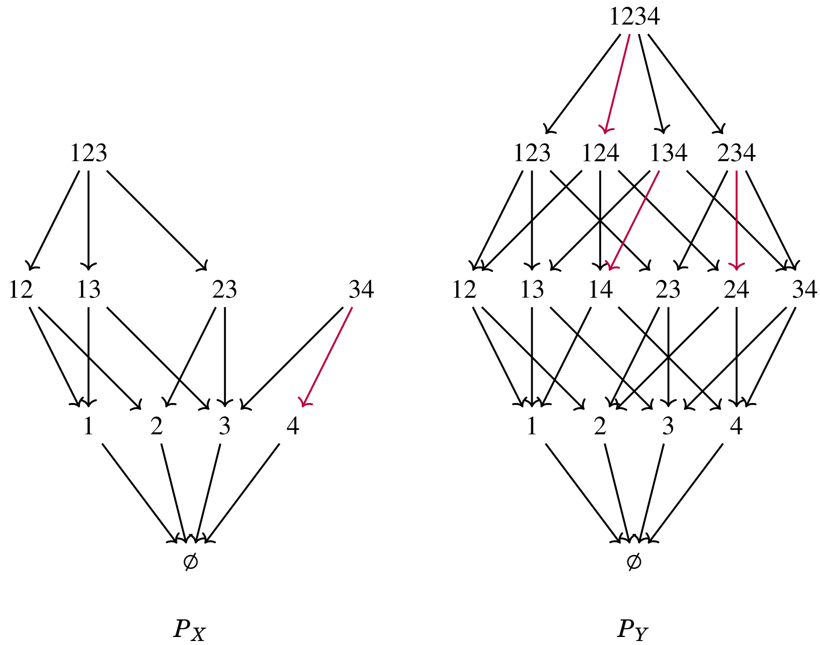


Figure 2.10. Face posets of the cell complexes in Figure 2.9.

go down in the degrees. There is an edge from $N_{i,j}$ to $N_{i-1,j'}$ if $\partial(N_{i,j}) \cap N_{i-1,j'}$ is not empty. Denote by $\partial_{j,k}$ the component of the differential corresponding to an edge from $N_{i,k}$ to $N_{i-1,j}$. The index i is not denoted in $\partial_{j,k}$ as it will be clear of the context. Note that the graph depends on the decomposition chosen for the N_i in the chain complex.

Example 2.5.6. This example showcases the different directed graphs associated to a resolution, in this case the resolution of S/I where $I = (xy, xz, xw, yzw)$. As seen in Example 2.2.1 a resolution is given by

$$S(-0) \xleftarrow{\partial_1} S(-2)^3 \oplus S(-3) \xleftarrow{\partial_2} S(-3)^3 \oplus S(-4) \xleftarrow{\partial_3} S(-4) \leftarrow 0$$

If the direct sum decomposition is taken to be the decomposition into the components $S(-i)$, so that the associated graph for this chain complex is in Figure 2.11(a). Alternatively, if the decomposition is taken to be such that each generator degree forms its own module, then the associated directed graph is different from the previous one and it is in Figure 2.11(b).

Definition 2.5.7. *A* Morse matching *on the graph* Γ_N *is a matching* M *on* Γ_N*, satisfying that there are no directed cycles in the graph* Γ_N^M*, which is* Γ_N *with the edges from* M *reversed, and that the maps in* $\mathbf{N}$ *corresponding to the edges in* M *are isomorphisms.*

In the algebraic setting of Morse theory, the condition on the selected edges being isomorphisms is essential. Moreover, it facilitates the definition of necessary maps

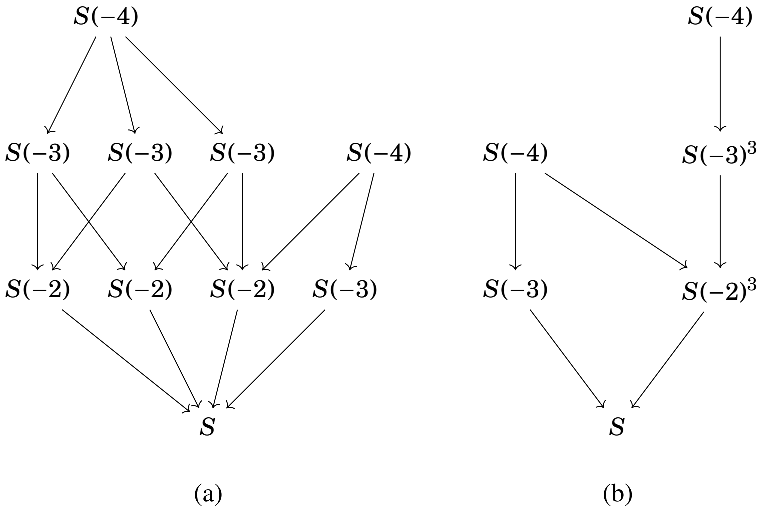


Figure 2.11. Directed graphs associated to different direct sum decompositions of Example 2.5.6.

to construct a homotopic cell complex. These have been collected into a single proposition from the smaller results in [50, Chapter 2].

Proposition 2.5.8. *The Morse matching M gives a graded map $\varphi : \mathbf{N} \to \mathbf{N}$. If j is minimal with respect to the partial order $\prec$ and $x \in N_{i,j}$, the map is given by*

$$\varphi(x) = \begin{cases} \partial_{j,k}^{-1}(x) & \exists \text{ an edge from } N_{i,k} \text{ to } N_{i-1,j} \text{ for some } k \in M \\ 0 & \text{otherwise} \end{cases}$$

If j is not minimal then φ is given by

$$\varphi(x) = \begin{cases} \partial_{j,k}^{-1}(x) - \sum \varphi \partial_{m,k} \partial_{j,k}^{-1}(x) & \exists \text{ an edge from } N_{i,k} \text{ to } N_{i-1,j} \text{for some } k \in M \\ 0 & \text{otherwise} \end{cases}$$

where the sum is over all edges from $N_{i,k}$ to $N_{i-1,m}$. The map φ is a splitting homotopy as it satisfies $\varphi^2 = 0$ and $\varphi \circ \partial \circ \varphi = \varphi$.

Let $\pi : \mathbf{N} \to \mathbf{N}$ be the chain map given by $\pi = id - (\partial \circ \varphi + \varphi \circ \partial)$. Then $\pi(v) = 0$ if v is a vertex incident to an edge in the partial matching M.

Theorem 2.5.9 ([50], Theorem 1)**.** *Let M be a Morse matching on the complex $\mathbf{N}$. Then the complexes $\mathbf{N}$ and $\pi(\mathbf{N})$ are homotopy equivalent. Furthermore, for each n there is an isomorphism of modules $\pi(N_n) \cong \bigoplus_{\alpha \in M_n^0} N_\alpha$, where M_n^0 denotes the intersection of $\mathscr{I}_n$ and the M-critical vertices.*

Remark 2.5.10. Instead of $\pi(\mathbf{N})$, we can look at the chain complex $\overline{\mathbf{N}}$ given by

$$\overline{N}_i = \bigoplus_{N_{ij} \text{ is unmatched in } M} N_{i,j}.$$

Let ρ be the projection from $\mathbf{N} = \bigoplus_i N_i$ to $\overline{\mathbf{N}}$. The differential $\overline{\partial}$ can be defined as $\overline{\partial} = \rho(\partial - \partial\varphi\partial)$. The complex $\overline{\mathbf{N}}$ is then also homotopy equivalent to $\mathbf{N}$.

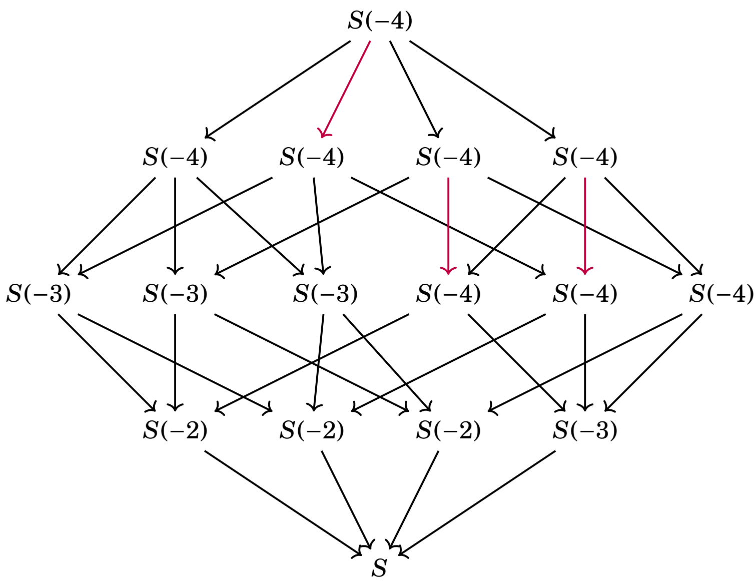


Figure 2.12. The directed graph associated to the Taylor resolution of S/I with $I = (xy, xz, xw, xyz)$.

Example 2.5.11. Let us consider the resolution from Example 2.5.6 again. One cannot choose a Morse matching on the associated graph in Figure 2.11(a) since there are no isomorphisms. Next, consider the cellular resolution of the same ideal, but coming from the Taylor complex instead. The resolution is given by

$$S(-0) \leftarrow S(-2)^3 \oplus S(-3) \leftarrow S(-3)^3 \oplus S(-4)^3 \leftarrow S(-4)^4 \leftarrow S(-4) \leftarrow 0$$

and the directed graph of this chain complex is in Figure 2.12 with the edges corresponding to isomorphism coloured. The coloured edges also correspond to the cells in the cell complex that have the same label. Moreover, the edges that correspond to isomorphisms include the chosen matching of Example 2.5.3. Then taking this matching, we can produce a homotopic chain complex given by the Remark 2.5.10:

$$S(-0) \xleftarrow{\partial_1} S(-2)^3 \oplus S(-3) \xleftarrow{\partial_2} S(-3)^3 \oplus S(-4) \xleftarrow{\partial_3} S(-4) \leftarrow 0.$$

This chain complex is the resolution of the ideal $I = (xy, xz, xw, yzw)$.

The observation in Example 2.5.11 on the edges corresponding to the same labels in the cell complex, and the resulting cellular resolution is not just a special case of the particular example. In general, the label requirement on the discrete Morse theory side is sufficient to keep cellular resolution structure, and this was proven by Batzies and Welker.

Theorem 2.5.12 ([3], Theorem 1.3)**.** *Let X be a complex that supports a cellular resolution, and let M be a Morse matching on this complex. If M only matches cells with the same labels, then the Morse complex $\tilde{X}$ also supports a cellular resolution of the same module.*

2.6 Representation stability

We turn our attention to representations of categories. Often, and where many of the concepts of representation stability originate, we can think of a representation of a category as a sequence of classic representations with maps between them. Representation stability then studies the properties of these representations of categories, or sequences, and in particular stability of some properties as the name suggests. The name for this phenomena was given in a paper by Church and Farb [15], where they combined homological stability with the study of sequences of representations. Later work of Church, Ellenberg and Farb [13] and their work with Nagpal [14] expanded on the study of representation stability and gave birth to the study of FI-modules that is still one of the most studied parts of representation stability. The ideas of representation stability allowed for work on representation stability of cohomology by Wilson [58] and Church, Ellenberg and Farb [12] and of configuration spaces of manifolds by Church [10] among others. There exists a good survey by Benson Farb [28] on the topic of representation stability.

The different directions of representation stability share many similarities, which motivated the categorical representation stability of Sam and Snowden [48] that was being worked on at the same time as the representation stability work mentioned above. Their main idea was to generalise the setting of individual cases to general categories and find combinatorial conditions on the categories which imply algebraic properties for the representations. The work of Sam and Snowden can be seen as a generalisation of representation stability ideas presented by Church and Farb and taking the ideas towards more of a commutative algebra setting.

In [48], they show that their methods recover the known results on representation stability and solved some open conjectures. A significant point is also using the newly defined setting to improve on the theory of Δ-modules from earlier work of Snowden [51]. The methods were later used to study FI-modules further, and these are one of the leading concrete applications of the abstract theory in [48]. The tools proposed by Sam and Snowden also show the connections between different categories, the main example being that modules over twisted commutative algebras are equivalent to the category of representations for the FI-modules [48, Proposition 7.2.5.]. Sam and Snowden have continued to work on the topic and use representation stability to prove results on modules over polynomial rings with infinitely many variables [46] and answering the Stembridge conjecture on Kronecker coefficients [47] for example. The defined concepts of representation stability in the very general category-theoretic setting has also created a plethora of further work by other authors and opened up representation stability to a wider variety of topics, including combinatorics [27, 45], noetherianity of representations of rooted trees [2, Chapter 5] and also applications of variations of finite sets with specific morphisms, like surjections that have been used to study stability in moduli spaces by Tosteson [53]. We will make use of these tools from representation stability as presented by Sam and Snowden in studying cellular resolutions. As far as we know, there has not been other work in the direction of cellular resolutions and representation stability. The closest results to cellular resolutions is the direction of representation stability to study homological invariants of the representations, like the homology of FI-modules [11]. Many of the results in this area are by Gan and Li [32, 33, 31].

Let R be a commutative noetherian ring and let Mod_R be the category of R-modules. Throughout this section, we assume the category $\mathscr{C}$ to be essentially small. Recall that this means the category $\mathscr{C}$ is equivalent to some small category; alternatively, it is locally small and has a small number of isomorphism classes as objects (assuming the axiom of choice). We want the category $\mathscr{C}$ to be of "combinatorial nature", which informally means objects are finite sets, possibly with some extra structures and morphisms are functions with extra structure allowed.

Remark 2.6.1. One should not confuse a category of combinatorial nature with a combinatorial category. There does exist a definition of *combinatorial category* [37], however, it is not the required condition, and it is a lot stronger than being of combinatorial nature.

Definition 2.6.2. *Let $\mathscr{C}$ be an essentially small category. A* representation*, or a $\mathscr{C}$-module, over R is a functor*

$$\mathscr{C} \to \mathrm{Mod}_R .$$

The representations of $\mathscr{C}$ form a category denoted by $\mathrm{Rep}_R(\mathscr{C})$. This is an abelian functor category with the morphisms between representations given by natural transformations.

Next let us consider some definitions related to the properties of individual representations (or modules). Let M be a representation of $\mathscr{C}$. A *subrepresentation* N of M is a subfunctor of M. Let M be a representation of $\mathscr{C}$. An *element* of M is an element of $M(x)$ for some $x \in \mathscr{C}$.

Having defined an element one can then talk about the generating sets for representations.

Definition 2.6.3. *Let M be a representation and let S be any set of elements of M. The smallest subrepresentation of M containing S is said to be* generated by S. *The representation M is said to be* finitely generated *if it is generated by some finite set of elements.*

The following representation is one of the main tools used to study noetherianity for representations.

Definition 2.6.4. *The* principal projective *representation for an element x is the functor P_x given by $P_x(y) = R[Hom(x, y)]$.*

Remark 2.6.5. In the paper of Sam and Snowden, they do not explicitly give the morphism part of the principal projective. The natural choice of maps between the Hom sets in the principal projective are post compositions, so this gives then a morphism between $P_x(y)$ and $P_x(z)$ if we have a morphism $f : y \to z$.

An important fact about principal projectives is that a representation of $\mathscr{C}$ is finitely generated if and only if it is a quotient of a finite direct sum of principal projectives.

Definition 2.6.6. *Let $M \in \mathrm{Rep}_R(\mathscr{C})$, then M is* noetherian *if every ascending chain of subobjects stabilises, or equivalently every subrepresentation is finitely generated. The category* $\mathrm{Rep}_R(\mathscr{C})$ *is* noetherian *if every finitely generated representation in it is noetherian.*

Proposition 2.6.7 ([48], Prop 3.1.1.)**.** *The category $Rep_R(\mathscr{C})$ is noetherian if and only if every principal projective is noetherian.*

One of the ways to study the representations is to use pullback functors. Given a functor $\Phi : \mathscr{C} \to \mathscr{C}'$ there is a *pullback functor* $\Phi^* : \mathrm{Rep}_R(\mathscr{C}') \to \mathrm{Rep}_R(\mathscr{C})$. The idea behind using pullback functors is that perhaps the representations we are interested in are not easy to study, but the representations of another category are, and thus we want to pull back the properties of interest. One of the desirable characteristics of a representation is finite generation, and thus, the main result on pullback functors is when do they map finitely generated objects to finitely generated objects. A sufficient condition for this is the property (F).

Definition 2.6.8. *Let $\Phi : \mathscr{C} \to \mathscr{C}'$ be a functor. Then Φ satisfies the* property (F) *if given any object $x \in \mathscr{C}'$ there exists finitely many $y_1, y_2, \dots, y_n \in \mathscr{C}$ and morphisms $f_i : x \to \Phi(y_i)$ such that for any $y \in \mathscr{C}$ and any morphism $f : x \to \Phi(y)$ there exists a morphism $g : y_i \to y$ such that $f = \Phi(g) \circ f_i$.*

Proposition 2.6.9 ([48],Prop 3.2.3.)**.** *A functor $\Phi : \mathscr{C} \to \mathscr{C}'$ satisfies the property (F) if and only if $\Phi^* : \mathrm{Rep}_R(\mathscr{C}') \to \mathrm{Rep}_R(\mathscr{C})$ takes finitely generated objects to finitely generated objects.*

Classically, in computational commutative algebra a Gröbner basis is a generating set for an ideal I such that the ideal generated by the leading terms of I equals the ideal generated by the leading terms of the Gröbner basis with respect to some monomial order [23, Chapter 15]. To explicitly compute a Gröbner basis one generally employs an algorithm, like the Buchberger's algorithm. One of the main results in [48] is the analogous definition of Gröbner basis for a representation of a category.

Let $S : \mathscr{C} \to Set$ denote a fixed functor to sets and let $S_x : \mathscr{C} \to Set$ be the functor given by $S_x(y) = \mathrm{Hom}(x, y)$. A *principal subfunctor* is a subfunctor of S generated by a single element.

Definition 2.6.10. *The poset $|S|$ is the set of principal subfunctors of S that is partially ordered by reverse inclusion.*

Let P denote the free module $R[S]$, where S is the functor $S : \mathscr{C} \to Set$, and write e_f for the element of $P(x)$ corresponding to $f \in S(x)$. An element of $P(x)$ is *monomial* if it is of the form λe_f for some $\lambda \in R$. A subrepresentation M is *monomial* if it is spanned by the monomials it contains.

One of the important properties a poset can have in this setting is noetherianity. Recall from Section 2.2 that a ring is noetherian when the ideals satisfy the ascending chain condition. This idea can be lifted to posets by defining an ideal in a poset P to be a subset I such that if $x \in I$ and $x \leq y$ then $y \in I$. Then the ideals form a new poset, and P is noetherian if the poset of ideals satisfies the ascending chain condition. Noetherian posets can also be defined directly by the poset itself: a poset P is noetherian if it satisfies the descending chain condition, descending chains stabilise, and has no infinite antichains, that is, subsets of elements where no two elements are comparable.

To define a Gröbner basis, it is required to have a concept of initial representations and terms. The functor S is *orderable* if there is a choice of a well-order on each $S(x)$ such that the induced map $S(x) \to S(y)$ is strictly order preserving for every $x \to y$.

Suppose S has ordering $\preceq$ on it. Then the *initial term* of an object $\alpha \in P(x)$ is $\text{init}(\alpha) = \lambda_g e_g$, where $g = \max_{\preceq}\{f | \lambda_f \neq 0\}$ and α is a direct sum of monomials. Let M be a subfunctor of P. The *initial representation* of M consists of $\text{init}(M)(x)$ that is the R-span of $\text{init}(\alpha)$ for a non-zero $\alpha \in M(x)$.

Definition 2.6.11. *Let M be a subrepresentation of P. A set of elements G is a Gröbner basis of M is* $\{\text{init}(\alpha) | \alpha \in G\}$ *generates* $\text{init}(M)$.

Theorem 2.6.12 ([48], Thm 4.2.4.). *Let S be orderable and $|S|$ be noetherian. Then every subrepresentation of P has finite a Gröbner basis. In particular, P is a noetherian object of* $\text{Rep}_R(\mathscr{C})$.

Definition 2.6.13. *Let $\mathscr{C}$ be an essentially small category. Then $\mathscr{C}$ is called* Gröbner *if for all $x \in \mathscr{C}$ the functor S_x is orderable and the poset $|S_x|$ is noetherian.*

The category $\mathscr{C}$ is quasi-Gröbner *if there exists some Gröbner category $\mathscr{C}'$ such that there is a functor $\Phi : \mathscr{C}' \to \mathscr{C}$ that is essentially surjective and satisfies property (F).*

The definition of Gröbner categories is the combinatorial condition providing us with algebraic properties of representations, and one of the main results concerns the noetherianity of representations.

Theorem 2.6.14 ([48], Thm 4.3.2.). *Let $\mathscr{C}$ be quasi-Gröbner, then* $\text{Rep}_R(\mathscr{C})$ *is noetherian.*

In the case the category is directed and small we can use the following proposition to determine if it is Gröbner. First note that an *admissible order* is a well-order on a set that also satisfies the following: if any two elements $u \le v$ then for any third element t, for which ut and vt make sense, we have $ut \le vt$.

Proposition 2.6.15 ([48], Prop 4.3.3.). *If $\mathscr{C}$ is a directed category, then as posets $|\mathscr{C}_x| \cong |S_x|$ for all objects x. In particular, $\mathscr{C}$ is Gröbner if and only if for all x the set $|\mathscr{C}_x|$ admits an admissible order and is noetherian as a poset.*

2.7 Graph theory and edge ideals

All graphs are assumed to be simple, meaning that the edges have no direction and there are no multiple edges or loops. The *degree* of a vertex is the number of edges incident to it. It is common to discuss the degree sequence of a graph, that is, the degrees sorted downwards. We will use a different strategy.

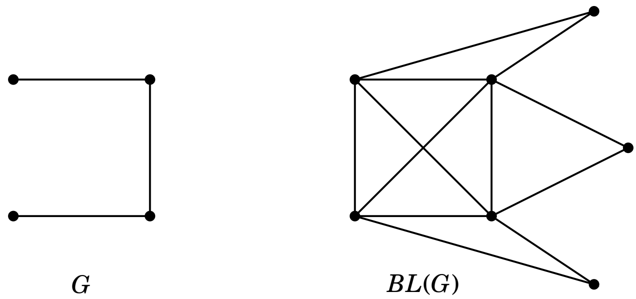


Figure 2.13. A graph G and its Booth–Lueker graph $BL(G)$.

Definition 2.7.1. *The* degree vector *or* degree statistics *of a graph G on n vertices is the column vector*

$$\mathbf{d}_G := (d_0, d_1, \ldots, d_{n-1})^T$$

where d_i is the number of vertices of degree i in G.

The complement of a graph G is denoted by $\overline{G}$ and the induced subgraph of G on the set of vertices W by $G[W]$.

Definition 2.7.2. *A graph G is said to be* chordal *if every cycle of length greater than three has a chord.*

Definition 2.7.3. *For any graph G let $BL(G)$ be the graph with vertex set $V(G) \cup E(G)$ and edges uv for every pair of vertices in G and ue for every vertex u incident to an edge e in G. We call $BL(G)$ the* Booth–Lueker graph *of G.*

See Figure 2.13 for an example of the Booth–Lueker graph construction. Both $BL(G)$ and its complement are chordal, for every graph G. They are split graphs, that is, the vertices can be divided into a connected set and an independent set. Thus, we actually have two interesting ideals to define.

Definition 2.7.4. *Given a graph G, we denote by $I_G := (x_i x_j \mid ij \in E(G))$ its* edge ideal *in a polynomial ring $S := k[x_1, \ldots, x_n]$ with as many variables as the vertices of G, where k is a field.*

Remark 2.7.5. We are interested in the betti numbers of the edge ideals of some chordal graphs. Corollary 5.10 of [34] shows that the characteristic of the field k does not affect these betti numbers.

The main results of Publication III are on the Boij-Söderberg coefficients of the edge ideals of Booth-Lueker graphs. Boij–Söderberg theory deals with writing the betti table of a finitely generated graded S-module as a sum of simpler pieces, coming from the so-called "pure betti tables": to each sequence $\mathbf{n} = (n_0, \ldots, n_s)$ of strictly increasing non-negative integers, we associate the table $\pi(\mathbf{n})$ with entries

$$\pi(\mathbf{n})_{i,j} := \begin{cases} \prod_{k \neq 0,i} \left| \frac{n_k - n_0}{n_k - n_i} \right| & \text{if } i \geq 0, j = n_i, \\ 0 & \text{otherwise.} \end{cases}$$

This is called the *pure betti table* associated to $\mathbf{n}$. There is a partial order to such sequences by setting

$$(n_0,\ldots,n_s) \geq (m_0,\ldots,m_t)$$

whenever $s \leq t$ and $n_i \geq m_i$ for all $i \in \{0,\ldots,s\}$.

Theorem 2.7.6 ([29], Theorem 5.1)**.** *For every finitely generated graded S-module M, there is a strictly increasing chain $\mathbf{n}_1 < \cdots < \mathbf{n}_p$ of strictly increasing sequences of $n+1$ non-negative integers and there are numbers $c_{\mathbf{n}_1},\ldots,c_{\mathbf{n}_p} \in \mathbb{Q}_{\geq 0}$ such that the betti table is given by*

$$\beta(M) = c_{\mathbf{n}_1}\pi(\mathbf{n}_1) + \cdots + c_{\mathbf{n}_p}\pi(\mathbf{n}_p).$$

Definition 2.7.7. *The non-negative rational numbers $c_{\mathbf{n}_1},\ldots,c_{\mathbf{n}_p}$ as in the theorem above are called* Boij–Söderberg coefficients *of M.*

Example 2.7.8. Let $I = (xy,xz,xw,yzw) \subset S = k[x,y,z,w]$. Then one can compute that

$$\begin{aligned}
\beta(S/I) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\qquad + \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 3 & 0 \end{pmatrix} \\
&= \frac{1}{3}\pi(0,2,3,4) + \frac{1}{6}\pi(0,2,3) + \frac{1}{4}\pi(0,2,4) + \frac{1}{4}\pi(0,3,4).
\end{aligned}$$

Hence $\mathbf{n}_1 = (0,2,3,4)$, $\mathbf{n}_2 = (0,2,3)$, $\mathbf{n}_3 = (0,2,4)$, $\mathbf{n}_4 = (0,3,4)$, and the Boij–Söderberg coefficients are $c_{\mathbf{n}_1} = 1/3$, $c_{\mathbf{n}_2} = 1/6$, $c_{\mathbf{n}_3} = 1/4$, and $c_{\mathbf{n}_4} = 1/4$.

One of the reasons why Boij-Söderberg theory is worth studying is that it provides a powerful tool to study the betti numbers of ideals. One of the big open questions relating to betti numbers is what betti tables are possible, and Boij-Söderberg theory provides a partial answer by being able to tell if a multiple of a betti table is possible.

Definition 2.7.9. *A* 2-linear resolution *of a graded S-module M is a resolution where $\beta_{i,j}(M) = 0$ if $j \neq i+1$. An ideal with a 2-linear resolution is called a* 2-linear ideal.

Often we refer to 2-linear ideals, this means that the betti table of S/I is of the form

$$\beta(S/I) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \beta_{1,2} & \beta_{2,3} & \cdots & \beta_{p,p+1} \end{pmatrix}.$$

By Boij–Söderberg theory, such a betti table will be the weighted average of certain pure tables of the form $\pi(0,2,3,\ldots,s,s+1)$. For instance

$$\pi(0,2) = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \end{pmatrix}$$

or

$$\pi(0,2,3) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 2 & 0 & \cdots \end{pmatrix}.$$

The edge ideals of Booth-Lueker graphs form a collection of examples of 2-linear ideals.

3. On categorical structures of cellular resolutions and their stability

In this chapter, we present an overview of the results in Publications I and II. The chapter has been divided into sections corresponding to Publications I and II. We will use S to refer to a polynomial ring as in the earlier chapters and by a cell complex will mean a regular CW-complex.

3.1 The category of cellular resolutions

As mentioned in the introduction, cellular resolutions are well known for specific families and particular types of ideals. The structure of cellular resolutions as a whole has not been investigated, though there exists a few proposed open problems pointing towards this kind of direction. Understanding more about the general structure of cellular resolutions can also let us build new cellular resolutions from the existing ones. Another the motivation for Publication I comes from wanting to apply representations of categories to cellular resolutions. Thus, one needs a category of cellular resolutions.

3.1.1 Definitions for the category

Let us fix a polynomial ring S. Recall that a cellular resolution is a cellular chain complex of a labelled cell complex that is acyclic. Alternatively, a cellular resolution can be thought of as a resolution of a monomial module that is supported on a cell complex. We have defined cellular resolutions on a connected cell complex. If we consider a disjoint cell complex, see Figure 3.1 for example, we note that the cellular complex would have to be disjoint too, in the sense that the free modules F_i for $i \geq 1$ can be decomposed to a direct sum of free modules each corresponding to one connected component. This follows from the definition. Moreover, it is the direct sum of two cellular resolutions coming from each piece apart from the F_0 module. In algebra the direct sum of two resolutions is well defined as a direct sum of chain complexes. The chain complex $S/I \oplus S/J$ is not a resolution of a single ideal, but if we consider the disjoint cell complex to contain an empty cell in each connected component, the direct sum is then the cellular complex of the disjoint cell complex, and this is consistent with the homology of cell complexes, too. We want these direct

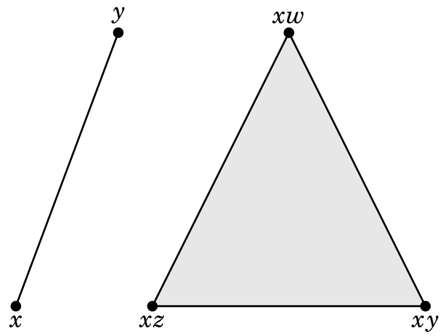


Figure 3.1. A labelled cell complex that is not connected.

sum resolutions to be included in cellular resolutions as they will provide desirable properties to the category. Moreover, the standard properties of cellular resolutions still hold for these direct sum resolutions. As an example Proposition 2.4.4 from Chapter 2 has a multi-component version.

Morphisms play a fundamental role in category theory and representation stability applications. Previous work on cellular resolutions has not included discussion on the maps between cellular resolutions. The examples of maps that have been used in the literature are Morse maps [36], multiplication by a monomial [20], and embedding of a minimal resolution to a non-minimal one as the direct summand by the definition of a direct summand [24, Thm 1.6]. Our definition of morphisms of cellular resolutions should, and will, capture all of the above cases. One could, of course, try using a chain map between cellular resolutions, yet this does not preserve the topological data a cellular resolution contains. On the other hand, there are cellular maps between cell complexes, but again these require a lifting to a chain map, and it might ignore some algebraic properties of cellular resolutions. Our goal is to have a morphism of cellular resolutions that would preserve both algebraic and topological data of it. For this purpose, it is useful to consider a cellular resolution as a pair $(\mathbf{F}, X)$ where $\mathbf{F}$ is a free resolution and X is the labelled cell complex supporting this resolution. Note that only taking one of the two allows us to recover the other. The main idea of the morphism construction is to find a chain map and a cellular map that "do the same thing". More precisely, this means we find a chain map that maps the generators of the free modules in the same way as the cellular map maps the corresponding cells of the generators. Formally this can be defined as follows:

Let $g : X \to Y$ be a cellular map between two labelled cell complexes X and Y with label ideals I and J, respectively. Define a map $\varphi_g : I \to J$ on the set of the labels by the action of g, that is, the label $m_x \in I$ maps to $m_y \in J$ if and only if the face x labelled by m_x maps to the faces $y_1, \dots, y_r$ labelled by $m_{y_1}, \dots, m_{y_r}$ with $m_y = \operatorname{lcm}(m_{y_1}, \dots, m_{y_r})$ under g. If φ_g is well-defined it is called the *label map* of g.

Definition 3.1.1. *Let X and Y be labelled cell complexes and $\mathbf{F}_X$, and $\mathbf{F}_Y$ be the cellular resolutions coming from the labelled cell complexes, respectively. The cellular map $g: X \to Y$ is compatible with a chain map $\mathbf{f}: \mathbf{F}_X \to \mathbf{F}_Y$ if they satisfy the following conditions:*

(i) φ_g is well-defined and the equality $f_0(x) = \varphi_g(x)$ holds for all labels $x \in I$, and

(ii) f_i maps the generator e_x, associated to face $x \in X$, in $\mathbf{F}_{X,i}$ to some linear combination of the generators e_{y_i}, $i \in \{1,2,\dots,r\}$, associated to $y_i \in Y$ with the coefficients in S if and only if g maps x to the union of $y_1, y_2, \dots, y_r$.

The compatible pair of a chain map $\mathbf{f}$ and a cellular map g are denoted by the pair notation $(\mathbf{f}, g)$ or by the fraktur letters, for example $\mathfrak{f}$.

Example 3.1.2. Let us consider the cellular resolutions of Example 2.4.3 and its cell complex X, and the cellular resolution of the ideal

$$I = (x^2yz, x^2zw, xy^2z, xyzw, xyz^2, y^2z^2w)$$

supported on the cell complex Y in Figure 3.2. The cellular resolution is

$$S \xleftarrow{\partial_1} S^6 \xleftarrow{\partial_2} S^7 \xleftarrow{\partial_3} S^2 \leftarrow 0$$

with the maps

$$\partial_1 = \begin{bmatrix} x^2yz & x^2zw & xy^2z & xyzw & xyz^2 & y^2z^2w \end{bmatrix},$$

$$\partial_2 = \begin{bmatrix} -w & -y & 0 & 0 & 0 & 0 & 0 \\ y & 0 & -y & 0 & 0 & 0 & 0 \\ 0 & x & 0 & -w & -z & 0 & 0 \\ 0 & 0 & x & y & 0 & -z & 0 \\ 0 & 0 & 0 & 0 & y & w & -yw \\ 0 & 0 & 0 & 0 & 0 & 0 & x \end{bmatrix},$$

and

$$\partial_3 = \begin{bmatrix} y & 0 \\ -w & 0 \\ y & 0 \\ -x & z \\ 0 & -w \\ 0 & y \\ 0 & 0 \end{bmatrix}.$$

Purely as cell complexes, X embeds into Y. This gives that φ_g is a multiplication by the monomial yz. Then we can find a map between the chain complexes with

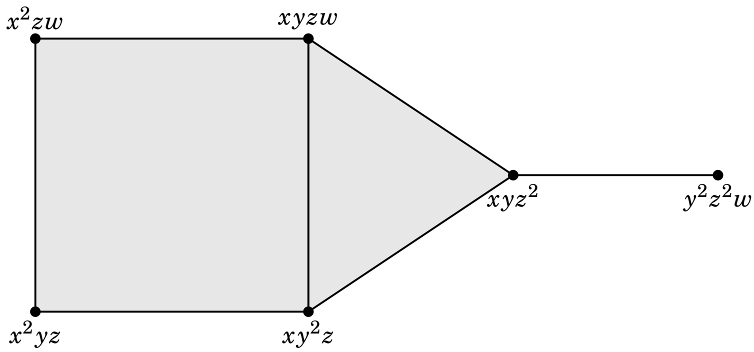


Figure 3.2. A cell complex supporting the resolution of the module S/I of Example 3.1.2.

$f_0 = [yz]$. It gives is the module homomorphisms

$$f_1 = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}, f_2 = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}, \text{ and } f_3 = \begin{bmatrix} 0\\ 1 \end{bmatrix}.$$

Next, we want to check that the chain map defined by $\{f_i\}$ is compatible with the embedding of cell complexes. The condition $f_0 = \varphi_g$ holds since both maps are multiplications by the same monomial. Moreover, the maps are all taking a single generator to a single generator, and these are precisely the ones the embedding gives.

The definition of compatible maps allows cellular resolutions to have multiplication by a monomial, embeddings, identity maps, and Morse maps as morphisms. However, the definition is very restrictive, and some things one might expect to give a morphism are not morphisms. For example, permutations or change of variables within the ring do not give maps between cellular resolutions. Despite the limitations, the defined compatible pairs provide desired properties for the morphisms and the category of cellular resolutions that is defined with them.

Definition 3.1.3. *Let* $S = k[x_1, x_2, \ldots, x_n]$. *We define* **CellRes** *over* S *to be the following category:*

- *Objects are cellular resolutions, coming from any regular CW-complex labelled with monomials from* S*, and their direct sums.*
- *A set of morphisms for any pair of objects* $\mathbf{F}_X$ *and* $\mathbf{F}_Y$ *with individual maps given by the compatible pairs* $(\mathbf{f}, f)$ *of a chain map of* S*-modules and a cellular map.*

3.1.2 Properties of the category of cellular resolutions

One of the desired properties for the category of cellular resolutions is to have forgetful functors to both **Top** and $C_\bullet(\mathrm{Mod}_S)$. The forgetful functor $\Phi_{CC}: \mathbf{CellRes} \to C_\bullet(\mathrm{Mod}_S)$ takes a cellular resolution to the chain complex that is the resolution and takes a morphism $(\mathbf{f}, g)$ to the chain map $\mathbf{f}$. The other forgetful functor $\Phi_{\mathbf{Top}}: \mathbf{CellRes} \to \mathbf{Top}$ takes a cellular resolution supported on cell complex X to the unlabelled cell complex given by X and it takes a morphism $(\mathbf{f}, g)$ to the cellular map g. The functor $\Phi_{\mathbf{Top}}$ does take cellular resolutions to the subcategory of regular CW-complexes.

For **CellRes** to have the desired forgetful functors does not yet tell us much about the structure or properties of this category. Studying some of the properties of the category **CellRes** was the goal of Publication I. Both **Top** and $C_\bullet(\mathrm{Mod}_S)$ are nice categories in the sense that they have desirable properties like being (co)complete, abelian, and other properties. One would, of course, hope that these properties appear in **CellRes** as well. Unfortunately, this is not always the case; however, some properties do lift from **Top**, $C_\bullet(\mathrm{Mod}_S)$, or both, to **CellRes**.

On the types of objects the category **CellRes** contains, it has an initial object that is the resolution $0 \leftarrow S \leftarrow 0$ that is supported on the empty cell complex with a label 1 and algebraically it is the resolution of the module $S/(1)$.

The category **CellRes** is more abundant on the properties coming from the morphisms than the objects, that is, we have more results that are about the morphisms than about the objects. Sometimes the chosen morphisms do not behave as well as one would wish and in general, the kernels and cokernels of the morphisms do not exist. This implies that **CellRes** is not an abelian category. The structure of morphisms allows us to lift many concepts from **Top** and $C_\bullet(\mathrm{Mod}_S)$. We can define homotopy for the morphisms in **CellRes** following the definitions in **Top** and $C_\bullet(\mathrm{Mod}_S)$.

Definition 3.1.4. *Let* $(\mathbf{f}, f), (\mathbf{g}, g): \mathbf{F} \to \mathbf{F}'$ *be cellular resolution morphisms, then* $(\mathbf{f}, f)$ *is* homotopic *to* $(\mathbf{g}, g)$*, denoted by* $(\mathbf{f}, f) \sim (\mathbf{g}, g)$*, if the components are homotopic, meaning that* $\mathbf{f}$ *is homotopic to* $\mathbf{g}$ *as chain maps and* f *is homotopic to* g *as continuous topological maps.*

Homotopies form a class of morphisms in **CellRes** that inherit properties from **Top** thanks to the cellular map component of the morphisms. Hence they satisfy properties that make **CellRes** a homotopical category with the homotopies as the weak equivalences. Another property that almost directly applies to **CellRes** is the enrichment by simplicial sets. Since the cellular map is a part of the morphism it provides the necessary structure to enrich **CellRes** with simplicial sets.

It is known that mapping cones can be used to construct minimal resolutions of particular ideals when starting with a resolution of a monomial module defined by at least a single monomial [20]. This is a way to build cellular resolutions from existing ones, and if we relax the condition on minimality, one can naturally ask if the mapping cone, or the mapping cylinder that is a similar construction, gives a cellular resolution. Noting that the ways to construct mapping cones and cylinders in **Top** and $C_\bullet(\mathrm{Mod}_S)$

are essentially the same, one would expect these to work for cellular resolutions as well, and the identification of the two being practically the same can be found in [56, pp. 20-21] for example. Indeed, both mapping cones and mapping cylinders construct a cellular resolution in **CellRes**, as shown in Publication I. Moreover, for practical computations of mapping cones and cylinders, one can either use the chain complex construction of the resolution, or the topological construction on the labelled cell complex supporting the resolution and get the same cellular resolution at the end.

Let $\mathbf{F}$ and $\mathbf{F}'$ be cellular resolutions supported on the cell complexes X and Y respectively, and suppose that there is a map $\mathfrak{f}: \mathbf{F}' \to \mathbf{F}$. For the mapping cone $C(\mathfrak{f})$ the process can be viewed as adding a point to the cell complex X, that is, then attached to the cell complex by some suitable k-cell. The work of Dochterman and Mohammadi on the mapping cones of cellular resolutions of ideals with linear quotients and regular decomposition functions uses this adding a point to build a larger minimal cellular resolution from an easier smaller one [20]. In general, the new point will often have label 1, implying that algebraically the resolution becomes trivial.

On the other hand, with mapping cylinders, we do not face the issue of creating vertices with label 1, but the mapping cylinders are not providing us with minimal resolutions in most cases. Disregarding the minimality of a resolution one can use mapping cylinders to build new cellular resolutions, and one of the main results of Publication II is on gluing them into a diagram. A special case of the gluing is the following: Let $\mathbf{F}$ and $\mathbf{F}'$ be cellular resolutions, such that both contain the sub-resolution $\mathbf{G}$. Then gluing $\mathbf{F}$ and $\mathbf{F}'$ together along $\mathbf{G}$, by identifying the $\mathbf{G}$ in $\mathbf{F}$ with the $\mathbf{G}$ in $\mathbf{F}'$, gives a cellular resolution.

Proposition 3.1.5. *Let D be a finite diagram of cellular resolutions. Then gluing mapping cylinders into D, gives a new cellular resolution.*

With the special case of the gluing in mind, showing that gluing in mapping cylinders to the diagram gives a cellular resolution is easy when the gluing components are disjoint. In the case of the given diagram having any loops, one must pay more detailed attention to the gluing, which requires gluing the cell complex to itself. To avoid building holes in the cell complex or homology in the resolution, we must identify some of the mapping cylinders and add cells between them.

Example 3.1.6. This example illustrates the gluing of mapping cylinders into a diagram and the need to pay attention to holes. Let $S = k[x,y]$ and let D be the diagram of cellular resolutions with a indexing category shown in Figure 3.3. Take the resolution of $S/(x,y)$ for each of the cellular resolutions in the diagram and let the maps be identity maps. Then each of the mapping cylinders is supported on a square. Gluing it all together forms a cell complex that has holes, and thus we know it will not support an acyclic cellular chain complex. One would like to fill in the hole that the gluing process has created, and this can be done by identifying the mapping cylinders that correspond to the same map. This means we add a cell to fill in the hole where the boundaries are formed by the corresponding mapping cones. These cases are illustrated in Figure 3.4.

The next part of the work in Publication II focuses on studying the products and

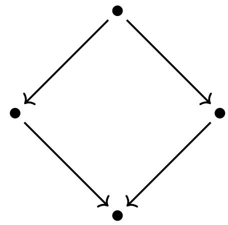


Figure 3.3. Diagram of the Example 3.1.6.

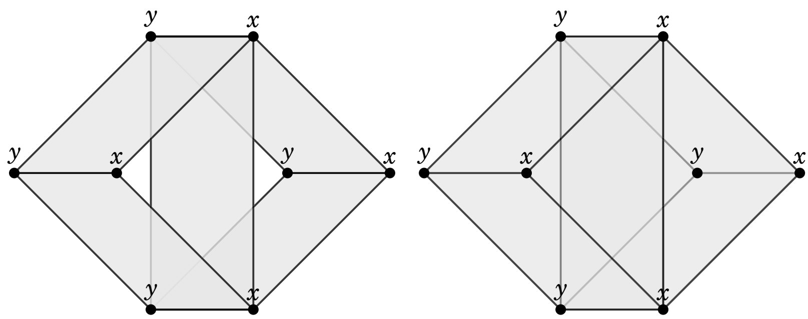


Figure 3.4. Labelled cell complexes obtained by gluing in mapping cylinders (i) without identification of corresponding mapping cylinders and (ii) after identification.

coproducts, followed by limits and colimits. We will only consider finite cases as the infinite case of any of them would produce an infinite cellular resolution which is not defined. Products both on **Top** and $C_\bullet(\mathrm{Mod}_S)$ are well defined, yet we run into problems trying to just lift definitions as before since we now have two different types of products. In $C_\bullet(\mathrm{Mod}_S)$ products are just direct sums of the chain complexes, and in **Top** a product is a topological space with the underlying set being a cartesian product of the spaces in the product. Trying to apply one of these to the resolution and one to the supporting cell complex would not give the same cellular resolution at the end. The implication of this is that neither of these two definitions satisfies the conditions of a product. In the case one of the cellular resolutions is the resolution of $S/(1)$ on some trivially labelled cell complex, the topological product can be lifted, and we get what we call "product up to homotopy".

The coproduct, denoted by $\mathbf{F} \sqcup \mathbf{F}'$, in **CellRes**, is a better behaving case. The coproduct is a direct sum both in $C_\bullet(\mathrm{Mod}_S)$ and **Top**, so defining a coproduct in **CellRes** as the direct sum is a fair assumption. The direct sum of cellular resolutions does satisfy the category-theoretic definition of a coproduct. The category **CellRes** has all finite coproducts. This is partially implied already by the chosen definition of the objects.

Limits of a diagram D do not, in general, exist in **CellRes**. This follows from not having well-defined products, and since the product is a special case of a limit, it implies that they do not exist in general. A *finite inverse system* is a diagram indexed by a directed poset. The only type of a limit **CellRes** has for all given diagrams of a particular type are those of finite inverse systems. Similarly to coproducts, the colimits are well defined, and they can be either computed as the colimit of resolutions, that is, using the chain complex definition, or by computing the colimit of the labelled cell complexes, that is, the topological definition.

Theorem 3.1.7. *The category* **CellRes** *has all finite colimits.*

Also having all finite colimits implies that **CellRes** is a finitely cocomplete category by definition.

Recall that tensor products are well defined for chain complexes, and they are used to define specific resolutions like the Koszul resolution and at times used in computing invariants like betti numbers via Tor-functor. Given two cellular resolutions $\mathbf{F}$ and $\mathbf{F}'$, taking their tensor product $\mathbf{F} \otimes \mathbf{F}'$ as chain complexes, often does not give a cellular resolution. The maps in the tensor product do not have the right degree to give a cellular resolution, but one can slightly modify the definition to have suitable maps in the chain complex. This gives a bifunctor $\otimes : \mathbf{CellRes} \times \mathbf{CellRes} \to \mathbf{CellRes}$, and a monoidal structure to the category. As with many of the earlier definitions, the tensor product of cellular resolutions can be computed using only the labelled cell complexes. If the cellular resolutions are supported on the complexes X and X', then the tensor product is supported on the join of X and X'.

A typical pattern appearing in the category of cellular resolutions is that if something is well defined in both **Top** and $C_\bullet(\mathrm{Mod}_S)$ such that the two definitions are essentially the same, then the definition lifts to **CellRes** and inherits many properties from the categories **Top** and $C_\bullet(\mathrm{Mod}_S)$. In the case when the definitions are too different, it may not lift to **CellRes** like with the products. This, somewhat expected behaviour of the category **CellRes** with relation to the other two categories is recorded in the Table 3.1.

3.1.3 Homotopy colimits and Morse theory in CellRes

Homotopy colimits were introduced for topological spaces in Chapter 2. Since the colimit lifts well from **Top** and homotopy is well defined for morphisms in **CellRes** it is sensible to wonder if homotopy colimits lift to **CellRes**. Homotopy colimits can also offer a way to build new resolutions from having a diagram of cellular resolutions and thus it is considered one of the important results of Publication II. One way to lift homotopy colimits is to define the *homotopy colimit* as the cellular resolution obtained by gluing in mapping cylinders into the coproduct of the diagram. Alternatively, we can define the *geometric realisation* of a simplicial set as the associated simplicial complex labelled with 1 on each vertex.

Definition 3.1.8. *Let $\mathscr{I}$ be a finite small category and let D be a diagram in* **CellRes**. *Let D^i and D^j be resolutions in D, and let the morphism between them be denoted by $\mathfrak{f}_{ij} = (\mathbf{f}^{ij}, f^{ij})$ if there is a map $\psi : i \to j$ in $\mathscr{I}$. Then define the* homotopy colimit of the diagram D, $\mathrm{Hocolim}(D)$*, to be the direct sum*

$$\mathrm{Hocolim}(D) = \sqcup B(i \downarrow \mathscr{I}) \times D^i$$

quotient by a relation $\sim$. Here $B(i \downarrow \mathscr{I})$ is the geometric realization of the nerve of the category under i, $(\mathscr{I} \downarrow i)$, and D^i is the element in the diagram D associated to the

Category	**CellRes**	**Top**	$C_\bullet(\mathrm{Mod}_S)$
product	does not exist in general, sometimes can be lifted from **Top** to give a product up to homotopy	Cartesian product of the underlying sets with a suitable topology	direct sum of chain complexes
coproduct	direct sum, all finite ones exist	direct sum, has all coproducts	direct sum, has all coproducts
limit	do not exist in general	has all limits	has all limits
colimit	has all finite colimits	has all colimits, glue the diagram	has all colimitss
tensor product	"reduced" tensor product on the resolution, corresponds to a join on the labelled cell complexes	join behaves computationally similar to the tensor product	tensor products exist

Table 3.1. Summary of properties of **CellRes**.

element $i \in \mathscr{I}$. *The maps* $\mathfrak{a} : B(j \downarrow \mathscr{I}) \times D^i \to B(j \downarrow \mathscr{I}) \times D^j$ *and* $\mathfrak{b} : B(j \downarrow \mathscr{I}) \times D^i \to B(i \downarrow \mathscr{I}) \times D^i$ *are given by*

$$\mathfrak{a}(p,x) = (p, \mathfrak{f}_{ij}(x))$$

and

$$\mathfrak{b}(p,x) = (\delta_{ji}(p), x)$$

for every map $i \to j \in \mathscr{I}$, *where* $\delta_{ji} : B(j \downarrow \mathscr{I}) \hookrightarrow B(i \downarrow \mathscr{I})$. *Then the quotient is given by the relation* $\mathfrak{a}(p,x) \sim \mathfrak{b}(p,x)$.

Since the homotopy colimit is a cellular resolution, it must belong to a module. We can describe the generating set of the ideal using the definition of gluing mapping cylinders, the labels on $\mathrm{Hocolim}(D)$ are the union of the labels in the cell complexes of D^i. It is very likely that some labels are repeated or redundant for the ideal the union creates, therefore $\mathrm{Hocolim}(D)$ is often a non-minimal resolution.

These two definitions of homotopy colimit are equivalent. There is a way to define homotopy colimits via derived functors as noted in Chapter 2; however, the theory is written for Quillen model categories, which **CellRes** is not. This creates an open question on the derived functors, homotopy colimits and **CellRes** to which we return in Section 3.3.

Finally, we come to the last part of Publication II concerning Morse theory on cellular resolutions. If one looks at the definitions and theorems given in Chapter 2 for algebraic and discrete Morse theory on cellular resolutions, they resemble each other, and one would expect them to "match up" on cellular resolutions. Again our expectations are met, and applying discrete Morse theory on the cell complex or algebraic Morse theory on the resolution gives the same result assuming the Morse matching is the same. This also allows us to define a Morse map between cellular resolutions as a morphism.

Theorem 3.1.9. *Let* $\mathbf{F}$ *be a cellular resolution with a labelled cell complex* X, *and let* M *be a Morse matching on both the face poset of* X *and the associated directed graph of* $\mathbf{F}$. *Let* $\mathbf{f}$ *be the chain map from* $\mathbf{F}$ *to* $\tilde{\mathbf{F}}$, *and let* f *be the cellular strong deformation retract of* X *coming from Morse theory for the matching* M. *Then the pair* $(\mathbf{f}, f)$ *formed of the Morse maps is a morphism in* **CellRes**.

Example 3.1.10. For an example of Morse maps in the category **CellRes** we will look at a special case of the example of powers of the maximal homogeneous ideal from Batzies and Welker [3, Section 5]. This same example is also referred to in [36, pp. 33-35] and also expanded to include products of these ideals. For convenience, we will refer to the example in [3] by BW-example.

The BW-example gives an explicit construction of a cell complex and a cellular resolution that supports the maximal homogenous ideal of a polynomial ring $S = k[x_1, x_2, \dots, x_d]$, and then describes the matching on the face poset that will give a minimal resolution after applying discrete Morse theory.

The case for this example is the third power of the ideal $I = (x_1, x_2, x_3)$ in the ring $S = k[x_1, x_2, x_3]$. The cellular resolution of this ideal is given by

$$\mathbf{F} : S \xleftarrow{\partial_1} S^{10} \xleftarrow{\partial_2} S^{18} \xleftarrow{\partial_3} S^9 \leftarrow 0$$

with maps in the resolution are given by

$$\partial_1 = \begin{bmatrix} x_1^3 & x_1^2x_2 & x_1^2x_3 & x_1x_2^2 & x_1x_2x_3 & x_1x_3^2 & x_2^3 & x_2^2x_3 & x_2x_3^2 & x_3^3 \end{bmatrix},$$

$$\partial_2 = \begin{bmatrix}
x_2 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-x_1 & 0 & x_3 & x_2 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -x_1 & -x_2 & 0 & 0 & x_2 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -x_1 & 0 & 0 & 0 & x_3 & x_2 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -x_1 & -x_1 & 0 & -x_2 & 0 & 0 & x_3 & x_2 & x_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -x_1 & 0 & 0 & 0 & -x_2 & 0 & 0 & x_2 & x_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & 0 & -x_1 & 0 & 0 & 0 & -x_2 & x_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & -x_1 & 0 & 0 & -x_2 & x_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & 0 & 0 & -x_2
\end{bmatrix},$$

and

$$\partial_3 = \begin{bmatrix} x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & x_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 \end{bmatrix},$$

On the side of cell complexes, this resolution is supported on a subdivided triangle denoted by C_3^3. The cell complex is drawn in Figure 3.6, and the corresponding face poset is presented in Figure 3.5. Let $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{N}^3$, $J \subseteq \{1,2,3\}$, $|\mathbf{a}| = \sum_{i=1}^{3} a_i$, and $\mathbf{e}_i$ is the i-th unit vector in $\mathbb{R}^3$. The BW-example shows that the individual cells can be described by

$$C_{\mathbf{a},J} = \operatorname{conv}\left(\mathbf{a} + \sum_{j \in J} \epsilon_j \mathbf{e}_j \,|\, \epsilon_j \in \{0,1\}, \sum_{j \in J} \epsilon_j = 3 - |\mathbf{a}| \right),$$

where conv denotes the convex hull, and one of the two conditions is satisfied: $|\mathbf{a}| = 3$ and $J = \emptyset$, or $1 \le 3 - |\mathbf{a}| \le |J| - 1$.

The first condition gives all the vertices of the cell complex, and in this case they correspond to the integer points $(3,0,0), (0,3,0), (0,0,3), (2,1,0), (2,0,1), (1,2,0), (1,0,2),$ $(1,1,1), (0,2,1)$, and $(0,1,2)$. These have a natural labelling by taking the label of the point (k,l,m) to be $x_1^k x_2^l x_3^m$. For the higher cells the conditions become $|J| = 2$ and $|\mathbf{a}| = 2$, or $|J| = 3$ and $|\mathbf{a}| = 1$ or 2. In the first case we have three possible sets J and six different vectors $\mathbf{a}$, thus is gives 18 cells. These are the 1-cells or edges in the cell complex. In the second case, there is only one possible set $J = \{1,2,3\}$ and nine different vectors $\mathbf{a}$, producing nine cells. These are the 2-cells. The labelling for the 1- and 2-cells is obtained from vertices they contain.

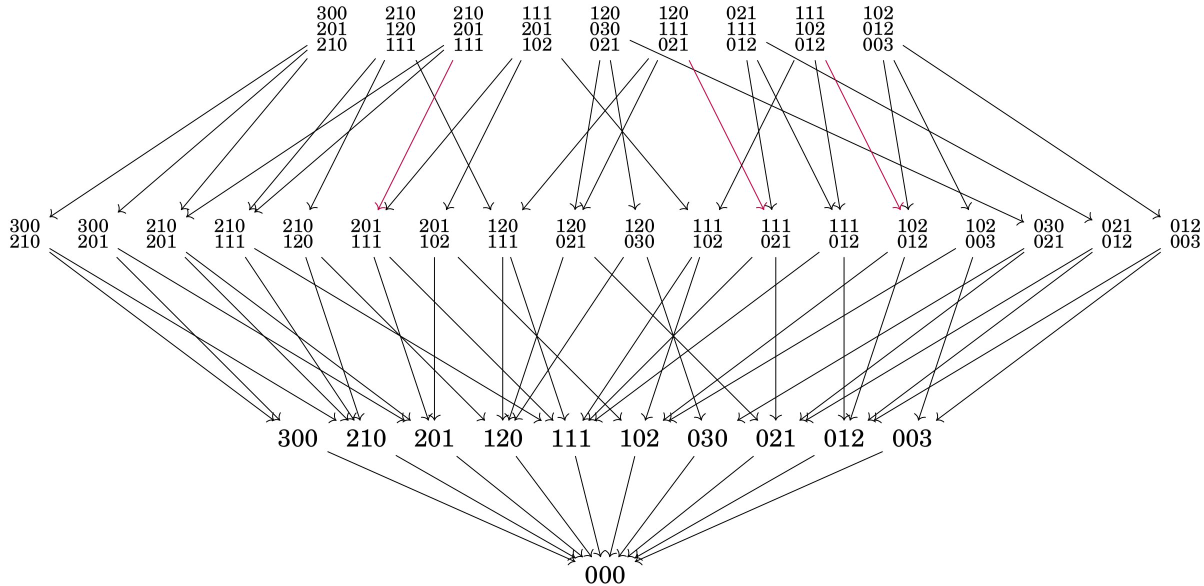


Figure 3.5. The face poset of the cell complex in Example 3.1.10. The vertices are denoted by $a_1a_2a_3$ corresponding to the exponent vector $\mathbf{a} = (a_1, a_2, a_3)$ defining them and higher dimensional cells are marked by the vertices they contain. Purple edges correspond to the Morse matching.

The Morse matching giving a minimal cellular resolution is also explicitly given in the BW-example. The matching is presented in the face poset in Figure 3.5 and the corresponding cells on the cell complex have been coloured. The matching can also be formulated as conditions on $\mathbf{a}$ and J [3, p.15 paragraph 1]. The critical cells are given by $C_{\mathbf{a},J}$ such that $2 \leq 3 - \mathbf{a} \leq |J| - 1$ and $\max J \geq \max\{i \in \{1,2,3\} \mid a_i \neq 0\}$. The critical cells then describe the resulting cellular resolution, and the BW-example provides the differential maps, too. Let $\widetilde{\mathbf{F}}$ denote the cellular resolution obtained from this matching. It is given by

$$\widetilde{\mathbf{F}} : S \xleftarrow{\partial_1'} S^{10} \xleftarrow{\partial_2'} S^{15} \xleftarrow{\partial_3'} S^6 \leftarrow 0$$

with differential maps given as

$$\partial_1' = \begin{bmatrix} x_1^3 & x_1^2x_2 & x_1^2x_3 & x_1x_2^2 & x_1x_2x_3 & x_1x_3^2 & x_2^3 & x_2^2x_3 & x_2x_3^2 & x_3^3 \end{bmatrix},$$

$$\partial_3' = \begin{bmatrix} x_3 & 0 & 0 & 0 & 0 & 0 \\ -x_2 & 0 & 0 & 0 & 0 & 0 \\ x_1 & 0 & z & 0 & 0 & 0 \\ 0 & x_3 & 0 & 0 & 0 & 0 \\ 0 & -x_2 & -x_3 & 0 & 0 & 0 \\ 0 & 0 & -x_2 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & -x_2 & -x_3 & 0 \\ 0 & 0 & x_1 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & -x_2 & -x_3 \\ 0 & 0 & 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 \end{bmatrix},$$

and

$$
\partial_2' = \begin{bmatrix}
x_2 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-x_1 & 0 & x_3 & x_2 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -x_1 & -x_2 & 0 & 0 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -x_1 & 0 & 0 & x_3 & x_2 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -x_1 & 0 & -x_2 & 0 & 0 & x_3 & x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -x_1 & 0 & 0 & 0 & -x_2 & 0 & x_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & 0 & 0 & 0 & 0 & x_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & 0 & 0 & 0 & -x_2 & x_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & 0 & 0 & -x_2 & x_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & 0 & 0 & -x_2
\end{bmatrix},
$$

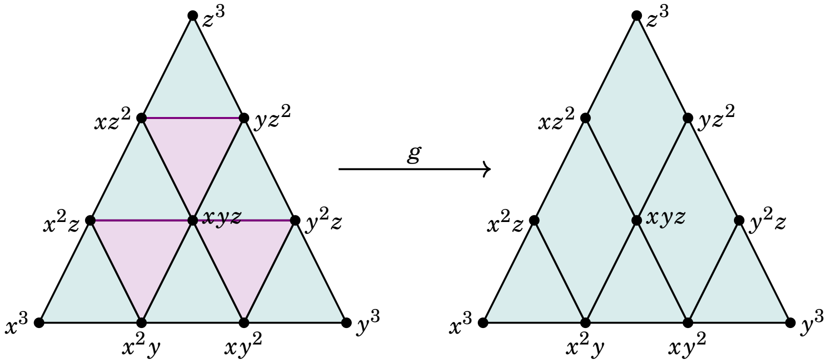


Figure 3.6. The non-minimal and minimal cell complex of Example 3.1.10.

in the matrix form. The supporting cell complex is presented in Figure 3.6.

Then the map $g : C_3^3 \to \widetilde{C_3^3}$ can be described as the cellular map that acts as an identity on all the critical cells, and each pair of a 2-cell and a 1-cell that correspond to a matching get mapped to the other two boundary 1-cells of the 2-cell, thus resulting in the lozenge shapes in Figure 3.6. The map between the cell complexes can be written out as the following matrices

$$f_0 = \begin{bmatrix} 1 \end{bmatrix}, f_1 = \begin{bmatrix} 1 \end{bmatrix},$$

$$f_2 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},$$

and

$$f_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The label map from g is the identity, and from this we see directly that condition (i) for compatible maps holds. Checking the second condition, we note that the cells and modules corresponding to critical vertices map by an identity map in both maps, so for those the condition (ii) holds. Thus it remains to check the "deleted" parts. These match up, too, and so we get that the pair forms a cellular resolution morphism.

For other explicit cases see [36, pp.33] for C_2^2 and [3, pp.14,16] for beautiful figures of the case C_4^3. In the general setting, the description of the differentials in the minimal resolution in the BW-example is based on how the cells behave, and the compatibility can be inferred from the general formulation as well.

3.2 Families of cellular resolutions

The category of cellular resolutions as a whole forms a large class of objects, and this may often be too large to study. Moreover, algebraically it is often interesting to study smaller families of cellular resolutions. Many of the existing results on cellular resolutions focus on a specific type of cellular resolutions. This can be done by restricting the ideal, construction, or just focusing on a specific family. These can all be viewed as subcategories of **CellRes**, if we can assume the number of variables has an upper bound. The focus of Publication II is on families of cellular resolutions. Families form an optimally structured category to apply tools from representation stability.

A *family $\mathscr{F}$ of cellular resolutions* is an infinite sequence of cellular resolutions with maps in between the cellular resolutions. The maps in the family can be taken to be all possible cellular resolution maps between the resolutions, no maps at all, or any variation between the two. A family of cellular resolutions is also a subcategory of **CellRes**. The sets of morphisms in the family are denoted by Hom($\mathbf{F}_i,\mathbf{F}_j$), and we refer to all of the sets of morphisms as Hom sets.

In the situation when we are interested in the algebraic properties of the family, it is useful to reduce the morphisms to only have a single pair of compatible maps for each chain map. This method of reducing maps is used throughout the chapter.

3.2.1 Linear families of cellular resolutions

A typical example of a family of cellular resolutions is the family of powers of an ideal. In particular, the family of powers of the maximal ideal $I = (x,y,z)$ in the polynomial ring $S = k[x,y,z]$ was a motivating example of families where we can apply representation stability tools, and it motivates the following definition.

Definition 3.2.1. *Let $\mathscr{F} : \mathbf{F}_1 \to \mathbf{F}_2 \to \ldots \to \mathbf{F}_i \to \ldots$ be a family of cellular resolutions. The family $\mathscr{F}$ is* linear *if there is at least one morphism $f_{i,i+1} : \mathbf{F}_i \to \mathbf{F}_{i+1}$ between consecutive cellular resolutions, and the other morphisms are compositions of those, that is, for any $f_{i,i+k} : \mathbf{F}_i \to \mathbf{F}_{i+k}$ there exists some consecutive morphisms such that*

$$f_{i,i+k} = f_{i+k-1,i+k} \circ f_{i+k-2,i+k-1} \circ \ldots \circ f_{i+1,i+2} \circ f_{i,i+1},$$

or maps from the cellular resolution to itself $\mathbf{F}_i \to \mathbf{F}_i$.

Linear families have many useful properties, and many of the types of families we are interested in studying are included in them.

We can, of course, define a *representation* for a family $\mathscr{F}$ as a functor from $\mathscr{F}$ to Mod_S, as introduced in Chapter 2. Noetherianity of the representation category is one of the desired properties as then we have that subrepresentations of finitely generated representations are finitely generated as well. Linear families, and families that are eventually linear, in particular, satisfy this condition thanks to their structure, forcing all principal projectives to be noetherian.

Theorem 3.2.2. *Let $\mathscr{F}$ be a linear family of cellular resolutions with finitely generated* Hom *sets. Then $\mathrm{Rep}_S(\mathscr{F})$ is noetherian.*

Gröbner categories are among the main results in [48] and there exist power families of cellular resolutions that are Gröbner. Despite being a very powerful definition in the results of Publication II, linearity is often more natural to show for the families, and it is enough to reach the desired properties.

The *p-th module representation* is a representation of a family $\mathscr{F}$

$$s_p : \mathscr{F} \to \mathrm{Mod}_R$$

such that $s_p(\mathbf{F}_i) = p$-th free module in the resolution and $s_p(\mathbf{F}_i \to \mathbf{F}_j)$ is the restriction of the chain map from $\mathbf{F}_i$ to $\mathbf{F}_j$ on the p-th component. This is related to the syzygy module representation.

Definition 3.2.3. *Let $\mathscr{F}$ be a family of cellular resolutions. The p-th syzygy functor, or syzygy representation,*

$$\sigma_p : \mathscr{F} \to \mathrm{Mod}_R$$

is defined by taking $\mathbf{F} \in \mathscr{F}$ to its p-th syzygy module. The morphisms are restrictions of the chain maps.

Proposition 3.2.4. *The representation σ_p is a subrepresentation of s_p.*

A fundamental observation relating to the above representations is that the generators of $s_p(\mathbf{F})$, where $\mathbf{F}$ is some element in the family, are in one-to-one relation with the cells of dimension p of the labelled cell complex supporting $\mathbf{F}$. This then allows one to define coverings for cell complexes.

Let $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_r$ be cellular resolutions mapping to G. Let X_i be the cell complex supporting $\mathbf{F}_i$ and Y be the cell complex supporting G. Then there is a *d-covering* of Y by $X_1, X_2, \ldots, X_r$ if the images of d-cells of X_i cover d-cells of Y under the maps $f \in \operatorname{Hom}(\mathbf{F}_i, G)$,

$$\bigcup_{f \in \operatorname{Hom}(\mathbf{F}_i, G)} f(X_i) = Y.$$

If there is a covering for all dimensions d that exist in the cell complexes then we call it a *covering*.

Now the observation on the relation of generators and cells can be written out as the following lemma.

Lemma 3.2.5. *The p-th module representation is finitely generated if and only if there is a $(p-1)$-covering of X_i by finitely many X_j's with $j < i$ for all i large enough.*

Note that this applies to a specific cellular resolution and it is possible for a single module S/I to have one cellular resolution that satisfies the covering conditions and another one that does not. This creates a base for the proof in the main theorem of Publication II.

Theorem 3.2.6. *Let $\mathscr{F}$ be a family of cellular resolutions with noetherian representation category $\operatorname{Rep}_S(\mathscr{F})$ such that the cell complex supporting $\mathbf{F}_i$ is covered by the cell complexes supporting $\mathbf{F}_j$, $j < i$, for all i large enough. Then the syzygy representation σ_p is finitely generated for all p.*

3.2.2 Explicit examples of families with finitely generated syzygies

Several classes of explicit families are linear and have finitely generated syzygies as a result of the above theorem. The first of these are the families given by powers of maximal ideals.

Let S be a polynomial ring in n variables, $\mathfrak{m}$ the maximal monomial ideal, and consider the family generated by cellular resolutions of $S/\mathfrak{m}^k$. These resolutions are supported on a subdivided n-simplices, for example, in the $n = 3$ case triangles that are shown in Figure 3.7, and these minimal cell complexes are explicitly defined in [3]. Adding cells into the subdivision still gives a cellular resolution of the same module and cell complexes where the covering with maps is easy to see. Moreover, since we are dealing with a power family, it is also a linear family, and thus we get that by applying Theorem 3.2.6 these families have finitely generated syzygies.

These families are also great examples to highlight the importance of morphisms in the family for getting finite generation. If we only allow one map between cellular resolutions, the family is not finitely generated as we will not be able to reach all generators from a finite set.

The second example of families with finitely generated syzygies that can be shown using our method is the "cube" ideals. Let S be a polynomial ring in $2n$ variables and let $I_{\mathscr{P}} = \{x_{i_1} x_{i_2} \ldots x_{i_n} | i_j \in P_j \text{ for } j = 1, \ldots, n\}$ be an ideal where $\mathscr{P}$ is a pairing of the variables that consists of $P_1, P_2, \ldots, P_n$ where P_i is the set of indices of the pair.

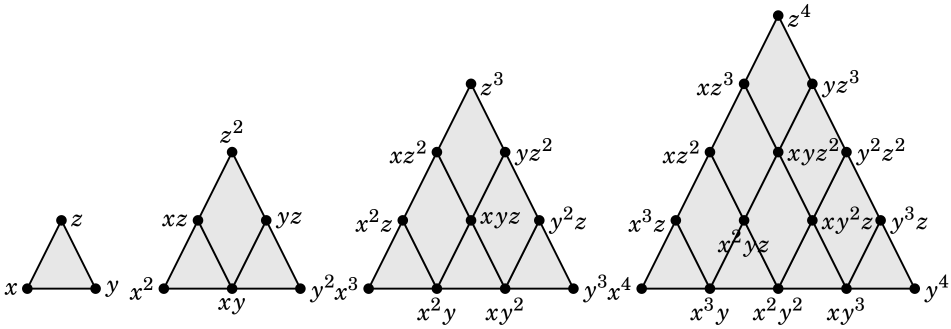


Figure 3.7. Cell complexes supporting the powers of the maximal ideal for $n = 3$.

The resolutions of $S/I_{\mathscr{P}}$ are supported on an n-cube, and the powers are supported on an n-cube subdivided to n-cubes. Again this family of powers is linear, and it has sufficiently many maps to have a covering of the cubes. Then application of Theorem 3.2.6 shows that the syzygies are finitely generated for these families, too.

The third type looked at in Publication II is the equigenerated ideals supported on a cell complex coming from the maximal ideal. A monomial ideal is *equigenerated* if all the monomials have the same degree.

Proposition 3.2.7. *If I is an equigenerated ideal in n variables and degree d such that I has a cellular resolution supported on X_I^d and the powers of I are supported on $X_{I^m}^{md}$, then the family of cellular resolutions given by them has finitely generated syzygies.*

Examples of families that satisfy these conditions are those bound by some vector $\mathbf{b}$ in the first power and $k\mathbf{b}$ in the k-th power.

The last examples of powers of ideals are types of edge ideals. Recall that an edge ideal is an ideal defined by a graph G. Firstly we consider the family of cellular resolutions for edge ideals of paths as defined in [26]. Engström and Norén give an explicit description of the cell complexes, subdivided simplices, supporting a non-minimal resolution of the k-th power of a path edge ideal $I_{P_n}^d$. Subdividing the given cell complex a bit further with hyperplanes still supports a cellular resolution of the ideal $I_{P_n}^d$, and allows us to use results for coverings of the subdivided simplices of the maximal ideals to show covering in this case. Powers of edge ideals of complete graphs form the second edge ideal power example. The edge ideals of complete graphs are all the square-free subsets of $\mathfrak{m}^2$ in a given polynomial ring S. This ideal can be defined by bounding the exponent vectors of elements in $\mathfrak{m}^2$ by the vector $(1,1,\ldots,1)$. Therefore the cell complex of $\mathfrak{m}^2$ bounded by this vector supports the resolution of the edge ideal of a complete graph and applying the results on equigenerated ideals we get that the syzygies of the power family are finitely generated.

3.2.3 Booth-Lueker ideals and unrestricted families

Previous work on Booth-Lueker ideals has shown that they have computable formulas for their invariants like betti numbers [25] and the resolutions are known to be 2-linear. These results are presented in Publication III of this thesis. The known results for this class of edge ideals was the motivation behind studying them from the point of view of cellular resolutions and trying to apply the tools from [48] to families of these ideals.

Proposition 3.2.8. *If I is an ideal of a Booth-Lueker graph, it has a minimal cellular resolution coming from the mapping cone resolution construction.*

Definition 3.2.9. *Let* $\mathrm{CellRes}_E(n)$ *denote the category of cellular resolutions coming from edge ideals of graphs with at most n vertices and m edges in $k[x_1,\ldots,x_n]$. Then define the functor*

$$\mathrm{BL}: \mathrm{CellRes}_E(n) \to \mathrm{CellRes}_E(n + \frac{n(n-1)}{2})$$

by sending the cellular resolution F_G to a minimal resolution of the Booth-Lueker edge ideal of G, $F_{BL(G)}$. The functor BL *takes an embedding of cellular resolutions to an embedding of the Booth-Lueker resolutions.*

This functor can be restricted to families of cellular resolutions, in which case it takes a family with individual resolutions belonging to $\mathrm{CellRes}_E(n)$ to a family where the resolutions are in $\mathrm{CellRes}_E(n + \frac{n(n-1)}{2})$.

Next recall that in Section 2.6, we defined a property for functors called property (F).

Proposition 3.2.10. *The restriction of the functor* BL *between families satisfies property (F).*

Satisfying property (F) implies that we can pull back desirable properties of representations from the Booth-Lueker resolutions to the representations of edge ideals. However, since we are working over a single polynomial ring, there is a maximal number of vertices a graph can have. Implications of this are that we have only finitely many ideals that can give the cellular resolutions in the family and thus answers to any questions about finite generation of syzygies immediately. This naturally raises a question if we can have a family of cellular resolutions such that each resolution is defined over its own ring and we propose a what can be called a naive solution for this problem.

Definition 3.2.11. *Let $\mathscr{F}$ be a family of cellular resolutions such that each resolution $\mathbf{F}_i$ is over a polynomial ring S_i. We call such a family the* unrestricted family of cellular resolutions.

The unrestricted family forms a category with the objects being the individual resolutions and morphisms are compositions of change of ring maps and a cellular resolution morphisms.

As a category, the unrestricted families behave much in a similar way as the previously defined families of cellular resolutions and thanks to this behaviour we can lift all the previous definitions to these unrestricted families as well. The main difference to the previous case is that instead of having representations over the ring S the ring in question is now the polynomial ring with infinitely many variables S_∞. Nonetheless, one can still prove an analogue of our previous theorem for the unrestricted families:

Theorem 3.2.12. *If $\mathscr{F}$ is an unrestricted family of cellular resolutions such that it is linear and the cell complexes have a t-covering for all i large enough, then the syzygy functor σ_t^∞ is finitely generated.*

3.3 Open questions arising from Publications I and II

Publications I and II establish a more category-theoretic approach to studying cellular resolutions and also provide some concrete results for cellular resolutions and edge ideals of Booth-Lueker graphs. Another important outcome of the research in this thesis is the open questions and conjectures it produces. Hence we will present a few of these coming from Publications I and II.

Many results deal with the structure of cellular resolutions or a smaller set of them, but also constructing cellular resolutions is a central theme in Publication I. The product construction from topology can lift at times to cellular resolutions, but most of the time it does not satisfy the category-theoretic conditions of being a product.Moreover, taking the product of cell complexes does not usually give a new cellular resolution if the product contains any repeated labels on the vertices. One could then ask if there is a modification that can be made to give a cellular resolution. Preliminary computations suggest that removing or contracting cells such that there are no repeated labels gives a cell complex supporting a cellular resolution. Whether this always holds is a valid question and one can ask if it can be used to compute the resolutions of powers or products of ideals as the corresponding labels would be as desired.

Conjecture 3.3.1. *Let X and Y be two labelled cell complexes supporting cellular resolutions with label ideals I an J. A product of X and Y supports a cellular resolution with a label ideal IJ after all repeat labels in the vertices have been removed.*

Taking the product of a labelled cell complex with itself would produce a cell complex with repeated labels. This poses a question on how to remove algebraically redundant vertices from a cell complex and what is the most efficient way for it. Relating to the removing redundant vertices, one can ask what happens if we use discrete Morse theory to remove vertices from a labelled cell complex that supports the resolution and whether this would result in a cellular resolution, or under what conditions it is still a labelled cell complex supporting a cellular resolution.

Conjecture 3.3.2. *Let X be a labelled cell complex supporting a cellular resolution. Then removing a vertex from X by an elementary strong collapse gives a cell complex supporting a cellular resolution.*

Further investigation can also be done on general structures of cellular resolutions. For example, we do not have homotopy colimits as derived functors for **CellRes** and investigating the categorical structures relating this could bring up new insights. Questions of classifying or parametrising spaces have been posed in other works as well. One class that could be studied with this is possible minimal cellular resolutions that a single monomial module can have. A variation of this is asked in [20] about mapping cone resolutions. The choices appearing in the maping cone construction appear to be such that they can be combined to build a resolution containing the different choices of cells.

Conjecture 3.3.3. *Mapping cone resolutions of an ideal I are simple homotopy equivalent, and there exists a non-minimal cellular resolution that can be reduced to any of the mapping cone resolutions using discrete Morse theory.*

Another possible approach is to study different subcategories of **CellRes**, and we would expect that some of them will have interesting properties. Families are a type of subcategories, and Publication II raises plenty of open questions about them. In all of our examples, the families of cellular resolutions have been linear, so a natural question would be to ask what about non-linear families. Even the existence of non-linear families that have finitely generated syzygies or satisfy other properties like noetherian representation category, or any other interesting properties is not certain, though one would expect there to be some examples. In all the examples in Publication II noetherianity was used to prove finite generation of syzygies, and often the noetherianity of the representation category is inherited from the structure the family has, which in the first place suggested finite generation of syzygies.

Question 3.3.4. *Does there exist a family of cellular resolutions that has finitely generated syzygies but does not have a noetherian representation category?*

Publication II is focused on the syzygies of the families of cellular resolutions; thus, one can ask if the representations are used to study other properties than syzygies for the families. Another open direction coming from Publication II is that the Gröbner property was not fully utilised to study the families and this provides an interesting direction for further work. Moreover, the paper of Sam and Snowden contains other structures, such as lingual structures, that have not been addressed in this thesis for the family of cellular resolutions setting.

The more concrete results in Publication II still point to new questions, in particular with the equigenerated ideals. A *connected equigenerated ideal* is one where every generator differs at most by a single variable from at least one other generator. Alternatively, these are ideals where the generators are sufficiently close to each other.

Conjecture 3.3.5. *Connected equigenerated ideals have resolutions supported on cell complexes coming from the cell complex of a maximal ideal.*

The "coming from a cell complex" here means that there is a way to delete vertices and contract unnecessary cells to reach the desired complex. There is some overlap with this conjecture and Conjecture 3.3.2. However, preliminary computations show that not all cases of connected equigenerated ideals can be dealt with by the strong elementary collapses.

The final part of the open questions coming from Publication II relates to the unrestricted case of families. The approach we have proposed could be considered the naive way to deal with the requirement of having cellular resolutions over different rings, and we have not dwelt very deeply into it. The theory of modules over the polynomial ring with infinitely many variables could offer tools to work further with the proposed setting and also make use of different representations for these families. Another direction to take with these are the cases where there is a polynomial ring with a maximal number of variables, in which case the modules can be taken over that ring. This mixed finite case also allows different permutations of variables within the same ring, which are not morphisms of cellular resolutions in the fixed ring case. This would allow larger automorphisms groups, and based on what happens with graph automorphisms, we would expect that in this setting, we can find a cellular resolution with the automorphism group being any group we want.

The unrestricted families are also tied to an example of the family of the edge ideals of paths. That is a family where the i-th cellular resolution comes from an edge ideal of a path of length i. The increasing length requires a new variable at each step. These (non-minimal) cellular resolutions can be supported on simplices, but then there are not enough maps to give a covering. However, we still would expect this family to have finitely generated syzygies. We conjecture that there exists a Morse map that removes all the non-covered cells in the cell complexes, and the reduced family would then have a covering. Generally, if a non-minimal family $\mathscr{F}$ has a Morse map for each cellular resolution, the family $\widetilde{\mathscr{F}}$, obtained applying discrete or algebraic Morse theory, is a subobject of $\mathscr{F}$ in the category of families. This does not say much about the maps in $\widetilde{\mathscr{F}}$, but one would expect them to reflect the behaviour of those in $\mathscr{F}$.

Conjecture 3.3.6. *Let $\mathscr{F}$ be a non-minimal family of cellular resolutions. Suppose that there is a Morse map from each cellular resolution $\mathbf{F}_i$ in $\mathscr{F}$ to a reduced cellular resolution $\widetilde{\mathbf{F}_i}$ in a family $\widetilde{\mathscr{F}}$ with corresponding maps. Then the squares formed by the maps*

$$\begin{array}{ccccccc} \longrightarrow & \mathbf{F}_i & \longrightarrow & \mathbf{F}_{i+1} & \longrightarrow \\ & \downarrow & & \downarrow & \\ \longrightarrow & \widetilde{\mathbf{F}}_i & \longrightarrow & \widetilde{\mathbf{F}}_{i+1} & \longrightarrow \end{array}$$

commute.

4. Combinatorial formulas for algebraic invariants of Booth-Lueker ideals

This chapter presents the results of Publication III. The approach to studying monomial modules in Publication III is based on restricting the ideals and then making use of the generous amount of combinatorial data the edge ideals under investigation contain. The motivation for studying the edge ideals coming from Booth-Lueker graphs is not only on the algebraic side, to understand this class of monomial ideals, but also graph-theoretic. The Booth-Lueker graph connects to the graph isomorphism problem, and in a quest to get closer to the solution, we considered the role of algebraic invariants in these ideals. Alas, the Betti numbers and Boij-Söderberg coefficients do not contain information on the isomorphisms of graphs, but the underlying combinatorial nature gives the explicit formulas to compute these. The results of Publication III also were a significant motivation to a part of the results in Publication II, and they form good examples to look at.

4.1 Booth-Lueker ideals and algebraic invariants of their resolutions

For an S-module S/I with a 2-linear minimal resolution of length n, we denote the non-trivial part of its betti table as

$$\omega(S/I) := (\beta_{1,2}, \beta_{2,3}, \ldots, \beta_{n,n+1})$$

and call it the *reduced betti vector* of S/I. If the ideal I is the edge ideal of a graph G, we just write $\omega(G)$.

The Booth-Lueker graph $BL(G)$ of G is chordal. Thus there are relevant results one can use for the betti numbers associated with it. Combining these with the combinatorial data of the edge graph we get the following formula for betti numbers from the degree vector.

Proposition 4.1.1. *Let G be a graph on n vertices and m edges, let A be the matrix of size $(n+m-1)\times n$ defined by $A_{ij} = \binom{j+n-2}{i}$, and let v be the column $(n+m-1)$-vector defined by $v_i = \binom{n}{i+1}$. Then*

$$\omega(BL(G)) = A\mathbf{d}_G - v, \tag{4.1}$$

where $\mathbf{d}_G = (d_0, d_1, \ldots, d_{n-1})^T$ *is the degree vector of* G.

The degree vector can also be obtained from the Betti numbers.

Proposition 4.1.2. *Let* $\Delta(G)$ *be the largest vertex degree in* G. *Let* A *be as in Proposition 4.1.1 and let* B *be the square submatrix of* A *obtained by taking the first* $\Delta(G)+1$ *columns and the rows from* $n-1$ *to* $n+\Delta(G)-1$. *Then we have* $(B^{-1})_{ij} = (-1)^{i+j}B_{ij}$ *and*

$$\mathbf{d}_G = B^{-1}(\beta_{n-1,n+1}, \beta_{n,n+1}, \ldots, \beta_{n+\Delta(G)-1,n+\Delta(G)}).$$

That is, we can compute the degree vector in terms of the (last non-zero) Betti numbers.

Having the formulas for betti numbers of S/I_G, we can then start looking at the Boij-Söderberg coefficients related to the graph. Combining the above results with known results of the Boij-Söderberg coefficients of specific 2-linear resolutions provides the following theorem after some application of the combinatorial results presented in Chapter 2.

Theorem 4.1.3. *Let* G *be a graph with* n *vertices and* $m \geq n$ *edges, and let* $\mathbf{d}_G = (d_0, d_1, \ldots, d_{n-1})$ *be its degree vector. Then the* j*-th Boij–Söderberg coefficient of* $BL(G)$ *is*

$$c_j = \begin{cases} 0 & \text{if } j \leq n-2, \\ \frac{d_0}{j} + \frac{\sum_{i=1}^{n-1} d_i}{j(j+1)} - \frac{n}{j(j+1)} = \frac{d_0}{n} & \text{if } j = n-1, \\ \frac{d_{j-n+1}}{j} + \frac{\sum_{i=j-n}^{n-1} d_i}{j(j+1)} & \text{if } n-1 < j \leq 2n-2, \\ 0 & \text{if } j > 2n-2. \end{cases}$$

A sequence of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that

$$t \geq \frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \cdots \geq \frac{\lambda_n}{n} \geq 0$$

is called an *anti-lecture hall composition* of length n bounded above by t. If the graph G satisfies the condition $m \geq n-1$, we can use the number of vertices in degree k, denoted by d_k, to obtain an anti-lecture hall composition associated to $BL(G)$. This sequence of integers is given by

$$\lambda_j = \begin{cases} j & \text{for } j = 1, \ldots, n, \\ d_{n-1} + d_{n-2} + \cdots + d_{j-n+1} & \text{for } j = n, \ldots, 2n-2, \\ 0 & \text{for } j > 2n-2. \end{cases}$$

4.2 Invariants of the complement of the Booth-Lueker graph

Following the results of these invariants on $BL(G)$, one can then ask if similar formulas in terms of edges and vertices can be obtained for the complement $\overline{BL(G)}$. The remaining results of Publication III are precisely the formulas for the complement.

Proposition 4.2.1. *Let G be a graph with n vertices and m edges. Then, for every integer $j \geq 1$ we have*

$$\beta_{j,j+1}\left(\overline{BL(G)}\right) = m\binom{m+n-3}{j} - \binom{m}{j+1}.$$

The proof for these betti numbers is more involved than for the previous case, based on expanding the Booth-Lueker construction to multi-graphs and showing that the formula holds for a particular multi-graph, with all edges between two vertices, that can be reached from any of the other multi-graphs. Following this, one can combine the results in the same manner as in the earlier results, and we obtain the main theorem for the complements.

Theorem 4.2.2. *Let G be a graph with n vertices and m edges. Then the i-th Boij–Söderberg coefficient of $\overline{BL(G)}$ is*

$$c_i = \begin{cases} 0 & \text{if } i < m, \\ \frac{m}{(i+1)i} & \text{if } m \leq i \leq m+n-4, \\ \frac{m}{i} & \text{if } i = m+n-3, \\ 0 & \text{if } i > m+n-3. \end{cases}$$

The anti-lecture hall composition $\lambda = (\lambda_1, \ldots, \lambda_{n+m-1})$ for a simple graph G can in this setting be expressed as

$$\lambda_j = \begin{cases} j & \text{if } j \leq m, \\ m & \text{if } m < j \leq m+n-3, \\ 0 & \text{if } j > m+n-3. \end{cases}$$

The Booth–Lueker construction can be written purely for the edge ideals and thus seen as a map that takes a square-free monomial ideal generated in degree 2 and returns a new such ideal, in a larger polynomial ring, with a linear resolution. This map nature is also visible in the Booth-Lueker functor presented in Publication II. Viewing the construction as a map, it can be expanded to include other ideals, as was done by Orlich in [44]. There he defines a construction called linearization, that is, a generalisation of the Booth-Lueker ideal construction, which associates to any monomial ideal a new monomial ideal with a linear resolution.

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Aalto-DD 17/2021

ISBN 978-952-64-0263-5 (printed)
ISBN 978-952-64-0264-2 (pdf)
ISSN 1799-4934 (printed)
ISSN 1799-4942 (pdf)

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