Department of Mathematics and Systems Analysis

Regularity for parabolic quasiminimizers in metric measure spaces

Mathias Masson

Aalto University

DOCTORAL DISSERTATIONS

Aalto University publication series
DOCTORAL DISSERTATIONS 89/2013

Regularity for parabolic quasiminimizers in metric measure spaces

Mathias Masson

A doctoral dissertation completed for the degree of Doctor of Science to be defended, with the permission of the Aalto University School of Science, at a public examination held at the lecture hall N of the Aalto University School of Science main building on the 30th of May, 2013 at 13.

Aalto University
School of Science
Department of Mathematics and Systems Analysis

Supervising professor
Juha Kinnunen

Thesis advisor
Juha Kinnunen

Preliminary examiners
Masashi Misawa, Kumamoto University, Japan
Shulin Zhou, Peking University, China

Opponent
Per-Anders Ivert, Lund University, Sweden

Aalto University publication series
DOCTORAL DISSERTATIONS 89/2013

ISBN 978-952-60-5184-0 (printed)
ISBN 978-952-60-5185-7 (pdf)
ISSN-L 1799-4934
ISSN 1799-4934 (printed)
ISSN 1799-4942 (pdf)
http://urn.fi/URN:ISBN:978-952-60-5185-7

Unigrafia Oy
Helsinki 2013

Finland

441 697
Printed matter

Aalto University

Aalto University, P.O. Box 11000, FI-00076 Aalto www.aalto.fi

Abstract

Author
Mathias Masson

Name of the doctoral dissertation
Regularity for parabolic quasiminimizers in metric measure spaces

Publisher School of Science

Unit Department of Mathematics and Systems Analysis

Series Aalto University publication series DOCTORAL DISSERTATIONS 89/2013

Field of research Mathematical analysis

Manuscript submitted 28 February 2013 **Date of the defence** 30 May 2013

Permission to publish granted (date) 29 April 2013 **Language** English

☐ **Monograph** ☒ **Article dissertation (summary + original articles)**

Abstract

In this thesis we study in the context of metric measure spaces, some methods which in Euclidean spaces are closely related to questions concerning regularity of nonlinear parabolic partial differential equations of the evolution p-Laplacian type and of the doubly nonlinear type. To be more specific, we are interested in methods which are based only on energy type estimates.

We take a purely variational approach to parabolic partial differential equations, and use the concept of parabolic quasiminimizers together with upper gradients and Newtonian spaces, to develop regularity theory for nonlinear parabolic partial differential equations in the context of general metric measure spaces. The underlying metric measure space is assumed to be equipped with a doubling measure and to support a weak Poincaré inequality.

We define parabolic quasiminimizers in metric measure spaces and establish some preliminary results. Then we prove several regularity results for parabolic quasiminimizers in metric measure spaces, using energy estimates and the properties of the underlying metric measure space. The results we present are previously unpublished.

We prove local Hölder continuity in metric measure spaces for locally bounded parabolic quasiminimizers related to degenerate evolution p-Laplacian equations. We prove a scale and location invariant weak Harnack estimate in metric measure spaces for parabolic minimizers related to the doubly nonlinear equation in the general case, where p is strictly between one and infinity. We prove higher integrability results in metric measure spaces, both in the local case and up to the boundary, for parabolic quasiminimizers related to the heat equation. Lastly, we prove a comparison principle in metric measure spaces for parabolic super- and subminimizers, and a uniqueness result for minimizers related to the evolution p-Laplacian equation in the general case, where p is strictly between one and infinity.

The results and the methods used in the proofs are discussed in detail, and some related open questions are presented.

Keywords partial differential equations, parabolic, nonlinear analysis, evolution p-Laplacian, doubly nonlinear, regularity theory, calculus of variations, energy estimates, quasiminimizers, metric spaces, doubling measure, Poincaré inequality, upper gradients, Newtonian spaces, Hölder continuity, Harnack estimate, higher integrability, boundary regularity, comparison principle, uniqueness

ISBN (printed) 978-952-60-5184-0 **ISBN (pdf)** 978-952-60-5185-7

ISSN-L 1799-4934 **ISSN (printed)** 1799-4934 **ISSN (pdf)** 1799-4942

Location of publisher Espoo **Location of printing** Helsinki **Year** 2013

Pages 190 **urn** http://urn.fi/URN:ISBN:978-952-60-5185-7

Aalto-yliopisto

Aalto-yliopisto, PL 11000, 00076 Aalto www.aalto.fi

Tiivistelmä

Tekijä
Mathias Masson

Väitöskirjan nimi
Parabolisten kvasiminimoijien säännöllisyys metrisissä mitta-avaruuksissa

Julkaisija Perustieteiden korkeakoulu

Yksikkö Matematiikan ja systeemianalyysin laitos

Sarja Aalto University publication series DOCTORAL DISSERTATIONS 89/2013

Tutkimusala Matemaattinen analyysi

Käsikirjoituksen pvm 28.02.2013 **Väitöspäivä** 30.05.2013

Julkaisuluvan myöntämispäivä 29.04.2013 **Kieli** Englanti

☐ **Monografia** ☒ **Yhdistelmäväitöskirja (yhteenveto-osa + erillisartikkelit)**

Tiivistelmä

Tässä työssä tutkimme metristen avaruuksien kontekstissa menetelmiä, jotka euklidisissa avaruuksissa liittyvät läheisesti epälineaaristen parabolisten osittaisdifferentiaaliyhtälöiden säännöllisyysteoriaan. Paraboliset prototyyppiyhtälöt, joihin menetelmät liittyvät ovat evoluutio p-Laplace-yhtälö sekä kahdesti epälineaarinen yhtälö. Olemme kiinnostuneita menetelmistä, jotka perustuvat kokonaan energiaestimaatteihin.

Tarkastelemme epälineaaristen parabolisten osittaisdifferentiaaliyhtälöiden teoriaa puhtaasti variaationäkökulmasta. Käyttämällä parabolisten kvasiminimoijien käsitettä yhdessä ylägradienttien ja Newtonin avaruuksien kanssa, kehitämme parabolisten epälineaaristen osittaisdifferentiaaliyhtälöiden säännöllisyysteoriaa yleisissä metrisissä mitta-avaruuksissa. Oletamme, että tarkastelumme taustalla oleva metrinen mitta-avaruus on varustettu tuplaavalla mitalla ja siinä toteutuu heikko Poincarén epäyhtälö.

Määrittelemme parabolisten kvasiminimoijien käsitteen metrisissä avaruuksissa, ja käymme läpi tarvittavia esitietoja. Tämän jälkeen todistamme useita säännöllisyystuloksia kvasiminimoijille metrisissä avaruuksissa. Tulokset ovat aiemmin julkaisemattomia.

Todistamme lokaalin Hölder-jatkuvuuden degeneroituneisiin evoluutio p-Laplace-yhtälöihin liittyville kvasiminimoijille metrisissä avaruuksissa. Todistamme paikasta ja mittakaavasta riippumattoman Harnack-estimaatin kahdesti epälineaarisiin yhtälöihin liittyville minimoijille metrisissä avaruuksissa. Todistamme korkeampaan integroituvuuteen liittyviä tuloksia lämpöyhtälöön liittyville kvasiminimoijille metrisissä avaruuksissa, sekä lokaalisti että alueen reunalle asti. Todistamme vertailuperiaatteen metrisissä avaruuksissa evoluutio p-Laplace-yhtälöihin liittyville super- ja subminimoijille, sekä yksikäsitteisyyslauseen minimoijille.

Käymme yksityiskohtaisesti läpi todistuksissa käytetyt menetelmät ja tekniikat, ja esitämme joitakin niihin liittyviä avoimia kysymyksiä.

Avainsanat osittaisdifferentiaaliyhtälöt, parabolinen, epälineaarinen analyysi, evoluutio p-Laplace, kahdesti epälineaarinen, säännöllisyysteoria, variaatiolaskenta, energiaestimaatit, kvasiminimoijat, metriset avaruudet, tuplaava mitta, Poincarén epäyhtälö, ylägradientit, Newtonin avaruudet, Hölder jatkuvuus, Harnack estimaatti, korkeampi integroituvuus, reunasäännöllisyys, vertailuperiaate, yksikäsitteisyys,

ISBN (painettu) 978-952-60-5184-0 **ISBN (pdf)** 978-952-60-5185-7

ISSN-L 1799-4934 **ISSN (painettu)** 1799-4934 **ISSN (pdf)** 1799-4942

Julkaisupaikka Espoo **Painopaikka** Helsinki **Vuosi** 2013

Sivumäärä 190 **urn** http://urn.fi/URN:ISBN:978-952-60-5185-7

Preface

I wish to express sincere gratitude towards my advisor professor Juha Kinnunen for his teaching, advice and guidance throughout my doctorate studies. I would also like to thank him for providing excellent working conditions in the nonlinear PDE research group at Aalto University.

I want to thank my co-authors Juha Kinnunen, Niko Marola, Michele Miranda jr, Fabio Paronetto, Mikko Parviainen and Juhana Siljander for their input, insight and collaboration in preparing the articles that constitute this thesis.

For the financial support I am indepted to the Magnus Ehrnrooth foundation and to the Finnish National Graduate School in Mathematical Analysis and its Applications.

Finally, I would like to thank Laura Lehvonen for her love and support, and for all the beautiful moments she has given me during the preparation of this thesis.

Helsinki, May 6, 2013,

Mathias Masson

Contents

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I Mathias Masson and Juhana Siljander. Hölder regularity for parabolic De Giorgi classes in metric measure spaces.
Manuscripta Mathematica,
DOI 10.1007/s00229-012-0598-2,
November 2012.

II Niko Marola and Mathias Masson. On the Harnack inequality for parabolic minimizers in metric measure spaces.
To appear in Tohoku Mathematical Journal,
January 2013.

III Mathias Masson, Michele Miranda jr, Fabio Paronetto and Mikko Parviainen. Local higher integrability for parabolic quasiminimizers in metric spaces.
Ricerche di Matematica,
DOI 10.1007/s11587-013-0150-z,
April 2013.

IV Mathias Masson and Mikko Parviainen. Global higher integrability for parabolic quasiminimizers in metric measure spaces.
To appear in Journal d'Analyse Mathématique,
February 2013.

V Juha Kinnunen and Mathias Masson. Parabolic comparison principle and quasiminimizers in metric measure spaces.
To appear in Proceedings of the American Mathematical Society,
January 2013.

Author's Contribution

Publication I: "Hölder regularity for parabolic De Giorgi classes in metric measure spaces"

The author has played a central role in preparing the article.

Publication II: "On the Harnack inequality for parabolic minimizers in metric measure spaces"

The author has played a central role in preparing the article.

Publication III: "Local higher integrability for parabolic quasiminimizers in metric spaces"

The author has played a central role in preparing the article.

Publication IV: "Global higher integrability for parabolic quasiminimizers in metric measure spaces"

The author has played a central role in preparing the article.

Publication V: "Parabolic comparison principle and quasiminimizers in metric measure spaces"

The author is responsible for a substantial part of the article.

1. Introduction

In this thesis we study in the context of metric measure spaces, some methods which in Euclidean spaces are closely related to questions concerning the regularity of nonlinear parabolic partial differential equations of the evolution p-Laplacian type

$$-\frac{\partial u}{\partial t} + \operatorname{div}(|Du|^{p-2}Du) = 0, \quad 1 < p < \infty,$$

and of the doubly nonlinear type

$$-\frac{\partial(|u|^{p-2}u)}{\partial t} + \operatorname{div}(|Du|^{p-2}Du) = 0, \quad 1 < p < \infty.$$

For the special case $p = 2$ the two equations coincide and we recover the classical heat equation. The evolution p-Laplacian equation has been extensively studied in the literature, and in recent years renewed interest has grown in studying also the doubly nonlinear equation. For an expository treatment on the doubly nonlinear equation we refer the reader to [K] and the references therein.

In many cases, when proving regularity results for weak solutions of these nonlinear parabolic equations, a method can be applied for a larger class of functions than weak solutions. It turns out that the method is based on an energy type estimate, and is applicable to all functions that satisfy the specific energy estimate.

In Euclidean spaces, the above parabolic equations can be formulated into equivalent variational problems. A function is a weak solution of the parabolic equation if and only if it is a minimizer to the corresponding variational problem. We then say that a function is a parabolic minimizer related to the parabolic equation.

The variational minimizing condition associated with the parabolic equation can be relaxed to define a larger class of functions, which contains weak solutions as a proper subclass, but at the same time retains the variational properties needed for establishing energy type estimates. This is the main idea behind parabolic quasiminimizers, defined in 1987 by Wieser [Wie].

Parabolic quasiminimizers form a natural starting point for studying from a purely variational perspective, methods used in the regularity theory of parabolic equations. As the class of parabolic quasiminimizers is strictly larger than weak solutions, regularity methods established for parabolic quasiminimizers are then known to only rely on energy estimates. Parabolic quasiminimizers also offer a unifying aspect, in the sense that several parabolic equations with similar elliptic growth conditions fall under the same class of quasiminimizers.

Moreover, an important advantage is that the elliptic part of the definition of parabolic quasiminimizers contains only moduli of gradients, instead of the gradients present in the definition of weak solutions. This opens up the possibility to generalize parabolic quasiminimizers to general metric measure spaces. Indeed, partial derivatives cannot be defined in the metric context, but moduli of gradients and Sobolev's spaces can be generalized by using well known concepts such as minimal upper gradients and Newtonian spaces, see [Sh1, Sh2] and the references therein.

This way the theory of nonlinear parabolic partial differential equations can be developed and studied in the metric space context, thus combining analysis of nonlinear partial differential equations with the robustness of analysis in metric spaces.

In this thesis we use this approach to establish regularity theory for nonlinear parabolic equations in general metric measure spaces. We assume the underlying metric space to be equipped with a doubling measure and to support a weak Poincaré inequality. The purely variational approach and the general doubling metric space setting cause several complications to known arguments for weak solutions, as many of the usual techniques associated with weak solutions, gradients or the Lebesgue measure are not available, and instead have to be replaced with a more general approach.

An alternative way to study the theory of partial differential equations in metric measure spaces also exists, by using a stochastic point of view. For more on this approach we refer the reader to recent articles by Kumagai and his co-authors [CK, CKK].

Already in the context of Euclidean spaces with Lebesgue's measure, the available literature on parabolic quasiminimizers is very limited, and several interesting questions still remain open. As far as we know, the available literature is as follows: In 1987 Wieser [Wie] introduced parabolic quasiminimizers related to the evolution p-Laplacian equation, and proved that they are locally Hölder continuous in the quadratic case. In the early 90's Zhou [Z1, Z2] extended this result by proving that parabolic quasiminimizers are locally Hölder continuous in the general degenerate case, and established the result also for parabolic quasiminimizers related to equations of Newtonian as well as non-Newtonian filtrations. In 2008 Parviainen [P1] proved higher integrability up to the boundary for the quadratic case. In the context of metric spaces, in 2012 Kinnunen, Marola, Miranda and Paronetto [KMMP] have proved a scale and location invariant Harnack inequality for parabolic quasiminimizers related to the heat equation.

One motivation for studying the regularity theory in general metric measure spaces is the following. Grigor'yan and Saloff-Coste [Gri, Sa1, Sa2] have observed for the heat equation that the doubling condition and the Poincaré inequality are sufficient and necessary conditions for a scale invariant parabolic Harnack principle on Riemannian manifolds. In this thesis, we extend this result by establishing sufficiency in metric measure spaces for the general case $1 < p < \infty$. It would be very interesting to find out to what extent also necessity holds in the metric space setting, as this would show that assuming a doubling measure and a weak Poincaré inequality are the natural assumptions for studying the regularity theory of parabolic partial differential equations in the metric context.

The text is organized in the following way. In Chapter 2 we present the concepts and preliminary results needed in the proofs of our regularity results. In Chapters 3–6 we present the main results of articles I–V, and discuss in detail the methods, techniques and ideas behind them. We also present some open questions related to our results. The last part of this thesis contains the five original articles.

2. Basic concepts and preliminary results

2.1 The variational approach to partial differential equations

Several nonlinear parabolic partial differential equations can be formulated as equivalent variational problems. For example, finding a weak solution u to the evolution p-Laplacian equation in $\Omega \times (0,T)$,

$$-\frac{\partial u}{\partial t} + \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \qquad 1 < p < \infty, \tag{2.1.1}$$

is equivalent to finding a function u, such that

$$p\int_{\{\phi\neq 0\}} u\frac{\partial \phi}{\partial t}\,dx\,dt + \int_{\{\phi\neq 0\}} |\nabla u|^p\,dx\,dt \le \int_{\{\phi\neq 0\}} |\nabla u + \nabla\phi|^p\,dx\,dt, \tag{2.1.2}$$

for every $\phi \in C_0^\infty(\Omega\times(0,T))$. Here Ω denotes a domain in $\mathbb{R}^d$ and $0 < T < \infty$. From now on we will denote $\Omega_T = \Omega \times (0,T)$.

The equivalence can be seen the following way. Let u be a weak solution of (2.1.1). Take a compactly supported $\phi \in C_0^\infty(\Omega_T)$. By the definition of a weak solution we can write

$$\begin{aligned}\int_{\{\phi\neq 0\}} |\nabla u|^p\,dx\,dt &= \int_{\{\phi\neq 0\}} |\nabla u|^{p-2}\nabla u\cdot\nabla u\,dx\,dt\\ &= \int_{\{\phi\neq 0\}} |\nabla u|^{p-2}\nabla u\cdot(\nabla u+\nabla\phi)\,dx\,dt - \int_{\{\phi\neq 0\}} u\frac{\partial\phi}{\partial t}\,dx\,dt.\end{aligned}$$

This implies that

$$\begin{aligned}\int_{\{\phi\neq 0\}} u\frac{\partial\phi}{\partial t}\,dx\,dt &+ \int_{\{\phi\neq 0\}} |\nabla u|^p\,dx\,dt\\ &\le \int_{\{\phi\neq 0\}} |\nabla u|^{p-2}\nabla u\cdot(\nabla u+\nabla\phi)\,dx\,dt\\ &\le \left(1-\frac{1}{p}\right)\int_{\{\phi\neq 0\}} |\nabla u|^p\,dx\,dt + \frac{1}{p}\int_{\{\phi\neq 0\}} |\nabla u+\nabla\phi|^p\,dx\,dt,\end{aligned}$$

where in the last step we use Young's inequality. After rearranging terms we obtain (2.1.2). Let then u be satisfy (2.1.2). Let $\phi \in C_0^\infty(\Omega_T)$. For every $\varepsilon > 0$ we have $\varepsilon\phi \in C_0^\infty(\Omega_T)$ and $\{\varepsilon\phi \neq 0\} = \{\phi \neq 0\}$. By (2.1.2), we have for every $\varepsilon > 0$

$$\varepsilon p\int_{\{\phi\neq 0\}} u\frac{\partial\phi}{\partial t}\,dx\,dt + \int_{\{\phi\neq 0\}} |\nabla u|^p\,dx\,dt \le \int_{\{\phi\neq 0\}} |\nabla u+\varepsilon\nabla\phi|^p\,dx\,dt,$$

which can be written as

$$p \int_{\{\phi \neq 0\}} u \frac{\partial \phi}{\partial t}\, dx\, dt + \int_{\{\phi \neq 0\}} \frac{1}{\varepsilon} \left(|\nabla u|^p - |\nabla u + \varepsilon \nabla \phi|^p \right) dx\, dt \leq 0.$$

As $\varepsilon \to 0$, we have

$$\frac{1}{\varepsilon} \left(|\nabla u|^p - |\nabla u + \varepsilon \nabla \phi|^p \right) \to -p |\nabla u|^{p-1} \frac{\nabla u}{|\nabla u|} \cdot \nabla \phi$$

pointwise. Hence, by the dominated convergence theorem we obtain

$$p \int_{\{\phi \neq 0\}} u \frac{\partial \phi}{\partial t}\, dx\, dt - p \int_{\{\phi \neq 0\}} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi\, dx\, dt \leq 0.$$

Choosing $-\varepsilon\phi$ yields the reverse inequality.

Analogously, for the doubly nonlinear parabolic equation

$$-\frac{\partial (|u|^{p-2} u)}{\partial t} + \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0, \quad 1 < p < \infty,$$

finding a weak solution is equivalent to finding a function u such that

$$p \int_{\{\phi \neq 0\}} |u|^{p-2} u \frac{\partial \phi}{\partial t}\, dx\, dt + \int_{\{\phi \neq 0\}} |\nabla u|^p\, dx\, dt \leq \int_{\{\phi \neq 0\}} |\nabla u + \nabla \phi|^p\, dx\, dt.$$

2.2 Parabolic quasiminimizers in Euclidean spaces

The concept of parabolic K-quasiminimizers related to the p-parabolic evolution equation was first defined by Wieser in 1987 [Wie]. Following his work, we consider a Carathéodory function

$$F = F(x, t, \xi) : \Omega \times (0, T) \times \mathbb{R}^d \to \mathbb{R}$$

satifying a growth condition

$$c_1 |\xi|^p \leq F(x, t, \xi) \leq c_2 |\xi|^p,$$

with positive constants c_1, c_2, and $1 < p < \infty$. In Euclidean spaces, a function $u \in L^p_{\mathrm{loc}}(0, T; W^{1,p}_{\mathrm{loc}}(\Omega)) \cap L^2(\Omega_T)$ is called a parabolic K-quasiminimizer related to the evolution p-Laplacian equation, $K \geq 1$, if for every $\phi \in C_0^\infty(\Omega_T)$,

$$\int_{\{\phi \neq 0\}} u \frac{\partial \phi}{\partial t}\, dx\, dt + \int_{\{\phi \neq 0\}} F(x, t, \nabla u)\, dx\, dt \leq K \int_{\{\phi \neq 0\}} F(x, t, \nabla (u + \phi))\, dx\, dt.$$

By the growth condition this is equivalent to stating that there exists constants $\alpha > 0$ and $K \geq 1$ such that

$$\begin{aligned} \alpha \int_{\{\phi \neq 0\}} u \frac{\partial \phi}{\partial t}\, dx\, dt + \int_{\{\phi \neq 0\}} |\nabla u|^p\, dx\, dt & \\ \leq K \int_{\{\phi \neq 0\}} |\nabla u + \nabla \phi|^p\, dx\, dt, & \end{aligned} \tag{2.2.1}$$

for every $\phi \in C_0^\infty(\Omega_T)$. We say that u is a parabolic quasiminimizer, if (2.2.1) is true for some $K \geq 1$.

From the discussion in the previous section, we see that u is a weak solution of (2.1.1) if and only if u is a parabolic K-quasiminimizer with $K = 1$ and $\alpha = p$. We say that u is a parabolic quasiminimizer related to (2.1.1), or borrowing from the vocabulary used in calculus of variations, we say that (2.1.1) is the parabolic Euler-Lagrange type equation of the quasiminimizer u.

By an analogous reasoning, for the doubly nonlinear parabolic equation

$$-\frac{\partial(u^{p-1})}{\partial t} + \operatorname{div}(|\nabla u|^{p-2}\nabla u)) = 0,$$

the related K-quasiminimizer is defined to be a function $u \in L^p_{\mathrm{loc}}(0, T; W^{1,p}_{\mathrm{loc}}(\Omega))$ which satisfies for some $\alpha > 0$ and $K \geq 1$

$$\alpha \int_{\{\phi \neq 0\}} |u|^{p-2} u \frac{\partial \phi}{\partial t}\, dx\, dt + \int_{\{\phi \neq 0\}} |\nabla u|^p \, dx\, dt \leq K \int_{\{\phi \neq 0\}} |\nabla u + \nabla \phi|^p \, dx\, dt,$$

for all $\phi \in C_0^\infty(\Omega_T)$. For now we return to discussing quasiminimizers related to the evolution p-Laplacian equation.

Like much of the parabolic theory, the historical background of parabolic quasiminimizers lies in the study of elliptic problems. In 1982 and 1984 Giaquinta and Giusti [GG1, GG2] introduced the notion of elliptic quasiminimizers as a unifying approach to the study of elliptic equation and systems, of minima of variational integrals and of quasiregular mappings. A function $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is called an elliptic K-quasiminimizer, $K \geq 1$, related to the p-Laplacian elliptic partial differential equation in $\Omega \subset \mathbb{R}^d$,

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0. \tag{2.2.2}$$

if for every $\phi \in C_0^\infty(\Omega)$ we have

$$\int_{\{\phi \neq 0\}} |\nabla v|^p \, dx \leq K \int_{\{\phi \neq 0\}} |\nabla v + \nabla \phi|^p \, dx. \tag{2.2.3}$$

In the elliptic 1-dimensional case one can show [GG2] that a function $v(x)$ defined on the interval $(a, b) \subset \mathbb{R}$ is an elliptic K-quasiminimizer if and only if

$$\int_{a'}^{b'} |v'(x)|^p \, dx \leq K \frac{|v(b') - v(a')|^p}{(b' - a')^{p-1}}, \tag{2.2.4}$$

for every $a \leq a' < b' \leq b$. On the other hand, assuming any elliptic K-quasiminimizer v, by defining $u(x, t) = v(x)$ for each $t \in (0, T)$, we obtain a parabolic K-quasiminimizer with the same constant K.

These observations provide us a way to use 1-dimensional elliptic quasiminimizers to construct examples of parabolic quasiminimizers. Indeed, with this procedure, for any $K > K' \geq 1$, it is fairly simple to construct a function which in Ω_T is a parabolic K-quasiminimizer but not a parabolic K'-quasiminimizer. In particular, we see that the class of weak solutions is a proper subclass of parabolic quasiminimizers.

Being a weak solution is a locally determined property, in the sense that if a function is a weak solution in every compactly contained open subset of a set, then it is also a weak solution in the whole set. The analogous statement is not true for parabolic quasiminimizers. Being a parabolic quasiminimizer in every compactly contained open subset of a set does not imply being a parabolic quasiminimizer in the whole set. This can be seen from the following example: Consider the elliptic function $v : (0, \infty) \to \mathbb{R}$, defined by setting

$$v(x) = \sum_{j=1}^{k-1} \frac{1}{j} + \frac{(x - (k-1))}{k}, \quad \text{when} \quad k-1 < x \leq k, \quad k \in \mathbb{N}.$$

For v on the interval $(m-1, n)$, $m, n \in \mathbb{N}$, inequality (2.2.4) is

$$\sum_{j=m}^{n} \frac{1}{j^p} \leq K \left(\frac{1}{(n-m)^{1-1/p}} \sum_{j=m}^{n} \frac{1}{j} \right)^p . \tag{2.2.5}$$

From this we see that for any $m, n \in \mathbb{N}$ the function v is an elliptic K-quasiminimizer on the interval $(m-1, n)$ with some large enough $K > 1$. However, as n tends to ∞, the right hand side of (2.2.5) tends to zero. It follows that in the set $(0, \infty)$, v is not an elliptic K-quasiminimizer with any $K \geq 1$. Setting now $u(x, t) = v(x)$ gives a function which is locally a parabolic quasiminimizer, but not in the whole set $(0, \infty) \times (0, T)$.

When $K > 1$, parabolic quasiminimizers are not uniquely determined by their behaviour on the parabolic boundary. As a simple example of this, consider solutions to the two problems

$$\begin{cases} \frac{\partial u_i(x,t)}{\partial t} - \lambda_i \frac{\partial^2 u_i(x,t)}{\partial x^2} = 0, & (x,t) \in (0,1) \times (0,1), \\ u_i(0,t) = u_i(1,t) = 0, & t \in (0,1), \\ u_i(x,0) = 1, & x \in (0,1), \end{cases}$$

where $\lambda_1 = 1$ and $\lambda_2 = a$, where $a > 1$. The solutions u_1 and u_2 are both parabolic quasiminimizers with $p = 2$ and $K = a^2/(2a-1)$ in the set $(0,1) \times (0,1)$, but are not identical. For more details see Example 4.3 in article V.

2.3 Doubling measures

In this thesis we investigate parabolic quasiminimizers in the general metric measure space setting (X, d, μ). We assume (X, d) to be complete, and that the measure μ is a complete positive Borel measure which is doubling.

A positive Borel measure μ is called doubling, if there exists a positive constant C, such that for metric balls

$$B(x, r) = \{ y \in X : d(x, y) < r \},$$

we have

$$0 < \mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty,$$

for any $x \in X$ and $r > 0$. We then say that the constant C is a doubling constant related to μ.

The doubling condition ensures that the measure is regular in any bounded subset of X, and that any non empty open set has non zero measure. Also, as a consequence of the doubling property, every bounded measurable set has finite measure.

Other than the doubling condition, we do not assume any more specific scaling properties for μ. In particular, we do not assume $\mu(B(x,r))$ to be independent of x or to be for example continuous with respect to x or r.

The only exception to only assuming μ is doubling will be made when proving local Hölder continuity for parabolic quasiminimizers in metric measure spaces. Then we make the additional assumption that μ also has the so called α-annular decay property. A measure μ is said to have the α-annular decay property, if there exists a positive constant α such that

$$\mu(\, B(x,r) \setminus B(x,(1-\delta)r)\,) \leq \delta^{\alpha} B(x,r),$$

for every $x \in X$, $r > 0$ and $0 < \delta < 1$. This says that the measure of an annulus with fixed outer surface decays in a control way as the thickness of the annulus tends to zero.

2.4 Parabolic upper gradients and parabolic Newtonian spaces

From now on, in cases where we integrate both in the spatial and time variable, for the sake of brevity we often denote the product measure as $d\nu = d\mu\, dt$.

In general metric spaces the concept of direction cannot be defined, and so classical partial derivatives have no meaning. Nevertheless, several ways to generalize Sobolev's spaces to metric measure spaces can be found in the literature [C, Haj, HeK, Sh1]. In this thesis, we follow the definition introduced by Shanmugalingam in 2000 [Sh1], where the generalization of Sobolev's spaces to metric spaces, called Newtonian spaces, is based on the notion of so called upper gradients, and more precicely on the concept of minimal p-weak upper gradients.

A nonnegative Borel measurable function g is said to be an upper gradient of the function $u : X \to [-\infty, \infty]$, if for all compact rectifiable arc length parametrized paths γ joining x and y we have

$$|u(x) - u(y)| \leq \int_{\gamma} g\, ds. \tag{2.4.1}$$

Clearly $|\nabla u|$ is an upper gradient of u in Euclidean spaces. However, from the point of view of generalizing Sobolev's spaces to metric measure spaces, a major drawback of upper gradients is that their integrability is not controlled by u. Indeed, if g is an upper gradient of u, then adding any non-negative measurable function to g is a again an upper gradient of u. Avoiding this drawback gives a natural motivation for the concept of minimal p-weak upper gradients.

For $1 \leq p < \infty$, the p-modulus of a family of paths Γ in X is defined to be

$$\inf_{\rho} \int_X \rho^p \, d\mu,$$

where the infimum is taken over all non-negative Borel measurable functions ρ such that for all rectifiable paths γ which belong to Γ, we have

$$\int_{\gamma} \rho\, ds \geq 1.$$

A property is said to hold for p-almost all paths, if the set of non-constant paths for which the property fails is of zero p-modulus. If (2.4.1) holds for p-almost all paths γ in X, then g is said to be a p-weak upper gradient of u.

From these definitions we see that the class of p-weak upper gradients of u is larger than the class of upper gradients of u. This enlargement of the class makes it possible to show the following: Whenever $1 < p < \infty$ and $u \in L^p(X,\mu)$ has an $L^p(X,\mu)$ integrable p-weak upper gradient, then there exists a minimal p-weak upper gradient of u, denote it by g_u, in the sense that g_u is a p-weak upper gradient of u and for every p-weak upper gradient g of u it holds $g_u \leq g$ μ-almost everywhere in X.

It can now be shown that g_u is μ-almost everywhere uniquely determined by u. On the other hand, in Euclidean spaces $|\nabla u|$ is exactly the minimal p-weak upper gradient of u, and so g_u generalizes the modulus of the gradient of u to metric measure spaces.

The minimal p-weak upper gradient is then used to define

$$\|u\|_{1,p,X}^p = \|u\|_{L^p(X,\mu)}^p + \|g_u\|_{L^p(X,\mu)}^p,$$

with the covention $\|u\|_{1,p,X} = \infty$ in case g_u does not exist. The Newtonian space is defined to be the quotient space

$$N^{1,p}(X) = \{u \,:\, \|u\|_{1,p,X} < \infty\}/\sim,$$

equipped with the norm $\|\cdot\|_{1,p,X}$, where the equivalence relation is defined by saying that $u \sim v$ if

$$\|u - v\|_{1,p,X} = 0.$$

Defined this way, $N^{1,p}(X)$ is a complete normed vector space, which generalizes the usual Sobolev space $W^{1,p}(\mathbb{R}^d)$ to metric measure spaces. The Newtonian space with zero boundary values is defined as

$$N_0^{1,p}(\Omega) = \{u|_\Omega : u \in N^{1,p}(X), u = 0 \text{ in } X \setminus \Omega\}.$$

In practice, this means that a function belongs to $N_0^{1,p}(\Omega)$ if and only if its zero extension to $X \setminus \Omega$ belongs to $N^{1,p}(X)$. For more properties of Newtonian spaces, see [Sh1, KKM, BB, He].

For a time-dependent function $u(x,t)$, whenever t is such that $u(\cdot,t) \in N^{1,p}(\Omega)$, we define the parabolic minimal p-weak upper gradient of u in a natural way by setting

$$g_u(x,t) = g_{u(\cdot,t)}(x),$$

at ν-almost every $(x,t) \in \Omega_T$. For the sake of brevity we refer to the parabolic minimal p-weak upper gradient as just the upper gradient.

Finally, we define the parabolic Newtonian space $L^p(0,T;N^{1,p}(\Omega))$ to be the space of functions $u(x,t)$ such that for almost every $0 < t < T$ the function $u(\cdot,t)$ belongs to $N^{1,p}(\Omega)$, and

$$\int_0^T \|u(\cdot,t)\|_{1,p,\Omega}^p \, dt < \infty.$$

We say that $u \in L^p_{\mathrm{loc}}(0,T;N^{1,p}_{\mathrm{loc}}(\Omega))$ if for every $0 < t_1 < t_2 < T$ and $\Omega' \subset\subset \Omega$ we have $u \in L^p(t_1,t_2;N^{1,p}(\Omega'))$. We say that $u \in L^p_c(0,T;N^{1,p}(\Omega))$ if for some $0 < t_1 < t_2 < T$, we have $u(\cdot,t) = 0$ outside $[t_1,t_2]$.

2.5 Poincaré inequalities and Sobolev embeddings

A key assumption we make is that (X, d, μ) is a metric measure space which supports a weak $(1, p)$-Poincaré inequality. A metric measure space is said to support a weak $(1, p)$-Poincaré inequality if there exist constants $C > 0$ and $\lambda \geq 1$ such that

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq Cr \left(\fint_{B(x,\lambda r)} g_u^p \, d\mu \right)^{1/p},$$

for every $u \in N^{1,p}(X)$ and $B_\rho(x) \subset X$. In case $\lambda = 1$, we say a $(1, p)$-Poincaré inequality is in force. Here we have denoted

$$u_{B(x,r)} = \fint_{B(x,r)} u \, d\mu = \frac{1}{\mu(B(x, r))} \int_{B(x,r)} u \, d\mu.$$

The weak Poincaré inequality relates the oscillation of u to its minimal p-weak upper gradient, via the measure μ. This has far reaching consequences, as it gives a first connection between integrals of u and of g_u.

In the case of a Euclidean space equipped with the Lebesgue measure, for any $1 < p < \infty$ a $(1, p)$-Poincaré's inequality follows from the properties of the Euclidean space and the Lebesgue measure. However, when μ is only assumed to be a doubling measure, a similar implication result is not known, and so already then the weak $(1, p)$-Poincaré is explicitely assumed.

A motive for assuming only a weak $(1, p)$-Poincaré inequality instead of a $(1, p)$-Poincaré, is that in a general metric measure space setting it is of interest to have assumptions which in the Euclidean special case are invariant under bi-Lipschitz continuous coordinate changing mappings. The weak $(1, p)$-Poincaré inequality has this property.

One important implication of assuming a doubling measure μ and a weak $(1, p)$-Poincaré inequality, is that together they imply a Sobolev embedding. This is a result established in 1995 by Bakry, Coulhon, Ledoux and Saloff-Coste [BCLS] and also by Hajlasz and Koskela [HaK], which says that if X is a metric measure space equipped with a doubling measure μ and supports a weak $(1, p)$-Poincaré inequality, then there exists positive constants $C > 0$ and $\lambda \geq 1$ such that

$$\left(\fint_{B(x,r)} |u - u_{B(x,r)}|^\kappa \, d\mu \right)^{1/\kappa} \leq Cr \left(\fint_{B(x,\lambda r)} g_u^p \, d\mu \right)^{1/p},$$

where $\kappa > p$. This Sobolev embedding with $\kappa > p$ turns out to be a cornerstone when building the proofs of the regularity results in articles I–IV.

Another deep result we use is a self improving principle for the Poincaré - inequality, established by Keith and Zhong in 2008 [KZ]. This principle says that if a complete metric space X is equipped with a doubling measure μ and supports a weak $(1, p)$-Poincaré inequality, then for some $1 < q < p$ a weak $(1, q)$-Poincaré inequality is also supported.

Combining the Sobolev embedding together with the self improving principle, it follows that for some $1 < q < p$, a weak (q, q)-Poincaré inequality

$$\left(\fint_{B(x,r)} |u - u_{B(x,r)}|^q \, d\mu \right)^{1/q} \leq Cr \left(\fint_{B(x,\lambda r)} g_u^q \, d\mu \right)^{1/q},$$

holds. As we shall see, this inequality is a key element for example in a parabolic version of De Giorgi's method in article I, and also when proving higher integrability in articles III and IV.

2.6 The variational capacity

For a measurable set $E \subset \Omega$, the variational capacity is defined to be

$$\operatorname{cap}_p(E, \Omega) = \inf_u \int_\Omega g_u^p \, d\mu,$$

where the infimum is taken over all $u \in N_0^{1,p}(\Omega)$ such that $u \geq 1$ on E. If there are no such functions, then we consider the variational capacity to be ∞. One can show [Bj] that if the underlying space X is equipped with a doubling measure and supports a weak p-Poincaré inequality, then there exists a positive constant C such that

$$\frac{\mu(E)}{Cr^p} \leq \operatorname{cap}_p(E, B(x, 2r)) \leq \frac{C\mu(B(x, r))}{r^p} \tag{2.6.1}$$

when $E \subset B(x, r)$. This gives us a tool to estimate the variational capacity of a set.

The variational capacity can be used to give a sort of regularity condition for the boundary of a set without actually having to define the boundary as a curve. Namely, one can define a so called thickness condition, by saying that a set E is uniformly p-thick provided there exists positive constants δ and ρ_0 such that

$$\operatorname{cap}_p(E \cap B_\rho(x), B_{2\rho}(x)) \geq \delta \operatorname{cap}_p(B_\rho(x), B_{2\rho}(x)),$$

for every $x \in E$ and $0 < \rho < \rho_0$.

In the context we work in, it is known that the uniform p-thickness satisfies the following deep self improving property established by Lewis in 1988 [L] for the Euclidean case and generalized to the metric setting by Björn, Macmanus and Shanmugalingam in 2001 [BMS]:

Let X be proper, linearly locally convex and equipped with a doubling measure. If a set $E \subset X$ is uniformly p-thick with $p > 1$, and X supports a weak $(1, p)$-Poincaré inequality, then E is also uniformly q-thick with some $1 < q < p$.

A space X is called proper if closed and bounded sets in X are compact. It can be shown, see Lemma 4.4 in [ATG], that a complete metric space equipped with a doubling measure is proper. A space X is called linearly locally convex if there exists constants $C_1 > 0$ and $r_1 > 0$ such that for all balls $B(x, r)$ in X with radius at most r_1, every pair of distinct points in the annulus $B(x, 2r) \setminus \overline{B(x, r)}$ can be connected by a curve lying in the annulus $B(x, 2C_1 r) \setminus \overline{B(x, C_1^{-1} r)}$. For example, it can be shown [BMS, HeK] that Ahlfors p-regular spaces supporting a $(1, p)$-Poincaré inequality are linearly locally convex.

The concepts presented in this section are used in article IV, where we prove higher integrability up to the boundary. For more details we refer the reader to [BB, BMS] and the references therein.

2.7 Parabolic quasiminimizers in metric measure spaces

The definition of parabolic quasiminimizers in Euclidean spaces only involves moduli of gradients. Therefore, the concept of p-weak minimal upper gradients and Newtonian spaces can be used to generalize the concept of parabolic quasiminimizers to metric spaces.

In metric spaces, we say that $u \in L^p_{\mathrm{loc}}(0, T, N^{1,p}_{\mathrm{loc}}(\Omega)) \cap L^2(\Omega_T)$ is a parabolic K-quasiminimizer related to the evolution p-Laplacian equation, if there exists constants $\alpha > 0$ and $K \geq 1$ such that

$$\alpha \int_{\{\phi \neq 0\}} u \frac{\partial \phi}{\partial t} \, d\nu + \int_{\{\phi \neq 0\}} g_u^p \, d\nu \leq K \int_{\{\phi \neq 0\}} g_{u+\phi}^p \, d\nu, \tag{2.7.1}$$

for all $\phi \in \mathrm{Lip}_c(\Omega_T)$. If $K = 1$, we say that u is a parabolic minimizer. If (2.7.1) holds for every non-negative $\phi \in \mathrm{Lip}_c(\Omega_T)$, we say that u is a K-quasisuperminimizer. If (2.7.1) holds for every non-positive $\phi \in \mathrm{Lip}_c(\Omega_T)$, we say that u is a K-quasisubminimizer.

Analogously, we define that in metric spaces a function $u \in L^p_{\mathrm{loc}}(0, T, N^{1,p}_{\mathrm{loc}}(\Omega))$ is a parabolic K-quasiminimizer related to the doubly nonlinear equation, if there exists constants $\alpha > 0$ and $K \geq 1$ such that

$$\alpha \int_{\{\phi \neq 0\}} |u|^{p-2} u \frac{\partial \phi}{\partial t} \, d\nu + \int_{\{\phi \neq 0\}} g_u^p \, d\nu \leq K \int_{\{\phi \neq 0\}} g_{u+\phi}^p \, d\nu, \tag{2.7.2}$$

for all $\phi \in \mathrm{Lip}_c(\Omega_T)$.

The constant α present in both definitions originates from taking into account in the Euclidean setting the growth conditions assumed for the elliptic parts of parabolic equations related to the evolution p-Laplacian type or to the doubly nonlinear type. For details see for example Section 3.1 in article V.

Defined this way, in the special case where the underlying space is a Euclidean space and the measure is the usual Lebesgue measure, the class of parabolic K-quasiminimizers is closed under bi-Lipschitz continuous coordinate changing mappings. The analogous property cannot be said to hold for the class of weak solutions.

We invite the reader to take note how the class of parabolic quasiminimizers related to the evolution p-Laplacian equation is also closed with respect to substracting a constant: if u is a parabolic quasiminimizer with constants α and K, then so is $u - k$ for every $k \in \mathbb{R}$. On the other hand a parabolic quasiminimizer related to the doubly nonlinear equation is scalable: If u is a parabolic quasiminimizer, then so is ku for every nonnegative constant k. When $p \neq 2$, one cannot a priori say that a quasiminimizer related to the evolution p-Laplacian equation is scalable, nor can one say that the class of quasiminimizers related to the doubly nonlinear equation is closed with respect to substracting a constant. However, in the quadratic case $p = 2$ where the two definitions for quasiminimizers coincide we have have both scalability and invariance with respect to substractiong a constant. As we shall see, these properties play an important role when proving regularity results.

Elliptic quasiminimizers were first introduced in metric measure spaces by Kinnunen and Shanmugalingam in 2001 [KS]. In 2012 Kinnunen, Marola, Miranda and Paronetto [KMMP] introduced parabolic quasiminimizers in metric measure spaces.

Since their introduction, elliptic quasiminimizers have been extensively studied, both in the Euclidean case and more recently in metric measure spaces. In contrast, for parabolic quasiminimizers already in the Euclidean case the available literature is significantly more scarce in number.

As an illustration of this, unlike for elliptic quasiminimizers, see the work by Kinnunen and Martio [KM], it is an open question already in the Euclidean setting to find out to what extent the class of parabolic quasiminimizers is closed with respect to basic structure operations. For instance, it is not yet known if taking the minimum of a parabolic minimizer and a constant produces a parabolic quasisuperminimizer.

2.8 Energy estimates

The first and in many ways the most important step in the proofs of the regularity results presented in this thesis, is to establish an energy estimate for the quasiminimizer (or quasisuperminimizer etc. depending on the case) under investigation. The regularity results are then proved based only on this energy estimate, and on the assumptions made on the underlying metric measure space. An example of an energy estimate is

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\tau_1<t<\tau_0} \int_{B(x,r_1)} (u(x,t)-k)_+^2 \, d\mu + \int_{\tau_1}^{\tau_0} \int_{B(x,r_1)} g_{(u-k)_+}^p \, d\mu \, dt \\
&\le \frac{C}{(r_2-r_1)^p} \int_{\tau_2}^{\tau_0} \int_{B(x,r_2)} (u-k)_+^p \, d\mu \, dt \\
&\qquad + \frac{C}{(\tau_1-\tau_2)} \int_{\tau_2}^{\tau_1} \int_{B(x,r_2)} (u-k)_+^2 \, d\mu \, dt,
\end{aligned}$$

which is used in article I, for proving local Hölder continuity of locally bounded parabolic quasiminimizers in metric measure spaces. Each regularity proof uses its own energy estimate, but the common feature is that potential energy type terms on time slices

$$\int_{B(x,r_1)} (u(x,t)-k)_+^2 \, d\mu,$$

and a kinetic energy type term

$$\int_{\tau_1}^{\tau_0} \int_{B(x,r_1)} g_{(u-k)_+}^p \, d\mu \, dt,$$

are estimated from above by a sort of possibly inhomogeneous potential energy of u. By saying inhomogeneous we refer to the case $p \neq 2$, where integrands of different degree are present in the inequality. Even in the homogeneous case, the spatial and time scales of the parabolic cylinders where the energy estimate is established are chosen in such a way that the terms $(r_2-r_1)^{-p}$ and $(\tau_2-\tau_1)^{-1}$ are of the same degree.

In the regularity proofs presented in this thesis, the energy estimate is exploited as an inequality which contains in one package both a Caccioppoli type inequality, and after using a weak Poincaré type inequality on the right hand side, also a parabolic Poincaré type inequality where time slice integrals of u are estimated with an integral of g_u over a parabolic cylinder. The possible inhomogeneity of the energy estimate causes complications, as the inhomogeneity has to be taken care of somehow before we can combine the integral terms of different degree to obtain a Caccioppoli or Poincaré inequality.

In some cases the energy estimate is also used to obtain a measure estimate, either by estimating the integrands by constants or by directly building an energy estimate where one or several integrands are of degree zero, and so the corresponding integrals become measures over sets.

Each energy estimate is obtained by testing the quasiminimizer (or quasisuperminimizer etc. depending on the case) with a suitably chosen test function, and then using real analytic techniques such as integration by parts, Lebesgue's differentiation theorem and the hole filling iteration [Wid] (or a similar real analytic lemma, see Lemma 2.1.4 in [WZYL]) to extract the energy estimate.

There is a technical difficulty present when establishing energy estimates for parabolic quasiminimizers. A common feature of the test functions used for proving energy estimates, is that they depend on the function u itself. However, it is not clear that the time regularity of a parabolic quasiminimizer u is a priori sufficient for placing u as the test function, and performing the usual techniques used for obtaining an energy estimate. Indeed, as $u \in L^p_{\mathrm{loc}}(0,T;N^{1,p}_{\mathrm{loc}}(\Omega))$, it is not evident that $\partial u/\partial t$ exists almost everywhere in such a sense that the real analytic techniques can be carried out.

Wieser [Wie] has shown that for $u \in L^2_{\mathrm{loc}}(0,T;N^{1,p}_{\mathrm{loc}}(\Omega))$, the quasiminimizing property implies that $\partial u/\partial t$ exists in the sense of distributions, and that $\partial u/\partial t \in L^2(0,T;(N^{1,p}_0(\Omega))')$, where $(N^{1,p}_0(\Omega))'$ denotes the dual space of $N^{1,p}_0(\Omega)$. Wieser's result relies on being able to test u with both positive and negative test functions, and hence does not apply to situations where we want to establish energy estimates for super- or subquasiminimizers.

We treat the issue of time regularity by using a mollification technique, where functions used in the test function are mollified with respect to time, and also the inequality from the definition of a quasiminimizer is manipulated to become an inequality for the mollification of u. For example, the rigorous version of the test function

$$\phi(x,t) = (u(x,t)-k)_+\varphi_1(x)\varphi_2(t),$$

where $\varphi_1(x)$ and $\varphi_2(t)$ are smooth enough cutoff functions with respect to space and time, is

$$\phi(x,t) = (u_\varepsilon(x,t)-k)_+\varphi_1(x)\varphi_2(t).$$

Here u_ε denotes the standard time mollification of u. Roughly speaking the idea of the technique we use is to then deduce the energy estimate for u_ε, and finally to establish the same estimate at the limit $\varepsilon \to 0$.

In the metric space setting, one runs into unexpected difficulties when taking the limit $\varepsilon \to 0$. To establish convergence of the estimate, one needs to show that $g_{u_\varepsilon - u}(x,t)$ tends to zero as $\varepsilon \to 0$. In the Euclidean case this poses no difficulties as we can use the linearity of taking a gradient to write $\nabla(u_\varepsilon - u) = (\nabla u)_\varepsilon - \nabla u$, and the convergence as $\varepsilon \to 0$ then follows from the integrability of ∇u. With the minimal p-weak upper gradients $g_{u_\varepsilon - u}$, the situation is not as simple, as taking an upper gradient does not preserve linearity, and so we cannot use the same argument as for gradients. It turns out to be problematic to establish the convergence by using only the theory of upper gradients.

In article I, we circumvent this difficulty by introducing so called Cheeger derivatives [C] from the literature, which are comparable to upper gradients, and preserve linearity. For the Cheeger derivatives the convergence as $\varepsilon \to 0$ can be established, which then implies convergence also for the minimal p-weak upper gradients. It would be interesting to know if one can show using only the theory of upper gradients, to what extent and in what sense $g_{u_\varepsilon - u}(x,t) \to 0$ as $\varepsilon \to 0$.

On the other hand, in article III we establish that each compactly supported function in $L^p(0,T;N^{1,p}(\Omega)) \cap L^2(\Omega_T)$ can be approximated by compactly supported functions in $\mathrm{Lip}(\Omega_T)$ in such a way, that roughly speaking the time mollification of a parabolic quasiminimizer can be tested with any compactly supported function in $L^p(0,T;N^{1,p}(\Omega)) \cap L^2(\Omega_T)$. For more details we refer the reader to Lemmas 2.3 and 2.7 in article III. This result helps in the sense that we only need to perform the time mollification for u, while the rest of the test function does not need to have much time regularity. The result also enables us to establish some of the characterizations of parabolic quasiminimizers, presented in article V.

Next we move on to discuss in more detail the results established in articles I–V.

3. Hölder continuity for parabolic quasiminimizers in metric measure spaces

In the mid 1950's De Giorgi [DeG] showed that elliptic functions satisfying a certain energy estimate type condition are Hölder continuous. The class of these functions was later named the De Giorgi class. In particular, the De Giorgi class contains weak solutions of the linear second order elliptic equation

$$\sum_{i,j=1}^{d}(a_{ij}(x)u_{x_j})_{x_i}=0, \qquad \text{in} \qquad \Omega\subset\mathbb{R}^d,$$

where the coefficients a_{ij} are only assumed to be bounded and measurable and to satisfy related growth conditions. The proof of De Giorgi used a novel approach, which did not rely on the linearity of an underlying equation. This allowed Ladyzhenskaya and Uralt'seva [LU] to extend De Giorgi's approach in the mid 1960's to prove Hölder continuity for weak solutions of elliptic quasi-linear equations

$$\operatorname{div}\mathbf{a}(x,u\nabla u)=0, \qquad \text{in} \qquad \Omega\subset\mathbb{R}^d,$$

with nonlinear structure assumptions of the type

$$\begin{cases}\mathbf{a}(x,u,\nabla u)\cdot\nabla u\geq C_1|\nabla u|^p-C\\ |\mathbf{a}(x,u,\nabla u)|\leq C_2(|\nabla u|^{p-1}+1),\end{cases} \tag{3.0.1}$$

with $1<p<\infty$, and constants $C_1>0$ and $C_2\geq 0$. In particular, these structure assumptions are satisfied by the p-Laplacian equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u)=0,$$

both in the denerate case $2<p<\infty$, in the quadratic case $p=2$, and in the singular case $1<p<2$.

In 1964 Moser [Mo] proved Hölder continuity for weak solutions of the parabolic equation

$$-u_t+\sum_{i,j=1}^{d}(a_{ij}(x,t)u_{x_j})_{x_i}=0, \qquad \text{in} \qquad \Omega_T,$$

where the coefficients a_{ij} are assumed to be bounded and measurable. His proof established Hölder continuity via Harnack's inequality. Again, the linearity did not play a role in the proof, and so it was plausible to expect that similarly to the elliptic case, the proof would be generalized to cover weak solutions of parabolic quasi-linear equations

$$-\frac{\partial u}{\partial t} + \operatorname{div} \mathbf{a}(x, t, u, \nabla u) = 0, \quad \text{in} \quad \Omega_T,$$

with non-linear structure assumptions analogous to (3.0.1). Success was obtained with growth conditions related to the quadratic case $p = 2$, but it turned out that when $p \neq 2$, the methods of De Giorgi and Moser could not be extended to the parabolic case. As a special case of this, whenever $p \neq 2$ it remained unknown if the evolution p-Laplacian equation

$$-\frac{\partial u}{\partial t} + \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0, \quad \text{in} \quad \Omega_T,$$

was Hölder continuous.

These questions remained open until 1986, when DiBenedetto published his celebrated article [DiB], in which he proved that in the degenerate case $p > 2$ bounded weak solutions of parabolic quasilinear equations are indeed locally Hölder continuous. In 1993 Chen and DiBenedetto [CD] extended this result to also handle the singular case $1 < p < 2$. Moreover, the method presented by DiBenedetto could be adapted to establish local Hölder continuity for any local solutions of quasilinear porous medium type equations. Recently, DiBenedetto's method has been modified by Kuusi, Siljander and Urbano [KSU] to prove local Hölder continuity for nonnegative weak solutions of the doubly nonlinear equation.

In a series of recent articles DiBenedetto, Gianazza and Vespri [DGV1, DGV2, DGV3] have established an intrinsic Harnack inequality for nonnegative solutions of degenerate and singular parabolic PDEs with the full quasi-linear structure. This result provides an alternative approach to proving Hölder continuity as an a posteriori estimate, by using Harnack's inequality.[U]

From the point of view of variational calculus, investigating Hölder regularity in the parabolic setting was initiated by Wieser in 1987 [Wie], when he introduced in Euclidean spaces the notion of parabolic quasiminimizers related to the evolution p-Laplacian equation and proved Hölder continuity in the case of growth conditions related to the quadratic case $p = 2$. In 1993 Zhou [Z1, Z2] adapted DiBenedetto's method to prove that bounded parabolic quasiminimizers in Euclidean spaces are locally Hölder continuous in the degenerate case $p > 2$, and also adapted this approach to include weak solutions to equations of Newtonian as well as non-Newtonian filtrations.

In article I, we show that in the degenerate case $p > 2$, locally bounded parabolic quasiminimizers in metric measure spaces are locally Hölder continuous. We do this by adapting DiBenedetto's method to work in the metric setting with upper gradients and a doubling measure. This establishes that Hölder continuity and the techniques used in DiBenedetto's method do not depend on having a linear structure or on properties of the underlying measure such as translation invariance or continuity.

3.1 Establishing the energy estimate

For showing Hölder's continuity for parabolic quasiminimizers related to the evolution p-Laplacian equation, the energy estimate employed is of the form

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{\tau_1<t<\tau_0} \int_{B(x,r_1)} (u(x,t)-k)_\pm^2 \, d\mu + \int_{\tau_1}^{\tau_0} \int_{B(x,r_1)} g_{(u-k)_\pm}^p \, d\mu \, dt \\
&\quad \le \frac{C}{(r_2-r_1)^p} \int_{\tau_2}^{\tau_0} \int_{B(x,r_2)} (u-k)_\pm^p \, d\mu \, dt \\
&\quad + \frac{C}{(\tau_1-\tau_2)} \int_{\tau_1}^{\tau_2} \int_{B(x,r_2)} (u-k)_\pm^2 \, d\mu \, dt.
\end{aligned} \tag{3.1.1}
$$

The energy estimate is obtained by testing u with the test function

$$\phi(x,t) = \pm (u-k)_\pm \varphi_1(x) \varphi_2(t).$$

where φ_1 and φ_2 are cutoff-functions with respect to space and time.

At the time of writing article I, it was not completely clear which space of test functions would be technically the most natural choice, as the proof for the density of $\operatorname{Lip}(\Omega_T)$ in $L^p(0,T;N_0^{1,p}(\Omega))$ in such a sense that also the support of the function is approximated in measure, see Lemma 2.3 in article III, was not yet established. This is the reason why in article I we have chosen the space of compactly supported functions in $C^\infty(0,T;N^{1,p}(\Omega))$ as the space of test functions, instead of $\operatorname{Lip}_c(\Omega_T)$ used in the other articles of this thesis.

Also, when proving the energy estimate for Hölder continuity, an essential technical detail for the argument to work is that the integrations in the definition of a parabolic quasiminimizer are taken over the set $\{\phi \neq 0\}$ instead of $\operatorname{supp}\phi$. Indeed, as the measure μ is only assumed to be doubling, the integral

$$\int_{\operatorname{supp}(\phi)} g_{u-\phi}^p \, d\nu$$

may well differ significantly from the same integral over the set $\{\phi \neq 0\}$. For more details on this we refer the reader to the proof of Theorem 6.11 in I. If the definition of parabolic quasiminimizers would be written using integrals over $\operatorname{supp}\phi$ instead of the set $\{\phi \neq 0\}$, then one would in any case first have to show that the definition using integrals over the support of ϕ implies the definition using integrals over the set $\{\phi \neq 0\}$. This is the origin of why in articles I and III–V we have used the convention of testing quasiminimizers over the sets $\{\phi \neq 0\}$ instead of supports.

In light of the the density results presented in article III and the characterizations of parabolic quasiminimizers presented in article V, the choice of the definition for parabolic quasiminimizers and the space of test functions become largely a matter of taste, as the different definitions for quasiminimizers can be shown to be equivalent.

An important property of energy estimate (3.1.1), is that it is invariant under substracting a constant. If u satisfies the energy estimate, then $u-k$ satisfies the same energy estimate with any constant k. This property is inherited from quasiminimizers related to the evolution p-Laplacian equation, and plays an fundamental role in DiBenetto's method, described next.

3.2 DiBenedetto's method

The purpose of DiBenedetto's method is to establish a reduction of oscillation for u, as the underlying parabolic cylinder converges to a point. By this we mean that if we have a parabolic cylinder, and a parabolic quasiminimizer u has essential oscillation ω in this cylinder, then by diminishing the measurements of the cylinder around a point of the cylinder in a predetermined fashion, also the oscillation of u diminishes by a fixed factor $\alpha < 1$, so that the oscillation in the smaller cylinder is at most $\alpha\omega$. The needed diminishment of the cylinder and the corresponding diminishment factor α for the oscillation of u do not depend on the scale of the initial cylinder or on the magnitude of the oscillation of u, and so we may repeat this reduction of oscillation in the smaller cylinder. Local Hölder continuity then follows by a standard iteration procedure from real analysis, see for example Section 4.4. in [U].

Fixing the center point of the upper time level face of the parabolic cylinder (see article I) and then seeking to show reduction of oscillation when diminishing the cylinder around the center point leads us to two alternatives. Indeed, if Q is the initial parabolic cylinder and Q' is the diminished subcylinder contained in Q, then showing that $(\operatorname{ess\,osc}_{Q'} u) < (\operatorname{ess\,osc}_{Q} u)$ is equivalent to showing that either there exists a constant $(\operatorname{ess\,inf}_Q u) < k^- < (\operatorname{ess\,sup}_Q u)$ such that

$$(u - k^-)_- = 0 \text{ almost everywhere in } Q', \tag{3.2.1}$$

or there exists a constant $(\operatorname{ess\,inf}_Q u) < k^+ < (\operatorname{ess\,sup}_Q u)$ such that

$$(u - k^+)_+ = 0 \text{ almost everywhere in } Q'. \tag{3.2.2}$$

DiBenedetto's method is based on assuming two mutually complementing measure theoretic alternatives in the initial cylinder Q, and then showing that the first one of the alternatives implies (3.2.1) and the second one implies (3.2.2), thus showing reduction of oscillation for u when moving from Q to Q'. Before describing the two measure theoretic alternatives, we discuss the techniques on which the method is based.

3.3 Moser's iteration

A key technique for showing the implication leading from the measure theoretic alternatives to (3.2.1) or (3.2.2) is to be able to construct a countably infinite sequence of nested cylinders $Q_0 \supset \ldots\ldots \supset Q_\infty$ and levels $k_0 > k_1 > \cdots > k_\infty$, such that for each finite j we have a measure estimate of the form

$$\frac{\nu(\{Q_{j+1} : (u - k_{j+1})_- > 0\})}{\nu(Q_{j+1})} \leq C^j \left(\frac{\nu(\{Q_j : (u - k_j)_- > 0\})}{\nu(Q_j)} \right)^\gamma, \tag{3.3.1}$$

where $\gamma > 1$. For each j these measure estimates can then be chained together in a fashion similar to Moser's iteration, and since $\gamma > 1$, it follows from a standard real analytic lemma, that if the initial fraction

$$\frac{\nu(\{Q_0 : (u - k_0)_- > 0\})}{\nu(Q_0)}$$

is small enough, then iterating the estimate (3.3.1) implies that

$$\frac{\nu(\{Q_\infty : (u-k_\infty)_- > 0\})}{\nu(Q_\infty)} = 0.$$

which in turn implies that $(u-k_\infty)_- = 0$ almost everywhere in Q_∞. For showing the implication leading from the second measure theoretic alternative to (3.2.2), the main technique is analogous, only this time one uses levels $k_0 < k_1 < \cdots < k_\infty$ and functions $(u-k_j)_+$ in place of $(u-k_j)_-$. In what follows we conduct explanations using $(u-k)_-$, but the situation is completely analogous for $(u-k)_+$.

Establishing measure estimates of the type (3.3.1) is done using integral averages of $(u-k)_-$, by combining Sobolev's inequality together with the energy estimate. Loosely speaking, the following scheme is applied:

$$\frac{\nu(\{Q_{j+1} : (u-k_{j+1})_- > 0\})}{\nu(Q_{j+1})} \leq \Big[\text{Hölder's and Sobolev's inequality}\Big],$$

$$\Big[\text{energy estimate (and intrinsic scaling)}\Big] \leq C^j \left(\frac{\nu(\{Q_j : (u-k_j)_- > 0\})}{\nu(Q_j)}\right)^\gamma, \tag{3.3.2}$$

where $\gamma > 1$. Sobolev's inequality is used to estimate the integral average of $(u-k)_-$ from above with the integral average of the upper gradient g_u, and the energy estimate is then used in a Caccioppoli estimate like fashion to revert back to the integral average of $(u-k)_-$. Finally the obtained inequality is estimated on both sides and divided by a suitable power of $(\operatorname{ess\,osc}(u-k)_-)$, resulting in measure estimate (3.3.1).

What is important is that the constants in the resulting measure estimate depend only on the data and in particular are independent of u. Here we also point out that the key component for obtaining $\gamma > 1$, a crucial property for making the measure estimate useful for the Moser type iteration, is Sobolev's inequality, as it enables to pass from a power to a lesser power. In the above scheme, when passing from integral averages over the cylinder Q_{j+1} to integral averages over the cylinder Q_j, the dilatation in measure of the underlying cylinder is handled by using the doubling property of the measure. This is a modification to DiBenedetto's original method, where instead of using integral averages, the explicit scaling properties of the Lebesgue measure are needed.

3.4 Intrinsic scaling

When $p \neq 2$, energy estimate (3.1.1) is inhomogeneous. This is a consequence originating from the evolution p-Laplacian equation, as in the related quasiminimizer the degree of the time derivative part is not equal to the degree of the elliptic part. It turns out that the above described scheme (3.3.2) cannot be pushed through if the inhomogoneity in the energy estimate is not somehow taken care of, as otherwise it is problematic to recover a measure estimate where the constants are independent of u. This is also the reason why in the case $p \neq 2$ it was originally problematic to adapt DeGiorgi's method to the parabolic case and combine it with Moser's method.

The breakthrough idea of DiBenedetto was to eliminate the inhomogeneity in the measure estimate scheme by introducing a technique called intrinsic scaling. Roughly speaking, intrinsic scaling consists of choosing the ratio between the time scale and the spatial scale of the parabolic cylinders in such a way that the ratio cancels out the inhomogeneity between $(\operatorname{ess\,osc}(u-k)_-)^2$ and $(\operatorname{ess\,osc}(u-k)_-)^p$. Using the notations of (3.1.1), this is done by choosing the ratio between the time and spacial scales in such a way that

$$\frac{(r_2-r_1)^p}{(\tau_2-\tau_1)} \le C(\text{ess osc}\,(u-k)_-)^{p-2}.$$

This way the power 2 on the right hand side of (3.1.1) can in essence be eliminated and we obtain an estimate homogeneously related to the power p. We note in passing that we choose to transform the power 2 into the power p and not vice versa, because of the assumption that the situation is degenerate, i.e. that $p > 2$.

As a result of intrinsic scaling, the ratio between the time and space scales of the parabolic cylinder where the Moser type iteration is carried out is dictated by the behaviour of u and the choice of the level k. The closer the level k is chosen to the extremal value of u (for example in the case of $(u-k)_-$ the extremal value is the essential infimum of u), the more 'elongated' the relative proportions of the underlying parabolic cylinder have to be. It turns out however that as long as the underlying parabolic cylinder is intrinsically scaled according to the choice of k, the threshold value of the initial fraction needed for initiating the Moser type iteration depends only on the data. This property is crucial for the success of DiBenedetto's method.

Originally DiBenedetto carries out the intrinsic scaling by a change variable and by using the explicit scaling properties of the Lebesgue measure. We avoid these techniques by working with integral averages and relying only on the doubling property of the measure when performing intrinsic scaling.

Here we digress to point out that in order to be able to carry out the method successfully, we need to be able to scale the initial parabolic cylinder intrinsically, while still staying inside Ω_T. Although it turns out that the space-time proportions caused by intrinsic scaling ultimately only depend on the data, the global scale of the initial cylinder inevitably depends on how close we are to the complement of Ω_T. This affects the constants in the final Hölder continuity estimate in a way that causes the result to be local.

3.5 Forwarding in time

Another key technique in DiBenedetto's method is forwarding in time. Starting from the energy estimate (3.1.1), by taking the limit $\tau_2 \to \tau_1$ and using Lebesgue's differentiation theorem on the right hand side, we obtain an estimate of the form

$$\begin{aligned}\operatorname*{ess\,sup}_{\tau_1<t<\tau_0}\int_{B(x,r_1)}(u(x,t)-k)_-^2\,d\mu \le \frac{C}{(r_2-r_1)^p}&\int_{\tau_1}^{\tau_0}\int_{B(x,r_2)}(u-k)_-^p\,d\mu\,dt\\ &+C\int_{B(x,r_2)}(u(x,\tau_1)-k)_-^2\,d\mu.\end{aligned}\tag{3.5.1}$$

From this expression, using intrinsic scaling and the fact that $p > 2$ one can show that if

$$\mu(\{\,x\,:\,(u(x,\tau_1)-k)_- > 0\,\}) < \delta < 1,$$

then for every $\varepsilon > 0$ and almost every $\tau_1 < \tau' < \tau_0$ there exists a k', closer to the extremal value of u, for which we have

$$\mu(\{\, x \,:\, (u(x,\tau') - k')_- > 0 \,\}) < \delta + \varepsilon.$$

Thus a spatial measure estimate at a time level can be forwarded in time. How much closer k' has to be chosen to the extremal value of u depends among other things on how much forward in time we want to forward the measure estimate. Using the α-annular decay property, one can show that no dilatation is necessary in the spatial direction when forwarding a spatial measure estimate in time. In some situations this property is needed, as we want to avoid having to dilate the cylinder in the spatial direction while forwarding information in time.

Another novelty of our proof is that we prove forwarding in time directly from the energy estimate, where as DiBenedetto does this using a separate logarithmic type energy estimate. In the technique we use, it is essential that $p > 2$.

3.6 De Giorgi type method

The third key technique in DiBenedetto's method is to combine the self improving principle for the Poincaré inequality together with the energy estimate (3.5.1) and intrinsic scaling, to obtain what is essentially De Giorgi's elliptic method applied uniformly at every time level of a parabolic cylinder. This way we obtain that if we have

$$\mu(\{x \,:\, (u(x,t) - k)_+ > 0\}) < \delta < 1,$$

for almost every t, in an intrinsically scaled parabolic cylinder, then for any $\varepsilon > 0$ there exists a k' closer to the extremal value of u, in this case the essential supremum of u, such that

$$\mu(\{x \,:\, (u(x,t) - k')_+ > 0\}) < \varepsilon,$$

for almost every t in the same cylinder. The main difference to the purely elliptic De Giorgi method is that here the energy estimate being used is parabolic, and hence intrinsic scaling is necessary to overcome its inherent inhomogeneity when $p \neq 2$. Also, here we point out that the self improving principle for the Poincaré inequality, see Section 2.5, plays a key role in making the De Giorgi type iteration effective, analogously to Sobolev's inequality when proving (3.3.1).

3.7 The measure theoretic alternatives

Now that the key techniques used in DiBenedetto's method have been covered, we describe the two measure theoretic alternatives.

In the first alternative, it is assumed that inside the initial cylinder Q there exists a parabolic intrinsically scaled cylinder Q_0 for which

$$\frac{\nu(\{Q_0 \,:\, (u - \frac{1}{2}\operatorname{ess\,osc}_Q u)_- > 0\})}{\nu(Q_0)} \tag{3.7.1}$$

is so small that the Moser type iteration inside Q_0 can be initiated. Since the cylinder Q_0 is assumed to be intrinsically scaled, this threshold value turns out to only depend on the data. From the Moser type iteration it follows that in a smaller cylinder inside Q_0 we have a reduction of oscillation. As it may well be that the upper time level of Q_0 does not coincide with the upper time level of the initial cylinder, the next step is to use forwarding in time. This way, by choosing a k' close enough to the essential infimum of u in the initial cylinder, we obtain that in a cylinder which coincides in upper time level with the initial cylinder, the conditions to initiate the Moser type iteration are also met. We end up with (3.2.1).

The second alternative is the complement of the first alternative. We assume that for every Q_0 inside the initial cylinder, the value of (3.7.1) is so large that the Moser type iteration cannot be initiated. It follows that there exists a $\delta < 1$ such that at time levels with predetermined maximal length between each other, we have

$$\mu(\{\, x \,:\, (u(x,t) - \frac{1}{2}\operatorname*{ess\,osc}_{Q} u)_{+} > 0 \,\}) < \delta. \tag{3.7.2}$$

Since the maximal length between these time levels is known, we can use forwarding in time to obtain that (3.7.2) holds with some predetermined $\delta < \delta' < 1$ at almost every time level of the initial cylinder. It follows that we can use the De Giorgi type method to obtain that for some k' close enough to the essential supremum of u, the value of

$$\frac{\nu(\{Q \,:\, (u-k')_{+} > 0\})}{\nu(Q)}$$

is smaller than the threshold value used in the first alternative. Since the threshold value only depends on the data, so does k'. Hence we may assume that the initial cylinder has been intrinsically scaled according to k'. It follows that the Moser type iteration can be initiated in the initial cylinder, and we arrive at (3.2.2).

Thus we see that both measure theoretic alternatives lead to a reduction of oscillation, and the proof is done.

4. Harnack inequality for parabolic minimizers in metric measure spaces

The doubly nonlinear equation was first studied by Trudinger [T] in the late 1960's, when he proved a Harnack inequality for nonnegative weak solutions, by using Moser's method [Mo] and proving a parabolic version of the John–Nirenberg inequality. The proof of this parabolic John–Nirenberg inequality was simplified by Fabes and Garofalo [FG] in their work twenty years later, but still the proof remained technically demanding. In 2007 Kinnunen and Kuusi [KKu] have given a proof for Harnack's inequality for positive weak solutions using the approach of Moser, but replacing the parabolic John–Nirenberg lemma with an abstract lemma due to Bombieri and Giusti [BG].

In article II we generalize this result by proving a scale and location invariant Harnack's inequality in metric measure spaces for positive parabolic minimizers related to the doubly nonlinear equation with $1 < p < \infty$. We assume that the parabolic minimizers are positive, locally bounded and locally bounded away from zero, and that the underlying metric space is geodesic.

Grigor'yan and Saloff-Coste observed independently [Gri, Sa1, Sa2], that for the heat equation, assuming a doubling measure and the Poincaré inequality is not only sufficient but also a necessary condition for obtaining a scale invariant parabolic Harnack principle on Riemannian manifolds. In our work we show the sufficiency in geodesic metric spaces for the case $1 < p < \infty$. It would be interesting to find out to what extent also the necessity holds in the metric space setting. The doubling condition and a weak Poincaré inequality are rather standard assumptions in analysis on metric spaces, and establishing necessity would imply that they are also in some sense the natural assumptions when generalizing the regularity theory of parabolic differential equations to metric spaces.

We use a similar proof to Kinnunen and Kuusi, but the purely variational approach and the metric setting cause several differences in the techniques needed. Kinnunen and Kuusi make use of the fact that if u is a weak supersolution bounded away from zero, then u^{-1} is a weak subsolution of the same equation. In the variational calculus approach such a technique is not available, and instead we have to prove energy estimates for both super- and subminimizers.

The parabolic minimizer under investigation is a positive $u \in L^p_{\text{loc}}(0,T; N^{1,p}_{\text{loc}}(\Omega))$, such that for some $\alpha > 0$ we have

$$\alpha \int_{\{\phi \neq 0\}} u^{p-1} \frac{\partial \phi}{\partial t} \, d\nu + \int_{\{\phi \neq 0\}} g_u^p \, d\nu \leq \int_{\{\phi \neq 0\}} g_{u+\phi}^p \, d\nu, \tag{4.0.1}$$

for every $\phi \in \mathrm{Lip}_c(\Omega_T)$. In article II, for the sake of brevity we have assumed that $\alpha = p$, and so the parabolic minimizer is the one that in Euclidean spaces is a weak solution to the doubly nonlinear equation. However, it is easy to check that our proof is valid with any positive fixed α.

4.1 Energy estimate for superminimizers

For our proof, a fundamentally important property of parabolic quasiminimizers related to the doubly nonlinear equation, is that if u satisfies condition (4.0.1), then for any $k > 0$ the same is true for ku.

As a consequence of this scalability, it turns out that a parabolic superminimizer related to the doubly nonlinear equation satisfies an energy estimate roughly of the form

$$\begin{aligned} &\operatorname*{ess\,sup}_t \int u^{p-1-\varepsilon}\,d\mu + \int\int g_u^p u^{-1-\varepsilon}\,d\mu\,dt \\ &\le \frac{C}{(r-r')^p}\int\int u^{p-1-\varepsilon}\,d\mu\,dt + \frac{C}{(\tau-\tau')}\int\int u^{p-1-\varepsilon}\,d\mu\,dt, \end{aligned} \tag{4.1.1}$$

for every positive $\varepsilon \neq p-1$. Roughly speaking, this implies that for every negative exponent q, by choosing a suitable $\varepsilon > p-1$ such that $q = p-1-\varepsilon$, we obtain an essentially homogeneous energy estimate for u^q.

Energy estimate (4.1.1) is proved by testing the parabolic minimizer with a test function of the form

$$\phi = u^{-\varepsilon}\varphi_1(x)\varphi_2(t),$$

where φ_1 and φ_2 are cutoff-functions with respect to space and time. Establishing energy estimate (4.1.1) using this test function is based using convexity properties, which in turn necessitate being able to scale u to be locally almost everywhere large enough. In order to be able to do this, we assume the superminimizer to be locally bounded away from zero. We note that here exceptionally, we are forced to extend the analysis all the way to the pathwise properties of upper gradients. Also, preserving the scalability for the energy energy estimate necessitates being able to cancel upper gradient terms sidewise in the proof. In order to be able to do this, one needs the property that u is a parabolic superminimizer, not just a quasisuperminimizer. For more details on these observations we refer the reader to the proof of Lemma 3.1 in article II.

An important property of energy estimate (4.1.1) is that it is essentially homogeneous, in the sense that it is scalable. If u satisfies the energy estimate, then also ku satisfies it, where k is a nonnegative constant. This is a fundamental difference with energy estimate (3.1.1), proved for quasiminimizers related to the evolution p-Laplacian equation, where the energy estimate is inhomogeneous and thus non scalable when $p \neq 2$. On the other hand however, energy estimate (3.1.1) is invariant with respect to substracting a constant. Energy estimate (4.1.1), which corresponds to the doubly nonlinear equation, does not have this property when $p \neq 2$.

Thus we see that the properties of scalability and invariance under substracting a constant are inherited all the way from the partial differential equations to the energy estimates for the corresponding quasiminimizers (or minimizers in this case). This seems incontournable, and greatly affects which regularity results are difficult to establish for each type of quasiminimizers.

As we have noted, the proof we present requires the strict minimization property, and does not apply for quasiminimizers. Recently Kinnunen, Marola, Miranda and Paronetto [KMMP] have proved a scale and location invariant Harnack inequality for functions belonging to the parabolic De Giorgi class with $p = 2$, which contains parabolic quasiminimizers related to the heat equation. Their proof relies on the homogeneity of the energy estimate. It would be interesting to find out if and how one can prove a scale and location invariant Harnack inequality for parabolic quasiminimizers related to the evolution p-Laplacian equation in the general case $1 < p < \infty$.

4.2 Moser's iteration

For minimizers related to the doubly nonlinear equality, the homogeneity in the energy estimate makes it possible to obtain a reverse Hölder inequality directly, without the need for techniques such as intrinsic scaling. Indeed, we can combine Sobolev's inequality with the energy estimate to obtain a decrease in the integration exponent. We use the following scheme:

$$\begin{aligned}\left(\fint_{Q'} u^{-\gamma p}\,d\nu\right)^{1/\gamma p} &\le \Big[\text{Hölder's and Sobolev's inequality}\Big], \\ &\Big[\text{energy estimate (4.1.1) for negative exponents}\Big] \\ &\le \left(\frac{C}{(\alpha-\alpha')^p}\fint_Q u^{-p}\,d\nu\right)^{1/p},\end{aligned} \tag{4.2.1}$$

where $\gamma > 1$. We then construct an countably infinite sequence of parabolic cylinders $Q \supset Q_1 \supset \cdots \supset Q'$, and iterate the reverse Hölder inequalities for each pair of cylinders in the sequence, to obtain that for parabolic superminimizers

$$\operatorname*{ess\,sup}_{Q'}(u^{-1}) \le \left(\frac{C}{(\alpha-\alpha')^\theta}\fint_Q (u^{-1})^s\,d\nu\right)^{1/s}, \tag{4.2.2}$$

for every $0 < s < p$, or writing this same expression in another form

$$\left(\frac{C}{(\alpha-\alpha')^\theta}\fint_Q u^s\,d\nu\right)^{1/s} \le \operatorname*{ess\,inf}_{Q'} u,$$

for every $-p < s < 0$.

The same energy estimate (4.1.1) allows also to be used for a segment of positive exponents near zero. By constructing a finite sequence of nested parabolic cylinders, and using the above scheme (4.2.1) with positive exponents, we obtain a reverse Hölder inequality of the form

$$\left(\fint_{Q'} u^q\,d\nu\right)^{1/q} \le \left(\frac{C}{(\alpha-\alpha')^\theta}\fint_Q u^s\,d\nu\right)^{1/s} \tag{4.2.3}$$

for every $0 < s < q < (p-1)(2-p/\kappa)$.

4.3 Bombieri's and Giusti's lemma

The tricky part of establishing Harnack's inequality is now to extend the reverse Hölder inequality over the value $s = 0$. We wish to obtain, by glueing inequalities (4.2.2) and (4.2.3) together, for parabolic superminimizers a weak version of Harnack's inequality. By this we mean an inequality of the form

$$\left(\fint_Q u^q \, d\nu \right)^{1/q} \leq \operatorname*{ess\,inf}_{Q'} u, \tag{4.3.1}$$

for some $q > 0$. In order to do this, we use an abstract lemma by Bombieri and Giusti, which roughly says that if a reverse Hölder inequality of the form

$$\left(\fint_{Q'} f^q \, d\nu \right)^{1/q} \leq \left(\frac{C}{(\alpha - \alpha')^\theta} \fint_Q f^s \, d\nu \right)^{1/s},$$

where $0 < q \leq \infty$, holds for every $0 < s < q$ close enough to zero, and if we have an estimate for the level sets of the logarithm of f of the form

$$\nu\Big(\{ x \in Q : \ \log f > \lambda \} \Big) \leq \frac{A\nu(Q')}{\lambda^\gamma}, \tag{4.3.2}$$

then we have

$$\left(\fint_{Q'} f^q \, d\nu \right)^{1/q} \leq C, \tag{4.3.3}$$

where C does not depend on f. The key idea in our proof is now to use the Bombier–Giusti lemma, separately on one hand for positive powers of u and on the other hand for positive powers of u^{-1}. The boundedness (4.3.3) for both cases implied by the Bombieri–Giusti lemma can then be used to glue the two estimates together, thus obtaining the weak Harnack inequality. We already have the reverse Hölder inequality conditions for both cases (4.2.2) and (4.2.3), but the corresponding measure estimates of type (4.3.2) for $\log u$ and $\log u^{-1}$ remain to be established.

4.4 Measure estimate around a time level

Establishing these measure estimates is based on a logarithmic energy estimate for parabolic superminimizers. The logarithmic energy estimate is obtained by testing the parabolic superminimizer with a nonnegative function of the form

$$\phi = u^{-(p-1)} \varphi_1(x) \varphi_2(t).$$

Again in the proof we need to be able to scale u to be as large as needed in a compact set, in order to take advantage of convexity. In order to be able to do this, we assume u to be locally bounded away from zero. We also need the strict minimizing property for the upper gradient terms to cancel out, and so it is not enough to assume u to be a quasisuperminimizer. The exponent $-(p-1)$ then plays together with convexity in such a way that the logarithmic energy estimate we obtain is of the form

$$\int\!\!\int g_{\log u}^p d\mu \, dt - p \left[\int \log u(x,t) \, d\mu \right]_{t=\tau_1}^{\tau_2} \leq \frac{C}{(r_2 - r_1)^p} \nu(Q).$$

What is significant about this estimate, is that u does not appear on the right hand side. We can now combine this estimate with a weighted Poincaré inequality, to obtain that the weighted spatial integral of $\log u$ is a monotonous function in time around the time level t_0.

We note in passing that the weighted Poincaré inequality used here is based on being able to connect two arbitrary points in space with a finite chain of balls. In order to be able to do this, one needs the assumption that the underlying space is geodesic. This is the reason why in article II we make the extra assumption that X is a geodesic metric space.

From the monotonicity of the weighted integral of $\log u$, after using energy estimate (4.1.1), we then obtain estimates of the type (4.3.2) for the level sets of $\log u$ when $t \geq t_0$ and for the level sets of $\log u^{-1}$ when $t \leq t_0$. Here we point out, that it is this motonicity property around the time level t_0 which leads us to use the Bombieri–Giusti lemma in two parabolic cylinders, adjacent to each other around the time level t_0. Because the constant in (4.3.3) only depends on the data of the setting, we are still able to compare the estimates obtained in the two adjacent cylinders.

Having established all the prerequisites for using the Bombieri–Giusti lemma, we obtain the weak Harnack inequality (4.3.1) by using the lemma on both sides of the time level t_0. Because the Bombieri–Giusti lemma requires dilatation of the parabolic cylinder and we use the lemma in the adjacent cylinders on both sides of the time level t_0 separately, the Harnack estimate unavoidably ends up containing a waiting time around t_0.

4.5 Energy estimate for subminimizers

After having obtained the weak Harnack inequality, the rest of the proof leading to the Harnack inequality is relatively straightforward. We establish the analogue of reverse Hölder inequality (4.2.2), but for parabolic subminimizers and positive exponents. For this we begin by establishing for subminimizers an energy estimate of the form

$$\begin{aligned} &\operatorname*{ess\,sup}_t \int u^{p-1+\varepsilon}\,d\mu + \int\int g_u^p u^{-1+\varepsilon}\,d\mu\,dt \\ &\leq \frac{C}{(r-r')^p}\int\int u^{p-1+\varepsilon}\,d\mu\,dt + \frac{C}{(\tau-\tau')}\int\int u^{p-1+\varepsilon}\,d\mu\,dt. \end{aligned} \tag{4.5.1}$$

This estimate is obtained by testing a parabolic subminimizer u with a function of the form

$$\phi = -u^\varepsilon \varphi_1(x)\varphi_2(t).$$

This time, in order to recover the above energy estimate, one needs to be able to scale u to be uniformly small enough on a compact set. This is why we assume the subminimizer to be locally bounded. Again, similarly as was done for superminimizers, we build a sequence of nested parabolic cylinders and use the homogeneity of the energy estimate to establish reverse Hölder inequalities. As before, by using the Moser iteration we obtain that for a locally bounded subminimizer u

$$\operatorname*{ess\,sup}_{Q'} u \leq \left(\frac{C}{(\alpha-\alpha')^\theta}\fint_Q u^s\,d\nu\right)^{1/s}, \tag{4.5.2}$$

for every $0 < s < p$.

The final Harnack estimate for parabolic minimizers related to the doubly nonlinear equation is now obtained by noting that a parabolic minimizer is both a super- and subminimizer, and then combining the weak Harnack inequality (4.3.1) together with the reverse Hölder inequality for subminimizers (4.5.2).

For supersolutions, one can show that if u is a positive weak supersolution locally bounded away from zero, then u^{-1} is a locally bounded weak subsolution. From this it follows that when working with the equation, one can show the Harnack inequality assuming only a positive weak supersolution which is locally bounded away from zero. In our case, since we cannot claim that if u is a superminimizer then u^{-1} is a subminimizer, we need the assumption that u is both a super- and subminimizer.

Note also, that one could prove Harnack's inequality for a minimizer with a shorter proof than what we have done, by using the Bombieri–Giusti lemma directly for the reverse Hölder inequalities (4.2.2) and (4.5.2), thus bypassing the weak Harnack estimate for parabolic superminimizers. However, the shorter proof is not used here, because the weak Harnack estimate is an interesting result in itself, as it only requires the superminimizing property instead of minimizing.

It would be interesting to find out if and, in case the answer is positive, how one can get rid of the assumption that the parabolic minimizer is locally bounded and locally bounded away from zero in our proof.

5. Higher integrability for parabolic quasiminimizers in metric measure spaces

In the elliptic setting the first higher integrability results date back to 1957, to an article by Bojarski [Boj]. Almost twenty years later in 1975, Elcrat and Meyers proved local higher integrability for nonlinear elliptic systems [EM]. In 1982, Granlund [Gra] showed that an elliptic minimizer has the higher integrability property if the complement of the domain satisfies a certain measure density condition. Later, Kilpeläinen and Koskela [KKo] generalized this result to a uniform capacity density condition.

In the parabolic setting higher integrability results were first proved by Giaquinta and Struwe in 1982 [GS], when they proved reverse Hölder inequalities and local higher integrability in the case $p = 2$, for weak solutions of parabolic second order systems of p-growth. Arkhipova has considered global integrability questions for parabolic systems, see [A1, A2]. In 2000 Kinnunen and Lewis [KLe] extended this local result to the general degenerate and singular case $p \neq 2$. Recently, several authors have worked in the parabolic setting on questions concerning local and global higher integrability and reverse Hölder inequalities, see [Mis], [AM], [Bö1, Bö2], [P2, P3], [BP], [BDM],[F], and in particular for quasiminimizers in the Euclidean setting see [P1].

Already in the Euclidean setting, it would be interesting to find out if one can establish higher integrability for weak solutions of the doubly nonlinear equation. Since quadratic parabolic quasiminimizers are a special case of quasiminimizers related to the doubly nonlinear equation with $1 < p < \infty$, showing higher integrability for quadratic parabolic quasiminimizers can be regarded as an initial step in investigating the variational approach to higher integrability of the doubly nonlinear equation. The quadratic case is the simplest to treat, as the time derivative term of the parabolic quasiminimizer is linear and the energy estimate is homogeneous. The proofs we use take advantage of this, both in the local and global case.

The local case

In article III we prove local higher integrability for minimal p-weak upper gradients of quadratic parabolic quasiminimizers. More specifically, we show that if u is a parabolic quasiminimizer with $p = 2$, then there exists a $\varepsilon > 0$, such that

$$\left(\fint_{Q_R} g_u^{2+\varepsilon} \, d\nu \right)^{\frac{1}{2+\varepsilon}} \leq C \left(\fint_{Q_{2R}} g_u^2 \, d\nu \right)^{\frac{1}{2}},$$

for every $Q_{2R} \subset \Omega_T$. The resulting higher integrability estimate depends on being able to dilate the cylinder Q_R with a fixed dilatation constant inside Ω_T. As a consequence of this the final higher integrability result is local in nature.

5.1 Establishing the energy estimate

As before, the first step of the proof is to etablish an energy estimate. Here the energy estimate is roughly (after using a hole filling type argument) of the form

$$\operatorname*{ess\,sup}_{t\in\Lambda'} \int_{B'} |u - u_B(t)|^2 \, d\mu + \int_{Q'} g_u^2 \, d\nu \leq \frac{C}{(r-r')^2} \int_Q |u - u_B(t)|^2 \, d\nu, \quad (5.1.1)$$

where we denote $Q' = B' \times \Lambda'$ and $Q = B \times \Lambda$. The energy estimate is obtained by testing the parabolic quasiminimizer u with a function of the form

$$\phi = -(u(x,t) - u_{\phi_1}(t))\phi_1(x)\phi_2(t),$$

where ϕ_1 and ϕ_2 are cutoff-functions with respect to space and time, and $u_{\phi_1}(t)$ is the weighted spatial mean value of u with $\phi_1(x)$ as the weight. The idea behind this choice of test function is that this way we obtain the property

$$\int u_{\phi_1}(t) \frac{\partial \phi}{\partial t} \, d\nu = 0,$$

and on the other hand, roughly speaking, we have the comparability

$$C' \int_B |u - u_B(t)|^2 \, d\mu \leq \int_B |u - u_{\phi_1}(t)|^2 \, d\mu \leq C \int_B |u - u_B(t)|^2 \, d\mu.$$

These two properties, together with a hole filling iteration type argument make it possible to obtain (roughly, see article III for the exact argument) estimate (5.1.1). Note how here again the energy estimate is homogeneous. This is a consequence of the fact that we consider the case of quadratic quasiminimizers, where $p = 2$.

5.2 Establishing a reverse Hölder inequality

Once the energy estimate is obtained, because of its homogeneity, obtaining the reverse Hölder inequality type estimate is relatively simple. We use a similar scheme as when proving Hölder continuity and Harnack's inequality, but since we want to establish a reverse Hölder inequality for upper gradients g_u instead of u, we reverse the order in which the energy estimate and Sobolev's inequality are combined together. The scheme we use is the following:

$$\begin{aligned}
&\fint_{Q'} g^2 \, d\nu \leq \Big[\text{Caccioppoli inequality from the energy estimate} \Big], \\
&\Big[\text{energy estimate, Poincaré's inequality} \Big], \Big[(2,q)\text{-Sobolev's inequality} \Big], \\
&\Big[\text{the } \varepsilon\text{-Young inequality} \Big] \leq \varepsilon C \fint_Q g_u^2 \, d\nu + \varepsilon^{-1} C \left(\fint_Q g_u^q \, d\nu \right)^{\frac{2}{q}},
\end{aligned} \qquad (5.2.1)$$

where $1 < q < 2$, and $\varepsilon > 0$. Here the key component for obtaining the smaller exponent $1 < q < 2$ on the right hand side is a weak $(2, q)$-Poincaré inequality obtained by combining Keith and Zhong's self improving result with Sobolev's inequality. In this reverse Hölder inequality, we demand that the larger cylinder on the right hand side of the estimate, denoted by Q above, is contained in the parabolic cylinder Ω_T, but other than that the local nature of the estimate does not appear.

5.3 A modification of Gehring's lemma

The remaining part of establishing local higher integrability relies only on the obtained reverse Hölder inequality and on properties of the underlying doubling metric measure space. We use Gehring's famous lemma, see for instance [Ma] and the references therein, modified in such a way that the reverse Hölder inequality (5.2.1) implies higher integrability.

In the proof of this modification, the initial cylinder is divided into a good set where g_u is bounded and into a bad set where g_u is unbounded. At each point of the bad set, in some small enough cylinder centered at this point, we have by our previous results a reverse Hölder inequality. These cylinders are then used to form a Vitali covering of the bad set, so that we obtain the reverse Hölder inequality over to whole bad set. Finally, by an argument involving Fubini's theorem, the reverse Hölder inequality is used to establish higher integrability over the bad set. In the good set higher integrability turns out to be a direct consequence of cleverly choosing the cutoff value by which we divide the initial cylinder into the good and bad set. This way we obtain higher integrability in the whole initial cylinder.

The global case

In article IV we prove higher integrability up to the boundary for minimal p-weak upper gradients of quadratic parabolic quasiminimizers that satisfy a Dirichlet type boundary condition on the parabolic boundary of Ω_T, where Ω is assumed to be regular in the sense that $X \setminus \Omega$ is uniformly 2-thick. For the definition uniform thickness see Section 2.6, and for a survey on boundary regularity see Section 8 of [Mik].

To be precise, we show that if u is a parabolic quasiminimizer with $p = 2$, if $X \setminus \Omega$ is uniformly 2-thick and if there exists a function η such that

$$u(x,t) - \eta(x,t) \in N_0^{1,2}(\Omega), \text{ for almost every } t \in (0,T),$$
$$\frac{1}{h}\int_0^h \int_\Omega |u(x,t) - \eta(x,t)|^2 \, d\mu \, dt \to 0, \text{ as } h \to 0,$$

then the minimal p-weak upper gradient g_u is globally integrable to a slightly higher power than initially assumed, in the sense that there exists a $\varepsilon > 0$

such that

$$
\begin{aligned}
\left(\frac{1}{\nu(Q_R)}\int_{Q_R\cap\Omega_T} g_u^{2+\varepsilon}\,d\nu\right)^{\frac{1}{2+\varepsilon}} &\le \left(\frac{c}{\nu(Q_{2R})}\int_{Q_{2R}\cap\Omega_T} g_u^{2}\,d\nu\right)^{\frac{1}{2}} \\
+\left(\frac{c}{\nu(Q_{2R})}\int_{Q_{2R}\cap\Omega_T} g_\eta^{2+\varepsilon}\,d\nu\right)^{\frac{1}{2+\varepsilon}} &+ \left(\frac{c}{\nu(Q_{2R})}\int_{Q_{2R}\cap\Omega_T} \left|\frac{\partial\eta}{\partial t}\right|^{2+\varepsilon}\,d\nu\right)^{\frac{1}{2+\varepsilon}} \\
+\left(\frac{c}{\mu(B_{2R})}\int_{B_{2R}\cap\Omega} g_\eta^{q+\varepsilon}(x,0)\,d\mu\right)^{\frac{1}{q+\varepsilon}}&,
\end{aligned}
$$

for every $Q_{2R}\subset X\times\mathbb{R}$. We assume X to be a complete linearly locally convex metric measure space equipped with a doubling measure and supporting a weak $(1,2)$-Poincaré inequality. For the definition of linearly locally convex see Section 2.6.

Establishing global higher integrability is based on obtaining a reverse Hölder inequality type estimate up to the boundary, and then using it together with a Caldéron–Zygmund type decomposition and a Vitali covering to obtain integrability at some slightly higher exponent than initially assumed, in the whole set Ω_T. The starting point for showing the reverse Hölder inequality for a parabolic quasiminimizer is an energy estimate over two concentric parabolic cylinders with different radii, Q' and Q, where $Q'\subset Q$. As before, this energy estimate is extracted from the definition of parabolic quasiminimizers by choosing a suitable test function.

In the global case, when choosing the test function, we are faced with two qualitatively different situations. Depending on the center point and radii of the concentric cylinders, the larger cylinder Q, may or may not overlap the lateral boundary of Ω_T. These two alternatives cause a difference in how we build the test function, and consequently lead to different energy estimates.

5.4 Estimates away from the lateral boundary

In case Q does not overlap the lateral boundary of Ω_T, we can construct the test function, much like in the local case, by using only the geometry of the cylinders Q' and Q, without having to take into consideration the lateral boundary of Ω_T. Similarly to the local case, we use a weighted mean value. The test function is of the form

$$\phi = -(u(x,t)-u_{\phi_1}(t))\phi_1(x)\phi_2(t).$$

The difference to the local case is however, that when determining boundary terms created by time-wise partial integration, we have to take into account the possibility that Q overlaps also the lower time level boundary $t=0$ of Ω_T. This is reflected by having to consider the initial condition for u near the boundary $t=0$. Accordingly, the initial condition is visible in the energy estimate, which ends up being of the form

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{t\in\Lambda'\cap(0,T)}\int_{B'}|u-u_\sigma(t)|^2\,d\mu+\int_{Q'\cap\Omega_T} g_u^2\,d\nu \\
&\quad\le \frac{C}{(r-r')^2}\int_{Q\cap\Omega_T}|u-u_\sigma(t)|^2\,d\nu + C\int_B|\eta(x,0)-\eta_B(0)|^2\,d\mu,
\end{aligned}
$$

where $Q' = B' \times \Lambda'$ and $Q = B \times \Lambda$. Once the energy estimate has been obtained, the reverse Hölder inequality is proved using a similar scheme as in the local case (5.2.1), except that we use the weak $(2,q)$-Poincaré inequality also for the term caused by the initial value condition. The reverse Hölder inequality type estimate we obtain is then roughly of the form

$$
\begin{aligned}
\frac{1}{\nu(Q')}\int_{Q'\cap\Omega_T} g_u^2\,d\nu &\le \frac{\varepsilon C}{\nu(Q)}\int_{Q\cap\Omega_T} g_u^2\,d\nu \\
&+\varepsilon^{-1}\left(\frac{C}{\nu(Q)}\int_{Q\cap\Omega_T} g_u^q\,d\nu\right)^{\frac{2}{q}} + \varepsilon C\left(\fint_B g_\eta^q(x,0)\,d\mu\right)^{\frac{2}{q}}.
\end{aligned}
\tag{5.4.1}
$$

for every $0<\varepsilon<1$.

5.5 Estimates near the lateral boundary

In case Q overlaps the lateral boundary of Ω_T, we take the lateral boundary of Ω_T into consideration when building the test function for obtaining the energy estimate. Indeed, instead of relying solely on a cutoff function using the geometry of Q' and Q, we also make use of the lateral boundary condition by setting the test function to be of the form

$$
\phi = -(u-\eta)\varphi_1(x)\varphi_2(t),
$$

where φ_1 is a spatial cutoff function such that $\operatorname{supp}(\varphi_1)\subset B$ and $\operatorname{supp}(\varphi_2)\subset (0,T)$. We invite the reader to note how the lateral boundary condition is built into the test function in such a way that ϕ vanishes at the parabolic boundary, even if $Q = B\times\Lambda$ overlaps the parabolic boundary of Ω_T. This way the energy estimate we obtain with this choice of test function is up to the lateral boundary of Ω_T. As the test function vanishes also on the lower time boundary $t=0$ of Ω_T, also the initial condition for u is taken into account in the estimate.

After using techniques such as partial integration and the hole filling iteration, the energy estimate we arrive to is essentially of the form

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{t\in\Lambda'\cap(0,T)}\int_{B'\cap\Omega}|u(x,t)-\eta(x,t)|^2\,d\mu + \int_{Q'\cap\Omega_T} g_u^2\,d\nu \\
&\quad\le \frac{C}{(r-r')^2}\int_{Q\cap\Omega_T}|u-\eta|^2\,d\nu + C\int_{Q\cap\Omega_T}\left(g_\eta^2 + \left|\frac{\partial\eta}{\partial t}\right|^2\right)d\nu.
\end{aligned}
$$

Since now in the estimate we have the integral of $|u-\eta|^2$ in place of the integral of $|u-u_B|^2$, we cannot directly combine the weak Poincaré inequality with the energy estimate, to prove a reverse Hölder inequality type estimate. This is were the regularity assumption concerning $X\setminus\Omega$ comes into play. Indeed, one can show that for any Newtonian function, and in particular for $u(\cdot,t)-\eta(\cdot,t)$ (after continuing it by zero in the set $X\setminus\Omega$), we have a weak Poincaré type inequality with capacity of the form

$$
\left(\fint_{B'}|u-\eta|^2\,d\mu\right)^{\frac{1}{2}} \le \left(\frac{C}{\operatorname{cap}_q(N_{B'}(u-\eta),B)}\int_B g_{u-\eta}^q\,d\mu\right)^{\frac{1}{q}}, \tag{5.5.1}
$$

where $N_{B'}(u-\eta) = \{\, B' : (u-\eta)(x) = 0\,\}$ and $1<q<2$. By the uniform thickness assumption for $X\setminus\Omega$ and by the self improving principle for the uniform p-thickness, see Section 2.6, we have

$$\operatorname{cap}_q(X \setminus \Omega \cap B', B) \geq \delta \operatorname{cap}_q(B', B),$$

for some $1 < q < 2$. By the fact that $u(,t) - \eta(,t) = 0$ in $X \setminus \Omega$, this implies that

$$\operatorname{cap}_q(N_{B'}(u - \eta), B) \geq \delta \operatorname{cap}_q(B', B).$$

Plugging these into (5.5.1) , we obtain a $(2, q)$-Sobolev inequality for $u-\eta$, up to the boundary when Q overlaps the lateral boundary of Ω_T.

Here we note that being able to use the self improving principle for the uniform p-thickness is the reason why in article IV we have made the additional assumption that X is linearly locally convex.

Having obtained a Sobolev inequality and the energy estimate, we have the building blocks to push through a scheme similar to (5.2.1), where we combine the Sobolev inequality together with a Caccioppoli type inequality extracted from the energy estimate, and the ε-Young inequality. We establish a reverse Hölder inequality type estimate of the form

$$\begin{aligned}\frac{1}{\nu(Q')} \int_{Q' \cap \Omega_T} g_u^2 \, d\nu &\leq \frac{\varepsilon C}{\nu(Q)} \int_{Q \cap \Omega_T} g_u^2 \, d\nu \\ &+ \varepsilon^{-1} \left(\frac{C}{\nu(Q)} \int_{Q \cap \Omega_T} g_u^q \, d\nu \right)^{\frac{2}{q}} + \frac{\varepsilon^{-1} C}{\nu(Q)} \int_{Q \cap \Omega_T} \left(g_\eta^2 + \left| \frac{\partial \eta}{\partial t} \right|^2 \right) d\nu\end{aligned} \tag{5.5.2}$$

for $0 < \varepsilon < 1$, when Q' overlaps the lateral boundary of Ω_T. By summing the right hand sides of (5.4.1) and (5.5.2), we obtain a reverse Hölder inequality type estimate which covers every $Q \subset X \times \mathbb{R}$, for every small enough $0 < \varepsilon < 1$.

Once the reverse Hölder inequality type estimate up to the boundary has been established, as in the local case, we prove global higher integrability by using a modification of Gehring's lemma. This time the modification of Gehring's lemma is adapted to take into account the terms containing η.

6. Comparison principle for parabolic minimizers in metric measure spaces

In article V we show by a counterexample that parabolic quasiminimizers related to the evolution p-Laplacian equation do not satisfy the comparison principle, and that a boundary value problem does not have a unique solution, in general. However, if we restrict our attention to parabolic minimizers, then a parabolic comparison principle for super- and subminimizers holds, and as a consequence a uniqueness result for minimizers is obtained.

More precisely, we assume a parabolic superminimizer $u \in L^p(0,T;N^{1,p}(\Omega)) \cap L^2(\Omega_T)$ and a parabolic subminimizer $v \in L^p(0,T;N^{1,p}(\Omega)) \cap L^2(\Omega_T)$ related to the evolution p-Laplacian equation. We assume that $v \leq u$ on the parabolic boundary of Ω_T, in the sense that for almost every $0 < t < T$

$$(v(x,t) - u(x,t))_+ \in N_0^{1,p}(\Omega),$$

and that

$$\frac{1}{h}\int_0^h \int_\Omega (v-u)_+^2 \, d\nu \to 0$$

as $h \to 0$. We prove that then $v \leq u$ μ-almost everywhere in Ω_T, and hence satisfy the comparison principle.

The proof for this is initiated by first testing the parabolic superminimizer u with a test function of the form

$$\phi = (v-u)_+ \chi_{(0,t')},$$

and then testing the parabolic subminimizer v with the test function $-\phi$. Next we substract the resulting two expressions from each other, to obtain an expression of the form

$$\begin{aligned}\frac{\alpha}{2}\int_0^{t'} \int_\Omega \frac{\partial}{\partial t}(v-u)_+^2 \, d\mu \, dt + \int_{\phi\neq 0} g_u^p \, d\nu + \int_{\phi\neq 0} g_v^p \, d\nu \\ \leq \int_{\phi\neq 0} g_v^p \, d\nu + \int_{\phi\neq 0} g_u^p \, d\nu.\end{aligned}$$

The upper gradient terms now cancel each other, and on the other hand we can perform the time integration for the first term on the left hand side of the above expression. This way we obtain

$$(v-u)_+(x,t) \leq 0$$

almost everywhere in Ω_T, which completes the proof. We note that in the proof, to obtain the cancellation of the upper gradient terms, we need the strict minimization property. Also, the linearity of the time derivative term, which distinguishes the evolution p-Laplacian equation from the doubly nonlinear equation is taken advantage of to write the first integrand on the left hand side as the time derivative of $(v-u)_+^2$.

Lastly we note that if u and v are both parabolic minimizers, then the argument can be repeated with the roles of u and v inversed. This implies a uniqueness result for parabolic minimizers in metric measure space, extending a result by Wieser established for the Euclidean case, see [Wie].

We end this section by noting that in the Euclidean setting with the Lebesque measure, the comparison principle is known to be a sufficient and necessary condition for a function to be a parabolic superminimizer. For more details we refer the reader to [KLi], [KKP] and [KKS]. In the metric space context however, the theory for parabolic obstacle problems has not yet been studied much.

Bibliography

[AM] E. Acerbi and G. Mingione. Gradient estimates for a class of parabolic systems, *Duke Mathematical Journal*, 136:285–320, 2007.

[A1] A. A. Arkhipova. L^p-estimates for the gradients of solutions of initial value problems to quasilinear parabolic systems (Russian). *St. Petersburg State Univ., Problems of Math. Analysis.* 13:5–18, 1992.

[A2] A. A. Arkhipova. L^p-estimates for solutions of nonlinear elliptic and parabolic boundary value problems. *Nonlinear evolution equations, Amer. Math. Soc. Transl. Ser.* 164(2):15–42, 1995.

[ATG] P. Auscher, T. Coulhon, A. Grigoryan. Heat Kernels and Analysis on Manifolds, Graphs, and Metric Spaces. *American mathematical society*, 2003.

[BCLS] D. Bakry, T. Coulhon, M. Ledoux and L. Saloff-Coste. Sobolev inequalities in disguise. *Indiana Univ. Math. J.*, 44:1033-1074, 1995.

[Bj] J. Björn. Boundary continuity for quasiminimizers on metric spaces. *Illinois J. Math.*, 46:383–403, 2002.

[BB] A. Björn and J. Björn. Nonlinear Potential Theory on Metric Spaces, volume 17 of *EMS Tracts in Mathematics. European Mathematics Society (EMS), Zürich*, 2011.

[BMS] J. Björn, P. Macmanus, and N. Shanmugalingam. Fat sets and pointwise boundary estimates for p harmonic functions in metric spaces. *J. Anal. Math.*, 85:339–369, 2001.

[Boj] B. V. Bojarski. Generalized solutions of a system of differential equations of first order and elliptic type with discontinuous coefficients (Russian). *Mat. Sb. N.S.* 43(85):451–503, 1957.

[BG] E. Bombieri and E. Giusti. Harnack's inequality for elliptic differential equations on minimial surfaces. *Invent. Math.* 15:24–46, 1972.

[Bö1] V. Bögelein. Higher integrability for weak solutions of higher order degenerate parabolic systems. *Ann. Acad. Sci. Fenn. Math.*, 33(2):387–412, 2008.

[Bö2] V. Bögelein. Partial regularity and singular sets of solutions of higher order parabolic systems. *Ann. Mat. Pura Appl. (4)*, 188(1):61–122, 2009.

[BDM] V. Bögelein, F. Duzaar, and G. Mingione. Degenerate problems with irregular obstacles. *J.Reine Angew. Math*, 650:107–160, 2011.

[BP] V. Bögelein and M. Parviainen. Self improving property of nonlinear higher order parabolic systems near the boundary. *Non Linear Differential Equations and Applications NoDEA*, 17(1):21–54, 2010.

[C] J. Cheeger. Differentiability of Lipschitz functions on metric measure spaces *Geom. Funct. Anal.* 9: 428–517, 1999.

[CKK] Z.–Q. Chen, P. Kim and T. Kumagai. On Heat kernel estimates and parabolic Harnack inequality for jump processes on metric measure spaces. *Acta Math. Sin. (Engl. Ser.)* 25: 1067–1086, 2009.

[CK] Z.–Q. Chen and T. Kumagai. Heat kernel estimates for jump processes of mixed types on metric measure spaces. Probab. Theory Relat. Fields. 140:277–317, 2008.

[CD] Y.Z. Chen and E. DiBenedetto. On the local behaviour of solutions of singular parabolic equations. *Arch. Ration. Mech. Anal.*103: 319–345, 1988.

[DeG] E. De Giorgi. Sulla differenziabilitá e l'analiticitá delle estremali degli integrali multipli regolari *Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* 3:25–43, 1957.

[DiB] E. DiBenedetto. On the local behaviour of solutions of degenerate parabolic equations with measurable coefficients. *Ann. Sc. Norm. Super. Pisa* 13: 487–535, 1986.

[DGV1] E. DiBenedetto, U. Gianazza and V. Vespri Intrinsic Harnack Estimates for Nonnegative Solutions of Quasilinear Degenerate Parabolic Equations. *Acta Math.*200:181–209, 2008.

[DGV2] E. DiBenedetto, U. Gianazza and V. Vespri Sub-Potential Lower Bounds for Non-Negative Solutions to Certain Quasi-linear Degenerate Parabolic Equations. *Duke Mathematical Journal*143(1):1–15, 2008.

[DGV3] E. DiBenedetto, U. Gianazza and V. Vespri Forward, Backward and Elliptic Harnack Inequalities for Non-Negative Solutions to Certain Singular Parabolic Differential Equations. *Ann. Scuola Norm. Sup. di Pisa Cl. Sc.*(5) Vol. IX,:385–422, 2010.

[EM] A. Elcrat and N. G. Meyers. Some results on regularity for solutions of non-linear elliptic systems and quasi-regular functions. *Duke Math. J.* 42:121–136, 1975.

[FG] E. Fabes and N. Garofalo. Parabolic B.M.O. and Harnack's inequality. *Proc. Amer. Math. Soc.* 50(1):63–69,1985.

[F] A. Fugazzola. Reverse hölder inequalities and higher integrability for subcritical parabolic equations. *Adv. Differential Equations*, 17.

[GG1] M. Giaquinta and E. Giusti. On the regularity of the minima of variational integrals. *Acta Math.*, 148:31–46, 1982.

[GG2] M. Giaquinta and E. Giusti. Quasi-minima. *Ann. Inst.H. Poincaré Anal. Non Linéaire*, 1(2):79–107, 1984.

[GS] M. Giaquinta and M. Struwe. On the partial regularity of weak solutions of nonlinear parabolic systems. *Math. Z.*, 179(4):437.

[Gra] S. Granlund. An L^p-estimate for the gradient of extremals. *Math. Scand.* 50(1):66–72, 1982.

[Gri] A. A. Grigoryan. The heat equation on noncompact Riemannian manifolds. *Mat. Sb.*182:55–87,1991.

[Hab] J. Habermann. Parabolic quasi-minimizers. *Habilitation thesis, in preparation.*

[Haj] P. Hajlasz. Sobolev spaces on an arbitrary metric space. *J. Potential Anal.* 5:403–415, 1995.

[HaK] P. Hajlasz and P. Koskela. Sobolev meets Poincaré. *C. R. Acad. Sci. Paris*, 320(1):1211–1215, 1995.

[He] J. Heinonen. Lectures on analysis on metric spaces. *Universitext. Springer-Verlag, New York* 2001.

[HeK] J. Heinonen and P. Koskela. Quasiconformal maps in metric spaces with controlled geometry. *Acta Math.* 181: 1–61, 1998.

[KZ] S. Keith and X. Zhong. The Poincaré inequality is an open ended condition. *Ann. of Math. (2)*, 167(2):575–599, 2008.

[KKM] T. Kilpeläinen, J. Kinnunen and O. Martio. Sobolev spaces with zero boundary values on metric spaces. *Potential Anal.*, 12:233–247, 2000.

[KKo] T. Kilpeläinen and P. Koskela. Global integrability of the gradients of solutions to partial differential equations. *Nonlinear anal.* 23(7):899–909, 1994.

[K] J. Kinnunen. Regularity for a doubly nonlinear parabolic equation, RIMS Kokyuroku. *Research Institute of Mathematical Sciences, Kyoto University, to appear.*

[KKu] J. Kinnunen and T. Kuusi. Local behaviour of solutions to doubly nonlinear parabolic equations. *Math. Ann.*337:705–728, 2007.

[KLe] J. Kinnunen and J. L. Lewis. Higher integrability for parabolic systems of p-Laplacian type. *Duke Math.J.*, 102(2):253–271, 2000.

[KLi] J. Kinnunen and P. Lindqvist. Pointwise behaviour of semicontinuous supersolutions to a quasilinear parabolic equation. *Ann. Mat. Pura Appl. (4)*, 185(3): 411–435, 2006.

[KMMP] J. Kinnunen, N. Marola, M. Miranda jr., and F. Paronetto. Harnack's inequality for parabolic De Giorgi classes in metric spaces. *Adv. Differential Equations*, 17:801–832, 2012.

[KM] J. Kinnunen and O. Martio. Potential theory of quasiminimizers. *Ann. Acad. Sci. Fenn. Math.*, 28:459–490, 2003.

[KS] J. Kinnunen and N. Shanmugalingam. Regularity of quasi-minimizers on metric spaces. *Manuscripta Math.*, 105(3):401–423, 2001.

[KKP] R. Korte, T. Kuusi and M. Parviainen. A connection between a general class of superparabolic functions and supersolutions *J. Evol. Eq.*, 10(1):1–20, 2010.

[KKS] R. Korte, T. Kuusi and J. Siljander. Obstacle problem for nonlinear parabolic equations. *J. Differential Equations*, 249(9): 3668-3680, 2009.

[KSU] T. Kuusi, J. Siljander and J. M. Urbano. Local Hölder continuity for doubly nonlinear parabolic equations. *To appear in Indiana Univ. Math. J.*

[LU] O.A. Ladyzhenskaja and N.N. Uralceva. Linear and Quasilinear Elliptic Equations *Academic Press, New York* 1968.

[L] J. L. Lewis. Uniformly fat sets. *Trans. Amer. Math. Society*, 308(2):177–196, 1988.

[Ma] O.–E. Maasalo. The Gehring Lemma in Metric Spaces. *Arxiv.org.* http://arxiv.org/abs/0704.3916v3, 2006.

[Mis] M. Misawa. L^q-estimates of gradients for evolutional p-Laplacian systems. *J. Differential Equations*, 219:390-420, 2006.

[Mik] P. Mikkonen. On the Wolff potential and quasilinear elliptic equations involving measures. *Ann. Acad. Sci. Fenn. Math. Diss.*, 104:1–71, 1996.

[Mo] J. Moser. A Harnack inequality for parabolic differential equations. *Comm. Pure App. Math.*17: 101–134, 1964.

[P1] M. Parviainen. Global higher integrability for parabolic quasiminimizers in non-smooth domains. *Calc. Var. Partial Differential Equations*, 31(1):75–98, 2008.

[P2] M. Parviainen. Global gradient estimates for degenerate parabolic equations in nonsmooth domains. *Ann. Mat. Pura Appl. (4)*, 188(2):333–358, 2009.

[P3] M. Parviainen. Reverse hölder inequalities for singular parabolic equations near the boundary. *J. Differential Equations*, 246:512–540, 2009.

[Sa1] L. Saloff-Costé. A note on Poincaré, Sobolev and Harnack inequalities. *Duke Math. J.*2:27–38, 1992.

[Sa2] L. Saloff-Costé. Aspects of Sobolev-type inequalities. *London Mathematical Society Lecture Note Series 289, Cambridge University Press.*, 2002.

[Sh1] N. Shanmugalingam. Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. *Rev. Mat. Iberoamericana*, 16(2):243–279, 2000.

[Sh2] N. Shanmugalingam. Harmonic functions on metric spaces. *Illinois J. Math.*, 45:1021–1050, 2001.

[T] N.S. Trudinger. Pointwise estimates and quasilinear parabolic equations. *Comm. Pure App. Math.* 21:205–226, 1968.

[U] J.M. Urbano. The Method of Intrinsic Scaling. *Springer-Verlag*, 2008.

[Wid] K.-O. Widman. Hölder continuity of solutions of elliptic systems *Manuscripta Math.*5:299–308, 1971.

[Wie] W. Wieser. Parabolic Q-minima and minimal solutions to variational flow *Manuscripta Math.*, 59:63–107,1987.

[WZYL] Z. Wu, J. Zhao, J. Yin and H. Li. Nonlinear diffusion equations. *World Scientific, Singapore*, 2001.

[Z1] S. Zhou. On the local behaviour of parabolic Q-minima. *J. Partial Differential Equations.* 6(3):255–272, 1993.

[Z2] S. Zhou. Parabolic Q-minima and their application. *J. Partial Differential Equations.* 7(4):289–322, 1994.

Aalto-DD 89/2013

In this thesis we study in the context of metric measure spaces, some methods which in Euclidean spaces are closely related to questions concerning regularity of nonlinear parabolic partial differential equations of the evolution p-Laplacian type and of the doubly nonlinear type. To be more specific, we are interested in methods which are based only on energy type estimates.

We take a purely variational approach to parabolic partial differential equations, and use the concept of parabolic quasiminimizers together with upper gradients and Newtonian spaces, to develop regularity theory for nonlinear parabolic partial differential equations in the context of general metric measure spaces.

ISBN 978-952-60-5184-0
ISBN 978-952-60-5185-7 (pdf)
ISSN-L 1799-4934
ISSN 1799-4934
ISSN 1799-4942 (pdf)

Aalto University
School of Science
Department of Mathematics and Systems Analysis
www.aalto.fi