UNITARY ROOT-MUSIC TECHNIQUE FOR UNIFORM CIRCULAR ARRAY

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ABSTRACT

In this paper, unitary root-MUSIC algorithm for direction of arrival estimation is proposed for Uniform Circular Array. Uniform circular array provides uniform performance in any direction and simultaneous azimuth and elevation angle estimates. The proposed algorithm has low computational complexity because eigenvalue decomposition for real number may be used. It also provides low variance estimates because the number of observations is doubled in comparison to conventional MUSIC algorithm. Also, a robust extension to the method is introduced based on multivariate extensions of nonparametric statistics. It gives highly reliable results in the face of heavy-tailed noise and interference. The additional computational cost is negligible.

1. INTRODUCTION

In this paper the problem of estimating the Direction of Arrival (DoA) using uniform circular arrays (UCA) is considered. An unitary root-MUSIC algorithm is derived for circular array configuration. Circular arrays are of interest because they occupy less space than linear arrays. This is of importance in particular in mobile terminals with multiple antennas, including MIMO systems. Moreover, UCA's have uniform performance regardless of the angle of arrival and they estimate both azimuth and elevation angles simultaneously.

The proposed method stems from the real Beamspace root-MUSIC algorithm [1] and unitary subspace methods originally developed in [3] and further studied in [2]. The main difference to [1] is in the unitary transforms that allow processing of real-valued data without approximations. The method provides consistently better performance in terms of lower mean-square error because the unitary transforms employed effectively double the number of snapshots. This is desirable property in particular at low SNR region. Also real-valued eigenvalues decomposition (singular value decomposition) may be used which lowers the computational complexity. In this paper, a robust extension of the proposed algorithm is introduced. The method stems from the nonparametric estimator derived in [5].

The paper is organized as follows. First, the signal model is presented. In section 3, the unitary root-MUSIC algorithm for UCA's is derived. In Section 4, the robust extension of the algorithm is briefly introduced. Finally, in Section 4, simulation results demonstrating improved performance are presented.

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2. SIGNAL MODEL

Let us have a Uniform Circular Array of N sensors. There are P ($P < N$) uncorrelated narrow-band source signals lying on the array plane and impinging the array from directions $\phi_1, \phi_2, \dots, \phi_P$ (ϕ is the azimuth angle). Furthermore we assume that K snapshots are observed by the array. The $N \times K$ array output matrix is modeled as

$$\mathbf{X} = \mathbf{AS} + \mathbf{N},$$

where $\mathbf{X}$ is the $N \times K$ element-space data matrix, $\mathbf{A}$ is the $N \times P$ element-space array manifold, $\mathbf{S}$ is the $P \times K$ source matrix and $\mathbf{N}$ is the $N \times K$ noise matrix. The noise is modeled as a stationary, second-order ergodic, zero-mean spatially and temporally white circular Gaussian process. The $N \times P$ element-space array manifold

$$\mathbf{A} = [\mathbf{a}_1(\zeta, \phi), \mathbf{a}_2(\zeta, \phi), \dots, \mathbf{a}_P(\zeta, \phi)]$$

has each column modeled as

$$\mathbf{a}(\zeta, \phi) = [e^{j\zeta \cos(\phi - \gamma_0)}, e^{j\zeta \cos(\phi - \gamma_1)}, \dots, e^{j\zeta \cos(\phi - \gamma_{(N-1)})}]^T$$

for $p = 1, 2, \dots, P$. Here, $\zeta = \kappa r \sin\theta$, r is the radius, $\kappa = \frac{\omega}{c}$ is the wavenumber, c is the speed of light, $\omega = 2\pi f$ is the angular frequency and $\gamma_n = \frac{2\pi n}{N}$ $(n = 1, \dots N - 1)$ is the sensor location. The elevation angle θ is measured down from the z-axis and ϕ is the azimuth angle measured counterclockwise from the x-axis. Since $\theta = 90°$ is fixed here, the UCA array manifold depends on the azimuth angle ϕ only, namely $\mathbf{a}(\phi)$.

We will address the following problem: given K snapshots observed by uniform circular array (UCA) with N sensors, we want to estimate the P angles of arrival ϕ_p, $p = 1, \dots, P$ in azimuth. The DoA estimation algorithms for linear arrays employ the Vandermonde structure of steering vector matrix. The UCA array manifold has no such structure. Hence, the methods derived for ULA's, such as root-MUSIC, Forward/Back averaging, *etc.* can not be directly applied. Some preprocessing steps are required instead [1].

3. UNITARY ROOT-MUSIC FOR UCA

The Beamspace transformation has been proposed in [1] as method for modifying the UCA manifold structure in a ULA-like manifold. The transformation is done by employing a beamformer $\mathbf{F}_e^H$ (see [1] for the construction) that include the concept of Phase-Mode excitation. The maximum number of excitation modes is $M \approx \kappa r$, namely the smallest integer that is close or equal to κr. Hence, the modes that can be excited are $m \in [-M, M]$ and $\mathcal{M} = 2M + 1$ is the total number of excited modes. Multiplying the $N \times 1$ element-space UCA manifold $\mathbf{a}(\phi)$ by the beamformer we obtain a $\mathcal{M} \times 1$ virtual array manifold

$$\mathbf{a}_e(\phi) = \mathbf{F}_e^H \mathbf{a}(\phi) = \mathbf{C}_v \mathbf{V}^H \mathbf{a}(\vartheta) \approx \sqrt{N} \mathbf{J}_\zeta \mathbf{v}(\phi)$$

that has a ULA-like structure, where,

$$\mathbf{v}(\phi) = [e^{-jM\phi}, \ldots, e^{-j\phi}, 1, e^{j\phi}, \ldots, e^{jM\phi}]^T$$

is a Vandermonde matrix that depends on the azimuth angle. The matrix $\mathbf{J}_\zeta$ is composed of Bessel functions and it is implicitly dependent on the elevation angle. The $\mathcal{M} \times N$ beamformer $\mathbf{F}_e^H$ is applied over the $N \times K$ element-space data matrix $\mathbf{X}$, so that the $\mathcal{M} \times K$ Beamspace data elements get the form as if they were created by a virtual array with ULA-like properties:

$$\mathbf{Y} = \mathbf{F}_e^H \mathbf{X}$$

$$= \mathbf{F}_e^H \mathbf{A}\mathbf{S} + \mathbf{F}_e^H \mathbf{N}.$$

The virtual array covariance matrix will be

$$\mathbf{R}_y = E\{\mathbf{y}(n)\mathbf{y}^H(n)\}$$

$$= \mathbf{F}_e^H \mathbf{R}_x \mathbf{F}_e = \mathbf{A}_e \mathbf{R}_s \mathbf{A}_e^H + \sigma_\eta^2 \mathbf{I},$$

where, $\mathbf{A}_e = \mathbf{F}_e^H \mathbf{A}$ is the $\mathcal{M} \times P$ Beamspace steering matrix, $\mathbf{R}_y$ has size $\mathcal{M} \times \mathcal{M}$, $\mathbf{R}_x$ and $\mathbf{R}_s$ represent the $N \times N$ element-space and the $P \times P$ signal covariance matrix and the Beamspace noise vector is still white due to unitary transformation $\mathbf{F}_e^H$, in fact $\mathbf{F}_e^H \mathbf{F}_e = \mathbf{I}$. Unitary transformation procedure employs a square unitary column conjugate symmetric matrix $\mathbf{Q}$ [3] for transforming a Centro-Hermitian matrix where the elements are real-valued. This approach has several advantages. First, a real-valued eigenvalue analysis may be used which lowers the computational cost. Second, the transformation requires that the matrix that we want to employ is Centro-Hermitian, so that, by operating on the Beamspace data element $\mathbf{Y}$, we can double the number of available data snapshots. In the simulation section, will demonstrate these performance gains. UCA Unitary root-MUSIC differs from UCA-RB-root-MUSIC method [1] because it allows the real-valued eigenvalues decomposition and uses a different unitary column conjugate symmetric matrix. The values of the elements in $\mathbf{Q}$ do not depend on the size $\mathcal{M}$ of the virtual array as in [1]. The $\mathbf{Q}$ matrix is defined as

$$\mathbf{Q}_{2n} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_n & j\mathbf{I}_n \\ \tilde{\mathbf{I}}_n & -j\tilde{\mathbf{I}}_n \end{bmatrix}$$

$$\mathbf{Q}_{2n+1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_n & \mathbf{0} & j\mathbf{I}_n \\ \mathbf{0}^T & \sqrt{2} & \mathbf{0}^T \\ \tilde{\mathbf{I}}_n & \mathbf{0} & -j\tilde{\mathbf{I}}_n \end{bmatrix}$$

where $\mathbf{Q}_{2n}$ denotes the even size matrix and $\mathbf{Q}_{(2n+1)}$ an odd size, $\mathbf{I}_n$ and $\tilde{\mathbf{I}}_n$ are the n size identify and permutation matrix respectively and j is the imaginary unit. The $N \times 1$ element-space UCA manifold $\mathbf{a}(\phi)$ gets first mapped into the $\mathcal{M} \times 1$ Beamspace manifold $\mathbf{a}_e(\phi)$ and then transformed into

$$\mathbf{a}_u(\phi) = {\mathbf{Q}_{Me}}^H {\mathbf{F}_e}^H \mathbf{a}_e(\phi)$$

that is the $\mathcal{M} \times 1$ real-valued manifold [2], called Unitary Beamspace manifold. The Beamspace data $\mathbf{Y}$ are transformed into a real-valued version first by constructing $\mathcal{M} \times 2K$ Centro-Hermitian matrix $\mathbf{Y}_{CH} = [\mathbf{Y} \vdots \tilde{\mathbf{I}}_N \mathbf{Y}^* \tilde{\mathbf{I}}_K]$; in this step we have doubled the number of available snapshots [3]. By employing the Unitary transformation, the data are finally mapped in a real-valued space as

$$\mathbf{Y}_{real} = \mathbf{Q}_N^H \mathbf{Y}_{CH} \mathbf{Q}_{2K} = \mathbf{Q}_N^H [\mathbf{Y} \vdots \tilde{\mathbf{I}}_N \mathbf{Y}^* \tilde{\mathbf{I}}_K] \mathbf{Q}_{2K}.$$

The orthogonal basis for the Beamspace signal and noise subspaces may be found using EVD or SVD. The method using SVD of the data matrix instead of EVD of covariance matrix is more efficient since it does not square the error.
Let $\mathbf{G}_s = [\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_P]$ and $\mathbf{G}_\eta = [\mathbf{g}_{P+1}, \mathbf{g}_{P+2}, \ldots, \mathbf{g}_N]$ be contain the vectors that span the Beamspace signal and noise subspace, respectively. Here, $\mathbf{G}_s$ and $\mathbf{G}_\eta$ have sizes $N \times P$ and $N \times (N - P)$. Note that SVD is computed for a real matrix. The Unitary MUSIC pseudo-spectrum is then given by

$$S_{uni}(\phi) = \frac{1}{\mathbf{a}_u^H(\phi) \mathbf{G}_\eta \mathbf{G}_\eta^T \mathbf{a}_u(\phi)}$$

$$\cong \frac{1}{\mathbf{v}^H(\phi)\sqrt{N}\mathbf{J}_\zeta \mathbf{Q}_{Me} \mathbf{G}_\eta \mathbf{G}_\eta^T {\mathbf{Q}_{Me}}^H \sqrt{N} \mathbf{J}_\zeta \mathbf{v}(\phi)} \quad (1)$$

where the azimuth dependent Vandermonde vector is denoted by $\mathbf{v}(\phi)$. By following the root-MUSIC approach, we get advantage from the Vandermonde structure of $\mathbf{v}(\phi)$ and we transform the denominator (known as null spectrum) of the Unitary MUSIC pseudo-spectrum (1) into a polynomial equation. The problem of finding DoA's may be formulated as an eigenfilter problem that can be solved using polynomial rooting as in the original root-MUSIC algorithm [4].

4. ROBUST METHOD

Most high-resolution array processing algorithms are based on array covariance matrix and subspaces spanned by its eigenvectors. Often the covariance matrix is estimated by the sample covariance matrix which, however, is known to have poor performance under non-Gaussian and heavy-tailed noise environments, i.e. it is not robust. In order to robustify the original method, we use the following preprocessing step that generalizes the sign function to multivariate cases. For a N-variate complex data set either in element space or beamspace, we define the *spatial sign function* [5]

$$\mathbf{s}(\mathbf{x}) = \begin{cases} \frac{\mathbf{x}}{||\mathbf{x}||}, & \mathbf{x} \neq \mathbf{0} \\ \mathbf{0} & \mathbf{x} = \mathbf{0} \end{cases}$$

with $||\mathbf{x}|| = (\mathbf{x}^H \mathbf{x})^{1/2}$. Notice, the spatial sign function is rotation equivariant, i.e., $\mathbf{s}(\mathbf{U}\mathbf{x}) = \mathbf{U}\mathbf{s}(\mathbf{x})$ for any unitary matrix $\mathbf{U}$ [5]. Using the spatial sign function we can define a robust estimate of covariance called spatial sign covariance matrix as follows [5]:

$$R_{SCM} = \frac{1}{K} \sum_{i=1}^{K} \mathbf{s}(\mathbf{x}_i)\mathbf{s}(\mathbf{x}_i)^H.$$

Due to the equivariance property, the signal and noise subspaces are estimated in a convergent manner using the spatial sign covariance matrix. Consequently, due to the fact that the beamformer $\mathbf{F}_e$ used in the proposed algorithm is unitary [1], we can apply this step robustifying the estimator either in element-space or in beamspace domain. Then the conventional covariance matrix in the unitary rootMUSIC algorithm may be just replaced by this robust estimated of covariance. The large sample properties of the estimator are established for ULA's in [5] and they may be extended to UCA's.

5. SIMULATION EXAMPLES

In this section we present some simulation studies where the proposed UCA unitary root-MUSIC is compared to UCA-RB-root-MUSIC [1]. In Fig. 1 and 2, the RMSE is shown as a function of number of snapshots. The Cramer-Rao Lower Bound (CRB) for stochastic signal model (see, e.g. [2]) to illustrate the asymptotic optimum for unbiased estimator. We observe that UCA unitary root-MUSIC constantly outperforms UCA-RB-rootMUSIC giving lower root mean square error (RMSE) for different UCA configurations. In case of small number of array elements neither of the methods reaches the CRB. This is due to the approximation error (bias) introduced by phase mode excitation based transform. It takes place in approximating continuous circular array with discrete circular array with few elements. With larger arrays (e.g. $N = 19$) we get very close to the the CRB.

The RMSE is plotted as a function of SNR in Fig 3 and 4, UCA unitary root-MUSIC shows a gain especially at low SNR region. At high SNR's the performances are practically similar. The performance of the proposed method is consistently better in all cases although the improvement is not very large. Notice, due to the approximation error (bias) during the beamspace transformation, in Fig 3 we can clearly see that both the algorithms do not achieve the CRB.

Next, the robustness of the method will be demonstrated. The additive noise is here assumed to be Cauchy which corresponds to complex symmetric alpha stable ($S\alpha S$) noise with characteristic exponent equal to unity. The dispersion parameter is such that the SNR in Gaussian case (characteristic exponent = 2, dispersion is kept constant) would be 10 dB. This is very demanding scenario since the moments of the noise distribution are not even defined. By using the proposed *spatial sign covariance matrix* [5] instead of conventional covariance matrix, highly reliable performance is achieved even in this very demanding noise environment. Simultaneously, the method is highly efficient (close to optimal) in the nominal Gaussian noise environment.The zeros of the unitary root-MUSIC for UCA are plotted in Fig 5 using conventional covariance matrix and robust spatial sign covariance matrix estimate. The DoA is $\phi = 280°$ and the robust method finds it very reliably whereas the conventional estimator completely fails.

6. CONCLUSION

In this paper, unitary root-MUSIC algorithm for UCA's was proposed. The algorithm extends the work in [3, 2] to circular arrays. The algorithm allows for real-valued eigenvalue analysis and basically doubles the number of observations which leads to lower variance. A robust version of the algorithm was introduced as well and it's robustness was demonstrated in highly demanding complex Cauchy noise environment where the moments of the noise are not even defined. Future work includes establishing the large sample properties of the robust estimator.

7. REFERENCES

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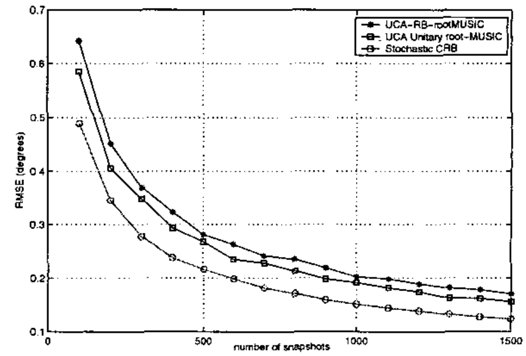

Fig. 1. RMSE as function of the number of snapshots; $N = 8$ UCA elements, radius $r = \frac{\lambda}{2.6}$, SNR=5 dB and angular position of the source $\phi = 250°$.

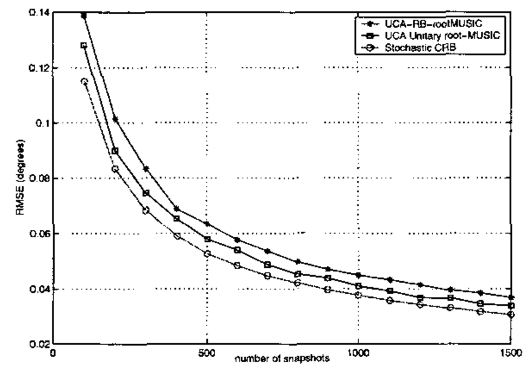

Fig. 2. RMSE as function of the number of snapshots; $N = 19$ UCA elements, $r = \lambda$, SNR=5 dB and angular position of the source $\phi = 80°$.

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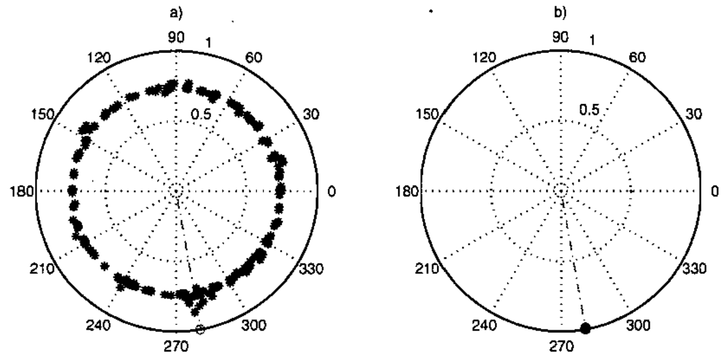

Fig. 5. Zeros of the unitary root-MUSIC algorithm for UCA in Cauchy noise environment: (left) results using conventional sample covariance matrix estimator and (right) result using robust spatial sign covariance matrix estimator. The robust method finds the angle of arrival reliably and the roots are clustered about the true angle of arrival. The conventional covariance matrix based estimator completely fails since the root are scattered all over the unit circle. Total of 200 snapshots are used, $r = \lambda$, $N = 19$ UCA elements.

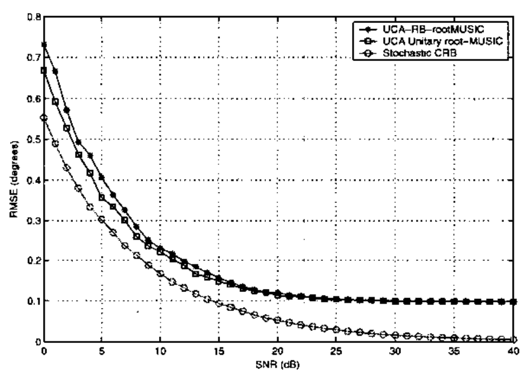

Fig. 3. RMSE as function of the SNR; $N = 8$ UCA elements, $r = \frac{\lambda}{2.6}$, $K = 256$ snapshots and angular position of the source $\phi = 50^\circ$. The effect of the bias caused by the approximation error in phase mode based transformation is clearly visible at high SNR regime.

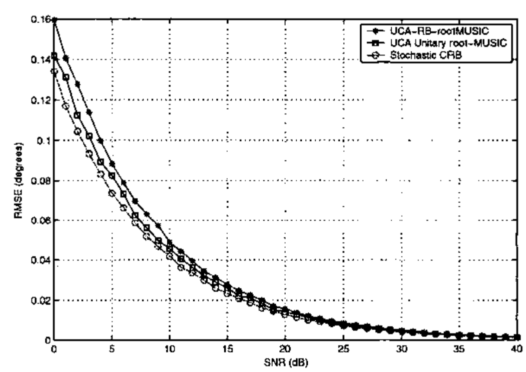

Fig. 4. RMSE as function of the SNR; $N = 19$ UCA elements, $r = \lambda$, $K = 256$ snapshots and angular position of the source $\phi = 80^\circ$. For larger array the bias becomes negligible.