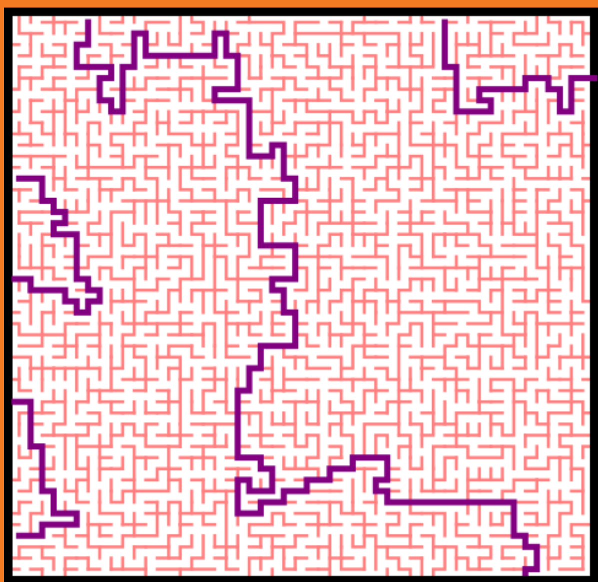
Department of Mathematics and Systems Analysis

Conformally invariant scaling limits of random curves and correlations

Alex Karrila



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Alex Karrila

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Aalto University
School of Science
Department of Mathematics and Systems Analysis
Mathematical physics group

Supervising professor
Associate professor Kalle Kytölä, Aalto University, Finland

Preliminary examiners
Associate professor Fredrik Viklund, Kungliga Tekniska högskolan, Sweden
Ph.D. Christian Hagendorf, Université catholique de Louvain, Belgium

Opponent
Ph.D. Vincent Beffara, Université Grenoble Alpes, France

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Abstract

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Abstract

This thesis studies scaling limits of critical random models on planar graphs, when a fine-mesh graph approximates a planar domain. Such studies are motivated by quantum field theoretic predictions that suggest the emergence of intricate, conformally invariant structures from simple combinatorial models. We take a mathematical approach, formalizing our results in terms of SLE type conformally invariant random curves or certain expected values, called boundary correlation functions in physics.

In the first publication of this thesis, we study two related random models, the uniform spanning tree (UST) and the loop-erased random walk (LERW). We obtain conformally covariant expressions for the scaling limit probabilities of certain UST branch connectivities and of LERW boundary visits. These expressions are solutions to partial differential equations (PDEs) of second and third order, respectively, and such solutions appear in Conformal field theory (CFT) as boundary correlation functions. CFT predicts such PDEs of arbitrarily high order, and this is among the first verifications of higher-than-second order PDEs.

The PDE solutions from the first publication can also be interpreted as weights that conjecturally convert the SLE(2) random curve measure, the scaling limit of a UST branch and a LERW, to multiple or boundary-visiting SLE(2). In the second publication, we elaborate this connection by finding an explicit relation between certain multiple SLE(k) weights, called pure partition functions, and certain CFT boundary correlation functions, called conformal blocks.

The third and fourth publication concern the weak convergence of the joint law of multiple lattice curves to multiple SLE type random curves. We first provide a result that guarantees at least subsequential convergence to some limiting random curves, given certain standard crossing estimates in the lattice models. Second, we show how such limits can be described by iteratively sampling the curves one by one from the weighted one-curve SLE(k) measures described above. These tools are applied to characterize the scaling limits of multiple curves in various random models, such as multiple UST branches.

Keywords scaling limits, lattice models, random curves, SLE, Conformal field theory, loop-erased random walk (LERW)

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Tekijä
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Väitöskirjan nimi
Satunnaiskäyrien ja korrelaatioiden konformi-invariantteja skaalausrajoja

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Tiivistelmä

Väitöskirjassa tutkitaan tasoverkoilla määriteltyjen kriittisten satunnaismallien skaalausrajoja, joissa pienenevän hilavälin verkot approksimoivat tasoaluetta. Kvanttikenttäteorian keinoin on ennustettu yksinkertaisten kombinatoristen mallien skaalausrajoille monimutkaisia konformi-invariantteja rakenteita. Väitöskirjassa skaalausrajoja tutkitaan matemaattisesta näkökulmasta ja tulokset muotoillaan joko SLE-tyyppisten konformi-invarianttien satunnaiskäyrien tai eräiden fysiikassa reunakorrelaatiofunktioiksi kutsuttujen odotusarvojen kautta.

Väitöskirjatyön ensimmäisessä julkaisussa tutkitaan kahta läheisesti toisiinsa liittyvää satunnaismallia: tasajakautunutta virittäjäpuuta ja silmukkapyyhittyä satunnaiskävelyä. Eräille tasajakautuneen virittäjäpuun oksien yhdistymistodennäköisyyksille sekä silmukkapyyhityn satunnaiskävelyn reunavierailutodennäköisyyksille johdetaan skaalausrajalla konformikovariantit lausekkeet. Nämä lausekkeet ovat toisen ja kolmannen kertaluvun osittaisdifferentiaaliyhtälöiden ratkaisuja, joita kutsutaan konformikenttäteoriassa reunakorrelaatiofunktioiksi. Konformikenttäteorian keinoin on ennustettu mielivaltaisen korkean kertaluvun osittaisdifferentiaaliyhtälöitä, ja työssä saatu tulos on ensimmäisiä matemaattisia todistuksia korkeamman kuin toisen kertaluvun differentiaaliyhtälöille.

Yllä mainitut osittaisdifferentiaaliyhtälöiden ratkaisut voidaan myös tulkita painoina, joiden odotetaan muuntavan tasajakautuneen virittäjäpuun oksan sekä silmukkapyyhityn satunnaiskävelyn SLE(2)-skaalausrajan moni-SLE(2)-kokoelman satunnaiskäyräksi tai reunavierailevaksi SLE(2)-satunnaiskäyräksi. Väitöskirjatyön toisessa julkaisussa tutkitaan tätä konformikenttäteorian ja satunnaisgeometrian välistä yhteyttä. Siinä eräiden puhtaiksi partitiofunktioiksi kutsuttujen moni-SLE(k)-painofunktioiden sekä eräiden konformiblokkifunktioiksi kutsuttujen konformikenttäteorian reunakorrelaatiofunktioiden välille johdetaan eksplisiittinen yhteys.

Väitöskirjatyön kolmas ja neljäs julkaisu käsittelevät tasoverkkojen satunnaismalleista saatavien käyräkokoelmien heikkoa suppenemista moni-SLE-tyyppisiin satunnaiskäyräkokoelmiin. Ensimmäiseksi johdetaan tulos, joka takaa verkon satunnaskäyräkokoelmien heikkojen osajonorajojen olemassaolon eräiden vakiintuneiden ylitystodennäköisyysestimaattien ollessa voimassa diskreeteille käyrille. Tämän jälkeen näytetään, miten näin saatavat rajat voidaan kuvailla iteroidulla satunnaisotannalla, jossa käyrät kasvatetaan yksi kerrallaan yllä mainituista painotetuista yhden satunnaiskäyrän SLE(k)-mitoista. Näitä kahta tulosta sovelletaan satunnaiskäyräkokoelmien skaalausrajojen karakterisoimiseen useille tasoverkkojen satunnaismalleille, kuten tasajakautuneen virittäjäpuun oksille.

Avainsanat skaalausrajat, hilamallit, satunnaiskäyrät, SLE, konformikenttäteoria, silmukkapyyhitty satunnaiskävely

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Preface

First of all, I wish to thank my doctoral advisor Kalle Kytölä, who took me as the first doctoral student in his newly-founded mathematical physics group at Aalto University in 2015. He introduced me to the research field in our joint projects, provided ideas, challenges, and support throughout my doctoral studies, and was enjoyable company in the spare moments.

I cordially thank Vincent Beffara for acting as my opponent in the public defence of this thesis, and Christian Hagendorf and Fredrik Viklund for a careful preliminary examination and useful comments on the thesis.

I am grateful to Eveliina Peltola for her efforts in our joint research, as well as many useful and interesting discussions. I also thank the numerous other colleagues with whom I have had fruitful conversations.

I had the privilege to spend the spring term of 2016 with the SwissMAP Master class in planar statistical physics at Université de Genève. I wish to thank the organizers, lecturers, and friends in Geneva for this opportunity and experience.

I gratefully acknowledge the financial support for my doctoral studies and research by the Vilho, Yrjö and Kalle Väisälä Foundation, the Academy of Finland, Aalto University, and Université de Genève.

Last but not least, I wish to thank my family and friends for all the support along the way. My greatest gratitude and love belong to my wife Vera-Maria and our children Varpu and Aarre. They are, and will remain, the most important for me.

Otaniemi, June 14, 2019,

Alex Karrila

Contents

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I A. Karrila, K. Kytölä, and E. Peltola. Boundary correlations in planar LERW and UST. arXiv:1702.03261. *Submitted journal article*, 70 pages, April 2019.

II A. Karrila, K. Kytölä, and E. Peltola. Conformal blocks, q-combinatorics, and quantum group symmetry. *Annales de l'Institut Henri Poincaré D*, online publication; to appear in print, 39 pages, April 2019.

III A. Karrila. Limits of conformal images and conformal images of limits for planar random curves. arXiv:1810.05608. *Submitted journal article*, 36 pages, June 2019.

IV A. Karrila. Multiple SLE type scaling limits: from local to global. arXiv:1903.10354. *Submitted journal article*, 64 pages, June 2019.

Author's Contribution

Publication I: "Boundary correlations in planar LERW and UST"

All three authors of the article contributed equally.

Publication II: "Conformal blocks, q-combinatorics, and quantum group symmetry"

All three authors of the article contributed equally.

Publication III: "Limits of conformal images and conformal images of limits for planar random curves"

The article is independent work by the author of this thesis.

Publication IV: "Multiple SLE type scaling limits: from local to global"

The article is independent work by the author of this thesis.

1. Introduction

Scaling limits of critical planar lattice models are predicted by quantum field theoretic arguments to be conformally invariant [BPZ84a, BPZ84b, Car88, Car96]. In mathematical physics, conformal invariance is commonly approached via either expected values of observables, i.e., deterministic numbers, or interfaces in the lattice model, i.e., random geometric objects. A breakthrough in the latter approach was the introduction of Schramm–Loewner evolution (SLE) curves as the natural scaling limit candidate [Sch00, RS05]. This thesis studies scaling limits of critical lattice models in terms of certain expectations, called boundary correlation functions, and multiple interacting SLE curves.

The compiling part of the thesis is organized as follows. In Section 2, we explain the general concepts of lattice models, scaling limits, and criticality. In Section 3, we briefly outline the physics ideas and the mathematical approaches to study the emergent conformally invariant structure in such scaling limits. Section 4 is an introduction to SLE curves. Finally, in Section 5, we summarize the main results of the publications and discuss some closely related research in more detail. The purpose of Sections 2–4 is to provide a short introduction to the field, sufficient to put the results of this thesis into context but accessible for a general mathematics or theoretical physics audience. References to textbooks and classic and contemporary articles are provided to satisfy an ambitious or expert reader.

2. Lattice models and criticality

2.1 Lattice models

Lattice models are used in physics and mathematics to model randomness in a variety of phenomena, for instance magnetism, porosity, diffusion aggregation of particles, or spatial geometry of polymers. They rely on the discretization of possible spatial locations and typically the random model or its important features are formulated in terms of local properties on the discretized space. This discrete and local structure makes lattice models approachable by both exact and numerical methods.

An archetypal example of a lattice model is the *Ising model* of ferromagnetism. On a finite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, it is a probability measure on the state space $\mathcal{S} = \{+1, -1\}^{\mathcal{V}}$. In other words, the states are vectors $\boldsymbol{\sigma}$ whose components σ_v, called spins, are labelled by the vertices $v \in \mathcal{V}$, each spin taking either the value $+1$ or -1.

To a state vector $\boldsymbol{\sigma}$ one associates the Hamiltonian $\mathcal{H}$

$$\mathcal{H}(\boldsymbol{\sigma}) = - \sum_{e=\langle u,v\rangle \in \mathcal{E}} J_e \sigma_u \sigma_v - \sum_{u \in \mathcal{V}} h_u \sigma_u,$$

where $J_e > 0$ are coupling constants of the edges $e \in \mathcal{E}$ and $h_u \in \mathbb{R}$ are the external magnetic field at locations $u \in \mathcal{V}$. The Hamiltonian depicts the energy of the system; for a lower energy the spins should thus align with their neighbouring spins and with the external magnetic field.

Finally, the Ising model is the Gibbs measure on the state space $\mathcal{S}$ with the Hamiltonian $\mathcal{H}$. This is the probability measure $\mathbb{P}$ on $\mathcal{S}$ determined by

$$\mathbb{P}[\boldsymbol{\sigma}] = \frac{1}{Z} e^{-\beta \mathcal{H}(\boldsymbol{\sigma})},$$

where $\beta > 0$ is the inverse temperature and Z is the partition function that normalizes $\mathbb{P}$ to a probability measure,

$$Z = \sum_{\boldsymbol{\sigma} \in \mathcal{S}} e^{-\beta \mathcal{H}(\boldsymbol{\sigma})}.$$

Let us exemplify here some other lattice models studied extensively in this thesis. For yet more examples, see Publication IV, Section 6.

The *simple random walk* on a locally finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is the discrete time Markov process on the vertices $\mathcal{V}$, whose next state vertex is always a uniform random neighbour of the current state vertex.

On a finite connected graph $\mathcal{G}$, the *loop-erased random walk* from $a \in \mathcal{V}$ to a target set $B \subset \mathcal{V}$ is a random path on the graph $\mathcal{G}$, obtained from the trajectory of the simple random walk launched from a and stopped upon hitting B, by erasing loops in their order of creation.

A spanning tree of a finite connected graph $\mathcal{G}$ is a connected acyclic subgraph of $\mathcal{G}$ that covers all the vertices of $\mathcal{G}$. The *uniform spanning tree* on $\mathcal{G}$ is a uniform random spanning tree.

2.2 Scaling limits

The lattice models listed above were defined purely combinatorially and for a wide range of graphs $\mathcal{G}$. Nevertheless, the motivation for studying these models often lies in cases where $\mathcal{G}$ is a small-mesh spatial discretization of some continuum shape. The scaling limit refers to the situation where a sequence of graphs $\mathcal{G}^{(n)}$, all embedded in the euclidean space $\mathbb{R}^d$, approximate some shape $\Lambda \subset \mathbb{R}^d$ and the mesh size of the spatial discretization tends to zero. For instance, $\mathcal{G}^{(n)}$ could be the largest subgraphs of $\frac{1}{n}\mathbb{Z}^d$ (equipped with a graph structure with nearest-neighbour edges) that fit inside a bounded domain Λ.

In other words, we are dealing with a sequence of probability measures $\mathbb{P}^{(n)}$, obtained from the lattice model on the sequence of graphs $\mathcal{G}^{(n)}$. A simple interesting question would be if some numbers, for instance the correlation of two spins at certain locations in the Ising model, or the expected x-coordinate of a simple random walk when first hitting the boundary of $\mathcal{G}^{(n)}$, converge as $n \to \infty$. More ambitiously, one can study the convergence of $\mathbb{P}^{(n)}$ as measures. For instance, does the law of the loop-erased random walk from a particular point to the boundary of $\mathcal{G}^{(n)}$, a random curve on $\frac{1}{n}\mathbb{Z}^d$, converge weakly to the law of some random curve in $\mathbb{R}^d$?

2.3 Criticality

Our prime example of a lattice model, the Ising model, contained several input parameters: the coupling constants J_e, the external magnetic field values h_v, and the inverse temperature β. It is clear that the answers to typical scaling limit questions, for instance the correlation of two spins, depend on the values of these parameters.

An important question thus arises: when do nontrivial scaling limits exist? The concept of criticality addresses this question.

In the mathematical approach, criticality for the Ising model can defined as follows. In the scaling limit setup of the previous subsection, take Λ to be the hypercube $(-1/2, 1/2)^d$. Consider the Ising model on $\mathcal{G}^{(n)}$ obtained by intersecting $\frac{1}{n}\mathbb{Z}^d$ with Λ, with $J_e = 1$ for all $e \in \mathcal{E}^{(n)}$ and $h_v = 0$ for all $v \in \mathcal{V}^{(n)}$, and β taking the same value for all n. Impose $+$ boundary conditions, i.e., condition the Ising model on all spins on the boundary of $\mathcal{G}^{(n)}$ being $+1$. Denote by $\mathbb{P}_+^{(n)}$ this conditional measure, and by σ_0 the spin of the vertex at the origin. If $\liminf_{n\to\infty} \mathbb{E}_+^{(n)}[\sigma_0] > 0$, the model is said to exhibit spontaneous magnetization. The infimum of β:s with spontaneous magnetization is called the inverse critical temperature β_c, and the model is said to undergo an order–disorder phase transition at β_c, if such $\beta_c \in \mathbb{R}$ exists. If the model at β_c does not exhibit spontaneous magnetization, the phase transition is said to be continuous; also this property is important for the existence of nontrivial scaling limits.

The Ising model on $\mathbb{Z}^d$ is nowadays known to undergo a continuous order–disorder phase transition in all dimensions $d \geq 2$; see [Pei36, Ons44, Yan52] for historical notes and [AF86, ADCS15] for more recent results establishing the continuity for $d \geq 3$. For modern mathematical text books on the topic, see [FV17, DC17].

For many models similar to the Ising model, criticality can be defined quite analogously. To name a few examples, such models include the Potts model, the FK random cluster model, and percolation. On the phase transitions of such models, see [BDC12, DCST17, DCGHMT16] for instance.

Let us yet return to our example question about the correlation of spins σ_x and σ_y at locations x and y in the Ising model on $\mathbb{Z}^d$, $d \geq 2$. It has recently been proven that outside of the critical temperature, such correlations decay exponentially in the distance $d(x, y)$ [ABF87, DCGR18]. This indicates that nontrivial scaling limits may only be obtained at criticality.

3. Conformal invariance in planar scaling limits

3.1 Physics approach: Conformal field theory

Scaling limits of critical lattice models have gained a lot of physicists' and mathematicians' interest since the 1980s. We now briefly discuss two important physics ideas behind this interest.

First, a scaling limit should naturally be scaling invariant. A principle in quantum field theories suggests that a scaling invariant quantum field theory is actually conformally invariant [Pol70], an *a priori* much stronger property. Conformal invariance is a particularly strong property in two dimensions, which led to the study of two dimensional Conformal field theory (CFT) [BPZ84a, BPZ84b, Car88]. Remarkable predictions on scaling limits of critical lattice models have been made by CFT, mainly relying on representation theoretic methods.

Second, the physics principle of universality suggests that there are universality classes of critical models, inside which scaling limits are the same, see [Car96] for instance. In particular, one would expect to see the same scaling limit for simpler and more complex lattice models depicting the same physical phenomenon, such as the Ising ferromagnetism model on $\mathbb{Z}^2$ with only nearest neighbour interactions, or a more realistic ferromagnetism model with interactions over a longer range.

These two principles motivate the study simple planar lattice models and their scaling limits, to which this thesis is devoted.

3.2 Mathematical approaches

Let us now list some mathematical ways of proving conformal invariance properties in the scaling limits of critical planar lattice models.

The simplest way is to study scaling limits of observables, i.e., functions $f_{v_1,\ldots,v_m}(\boldsymbol{\sigma})$ of the state $\boldsymbol{\sigma}$ of the lattice model where some lattice vertices $v_1, \ldots, v_m$ (or edges) play a special role. For instance, in the scaling limit setup of Section 2.2 and with the vertices $v_1, \ldots, v_m$ approximating some points $z_1, \ldots, z_m \in \Lambda$, the expectations of such observables should converge to a conformally covariant function $g_\Lambda(z_1, \ldots, z_m)$ of $z_1, \ldots, z_m \in \Lambda$. Conformal covariance means essentially that given a conformal map $\phi : \Lambda \to \Lambda'$, we can express $g_{\Lambda'}$ in terms of g_Λ and ϕ. (For instance for simply-connected domains Λ and Λ', such conformal maps ϕ exist by the Riemann mapping theorem [Ahl79].)

Famous examples of observable convergence results in the critical planar Ising model address the energy density [HS13], spin correlations [Hon10, CHI15], and the more general local spin pattern probabilities [GHP19]. The simplest but perhaps the most important observable convergence results in the study of conformal invariance are related to the simple random walk. Essentially all other observable convergence results rely on discrete harmonic functions, which converge to conformally invariant harmonic functions and are typically expressed and studied in terms of random walk models, see [CFL28, CS11, Che16].

An important question relating the observable convergence more directly to CFT are the partial differential equations (PDEs) predicted by CFT for correlation functions of primary fields [BPZ84a, BPZ84b, BSA88]. Primary fields $\psi(z)$ are characterized by their conformal covariance rule, and correlation functions are, naively speaking, expected values of products of field values $\psi_1(z_1) \cdot \cdots \cdot \psi_m(z_m)$. In particular, having found the correct covariance rule for the scaling limit $g_\Lambda(z_1, \ldots, z_m)$ of some observable, one hopes to also verify the associated PDEs of CFT.

A third mathematical approach to conformal invariance is to characterize the scaling limit of some random geometric object in the lattice model in terms of conformally invariant random geometry. Such geometric objects could be, e.g., boundaries of spin clusters in the Ising model, the trajectory of a simple or loop-erased random walk, or collections of paths between given vertices in the uniform spanning tree (any two vertices are connected by a unique simple path in a spanning tree). We will detail the random geometry approach in the next section.

New conformal invariance results of all the three types listed above are proven in the publications, see Section 5 for a summary of main results.

4. SLE type curves

Schramm–Loewner evolution (SLE) type curves are conformally invariant planar random curves that often appear as the only natural scaling limit candidates for random curves in critical lattice models. In this section, we briefly introduce the best-known variant, the chordal SLE; for multiple SLEs, see Publication IV, Section 2.

4.1 Motivation: the domain Markov property

Let us start with the observation about lattice models that led to the introduction of SLEs in [Sch00]. This is simplest to illustrate in the uniform spanning tree on $\mathcal{G}$. Condition the obtained random spanning tree $\mathcal{T}$ to contain a subtree t. It is elementary to check that the remainder of the tree $\mathcal{T}$ under this condition is distributed as a uniform spanning tree on the graph obtained $\mathcal{G}/t$ from $\mathcal{G}$ by contracting t to a single vertex.

In many cases, discrete random curves obtained as natural interfaces in lattice models satisfy an analogous property, called the *discrete domain Markov property*. Informally, this means that conditioning on an initial segment of the curve is equivalent to reducing the graph by it — for a formal definition, see Publication IV, Section 3. As detailed soon in Section 4.3, SLEs are the unique curve growth processes satisfying conformal invariance and a continuum version of the domain Markov property.

4.2 The Loewner equation

SLEs are random curves defined via Loewner's equation. In this subsection, we present some necessary standard facts about the *deterministic* Loewner equation. The proofs omitted here and a more thorough introduction can be found, e.g., in the textbooks [Law05, BN14, Kem17].

From Loewner equation to growing hulls

The *Loewner differential equation* in the upper half-plane $\mathbb{H}$ determines a family of complex analytic mappings g_t, $t \geq 0$ by

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \qquad g_0(z) = z \in \mathbb{H}, \tag{4.1}$$

where $W_\cdot : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a given continuous function, called the *driving function*. For a given $z \in \mathbb{H}$, the solution $g_t(z)$ of this equation is defined up to a possibly infinite explosion time $\tau(z) = \sup_{\varepsilon > 0} T_\varepsilon(z)$, where $T_\varepsilon(z)$ is the first time the process $|g_t(z) - W_t|$ takes a value at most ε. The set where the solution g_t is not defined is denoted by

$$K_t = \{z \in \mathbb{H} : \tau(z) \leq t\}.$$

The sets K_t are growing in t, and for all t they turn out to be *hulls*, i.e., K_t are bounded and closed in $\mathbb{H}$, and $H_t := \mathbb{H} \setminus K_t$ is simply-connected. Furthermore, g_t is a conformal map $H_t \to \mathbb{H}$ such that

$$g_t(z) = z + \frac{2t}{z} + O(1/z^2) \qquad \text{as } z \to \infty.$$

In this sense, the Loewner differential equation maps a driving function $W_\cdot$ to a growing family of hulls $K_\cdot$. If there exists a curve $\gamma_{\mathbb{H}} : \mathbb{R}_{\geq 0} \to \overline{\mathbb{H}}$ such that H_t is the unbounded component of $\mathbb{H} \setminus \gamma_{\mathbb{H}}([0,t])$ for all t, we say that the hulls $K_\cdot$ are generated by the curve $\gamma_{\mathbb{H}}$.

From growing hulls to Loewner equation

Suppose now that instead of a driving function we are given a collection of hulls $(K_t)_{t \geq 0} \subset \mathbb{H}$ that are growing, $K_t \subsetneq K_s$ for all $t < s$. For each t, let g_t be the unique conformal maps $H_t \to \mathbb{H}$ such that

$$g_t(z) = z + \frac{b(t)}{z} + O(1/z^2) \qquad \text{as } z \to \infty.$$

The function $b(t)$ turns out to be real, positive, and strictly increasing in t, and it is called the (half-plane) capacity $\mathrm{hcap}(K_t)$ of the set K_t.

Let $t < s$ and denote $K_{t,s} = g_t(K_s \setminus K_t)$. We say that the growing hulls $(K_t)_{t \geq 0}$ satisfy *the local growth property* if for all $T \in \mathbb{R}$

$$\mathrm{diam}(K_{t,t+h}) \downarrow 0 \qquad \text{as } h \downarrow 0, \qquad \text{uniformly over } t \in [0,T].$$

It holds true that any growing hulls $(K_t)_{t \geq 0}$ satisfying the local growth property can be represented by a Loewner equation in the following precise sense: first, the capacity function $b(t)$ is continuous, and reparametrizing the time variable t, we may assume that $b(t) = 2t$ if $b(t) \overset{t \to \infty}{\longrightarrow} \infty$ originally. Assume that this is done, and set $W_t = \cap_{h > 0} \overline{K_{t,t+h}}$. Then, $t \mapsto W_t$ is continuous and the conformal maps $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ will solve the Loewner equation (4.1) with the driving function $W_\cdot$.

Conversely, it holds true that the growing hulls $K_\cdot$ obtained from the Loewner equation with a continuous driving function $W_\cdot$ satisfy the local growth property. Thus, the Loewner equation provides a bijection between locally growing hulls $K_\cdot$ with $\mathrm{hcap}(K_t) \overset{t\to\infty}{\longrightarrow} \infty$ and driving functions $W_\cdot$.

4.3 The chordal SLE

We now wish to consider *random* growing hulls $K_\cdot$ with the local growth property and with $\mathrm{hcap}(K_t) \overset{t\to\infty}{\longrightarrow} \infty$. These hulls can equivalently be described by the random driving functions $W_\cdot$. Let us equip the space of such growing hulls $K_\cdot$ with the topology of uniform convergence over compacts of their driving functions. Denote by $\mathcal{F}_t$ the sigma algebra of the driving function $W_{\cdot \wedge t}$ up to time t.

Schramm's theorem, leading to the definition of SLE can now be stated as follows [Sch00, Kem17]. Consider random hulls $K_\cdot$ as above, starting from $\cap_{t>0}\overline{K_t} = 0$. Suppose that the distribution of the hulls is scaling invariant, and that for any $t \in \mathbb{R}_{\geq 0}$, the conditional law of the conformal image hulls $(K_{t,t+s})_{s\geq 0}$ given $\mathcal{F}_t$ is equal to the distribution of the hulls K_t, shifted horizontally to start from $\cap_{t>0}\overline{K_{t,t+s}} = W_t$. Then, $W_\cdot$ is a scaled standard Brownian motion, $W_t = \sqrt{\kappa}B_t$ for some $\kappa \geq 0$.

Definition 4.3.1. *The* chordal $\mathrm{SLE}(\kappa)$ from 0 to ∞ in $\mathbb{H}$ *is the random family of hulls* $(K_t)_{t\geq 0}$*, obtained from the Loewner equation* (4.1) *driven by a scaled standard Brownian motion* $W_t = \sqrt{\kappa}B_t$*.*

The chordal SLE in $(\mathbb{H}; 0, \infty)$ is a chordal curve in the precise sense that the hulls $K_\cdot$ are almost surely generated by a curve $\gamma_{\mathbb{H}} : \mathbb{R}_{\geq 0} \to \overline{\mathbb{H}}$ with $\gamma_{\mathbb{H}}(0) = 0$ and $\gamma_{\mathbb{H}}(t) \overset{t\to\infty}{\longrightarrow} \infty$, see [RS05]. For parameter values $\kappa \in [0, 4]$ the curve $\gamma_{\mathbb{H}}$ is almost surely simple and visits the boundary $\mathbb{R} \cup \{\infty\}$ only at its end points, for $\kappa \in (4, 8)$ the curve touches itself and $\mathbb{R}$ on a random Cantor set of times, and for $\kappa \geq 8$ it is a space-filling curve [RS05]. The Hausdorff dimension of the curve $\gamma_{\mathbb{H}}$ is $\min\{1 + \kappa/8, 2\}$, see [Bef08].

Take now a bounded simply-connected domain $(\Lambda; a, b)$ with two marked boundary points (more precisely, prime ends where radial limits exist, see Publication III, Section 4.4). Consider the image of an $\mathrm{SLE}(\kappa)$ curve $\gamma_{\mathbb{H}}$ in $(\mathbb{H}; 0, \infty)$ with $\kappa < 8$, when $(\mathbb{H}; 0, \infty)$ is mapped conformally to $(\Lambda; a, b)$. This image is almost surely a compact curve $\gamma_\Lambda : [0, 1] \to \overline{\Lambda}$, and the curve

γ_Λ is a measurable random variable (in the metric space of Publication III, Section 2.2) with respect to the sigma algebra of the Brownian motion generating the SLE, see Publication III, Proposition 5.2 and [GRS12]. The distribution of the obtained random curve γ_Λ does not depend on the chosen conformal map. We call the curve γ_Λ the *chordal* $\mathrm{SLE}(\kappa)$ *in* $(\Lambda; a, b)$.

The introduction of SLEs sparked a whole new branch of literature, studying SLEs or, if interpreted differently, studying the scaling limits of random curves *a posteriori*, see [LSW00, LSW03, RS05, Bef08, AS08, AS09, LS11, LZ13] for instance. In addition to the chordal and multiple SLEs studied in this thesis, there are various other SLE type random curves, see [Sch00, LSW03, Zha04, BBH05, Zha08, Izy17] for examples.

4.4 Convergence proofs

Proving the weak convergence of lattice curves to SLE always takes two steps: precompactness and identification. Precompactness means that any sequence of random curves on graphs of decreasing mesh sizes has a weakly convergent subsequence. The identification of any subsequential weak limit then proves weak convergence along the entire sequence.

In modern SLE convergence proofs, the precompactness relies on verifying certain "crossing estimates" for the lattice curves. These estimates guarantee the precompactness in the sense of both curves and driving functions, see Section 5.3. The identification part then starts from a weakly convergent (sub)sequence of functions $W^{(n)} \to W$, where $W^{(n)}$ are the driving functions of the lattice curves mapped conformally to $(\mathbb{H}; 0, \infty)$. The aim is to show $W_t = \sqrt{\kappa} B_t$. The nowadays established strategy of [Smi06] starts from a discrete martingale $M^{(n)}$ under growing lattice curve. One then promotes it to a continuous martingale $\mathcal{M}$ under growing the scaling limit curve; the martingale $\mathcal{M}$ is thus expressed in terms of the driving function $W_\cdot$, in the simplest case in the form $\mathcal{M}_t = f(W_t)$. This relies on an analogous expression for $M^{(n)}$ and, typically, convergence results on discrete harmonic functions [CS11]. Finally, computing the Itô drift term of $\mathcal{M}_t = f(W_t)$ then allows to deduce $W_t = \sqrt{\kappa} B_t$. For a concrete example of this strategy, see Publication IV, Section 6.4.

For celebrated convergence results of lattice interfaces to chordal SLEs, see [Smi01, LSW04, SS05, CN07, Zha08, SS09, CDCHKS14]; for other SLE type curves, see [LSW04, Zha08, HK13, Izy15, LV16, Izy17, GW18]; for multiple SLEs, see [Izy17, Wu18, KS18, BPW18] and Publication IV.

5. Main results and related work

5.1 Publication I

The first publication of the thesis studies the probabilities of certain connectivity events in the planar uniform spanning tree (UST) and boundary visits by loop-erased random walk (LERW); see Publication I, Section 1 for the precise descriptions.

First, in Theorem 3.12 and Corollary 3.13, we find a combinatorial formula solving these probabilities on any planar graph in terms of random walk excursion kernels. These results take three inputs that are quite well known in the field: Wilson's algorithm [Wil96], Fomin's formula [Fom01], and a combinatorial matrix inversion [KW11b, SZJ12]. The UST part of the result can also be found in [KW11a]–[KW11b] with a proof relying on somewhat different combinatorial tools.

The main results of the publication are Theorems 3.16 and 3.17, where we find conformally covariant formulas for the scaling limits of the UST and LERW probabilities. These scaling limits solve second and third order PDEs, respectively, as predicted for boundary correlation functions by CFT [BPZ84b, BSA88, JJK16]. They also satisfy asymptotics that have been predicted for such correlation functions in [JJK16, KP16], see Propositions 4.7 and 5.8. Such asymptotics define the boundary correlation functions, in the sense that they determine unique conformally covariant PDE solutions in the second-order (UST) case (see Publication I, Section 4.1.1) and conjecturally also in the third-order (LERW) case.

Let us now discuss results addressing analogous questions. Connectivity probabilities similar to the UST question have been computed in other lattice models in [Dub06a, KW11a, KS15, PW18]. On the other hand, the analogous *interior* visit probability for LERW is a well-known problem [Ken00, Law14, BLV16]. Combinatorics similar to Publication I appear in another LERW question in [Pon18].

As regards similar results on PDEs, to our knowledge the only higher-than-second order PDE proof in lattice model scaling limits is Watts' formula for percolation [Dub06b, SW11]. For SLEs, similar PDEs have appeared in [Dub15, LV18]. The proof in Publication I uses ideas from [Dub15].

5.2 Publication II

Conformally covariant solutions to second-order PDEs as in Publication I play a key role also in the definition of multiple SLEs. Such functions provide weights that transform a chordal SLE measure to the marginal measure of one curve in multiple SLEs, see [Dub07, KP16], and Publication IV, Section 2. In Publication II, we connect the CFT and SLE approaches to the solutions to these PDEs.

In some more detail, the space of conformally covariant solutions to these PDEs is (under a technical growth condition) a finite dimensional vector space. We examine two different bases of this space: the basis of multiple SLE pure partition functions and the basis of CFT conformal block functions. These can be regarded as the most fundamental bases in respective views, as detailed in Publication II, Sections 3 and 4.

The first main result of Publication II, Theorem A, is a combinatorially defined change of basis matrix between the two bases above. The matrix and related combinatorics are a weighted version of the matrix inversion of [KW11b, SZJ12] as used in Publication I. With the construction of multiple SLE pure partition functions in [KP16], this determines the conformal block functions. The second main result, Theorem B, provides an alternative, direct construction of conformal block functions based on the quantum group method of [KP14]. The latter construction follows the representation theoretic idea of conformal blocks, explained in Section 3 of Publication III.

The role of conformal blocks in CFT was realized early [BPZ84a]. The connection of SLEs and CFT was first explicated in [BB02, BB03], and was one of the motivations for studying multiple SLEs [BB03, BBK05, Gra07]. The space of covariant solutions to the PDEs has been studied in [FK15a]–[FK15d], in particular showing the finite dimension. Multiple SLE pure partition functions have been subsequently studied in [Wu18, PW19].

5.3 Publication III

Publication III provides a precompactness result related to SLE convergence proofs which plays an important role in Publication IV but is interesting also on its own right.

Let us explain the result via its main application. All SLE type curves are defined via Loewner's equation and conformal invariance: a random driving function $W_.$ yields a random curve in a reference domain, say the curve $\gamma_{\mathbb{D}}$ in the unit disc $\mathbb{D}$, and this is mapped conformally to our domain of interest Λ to yield the random curve of interest γ

$$W \overset{\text{Loewner}}{\longrightarrow} \gamma_{\mathbb{D}} \overset{\text{conformal}}{\longrightarrow} \gamma.$$

Consider now a lattice model on planar graphs $\mathcal{G}^{(n)}$, corresponding to domains Λ_n that approximate Λ. Suppose that we wish to prove the convergence of some discrete chordal curves $\gamma^{(n)}$ on Λ_n to an SLE type curve γ in Λ. By the very definition, the identification may only be done via driving functions. Thus, one has to map $\gamma^{(n)}$ and Λ_n conformally to $\gamma_{\mathbb{D}}^{(n)}$ and $\mathbb{D}$, and then by Loewner's correspondence to a driving function $W^{(n)}$,

$$\gamma^{(n)} \overset{\text{conformal}}{\longrightarrow} \gamma_{\mathbb{D}}^{(n)} \overset{\text{Loewner}}{\longrightarrow} W^{(n)}.$$

Then, one has to show that the weak convergence of $\gamma^{(n)}$ and $\gamma_{\mathbb{D}}^{(n)}$ follows from that of $W^{(n)}$, i.e., informally, establish the commutative diagram

$$\begin{array}{ccccc} \gamma^{(n)} & \xrightarrow{\text{conformal}} & \gamma_{\mathbb{D}}^{(n)} & \xrightarrow{\text{Loewner}} & W^{(n)} \\ \downarrow {\scriptstyle n\to\infty} & & \downarrow {\scriptstyle n\to\infty} & & \downarrow {\scriptstyle n\to\infty} \\ \gamma & \xleftarrow[\text{conformal}]{} & \gamma_{\mathbb{D}} & \xleftarrow[\text{Loewner}]{} & W. \end{array} \tag{5.1}$$

Certain crossing estimates for the random curves $\gamma^{(n)}$, known for many lattice models, guarantee most of Diagram (5.1), see [AB99, KS17]. However, the bottom left horizontal arrow seems not to have been explicated before and it is not trivial, see Publication III, Section 3.2 for warning examples. Theorem 4.4 of Publication III completes Diagram (5.1), assuming the crossing conditions of [KS17]. Moreover, Proposition 4.7 shows that under these conditions, all the horizontal arrows in (5.1) are two directional, i.e., weak convergences in all three topologies are equivalent.

Publication III provides a common, clean argument for Diagram (5.1), needed in all SLE convergence proofs. Alternative proofs exist if the boundary visits of $\gamma^{(n)}$ can be excluded (e.g., *a posteriori* if $\gamma_{\mathbb{D}}$ is $\mathrm{SLE}(\kappa)$ with $\kappa \leq 4$), or if the boundaries of Λ_n and Λ are nice enough. However, this cannot be assumed, e.g., when studying multiple SLE convergence, where the curves lie on subdomains restricted by other random curves.

5.4 Publication IV

Publication IV addresses the convergence of collections of N chordal lattice model curves to multiple SLE type scaling limits. First, we define a candidate for such limits by a "domain Markov extension" of the earlier introduced local multiple SLE curve initial segments [BBK05, Dub07, KP16] to full chordal curves. The main results of the paper are two *a priori* results suited for realizing such scaling limits: Theorem 4.1 is an analogue of the main result of Publication III for multiple curves, while Theorem 5.8 allows promoting a discrete domain Markov property to the scaling limit. These *a priori* results essentially only require of the lattice models the crossing conditions of [KS17] and the discrete domain Markov property. The use of these results is exemplified in Publication IV with convergence proofs in various lattice models.

Roughly speaking, two approaches to multiple SLE type curves exist in prior literature. Either one studies local multiple SLEs, i.e., $2N$ initial segments of curves defined via explicit Loewner evolutions [BBK05, Dub07, Gra07, KP16], or one fixes the way in which the random interfaces pair the $2N$ marked boundary points, and takes a more implicit definition of N full curves, called global multiple SLEs [KL07, Law09, PW19, BPW18]. The approach of Publication IV has the asset of defining full curves in terms of explicit iterated Loewner evolutions. However, the well-definedness of such random curve measures is proven relying either on realizing them as scaling limits or, for SLE parameters $\kappa \leq 4$, on global multiple SLE results [PW19].

The connection of the local and global multiple SLEs is unfortunately not immediate. For $\kappa \leq 4$, global multiple SLEs are local [PW19]. Publication IV, Section 6.2 provides a warning example of random curves that are local but not global multiple SLEs (with $\kappa = 6$ and $N \geq 3$). Addressing this caveat, we show in Theorem 5.2, Proposition 5.9, and Theorem 5.10 that the scaling limits obtained and described with the method of Publication IV also coincide with both local and global multiple SLEs.

The nontriviality of promoting the discrete domain Markov property to the scaling limit has been addressed in [BPW18, GW18]. The tools of Publication IV are suited for the SLE convergence proof strategy of [Smi06], see Section 4.4 and references therein. This strategy has been applied earlier for one-curve marginals in curve collections in [Izy17, KS18]; for other multiple SLE type convergence results, see [KS15, Wu18, BPW18].

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