



Sami Järvinen

ESSAYS ON PRICING
COMMODITY DERIVATIVES

HELSINKI SCHOOL OF ECONOMICS

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Sami Järvinen

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1 Introduction

In recent years, commodity contingent claims have become increasingly popular and important hedging instruments in the financial markets characterized by varying degree of volatility. Previously, companies have primarily focused on hedging interest rate and foreign exchange risks. During recent years, hedging the commodity price risk of the production process has received more attention. As a result, the over-the-counter market for commodity contingent claims has grown remarkably. Therefore, the modelling of price processes and the development of new contingent claims pricing approaches are important area of research today.

This thesis aims to contribute to the literature on commodity contingent claims pricing through four interrelated essays. The first essay studies the estimation of the term structure of forward prices from the prices of traded long-term commodity swaps. The second essay investigates empirically the factor structure of the term structure movements using data from oil and pulp derivatives markets. Whereas previous studies have used futures prices, the data used in this study consists of swap market quotes. The forward curves used in the empirical study are derived using the method developed in the first essay. In the third essay, the focus is moved to the pricing of Bermudan swaptions using a simple binomial model. In addition to presenting an algorithm for pricing interest rate swaptions, an extension to handle commodity swaptions is also provided. Here again, the forward values of the variables are derived using the method developed in the first essay. Finally, the fourth essay is a joint work with Harri Toivonen. We concentrate on the necessary modification of the Black -76 formula in order to price European commodity swaptions. The central theme in all of the essays is the orientation towards using market data efficiently and the simplification of the modelling approach. Market oriented approaches towards modelling of the commodity price processes is a relatively new avenue. The vital ingredient to enable full utilization of those techniques

is extraction of the term structure of forward prices as the zero coupon term structure is an essential input in pricing interest derivative securities.

The structure of this introductory chapter is as follows: Section 2 presents the central concepts underlying the analysis of commodity contingent claims and describes the functioning of commodity markets in general. Section 3 reviews the existing literature on the term structure estimation. This earlier material is concerned almost exclusively with the estimation of the term structure of interest rates, the results of which form the essential basis for the results derived in this thesis. In Section 4, the objective is to review the existing literature on the theoretical models for pricing commodity contingent claims and to introduce the theory of storage. Finally, Section 5 presents a summary of the four essays in this thesis, with an emphasis on their contribution and extensions to the previous research.

2 Specific Features of Commodities

Commodities differ from financial assets in a profound way; unlike most financial assets, commodities are continuously produced and consumed. Moreover, from an economic point of view, commodities form an interesting asset class as production and consumption do not have to match each other in every period as commodities can be stored in the form of inventories. As an exception to this general rule some commodities, e.g. electricity and bandwidth, are not storable. Moreover, there are also other types of commodity classes such as agricultural and animal products that are perishable and hence in practice storable only for some period of time.

In equilibrium, supply increases if the price of commodity is high, and as a consequence higher cost producers enter the marketplace. Correspondingly, supply decreases if the price is low as some of the higher cost producers exit the marketplace. This

economic activity is reflected by the well-known and documented phenomena of mean-reverting commodity prices. Moreover, seasonal effects introduced by the nature of the production process are typical for many commodity markets. In particular, seasonal effects can be observed in agricultural commodity and electricity markets.

For the purpose of understanding futures prices, it is convenient to divide commodity futures contracts into the following two categories according to the underlying asset: investment and consumption commodities. Investment commodities, e.g. gold and silver, are held for investment purposes by a significant number of investors. Consumption commodities, e.g. oil, are held primarily for consumption purposes. In the case of consumption commodities, it is not possible to obtain the futures price as a function of the spot price and other observable variables. Hence, a parameter known as convenience yield becomes important.

Most commodity markets also have a well-functioning and liquid futures markets. One question that is often raised is whether the future price of an asset is equal to the expected future spot price. When the futures price is below the expected spot price, the situation is known as normal backwardation. Strong backwardation is the situation in which the futures price is below the current spot price. The situation where the futures price is above the expected future spot price is known as contango.

3 Review of Literature on Term Structure Estimation

3.1 Estimating the Term Structure of Interest Rates

The theoretical models for term structure estimation have been developed especially for interest rate markets where the amount and quality of data available has been good

enough to facilitate the empirical comparison of the different methods. The first serious estimation method was proposed by McCulloch (1971). He developed a method for estimating the discount function by fitting a cubic spline to the data of government Bills and Bonds. The method proposed by McCulloch (1971) is especially appealing from the perspective of estimation: since the model is linear in the discount function, it can be estimated using simple linear regression. McCulloch (1975) applies the spline method to the estimation of the tax-adjusted yield curve. Carleton and Cooper (1976) estimated the discount function of the US Government securities, but without imposing any functional form. Their method produces a set of discrete discount bond prices that minimize the pricing error of the bonds. The estimation was carried out using linear regression.

Since the first steps in the estimation of the term structure of interest rates were taken, the research intensified and many new methodologies and extensions to the existing methods were proposed. One line of research extended the spline methodology to other functional forms. An important account in this respect is Vasicek and Fong (1982). The authors modified the spline estimation method so that the resulting discount function is exponential. The exponential form, the authors claim, avoids some of the problems associated with using polynomial forms in discount function estimation. Vasicek and Fong (1982) do not provide comprehensive evidence on the performance of their method. Shea (1985) criticized the exponential spline method and proved that it essentially produces a similar type of unstable forward rate functions as does the ordinary polynomial spline. Chambers, Carleton, and Waldman (1984) use exponential splines of varying lengths to estimate the present value functions. They let the sample decide which polynomial length best suits the given estimation problem. Maximum likelihood method is used to find the appropriate parameterization. Chambers, Carleton, and Waldman (1984) postulate that their method confirms that the use of a constant parameterization does not yield good results over heterogeneous samples. Their method,

however, is rather awkward to use in practical situations.

Research on improving the spline method, especially to yield smooth forward rate functions, is extended by Fisher, Nychka, and Zervos (1995) and Käppi (1997). These authors propose smoothing splines that impose a penalty on the target function which aims to produce a smoother forward rate function. The forward rate function is important especially for traders of derivative contracts, since the values of the underlying instrument often need to be derived from the forward rates (or prices). Fisher, Nychka, and Zervos (1995) claim, based on empirical comparisons against McCulloch (1971), that their method generally produces the most reliable forward curves and less biased results. The smoothing splines need to be estimated numerically.

Even though spline methods have received most of the attention, other methods for estimating the term structure of interest rates have also been proposed. One of the best known of these other approaches is Nelson and Siegel (1987). The proposed model expresses the instantaneous forward rates as a solution to the difference equation describing the movements of the spot rates. In the continuous time framework, the yields are just averages of these instantaneous forward rates, and the parameters of the solution equation can be estimated from, for example the yields of the government securities. Nelson and Siegel (1987) provide illustrations of their method using data on Treasury Bills. Depending on the chosen parameterization, there is a trade-off between fitting the front-end and back-end of the curve. Finally, the term structure of interest rates can be estimated using the so-called bootstrap method. This method is especially popular among traders of the derivatives contracts in the swap market, since par swap quotes provide ideal data to effectively bootstrap the term structure. As the quotes are always equally spaced, they do not move with the passage of time. Fama and Bliss (1987) is the best known academic treatment of the bootstrap methodology. They present two methods for estimating the discount functions: 1) Unsmoothed and 2) Smoothed. The

former produces a piecewise linear discount function and prices the bonds used for estimation exactly, whereas the latter first estimates the unsmoothed discount factors as in the former case, but then tries to smooth out the discount function by fitting an approximating function through the discount factors.

Bliss (1996) conduct an extensive empirical comparison between the various methods for term structure estimation. The methods he tested were: 1) Unsmoothed Fama-Bliss, 2) McCulloch spline, 3) Fisher-Nychka-Zervos smoothing spline, 4) Extended Nelson-Siegel and 5) Smoothed Fama-Bliss. The most important conclusions of his study are that although the Unsmoothed Fama-Bliss method was able to fit the long term structures best, the method suffered from over-parameterization. Of the other methods, the Fisher-Nychka-Zervos method somewhat surprisingly performed the worst. The other methods were approximately equal.

3.2 Estimating the Factor Structure of Forward Curve Movements

Research on the commodity forward curve dynamics has used data almost exclusively on short term, exchange traded, contracts. Of these, among the first is Cortazar and Schwartz (1994), who studied the dynamics of the futures price curve of copper futures. They found that the factor structure of the copper futures curve was surprisingly similar to the factor structure of yield curve movements. Moreover, the explanatory power of the first two principal components was 97 percent. By contrast, Litterman and Scheinkman (1988) found that the explanatory power of first two principal components of the government yield curve movements was remarkably lower, around 90 percent. Clewlow and Strickland (2000) studied the factor structure of NYMEX oil futures and they found that three factors explained over 98 percent of the variation of the futures price movements in the period from 1998 to 2000. A recent paper by Tolmasky and Hindanov (2002) inves-

tigated the dynamics of the petroleum futures contracts. They found that especially for heating oil, seasonality is an important variable driving the factor structure, however the statistical significance is somewhat unclear. Crude oil and petroleum markets were not found to be affected by seasonality. Koekebakker and Ollmar (2001) studied the forward curve dynamics using data from the Nord Pool electricity derivatives exchange, using fitted forward curves. The explanatory powers they report are fairly low in comparison with other studies. They argue that the most likely reason for low explanatory power is the extremely complex dynamics of the electricity prices.

4 Review of Literature on Pricing Commodity Contingent Claims

4.1 Theory of Storage and Convenience Yield

Two different models have been introduced as alternative perspectives for the price formation of commodity futures prices. Working (1949) and Kaldor (1939) developed the theory of storage, which explains the difference between contemporaneous spot and futures prices on the basis of interest foregone in storing commodity product, warehousing costs and a convenience yield on inventory. The alternative model views commodity futures price as a combination of an expected risk premium and a forecast of the future spot price, see Cootner (1960). It has been argued that if hedgers tend to hold short positions and speculators tend to hold long positions, the futures price will be below the expected future spot price. This is because speculators require compensation for the risks they are bearing. They will only trade if there is an expectation that the futures price will rise over time. Hedgers, on the other hand, are prepared to enter into contracts where the expected payoff is slightly negative because they primarily aim at

reducing the risk. Hence, if hedgers tend to hold long positions while speculators hold short positions, the futures price must be above the expected future spot price. To compensate speculators for the risk they are bearing, there must be an expectation that the futures prices will decline over time. Many studies on agricultural, wood, animal products and precious metals markets have shown results in favor of the former explanation, i.e. the theory of storage is able to explain the difference between the futures price and the contemporaneous spot price. This relation can be summarized into the following equation.

$$F(t, T) = (S(t) + W(t, T))e^{r(T-t)}$$

Where $F(t, T)$ is the futures price at time t , for delivery of a commodity at time T . $S(t)$ is the spot price at time t and $W(t, T)$ is the present value of all the storage costs that will be incurred during the life of a futures contract. If the storage costs per unit are proportional to the price of the commodity, $w(T, t)$, they can be regarded as providing a negative dividend yield

$$F(t, T) = S(t)e^{(r+w)(T-t)}$$

The theory of storage predicts that the return from purchasing the commodity at time t and selling it for delivery at time T , $F(t, T) - S(t)$ equals the interest foregone, plus the marginal storage cost $W(t, T)$, less the marginal convenience yield from an additional unit of inventory, $\delta(t, T)$. Or, equivalently

$$F(t, T) = S(t)e^{(r+w-\delta)(T-t)}$$

Where $F(t, T) - S(t)$, or $(F(t, T) - S(t)) / S(t)$ is commonly known as basis, and $r + w$, or equivalently for consumption commodities $r + w - y$ is referred to as the cost of carry. Often the above equation is written as in the form of net convenience yield, defined as

convenience yield minus storage costs.

$$F(t, T) = S(t)e^{(r-\delta)(T-t)}$$

Convenience yield, δ , of a commodity is defined as the flow of services which accrues to the owner of the physical inventory, i.e. spot commodity, but not to the owner of a contract for future delivery, Brennan (1958). For investment assets such as gold, the convenience yield must be zero otherwise there will be arbitrage opportunities. However, a consumption asset behaves like an investment asset that provides a return equal to the convenience yield. The marginal convenience yield arises due to the fact that inventory can have productive value. The most obvious reason for this is that the owner of the commodity is able to determine where the commodity will be stored and when to liquidate inventory. Moreover, considering time lost and the costs incurred in ordering and transporting a commodity from one location to another, marginal convenience yield reflects both the reduction in costs of acquiring inventory and the value of being able to profit from temporary local shortages of the commodity through ownership of a larger inventory. Profit may also arise either from local price variations or from the ability to maintain production process despite local shortages of raw materials.

The convenience yield reflects the markets expectation concerning the future availability of the commodity. Hence, the greater the possibility that shortages will occur during the life of the futures contract, the higher the convenience yield. If users of the commodity have high inventories, there is very little chance of shortages in the near future and the convenience yield tends to be low. On the other hand, low inventories tend to lead to high convenience yields. This is especially true on those commodity markets where there are agents in the market who do not buy commodities only for production process purposes. This speculative storage activity can smooth the price and supply in this commodity market as the speculative storers buy the commodity when it is cheap

and abundant, thus increasing the price, and then release their stores onto the market when the commodity is expensive and in short supply, thus reducing price. The convenience services yielded by particular inventory naturally depend upon the identity of the agent holding it. However, in theory, competition between potential storers will ensure that in equilibrium the convenience yield, net of any direct costs of storage, the marginal unit of inventory will be balanced across all storers. Then, assuming that there exists a positive inventory of the commodity, the relation between spot and futures price of the commodity will reflect this marginal net convenience yield.

In the case of Commodities with constant marginal costs of storage, the estimated convenience yield function is large and positive at low inventory levels and small and negative at high inventory levels, i.e. the theory of storage posits that the net rate of convenience yield depends upon the level of inventories. The economic reasoning behind this argument is as follows: Convenience yield is high because inventories are low and storage firms will tend to have an incentive to increase their investment in inventories which, in turn, will tend to reduce the convenience yield. Hence, the theory of storage predicts a negative relation between convenience yields and levels of inventories. Positive correlation between changes in the spot price and changes in the convenience yield of the commodity is induced by the level of inventories. As the inventories decrease, the spot price should increase due to the fact that the commodity is scarce, and the convenience yield should increase as the futures price will not increase as much as the spot price.

4.2 Summary of Pricing Models

Modelling commodity prices for the purposes of valuing derivative securities can be categorized into two distinct approaches, namely 1) convenience yield models and 2) forward price models. While this thesis concentrates exclusively on the latter, the former approach has been more common in the academic literature. It has sometimes been

named as the traditional approach. The ideological inspiration of the approach is derived from the theory of spot rate modelling, pioneered by Vasicek (1977) and Cox, Ingersoll, and Ross (1981). The forward price modelling is the more market-oriented approach, since the quantities traded in the market are prices and therefore, it is intuitively more straightforward to model the prices directly. Similar developments have already taken place in the study of interest rate modelling. So-called market models (see Miltersen, Sandmann, and Sondermann (1997), Brace, Gatarek, and Musiela (1997) for the earliest developments and Musiela and Rutkowski (1997) and Jamshidian (1997) for generalizations) have largely replaced the traditional models of Hull and White (1990) and Black, Derman, and Toy (1990).

In a seminal paper, Gibson and Schwartz (1990) develop a two-factor model in which the dynamics of the commodity prices are determined as a function of the stochastic spot price of the commodity, and the stochastic instantaneous convenience yield. Schwartz (1997) extends the previous model by introducing also the stochastic interest rates. Furthermore, he provides empirical evidence on three different models using copper, oil and gold data. Evidence suggests that for the commercial commodities (copper and oil), mean reversion is an important characteristic of price movements, whereas for gold it is not. Miltersen and Schwartz (1998) take this idea a step further and introduce a model for forward/future convenience yield. The approach is similar to the infamous Heath, Jarrow, and Morton (1992) approach for interest rates. Hilliard and Reis (1998) introduce jumps to the spot price process.

Whereas in interest rate derivatives research, the forward rate based models have been more popular objects of study than spot rate models among both academics and practitioners, in commodity derivatives research, little attention has been devoted to developing forward price based models. The few exceptions include Reisman (1991) and Cortazar and Schwartz (1994). Both model the forward price processes as multidimensional

mensional geometric Brownian motion. The processes have zero drift, since under the risk-adjusted measure, it can be shown that forward prices must be martingales. Using Ito's lemma, one can make a transformation from the defined forward price process to the forward cost of carry process and vice versa, Cortazar and Schwartz (1994). Even though the theoretic notion of convenience yield is appealing from an economic point of view, practitioners working in the commodity derivatives industry will certainly find modelling the forward prices directly more intuitive.

4.3 Simple Approaches to Valuation

Complex pricing models are required to handle exotic derivative structures, whereas if market data on the plain vanilla instruments is available with reasonable accuracy, it is often more convenient to use standard Black and Scholes (1973) and Black (1976) models instead. To handle interest rate swaptions, modifications to the Black (1976) model have been proposed by Neuberger (1990) and Smith (1991). Since then, these formulas have become the market standard for pricing plain vanilla interest rate swaptions. Moreover, the practitioners use model implied volatilities in order to calibrate more complex models to the market data and, therefore, the exotic products can be priced consistently with actively traded instruments. This is essential, since the plain vanilla instruments are used for hedging exposures arising from trading the exotic products. Initially, it seemed that the Black (1976) model was inconsistent when applied to pricing interest rate products, since it assumed constant discounting. However, the measure change techniques, see Geman, Karoui, and Rochet (1995), formed theoretical justification for these simple models. In addition, the research efforts turned away from the traditional short rate models to modelling the stochastic evolution of the underlying variables (Libor and Swap rates) instead.

Black and Scholes (1973) model was a breakthrough in derivatives research. The formulas, however, applied only to European options. Therefore, applicable numerical methods were needed in order to handle more general option types. Algorithms for solving the Black-Scholes partial differential equation do exist, and Monte Carlo methods can be used to simulate Black-Scholes model stochastic differential equation in order to calculate path-dependent option prices. One of the best known numerical methods for calculating Black-Scholes model prices is Cox, Ross, and Rubinstein (1979). The authors derive a binomial lattice algorithm that converges at the limit to the Black-Scholes continuous time model as steps are increased to infinity. The binomial algorithm provided not only option price, but also the replicating portfolio calculations. The binomial model of Cox, Ross, and Rubinstein (1979) (CRR) has become the most important pedagogical tool in discrete time finance. In addition, it is one of the most applied numerical methods in actual option pricing.

Since the publication of the article by Cox, Ross, and Rubinstein (1979) article was published, researchers have devoted a great deal of interest to improving and modifying the simple binomial model in order to handle more difficult option structures and different stochastic processes. Jarrow and Rudd (1983) parameterize the algorithm so that the probabilities for up and down movements in the tree are always $1/2$. This is sometimes a more convenient parameterization for constructing algorithms, for example if the interest rate and volatility parameters are not constant throughout the tree. Hull and White (1988) improve the accuracy of the binomial method by applying a control variate technique. In addition, Hull and White (1993) extend the binomial method in order to handle path-dependent options. Boyle (1988) and Boyle, Evnine, and Gibbs (1989) extend the CRR binomial model to handle options dependent on several assets. These and many more refinements and modifications to the original CRR binomial model have been proposed to improve pricing efficiency, handle complex exotic option contracts and deal with different distributional assumptions. One of the most important extensions

to the model is the so called "Implied Binomial Tree" method, proposed by Rubinstein (1994). With this algorithm, the binomial tree structure is made consistent with the prices of traded options. Therefore, market data is used efficiently and other, more complex option structures can be priced with maximum utilization of market information.

5 Summary of the Four Essays

The first essay in this thesis develops methods for deriving the term structure of forward prices, or the forward curve, from the price quotes of commodity swaps. The importance of developing methods for forward curve extraction derives from the fact that dealing in commodity swaps has increased dramatically over the last decade. Existing research on commodity contingent claims pricing has concentrated exclusively on using the data from the futures markets, since it is readily available in convenient form. By contrast, the data (broker quotes) on the swap market contracts only became available in the late 1990's, and only for the most liquid markets. The focus of this first essay is mainly on extending the methods for interest rate term structure estimation to the commodity world. Empirical data from the oil and pulp derivatives markets is utilized to illustrate and compare the methods. The literature on the interest rate term structure estimation is vast, one of the seminal works being by McCulloch (1971), who applied the cubic spline method to the derivation of the discount function.

The main conclusions of this first essay are that the bootstrap method produces a very saw-toothed forward curve, particularly in cases where the averaging frequency of the index values over the settlement periods is high, i.e. daily. An example of this would be oil derivatives market. For most practical purposes then, the bootstrap method is not a realistic method to use. The standard cubic spline produces a smooth curve, but often with very large pricing errors. Finally, the optimization of fit method produces

a smooth curve with acceptable pricing errors, so this method, or modifications of it, provides traders a practical means for deriving the forward curves for commodities.

The second essay examines the factor structure of the commodity term structure of forward price changes. This essay is an empirical examination, based on the estimation method developed in the first essay. To the best knowledge of the author, this is the first study on the commodity forward curve factor structure using swap contract quotes. There exists earlier studies on the commodity futures curve factor structure. Cortazar and Schwartz (1994) is one of the first studies examining the changes in the copper futures curve, and their findings were largely similar to corresponding studies done using the term structure of interest rate data. Koekebakker and Ollmar (2001) have studied the Nord Pool exchange's electricity contracts using a forward curve estimation method that is similar in spirit to the method used in this paper.

The empirical results of the second essay lead to a conclusion that the dynamics of the term structure of forward price changes is more complex when the data on the long-term derivatives contracts are used for estimation. Moreover, the dynamics of the front end of the curve is largely dictated by how the proxy for the spot index value is chosen in the forward curve estimation phase. It seems that bias is caused by the use of infrequently published index value data; if possible, one should therefore use the shortest future's quote as a proxy for the spot index value instead. In general, the results were similar to those found in earlier studies. The explanatory power of the first three principal components of Principal Components Analysis (PCA) applied to forward curve movements is around 89 percent for oil data and 84 percent for pulp data.

The data used in the first and second essays consists of weekly NBSK Risi pulp swap quotes covering the period from June 1998 to October 2001 and of monthly oil swap quotes (European Brent) covering the period from February 1997 to February 2002. The data was obtained from Nordea Bank Finland. The interest rate data used

in estimation consists of Libor rates and zero rates. The European interest rate data consists of EuroLibor quotes and zero rates until 1.1.1999 and from that date onwards of Euribor quotes and zero rates. The interest rate data was retrieved from DataStream.

The third essay in this thesis concentrates on the pricing of the derivatives contracts. In particular, a simple binomial method is presented for pricing Bermudan swaptions, an important subset of options, found especially in the interest rate markets. Existing algorithms and pricing methods are fairly complex to implement and calibrate to the prices of plain vanilla European Swaptions. This essay also extends the algorithm to be applicable in the commodity swaption markets. It is likely that in the near future, the Bermudan type swaptions will become interesting and popular hedging vehicles, as already is the case in the interest rate swaption market.

Finally, the fourth essay is a joint work with Harri Toivonen. We derive formulas for European commodity payer and receiver swaptions and provide extensions. The pricing formulas are direct modifications of the famous Black (1976) formula for options on the commodity futures. Similar formulas have been derived for the case of interest rate swaptions by Neuberger (1990).

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ESSAY 1:
Estimating the Forward Curve for Commodities

Abstract

This paper develops methods for estimating the term structure of forward prices and the term structure of convenience yields for general storable commodity classes on which tradable derivative contracts exist. The methodologies presented here can be used to infer inputs for the pricing of commodity contingent claims such as options. Another important area of application is the extraction of long-term convenience yields and forward prices for empirical studies on the stochastic processes of these variables. The estimation procedures are tested empirically using swap price quotes from the oil and pulp markets.

The forward curve is the fundamental input in pricing contingent claims on any underlying assets with an active market. To date, there has been very little research on the estimation of the forward curve for commodity contracts of longer maturities. By contrast, estimation of the term structure of interest rates from the prices of bonds has been an active area of research pioneered by McCulloch (1971) and later extended by many researchers. In this paper, the estimation of the forward curve for commodity contracts from prices of actively traded over-the-counter derivative contracts is investigated, both theoretically and empirically. The potential uses of the estimation methods include applying new pricing models, testing hypotheses of the term structure of forward prices and convenience yields, and projecting futures prices for capital budgeting purposes.

Various approaches have been presented for estimating the term structure of interest rates. The best known and actively quoted method, presented by McCulloch (1971), is based on estimating the discount function using the cubic spline specification. The method is amenable to the use of linear regression due to equations being linear in parameters. The spline method has been expanded and investigated by many authors, Vasicek and Fong (1982), Shea (1984) and Chambers, Carleton, and Waldman (1984), among others. Fisher, Nychka, and Zervos (1995) and Käppi (1997) have extended the spline method further by exploiting the smoothness penalty in the estimation of the term structure of forward rates. Yet another well known approach to estimation of the term structure of interest rates has been presented by Nelson and Siegel (1987). They devised a parsimonious parameterization of the yield curve that can be used to generate yield curves with many different shapes. In addition to the above mentioned methods that rely on fitting to the data, there is also the well-known bootstrap method¹, which is based on iteratively solving a system of equations. The bootstrap method has also been extended by many authors - see, for example Fama and Bliss (1987). In addition to studying some of these well known methods of forward curve fitting for commodity forward curve estimation, this paper proposes a non-parametric minimization of the sum

of squared pricing errors and the squared difference of two adjacent maturity forward prices for extracting the forward prices from the prices of swap contracts. This method has a close counterpart from the interest rate yield and forward curve estimation field: Delbaen and Lorimier (1992) propose a non-parametric optimization method where the squared difference between adjacent forward rates is minimized.

The growing importance of obtaining reliable forward curves for commodity contracts is due to the expanding over-the-counter market for long term commodity derivatives. At present, active swap markets can be found for oil and some metals. At the same time, the modelling of the stochastic processes tailored for pricing contingent claims are moving from the traditional spot convenience yield models towards more market oriented approaches.² This transition has already been observed in the interest rate markets. One of the best known new models developed for pricing commodity contingent claims is by Miltersen and Schwartz (1998). Their approach is based on the celebrated Heath, Jarrow, and Morton (1992) model for the stochastic behavior of the instantaneous forward rate curve. Miltersen and Schwartz (1998) derive a model where the starting points are the initial term structure of interest rates and commodity futures prices along with the spot price of the commodity. After obtaining the risk neutral processes for each stochastic quantity, it is possible to price other contingent claims such as futures options within the model framework. A similar model is also presented by Hilliard and Reis (1998). Their approach extends the Miltersen and Schwartz (1998) model by including a jump component to the stochastic process for the spot price. In addition to these models that are based on the term structure of convenience yields, Reisman (1991) and Cortazar and Schwartz (1994) have developed a model that directly specifies the evolution of the term structure of futures prices. Cortazar and Schwartz (1994) apply the principal components analysis to estimate the model using the futures price data from the copper market. Both types of term structure models, convenience yield and futures prices, can be applied to price longer maturity contracts, as soon as the initial curves are available.

The main contribution of this paper is to provide a theoretical framework and applicable methods for obtaining the forward price curve from the prices of swaps. Several methodologies are presented: Bootstrapped Extraction, non-parametric Optimization of Fit, Cubic Spline based method and Cubic Spline including seasonality factor. The forward curve can be put into a linear form that conveniently enables the use of regression methods for minimization of the sum of squared errors. The Bootstrap method provides output forward curves that are exceedingly saw-toothed in the presence of average price swaps. The non-parametric Optimization of Fit method presented in this paper is flexible to use, and similar in spirit to calibrating model prices of options to the market prices. The methods presented here are primarily designed for estimation of the forward price curves, but the equivalence between the forward price and the convenience yield facilitates translation between the two quantities. In addition to the theoretical developments, empirical evidence on the performance of various methods is also presented as well as graphical illustrations. In this paper, the fitting methods are compared using data from the pulp and oil swap markets.

This paper is primarily interested in the pricing efficiency of the methods presented. Another area in forward curve estimation that needs to be investigated in future studies is the economic realism of the chosen methods for extracting the forward curves for commodities. In fact, it is often quoted that commodity prices are mean-reverting, follow business cycles and / or exhibit seasonal patterns depending on the type of commodity. Of the methods analyzed in this paper, only the Cubic Spline method including sin-cos functions aims at producing seasonal behavior on the extracted forward prices. All other methods are designed primarily to produce minimal pricing errors, in the same way as is done in the interest rate market. Optimization and basic Cubic Spline methods trade-off some of the pricing effectiveness against the smoothness of the forward price curve.

Another aim of this paper is to study empirically the methodologies presented with

the data from oil and pulp markets. By definition, the Bootstrap method prices each of the swaps in the sample exactly, so the empirical section of this paper focuses only on comparing the fitting methods. Evidence suggests that among the fitting methods proposed, the Optimization of Fit provides the best alternative for extracting the forward curve from the market prices of swaps. Friedman Rank Statistic strongly rejects the null hypothesis of no difference between the models. It is also found that using internal knot points with the spline method significantly improves the overall performance of the spline method. However, no method is satisfactory for real trading situations at all times. Even the Optimization of Fit method has to be applied with caution, due to intolerably large pricing errors with some term structures. An applicable remedy to resolve the pricing error issue is to use weighting to assign importance to specific portions of the curve. All the methods presented here are based on linear interpolation between the observed and derived curve points: This provides efficiency for estimation and simplicity for the formulas. Future research is needed to investigate the effect of various interpolation techniques to the results.

The paper is organized into eight sections. Section 1 introduces the basic definitions and terminology of the analysis framework. Section 2 describes the Bootstrap methodology for forward curve estimation, with consideration of both forward price curve and convenience yield term structure specifications. Section 3 presents the Cubic Spline estimation method and Section 4 develops the Optimization of Fit method. The new methodologies are compared in Section 5 and in Section 6 the models are fitted to the data and empirical results reported. Section 7 discusses some possible extensions to the methodologies. Finally, section 8 concludes the paper.

1 Preliminaries

The aim of this paper is to derive theoretical commodity forward price curves from the prices of traded swap contracts. This chapter reviews the fundamental concepts of commodity swaps. The basic elements needed for the analysis are the par swap quotes $G(t, T)$ for all maturities T , $t \leq T$, the spot price of the underlying commodity S_t , the forward prices of the commodity, $F(t, T)$, for all maturities T , $t \leq T$. In addition, we set $F(t, t) = S_t$. Futures prices, $f(t, T)$, and the pure discount bond prices $P(t, T)$, for all maturities T , $t \leq T$, are also essential parts of the framework. $P(t, T)$ denotes the present value, at time t of one unit of money received at time T . The analysis assumes constant interest rates throughout, so theoretical forward and futures prices are equal, as shown by Cox, Ingersoll, and Ross (1981). Rather than theoretical futures prices, in the current setup the futures prices, $f(t, T)$, are taken to represent exchange traded futures contracts that can be utilized in the curve generation process.

The starting point of the analysis is the valuation formula for commodity swaps. Typically, the swap contracts quoted in the commodity derivatives markets are such that fixed payment streams are exchanged, in monthly or quarterly intervals, for floating price payments. The floating price payments are usually based on some index or spot value of the commodity, calculated and defined by a reliable market information provider. Additionally, sometimes the futures price quotes are used to determine the floating price settlements. The distinguishing feature of a typical commodity swap contract from a standard interest rate swap contract is the averaging of the floating prices in determination of the final settlement prices. In more liquid markets, such as the oil derivatives markets, the averaging frequency is usually daily, whereas in the less liquid markets the observations are typically made weekly or monthly. Single price observation swaps also exist, but they occur less frequently in the commodity derivatives markets.³

Consider a standard commodity swap contract, with maturity T_N and a fixed contract

price $G(t, T_N)$. The present value of this fixed leg of the swap, given the discount bond prices $P(t, T)$ is given by

$$V_{fx}^N = G(t, T_N) \sum_{i=1}^N P(t, T_i) \quad (1)$$

i.e, simply a sum of the present values of the payments. The other side of the swap contract, the floating leg, is composed of the present values of the yet unknown average prices of the commodity at settlement dates T_i , $i = 1, 2, \dots, N$ and $T_1 < T_2 < \dots < T_N$. In mathematical notation

$$V_{fl}^N = \sum_{i=1}^N F_A(t, T_i) P(t, T_i) \quad (2)$$

where A is used to indicate that the swap contract has average- based floating price settlement. If we know the forward price curve, the par swap prices for all maturities, T , $t \leq T$, could be obtained by setting the values of the floating and fixed legs to be equal. Solving for the par swap price $G(t, T_N)$ gives

$$G(t, T_N) = \frac{\sum_{i=1}^N F_A(t, T_i) P(t, T_i)}{\sum_{i=1}^N P(t, T_i)} \quad (3)$$

which shows that the par swap price is defined by the ratio of the discounted value of the floating price payments and an annuity. Therefore, as with interest rate swaps, the par swap price can be interpreted as the present value weighted average of the forward prices.

The traditional approach of modelling the forward prices of commodities is based on defining the net convenience yield, which is defined as the net "income", accruing to the owner of the physical commodity, Brennan (1958). The relation between the spot price of commodity, S_t , and the forward price of maturity, T , is given by the well known

formula

$$F(t, T) = S_t e^{(r(t, T) - \delta(t, T))(T-t)} \quad (4)$$

which can be written, using the familiar formula $e^{-r(t, T)(T-t)} = P(t, T)$, as follows:

$$F(t, T) = \frac{S_t e^{-\delta(t, T)(T-t)}}{P(t, T)} \quad (5)$$

showing that there is a one-to-one correspondence between the net convenience yields and forward prices. Therefore, either the term structure of forward prices or the term structure of convenience yields can be estimated from the prices of swaps (and futures).

2 Bootstrap Method

In this section, the whole procedure of deriving the implied average forward prices and direct forward prices (forward prices without averaging) is presented. The procedure is based on direct application of the net present value rule in order to iteratively solve the system of equations. Assume that par swap quotes $G(t, T_i)$ for maturities $T_i = 1, 2, \dots$ years are observed in the market. In addition, the spot price of the commodity, S_t , is observed. The procedure begins with interpolation of the par swap curve in order to find the par swap prices for maturities corresponding to the settlement dates of the swap payments. Interpolation of the swap curve is necessary because there are not enough swap quotes to cover all the settlement dates. For example, if the market convention is fixed price versus quarterly settled averages of daily price observations, then the par swap curve needs to be interpolated in order to find prices for three month, six month, nine month, etc par swaps. The first implied average forward price sets the initial seed for the Bootstrap method.

Having interpolated the par swap curve to find the swap price corresponding to the

first settlement date of the one year swap, we can use the standard net present value method to obtain the first implied average forward price. As at contract initiation, the value of a swap is zero, so it follows that

$$(G_A(t, T_1) - F_A(t, T_1))P(t, T_1) = 0$$

and solving for $F_A(t, T_1)$ gives

$$F_A(t, T_1) = G_A(t, T_1) \tag{6}$$

Hence, the first implied average forward price is just the value interpolated from the par swap curve. Therefore, the procedure can now be continued in order to find the second point. For this one can use the net present value with the six month par swap, that is:

$$(G_A(t, T_2) - F_A(t, T_1))P(t, T_1) + (G_A(t, T_2) - F_A(t, T_2))P(t, T_2) = 0$$

with the solution

$$F_A(t, T_2) = G_A(t, T_2) + (G_A(t, T_2) - F_A(t, T_1))\frac{P(t, T_1)}{P(t, T_2)} \tag{7}$$

The full procedure, generalizing up to $T_n \leq T_N$ -maturity par swap quote, can be expressed with a simple formula

$$F_A(t, T_n) = G_A(t, T_n) + \sum_{i=1}^{n-1} (G_A(t, T_n) - F_A(t, T_i))\frac{P(t, T_i)}{P(t, T_n)} \tag{8}$$

Using equation (8), the whole term structure of implied average forward prices can be recovered. Having obtained the implied average forward prices, the next step is to obtain the implied direct forward prices. This needs to be done because the term

structure of implied average forward prices is not flexible enough to be used for pricing general commodity derivative products, since there may be different averaging periods or no averaging at all in other derivatives structures.⁴ Let k denote the number of observations for the arithmetic average on a single settlement period. (In practice, k is the number of bank days on a single settlement period, if the averaging frequency is daily, though other conventions are also possible.) The following shows the steps for finding the implied direct forward prices. First it should be noted that there is a problem of non-uniqueness of the direct forward prices, since our task is to produce $F_A(t, T_i)$ from a set of possible direct forward prices, $F(t, T_i^j)$. The average can therefore be arrived at by any suitable combination of $F(t, T_i^j)$'s. Therefore, in order to force unique direct forward prices from a set of average prices, impose the linear interpolation constraint as follows

$$\begin{aligned} F_A(t, T_i) &= \frac{\sum_{j=1}^k F(t, T_i^j)}{k} \\ &= \frac{\sum_{j=1}^k ((F(t, T_i) - F(t, T_{i-1})) \frac{j}{k} + F(t, T_{i-1}))}{k} \end{aligned}$$

which can be rewritten

$$F_A(t, T_i) = \frac{F(t, T_i) \sum_{j=1}^k \frac{j}{k} + \left(k - \sum_{j=1}^k \frac{j}{k} \right) \times F(t, T_{i-1})}{k}$$

and using the fact that $\sum_{i=1}^k i = k(k+1)/2$, the equation can be simplified further:

$$F_A(t, T_i) = \frac{1}{2} \left[\left(1 + \frac{1}{k} \right) F(t, T_i) + \left(1 - \frac{1}{k} \right) F(t, T_{i-1}) \right] \quad (9)$$

This can now be solved inductively, so that a full term structure of direct forward

prices can be extracted. The first step is to solve for $F(t, T_1)$. This can be done, because we know that the first direct forward price is equal to the spot price (which is directly observable), so $F(t, T_0) = S_t$. Finally, use equation (9) and $F_A(t, T_1)$, which was already solved in the first phase when the implied average forward prices were obtained. This process is continued iteratively until the whole term structure of direct forward prices is derived. Figures 1 and 2 show examples of forward price curves using the method presented above. The data is from the Brent oil market in both upward and downward sloping swap price curve environments. Also note that the observation frequency of the index values for the settlement prices of the floating legs can be daily, weekly, monthly etc. If the payment frequency is equal to the observation frequency, then clearly $F_A(t, T) = F(t, T)$, and there is no need to complete the last step of the Bootstrap method.

The main problem with the Bootstrap method is that the forward curve it produces is exceedingly saw-toothed. This is even more so if the underlying swap market data consists of average settlement contracts. The average forward prices are solved for first with the Bootstrap method, i.e. the first phase of the algorithm derives a forward curve for implied average prices. In order to force the direct forward curve to yield those same average prices, it must attain extreme values. An example clarifies this: Consider that the spot index value is 100 units. The settlement frequency is daily with 22 observations in the current one month period. The implied average forward price for the first settlement is 105 units. Then applying formula (9) yields the direct forward price of 109.57 units.

Additionally, the forward price curve can also be derived directly in terms of the net convenience yields. In this case, one starts with the implied average forward price curve that has already been derived at this point. The term structure of cost of carry can then

Figure 1: OIL SWAP AND FORWARD CURVES FOR 3/31/99, bootstrap forward prices method

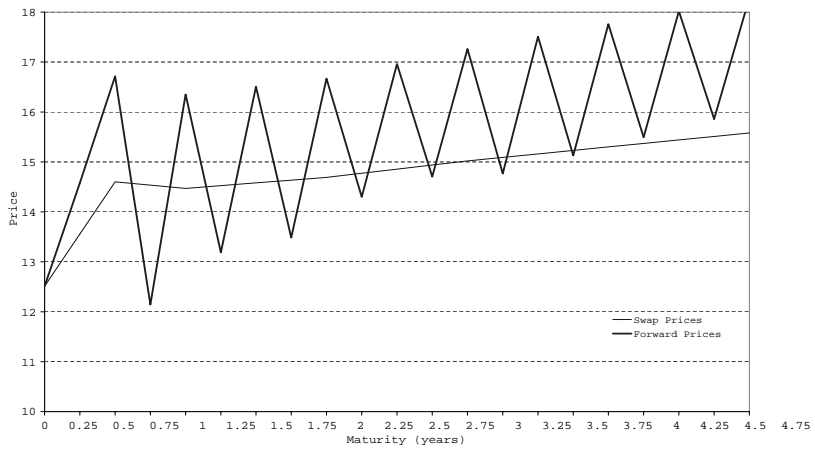
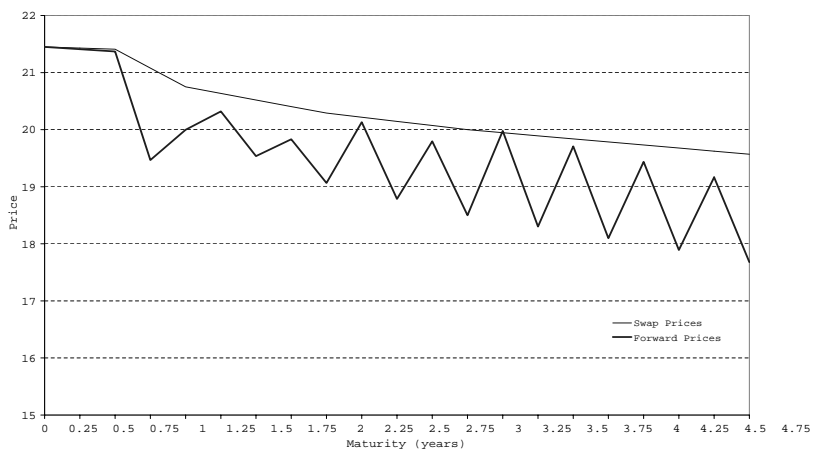


Figure 2: OIL SWAP AND FORWARD CURVES FOR 3/31/02, bootstrap forward prices method



be obtained by solving iteratively

$$F_A(t, T_i) = \frac{S_t \sum_{j=1}^k e^{c(t, T_i)(\Delta j + T_{i-1})}}{k} \quad (10)$$

where S_t is the spot price of the commodity, $c(t, T_i)$ is the cost of carry parameter for the period of average forward. Δ divides the period into intervals, from the start to the end of averaging. For example, if the observations are taken daily, then $\Delta = 1/365$. k again denotes the number of observations on a single settlement period from T_{i-1} to T_i . Equation (10) has to be solved iteratively for $c(t, T_i)$. That can be easily done using, for example the Newton-Raphson scheme. After the full term structure of cost of carry is obtained, various swap structures, including the standard swap products, can be priced.

Using the fact that $e^{c(t, T)} = e^{r(t, T) - \delta(t, T)} = e^{-\delta(t, T)} / P(t, T)$, together with (10), it is possible to derive the net convenience yield $\delta(t, T)$ term structure, using the formula given below

$$F_A(t, T_i) = \frac{S_t \sum_{j=1}^k \frac{e^{-\delta(t, T_i)(\Delta j + T_{i-1})}}{P(t, T_{i-1} + \Delta j)}}{k} \quad (11)$$

Note that the net convenience yield parameter is expressed in terms of continuously compounded return. This is due to the fact that the theoretical models of the net convenience yield are defined this way. If one wants to use some other compounding style, the adjustments to the formulas are then straightforward. However, the price average is always defined in terms of the arithmetic average. Therefore, it must be in the form presented above.

Figures 3 and 4 show examples of forward curves derived using the bootstrapped net convenience yield method.

Figure 3: OIL SWAP AND FORWARD CURVES FOR 3/31/99, bootstrap convenience yield method

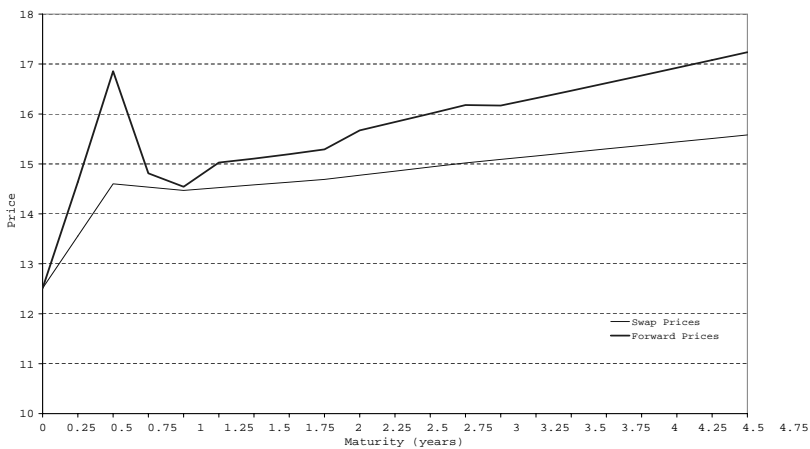
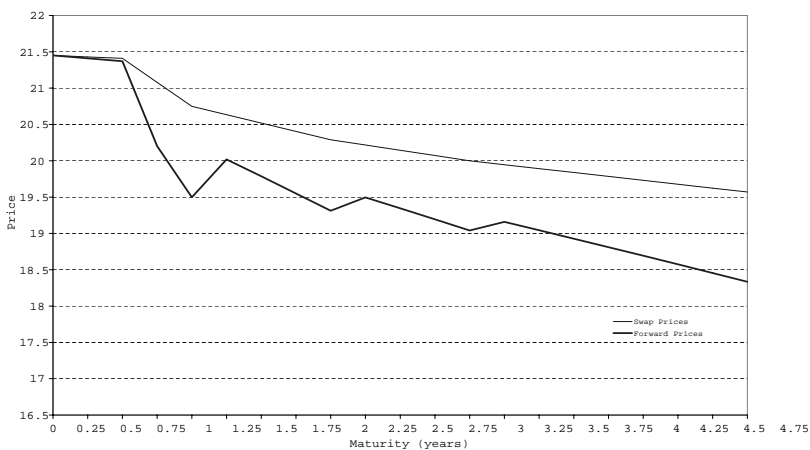


Figure 4: OIL SWAP AND FORWARD CURVES FOR 3/31/02, bootstrap convenience yield method



3 Spline Method

In this paper, the spline form used is the standard cubic functional, similar to that used by McCulloch (1971), though more general forms could be applied. Results are derived using general functional forms. It is demonstrated that the forward price term structure can be obtained using simple linear regression, analogously to discount function estimation. The net present value rule requires that the fixed leg and floating leg present values are equal, for each $n = 1, 2, \dots, N$

$$G(t, T_n) \sum_{i=1}^n P(t, T_i) = \sum_{i=1}^n F_A(t, T_i) P(t, T_i) \quad (12)$$

In order to fit a spline function to the forward curve by linear regression, k continuously differentiable $g_j(T)$ functions must be defined. In the following, it is assumed that $t = 0$. Hence, the forward curve can be represented as a constant term and a linear combination of the functions $g_j(T)$

$$F_A(0, T) = \alpha_0 + \sum_{j=1}^k \alpha_j g_j(T) \quad (13)$$

Since the forward price for $F(t, t)$ must equal S_t , it is therefore required that $\alpha_0 = S_t$. Combining equations (12) and (13), the function to be estimated becomes, for each $n = 1, 2, \dots, N$

$$\begin{aligned} V_{fl,n} &= \sum_{i=0}^n F_A(0, T_i) P(0, T_i) \\ &= \sum_{i=0}^n \sum_{j=0}^k \alpha_j g_j(T_i) P(0, T_i) \\ &= \sum_{j=0}^k \alpha_j G_{n+1,k+1} \end{aligned}$$

which can be expressed in vector notation as follows

$$\mathbf{V}_{fl} = \mathbf{G}\mathbf{A} \quad (14)$$

where

$$\mathbf{G} = \begin{pmatrix} 1 & g_1(0) & g_2(0) & \cdots & g_k(0) \\ P(0, T_1) & P(0, T_1)g_1(T_1) & P(0, T_1)g_2(T_1) & \cdots & P(0, T_1)g_k(T_1) \\ \sum_{i=1}^2 P(0, T_i) & \sum_{i=1}^2 P(0, T_i)g_1(T_i) & \sum_{i=1}^2 P(0, T_i)g_2(T_i) & \cdots & \sum_{i=1}^2 P(0, T_i)g_k(T_i) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^N P(0, T_i) & \sum_{i=1}^N P(0, T_i)g_1(T_i) & \sum_{i=1}^N P(0, T_i)g_2(T_i) & \cdots & \sum_{i=1}^N P(0, T_i)g_k(T_i) \end{pmatrix}$$

and \mathbf{A} is a vector of coefficients. The fact $P(t, t) = 1$ has been used to simplify the notation. Figures 5 and 6 show examples of forward curves derived using the standard Cubic Spline method.

The spline method can be made more flexible in order to handle difficult curve shapes by introducing knot points. Therefore, the spline method will be extended to piecewise polynomial functions for the empirical part of this paper. In the empirical study, only two internal evenly spaced knot points are used. It is left for further research to investigate the effect of knot positioning and the optimal amount of knots. Details of the effects on the estimation of the resulting regression are given by Suits, Mason, and Chan (1978).

In addition to the standard piecewise cubic spline, the sine and cosine functions are also used in order to capture possible seasonality in the data. It is often postulated that commodity prices exhibit mean reversion. Therefore, that possibility will also be considered in this study. In order to implement seasonality with curve estimation, a combination of cubic polynomial and an expansion of two sine and cosine functions are used. Since with polynomial, we have $k = 4$, it follows that there will be four additional

Figure 5: OIL SWAP AND FORWARD CURVES FOR 3/31/99, splining forward curve method

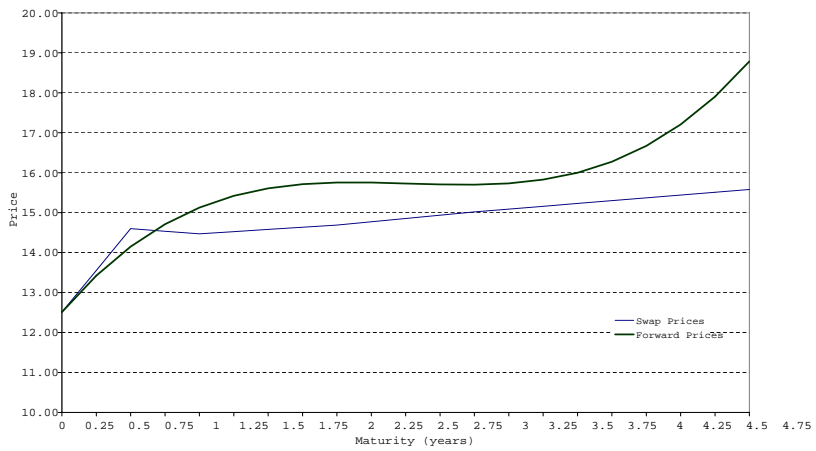
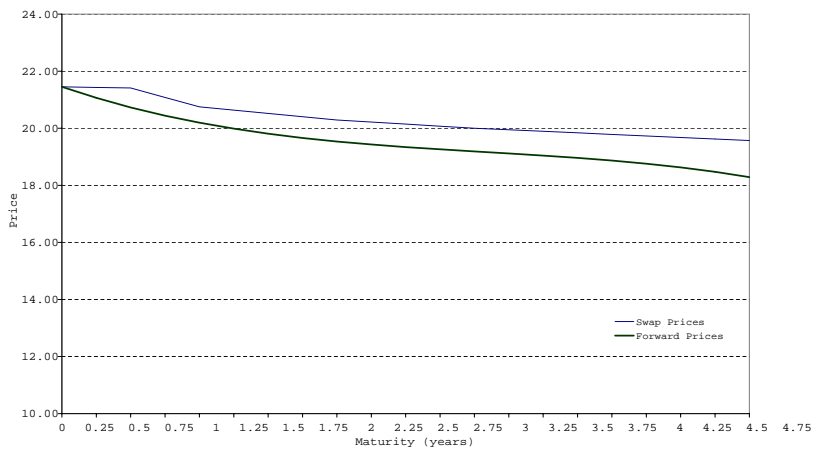


Figure 6: OIL SWAP AND FORWARD CURVES FOR 3/31/02, splining forward curve method



coefficients to be estimated

$$\begin{aligned}
 g_{k+1}(T) &= \cos\left(\frac{2 * T * \pi}{\beta}\right) \\
 g_{k+2}(T) &= \sin\left(\frac{2 * T * \pi}{\beta}\right) \\
 g_{k+3}(T) &= \cos\left(\frac{4 * T * \pi}{\beta}\right) \\
 g_{k+4}(T) &= \sin\left(\frac{4 * T * \pi}{\beta}\right)
 \end{aligned}$$

where β is the period factor. In principle, β can either be estimated from the historical data or used as one parameter in the fitting procedure.

As is the case with the bootstrap method, the direct forward prices $F(t, T)$ can be obtained by imposing the linear interpolation constraint and then using (9). The direct forward price curve will still be a saw-toothed function, however. If one wants to get a smooth direct forward price curve, then the floating price leg value has to be transformed to be dependent directly on the direct forward prices (instead of the implied average forward prices). An obvious choice is to use equation (9), to impose linear interpolation constraint. The spline can then also be configured to handle this situation.

4 Optimization Method

This section develops the Optimization of Fit method for extracting forward curve information from the prices of commodity swaps. The method is based on the minimization of the sum of squared errors. The aim is to fit the observed swap prices as precisely as possible while maintaining a reasonable degree of smoothness in the output forward

curve. The proposed target function of the minimization problem is

$$\min_{\{F_i\}_{i=1}^N} \left[\sum_{i=1}^N (V_{fl}(t, T_i) - V_{fx}(t, T_i))^2 + \lambda \sum_{i=1}^N (F(t, T_i) - F(t, T_{i-1}))^2 \right] \quad (15)$$

where the fundamental idea is to match the swap values from the forward curve to the observed swap values, and simultaneously controlling that consecutive forward prices are not too far apart. To motivate this choice, let us note that the desire of a trader is often to produce a realistic looking, or at least not excessively vibrating, forward curve with which to price other products. Similar ideas have been used in the interest rate market in the spline form, for example by Fisher, Nychka, and Zervos (1995) and Käppi (1997). Delbaen and Lorimier (1992) use similar methodology to estimate the forward rate curve from the prices of bonds. Their objective is to minimize the squared difference between two adjacent maturity forward rates. Additionally, they impose a constraint that the relative pricing errors lie within a given tolerance. The approach presented in this paper is closely related to the latter method. The empirical results show that the method works well in comparison with the spline method. This suggests that, especially in the commodity markets where rather extreme and awkward curve shapes are very common, it is beneficial to have a more flexible fitting method.

The parameter λ is introduced to adjust the weight put on the forward curve smoothness. Note that the formulation uses the direct forward prices, so the output curve from the optimization problem is immediately of desired form. Moreover, it follows from the linear interpolation constraint that the derivatives for the optimization problem are straightforward to calculate. Therefore, analytic derivatives can be used in the actual minimization - see Appendix for the calculation of the derivatives. The optimization problem can be solved using any suitable algorithm. The method utilized in this paper is based on a simple gradient search with constant step size and analytical derivatives. While not the fastest possible algorithm, the convergence properties are good and the

global minimum is almost sure to be attained for any curve shapes. For an excellent description on the various optimization algorithms, see Press, Teukolsky, Vetterling, and Flannery (1992).

In choosing λ , two points need to be mentioned. First, if one picks $\lambda = 0$, then the Optimization of Fit method will converge to the bootstrapped curve and the target function value will be equal to zero. I.e. exact match can be obtained through the Optimization method. Alternatively, if a high enough λ is chosen, then, starting from the spot value S_t , the final forward curve will be flat. Hence, a reasonably shaped and correctly pricing forward curve will result only if one picks a reasonable value for λ . For the purposes of this study, λ is set equal to one. The decision is based on experimenting with the algorithm, not explicitly on any objective criteria. In further research, implementing some penalty function like in Fisher, Nychka, and Zervos (1995) might yield interesting results. Figures 7 and 8 show examples of forward curves obtained using the Optimization of Fit method.

5 Comparison of Methods

Four methods for estimating the forward curve for commodities are developed. The Bootstrap methods use the analogy from the term structure of interest rate estimation research to iteratively solve a system of equations to obtain the forward prices either directly, or indirectly through convenience yield. Also, the Cubic Spline method derives analogy from the term structure of interest rates estimation literature to specify equations for commodity forward curve estimation. The Optimization of Fit method, however, is more like a calibration procedure, where the forward curve is derived by minimizing the sum of squared pricing errors. Weighting can also be utilized to express the relative importance of each swap price and the forward curve smoothness.

Figure 7: OIL SWAP AND FORWARD CURVES FOR 3/31/99, optimization of fit method

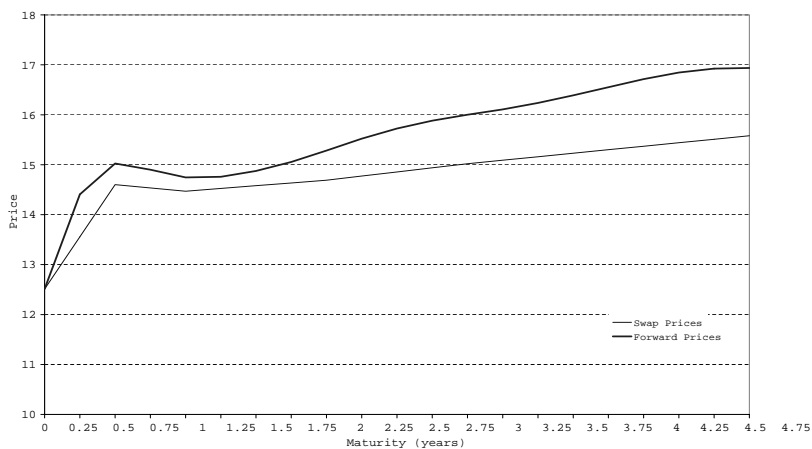
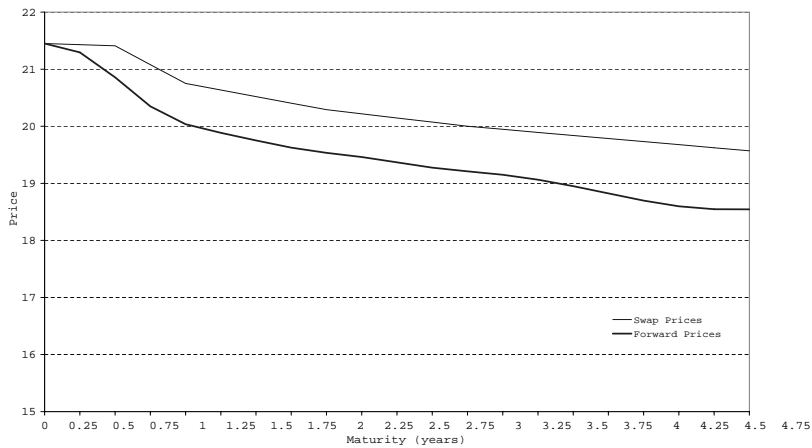


Figure 8: OIL SWAP AND FORWARD CURVES FOR 3/31/02, optimization of fit method



- **Bootstrap Forward Curve Method**

The Bootstrap forward curve method is a recursive method to solve for implied average forward prices and, further, direct forward prices. At each step, the next forward price is obtained from the swap pricing equation and previous forward prices. In order to find the swap prices for intermediate points, it is necessary to interpolate the swap prices since there are not enough quotes to span all periods. An easy choice is to interpolate linearly, but other choices are also possible. Moreover, it is also necessary to impose interpolation condition on the derived implied average prices, since an average forward price is composed of a set of direct forward prices. The Bootstrap forward curve method prices market swap quotes exactly, but the resulting curve is intolerably jagged.

- **Bootstrap Convenience Yield Method**

Bootstrapping the convenience yield term structure results in a forward price curve that behaves better than bootstrapping the forward prices directly. However, it is not as convenient to use in real applications, since the specification presented in this paper is such that the spot convenience yield is constant for each period. Therefore, if one interpolates the forward curve, as is customary in trading systems, the market swaps are no longer priced exactly. However, correct results can be obtained by appropriately constructing the algorithms used in this approach.

- **Cubic Spline Method**

There are many possibilities to specify splining of the forward price curve in order to find the best suited result for each particular need. One can put the spline on the implied average forward price curve or on the direct forward price curve according to what is considered suitable. Moreover, it is possible to make the spline more flexible by specifying knot points. Therefore, more difficult curve shapes can be dealt with efficiently. One particularly interesting possibility is to include sine

and cosine functions in order to capture cyclical (or seasonality) that is often encountered in the commodity markets. Furthermore, since the problem is linear, the standard linear regression methods can be used in estimation. By the very nature of the minimization of the squared errors, it is not possible to price market quoted swaps exactly.

- **Optimization of Fit Method**

The Optimization of Fit method is similar in many ways to the calibration methods used in the option markets. By expressing the swap prices as functions of direct forward prices and adding the condition that the following forward prices are not allowed to deviate too much from the previous forward prices, it is possible to define a well-formed problem that yields basic analytical formulas. The Optimization of Fit method handles many different curve shapes with little difference in goodness-of-fit results, and is therefore a good candidate for applied work and even for trading desks of investment banks. It is also possible to include futures in the optimization problem fairly easily or build the curve in such a way that the short-end of the curve is built using quoted futures prices and the long-end of the curve is build using the Optimization of Fit method. Hence, this method provides much respected flexibility.

Of the four methods (and their variants), two are constructed so that the market prices of swaps are recovered exactly. Therefore, it would be pointless to test the pricing efficiency of these methods; Moreover, the scarcity of data would render out-of-sample tests meaningless. Hence, the pricing efficiency tests are conducted only on the Cubic Spline method and the Optimization of Fit method.

6 Empirical Results

The empirical section of this study concentrates on comparing the pricing errors that result from fitting the forward curve to the data, using the Spline and the Optimization of Fit methods. To keep the scope of the study manageable, only the simple cubic spline is used. It is left for further studies to investigate the effect of using other functional forms, knot points and seasonality. The data used in this study consists of weekly NBSK Risi pulp data covering the period between June 1998 to October 2001 and of monthly Brent oil data covering the period between October 1997 to February 2002. The data is obtained from Nordea Bank. The interest rate data used in estimation consists of Libor rates and zero rates is retrieved from DataStream. The commodity market data is in the form of mid market prices only, so arbitrage violations are not considered in the present study. Figures 9, 10 and 11 show the evolution of the forward curves over the study period for the NBSK Risi pulp data.

For testing estimated forward curves, the fitted price errors are defined as

$$\mathbf{V}_{fx} - \hat{\mathbf{V}}_{fl} \tag{16}$$

That is, the difference between the value of the fixed price leg and the value of the floating price leg. The absolute values of the fitted price errors are then used as a criterion for the goodness-of-fit. Friedman Rank Test is used as a formal test for comparing the relative ranking of the proposed estimation methods.

Summary statistics for NBSK Risi pulp and Brent oil are shown in Tables 1 and 2. Inspection shows very clearly that there are great differences among the estimation methods in their ability to price market swaps. The sum of absolute fitted price errors is very large for the Cubic Spline method without knots. However, for the two other methods, the errors are much less significant, with the Optimization of Fit method

Figure 9: NBSKRISI PULP FORWARD CURVES ESTIMATED USING THE CUBIC SPLINE METHOD

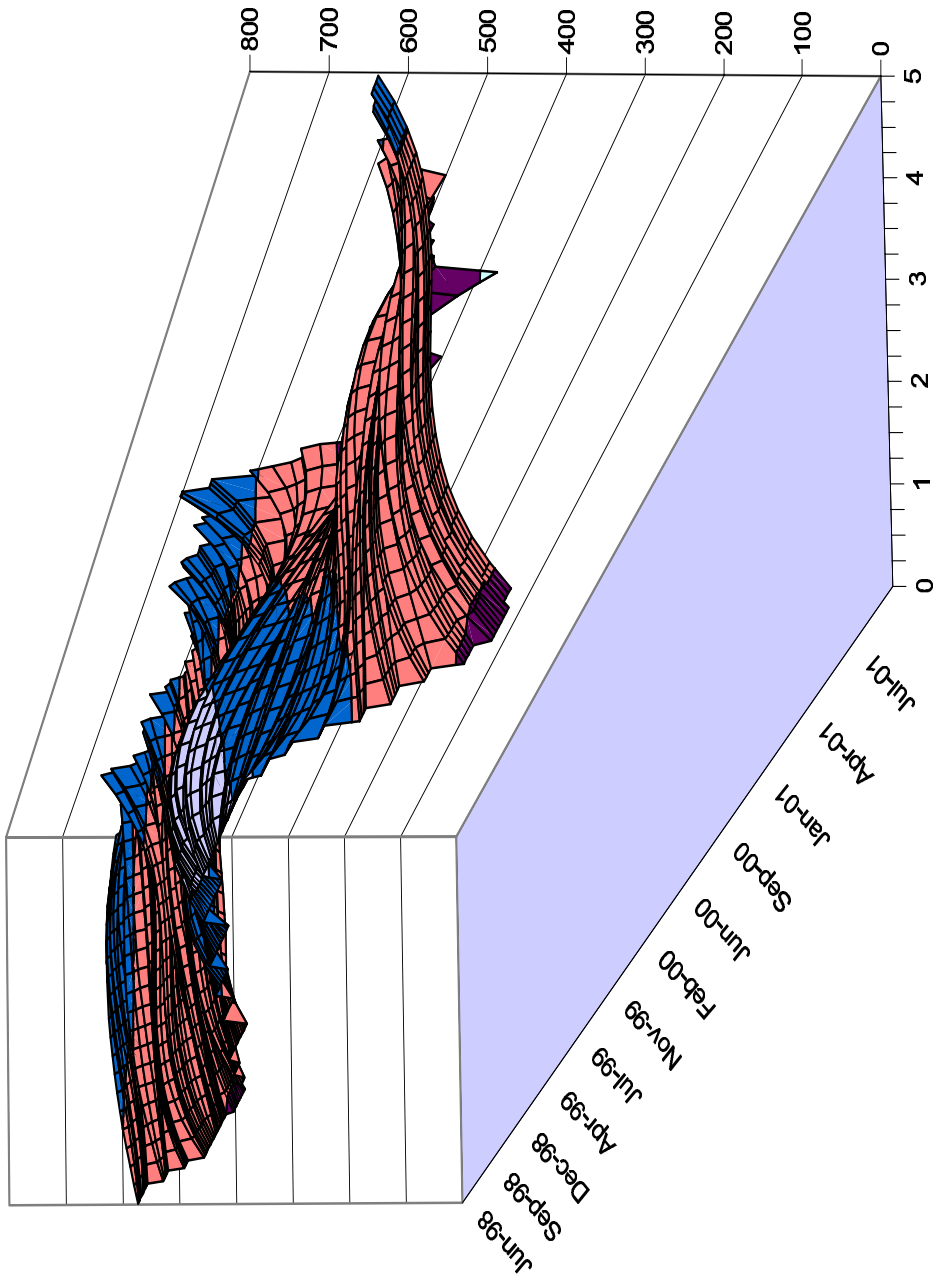


Figure 10: NBSKRISI PULP FORWARD CURVES ESTIMATED USING THE CUBIC SPLINE METHOD (with two equally spaced internal knot points)

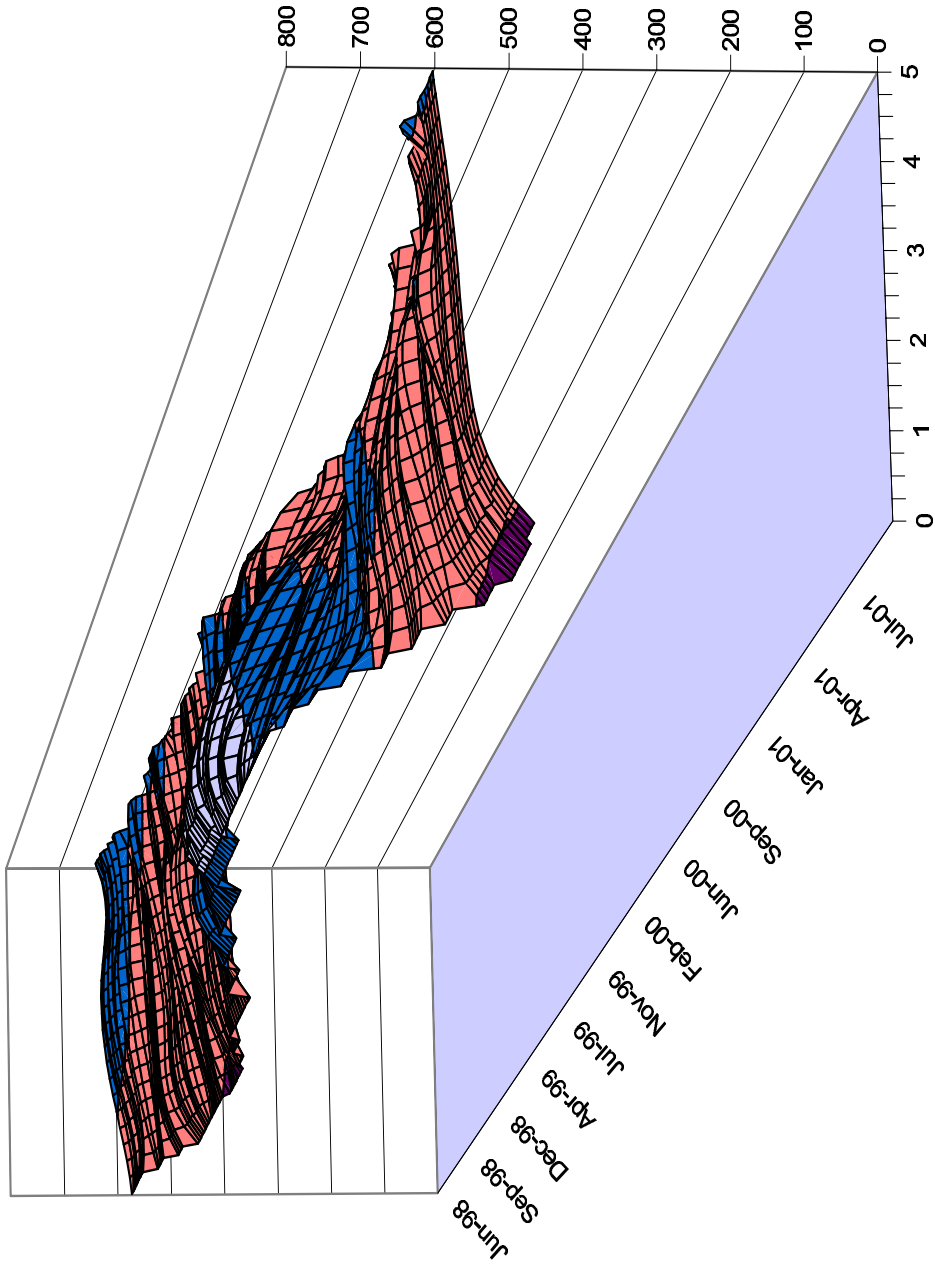


Figure 11: NBSKRISI PULP FORWARD CURVES ESTIMATED USING THE OPTIMIZATION OF FIT METHOD

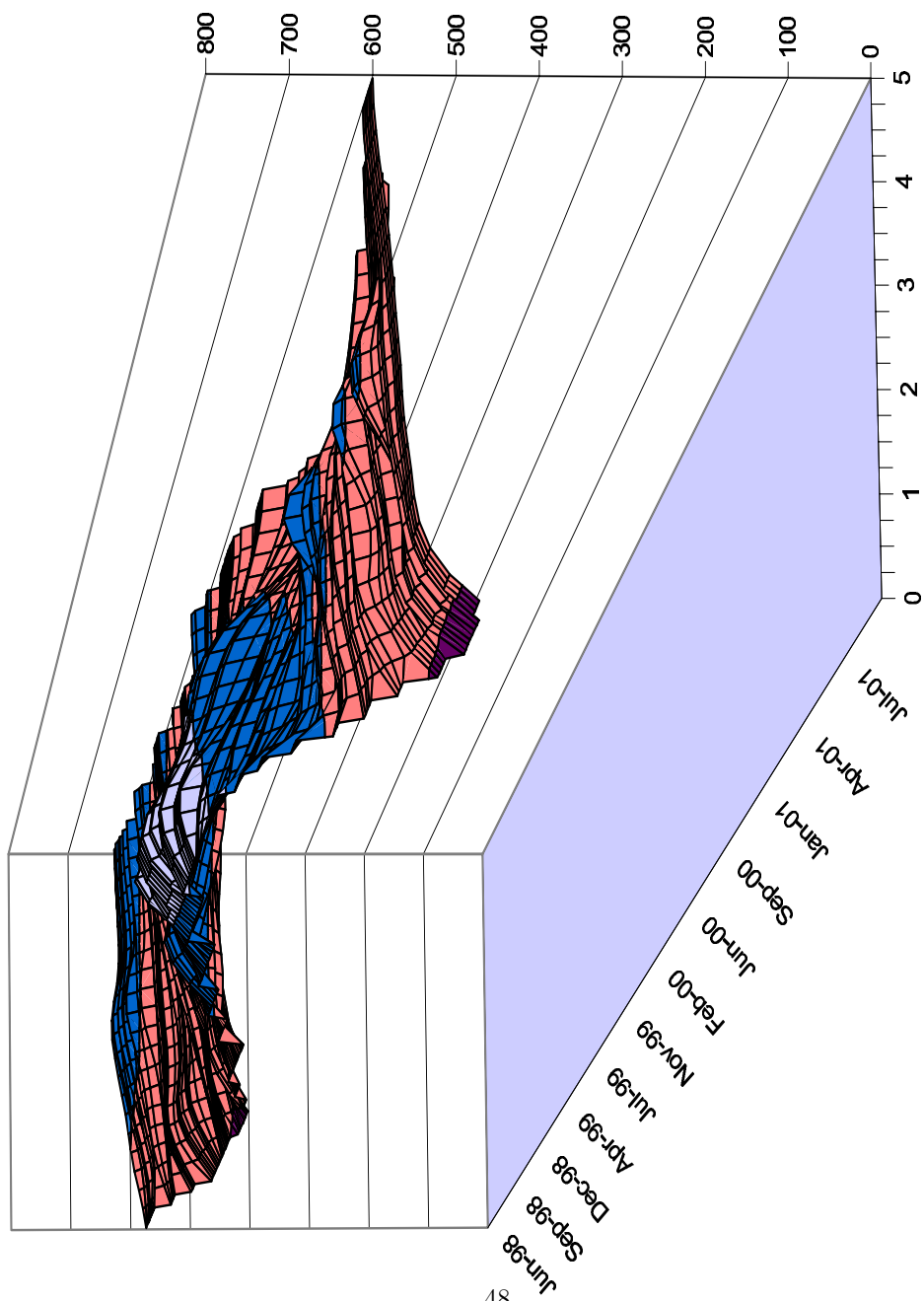


Table 1: Summary Statistics NBSK Risi Pulp

Statistic	Cubic Spline	Cubic Spline with Knots	Optimization of Fit
Mean	235.78	90.38	47.39
Standard Error	12.14	4.54	2.12
Median	199.16	77.67	39.89
Standard Deviation	161.91	60.63	28.30
Kurtosis	2.59	1.04	0.98
Skewness	1.43	1.05	1.04
Range	868.92	326.47	152.36
Minimum	14.22	6.71	5.73
Maximum	883.14	333.18	158.09
Count	178	178	178

- Reported statistics are for the Sum of Absolute Fitted Price Errors.
- Data consists of weekly observations for NBSK Risi Pulp swap quotes from June 1998 to October 2001.
- Models used are: standard Cubic Spline for direct forward prices, standard Cubic Spline for average forward prices with two internal knot points and Optimization of Fit method for direct forward prices.

giving the smallest total errors and standard deviations. Casual inspection immediately suggests that the standard Cubic Spline method without added flexibility is not suitable for estimating the forward curve for commodities.

In order to gain more insight into how the relative portions of the forward curve are fitted, the maturity spectrum is divided into five annual subperiods. The results are reported in Tables 3 and 4. Within these intervals, the absolute fitted price errors are weighted by the respective annuity factors in order to examine the effect on the actual price quotations derived from the analysis. The results indicate, once again, that the Optimization of Fit method yields implied price quotations that are reasonably close to the actual market prices that were used as inputs in the analysis. The standard Cubic Spline method, on the other hand, implies price quotes that are intolerably far from the actual market quotes: The pricing errors range from 50 cents to almost 5 dollars, with

Table 2: Summary Statistics Brent Oil

Statistic Brent oil	Cubic Spline	Cubic Spline with Knots	Optimization of Fit
Mean	13.57	4.69	4.58
Standard Error	1.14	0.44	0.43
Median	11.90	3.66	3.91
Standard Deviation	9.06	3.46	3.40
Kurtosis	0.66	0.75	0.44
Skewness	0.99	1.24	1.01
Range	37.17	13.59	14.13
Minimum	1.24	0.94	0.55
Maximum	38.41	14.52	14.68
Count	63	63	63

- Reported statistics are for the Sum of Absolute Fitted Price Errors.
- Data consists of monthly observations for Brent oil swap quotes from October 1997 to October 2002.
- Models used are: standard Cubic Spline for direct forward prices, standard Cubic Spline for average forward prices with two internal knot points and Optimization of Fit method for direct forward prices.

more severe errors observed at the short-end of the curve.

In general, it seems that the methods yield larger pricing errors at the short-end of the curve than at the long-end of the curve. This phenomenon was also observed by Bliss (1996), who studied the performance of term structure estimation methods to fit the prices of Treasury securities. The most likely reason for this is that the initial swap curves tend to be relatively flat for longer maturities but often very steeply upward or downward sloping for maturities below two years. As the long-end of the curve contains more observations than the short-end, the minimization of the squared errors will put more weight on fitting the long end correctly. A partial remedy for this would be to introduce more knots to the Spline method to make it more flexible in dealing with shorted maturities. The Optimization of Fit method could be made more flexible by making the λ -parameter time dependent, so that the smoothness penalty would have a

Table 3: Non-parametric Test of Mean Absolute Pricing Errors (NBSK Risi)

Method	0 - 1 year	1 - 2 years	2 - 3 years	3 - 4 years	4 - 5 years
Cubic Spline	4.85	2.19	1.45	0.60	0.68
Cubic Spline w Knots	4.42	0.58	0.47	0.27	0.15
Optimization of Fit	1.88	0.30	0.28	0.08	0.09

- Errors are defined as the mean error weighted with the annuity factor within the shown intervals

Table 4: Non-parametric Test of Mean Absolute Pricing Errors (Brent oil)

Method	0 - 1 year	1 - 2 years	2 - 3 years	3 - 4 years	4 - 5 years
Cubic Spline	0.438	0.123	0.069	0.031	0.032
Cubic Spline w Knots	0.246	0.047	0.47	0.008	0.004
Optimization of Fit	0.135	0.042	0.28	0.014	0.010

- Errors are defined as the mean error weighted with the annuity factor within the shown intervals

smaller effect on the short maturities. Of course, each presented method may also utilize simple weighting schemes.

The formal comparison of the relative performance of the methods is carried out using the non-parametric Friedman Rank test applied to the sum of absolute fitted price errors over the whole sample. Results are given in Table 5. For NBSK Risi pulp, the evidence strongly rejects the null of no difference between the methods. The test statistic 336.58 is significant at 1 % level. In fact, the standard Cubic Spline method without knots ranks the worst in all 178 cases, whereas the Optimization of Fit method ranks the best in all but eight cases. For Brent oil, the results are not quite so strong, in particular, there do not seem to be very big differences between the Optimization of Fit method and Spline method with knots. The test statistic for Brent oil data is 95.14, which is still significant at 1 % level. Now, the Optimization of Fit method provides the best fit in 36 cases, whereas the Spline method with knots is the best method in 27

Table 5: Non-parametric Test of Mean Absolute Pricing Errors

Statistic	Friedman Stat	Cubic Spline	Cubic Spline w Knots	Optimization of Fit
NBSK Risi Rank Sum	336.58 **	532	350	186
Brent Oil Rank Sum	95.14 **	189	99	90

* (**) indicates that the Friedman Statistic is significant at 5 % (1 %) level

- Reported statistics are for the Absolute Fitted Price Errors.
- Data consists of weekly observations for NBSK Risi Pulp swap quotes from June 1998 to October 2001 and of monthly monthly observations for Brent oil swap quotes from October 1997 to October 2002.
- Models used are: standard Cubic Spline for direct forward prices, standard Cubic Spline for average forward prices with two internal knot points and Optimization of Fit method for direct forward prices.

cases. The standard Cubic method without knots ranks worst in all cases.

In general, the performance of the fitting methods for estimation of the forward curve for NBSK Risi pulp indicates a strong preference for the non-parametric Optimization of Fit method over the Cubic Spline methods. Investigation shows that the short end of the curve is more demanding for the fitting methodologies to handle than the long end. This is true for all methods. From the data perspective, the NBSK Risi pulp data yields more demanding curve shapes overall than Brent oil data.

7 Extensions

This chapter discusses some possible extensions to the presented methods for handling forward curve estimation more flexibly. The most important extension is related to the

handling of the futures contracts as part of the curve generation process. There are, in general, two possibilities for doing this: The prices of futures contracts can be taken as an internal part of the optimization problem; alternatively the curve can, at least in some cases, be bootstrapped.

In real trading situations, the market participants have straightforward requirements for suitable forward curve derivation methodologies. First and foremost, the resulting forward curve has to be able to recover the prices of quoted swaps as accurately as possible. Usually, the errors have to be within the bid-ask spread, or else the proposed method is unsuitable and must be improved and extended. The empirical evidence shows that, in general, the Optimization of Fit method does a good job in recovering the market prices of longer term instruments. However, at times, the short end causes problems due to the often observed steep and concave form of the sequential quotes. The method can be modified if a liquid futures market exists by creating the short end of the curve from the futures price quotes and fitting only the long end of the curve using the Optimization of Fit method.

One possible extension would be to introduce Nelson and Siegel (1987) parameterization to the forward curve and investigate how well it would price the market swap quotes in different market scenarios ⁵.

8 Conclusions

Several methods for deriving the forward curve from the market prices of commodity swaps are developed. The methods can be classified into 1) bootstrapping methods and 2) fitting methods. By definition, the bootstrapping methods recover the market swap quotes exactly. However, the downside of these methods is the saw-toothed shape of the resulting forward curve. In particular, if the averaging frequency of the underlying

commodity swaps is daily, then the bootstrap method will result in a forward curve that is excessively badly shaped. By contrast, the fitting methods are able to generate smooth curves. However, the methods vary in terms of pricing performance, with all fitting methods performing worse than the Bootstrap method for recovering market swap quotes.

In addition to developing methods for forward curve estimation, two fitting methods were tested with data from pulp and oil swap markets. The evidence shows that the Optimization of Fit method is preferred to the Cubic Spline method (even with internal knot points). The Cubic Spline method performs so poorly in recovering the market prices of quoted swaps that its application to real trading situations cannot be considered. The Optimization of Fit method, by contrast, provides a reasonable alternative for traders, though its properties in different curve shape environments and extensions to the method's flexibility need to be explored further.

Extensions to the methodologies presented in this study are suggested. The most important issue is the inclusion of the futures prices into the curve building procedure, which calls for some modifications to the basic structure of the methods. In general, the most versatile way of including the futures contracts is to divide the curve into two parts where the short-end of the curve is built directly from the futures price quotes and the long-end of the curve is derived using the Optimization of Fit method. This approach provides a very appealing alternative for traders, since the liquid part is built from the most liquid instruments and the longer end is fitted to market prices of swaps. Empirical evidence suggests that the Optimization of Fit method performs very well especially in fitting the long maturity swaps. Finally, the paper presents also a simple application of the sine and cosine function in conjunction with the Cubic Spline method to introduce seasonality to the forward curve. The issue of bringing in more economic realism to the estimated forward curve is left for further studies to explore. This realism

is particularly important in the energy and agricultural markets where cyclical and seasonal components are generally very strong and identifiable.

Notes

¹The description of the bootstrap method can be found from any standard text book on fixed income markets.

²Some of the most well known traditional models include Brennan and Schwartz (1985), Gibson and Schwartz (1990) and Schwartz (1997), among many others.

³There exist many settlement procedures depending on market conventions and common practices. Mostly, the contracts are settled against an arithmetic average price and sometimes the nearby futures contract is used in place of the spot price in order to calculate the settlement price.

⁴This is to some extent analogous to the interest rate market, where the yields of benchmark bonds cannot readily be used to accurately price other bonds.

⁵This idea was taken up by Jari Käppi during a seminar session at Helsinki School of Economics

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Appendix

Implementing the optimization method requires the calculation of the gradient of the target function, i.e. equation (15). This can be done either numerically or analytically. It transpires that the derivatives of the target function are straightforward to obtain (in the calculations below; $\lambda = 1$).

$$\begin{aligned}
 \frac{\partial F_{ct}}{\partial F(t, T_i)} &= 2 \times \sum_{j=i+1}^N (V_{fl}(t, T_j) - V_{fx}(t, T_j)) \times (\gamma P(t, T_i) + (1 - \gamma)P(t, T_{i+1})) \\
 &+ 2 \times (V_{fl}(t, T_i) - V_{fx}(t, T_i)) \times \gamma P(t, T_i) \\
 &+ 2 \times (F(t, T_i) - F(t, T_{i-1})) - 2 \times (F(t, T_{i+1}) - F(t, T_i)) \\
 \frac{\partial F_{ct}}{\partial F(t, T_N)} &= 2 \times (V_{fl}(t, T_N) - V_{fx}(t, T_N)) \times \gamma P(t, T_N) \\
 &+ 2 \times (F(t, T_N) - F(t, T_{N-1}))
 \end{aligned}$$

where

$$\begin{aligned}
 V_{fl}(t, T_i) &= \sum_{j=1}^i (\gamma F(t, T_j) + (1 - \gamma)F(t, T_{j-1})) \times P(t, T_j) \\
 V_{fx}(t, T_i) &= X \sum_{j=1}^i P(t, T_j) \\
 \gamma &= (m + 1)/(2m)
 \end{aligned}$$

and m is the number of price observations on a settlement period, c.f. linear interpolation constraint. The futures contracts, f_k , can be included in the optimization by defining

$$\begin{aligned}
 \frac{\partial f_k}{\partial F(t, T_{i>k})} &= 2 \times \tau \times (\tau F(t, T_i) + (1 - \tau)F(t, T_{i-1}) - f_k) \\
 \frac{\partial f_k}{\partial F(t, T_{i<k})} &= 2 \times (1 - \tau) \times (\tau F(t, T_{i+1}) + (1 - \tau)F(t, T_i) - f_k)
 \end{aligned}$$

where τ is time (in years) from the previous forward price to the settlement date of the futures contract. Subscript k denotes the position of the futures price relative to the forward curve points. Additionally, the futures prices are assumed equal to forward prices. Therefore, the objective is to match the observed futures prices with the optimized forward price curve. As the linear interpolation again is assumed, only the forward prices

right before and after any f_k contribute to the result of the optimization.

Minimization Algorithm:

The derivatives of the target function with respect to the given set of forward prices can be utilized, for example, in the implementation of the method of steepest descent (as is done in this study). Below is a schematic description of the methodology used in this paper. This method is called a *simple gradient search*.

Choose the direction where the value of the target function, f decreases most quickly, which is in the direction opposite to the gradient, $\nabla f(\mathbf{x}_i)$. Start the search at an arbitrary point, or preferably with some reasonable initial value, and then move along the opposite direction of the gradient until the solution is reached (or is close enough). In other words, the iterative procedure is

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \lambda \nabla f(\mathbf{x}_k)$$

where λ can be also be chosen to be of optimal size; for details see Press, Teukolsky, Vetterling, and Flannery (1992). Alternatively, a constant value for λ can be used, then the method is much simpler and requires many more iterations. However, each iteration takes less time to compute and experimentation with the algorithm helps in choosing a suitable step size. Continue these iterations until $|\nabla f(\mathbf{x}_k)| < \epsilon$.

ESSAY 2:
Principal Components Analysis of Commodity Forward Curves:
Evidence from Oil and Pulp Swap Markets

Abstract

This paper investigates the factor structure of commodity forward curve dynamics using data from pulp and oil markets. The data used consists of swap contract quotes, from which forward curves are derived using an optimization algorithm. A three factor model explains 89% of the price variation of the oil forward curves and 84% of the price variation of the pulp forward curves. The factor structure, especially in pulp derivatives market, is more complex than found in many other studies conducted mainly using interest rate data. Possible reasons for this phenomenon are discussed.

The dynamics of the forward curve is important for practitioners pricing and hedging derivatives contracts and for economists studying stochastic movements of economic variables. Traditionally, modelling of the yield curve movements has been an active area of research, both theoretically and empirically. By contrast, research on commodity forward curve dynamics has been relatively scarce. This paper contributes to the research on commodity forward curve dynamics by presenting empirical evidence on the factor structure of long term forward prices estimated from the broker quotes of the swap contracts.

Analogously to the study of the interest rates, there has been two approaches to the study of commodity prices. The traditional method has focused on modelling the stochastic process for the spot price of the commodity and possibly other state variables, such as the convenience yield. Seminal research along these lines include Brennan and Schwartz (1985), Gibson and Schwartz (1990) and Schwartz (1997). More recently, the study of the stochastic movements in commodity prices has concentrated on modelling the whole term structure of either forward prices directly, Cortazar and Schwartz (1994), or convenience yields, see for example Miltersen and Schwartz (1998). This current paper builds on the whole term structure modelling principle.

Earlier research on the commodity forward curve dynamics has concentrated on modelling the factor structure of the short term contracts, i.e. exchange traded futures. Among the earliest of this research was Cortazar and Schwartz (1994), who studied the dynamics of the futures price curve constructed from copper futures. They found that the factor structure of the copper futures curve was surprisingly similar to the factor structure of yield curve movements. Moreover, the explanatory power of the two most important principal components was 97 percent. By contrast, Litterman and Scheinkman (1988) found that the explanatory power of the two most important principal components of the yield curve movements was remarkably lower, at around 90

percent. Clewlow and Strickland (2000) studied the factor structure of NYMEX oil futures and they found that three factors explained over 98 percent of the variation of the futures price movements during the period between 1998 to 2000. A recent paper by Tolmasky and Hindanov (2002) investigated the dynamics of the petroleum futures contracts. In particular, they focused on isolating the effect of seasonality in the factor structure of returns. They found that, especially for heating oil, seasonality is an important variable driving the factor structure, however it's statistical significance is somewhat unclear. Crude oil and petroleum markets were not found to be affected by seasonality. In a closely related line of research, Koekebakker and Ollmar (2001) studied the forward curve dynamics using data from the Nordpool electricity derivatives exchange. The explanatory powers they report are fairly low in comparison, the most likely reason being the extremely complex dynamics of the electricity spot and forward prices. Koekebakker and Ollmar (2001) use fitted curves as is also done in this current paper.

The results obtained in this study extend the results of the empirical investigations of the forward curve movements by presenting evidence using data from two distinct commodity markets. In particular, the focus is on modelling the dynamics of the long term forward prices that are implied, not actually observed, from the market quotes of the swap contracts. Using the swap market data significantly complicates the analysis and the results obtained are also less clear cut than those obtained using the data on the short term futures contracts. The data used consists of Brent crude oil swap quotes up to five years, covering the period between 1997 to 2002, and NBSK Risi pulp swap quotes, also up to five years, covering the period between 1998 to 2001. The former data is in the form of monthly observations and the latter is weekly sampled. Principal Components Analysis (PCA) applied to the implied forward curve movements reveals complex factor structures and the explanatory power of the first three principal components is around 89 percent for oil data and 84 percent for pulp data.

The paper is organized as follows. Section 1 introduces the underlying theoretical model. Section 2 presents and discusses the data used in the study. Section 3 introduces the method used for estimating the forward curves and Section 4 gives a description of the Principal Components Analysis method. In Section 5, the empirical results are discussed and comparisons are made with results obtained in earlier studies. Finally, Section 6 concludes the paper and gives suggestions for further research.

1 Model

The model studied here is similar to the model proposed by Reisman (1991) and Cortazar and Schwartz (1994). The main idea is to model the movements of the forward curve directly, instead of modelling the forward prices as a function of the spot price and convenience yield processes. Utilizing this approach, the principal components analysis is applied to the movements of the whole extracted term structure of the forward prices. The procedure for obtaining the term structures of forward prices is explained in the next section.

In the analysis that follows, the spot price of a commodity, at time t , is denoted by $S(t)$. A futures contract at time t , for delivery at time T , is denoted $F(t, T)$. The underlying market is assumed to be complete and frictionless. In the absence of arbitrage opportunities there exists an equivalent martingale measure \mathbb{Q} , under which all discounted asset prices are martingales (Harrison and Kreps (1979)).¹ This implies that, under the risk-neutral pricing measure, the instantaneous expected return on all financial assets is the instantaneous risk free rate and hence, the expected return on futures contracts is equal to zero. Therefore, the stochastic process for the futures (and forward) prices is given by

$$dF(t, T) = \sum_{j=1}^K \sigma_j(t, T) F(t, T) dW_j(t) \quad (1)$$

and in integrated form

$$F(t, T) = F(0, T) \exp \left(-\frac{1}{2} \sum_{j=1}^K \int_0^t \sigma_j^2(u, T) du + \sum_{j=1}^K \int_0^t \sigma_j(u, T) dW_j(u) \right) \quad (2)$$

where dW_1, dW_2, \dots, dW_K are independent increments of Brownian motions under the risk-neutral measure, and $\sigma_j(t, T)$'s are the volatility functions of the futures (or forward) prices. The volatilities are only functions of time to maturity ($T - t$). The analysis presented here is based on the assumption of constant interest rates, and therefore, the futures prices and forward prices are equivalent as shown by Cox, Ingersoll, and Ross (1981).

2 Data

The forward price model (1) describes the stochastic evolution of each of the forward prices along the forward curve under an equivalent martingale measure. However, observations are taken under the real world measure. This is not a problem, since only volatility functions are of interest here, and they are invariant with respect to the measure change ². Let $F(t_i, T_j)$ denote the forward price at time t_i , with maturity T_j ($t < T$), for all $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, M$. The instantaneous proportional change in the forward price is approximated by

$$\frac{dF(t_i, T_j)}{F(t_i, T_j)} = \frac{F(t_i, T_j) - F(t_{i-1}, T_j)}{F(t_{i-1}, T_j)} = x_{i,j} \quad (3)$$

The data set $\mathbf{X}_{(N,M)}$ is a matrix of returns (3) constructed from the fitted forward price curve and swap price curve

$$\mathbf{X}_{(N,M)} = \begin{pmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,M} \\ x_{2,1} & x_{2,2} & \dots & x_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N,1} & x_{N,2} & \dots & x_{N,M} \end{pmatrix} \quad (4)$$

The matrix entries are obtained in the following way. First, the forward curves are derived from the swap price data using Järvinen (2002). The forward curves are then used to find the individual forward price observations. From these observations, the percentage differences are calculated. This procedure is replicated until the whole matrix is filled.

The data used in this study consists of weekly NBSK Risi pulp data covering the period between June 1998 to October 2001 and of monthly oil data (European Brent) covering the period between February 1997 to February 2002. The data was obtained from Nordea Bank Finland. The interest rate data used in estimation consists of Libor rates and zero rates. The European interest rate data consists of EuroLibor quotes and zero rates until 1.1.1999; from that date onwards the data consists of Euribor quotes and zero rates. The interest rate data was retrieved from DataStream. The commodity data consists of mid market quotes. Tables 1 and 2 report the summary statistics for NBSK Risi pulp forwards and swaps, and Tables 3 and 4 report the summary statistics for Brent oil forwards and swaps.

Table 1: NBSK Risi Forwards Summary Statistics

Statistic	Index	6m	1y	2y	3y	5y
Mean	-0.0009	-0.0004	-0.0004	-0.0004	-0.0004	0.0000
Standard Deviation	0.0167	0.0109	0.0116	0.0098	0.0109	0.0184
Kurtosis	9.2818	1.9003	2.2834	1.7928	1.5060	4.7249
Skewness	-0.4417	0.6824	0.4424	0.0452	0.3460	0.2415
Range	0.1495	0.0688	0.0778	0.0646	0.0656	0.1514
Minimum	-0.0755	-0.0306	-0.0327	-0.0324	-0.0305	-0.0555
Maximum	0.0741	0.0382	0.0451	0.0323	0.0351	0.0958
Count	177	177	177	177	177	177

Table 2: NBSK Risi Swaps Summary Statistics

Statistic	Index	6m	1y	2y	3y	5y
Mean	-0.0009	-0.0006	-0.0005	-0.0005	-0.0005	-0.0004
Standard Deviation	0.0167	0.0099	0.0101	0.0080	0.0073	0.0059
Kurtosis	9.2818	1.8828	2.4779	1.7625	0.9910	2.2131
Skewness	-0.4417	-0.2175	0.4850	0.0794	-0.0438	0.3648
Range	0.1495	0.0626	0.0672	0.0537	0.0382	0.0343
Minimum	-0.0755	-0.0333	-0.0289	-0.0279	-0.0188	-0.0169
Maximum	0.0741	0.0293	0.0383	0.0258	0.0194	0.0175
Count	177	177	177	177	177	177

Table 3: Brent Forwards Summary Statistics

Statistic	Index	6m	1y	2y	3y	5y
Mean	0.0062	0.0024	0.0021	0.0051	0.0025	0.0008
Standard Deviation	0.1255	0.0650	0.0578	0.0547	0.0491	0.0630
Kurtosis	0.5542	0.4751	1.7357	1.3674	0.4552	0.5037
Skewness	0.3486	-0.1424	0.6150	0.5294	0.1060	0.5465
Range	0.6924	0.3386	0.3400	0.2950	0.2471	0.3165
Minimum	-0.3059	-0.1754	-0.1299	-0.1239	-0.1235	-0.1340
Maximum	0.3866	0.1631	0.2101	0.1712	0.1236	0.1825
Count	62	62	62	62	62	62

Table 4: Brent Swaps Summary Statistics

Statistic	Index	6m	1y	2y	3y	5y
Mean	0.0062	0.0031	0.0018	0.0023	0.0024	0.0016
Standard Deviation	0.1255	0.0709	0.0645	0.0434	0.0383	0.0390
Kurtosis	0.5542	1.3295	1.0097	0.3944	0.3283	0.4677
Skewness	0.3486	0.2551	-0.4186	0.0352	0.0610	0.1655
Range	0.6924	0.4248	0.3501	0.2161	0.1887	0.1889
Minimum	-0.3059	-0.1959	-0.2063	-0.1099	-0.0846	-0.0859
Maximum	0.3866	0.2290	0.1439	0.1062	0.1041	0.1030
Count	62	62	62	62	62	62

3 Curve Estimation

The first step in applying the principal analysis to the forward curve movements, is to estimate the forward curves from the par swap quotes. To introduce the idea, we consider a standard commodity swap contract, with maturity T_N and a fixed contract price $G(t, T_N)$. The present value of this fixed leg of the swap, given the discount bond prices $P(t, T)$ is

$$V_{fx}^N = G(t, T_N) \sum_{i=1}^N P(t, T_i) \quad (5)$$

i.e. a simple sum of the present values of the payments. The other side of the swap contract, the floating leg, is composed of the present values of the unknown forward prices of the commodity at settlement dates, $i = 1, 2, \dots, N$. That is,

$$V_{fl}^N = \sum_{i=1}^N F(t, T_i) P(t, T_i) \quad (6)$$

where forward prices may be based on average or single observation settlement values. If we know the forward price curve, the par swap prices for all maturities, $t \geq T$, can be obtained by setting the values of the floating and fixed legs equal. Solving for the par

swap price $G(t, T_N)$ gives

$$G(t, T_N) = \frac{\sum_{i=1}^N F(t, T_i) P(t, T_i)}{\sum_{i=1}^N P(t, T_i)} \quad (7)$$

which shows that the par swap price is defined as the ratio of the discounted value of the floating price payments and an annuity. Therefore, analogously with interest rate swaps, the par swap price can be interpreted as a present value weighted sum of forward prices.

The method for extracting $F(t, T)$'s is based on the Optimization of Fit method, introduced by Järvinen (2002). The method is applied by minimizing the sum of squared pricing errors, which is an analogous idea to the option model calibration to observed implied volatilities. The aim is to fit the observed swap prices as precisely as possible while maintaining a reasonable degree of smoothness in the output forward curve. The target function of the minimization problem is

$$\min_{\{F_i\}_{i=1}^N} \left[\sum_{i=1}^N (V_{fl}(t, T_i) - V_{fx}(t, T_i))^2 + \lambda \sum_{i=1}^N (F(t, T_i) - F(t, T_{i-1}))^2 \right] \quad (8)$$

where parameter λ is introduced to adjust the weight put on the forward curve smoothness. The forward prices that are the solution of the problem are direct forward prices, and not average-based. ³ For further of this procedure, see Järvinen (2002). The optimization problem can be solved using any suitable algorithm. The method utilized in this paper is based on a simple gradient search using analytic derivatives. For explanation on the various optimization algorithms, see Press, Teukolsky, Vetterling, and Flannery (1992).

In choosing λ , two points need to be mentioned. First, if one picks $\lambda = 0$, then the result of the optimization will converge to the bootstrapped curve and the target

function value will be equal to zero. In other words, an exact match can be obtained using this method. On the other hand, if a high enough λ is chosen, then the final forward curve will be flat, starting from the spot value $S(t)$. Obviously, a reasonable value from λ is such that a smooth, yet reasonably shaped and correctly pricing forward curve will result. For the purposes of this study, λ is set equal to one. The decision is based on experimenting with the algorithm, not explicitly on any measurable criteria.

4 Principal Components Analysis

Principal Components Analysis (PCA) is a statistical tool used to identify a structure within a set of interrelated variables. Applying (PCA) to the data, the number of orthogonal factors and the corresponding volatility coefficients (assuming that volatility functions depend only on the time to maturity (T-t)) can be estimated. The data consists of N observations of M different variables, i.e. percentage differences calculated from curve points. Hence, the data matrix is given by (4).

The sample covariance matrix is of order M , and is denoted by ω . The orthogonal decomposition of the covariance matrix is given by

$$\mathbf{\Omega} = \mathbf{C}\mathbf{\Lambda}\mathbf{C}' \quad (9)$$

where

$$\mathbf{C} = [\mathbf{c}_1, \mathbf{c}_2 \dots \mathbf{c}_M] = \begin{pmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,M} \\ c_{2,1} & c_{2,2} & \dots & c_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N,1} & c_{N,2} & \dots & c_{N,M} \end{pmatrix} \quad (10)$$

and

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_{1,1} & 0 & \dots & 0 \\ 0 & \lambda_{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{M,M} \end{pmatrix} \quad (11)$$

The $\mathbf{\Lambda}$ -matrix is a diagonal matrix and the diagonal elements are the eigenvalues $\lambda_{1,1}, \lambda_{2,2}, \dots, \lambda_{M,M}$. \mathbf{C} is an orthogonal matrix of order M . The columns of \mathbf{C} are the eigenvectors corresponding to $\lambda_{j,j}$. \mathbf{C}' denotes the transpose of \mathbf{C} . The matrix $\mathbf{P} = \mathbf{XC}$ is the matrix of principal components. The columns \mathbf{p}_j of \mathbf{P} are linear combinations of the columns of \mathbf{X} , i.e. the j :th principal component is

$$\mathbf{p}_j = \mathbf{Xc}_j = \mathbf{x}_1c_{1j} + \mathbf{x}_2c_{2j} + \dots + \mathbf{x}_M c_{Mj} \quad (12)$$

In order to explain all the variance in the sample, one needs to use all M principal components. However, the main idea behind using (PCA) is to reduce the dimensionality of the data. Therefore, the covariance structure is approximated by using $K < M$ largest eigenvalues. The larger the proportion of the explained variance, the better the objective is achieved. The criteria for selecting K is somewhat ambiguous, since there exists no clearcut statistical criterion for selecting significant eigenvalues, and hence, the number of factors ⁴. Therefore the conventional methodology, employed in the majority of the finance literature, is to add factors until the cumulative explained variance reaches a specified limit; this procedure is also used in this study. In addition, financial derivatives multifactor models usually aim at constructing a model that contains from two to four driving factors. The proportion of the total sample variance explained can be found using the following formula

$$CEV_K = \frac{\sum_{j=1}^K \lambda_j}{\sum_{j=1}^M \lambda_M} \quad (13)$$

Where CEV_K denotes *Cumulative explained variance of first K factors*.

Table 5: NBSK Risi Fwds Correlations

	Index	6y	1y	2y	3y	5y
Index	1	-0.01	-0.09	-0.06	-0.06	0.02
6m	-0.01	1	0.85	0.06	-0.01	0.01
1y	-0.09	0.85	1	0.17	-0.05	-0.12
2y	-0.06	0.06	0.17	1	0.64	-0.36
3y	-0.06	-0.01	-0.05	0.64	1	-0.03
5y	0.02	0.01	-0.12	-0.36	-0.03	1

Table 6: NBSK Risi Swap Correlations

	Index	6m	1y	2y	3y	5y
Index	1	0.48	0.23	0.15	0.07	0.07
6m	0.48	1	0.62	0.62	0.4	0.47
1y	0.23	0.62	1	0.83	0.6	0.51
2y	0.15	0.62	0.83	1	0.73	0.6
3y	0.07	0.4	0.6	0.73	1	0.67
5y	0.07	0.47	0.51	0.6	0.67	1

5 Empirical Results

Tables 9 and 10 show the results from the PCA analysis applied to both data sets. Empirical results are markedly different, a one factor model is able to explain 62 percent of the variation of returns of the Brent forward prices, whereas it can explain only 38 percent of the returns in the case of NBSK Risi prices. The explanatory power of one factor model, in particular for NBSK Risi pulp, is very low. The most likely reason for this is the stickiness of the reference index price. NBSK Risi index prices are published only monthly, whereas the swap quotes change, in principle, daily, and in most cases at least weekly. Inspection of the empirical correlation matrices (see Tables 5 and 6) reveal that movements of the NBSK Risi pulp prices are almost non correlated (particularly in the short term) whereas correlations of oil price movements are clearly stronger (Tables 7 and 8).

Table 7: Brent Fwd Correlations

	Index	6m	1y	2y	3y	5y
Index	1	0.36	0.14	-0.25	-0.13	-0.13
6m	0.36	1	0.84	-0.03	0.08	-0.02
1y	0.14	0.84	1	0.26	0.33	0.17
2y	-0.25	-0.03	0.26	1	0.86	0.71
3y	-0.13	0.08	0.33	0.86	1	0.91
5y	-0.13	-0.02	0.17	0.71	0.91	1

Table 8: Brent Swap Correlations

	Index	6m	1y	2y	3y	5y
Index	1	0.78	0.63	0.47	0.3	0.16
6m	0.78	1	0.87	0.74	0.53	0.33
1y	0.63	0.87	1	0.88	0.68	0.46
2y	0.47	0.74	0.88	1	0.93	0.77
3y	0.3	0.53	0.68	0.93	1	0.93
5y	0.16	0.33	0.46	0.77	0.93	1

If more factors are used, then the difference in explanatory power of PCA decreases considerably. A two factor model is able to explain 81 and 63 percent of variation in price returns and a three factor model can explain 89 and 84 percent. Using the eigenvalue criterion that only eigenvalues greater than one are considered significant, the analysis would select a three factor model for Brent and a four factor model for NBSK Risi pulp. For practical modelling purposes, 90 percent has often been considered a minimal threshold. If this criteria is used, then both data sets need a four factor model to explain enough variation of returns. The plotting of eigenvalues for NBSK Risi pulp and Brent oil forward prices are shown in Figures 1 and 2.

Inspection of the factor loadings, see Figures 3 and 4 (Tables 11 and 12), provides more insight on how the dynamics of the forward curves are determined. The most striking finding of the analysis is that contrary to the factor structure found in many other

Table 9: NBSK Risi Fwd Eigenvalues

Eigenval	Variance (% total)	Cumul Eigenval	Cumul %
7.610	38.051	7.610	38.051
5.001	25.006	12.611	63.057
4.278	21.390	16.889	84.447
1.656	8.280	18.545	92.727
0.669	3.346	19.215	96.073
0.108	0.539	19.322	96.612
0.058	0.289	19.380	96.901
0.048	0.240	19.428	97.141
0.048	0.240	19.476	97.380
0.048	0.238	19.524	97.619

Table 10: Brent Fwd Eigenvalues

Eigenval	Variance (% total)	Cumul Eigenval	Cumul %
12.378	61.888	12.378	61.888
3.842	19.210	16.220	81.098
1.627	8.135	17.847	89.233
0.932	4.659	18.778	93.892
0.330	1.651	19.108	95.542
0.217	1.083	19.325	96.625
0.053	0.266	19.378	96.891
0.050	0.248	19.428	97.139
0.048	0.241	19.476	97.380
0.048	0.239	19.524	97.619

Figure 1: Plot of Eigenvalues (NBSK Risi Fwd)

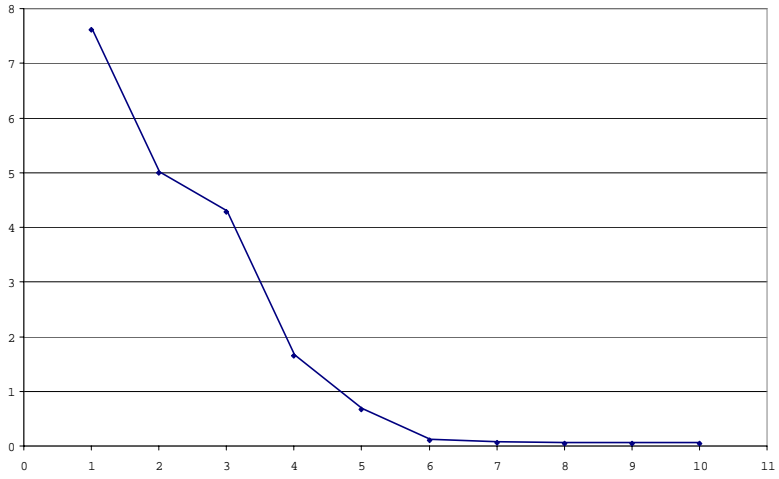
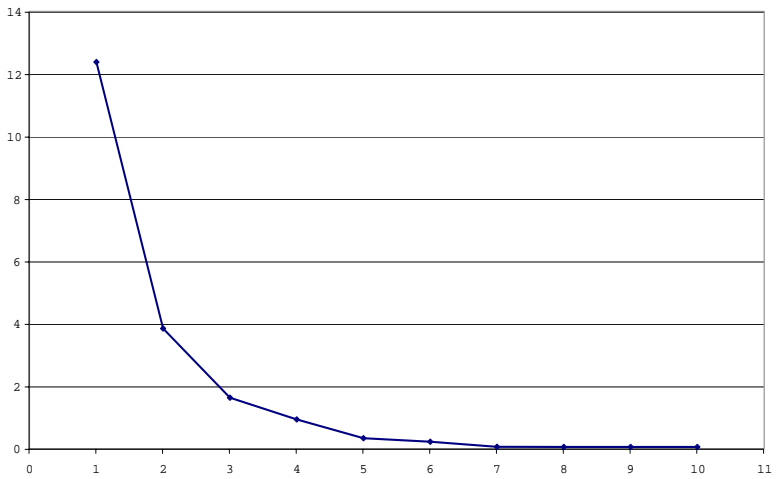


Figure 2: Plot of Eigenvalues (Brent Fwd)



studies, see for example Litterman and Scheinkman (1988) and Cortazar and Schwartz (1994), there seems to be no obvious level, slope and curvature factors. By contrast, the factors obtained here exhibit much more complex shapes. There is no general factor that would change the forward prices equally across the whole term structure. For example, in the case of NBSK Risi, a shock to the first factor changes the very short term forward prices upwards, albeit marginally, and the medium term forward prices downwards and finally, the long term forward prices upwards. An interesting note is also that only the first factor is important in explaining the variation in the very long forward prices whereas all four factors have an important impact in explaining the movements in the short end of the forward curve. The conclusion from the analysis of the factor loadings is that when long term contracts are included in the estimation of the dynamics of the forward curve, the complexity of curve changes increases dramatically. Possible explanations for these results could be the strongly mean-reverting nature of the commodity prices and the complex interplay between future demand and supply expectations and also the possibility of storing these commodities for speculating or hedging. Another possibility is that there are great differences in the liquidity of the contracts of different maturities, and that the reference index value changes non-synchronously with the prices of the swap contracts. It is likely that both these factors have an effect on the estimation results.

Table 11: NBSK Risi Fwd Factor Loadings

	Factor 1	Factor 2	Factor 3	Factor 4
3 month	0.1286	0.4087	-0.4804	-0.3098
6 month	0.0461	0.5391	-0.6561	-0.4472
9 month	-0.0444	0.5104	-0.6815	-0.4333
12 month	-0.0992	0.5437	-0.7116	-0.2477
15 month	-0.1437	0.6316	-0.6914	0.1920
18 month	-0.1907	0.6210	-0.4666	0.5726
21 month	-0.3033	0.6493	-0.1704	0.6459
24 month	-0.4914	0.7227	0.2408	0.3678
27 month	-0.5327	0.6310	0.5266	-0.0289
30 month	-0.4893	0.5516	0.6181	-0.1880
33 month	-0.4078	0.5487	0.6616	-0.2374
36 month	-0.1616	0.6409	0.6822	-0.2434
39 month	0.4938	0.6775	0.4853	-0.1397
42 month	0.8639	0.4303	0.1604	0.0042
45 month	0.9337	0.2831	0.0601	0.0524
48 month	0.9520	0.2124	0.0559	0.0628
51 month	0.9592	0.1691	0.0739	0.0635
54 month	0.9624	0.1388	0.0891	0.0630
57 month	0.9639	0.1200	0.0967	0.0630
60 month	0.9643	0.1135	0.0987	0.0632
Expl.Var	7.6102	5.0011	4.2780	1.6561
Prp.Totl	0.3805	0.2501	0.2139	0.0828

Table 12: Brent Fwd Factor Loadings

	Factor 1	Factor 2	Factor 3	Factor 4
3 month	-0.1198	-0.5095	0.5732	-0.5889
6 month	-0.0133	-0.8371	0.3798	-0.3023
9 month	0.1552	-0.9329	0.0182	0.1437
12 month	0.2625	-0.8840	-0.1496	0.3018
15 month	0.4091	-0.8136	-0.2359	0.2296
18 month	0.6779	-0.5946	-0.3486	-0.0340
21 month	0.8166	-0.1628	-0.4142	-0.2615
24 month	0.8263	0.0900	-0.3969	-0.2973
27 month	0.8764	0.1521	-0.3237	-0.2454
30 month	0.9237	0.1429	-0.1991	-0.1520
33 month	0.9266	0.1518	-0.0978	-0.0732
36 month	0.9329	0.1509	-0.0256	-0.0258
39 month	0.9578	0.0879	0.0582	0.0076
42 month	0.9644	-0.0006	0.1500	0.0383
45 month	0.9511	-0.0266	0.2126	0.0646
48 month	0.9389	0.0060	0.2484	0.0873
51 month	0.9280	0.0636	0.2702	0.1076
54 month	0.9153	0.1209	0.2859	0.1252
57 month	0.9023	0.1649	0.2982	0.1391
60 month	0.8926	0.1905	0.3068	0.1480
Expl.Var	12.3776	3.8420	1.6270	0.9317
Prp.Totl	0.6189	0.1921	0.0814	0.0466

Figure 3: Plot of Factor Loadings (NBSK Risi Fwd)

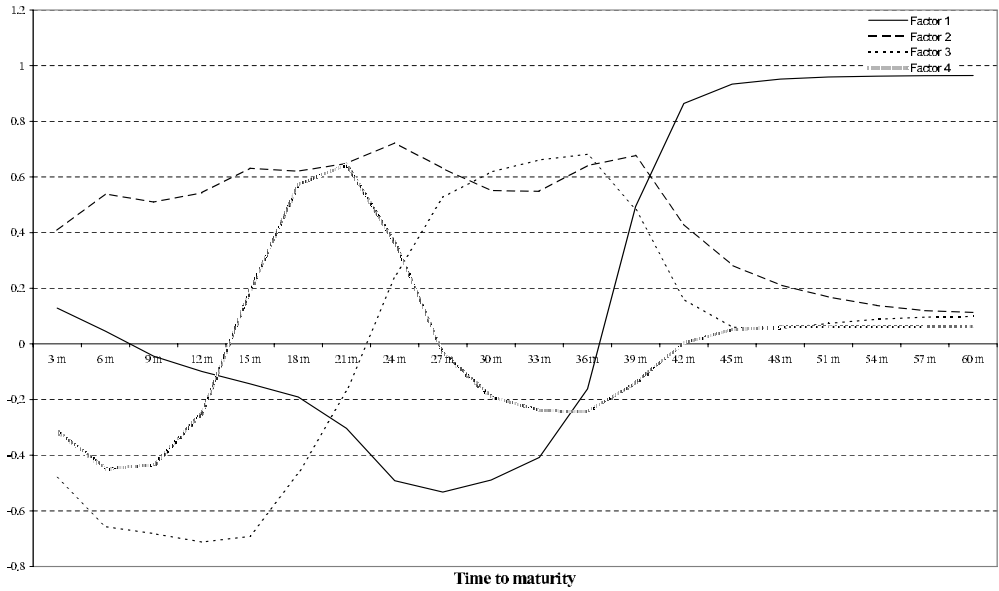
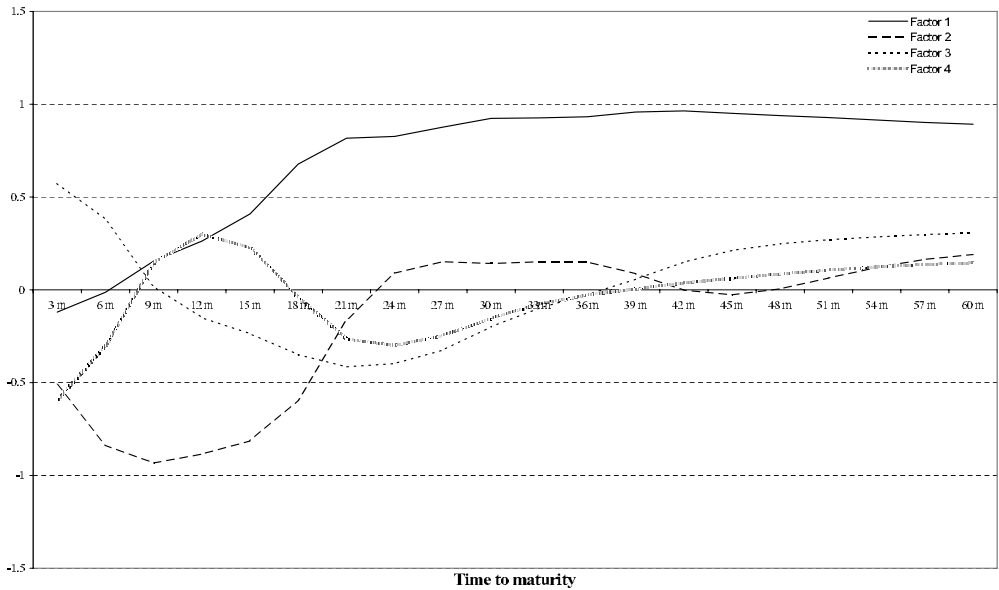


Figure 4: Plot of Factor Loadings (Brent Fwd)



6 Conclusions

This paper has investigated the dynamics of the commodity forward price curves using the principal components analysis on the Brent oil and NBSK Risi pulp data. The forward curves have been estimated using the fitting method, presented in Järvinen (2002). The data consists of broker swap quotes from 1998 to 2002 for Brent and from 1997 to 2002 for NBSK Risi. The longest maturity swaps are five years. The principal component analysis was conducted on the percentage changes of the forward prices from the fitted forward curves.

The main findings of the study are: At least three and even four factors are needed in order to adequately model the dynamics of the forward curves for both data series. Interestingly, the factor structures of the markets analyzed are markedly different, and furthermore do not resemble the structures found in earlier studies: level, slope and curvature. By contrast, the results derived in this paper reveal that the short-end of the curve is very demanding from the modelling perspective. In contrast, the long-end of the curve exhibited simpler dynamics. Possible reasons for these findings include: complex demand, supply and storage dynamics leading to non-synchronously moving forward prices, in particular, in the short-end of the curve; liquidity and reliability of the swap quotes from which the forward curves are derived; and stickiness of the reference index quotes that provide the pseudo spot index value for the fitting algorithm. Finally, the results are naturally dependent on the forward curve estimation algorithm. Since the forward curve points cannot be estimated uniquely from the swap data, a fitting method is necessary in order to obtain the forward curves.

This paper provides the first documented results of applying PCA to model the dynamics of the commodity forward curves using long term swap contract data. Earlier studies have used readily available futures data. As the OTC market for long term commodity derivative contracts is growing rapidly, more research is needed in order to

investigate the statistical behavior of the prices of these contracts. As new data arrives at an expanding rate, this provides the academic community fruitful opportunities to investigate the subject further. Moreover, since the futures markets are often very liquid and usually proxy the spot price better than reference index values, future studies might benefit from integrating these two markets in order to study the term structure dynamics.

Notes

¹The equivalent martingale measure is commonly called the risk-neutral measure, in particular in the applied research.

²This is strictly correct only in continuous time

³By direct forward prices we mean forward prices that apply to single settlement date values. On the contrary, average forward prices contain resets from the start of the forward period to the end of the forward period.

⁴Criteria discussed in the literature includes: 1) Scree plot test: the test is carried out by graphical inspection of the eigenvalue plot; eigenvalues are added until the plot levels off. 2) Eigenvalue criterion: eigenvalues that are greater than one are considered significant.

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ESSAY 3:

Simple Binomial Model for Bermudan Swaptions

Abstract

This paper presents a model for pricing Bermudan swaptions in a standard binomial tree. In particular, we adopt the equal probability specification of Jarrow and Rudd (1983). The model is extremely simple and efficient to implement, but the most obvious gain comes from its intuitive nature and its ability to incorporate market information without calibration. The methodology involves the computation of recombining Markov lattices for every floating leg underlying the swaption. An option tree is constructed and used to value a Bermudan swaption. We give examples of applying the model to price interest rate and commodity swaptions.

A BERMUDAN SWAPTION is an option which, at each predetermined date in the exercise schedule, gives the owner the right to enter into a swap, either to pay fixed (payer swaption) or receive fixed (receiver swaption). The importance of Bermudan type swaptions arises from the widespread use of call features in the issuance of corporate and government bonds. Bermudan swaptions provide hedges for these bonds and the Bermudan swaption pricing models can also be used to estimate the value of the call provision for the benefit of both the issuer and the investor.

The most popular models for pricing Bermudan swaptions are low-dimensional (typically one-factor, sometimes two factors). Familiar models include classics such as BDT Black, Derman, and Toy (1990) and HW Hull and White (1990) among many others. The popularity of these well-known one-factor models comes from their low-cost implementation in the binomial (or trinomial) lattices. Moreover, lattices are generally preferred to Monte Carlo methods in the pricing tasks that involve estimation of the early-exercise premium, since using lattices this can be done in a straightforward fashion¹. It is often claimed that the convenience of one-factor models comes at a cost, since we are restricted to evolve the whole spectrum of rates using only one driving Brownian motion. This severely limits the possible shapes that the yield curve may obtain in the future and also forces the yields to move in tandem. Whether this is a practically relevant, restrictive feature or not is still to be proved. Empirical evidence is, to date, weak and mixed.

In the implementation of the models above, the BDT model comes closest to the market practice, since under this model the interest rates follow the lognormal law. However, there is a drawback: The nodes in the binomial lattice have to be solved numerically, since analytical formulae do not exist². Furthermore, it is not enough to build the tree until option expiry. The tree has to be evolved up until the maturity of the underlying swap. By contrast, the HW model is analytically more tractable, but

problems arise due to underlying Gaussian law for the rates. The normal distribution of the interest rates is not what the market assumes and, moreover, it follows that the rates can be negative at positive probability in the HW model. Finally, the calibration to market data is less transparent than in the BDT model, since the volatilities of the rates in HW are absolute, not proportionate to the level of rates as the market implicitly assumes.

To introduce a more realistic framework for the modeling of yield curves, many authors have proposed the so-called market models Miltersen, Sandmann, and Sondermann (1997) and Brace, Gatarek, and Musiela (1997), LMM's (Libor Market Models) and their implementation specific variations. These models describe the evolution of the market quantities (Libor rates), rather than the instantaneous rates of preceding models. The change of modelling standpoint allows for lognormal forward Libor rates. This assumption coincides with the market practice of pricing caps. Therefore, interest rate caps can be priced using an analytical formula and, hence, calibration to market data is straightforward. The purpose of the LMM's is to model the yield curve in a consistent way, using multiple factors to better capture imperfect correlation properties of forward rates. The inclusion of many factors makes it harder to implement tree methods. Therefore, Monte Carlo techniques must be used instead. This field of research has grown tremendously during the past decade, see for example Broadie and Glasserman (1997), Barraquand and Martineau (1995), Carr and Yang (1997), and Longstaff and Schwartz (2001) for various methods to incorporate the value of early exercise into the Monte Carlo method. One of the major motivations behind the popularity of this area of research has been to enable pricing of Bermudan swaptions in the LMM's, see Andersen (1999) and Pedersen (1999).

While the field of yield curve modelling continues to flourish, there still exists the problem of finding a reliable way to model the curve evolution. The empirical evidence

has been discouraging and the intuition of the model is easily lost if the model contains multiple driving factors. While multifactor models are intuitively the hardest models to understand and use in practise, the one-factor models have their own problems too. The first and obvious one is the calibration. The HW model for example is Gaussian and must be fitted to a lognormal market, using the parameters of the stochastic differential describing the motion of the hypothetical short rate. BDT, on the other hand, is less analytical and more market oriented.

The method presented in this paper is based on a simple idea of pricing, in an arbitrage-free manner, the European swaptions underlying the Bermudan swaption. Each of the Europeans will be priced using their individual, broker quoted, volatilities. The forward swap rates are recovered exactly from the swap curve. Hence, there is no need for any kind of calibration whatsoever. Moreover, from an intuitive point of view, using the European swaption volatilities as inputs in the pricing along with the forward swap rates means the trader will have a greater degree of trust in the pricing procedure. The pricing procedure goes as follows: first we build trees of underlying forward floating legs of the swap, using the volatilities of the European swaptions. In fact, we only need to produce the end node values. After that we can compute the Bermudan option tree in a standard way of working from back to the root and checking for early exercise. The pricing method is very efficient, since along with the avoidance of any calibration, we can use a lattice similar to the famous Cox, Ross, and Rubinstein (1979) or Jarrow and Rudd (1983), the latter being more suitable for this type of implementation. In addition to presenting the algorithm for standard interest rate Bermudan swaptions, we extend the analysis to cover the case of the commodity Bermudan swaption. For applications of the swap market models, Jamshidian (1997) has proposed pricing Bermudan swaptions using an ordinary binomial lattice in order to get quick and accurate results. He does not, however, present details of the implementation.

The rest of this paper is organized as follows: Section 2 introduces the notation, the general ideas of the swap derivatives and the standard binomial pricing model. In Section 3, we develop the model and show how the pricing of the Bermudan swaptions can be done in this framework. Section 4 contains numerical examples of pricing and hedging of a Bermudan swaption within the model framework and compares the results with the BDT prices. In Section 5, an application to handle the commodity Bermudan swaptions is presented. Section 6 concludes the paper and suggests further research.

1 Notation, Swap Derivatives and Binomial Pricing

In this section, we review the fundamental background of swap and swap derivatives. We also present an application of the famous Cox, Ross, and Rubinstein (1979) (CRR) type binomial tree for the pricing of European swaptions. This will provide us with a tool for extending the model to price Bermudan type swaptions. We start by introducing the basic notation and concepts of swap derivatives, concentrating first on the interest rate swaption.

The fundamental building blocks of the interest rate derivatives are zero coupon bonds. The value at time t of a zero coupon bond maturing at T_i is denoted by $P(t, T_i)$. It is assumed that the zero coupon bond exists for every maturity T_i . The par swap rate is a fixed interest rate that equalizes the present values of the floating and fixed legs. The value of the floating leg at time t is given by

$$Fl_t = P(t, T_0) - P(t, T_N) \tag{1}$$

where the subscripts 0 and N denote the start date and the end date of the contract. The derivation of this result is straightforward and can be found in Rebonato (1996). The value of the fixed leg at time t is the value of an annuity with tenor structure τ ³.

In mathematical notation, this is expressed as

$$Fx_t = X\tau \sum_{i=1}^N P(t, T_i) \quad (2)$$

where X is the fixed rate of interest. Hence, we write the par (spot or forward) swap rate as

$$S_t = \frac{P(t, T_0) - P(t, T_N)}{\tau \sum_{i=1}^N P(t, T_i)} \quad (3)$$

So, the derivation of the values of the floating and fixed legs of any swap at any time t requires only knowledge of the discount function. Similarly, any forward swap rate is instantaneously recovered from the discount function. Note that in the calculation of the spot swap rate, the numerator becomes $P(t, t) - P(t, T_N) = 1 - P(t, T_N)$. The above definitions show that the fundamental quantities in the swap market are very easy to derive. The market practice for extracting the P 's is to bootstrap the par swap curve to get the zero coupon bonds for maturities 1, 2, 3 etc years. The intermediate zero coupon prices can be obtained via some interpolation method.

1.1 Standard European Swaption Formula

The underlying model used in this paper is a version of the famous Black (1976) formula applied to swaptions. In this formula, the floating leg of the swap is treated as a stochastic variable whereas the value of the fixed leg is a constant annuity (the discounting factor). The present value of basis point PVBP $\tau \sum_{i=1}^N P(t, T_i)$ is used as the martingale measure. Therefore, the floating leg discounted using this numeraire (i.e. the forward swap rate), will be a martingale. The stochastic differential of the forward swap rate, under the forward swap measure \mathbb{Q} , is

$$dS_t = \sigma S_t d\tilde{W}_t \quad (4)$$

where $d\tilde{W}_t$ is the standard Brownian motion under \mathbb{Q} . Hence, the European swaption (payer) price is given by the following well-known formula

$$Payer_t = Fl_t \mathbf{N}(h_1) - Fx_t \mathbf{N}(h_2) \quad (5)$$

where

$$h_1 = \frac{\ln(Fl_t/Fx_t) + 1/2\sigma^2 T}{\sigma\sqrt{T}}, h_2 = h_1 - \sigma\sqrt{T}$$

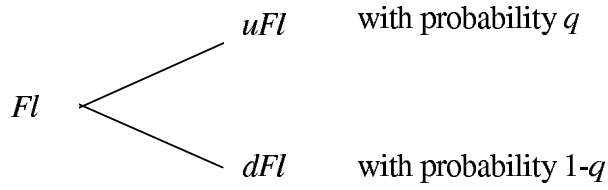
and $\mathbf{N}(\cdot)$ represents the cumulative standard normal distribution function. In this form, the European payer swaption formula has the most convenient representation. The formula for the receiver swaption can be obtained directly from the put-call parity. The Black (1976) model for European commodity swaptions is derived in Järvinen and Toivonen (2002).

1.2 Binomial Approximation

We now illustrate how the standard binomial model of Cox, Ross, and Rubinstein (1979) is used for calculating the price of a European swaption. We start by defining the risk-neutral probability for a futures market.

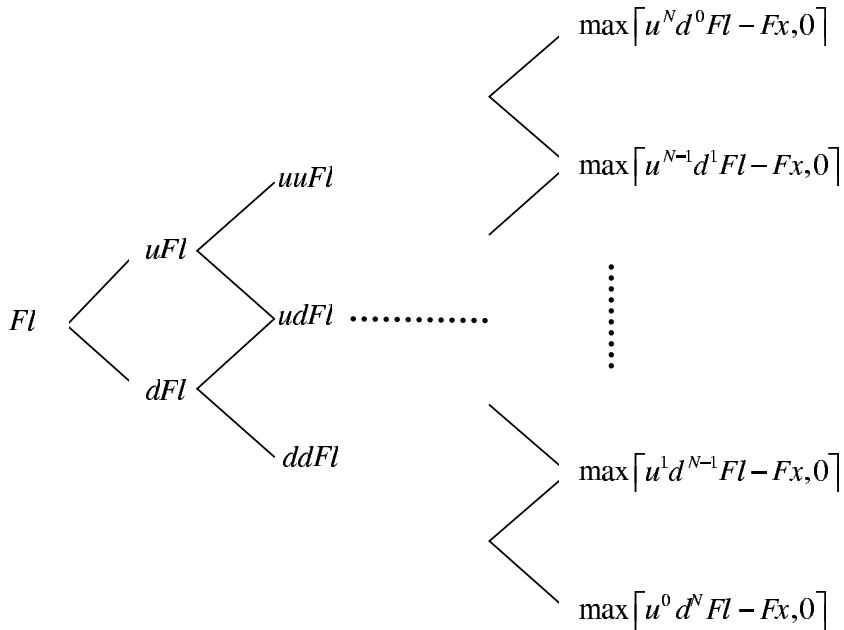
$$q = \frac{1-d}{u-d} = \frac{1-e^{-\sigma\sqrt{dt}}}{e^{\sigma\sqrt{dt}}-e^{-\sigma\sqrt{dt}}} \quad (6)$$

where σ is the volatility of the forward swap rate and dt equals time to maturity of the swaption divided by the number of periods in the binomial tree. Therefore, the floating leg of the swap evolves as follows



The requirement $u > 1 > d$ is enough to guarantee that the model is arbitrage-free. To enable calculations of option prices using the model, we construct a binomial tree for the value of the floating leg as follows

EXHIBIT 1 Standard Binomial Tree of the Floating Leg Value



In the binomial model above, index i runs from 0 to N (number of periods in the tree). It is a well-known fact that in order to price American options, we must calculate option values in every node to find out whether it pays off to exercise early or not (the optimal stopping problem). However, in the case of European options, the above tree reduces to the calculation of the probability-weighted average of the option values at the end nodes. The formula for European payer swaption is

$$Payer_t = \sum_{i=0}^N \frac{N!}{i!(N-i)!} q^i (1-q)^{N-i} \max [u^i d^{N-i} Fl - Fx, 0] \quad (7)$$

Note that there is no explicit discounting in the formula. The discounting is implicit, since Fl and Fx are already present values. This is how the original CRR model is applied to price European swaptions. In what follows, we will change the probabilities from q to $\frac{1}{2}$ in order to make the computations more efficient. This change of probability also affects both the up and the down factors. The details can be found in Jarrow and Rudd (1983).

2 The Binomial Model for Bermudan Swaptions

The object of analysis here is a Bermudan swaption with exercise dates T_1, T_2, \dots, T_N . We consider a time interval $0 \leq t \leq T_N$. At this interval, there exists a collection of European swaptions O_{t, T_i} and spot swaps S_{t, T_i} with maturity dates T_i . The economic setting is a frictionless financial market where a given set of swaps and European swaptions are traded on a given finite time interval (all time points mentioned below are assumed to fall within this interval). The interval between trades is of length $\Delta > 0$, measured in units of time. The total number of steps until the final maturity date is n and steps to each T_i (maturity date), is given by T_i/Δ .

In order to construct a feasible and efficient binomial tree, we set the probabilities of up and down movements equal to $1/2$. The continuous-time dynamics of the forward swap rates are given by (4) with the same Wiener process driving all the forward swaps, but with separate volatility parameters. There exists a zero coupon bond for each T_i , denoted by P_{t,T_i} . We start by approximating the dynamics of the forward floating leg processes using the discretization

$$Fl_{T_i,T_i,T_S} = Fl_{t,T_i,T_S} \exp \left(- \sum_{k=1}^{T_i/\Delta} \frac{\sigma_{T_i}^2}{2} \Delta + \sum_{k=1}^{T_i/\Delta} \sigma_{T_i} \rho_k \sqrt{\Delta} \right) \quad (8)$$

where T_S denotes the time of maturity of the swap underlying the option. The random variables ρ can take values in 1 and -1 with probability $1/2$ for each realization. The random variables are independent and identically distributed. In addition, they are the same for all forward floating legs. In the limit, the discrete process (8) converges to its continuous-time counterpart

$$Fl_{T_i,T_i,T_S} = Fl_{t,T_i,T_S} \exp \left(- \frac{\sigma_{T_i}^2}{2} T_i + \sigma_{T_i} W_{T_i} \right) \quad (9)$$

The volatility parameters in (8) and (9) can be obtained directly from the volatility matrix for European swaptions. Hence, we construct a binomial tree for each floating leg, Fl_{T_i} , using the market volatilities in equation (8). The initial, time 0, values for each Fl_{T_i} are calculated directly from the discount curve, using equation (1).

With the given set up, we can now proceed by building a binomial tree for Fl_{T_N} , i.e. the floating leg underlying the longest maturity European swaption. This represents the last exercise opportunity of a Bermudan swaption and therefore gives the end-node values for the binomial tree. To simplify notation somewhat, we drop the subscripts denoting the maturity of the floating and fixed legs. It should be clear that for each floating and fixed leg, the maturity is equal to the last exercise opportunity of the

Bermudan swaption plus the remaining term. Now, the final node values are given by ⁴

$$\max \left[Fl_{T_N}^j - Fx_{T_N}, 0 \right] \quad (10)$$

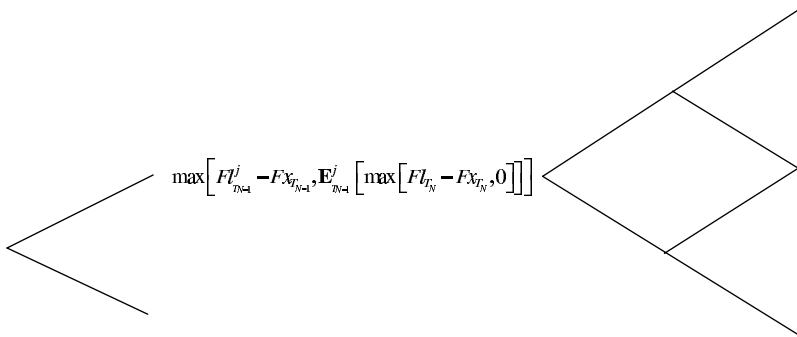
with $Fl_{T_N}^j$'s obtained from equation (8). Fx_{T_N} is the exercise price, taken from the initial discount curve. After the end node values have been calculated for each state j , using (10), the recursive valuation methodology can be applied. From the last exercise opportunity, we move on to consider exercising at date T_{N-1} . As the exercise opportunities of Bermudan swaptions occur usually annually or semi-annually, we do not have to consider each step, as in the American option case. Instead, it is enough to step straight to the time point where there is potentially an early exercise opportunity. The value of a Bermudan swaption at the second last exercise opportunity is given by

$$\max \left[Fl_{T_{N-1}}^j - Fx_{T_{N-1}}, \mathbf{E}_{T_{N-1}}^j \left[\max \left[Fl_{T_N} - Fx_{T_N}, 0 \right] \right] \right] \quad (11)$$

where the expectation is a conditional expectation evaluated at each particular node j .

The graph below illustrates this

EXHIBIT 2 Calculation of conditional expectations



The optimality of early exercise is checked using (11). The next step is to make the

analogous calculations for the preceding steps, the general formula for this is

$$\max \left[Fl_{T_{i-1}}^j - Fx_{T_{i-1}}, BO_{T_i}^j \right] \quad (12)$$

where $BO_{T_i}^j$ is the expected value, at time T_{i-1} of the Bermudan swaption at time T_i , i.e. at the next exercise opportunity. The expected value is calculated for every state of nature j . Therefore, we calculate the value of the Bermudan swaption using iterated conditional expectations so that the probabilities are set to 1/2. Moreover, the binomial distributions of the underlying floating leg values are generated using the market quoted volatilities (potentially different for each underlying floating leg) and the initial values of the floating legs are extracted directly from the discount curve. To find the present value of the Bermudan swaption, we finally calculate the expected value today of the Bermudan swaption at the first exercise opportunity, T_1 .

2.1 Market Data

To implement the binomial pricing algorithm for pricing Bermudan swaptions, we first need to supply the necessary market data. This data includes the par swap curve and the volatility matrix for European swaptions. In order to calculate the initial values of the floating leg and fixed leg (strikes) values, a bootstrapping method need to be used to find the discount bond prices. After that, the required variables are calculated using equations (1) and (2).

The volatility matrix for European swaptions gives the Black (1976) swaption volatilities for at-the-money swaptions. On the other axis are the maturities of the underlying swaps, whereas the other axis gives the swaption maturities. Therefore, in order to price a ten-year Bermudan swaption using the binomial model presented here, we need to read specific entries from the volatility matrix. Of course, the entries in this case are

1y/9y, 2y/8y, 3y/7y,...,9y/1y. Here the first and second numbers represent the swaption maturity and the underlying swap maturity, respectively. These volatilities are the volatilities of the European swaptions maturing at the time of exercise opportunities of the Bermudan swaption. Each European swaption has the same term for the underlying swap as the Bermudan has, should it be exercised.

Working with the volatility matrix presents two difficulties in practice. The first is a mismatch of the tenors of the European swaptions and the Bermudan swaption that should be valued. This problem is easily overcome by using a two-dimensional interpolation scheme. The second problem is due to the volatility matrix data being quoted for at-the-money European swaptions. Almost always, the Bermudan swaption has a constant strike yield. Therefore, at-the-money European swaption data is not fully appropriate for valuation. A reasonable remedy for this would be to obtain smile data directly from brokers, at least for the most valuable European swaptions.

2.2 Variable Strike Bermudan

Using the binomial algorithm in pricing a Bermudan swaption with a variable strike is straightforward. As the strikes in the OTC interest rate option market are quoted as rates, we only need to apply these strike rates in the calculation of fixed leg values for individual exercise opportunities. Obviously, if the strike rates deviate significantly from the quoted at-the-money rates, then the volatility matrix figures need to be adjusted.

2.3 Hedging Parameters

Derivation of the hedging parameters is similar to the standard binomial model. However, in this case, there is a freedom in the choice of the hedging instrument, i.e. which of the underlying forward swaps to use. From the available choices, there are two that

stand out. The first is to use the forward swap underlying the most valuable European swaption. Since the value of that instrument accounts for a large part of the value of the Bermudan swaption in total, it is likely that the hedging error is not going to be exceedingly large if that particular forward swap is used. The second good alternative is to use some combination of the underlying forward swaps. This reduces the effect of the yield curve risk in case of non-parallel movements. Conceptually this is similar to bucket management of risk. Analysis and management of risk based on bucketing is a widely-used approach in practice.

The formula for calculation of the Delta is

$$Delta_t = \frac{(BO_t^{up} - BO_t^{down})}{(u_{T_i} - d_{T_i})F_{t,T_i}} \quad (13)$$

where

$$\begin{aligned} u_{T_i} &= e^{-\frac{\sigma_{T_i}^2}{2}\Delta + \sigma_{T_i}\sqrt{\Delta}} \\ d_{T_i} &= e^{-\frac{\sigma_{T_i}^2}{2}\Delta - \sigma_{T_i}\sqrt{\Delta}} \end{aligned} \quad (14)$$

3 Numerical Examples

We illustrate the use of the binomial model by applying it to the pricing of a Bermudan interest rate swaption. In addition, we compare the prices given by the binomial model to BDT model values.

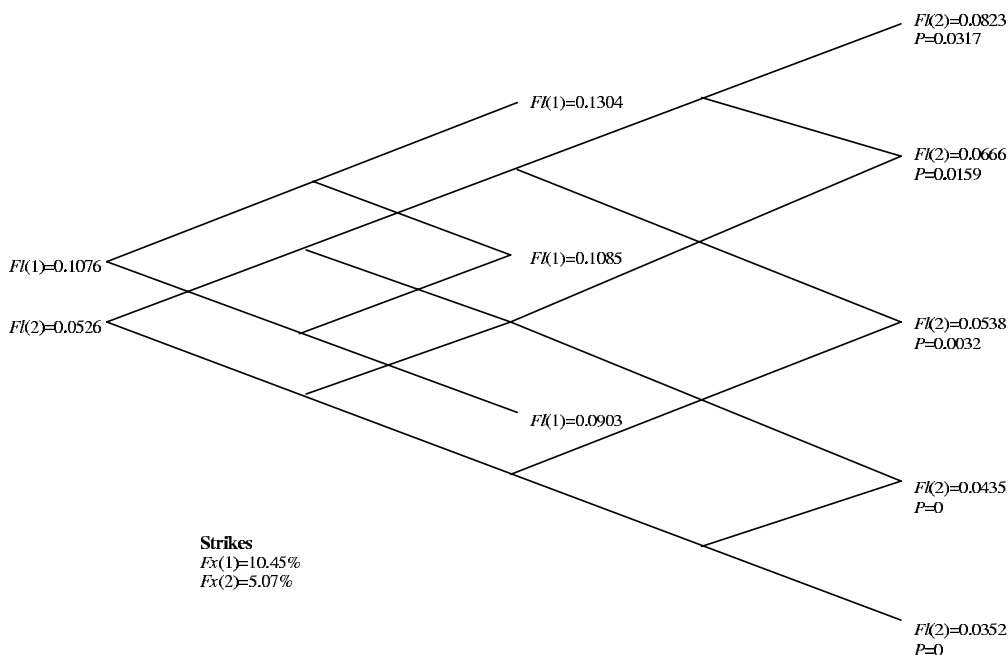
Further assume that we want to price a Bermudan payer swaption with the maturity of two years, exercisable after one or two years. Suppose that the initial data values are as follows and the strike rate is 6.00. As indicated previously, the probabilities in the tree are 1/2 throughout.

The data: $N = 2$, $K=1$, $D=0.5$, and

time	swap rate	volatility
$t = 0.5$	swap = 0.048	
$t = 1$	swap = 0.051	vol = 0.13 (swap 2y)
$t = 1.5$	swap = 0.054	
$t = 2$	swap = 0.056	vol = 0.15 (swap 1y)
$t = 2.5$	swap = 0.057	
$t = 3$	swap = 0.058	

From the initial swap market data, we calculate the values of the floating and the fixed legs (strike prices) and the corresponding value trees using the dynamics given by (8). The resulting trees with end-node option values are shown in Exhibit 3 below

EXHIBIT 3 Binomial Evolution of Floating Legs and End Node Values of a Payer Swaption with Strike 6.00%



Using the formula (7), the current price of the European payer swaption with time to maturity of two years and the underlying swap length of one year is

$$EO_t = 0.5^4 * (0.031 + 4 * 0.016 + 6 * 0.003)$$

$$= 0.0071$$

To price a Bermudan swaption, we need to check the possibility for early exercise. We start by observing the intrinsic values at the end nodes of the 2y/1y tree. Using these values we calculate the prices of Bermudan options at end-nodes of the 1/2 tree (the first exercise opportunity). According to equation (11), we find that in the upper node,

the value of the Bermudan swaption with strike rate 6.00% is

$$\max [0.1304 - 0.1044, 0.25 \times 0.0317 + 0.5 \times 0.0159 + 0.25 \times 0.0032] = 0.0260$$

Hence, early exercise is optimal. Similar calculations give: 0.0056 (no early exercise) for the middle-node and 0.0008 (no early exercise) for the lower-node. Finally, we can calculate the present value of the Bermudan swaption as follows

$$BO_t = (0.25 \times 0.0260 + 0.5 \times 0.0056 + 0.25 \times 0.0008) = 0.0095$$

In order to calculate the delta we need the up and down values for the Bermudan swaption. In addition, assume that we are going to implement the hedging with the two-year swap starting after one year. The Delta parameter in this case becomes

$$Delta_t = \frac{(0.0158 - 0.0032)}{(1.1009 - 0.9160) \times 0.1076} = 0.6332$$

The procedure described here illustrates how the Bermudan swaption values can be calculated using a very simple binomial algorithm. In practise, the number of steps between the exercise opportunities would usually be between 100 and 1000. The algorithm is very quick, so this does not present any problem.

To gain some insight on how the binomial model values Bermudan swaptions, we compare the simple binomial model with the BDT model that is widely used by market participants. We fitted the BDT model to market data using a tree with 100 steps. Three different yield curve environments were used: flat, linear upward and linear downward. The volatility structure was flat in all cases, but the level varied from 10 to 20 percent. In calculating the option values, we set the number of steps between exercise opportunities to 100. The results are shown in Exhibit 4 below

EXHIBIT 4 Simple Binomial Algorithm and BDT Model Compared

Tenor	BDT Vol	Black Vol	Curve	Calibration	BDT Price	Binomial
10 yr	14.90%	15.0%	Flat 5.00%	Atm fwd	352.19	346.70
10 yr	15.02%	15.0%	Up 4.00%-6.00%	Atm fwd	812.09	804.83
10 yr	14.79%	15.0%	Dn 6.00%-4.00%	Atm fwd	59.65	57.84
10 yr	9.87%	10.0%	Flat 5.00%	Atm fwd	234.62	231.66
10 yr	9.91%	10.0%	Up 4.00%-6.00%	Atm fwd	711.68	709.84
10 yr	9.82%	10.0%	Dn 6.00%-4.00%	Atm fwd	11.67	11.39
10 yr	20.12%	20.0%	Flat 5.00%	Atm fwd	471.03	461.06
10 yr	20.36%	20.0%	Up 4.00%-6.00%	Atm fwd	927.55	911.66
10 yr	19.89%	20.0%	Dn 6.00%-4.00%	Atm fwd	132.59	127.44
5 yr	14.69%	15.0%	Flat 5.00%	Atm fwd	135.43	134.65
5 yr	14.69%	15.0%	Up 4.00%-6.00%	Atm fwd	172.41	172.37
5 yr	14.68%	15.0%	Dn 6.00%-4.00%	Atm fwd	106.96	106.53
5 yr	9.79%	10.0%	Flat 5.00%	Atm fwd	90.43	89.85
5 yr	9.79%	10.0%	Up 4.00%-6.00%	Atm fwd	129.20	128.98
5 yr	9.78%	10.0%	Dn 6.00%-4.00%	Atm fwd	64.33	64.56
5 yr	19.62%	20.0%	Flat 5.00%	Atm fwd	180.51	179.24
5 yr	19.64%	20.0%	Up 4.00%-6.00%	Atm fwd	217.35	216.74
5 yr	19.61%	20.0%	Dn 6.00%-4.00%	Atm fwd	150.11	148.96

Exhibit 4 shows that, in general, the prices given by the simple binomial model and the BDT model are fairly similar. The shorter maturity swaption prices, in particular, are almost equal. With longer maturity swaptions, the divergence is more pronounced, increasing with the level of volatility. This may be due to the fact that the valuation tree is relatively scarcely sliced. Given that the binomial algorithm is far simpler to implement, calibrates automatically, doesn't need an iterative algorithm to solve for node values, it provides a good alternative to the traditional interest rate models for valuation of Bermudan type swaptions. In particular, if the aim is a quick and reliable valuation using market data, then the best method is to implement the simple binomial algorithm presented in this paper.

4 Application to Commodity

It is possible to extend this simple binomial model to handle commodity Bermudan swaptions as well. Although the model is overly simplistic in the assumed dynamics (i.e. one driving Brownian motion for the set of forward par swap prices, scaled by the individual constant volatility parameters) it is still an applicable method in the presence of suitable market data. The model implemented here closely resembles the ideologies put forth in the papers by Reisman (1991) and Cortazar and Schwartz (1994). That is, instead of modelling forward prices through the stochastic model for the spot price and net convenience yield, the dynamics of the prices is exogenously given. In that sense, the effect of a stochastic net convenience yield, which is able to generate mean-reversion to the prices and affect the volatility structure is ignored. It is fair to say that in the absence of reliable market data on both the forward prices and forward price volatilities, the model is not applicable in practice. However, if one does have good data to use and a liquid market to trade in, then even this simple one-factor implementation is able to produce useful results. A recent paper by Driessen, Klaassen, and Melenberg (2003) shows that the hedging performance of one-factor versus multi-factor models is approximately equal if one uses a many underlying instruments i.e. bucket hedging, which is a common practice among trading institutions. Even though they analyzed data on US interest rate cap and swaption prices, the general observation of the disappearing advantage of multi-factor models when implementing bucket hedging strategies is likely to be valid when applied to other markets and instruments as well.

In order to describe how the model is to be adjusted for handling commodity Bermudan swaptions, we need to redefine the underlying variables: the value of the floating leg and the value of the fixed leg. The present value of the floating price leg at time t is

given by

$$Fl_t = \sum_{i=1}^N F(t, T_i)P(t, T_i) \quad (15)$$

where $F(t, T_i)$'s are the implied forward prices from the forward curve. See Järvinen (2002) for various methods to handle the derivation of the forward curve from swap quotes. In commodity implementation, the value of the floating leg has to be calculated forward price by forward price and the sum of the discounted forward prices finally gives the present value of the floating leg of the swap. The value of the fixed leg at time t is given by

$$Fx_t = X \sum_{i=1}^N P(t, T_i) \quad (16)$$

Note that there are no day count conventions used, since the fixings of a commodity swap are defined as notional multiplied by the fixed price, X . Multiplying these fixing values by the appropriate discount factors gives the present value of the fixed leg.

Analogously with the analysis of the interest rate instruments, by setting the values of the floating price and fixed price equal we can solve for the par forward swap price, i.e. the forward swap that makes the present value of the contract zero. Solving for X gives the par swap price S_t

$$S_t = \frac{\sum_{i=1}^N F(t, T_i)P(t, T_i)}{\sum_{i=1}^N P(t, T_i)} \quad (17)$$

Moreover, we assume that the par swap price follows geometric Brownian motion with the drift rate of zero, i.e. a gaussian martingale process for the log par swap price. Hence, the European commodity swaption prices can be calculated by using the standard Black (1976) formula, i.e. equation (5). Järvinen and Toivonen (2002) presents detailed arguments for arriving at this result.

After the underlying variables have been calculated to enable the building of the

multilayer binomial tree, the rest of the analysis is analogous to the interest rate case. In contrast to the interest rate market, where data on European swaption volatilities on both maturity and strike axis is available, the data from the commodity derivatives market is definitely more difficult, if not impossible, to obtain. Therefore, the practical applicability of this pricing methodology is suspect, or at least varies across commodity types.

5 Conclusion

A simple binomial algorithm enables the calculation of Bermudan swaption values and hedges very efficiently. More importantly, it enables inclusion of the market volatilities and forward swap rate information instantly, without the need for any sort of calibration. The algorithm itself is based on the standard binomial model, with probabilities set to $1/2$. The underlying assumption is that there is only one source of randomness that drives the values of all the forward swaps (or forward floating legs) simultaneously, each weighted by their own Black (1976) market volatilities. Discounting is assumed constant.

It is demonstrated that the model prices do not differ much from the Bermudan swaption prices given by the BDT model fitted to the yield curve. This suggests that, in particular for those market parties who need a good and fast estimate of the value of Bermudan swaptions or multi-call bonds, the algorithm provides a cost efficient valuation method. Also traders, experienced with the Black (1976) model for European caps and swaptions, will find the simple binomial algorithm a useful tool for the analysis of Bermudan swaptions.

Notes

¹Recently, many feasible methods for adapting the Monte Carlo method to handle early-exercise problems have been proposed. One of the major motivations for these studies have been to find ways of implementing the multifactor Libor Market Models, LMM's (Brace, Gatarek, and Musiela (1997) and Miltersen, Sandmann, and Sondermann (1997)) to price also Bermudan swaptions.

²Bjerk Sund and Stensland (1996) have proposed an analytical approximation.

³There exists various methods for calculating interest rate day count factors, τ . Some of the most often used are act/360, act/act and 30/360, where act is the actual number of days applying to the interest accrual period.

⁴For a receiver option, this becomes $\max [Fx_{T_N} - Fl_{T_N}^j, 0]$

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ESSAY 4:

Pricing European Commodity Swaptions

Sami Järvinen and Harri Toivonen

Abstract

In this paper, we present formulas for commodity swaptions. By utilizing the forward price based approach we derive a simple closed form solution for European swaptions based on the assumption of deterministic volatility for lognormal variables. The formulas given result from applying the Margrabe (1978) exchange option concept to the present problem. A special case of constant volatility yields the Black (1976) formula that has been the market standard in the interest rate swaption markets for many years.

Futures markets have been the traditional vehicle for participating in the commodities markets, and exchange-traded futures contracts on commodities have a long history dating back to the 1800s. In fact, the first standardized derivative contracts were traded on commodities. In recent years, the OTC market for commodity derivatives has also expanded rapidly: BIS (2002) reports that the notional amount outstanding of OTC commodity derivative contracts was USD 590 billion in June 2001 ¹. The increasing financial importance of these products has also led to growing interest among academic researchers.

Modelling the stochastic behavior of underlying commodity prices is important for the valuation of contingent claims on commodities. Earlier studies on commodity derivatives have concentrated on modelling the joint stochastic process for net convenience yield and spot commodity price; a seminal paper along these lines is Gibson and Schwartz (1990),² who develop a two-factor model where the state variables are spot price and mean-reverting instantaneous net convenience yield. Schwartz (1997) extends this approach by introducing three alternative stochastic models, where the most sophisticated has three state variables: spot price of commodity, convenience yield and interest rate. Miltersen and Schwartz (1998) present a different approach and extend earlier models by utilizing information in the initial term structure of interest rates and futures prices, thereby obtaining a more market-oriented model in the spirit of the classic Heath, Jarrow, and Morton (1992) paper. In addition, by assuming normal distribution for instantaneous interest rates and convenience yields, Miltersen and Schwartz (1998) are able to derive simple closed form solutions for futures and forward prices. Hilliard and Reis (1998) present a three-factor model which allows for jumps in the spot price. In addition, they develop an equilibrium-based diffusion for the convenience yield process and an arbitrage free term structure process. Hilliard and Reis (1998) make this choice for applicability; convenience yield data is hard to obtain, but interest rate data is reliably available.

Hence, in practice, in order to price commodity derivatives, it is often more convenient to model the evolution of the forward variables instead of explicitly modelling the spot variables, especially the rather abstract convenience yield. This approach was pioneered by Black (1976). He priced options directly in terms of the forward price of the underlying asset with general stochastic dividends, though he kept the interest rates constant. Since then, the forward approach to arbitrage valuation of contingent claims has gained popularity. Bick (1998) developed a general forward price based valuation framework and replication strategy. More recently, the numeraire change based technique, formalized by Geman, Karoui, and Rochet (1995), has become the standard tool for derivatives valuation, and it has been used to give mathematically formal justifications for many important pricing models, e.g. the Black formula for interest rate swaptions, originally developed by Neuberger (1990) and later formalized by Jamshidian (1997).

Chance and Rich (1996) have previously introduced a methodology for pricing commodity swaptions. However, in order to use their pricing formula in practice, one needs to know the continuously compounded convenience yield. In this paper, we present a valuation formula for a European commodity swaption, based on the forward price approach, i.e. we specify the dynamics of the forward price exogenously. Within this framework, no information regarding the stochastic process of the net convenience yield is required in order to price commodity instruments. This approach simplifies the pricing of commodity swaptions considerably. To the best of our knowledge, to date, there have been no published studies where the forward based approach has been applied to the pricing of commodity swaptions. We contribute to the existing literature by presenting a general framework for pricing commodity swaptions using the change of numeraire technique by Geman, Karoui, and Rochet (1995). Use of the forward price process is arguably a more market-oriented approach than using the joint processes for spot variables. The results presented in this paper also include the effect on the pricing formula

if one allows for stochastic interest rates.

The paper is organized as follows: An exact definition of a commodity swaption is given in Section 1. The pricing formula for both the constant and stochastic interest rates are derived in Section 2. Extensions are presented in Section 3. Finally, Section 4 discusses the validity of the approach and Section 5 concludes the paper.

1 Basic Definitions

A commodity swaption is an option that gives the holder a right to pay (payer swaption) or receive (receiver swaption) a fixed price against the floating price of the underlying commodity. In many ways, the commodity swaption is similar to the standard interest rate swaption. However, there are some important differences: First, the notional amount of the commodity swaption is in tonnes or barrels or other units of the underlying commodity, whereas the notional amount of the interest rate swaption is a currency amount. In addition, the settlement prices of the underlying swap are often calculated from multiple price observations instead of a single observation that is typically the case with interest rate swaps. The underlying instrument in the commodity swaption, when it matures, is a spot commodity swap. However, it is often more convenient to cast the actual pricing of the contract into the framework where forward start commodity swap is used as the underlying instrument. Let us fix a collection of future dates $T_0 = T < T_1 < \dots < T_N$. The forward swap is a financial contract entered into at trade date $t < T_0$ with settlement dates $T_1 < \dots < T_N$. Often the spot price of the commodity or a proxy index, is observed daily or weekly between the settlement dates and that value is then used as the settlement price. At each settlement date of the swap, the other party pays the difference $S(T_i) - X$, where $S(T_i)$ is the floating price for settlement T_i and X is the agreed fixed price.

A forward swap contract is composed of two legs, namely the floating price leg and the fixed price leg. The present value of the floating price leg is given by

$$V_{float} = \sum_{i=1}^N F(t, T_i)P(t, T_i) \quad (1)$$

i.e. it is the sum of the discounted forward prices. Usually, the swap contracts exist for maturities far exceeding the longest dated available forward or futures contracts. By contrast, the spot swap prices are quoted for maturities of many years, typically at least five years. Therefore, the forward prices in (1) have to be implied from the swap market data using a suitable estimation algorithm - see Järvinen (2002) ³.

The present value of the fixed price leg can be obtained by the following formula

$$V_{fixed} = X \sum_{i=1}^N P(t, T_i) \quad (2)$$

So, the fixed price leg value is simply the sum of the discount factors for the settlement dates of the swap contract multiplied by the fixed price of the swap. The sum of discount factors is often called annuity. Both of the leg values are multiplied by notional per fixing to find the actual currency denominated values. In equations (1) and (2), the notional quantity is assumed to be one to simplify notation.

By setting the values of the floating price and fixed price leg values equal, we can solve for the par forward swap price, i.e. the forward swap which makes the present value of the contract zero. Solving for X gives

$$X = \frac{\sum_{i=1}^N F(t, T_i)P(t, T_i)}{\sum_{i=1}^N P(t, T_i)} \quad (3)$$

The payoff of the European commodity swaption can be defined as the maximum of the

values of the floating price leg and the fixed price leg. We have for the payer swaption

$$C(T_0) = \max [V_{float}(T_0) - V_{fixed}(T_0), 0] \quad (4)$$

and for the receiver swaption

$$P(T_0) = \max [V_{fixed}(T_0) - V_{float}(T_0), 0] \quad (5)$$

In effect, the payer swaption is a call on the floating price leg, and the receiver swaption is a put on the floating price leg. From the homogeneity of the payoff functions (4) and (5), it follows that we can rewrite the payoffs as

$$C(T_0) = \sum_{i=1}^N P(T_0, T_i) \times \max [S(T_0) - X, 0] \quad (6)$$

$$P(T_0) = \sum_{i=1}^N P(T_0, T_i) \times \max [X - S(T_0), 0] \quad (7)$$

In other words, the swaption payoffs can be rephrased as the annuitized difference between the spot swap price at date T_0 and the fixed strike price agreed upon the contract initiation. The rewritten payoff functions can be compared with the interest rate swaption payoffs. First, the spot swap price corresponds to the spot swap rate in the interest rate swaption with the fixed strike price having an analogous interpretation. Second, the price difference is annuitized, which corresponds to present valuing the rate difference in the interest rate swaption. Third, there are no day counts or annual notional amount. Instead, there is a notional amount per fixing.⁴ In the case of commodity swaps and swaptions, the notional amount is not a monetary unit, but rather it is the quantity of the underlying commodity's unit of quotation.

2 The Pricing Formulas

Valuing a commodity swaption requires the specification of the dynamics for the underlying variables, namely, the forward floating price leg value and the forward fixed price leg value.

We assume that the dynamics under the real-world probability \mathbb{P} are given by

$$dV_{float}(t) = \alpha_1(t)V_{float}(t)dt + \sigma_1(t)V_{float}(t)dW_t^1 \quad (8)$$

and

$$dV_{fixed}(t) = \alpha_2(t)V_{fixed}(t)dt + \sigma_2(t)V_{fixed}(t)dW_t^2 \quad (9)$$

where dW_t is the standard Brownian motion under \mathbb{P} and $\alpha_{1,2}(t)$ and $\sigma_{1,2}(t)$ are drift and volatility coefficients, respectively. In addition, the standard Brownian motions are correlated with instantaneous correlation given by $\rho(t)dt$. As in Bick (1998) we assume no specific processes for convenience yield and interest rates. Instead, the forward processes (8) and (9) are modelled as correlated geometric Brownian motions. This facilitates straightforward calculation of swaption values. Under the traditional risk-neutral measure \mathbb{Q} , the drift coefficients of the processes (8) and (9) become $r(t)$, which denotes the continuously compounded risk-free rate of return.⁵

In order to derive simple formulas, we model the forward floating leg and the forward fixed leg values by summing up all the associated elements $P(t, T_i)F(t, T_i)$ and $P(t, T_i)$, respectively. This way, assuming that the volatility of the forward swap price is deterministic, it then follows the lognormal law. In analyzing (6) and (7) we are effectively calculating the expected value of the option payoff using the forward fixed price leg divided by the strike price as the numéraire (this is also known as annuity). This is an application of the general numéraire change technique developed by Geman,

Karoui, and Rochet (1995). Our results are direct applications of the general theorems established in their paper. We get the following expression for the expectation

$$\frac{V_{float}(0)}{A_{N,T}(0)} = \mathbb{E}^A \left[\frac{V_{float}(t)}{A_{T_1,T_N}(t)} \right] \quad (10)$$

where \mathbb{E}^A denotes expectations under the annuity measure and $A_{T_1,T_N}(t) = \sum_{i=1}^N P(t, T_i)$. It is well known that under this change of numéraire, the forward swap price is a martingale and the expected spot swap price equals the forward swap price. In fact, given our assumptions, the problem of finding the value function for European commodity swaption is analogous to exchange option problem, introduced by Margrabe (1978). The strike price X can be interpreted as simply a scaling factor. The equation (10) generalizes to the contingent claims valuation function, where f denotes any attainable simple claim, having T_0 as maturity date

$$f(0) = A_{T_1,T_N}(0) \times \mathbb{E}^A \left[\frac{f(T_0)}{A_{T_1,T_N}(T_0)} \right] \quad (11)$$

Substituting the payer commodity swaption pricing formula in (11), we get

$$\begin{aligned} C(0) &= A_{T_1,T_N}(0) \times \mathbb{E}^A \left[\frac{A_{T_1,T_N}(T_0) \times \max[S(T_0) - X, 0]}{A_{T_1,T_N}(T_0)} \right] \\ &= A_{T_1,T_N}(0) \times \mathbb{E}^A [\max[S(T_0) - X, 0]] \end{aligned}$$

This is the payoff of the standard European call. From the assumption of deterministic volatility of the forward swap price, we have the familiar results

$$C(t) = \sum_{i=1}^N P(t, T_i) [S(t, T_0, T_N)N(d_1) - XN(d_2)] \quad (12)$$

and

$$P(t) = \sum_{i=1}^N P(t, T_i) [XN(-d_2) - S(t, T_0, T_N)N(-d_1)] \quad (13)$$

where

$$d_1 = \frac{\ln(S(t, T_0, T_N)/X) + \frac{1}{2} \int_t^{T_0} \|\sigma_1(u) - \sigma_2(u)\|^2 du}{\sqrt{\int_t^{T_0} \|\sigma_1(u) - \sigma_2(u)\|^2 du}}$$

$$d_2 = d_1 - \sqrt{\int_t^{T_0} \|\sigma_1(u) - \sigma_2(u)\|^2 du}$$

where $\|\cdot\|$ denotes Euclidean norm. Equations (12) and (13) are solutions to the call (payer) and put (receiver) prices for commodity swaptions under our restrictive assumptions. Similar results have been derived already by Merton (1973) for the case of equity options with stochastic interest rates. In case of constant volatility and correlation coefficients, we will have

$$\sqrt{\int_t^{T_0} \|\sigma_1(u) - \sigma_2(u)\|^2 du} = \sqrt{(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)(T_0 - t)} \quad (14)$$

and we have a version of the famous Margrabe (1978) formula for options to exchange one asset for another. Furthermore, if we set $\sigma_2(t) = 0$, then the Black (1976) formula is recovered. To interpret the formula, we note that if the stochastic interest rate environment is assumed, then the implied volatility of the asset is actually the implied volatility of the forward swap price. Only in the constant interest rate world is the interpretation of volatility being the implied volatility of the asset correct.

3 Extensions

Due to the simplicity of the assumed dynamics, it is straightforward to extend the model to include more driving factors providing one makes sure that the forward swap price has a deterministic volatility. Moreover, the model can easily be extended to price structures such as swaptions on swaps with spread, cross currency swaptions, quanto swaptions and many popular exotic swaptions. These results have already been established for the standard options assuming the Black and Scholes (1973) market. We give examples of pricing a swaption with spread and an amortizing notional swaption. A swaption on a swap with spread is similar the same as the swaption apart from the fact that there is a spread attached to the floating price leg of the swap underlying the swaption. As the spread is a fixed value, we therefore adjust the strike price to account for the spread and treat the floating price leg as a pure floating price leg. Hence, the spread L , is subtracted from the fixed price leg value

$$V_{fixed-adjusted} = (X - L) \sum_{i=1}^N P(t, T_i) \quad (15)$$

Now, the swaption payoff becomes

$$C_{T_0} = \max[V_{float}(T_0) - V_{fixed-adjusted}(T_0), 0] \quad (16)$$

and again, from linear homogeneity

$$C_{T_0} = \sum_{i=1}^N P(T_0, T_i) \times \max[S(T_0) - (X - L), 0] \quad (17)$$

Thus, the swaption with a spread can now be valued in the same way as the ordinary swaption with strike price replaced by $(X - L)$.

Amortizing notional can be handled as in Jamshidian (1997). Let us denote the

notional quantity of the underlying swap on settlement date T_i by $L(T_i)$. Then equations (1) and (2) become

$$V_{float} = \sum_{i=1}^N L(T_i)F(t, T_i)P(t, T_i) \quad (18)$$

and

$$V_{fixed} = X \sum_{i=1}^N L(T_i)P(t, T_i) \quad (19)$$

Now, the "break-even" forward swap price will be

$$X = \frac{\sum_{i=1}^N L(T_i)F(t, T_i)P(t, T_i)}{\sum_{i=1}^N L(T_i)P(t, T_i)} \quad (20)$$

and again, assuming that this "break-even" price has deterministic volatility, the results given earlier apply. As noted by Jamshidian (1997), the assumptions of deterministic volatility for different swap prices may not be consistent with each other, but individually they appear to be quite reasonable, robust and convenient.

4 Discussion

The standard approach to analyzing commodity derivatives has traditionally been based on stochastic models of the spot price, convenience yield and interest rates. The approach presented here relies on modelling the forward prices directly without making any statements on the stochastic behavior of convenience yield. Moreover, in order to establish a simple formula for pricing swaptions, we treat the ratio of the forward floating leg value and the forward fixed leg value as a stochastic variable with deterministic volatility. That is, instead of modelling the underlying forward swap price as a sum of weighted individual forward price dynamics, we first put the package together and only then define the stochastic model. This approach creates a modelling problem in the sense

that if the forward swap price follows the lognormal distribution, then the components (forward prices) can not be lognormal simultaneously. This problem is already known from the interest rate derivatives modelling, where the Libor market models (Miltersen, Sandmann, and Sondermann (1997)) and (Brace, Gatarek, and Musiela (1997)) cannot be valid simultaneously with the Swap market model (Jamshidian (1997)). Even Libor market models configured for certain tenor structure, i.e. quarterly, semi-annual, etc., are inconsistent with each other. We refer to Jamshidian (1997) for an excellent discussion of the merits of using different models for different products despite the apparent inconsistency between the model's assumptions.

In summary, the market for commodity derivatives is still rather young compared to, for example the market for interest rate derivatives, where modelling approaches have evolved from simple instantaneous short rate models to the Libor market models currently being the tool of analysis in the sophisticated investment banks. In contrast, the commodity markets are much more heterogeneous in terms of the institutional settings and details (such as the liquidity and quotation practices) of the market for the underlying commodities, as well as in the products themselves. It remains to be seen what the standard market models for these instruments will be in the future. This paper attacks the problem from the point of view of pricing in terms of the forward variables, and the validity of such an approach obviously depends upon the depth of the market for forward contracts, swaps and plain vanilla options. If there is enough liquidity in those contracts, then there is certainly a good argument for applying the forward price based approach.

5 Conclusions

This paper presents a Black-76 formula for commodity swaptions. The underlying stochastic variables are the values of the floating and fixed price legs. Using the annuity factor as the numéraire, the familiar closed form solution for the swaption price is obtained. Interest rate market participants have long used the interest rate swaption formula, derived initially by Neuberger (1990), to price plain vanilla interest rate swaptions. For commodity swaptions, a similar formula is shown to exist, with the same distributional assumptions. Hence, it is very likely that the traders, familiar with the Black-76 volatilities, will use the model as a base when the OTC-market for commodity swaptions is large enough to support regularly quoted market prices.

Notes

¹The figure comprises of OTC contracts for commodity forwards, swaps and options.

²Convenience yield is defined as the benefit derived from holding the physical commodity instead of the futures contract on the underlying. Net convenience yield is convenience yield net of holding period storage costs.

³The quotations for longer maturity swaps are usually average-based, which complicates the bootstrapping process.

⁴The formula can be specified so that it uses annual notional and year fractions instead

⁵The forward floating price leg and the forward fixed price leg do not provide income before the start date of the spot swap, since convenience yields are already capitalized into the forward prices. This can be shown using Ito's Lemma.

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00100 Helsinki
Puh. (09) 4313 8310, fax (09) 495 617
Sähköposti: kirjak@kyppari.hkkk.fi

Helsingin kauppakorkeakoulu
Julkaisutoimittaja
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00101 Helsinki
Puh. (09) 4313 8579, fax (09) 4313 8305
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