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Cube Edge Graph and Similar Graphs as Forbidden Structures

Matvei Zhukov

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Author Matvei Zhukov

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Advisor Ms Patricija Šapokaitė

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Abstract

In extremal graph theory, degenerate (bipartite) extremal graph problems occupy a special place due to Erdős–Simonovits theorem. One of the smallest graphs for which the asymptotic size of the largest graph avoiding it is unknown is a cube edge graph Q_8. Its symmetries and its membership in different classes, such as cycles with all diagonals and prism graphs allows to apply different methods to estimate its extremal numbers. In this thesis a review of known classical and modern methods in degenerate extremal graph theory is given with a focus on the graph Q_8 and similar graphs such as hypercube edge graphs and prism graphs. Applicabilities of these approaches are explored and their limitations are analysed.

Keywords Turan Numbers, Extremal graph theory, Probabilistic method

Preface

I want to thank my Supervisor Ragnar Freij–Hollanti for giving me freedom of choosing a thesis theme and useful comments. I would also like to thank my Advisor Patricija Šapokaitė for carefully reading previous versions of this thesis and providing useful suggestions on the writing style.

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Otaniemi, 24 July 2025

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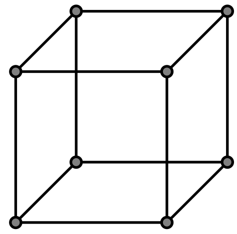

Figure 1: Cube edge graph Q_8.

1 Introduction

Extremal combinatorics is a broad field of mathematics that studies extremal structures. That is, the largest (or smallest) structures with a given property. In particular, forbidden subgraph theory investigates the largest graphs that do not contain a given graph as a subgraph. Extremal graph problems of this kind are often called Turán type problems, because Turán's theorem and his questions were the most influential roots of this field, see [1]. After Turán's theorem, Erdős, Simonovits and Sós had the most significant impact on this field, see [14]. Many of their results became the foundation of new methods in extremal graph theory and are presented in this thesis. A general forbidden subgraph problem can be formulated as follows.

Question 1. Let H be a graph. How many edges can a graph on n vertices have without containing H as a subgraph?

This number is called the extremal number of the graph H and denoted by $\mathrm{ex}(n, H)$. The answer to this question is known only for particular cases of H. The following Erdős–Stone theorem provides a bound for the extremal numbers.

Theorem 1.1 ([7])**.** *Let H be a graph, and its chromatic number be $\chi(H)$, that is the smallest natural number k such that $V(H)$ can be partitioned into k independent sets. Then,*

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\binom{n}{2}. \tag{1}$$

Theorem 1.1 answers the Question 1 asymptotically in the case $\chi(H) > 2$, implying that $\mathrm{ex}(n, H) = \Theta(n^2)$. In the case $\chi(H) = 2$, that is, when H is bipartite, the inequality (1) implies that $\mathrm{ex}(n, H) = o(n^2)$ and the exact asymptotics of $\mathrm{ex}(n, H)$ is known only for a certain set of bipartite graphs H. The exact order of magnitude of $\mathrm{ex}(n, H)$ is known for trees (see Theorem 3.6), for cycles of length 4, 6 and 10 (see Theorem 3.13 and [4, 5]), and for complete bipartite graphs $K_{a,b}$ with $b \geqslant 9^a a^{4a^{2/3}}$ [26].

The main obstacle in this field is the lack of methods for proving a lower bound for $\mathrm{ex}(n, H)$. A direct approach is to present an example of an extremal graph. However, it is usually hard to present an extremal graph with the right asymptotic behaviour. Successful cases include C_4, C_6, and C_{10} ([4, 6]) and $\theta_{4,3}$[22], that is, an internally disjoint union of three paths of length 4 between a pair of vertices. In these cases an example utilizes geometry, for example, vertices of a graph correspond to specific points in a certain finite vector space, and two vertices are adjacent if they are contained in a specific line. This method was used by Benson in [4] to construct graphs with no C_6 and C_{10}. Another method that yields a lower bound for extremal numbers is the probabilistic method, which will be discussed in the section 3.1. In recent years, a combination of probabilistic methods and geometric approaches has gained popularity, see [22, 26].

For establishing the upper bounds for extremal numbers, classical results such as the Erdős Reduction Theorem are presented, along with more recent methods, such as counting homomorphic copies of subgraphs and paths.

This thesis is organized as follows. In section 2 all necessary definitions and notation are given, along with the regularization tools. In Section 3 classic results of extremal graph theory are presented, including the best known bounds for $\mathrm{ex}(n, Q_8)$. In Section 4 a modern approach of counting the homomorphic copies of Q_8 is presented. Section 5 presents another probabilistic method, called dependent random choice. Section 6 presents the method of counting cycles in an augmented edge graph, recently developed by Gao, Janzer, Liu and Xu . The final Section 7 presents the comparison between methods and analysis of their limitations.

2 Background

Starting from the graph theory definitions and notation, followed by regularization lemmas, this section provides the necessary tools for theorems in the following sections.

2.1 Definitions

Definition 2.1. A *graph* G is a pair $(V(G), E(G))$ with $V = V(G)$ being a finite set, called the set of *vertices*, and $E = E(G)$ being a set of two-element subsets of V, called the set of *edges*. Notations $v \in V(G)$ and $v \in G$ for vertices are used interchangeably. An edge $\{u, v\} \in E$ is denoted as $uv \in E$. If $uv \in E$ then vertices u and v are *adjacent* to each other. The numbers of vertices and edges of a graph G are denoted by $v(G)$ and $e(G)$, respectively.

The following concepts are of central importance in graph theory:

- A *neighbourhood* of a subset of vertices $V' \subseteq V(G)$ is $N_G(V') = \{u \in V(G) \mid \forall v \in V' \colon uv \in E\}$. For $V' = \{v_1, v_2, \ldots, v_k\}$ the notation $N_G(v_1, v_2, \ldots, v_k)$ can be used.
- A *degree* of a vertex v is $d_G(v) = |N_G(v)|$, and a *codegree* of a pair of vertices u, v is $d_G(u, v) = |N_G(u, v)|$.
- A *minimum degree* of G is $\delta(G) = \min_{v \in V(G)} d_G(v)$.
- A *maximum degree* of G is $\Delta(G) = \max_{v \in V(G)} d_G(v)$.
- An *average degree* of G is $d(G) = (\sum_{v \in V} d_G(v))/v(G) = 2e(G)/v(G)$.
- An *edge density* of G is defined as $p(G) = 2e(G)/v(G)^2$.
- A graph G is called *$d-$regular* if $\Delta(G) = \delta(G) = d$.
- A graph G is called *$K-$almost regular* if $\Delta(G) \leqslant K\delta(G)$.

The following definitions contain basic concepts of graph homomorphisms, subgraphs and bipartite graphs.

Definition 2.2. Let H, G be graphs.

- A *graph homomorphism* from H to G is a map $f \colon V(H) \to V(G)$ such that $f(u)f(v) \in E(G)$ for each $uv \in H$. It is denoted as $f \colon H \to G$.
- *G contains H as a subgraph* if there exists an injective homomorphism $f \colon H \to G$. Alternatively, it can be said that G has a subgraph H, and H is called a subgraph of G. If G does not contain H as a subgraph, G is called an *$H-$free* graph. Alternatively, it can be said that G *avoids* H.
- Graphs H and G are called *isomorphic* if there exists a pair of injective graph homomorphisms $f \colon H \to G$ and $g \colon G \to H$.

The definitions below describe important types of graphs and graph properties.

Definition 2.3.

- A *path* of length k is a graph isomorphic to the graph $P_k = (V, E)$ with $V = \{1, 2, \ldots, k\}$ and $ij \in E \Leftrightarrow |i - j| = 1$.
- A *cycle* of length k is a graph isomorphic to the graph $C_k = (V, E)$ with $V = \{1, 2, \ldots, k\}$ and $ij \in E \Leftrightarrow i - j \equiv 1 \mod k$.
- A *complete* graph on k vertices is a graph $K_k = (V, E)$ with $V = \{1, 2, \ldots, k\}$ and $E = \{uv \mid u, v \in V\}$.
- A graph G is called *empty* if $e(G) = 0$.
- A graph G is called *connected* if for any two vertices $u, v \in G$ there exists a natural number k and a homomorphism $f\colon P_k \to G$ with $f(1) = u,\ f(k) = v$.

Definition 2.4. Let $G = (V, E)$ be a graph. For a subset of vertices $V' \subseteq V$ an *induced subgraph on* V' is the graph $G[V'] = (V', E')$ with $uv \in E'$ if and only if $u, v \in V'$ and $uv \in E$.

A graph G is called *bipartite* if there exists a partition $V(G) = A \sqcup B$ of vertices such that $G[A], G[B]$ are empty graphs. A *complete bipartite graph* with parts A, B is a graph $G = (V, E)$ with $V = A \sqcup B$, and $uv \in E$ for any $u \in A,\ v \in B$. With $|A| = m$ and $|B| = n$, it is denoted by $K_{m,n}$.

Let G be a graph. For a partition $V(G) = A \sqcup B$ of vertices the *bipartite induced subgraph* is a graph $G[A, B] = (V, E')$ with $uv \in E'$ if and only if u, v belongs to different parts of the partition.

With this background established, the extremal number of a graph can now be defined, along with a basic proposition, estimating the extremal number of a graph from extremal number of its subgraph.

Definition 2.5. An *extremal number* of a graph H is the largest number of edges in an n vertex graph which does not contain H as a subgraph. It is denoted as

$$\operatorname{ex}(n, H) = \max\{E(G) \mid v(G) = n,\ G \text{ does not contain } H \text{ as a subgraph}\}.$$

Proposition 2.6. *Let G be a graph and H be its subgraph. Then* $\operatorname{ex}(n, H) \leqslant \operatorname{ex}(n, G)$.

Proof. Let G' be a graph on n vertices with $e(G') > \operatorname{ex}(n, G)$. Then G' contains G and hence also H as a subgraph. Thus, $\operatorname{ex}(n, H) \leqslant \operatorname{ex}(n, G)$. □

2.2 Regularization Lemmas

Regularization lemmas are tools for finding somewhat large subgraphs of a given graph with required properties i.e., providing some regularity without losing too much from the structure of the initial graph. Suppose that G is a H-free graph on n vertices with $\operatorname{ex}(n, H)$ edges. What can be assumed about the structure of G without losing the asymptotics $e(G) = \Theta(\operatorname{ex}(n, H))$?

The following lemma shows that up to a constant factor 2 in $e(G)$, G may be assumed to be bipartite. This fact is helpful, since the subgraph H that G avoids, is also bipartite.

Lemma 2.7. *Let G be a graph with $e(G)$ edges. Then G has a bipartite subgraph G' with $e(G') \geqslant e(G)/2$. Moreover, for any $v \in V(G)$, the graph G' can be taken in such a way that $d_{G'}(v) \geqslant d_G(v)/2$.*

Proof. Let (A, B) be a partition of $V(G)$ such that $e(G[A, B])$ is maximal. For any vertex $v \in A$, if $d_G(v, A) > d_G(v, B)$, then the partition $(A \setminus \{v\}, B \cup \{v\})$ is more optimal, meaning that $e(A \setminus \{v\}, B \cup \{v\}) > e(G[A, B])$. Thus, $d_G(v, A) \leqslant d_G(v, B)$. Similarly, for any vertex $w \in B$ it holds that $d_G(w, B) \leqslant d_G(w, A)$. After summing the degrees of all vertices in A, it follows that

$$e(G[A, B]) = \sum_{v \in A} d_G(v, B) \geqslant \sum_{v \in A} d_G(v, A) = 2e(G[A]),$$

and similarly $e(G[A, B]) \geqslant 2e(G[B])$. Combination of these two inequalities implies that $e(G[A, B]) \geqslant e(G[A]) + e(G[B])$. Since $e(G) = e(G[A, B]) + e(G[A]) + e(G[B])$, it follows that $e(G[A, B]) \geqslant e(G)/2$. □

The following lemma provides a 'weak' degree regularity. By deleting not too many edges, the minimal degree of a graph may be assumed at least a fraction of the average degree.

Lemma 2.8. *Let G be a graph on n vertices. Then G has a subgraph G' with $e(G') \geqslant e(G)/2$ and $\delta(G') \geqslant d(G)/4$.*

Proof. If there is a vertex with degree less than $d(G)/4$, then delete it. Iterate this process until there are no more such vertices. Denote the remaining graph G'. Suppose that k vertices were deleted. Then at most $kd(G)/4$ edges were deleted. Since $k \leqslant n$, then $kd(G)/4 \leqslant nd(G)/4 = e(G)/2$. Hence, $e(G') \geqslant e(G)/2$. □

Finally, next lemma allows to restrict any degenerate extremal graph problem to a case of almost regular graphs. This lemma was originally proved by Erdős for his Reduction Theorem, see 3.17. It has been shown to be highly useful in various approaches to extremal graph theory.

Lemma 2.9 ([8])**.** *Let n be a sufficiently large natural number, let $0 < \alpha \leqslant 1$ be a constant and let G be a graph on n vertices with $e(G) \geqslant cn^{1+\alpha}$. Take $K = 10 \cdot 2^{\frac{1}{\alpha^2}+1}$. Then there exists a $K-$almost regular subgraph H of G with $v(H) \geqslant n^{\frac{\alpha^2-\alpha}{1+\alpha}}$ and $e(H) \geqslant \frac{2c}{5} v(H)^{1+\alpha}$.*

3 Classical approaches

This section presents the classical results of Erdős. It stars with the probabilistic method, defined in 3.1. Then it continues in 3.2 with extremal numbers for paths and cycles , which give a foundation for estimating the extremal numbers for more complex graphs. The last subsection is dedicated for the Erdős Reduction Theorem that is presented in 3.3. It proved itself to be a powerful tool for estimating extremal numbers and gives the best known upper bound for $\mathrm{ex}(n, Q_8)$.

3.1 Probabilistic method

One of the first approaches used in the extremal graph theory is the probabilistic method. That is, considering a random graph and proving that it contains a certain subgraph with positive probability. This method proves the existence of an extremal graph of certain size without actually finding its exact form. The main theorem in this method is the Erdős-Renyi First Moment method.

Theorem 3.1 ([3]). *Let $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ be a set of graphs, and let*

$$c_1 = c_1(\mathcal{H}) = \max_{j\in[k]} \min_{H\subseteq H_j} \frac{v(H)}{e(H)}, \quad c_2 = c_2(\mathcal{H}) = \max_{j\in[k]} \min_{H\subseteq H_j} \frac{v(H)-2}{e(H)-1}, \tag{2}$$

where the minimum is taken only for fractions with positive denominator, that is subgraphs H with $e(H) > 0$ for c_1 and $e(H) > 1$ for c_2. Let G_n be a graph on n vertices chosen uniformly at random among graphs with e_n edges.

1. *If for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $e_n < \delta n^{2-c_1}$ then the probability that G_n contains at least one of the graphs $H \in \mathcal{H}$ is at most ε.*

2. *If instead $e_n < \varepsilon n^{2-c_2}$ then then the probability that G_n contains at least $e_n/2$ copies of $H \in \mathcal{H}$ is at most ε.*

Note. For the purposes of this thesis, it is enough to prove only the second statement of this theorem only for $\mathcal{H} = \{H\}$. Moreover, only the case $c_2(\mathcal{H}) = c_2(H) = \frac{v(H)-2}{e(H)-1}$, that is minimum in (2), is obtained for H. Graphs H, for which $c_2(H) \leqslant c_2(H_j)$ for all $H_j \subseteq H$ are called balanced graphs. Originally, Erdős proved this theorem specifically for families of balanced graphs. Later, the proof was extended to arbitrary families by Győri, Rothschild, and Ruciński, see [10].

Proof. Let $G = G(n, p)$ be a random graph on n vertices with each edge taken with probability p. Then the expected value of number of edges in G is

$$\mathbb{E}[e(G)] = p\binom{n}{2}. \tag{3}$$

Let $k = k(H)$ be the number of subgraphs of $K_{v(H)}$ isomorphic to H. This number depends on the graph H since there is no order on vertices of H. For example, $k(K_{v(H)}) = 1$ since there is only one way to embed a complete graph into itself, while $k(P_{v(H)}) = \frac{1}{2}(v(H) - 1)!$. However, the exact value of k does not matter, but it is clear that $k \leqslant (v(H))!$. Let X be the number of subgraphs isomorphic to H in G. Then the expected value of X is

$$\mathbb{E}[X] = k\binom{n}{v(H)}p^{e(H)} \leqslant p^{e(H)}n^{v(H)}. \tag{4}$$

Let $p = cn^{-c_2}$ for a small constant $c \in (0, 1)$. Combining (3) and (4), the following estimation holds for n large enough and c small enough:

$$\mathbb{E}\left[\frac{1}{2}e(G) - X\right] \geqslant \frac{c}{5}n^{2-c_2} - c^{e(H)}n^{v(H)-c_2e(H)} \geqslant \frac{c}{6}n^{2-c_2} > 0.$$

The second inequality follows from the fact, that $e(H) > 1$, and $v(H) - c_2e(H) = 2 - c_2$. Thus, there exist a graph with $e(G) \geqslant 2c^{e(H)}n^{2-c_2}$ edges with at most $e(G)/2$ subgraphs isomorphic to H. □

Corollary 3.2. Theorem 3.1 implies that for $H \in \mathcal{H}$, $\text{ex}(n, H) = \Omega(n^{2-c_2})$, since at most half of the edges of a certain graph G_n with εn^{2-c_2} edges could be deleted to obtain an H-free graph.

Rem 3.3. A simple check of all induced subgraphs of Q_8 shows that for $\mathcal{H} = \{Q_8\}$, the constant $c_2(\mathcal{H}) = 6/11$, and $H = Q_8$ is the subgraph minimizing the fraction in (2). Thus, $\text{ex}(n, Q_8) = \Omega(n^{16/11})$.

3.2 Extremal numbers for paths and cycles

One approach of studying extremal number of a graph H, is studying the extremal numbers of its subgraphs first. Thus, it is useful to study the extremal numbers of some simple graphs. This section presents the study of extremal numbers of paths, cycles, and probably the easiest to analyse — star graphs.

Theorem 3.4. *Let $k \geqslant 2$ be an integer, and let G be a graph on n vertices that does not contain a star graph S_k, that is a tree on k vertices with one of the vertices being adjacent to all others. Then*

$$e(G) \leqslant \frac{1}{2}(k-2)n.$$

Proof. If G does not contain S_k as a subgraph then $\Delta(G) \leqslant k - 2$. Otherwise a vertex with maximum degree v and its $k - 1 \leqslant \Delta(G) = d_G(v)$ neighbours are isomorphic to S_k. The following estimation finishes the proof:

$$e(G) = \frac{1}{2}nd(G) \leqslant \frac{1}{2}n\Delta(G) \leqslant \frac{1}{2}(k-2)n.$$

□

Theorem 3.5 ([2])**.** *Let $k \geqslant 2$ be an integer, and let G be a graph on n vertices that does not contain P_k as a subgraph. Then*

$$e(G) \leqslant \frac{1}{2}(k-2)n,$$

This implies a bound $\text{ex}(n, P_k) = O(n)$.

Proof. The proof uses the method of mathematical induction. The base is the case $n \leqslant k-1$. Then, a graph G on n vertices trivially cannot contain P_k as a subgraph and has at most that many edges as the complete graph K_n. That is,

$$e(G) \leqslant e(K_n) = \frac{n(n-1)}{2} \leqslant \frac{n(k-2)}{2}.$$

Now suppose that the statement of the theorem holds for $n = 1, 2, \ldots, N-1$. Let G be a graph on N vertices with $\mathrm{ex}(N, P_k)$ edges, which does not contain P_k as a subgraph. Assume a contradiction, that is $e(G) > \frac{N(k-2)}{2}$. If G is a disjoint union of non-empty graphs $G_1, G_2, \ldots, G_m$, then $v(G_i) < v(G) = N$ for every $i \in [m]$. The induction hypothesis implies that $e(G_i) \leqslant \frac{k-2}{2} v(G_i)$, which results in a following inequality:

$$e(G) = \sum_{i=1}^{m} e(G_i) \leqslant \frac{k-2}{2} \sum_{i=1}^{m} v(G_i) = \frac{N(k-2)}{2}.$$

Thus, the graph G may be assumed to be connected. If G has a vertex of degree v at most $\frac{k-2}{2}$ then using an induction hypothesis for $G' = G[V(G) \setminus \{u, v\}]$, the number of edges in G can be estimated as

$$e(G) \leqslant e(G') + \frac{k-2}{2} \leqslant \frac{(N-1)(k-2)}{2} + \frac{k-2}{2} = \frac{N(k-2)}{2}.$$

Thus, it may be assumed that $\delta(G) \geqslant \frac{k-1}{2}$.

Let $v_1 v_2 \ldots, v_l$ be the longest path in G. By assumption, $l < k$. Suppose that G contains a cycle of length l, that is $w_1 w_2 \ldots w_l w_1$. If for any $p \in [l]$, the vertex w_p has a neighbour v not in the given cycle, then $w_{p+1} w_{p+2} \ldots w_l w_1 w_2 \ldots w_p v$ would be a path of length $l+1$ in G, which contradicts the choice of l. Otherwise, $\{w_1, w_2, \ldots, w_l\}$ is a connectivity component of G with $l < k \leqslant N$ vertices, which contradicts with G being connected. Thus, G contains no cycles of length l.

Since $v_1 v_2 \ldots v_l$ is the longest path in G, then v_1, v_l have no neighbours outside the path. Moreover, $v_1 v_l \notin E(G)$, since G contains no cycles of length l. Let $N_G(v_1) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$. By assumption, $r \geqslant \frac{k-1}{2}$. If for any $s \in [r]$ there is an edge between v_l and $v_{i_s - 1}$, then $v_{i_s} v_{i_s+1} \ldots v_l v_{i_s-1} v_{i_s-2} \ldots v_1 v_{i_s}$ is a cycle of length l in G; a contradiction. Hence, among $l-1$ possible neighbours of v_l there are r prohibited. Thus,

$$d_G(v_l) \leqslant l - 1 - r \leqslant k - 2 - \frac{k-1}{2} \leqslant \frac{k-3}{2};$$

a contradiction. Thus, $l \geqslant k$; a contradiction with G containing no P_k as a subgraph. $\square$

Example 3.6. Let $n = r(k-1) + q$, $q < k-1$. Then a disjoint union of r copies of K_{k-1} and a copy of K_q trivially contains no paths of length k. This graph has

$$\frac{r(k-1)(k-2) + q(q-1)}{2} = \frac{1}{2}(k-2)n - \frac{1}{2}q(k+1-1) \geqslant \frac{1}{2}(k-2)n - \frac{1}{8}k^2 \qquad (5)$$

edges. Thus, the bound in the theorem 3.5 is asymptotically sharp.

Rem 3.7. The example 3.6 provides a lower bound for any connected graph on k vertices. In particular, for any tree T_k of size k. It was conjectured by Erdős and Sós that the statement of the theorem 3.5 holds for any tree T_k instead of P_k.

Next theorem provides a bound on extremal numbers of even cycles. According to [21], the proof was discovered by Erdős, but has not been published. The first proof was published in [9].

Theorem 3.8. *Let $k \geqslant 2$ be a natural number. Then*

$$\mathrm{ex}(n, C_{2k}) = O(n^{1+1/k}). \tag{6}$$

The proof of this theorem requires some preparation. Lemma 3.10 provides some technical tools that would be used in the proof, while Lemma 3.11 helps finding a cycle in graphs with large $\delta(G)$.

Definition 3.9. Let $G = (V, E)$ be a graph and t be a natural number. A *$t-$periodic colouring* of G is a function $f : V(G) \to \mathbb{N}$, such that the endpoints of any path of length t have the same colour (image under f).

Lemma 3.10. *Let t be a natural number and G be a connected graph for which*

$$e(G) \geqslant 2tv(G). \tag{7}$$

Then, any $t-$periodic colouring of G has at most two colours.

Proof. The key step of the proof is to find two adjacent vertices, which are connected by two vertex-disjoint paths of length at least t.

Firstly, suppose that $\delta(G) \geqslant 2t$. Then consider a longest path in G and denote it $v_1v_2 \dots v_m$. Since it is a longest path, then v_1 is adjacent only to vertices in this path. Let the indices of these neighbours of v_1 be $i_1 < i_2 < \dots < i_{d_G(v_1)}$. By assumption, $d_G(v_1) \geqslant \delta(G) \geqslant 2t$. Then, trivially $i_{2t} \geqslant i_t + t \geqslant 2t$, thus v_1 and v_{i_t} are adjacent with two paths of length t connecting them, namely $v_1v_2 \dots v_{i_t}$ and $v_1v_{i_{2t}}v_{i_{2t}-1} \dots v_{i_t}$.

Now the statement can be proved by induction. Clearly, (7) implies that $v(G) > 4t$, which would be the base of induction. Suppose that any graph on at most n vertices satisfying (7) has such pair of vertices. Let G be a graph on $n+1$ vertices. It has either $\delta(G) \geqslant 2t$, and thus a required pair of vertices or there exists $v \in V(G) \colon d_G(v) < 2t$. Let $G' = G[V(G) \setminus \{v\}]$ be a graph with this vertex removed. Then $v(G') = n$ and $e(G') > e(G) - 2t \geqslant 2tn - 2t = 2t(n-1)$. Thus, G' contains the required pair of vertices by induction hypothesis.

This pair of adjacent vertices with two paths form three cycles C^1, C^2, C^3 with lengths l_1, l_2, l_3. Fix a $t-$periodic colouring f and let $t_1, t_2, t_3 \leqslant t$ be the minimal numbers, such that f is t_i-periodic on C^i. Notice that if one of the cycles is $k-$periodic for some k, then all of them are $k-$periodic and $k \mid l_i$. Thus, $t_1 = t_2 = t_3$. However, since $l_1 + l_2 - l_3 = 2$, provided that C^3 is the longest among three cycles, then $t_1 = t_2 = t_3 = 2$, so this parts of the graph is coloured on two colours.

Since G is connected, then any vertex $v \in V(G)$ is connected to C^3. Taking a path from v to C^3 and extending it along the cycle, so its length would be divisible by t, forces v to be one of two colours used in C^3. Hence, the colouring has at most two colours. $\square$

Lemma 3.11. *Let G be a bipartite graph on n vertices with $\delta = \delta(G) \geqslant \max(5ln^{1/l}, 50l)$ for some integer $l \geqslant 2$. Then G contains C_{2l} as a subgraph.*

Proof. Suppose that G does not contain C_{2l} as a subgraph. Fix an arbitrary vertex $v \in V(G)$. Let V_i be the set of vertices at distance i from v. Notice that since G is bipartite, then each V_i is an independent set. The key point is to show that $|V_i|$ grows with i, more precisely

$$\frac{|V_i|}{|V_{i-1}|} \geqslant \frac{\delta}{5l} \geqslant n^{1/l},\ 1 \leqslant i \leqslant l. \tag{8}$$

The inequality (8) would imply that $|V_l| \geqslant (n^{1/l})^l = n$; a contradiction with $v(G) = n$.

Proving (8) by induction on i, base holds trivially since $|V_1|/|V_0| = d_G(v) \geqslant \delta \geqslant \delta/5l$. Suppose that (8) holds for $i-1$.

Denote $H = G[V_{i-1} \cup V_i]$ and $H_1, H_2, \ldots, H_m$ be the connectivity components of H. Let $W_j = V(H_j) \cap V_{i-1}$. A path $v_1 v_2 \ldots v_k$ would be called *monotonic* if the distance between v and v_j is a monotonic function of j. Dealing with each H_j independently, the goal is to show that $e(H_j) < 4lv(H_j)$.

If $|W_j| = 1$, then $e(H_j) < v(H_j)$, thus it may be assumed that $|W_1| \geqslant 2$. Let $u \in V_p$ be the vertex satisfying the following two properties:

- there are two monotonic paths P^1, P^2 from u to W_j, that are vertex disjoint except for u;
- p is the minimal number for which such u exists.

Let P be the monotonic path from v to u. Let $w \in W_j$ be a vertex. There is a monotonic path P^3 from v to w. If P^3 does not intersect with P^1, then PP^1 intersects P^3 only in V_k, $k < p$. Thus, p is not minimal, which is required by (3.2). Hence, P^3 intersects with P^1 and there exists a monotonic path from u to w. Call all the vertices $w \in W_j$, that can be reached from u by monotonic path which is vertex-disjoint with P^2 (except for u), red. And call all other vertices in W_j blue. The goal is to show that any pair of differently coloured vertices can be reached from u by internally vertex-disjoint monotonic paths.

Take any pair of differently coloured vertices, red w_1 and blue w_2, and denote the corresponding monotonic paths from u as Q^1, Q^2. Let w' be their last intersecting vertex, and w'' be the last intersecting vertex of Q^2 and P^2 (which exists by definition of blue vertices). If w' is presented in Q^2 after w'', then w_2 is in fact red: let Q be a path from u to w' along Q^1 and then from w' to w_2 along Q^2. This path does not intersect with P^2; a contradiction. Hence, w'' is presented in Q^2 after w'. Let Q' be a path from u to w'' along P^2 and then from w'' to w_2 along Q^2. The path Q' does not intersect with Q^1. Hence, any pair of differently coloured vertices can be reached by internally vertex-disjoint monotonic paths from u. Assign the vertices from $V(H_j) \cap V_i$ to the green colour. Notice that this 3-colouring of H_j is in fact t−periodic for $t = 2(l - i + p + 1)$. Clearly, since t is even and H_j is bipartite, no path of length t can join any green and any differently coloured vertex. However, if red w_1 and blue w_2 are joined by a path of length t, then this path together with two internally disjoint paths from w_1 and w_2 to u create a cycle of length $2l$; a contradiction.

Hence, Lemma 3.10 implies that $e(H_j) < 2tv(H_j) < 4lv(H_j)$. Summing these inequalities for all $1 \leqslant j \leqslant m$, it follows that

$$e(H) \leqslant 4lv(H). \tag{9}$$

Let $H' = G[V_{i-1} \cup V_{i-2}]$. By induction hypothesis, it is known that $\frac{|V_{i-1}|}{|V_{i-2}|} \geqslant \delta/5l$ and, similarly to the (9), $e(H') < 4lv(H')$. Combining these two inequalities, it follows that

$$4l(v(H') + v(H)) = 4l(|V_{i-2}| + 2|V_{i-1}| + |V_i|) \geqslant e(H) + e(H') \geqslant \delta|V_{i-1}|,$$

and so

$$|V_i| \geqslant \frac{1}{4l}((\delta - 8l)|V_{i-1}| - 4l|V_{i-2}|) > \frac{1}{4l}\left(\delta - 8l - \frac{20l^2}{\delta}\right)|V_{i-1}| > \frac{\delta}{5l}|V_{i-1}|,$$

where the induction hypothesis is used for the first inequality, and estimation $\delta \geqslant 50l$ for the second one; the statement of the lemma is proved. □

The final lemma contains the statement that is in fact much stronger than the theorem 3.8. The theorem immediately follows from the lemma for G with at least $100kn^{1+1/k}$ edges.

Lemma 3.12. *Let G be a graph on n vertices and $l \geqslant 2$ be an integer such that*

$$l \leqslant \frac{e(G)}{n}, \quad ln^{1/l} \leqslant \frac{e(G)}{10n}. \tag{10}$$

Then G contains C_{2l} as a subgraph.

Proof. The proof is by induction on n with base $n = 1$ being trivial since (10) cannot hold for $n = 1$.

Suppose that lemma holds for all graphs on $n - 1$ vertices. Let G be a graph on n vertices and $l \geqslant 2$ satisfy (10). Take a bipartite subgraph G' of G provided by Lemma 2.7. By construction, for every $v \in V(G') = V(G)$ it holds that $d_{G'}(v) \geqslant d_G(v)$. If $\delta(G') \geqslant e(G)/2n$, then G' contains C_{2l} as a subgraph by Lemma 3.11. Otherwise, consider $G'' = G[V(G) \setminus v]$, where $v \in V(G)\colon d_G(v) \leqslant 2d_{G'}(v) < e(G)/n$. Notice that $e(G'') \geqslant \frac{n-1}{n}e(G)$. Hence, if l satisfies (10) for G then it satisfies analogous inequalities for G''. Thus, by induction hypothesis, G'' contains C_{2l} as a subgraph. □

of Theorem 3.8. Let G be a graph with $100kn^{1+1/k}$ edges. Since k satisfies both inequalities on l in (10), then Lemma 3.12 implies that G contains C_{2k} as a subgraph. □

Contrary to 3.6, the proof of Theorem 3.8 does not provide a construction for an extremal graph. However, for $k = 2, 3, 5$, the bound (6) is known to be sharp. Constructions of Benson [4] and Brown [5] rely on a finite projective geometry. Here, the construction of Wenger from [13] is presented.

Theorem 3.13.

$$\mathrm{ex}(n, C_{2k}) = \Omega(n^{1+1/k})$$

for $k = 2, 3$ and 5.

Proof. Let p be a prime. Fix $k \in \{2, 3, 5\}$ and define a bipartite graph $G = G_k(p)$ as follows. Let $V_1, V_2 \subset V(G)$ be the partition of G. Each vertex in V_1 and V_2 is an element of F_p^k, where $F_p = \{0, 1, \dots, p-1\}$ is the finite field of characteristics p. Thus, both V_1 and V_2 have p^k vertices. Vertices $u = (u_0, u_1, \dots u_{k-1}) \in V_1$ and $v = (v_0, v_1, \dots, v_{k-1}) \in V_2$ are adjacent if $v_j \equiv u_j - u_{j+1}v_{k-1} \mod p$ for $j = 0, \dots, k-2$.

For every $u \in V_1$ and any $a \in \{0, 1, \cdots p-1\}$ there is a unique $v \in V_2$, which is adjacent to u and $v_{k-1} = a$. Thus, every vertex in V_1 has degree p and $e(G) = p^{k+1} = (v(G)/2)^{1+1/k}$.

Suppose that G contains a cycle of length $2k$. Let u, v, u' be three consecutive vertices in this cycle with $u \in V_1$. By construction,

$$u_j - u_{j+1}v_{k-1} \equiv v_j \equiv u'_j - u'_{j+1}v_{k-1} \mod p, \ j = 0, \dots, k-2.$$

Thus, $u_j - u'_j \equiv (u_{j+1} - u'_{j+1})v_{k-1} \mod p, \ j = 0, \dots, k-2$. Iterating this congruency, it follows that

$$u_j - u'_j \equiv (u_{k-1} - u'_{k-1})v_{k-1}^{k-j-1}, j = 0, \dots, k-1. \tag{11}$$

Hence, $u_{k-1} = u'_{k-1}$ implies $u = u'$; a contradiction.

Denote the aforementioned cycle as $u_0v_0u_1v_1 \dots u_kv_k$, with $u_0 \in V_1$. Notice that now indices refers to different vertices of G and not coordinates. Let $s_i = (v_i)_{k-1}$ and $t_i = (u_i)_{k-1} - (u_{(i+1) \mod p})_{k-1}$. The congruence (11) implies that

$$\begin{aligned}
(u_0)_j - (u_1)_j &\equiv t_0 s_0^{k-j-1} \mod p,\\
(u_1)_j - (u_2)_j &\equiv t_1 s_1^{k-j-1} \mod p,\\
&\vdots\\
(u_{k-1})_j - (u_0)_j &\equiv t_{k-1} s_{k-1}^{k-j-1} \mod p,
\end{aligned}$$

for every $j = 0, \dots, k-1$. Summing this congruences for every j results in a following set of congruences:

$$\sum_{i=0}^{k-1} t_i s_i^{k-j-1} \equiv 0 \mod p, \ j = 0, \dots, k-1. \tag{12}$$

The matrix $S = (s_i^j)_{i,j}$ is a Vandermonde matrix over a field F_p. Since $t = (t_{k-1-i})_{i=0}^{k-1}$ is a solution to $St = 0$ in F_p and all coordinates are not zero, then each column of S is a linear combination of other columns due to properties of Vandermonde matrices, which implies that for any $i = 0, \dots, k-1$ there exists $i' \in 0, \dots, k-1$, such that $s_i = s_{i'}$.

If $k = 2$ then for any cycle $u_0v_0u_1v_1$ in $G_2(p)$, it holds that $(v_0)_1 = (v_1)_1$ with both v_0 and v_1 being adjacent to the same vertex u_0; a contradiction.

If $k = 3$ then for any cycle $u_0v_0u_1v_1u_2v_2$ in $G_3(p)$, it holds that $(v_0)_1 = (v_1)_1$ or $(v_0)_1 = (v_2)_1$. Both pairs (v_0, v_1) and (v_0, v_2) have non-zero codegree; a contradiction.

If $k = 5$ then similarly, there are three vertices in a cycle in V_2 with equal last coordinates. At least one pair of them has non-zero codegree; a contradiction. □

Note. Clearly, this construction fails for any other value of k. Vertices $v_1, v_2, \dots v_k$ can be partitioned in such a way that no consecutive pair of vertices is in the same part.

Rem 3.14. The graph described in the proof of Theorem 3.13 contains no 4-cycles, hence no copies of Q_8 as subgraphs. Thus, $\mathrm{ex}(n, Q_8) = \Omega(n^{3/2})$. Notice that this bound is sharper than the bound given by Theorem 3.1.

3.3 Erdős Reduction Theorem

The following theorem provides a tool for estimating the extremal numbers of graphs using known extremal numbers for its subgraph. Firstly a new notation is needed.

Definition 3.15. Let G_1, G_2 be bipartite graphs with partitions (A_1, B_1), (A_2, B_2). Denote

$$G = G_1 * G_2$$

a graph G obtained by completely joining A_1 to A_2 and B_1 to B_2, that is adding all edges between A_1 and B_2, and A_2 and B_1.

Example 3.16. Figure 2 shows graphs $K_{1,1} * P_6$ and $K_{1,1} * C_6$. In both graphs, green edges are the edges of the initial graph, red edge is the added $K_{1,1}$, and blue edges are the edges of the join.

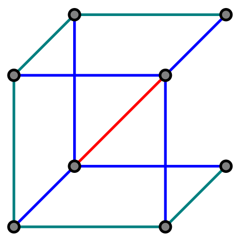

Figure 2: Bipartite joins of $K_{1,1}$ with P_6 and C_6.

Theorem 3.17 ([8])**.** *Let H be a bipartite graph with $\mathrm{ex}(n, H) = O(n^{2-a})$, $a \geqslant 1$, and let $t \geqslant 1$ be a natural number. Let $b > 0$ be defined by $\frac{1}{b} = \frac{1}{a} + t$. Then,*

$$\mathrm{ex}(n, K_{t,t} * H) = O(n^{2-b}) \tag{13}$$

Proof. Applying Lemma 2.9, it is enough to prove the theorem for $K-$almost regular graphs. Notice that proving the theorem for $t = 1$ would imply the theorem for any t, as the theorem for $t = 1$ can be applied any number of times. So the proof can be restricted to the case $t = 1$.

The idea of the proof is to find an edge $uv \in E(G)$, which is included in many copies of $K_{2,2}$. Then if the the induced graph on their neighbours $G[N_G(u) \cup N_G(v) \setminus \{u, v\}]$ contains H, this would imply that G contains $K_{1,1} * H$.

Since $a \geqslant 1$ then $b \leqslant 1/2$. Let G be a graph without a copy of $K_{1,1} * H$ as a subgraph. Assume that $e(G) > 1000n^{3/2}$. Then, the number of copies of $K_{2,1}$ in G can be estimated as

$$
\begin{aligned}
A &= \sum_{v\in G} \frac{d_G(v)^2 - d_G(v)}{2} \\
&\geqslant \frac{1}{2}\left(\frac{\left(\sum_{v\in G} d_G(v)\right)^2}{n} - \sum_{v\in G} d_G(v)\right) \\
&= \frac{1}{2}\left(\frac{4e(G)^2}{n} - e(G)\right) \\
&> \frac{e(G)^2}{n}.
\end{aligned} \tag{14}
$$

Here the Cauchy-Schwartz inequality was used for the first inequality. Second inequality holds due to the constraint on $e(G)$.

For a pair of distinct vertices $u, v \in G$, number of $K_{2,2}$ in G having u, v as its component, equals to $(d_G(u,v)^2 - d_G(u,v))/4$. The additional factor $\frac{1}{2}$ comes from the fact that each copy of $K_{2,2}$ is counted twice this way. On another hand, the number of $K_{2,1}$ in G having u, v as its component, equals to $d_G(u,v)$. Thus the number of copies of $K_{2,2}$ in G can be estimated as

$$
\begin{aligned}
B &= \frac{1}{2}\sum_{u,v\in V} \frac{d_G(u,v)^2 - d_G(u,v)}{2} \\
&\geqslant \frac{1}{4}\frac{2\left(\sum_{u,v\in V} d_G(u,v)\right)^2}{n(n-1)} - \sum_{u,v\in V} d_G(u,v) \\
&= \frac{1}{4}\left(\frac{2A^2}{n(n-1)} - A\right) \\
&> \frac{A^2}{3n^2}.
\end{aligned} \tag{15}
$$

The last inequality here follows from the estimation $A > 10^6 n$. Thus, the number of copies of $K_{2,2}$ in G at least $\frac{e(G)^4}{3n^4}$. Then, by the pigeonhole principle, there exists an edge $uv \in E(G)$, such that uv is an edge in at least $\frac{e(G)^3}{3n^4}$ copies.

Notice that, if an induced graph $G' = G[N_G(u) \cup N_G(v) \setminus \{u, v\}]$ on neighbours of u and v (excluding u, v) contains a copy of H as a subgraph, then G contains a copy of $K_{1,1} * H$ as a subgraph. Since G is $K-$almost regular, then the degrees of u and v are at most $\frac{2Ke(G)}{n}$. Thus, induced subgraph G' has at most $\frac{4Ke(G)}{n}$ vertices, and the following estimation holds:

$$
\frac{e(G)^3}{3n^4} \leqslant \mathrm{ex}\left(\frac{4Ke(G)}{n}, H\right) \leqslant C\left(\frac{4Ke(G)}{n}\right)^{2-a}.
$$

Moving n to the right and $e(G)$ to the left, the following holds:

$$
e(G)^{1+a} \leqslant C' n^{2+a},
$$

which is equivalent to $e(G) = O(n^{2-b})$ for $b = \frac{1}{1+1/a}$, which finishes the proof. □

This theorem implies two important bounds on extremal numbers of graphs close to Q_8.

Corollary 3.18. Let Q_8^- be a graph obtained from Q_8 by deleting an edge. Notice that Q_8^- is a subgraph of $K_{1,1} * P_6$. Then Theorems 3.5, 3.17 imply that $\mathrm{ex}(n, Q_8^-) = O(n^{3/2})$.

Corollary 3.19. Let Q_8^+ be a graph obtained from Q_8 by adding an edge between a pair of opposite vertices. Notice that $Q_8^+ = K_{1,1} * C_6$. Then Theorems 3.8, 3.17 imply that $\mathrm{ex}(n, Q_8^+) = O(n^{8/5})$, hence also $\mathrm{ex}(n, Q_8) = O(n^{8/5})$.

Corollary 3.20. Let $K_{a,b}$ be a complete bipartite graph with $a \leqslant b$. Then $\mathrm{ex}(n, K_{a,b}) = O(n^{2-1/a})$.

Proof. Notice that $K_{a,b} = K_{a-1,a-1} * S_{b-a+2}$. Theorems 3.4 and 3.17 imply the bound $\mathrm{ex}(n^{2-t})$ for $1/t = 1 + a - 1 = a$, thus $t = 1/a$.

This bound is believed to be sharp, however, the matching lower bound is known only for $a = 2$ (see Theorem 3.13), $a = 3$ and $b \gg a$ (see [26]). $\square$

4 Counting homomorphic copies

This section presents the method of counting homomorphic copies of a graph H in G, or alternatively homomorphisms from H to G. A lower bound for such homomorphisms of certain graphs H is provided by Sidorenko conjecture. On the other hand, an upper bound can be obtained by some type of counting.

4.1 Sidorenko conjecture

The Sidorenko conjecture provides a lower bound on a number of homomorphisms from a bipartite graph H to an arbitrary graph G. This conjecture was formulated by Sidorenko in [11] and proved for certain families of graphs, such as trees, cycles, complete bipartite graphs, and a few others.

Conjecture 1. Let H be a bipartite graph, and let G be a graph on n vertices with $pn^2/2$ edges. Then the number of homomorphisms $f \colon H \to G$ is at least $p^{e(H)}n^{v(H)}$.

This conjecture can be interpreted as follows: the density of homomorphic copies of a bipartite graph H is minimized by a random graph. If in a graph on n vertices each edge is taken with probability p, then among $v(H)-$tuples of vertices, the expected proportion of homomorphic copies of H equals to $p^{e(H)}$.

The following proposition states that the cube edge graph satisfies the Sidorenko conjecture.

Proposition 4.1. *The graph Q_8 satisfies the Sidorenko conjecture, that is for any graph G with n vertices and $pn^2/2$ edges the number of homomorphisms $f \colon Q_8 \to G$ is at least $p^{12}n^8$.*

Several proofs of this proposition exist, including the results of Sidorenko [12] and Hatami [19]. Each of them requires heavy machinery which is beyond the scope of this thesis.

4.2 Counting homomorphic copies of Q_8

This section analyzes the approach, presented by Janzer and Sudakov in [27]. If a graph G contains no H as a subgraph then there are no injective homomorphisms $f \colon H \to G$. For a non-injective homomorphism $f \colon H \to G$, the image $f(H) \subseteq G$ is a graph that is smaller than H. After using this approach several times, the initial problem can be reduced to a trivial one, that is counting homomorphic copies of Q_8 can be reduced to counting homomorphic copies of a star graph.

In this subsection vertices of a graph Q_8 are denoted as $\{0,1\}^3 = \{000, 001, \dots, 111\}$. Furthermore, for any $A \subseteq V(H)$, the number of graph homomorphisms f from H to G, such that $|f(A)| = 1$, is denoted as $\hom(H, G; A)$.

Lemma 4.2. *Let G be a graph. Then,*

$$\hom(Q_8, G; \{000, 011\})^2 \leqslant \hom(Q_8, G; \{000, 011, 101\}) \hom(Q_8, G). \tag{16}$$

In addition,

$$\hom(Q_8, G; \{000, 011, 101\})^2 \leqslant \hom(Q_8, G; \{000, 011, 101, 110\}) \hom(Q_8, G). \tag{17}$$

Proof. Let $\phi\colon Q_8[\{000,001,110,111\}] \to G$ be a homomorphism. Notice that $Q_8[\{000,001,110,111\}]$ is a disjoint union of two edges. Thus, ϕ basically takes two edges of G. Now let a_ϕ be the number of maps $\psi\colon \{010,011\} \to G$, such that ϕ and ψ together induce a homomorphism. By the symmetry of the graph Q_8, a_ϕ is equal to the number of maps $\psi\colon \{100,101\} \to G$, such that ϕ and ψ together induce a homomorphism. Additionally, let b_ϕ be the number of maps $\psi\colon \{010,011\} \to G$ such that ϕ and ψ together induces a homomorphism and $f(000) = g(011)$. Likewise, b_ϕ is equal to the number of maps $\psi\colon \{100,101\} \to G$ such that ϕ and ψ together induce a homomorphism. Now, the number of specific homomorphisms from Q_8 to G can be written as

$$\hom(Q_8,G) = \sum_\phi a_\phi^2,$$

$$\hom(Q_8,G;\{000,011\}) = \sum_\phi a_\phi b_\phi,$$

$$\hom(Q_8,G;\{000,011,101\}) = \sum_\phi b_\phi^2.$$

Here the summation is taken over all possible homomorphisms $\phi\colon Q_8[\{000,001,110,111\}] \to G$. This follows from the observation that the images of $\{100,101\}$ and $\{010,011\}$ can be chosen independently of each other, since there are no edges between these two pairs of vertices. The Cauchy-Schwartz inequality finishes the proof of the first inequality.

Now continuing with the second inequality, for any homomorphism $\phi\colon Q_8[\{100,101,010,011\}] \to G$, such that $\phi(101) = \phi(011)$, let c_ϕ be the number of maps $\psi\colon \{000,001\} \to G$ such that ϕ and ψ together induce a homomorphism. The symmetry of Q_8 implies that c_ϕ is equal to the number of maps $\psi\colon \{110,111\}$ such that ϕ and ψ together induce a homomorphism. Additionally, let d_ϕ be the number of maps $\psi\colon \{000,001\} \to G$ such that ϕ and ψ together induce a homomorphism and $\psi(000) = \phi(101) = \phi(011)$. The symmetry of Q_8 implies that d_ϕ is equal to the number of maps $\psi\colon \{110,111\}$ such that ϕ and ψ together induce a homomorphism and $\psi(110) = \phi(101) = \phi(011)$. Similarly, to the first inequality, the numbers of specific homomorphisms from Q_8 to G can be written as

$$\hom(Q_8,G) \geqslant \hom(Q_8,G;\{101,011\}) = \sum_\phi c_\phi^2,$$

$$\hom(Q_8,G;\{000,101,011\}) = \sum_\phi c_\phi d_\phi,$$

$$\hom(Q_8,G;\{000,101,011,110\}) = \sum_\phi d_\phi^2.$$

Here the summation is taken over all homomorphisms $\phi\colon Q_8[\{100,101,010,011\}] \to G$, such that $\phi(101) = \phi(011)$. Cauchy-Schwartz inequality finalizes the proof of the second inequality. □

Combining the inequalities (16) and (17), the following inequality holds.

Corollary 4.3. Let G be a graph. Then

$$\hom(Q_8, G; \{000, 011, 101, 110\}) \geqslant \frac{\hom(Q_8, G, \{000, 011\})^4}{\hom(Q_8, G)^3}.$$

The left-hand side of this inequality represents the number of homomorphisms from a star graph with four edges to G and the number of such homomorphisms can be easily estimated.

Theorem 4.4. *Let G be a graph on n vertices with edge density $p = Cn^{-3/8}$. Then G contains Q_8 as a subgraph provided that C is sufficiently large.*

Proof. Denote the maximal degree of G as $\Delta = \Delta(G)$. Similarly to the previous methods, Lemma 2.9 allows to restrict the problem to a $K-$almost regular graphs for some constant K. Suppose that G does not contain Q_8 as a subgraph. Furthermore, assume that $\hom(Q_8, G; \{000, 011\}) \geqslant \hom(Q_8, G)/12$. Then, Lemma 4.3 implies that

$$\hom(Q_8, G; \{000, 011, 101, 110\}) \geqslant \hom(Q_8, G)/12^4. \tag{18}$$

The left-hand side in (18) is the number of homomorphisms from the star graph to G and can be trivially bounded from above as

$$\hom(Q_8, G; \{000, 011, 101, 110\}) \leqslant n\Delta^4 \leqslant n(Kpn)^4 = K^4p^4n^5, \tag{19}$$

that is n choices of an image of 000 and at most Δ choices for each of the images of 001, 010, 100, 111. The right-hand side in (18) can be bounded below using Lemma 4.1 as

$$\hom(Q_8, G)/12^4 \geqslant 12^{-4}p^{12}n^8. \tag{20}$$

Combining the inequalities (18), (19) and (20), it follows that

$$K^4p^4n^5 \geqslant 12^{-4}p^{12}n^8 \iff p \leqslant (12K)^{1/2}n^{-3/8}, \tag{21}$$

which contradicts with the assumption $p = Cn^{-3/8}$ provided that C is a sufficiently large constant.

Hence, $\hom(Q_8, G; \{000, 011\}) < \hom(Q_8, G)/12$. However, any non-injective homomorphism $f\colon Q_8 \to G$ has a pair of vertices u, v of distance 2 such that $f(u) = f(v)$. There are 12 such pairs of vertices in Q_8. Symmetry of Q_8 implies that $\hom(Q_8, G; \{u, v\}) = \hom(Q_8, G; \{000, 011\})$. Combining these inequalities, it follows that

$$\hom(Q_8, G) \leqslant \sum_{u,v \in Q_8\colon\ \mathrm{dist}(u,v)=2} \hom(Q_8, G; \{u, v\}) = 12 \hom(Q_8, G; \{000, 011\}) < \hom(Q_8, G);$$

a contradiction. Thus, there exists an injective homomorphism from Q_8 to G, that is G contains Q_8 as a subgraph. □

Note. This method originally was extended by Janzer and others to deal with hypercube edge graphs. A hypercube edge graph Q_{2^n} is a graph with $V = \{0, 1\}^n$ and edges between vertices that differ only in one coordinate. This method results in a bound $\mathrm{ex}(n, Q_{2^n}) = O(n^{1-1/(n-1)-1/(n-1)2^{n-1}}$, improving the bound obtained by dependent random choice, see Corollary 5.4).

5 Dependent Random Choice

Dependent random choice is a powerful tool in combinatorics. It can be used for several problems in extremal graph theory, including Ramsey numbers and forbidden graph problem. Although it is hard to say who introduced it first, few of the earliest applications were by Rödl [16] and Sudakov [18]. The idea of this method is that in a dense enough graph G, there exists a set A of vertices with a large neighbourhood $N(A)$. The following lemma is a typical Dependent Random Choice result and exists in many variations. The presented variation is from the review by Fox and Sudakov.

Lemma 5.1 ([20])**.** *Let G be a graph on n vertices. Denote the average degree of G by $d = d(G)$. Let a, b and k be positive integers. If there exists a positive integer q, such that*

$$\frac{d^q}{n^{q-1}} - \binom{n}{b}\left(\frac{k}{n}\right)^q \geqslant a, \tag{22}$$

then G contains a set A of a vertices such that for any b vertices from A, they have at least k common neighbours.

Proof. Pick a set Q of q vertices from $V(G)$ uniformly with repetitions. Let $X = |N(Q)|$. Then the expectation of X can be estimated as

$$\mathbb{E}[X] = \sum_{v \in V(G)} \left(\frac{d_G(v)}{n}\right)^q = n^{-q} \sum_{v \in V(G)} d_G(v)^q \geqslant n^{1-q} \left(\frac{\sum_{v \in V(G)} d_G(v)}{n}\right)^q = \frac{d^q}{n^{q-1}},$$

where the last inequality follows from the Jensen inequality for the function $x \mapsto x^q$. Let Y be the random variable counting the number of subsets $P \subseteq N(Q)$ of size a with $|N(P)| < k$. For a given set of vertices P of size b, the probability of it being a subset of $N(Q)$ equals $\left(\frac{|N(P)|}{n}\right)^q \leqslant \left(\frac{k}{n}\right)^q$. Since there are $\binom{n}{b}$ such sets P, then

$$\mathbb{E}[Y] \leqslant \binom{n}{b}\left(\frac{k}{n}\right)^q.$$

Hence,

$$\mathbb{E}[X - Y] = \frac{d^q}{n^{q-1}} - \binom{n}{b}\left(\frac{k}{n}\right)^q \geqslant a,$$

and there exists a set Q for which $X(Q) - Y(Q) \geqslant a$. Delete one vertex from each subset of $N(Q)$ with fewer than k common neighbours. Then A is defined as the remaining part of $N(Q)$, and the estimation $|A| \geqslant X - Y \geqslant a$ holds. Clearly, A satisfies the desired property. □

In the forbidden subgraph problem, the main application of the Dependent Random Choice is to bound the extremal numbers of bipartite graphs with bounded degree in one of its parts. The following theorem is due to Alon, Krivelevich and Sudakov [17]. Firstly, a simple embedding lemma that connects the Lemma 5.1 and the forbidden subgraph problem, is presented.

Lemma 5.2. *Let H be a bipartite graph with parts A and B of sizes a and b, respectively. Suppose that the maximum degree of vertices in B is d. If a graph $G = (V, E)$ has a subset Q of vertices with $|Q| = a$ such that every subset $Q' \subseteq Q$ of size d satisfies $|N(Q')| \geqslant a + b$, then G contains H as a subgraph.*

Proof. Finding H as a subgraph of G is equivalent to finding an injective graph homomorphism $\phi\colon H \to G$. Define $\phi\colon A \to Q$ arbitrarily. Now, the embedding of $B = \{v_1, \ldots, v_b\}$ can be done one vertex at a time. Let v_j be the next vertex to be embedded. By assumption, $d_H(v_j) \leqslant d$. Thus, $|\phi(N_H(v_j))| \leqslant d$, and by choice of the set Q, $|N_G(\phi(N_H(v_j)))| \geqslant a + b$. Since the total number of embedded vertices is less than $a + b$, there exists a valid image of v_j. Hence, all vertices from B can be embedded. □

Theorem 5.3. *Let H be a bipartite graph with parts A and B and suppose that the maximum degree of vertices in B is Δ. Then,*

$$\mathrm{ex}(n, H) = O(n^{2-1/\Delta}).$$

Proof. Denote $a = |A|, b = |B|, c = \max\left(a^{1/d}, 3(a+b)/d\right)$ and $d = d(G)$. Suppose that G is a graph with n vertices and $cn^{2-1/d}$ edges. Taking $q = \Delta$ and utilizing the inequality $\Delta! \geqslant (\Delta/e)^\Delta$, it follows that

$$\frac{d^q}{n^{q-1}} - \binom{n}{\Delta}\left(\frac{a+b}{n}\right)^q \geqslant (2c)^\Delta - \frac{n^\Delta}{\Delta!}\left(\frac{a+b}{n}\right)^\Delta \geqslant (2c)^\Delta - \left(\frac{e(a+b)}{\Delta}\right)^\Delta \geqslant c^\Delta \geqslant a.$$

Hence, Lemma 5.1 implies that there exists a set of vertices Q with $|Q| = a$, such that any subset $Q' \subseteq Q$ of size Δ satisfies $|N(Q')| \geqslant a + b$. Now Lemma 5.2 finishes the proof. □

Corollary 5.4. For a hypercube edge graph Q_{2^n}, the extremal number can be bounded as $\mathrm{ex}(n, Q_{2^n}) = O(n^{2-1/n})$.

Note. Despite giving a relatively weak estimate for Q_8, that is $\mathrm{ex}(n, Q_8) = O(n^{5/3})$, Theorem 5.3 provides the best possible estimation in general, see [15]. Meaning that for any choice of Δ, there exists a graph H which satisfies the properties of the Theorem 5.3 and $\mathrm{ex}(n, H) = \Theta(n^{2-1/\Delta})$.

Dependent random choice gives a good bound for sparse graphs, while Erdős Reduction Theorem 3.17 gives a good bound for dense graphs. These two results can be combined for partly dense and partly sparse graphs.

Proposition 5.5. *Let $H = (A \sqcup B, E)$ be a bipartite graph with parts A and B. Suppose that there are at most k vertices in A with degree greater than d. Then $\mathrm{ex}(n, H) = O(n^{2-\frac{1}{k+d}})$.*

Proof. Denote by $A' \subseteq A$ the set of at most k vertices with degree at least d. Denote $H' = H[A' \sqcup B]$. Then Theorem 5.3 implies that $\mathrm{ex}(n, H') = O(n^{2-1/d}$. Since $H \subseteq K_{k,k} * H'$, then $\mathrm{ex}(n, H) = O(n^{2-\frac{1}{k+d}})$. □

6 Counting cycles in the edge graph

6.1 Tight bound for even prisms

This section presents the approach of Gao, Janzer, Liu and Xu from [24] for even prisms edge graphs. This approach is based on an equivalency of finding prism subgraphs in a graph and finding cycles in an augmented edge graph.

Firstly, define a *prismification* of a graph $H = (V, E)$, that is a graph $H^\square = (V \times \{0, 1\}, E')$ with $\{(u, i), (v, j)\} \in E'$ if and only if either $u = v$, or $i = j$ and $uv \in E$. Take any graph G and let $\mathcal{G}$ be a graph with $V(\mathcal{G}) = E(G)$ and $ef \in E(\mathcal{G})$ if there exist two edges $e', f' \in E(G)$ such that $ee'ff'$ is a 4-cycle in G. If G contains $H^\square$ as a subgraph, then $\mathcal{G}$ contains a copy of H. For a cube edge graph Q_8 this approach may be of interest since $Q_8 = C_4^\square$.

The main problem of this method is the lack of a reverse implication. If the graph $\mathcal{G}$ contains H as a subgraph, it does not imply that G contains a copy of $H^\square$.

Example 6.1. Graphs presented in the Figure 3 all induce a 4-cycle in their respective augmented graphs. The red edges correspond to the vertices in $\mathcal{G}$ that form a copy of C_4. However, only Q_8 contains $C_4^\square$ as a subgraph.

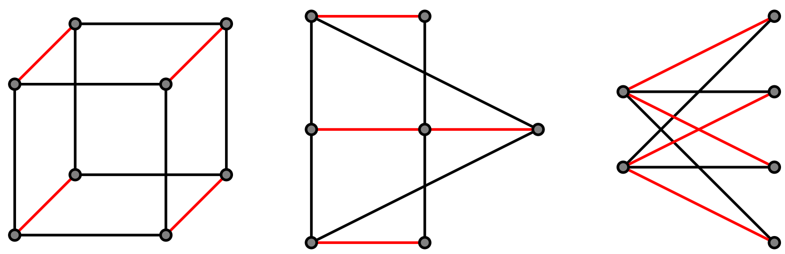

Figure 3: Graphs that induce C_4 in the augmented graph.

The following lemma helps to deal with this issue.

Lemma 6.2 ([25])**.** *Let $k \geqslant 2$ be an integer and G be a graph on n vertices. Let $\sim$ be a symmetric binary relation on $V(G)$ such that any vertices $u, v \in G$, vertex v has at most $\varepsilon d_G(v)$ neighbours w, such that $w \sim u$. If $\varepsilon < (2^{20}k^3(\log n)^4 n^{1/k})^{-1}$, then there exists an injective homomorphism $f : C_{2k} \to G$ such that $f(u) \nsim f(v)$ for all pairs $u, v \in V(C_{2k})$. In other words, G contains C_{2k} as a subgraph with no pairs of vertices being relatives.*

Note. The binary relation $\sim$ is not required to be transitive, thus in general there are no classes defined by it.

The next lemma provides an estimation of a number of cycles in a graph.

Lemma 6.3 ([28])**.** *Let G be a graph on n vertices with an average degree $d = d(G) \geqslant 2 \cdot 10^5 k^3 n^{1/k}$. Then G contains at least $\frac{1}{2}(2^{12}k)^{-k} d^{2k}$ subgraphs isomorphic to C_{2k}.*

Finally, the following lemma provides a bound for codegrees in $C_{2k}^\square$-free graphs.

Lemma 6.4 ([24]). *Fix an integer $k \geqslant 2$. Let G be a $C_{2k}^{\square}$-free K-almost regular graph on n vertices. There exists a constant $C = C(K,k)$ such that if $d(G) > Cn^{1/2}$, then for any edge $uv \in E(G)$ the number of 4-cycles $uvu'v'$ with $d_G(u,u') > Cd^{1/2}$ is at most Cd.*

Fix $k \geqslant 2$ and let G be a $C_{2k}^{\square}$−free graph on n vertices with an average degree $d = d(G) \geqslant \gamma(n)n^{-1/2}$, where $\gamma = \gamma(n)$ is a function to be chosen later, but at least a large constant. Due to lemmas 2.7 and 2.9, G may be assumed to be a bipartite K−almost regular graph. Let C be a constant from Lemma 6.4. Call a copy of C_4 $v_1v_2v_3v_4$ in G *thin,* if both codegrees of opposite pairs of vertices are at most $Cd^{1/2}$. Lemma 6.3 implies that there are at least cd^4 copies of C_4 in G.

The next step is to define an augmented edge graph $\mathcal{G}$. Suppose that there are at least $cd^4/2$ thin copies of C_4 in G. Define an augmented graph $\mathcal{G}$ with $V(\mathcal{G}) = \{(u,v) \mid u,v \in V(G),\ uv \in E(G)\}$ and $(u_1,v_1)(u_2,v_2) \in E(\mathcal{G})$ if $u_1v_1u_2v_2$ is a thin 4-cycle in G. Clearly, $v(\mathcal{G}) = 2e(G) = nd$ and $e(\mathcal{G}) \geqslant 2cd^4$, since for every thin 4-cycle there are 4 edges in $\mathcal{G}$. By Lemma 2.8, there exists a subgraph $\mathcal{G}'$ of $\mathcal{G}$ with $e(\mathcal{G}') \geqslant cd^4$ and

$$\delta(\mathcal{G}') \geqslant \frac{e(\mathcal{G}')}{4v(\mathcal{G}')} \geqslant \frac{e(\mathcal{G}')}{4v(\mathcal{G})} \geqslant \frac{cd^3}{4n}.$$

Let $m = nd = \gamma n^{3/2}$, then $e(\mathcal{G}') \geqslant cd^4 = c\gamma^4 n^2 = cm^{4/3}\gamma^{8/3}$ and $\delta(\mathcal{G}') \geqslant cm^{1/3}\gamma^{8/3}/4$.

Let $\sim$ be a binary relation on $V(H')$ such that $(u,v) \sim (u',v')$ if $\{u,v\} \cap \{u',v'\} \neq \varnothing$. Finding a copy of C_{2k} in $\mathcal{G}'$ with no related pairs of vertices, would imply that there is a copy of $C_{2k}^{\square}$ in G. To use Lemma 6.2, it is required to estimate the number of neighbours of $(u,v) \in V(\mathcal{G}')$ that are related. Let $w \in V(G)$ and $(u,v) \in V(\mathcal{G}')$. Fix the constant $D = \frac{8C}{c\gamma^2}$. Since opposite vertices of a thin copy of C_4 have codegrees at most $Cd^{1/2}$, then $\{u,v\}$ are in at most $2Cd^{1/2}$ thin copies of C_4 with w.

Combining this with $d_{\mathcal{G}'}((u,v)) \geqslant \delta(\mathcal{G}') \geqslant \frac{cd^4}{4n} = \frac{c\gamma^2 d}{4}$, it follows that (u,v) is adjacent to at most $Dd^{-1/2}d_{\mathcal{G}'}((u,v))$ vertices containing w. To use Lemma 6.2, it is enough to show that

$$Dd^{-1/2} \leqslant (2^{20}k^3\log^4 m \cdot m^{1/k})^{-1}.$$

Equivalently, expressing everything through γ and n,

$$\frac{8C}{c\gamma^2}\gamma^{-1/2}n^{-1/4} < \frac{\kappa}{\log^4(\gamma n^{3/2})(\gamma n^{3/2})^{1/k}},$$

with κ being some absolute constant. Since γ can be multiplied by any constant factor, it is enough to show that the function

$$f(\gamma,n) = \gamma^{\frac{1}{k}-\frac{5}{2}}n^{-\frac{1}{4}+\frac{3}{2k}}\log^4(\gamma n^{3/2}) \tag{23}$$

is asymptotically decreasing.

Finally, suppose there are less than $cd^4/2$ thin copies of C_4 in G. Without loss of generality, there are at least $cd^4/2$ copies of C_4 — $u_1u_2v_1v_2$ with $d_G(u_1,v_1) > Cd^{1/2}$. Using the pigeonhole principle, there exists an edge $u_1u_2 \in E(G)$ which can be extended to at least $\frac{cd^4}{4e(G)} = \frac{c\gamma^2 d}{8}$ such non-thin copies of C_4. Taking sufficiently large constant γ, Lemma 6.4 implies that G contains $C_{2k}^{\square}$ as a subgraph.

Notice that in (23) for $k \geqslant 7$, it is enough for γ to be a constant, implying the bound $\mathrm{ex}(n, C_{2k}^{\square}) = \Theta(n^{3/2})$. The lower bound is obtained by C_4-free graph with $\Theta(n^{3/2})$ edges described in Theorem 3.13.

For smaller k, the function $\gamma(n)$ can be taken equal to some positive power of n. In particular, for $k = 2$, the function $\gamma(n) = An^{1/4}\log^2(n)$ makes $f(\gamma, n)$ decreasing, implying $\mathrm{ex}(n, Q_8) = O(n^{7/4+o(1)})$. This bound is weaker than the bound from Corollary 3.19, namely $\mathrm{ex}(n, Q_8) = O(n^{8/5})$.

Rem 6.5. It is worth considering, what the limitations of this method are. One of the caveats is Lemma 6.2, which is used to find 'good' 4-cycles in $\mathcal{G}$, that are 4-cycles that correspond to Q_8 in G. Another is the structure of the graph $\mathcal{G}$. There is no assumptions about the graph $\mathcal{G}$, however, it is clear that not every graph can be an augmented edge graph. For example, such graph has an even number of edges. Without assuming anything about $\mathcal{G}$, what estimation can be achieved, if the existence of a 'good' 4-cycle is assumed, provided that there are 4-cycles in H at all?

Let G be a graph on n with an edge density p. Then the corresponding augmented graph $\mathcal{G}$ has $m = pn^2$ vertices and $s = c(pn)^4$ edges. If $\mathcal{G}$ is a C_4-free graph then G is clearly a Q_8-free graph. Theorem 3.8 gives a bound $s = O(m^{1+1/2})$, which is equivalent to

$$p = O(n^{-2/5}).$$

Thus, $e(G) = O(n^{8/5})$, same bound as in the Corollary 3.19. So without any assumptions on the graph $\mathcal{G}$, it would be impossible to prove a better bound.

6.2 Augmented graph structure

Let G be a K-almost regular bipartite graph with parts A and B, and $e(G) = \alpha n^{3/2}$ with α being at least a large constant. In this section, the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ refers to an augmented edge graph with $\mathcal{V} = E(G)$ and $(u_1, v_1)(u_2, v_2) \in \mathcal{E}$ for $u_1, u_2 \in A$ if $u_1v_1u_2v_2$ is a 4-cycle in G.

Let M be a bipartite adjacency matrix for G, that is, rows of M correspond to the vertices from A, columns of M correspond to the vertices from B, and $M(u, v) = 1$ if $uv \in E(G)$ and $M(u, v) = 0$ otherwise. Now ones can be interpreted as vertices of an edge graph $\mathcal{G}$. Notice that $\mathcal{G}$ contains no edges between vertices of the same row (or the same column). Moreover, two vertices of $\mathcal{G}$ are adjacent if and only if the corresponding 2×2 minor of M, containing these vertices is $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. The following lemma is due to D. Bradač, Janzer, Sudakov, and Tomon provides regularity for codegrees in G.

Lemma 6.6 ([23]). *Let G be a graph on n vertices with $\alpha n^{3/2}$ edges. Then G has a subgraph G' with at least $e(G)/2$ edges and for any $uv \in E(G')$, vertex u has at least $\frac{1}{8}\alpha n^{1/2}$ neighbours w in H with $d_H(v, w) \geqslant \alpha^2/32$.*

Notice that it is a sharper version of Lemma 2.8. It provides a lower bound not only for degrees in the graph G', but also for codegrees.

In terms of graph $\mathcal{G}$ Lemma 6.6 states that at least half of the vertices $(u, v) \in V(\mathcal{G})$ there are at least $\frac{1}{8}\alpha n^{1/2}$ rows (columns) of M with at least $\alpha^2/32$ neighbours of (u, v) in each. The proof of Erdős Reduction Theorem 3.17 finds a vertex $(u, v) \in \mathcal{G}$ with $\frac{e(G)^3}{3n^4} = \frac{1}{3}\alpha^3 n^{1/2}$ neighbours, while Lemma 6.6 also provides some structure of these neighbours. The graph $\mathcal{G}$ therefore may be assumed to contain only rich vertices after deleting at most half of its vertices.

Looking at neighbourhood of any vertex in $\mathcal{G}$ is essentially the same argument as in the proof of the Erdős Reduction Theorem 3.17.

Fix a rich vertex $(u, v) \in V(\mathcal{G})$. Its neighbourhood has at least $\frac{1}{8}\alpha n^{1/2}$ rows with at least $\alpha^2/32$ neighbours in each. If (u, v) has a neighbour (u', v') then there is an edge $u'v \in E(G)$. Since $d_G(u) \leqslant K\alpha n^{1/2}$, then (u, v) has neighbours in at most $K\alpha n^{1/2}$ columns. Suppose that (u, v) has neighbours in n^μ columns. The following observation is equivalent to the inclusion $Q_8 \subseteq K_{1,1} * C_6$.

Proposition 6.7. *If there are six neighbours $(u_1, v_1), (u_1, v_2), (u_2, v_2), (u_2, v_3), (u_3, v_1)$ for different $u_1, u_2, u_3, v_1, v_2, v_3$ then together with u and v these vertices create a copy of Q_8 in G.*

There are $n^\mu = O(\alpha n^{1/2})$ columns and $\frac{1}{8}\alpha n^{1/2}$ rows with $\alpha^2/32$ neighbours in each. Let $D(u, v)$ be the bipartite graph with one parts being columns with neighbours of (u, v) and other being these $\frac{1}{8}\alpha n^{1/2}$ rows. A row and a column are adjacent if there is a neighbour of (u, v) in the intersection. Then, on the one hand, $e(D(u, v)) \geqslant \frac{1}{8}\alpha n^{1/2} \cdot \frac{\alpha^2}{32}$, but on the other, it does not contain C_6 as a subgraph, see 6.7. Thus, it has at most $Cn^{4\mu/3}$ edges. Hence,

$$\begin{aligned} n^{4\mu/3} &= \Omega(\alpha^3 n^{1/2}), \\ \alpha^{4/3} n^{2/3} &= \Omega(\alpha^3 n^{1/2}), \\ \alpha^{5/9} &= O(n^{1/6}), \\ \alpha &= O(n^{1/10}), \end{aligned}$$

which is equivalent to $e(G) = O(n^{8/5})$.

7 Discussion and conclusions

This final section provides an outline of the considered methods. It shows their applicability to different graphs H and possible refinements, that can be studied in the future.

Erdős–Rényi First Moment method. This is a universal tool for calculating a lower bound for extremal numbers. However, it is believed to never provide a tight lower bound [21].

Algebraic Constructions. Algebraic (or geometric) construction provides examples of extremal H-free graphs. However, each graph H requires an individual construction, based on some type of algebraic objects. For example, Theorem 3.13 constructs an example for cycles of three specific lengths.

Erdős Reduction Theorem. This theorem is an effective tool for estimating the upper bound of extremal numbers. However, it works well only for graphs with vertices of maximum possible degree. Since, when the bipartite join $K_{1,1} * H$ is taken, two new vertices are connected to every vertex in the other part. Despite that, this theorem still provides the best known bound for $\mathrm{ex}(n, Q_8)$, even though Q_8 does not contain a vertex with the highest maximum possible degree.

Counting homomorphic copies. This method strongly relies on the symmetries of the excluded graph H. In particular, it provides the best known upper bounds for hypercube edge graphs Q_{2^n} and grid graphs $R_n = (V, E)$ with $V = [n] \times [n]$ and $(i_1, i_2)(j_1, j_2) \in E$ if, and only if, $|i_1 - j_1| + |i_2 - j_2| = 1$. However, Cauchy-Schwartz inequality, which is highly utilized in this method, makes the bound less tight. Counting homomorphic copies is applicable only for graphs with some symmetries.

Dependent Random Choice. This method is based on a probabilistic approach, and thus can be applied to any graph H. Due to its nature, it gives more optimal bounds for sparser graphs, for example, it gave the best known upper bound for hypercube edge graphs before Janzer and Sudakov developed the method of counting homomorphic copies. Since its area of applicability is somewhat opposite to the Erdős Reduction Theorem, these two methods can be combined to deal with partly dense, partly sparse graphs, see Proposition 5.5.

Counting cycles in the augmented graph. This method provides the best possible bounds for the extremal numbers of $C_{2k}^{\square}$ for $k \geqslant 7$. Quite possibly, it can be extended to work with any prism graph $H^{\square}$, however it would require estimations analogous to Theorems 6.3 and 6.2 for graphs H other than even cycles. A more detailed study of the structure of the augmented graph can also provide new bounds for extremal numbers. In the author's opinion, this approach is the most promising for the tight estimation of $\mathrm{ex}(n, Q_8)$.

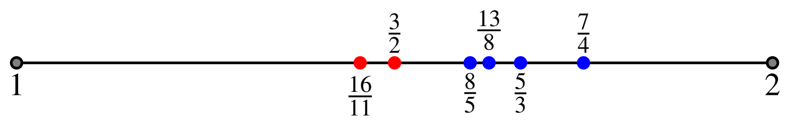

Figure 4: Comparison of the bounds on the $\mathrm{ex}(n, Q_8)$ obtained by different methods.

Finally, Figure 4 presents the comparison of all bounds on $\mathrm{ex}(n, Q_8)$, which are presented in this thesis. The bounds are given in the form $\lim_{n\to\infty} \frac{\log f(n)}{\log n}$, where $f(n)$ is the bound for $\mathrm{ex}(n, Q_8)$ obtained by a specific method. The red dots correspond to lower bounds obtained by

Erdős First Moment method and the direct C_4-free graph respectively. The blue dots correspond to the upper bounds. From left to right, these are the Erdős Reduction theorem, the counting homomorphic copies of Q_8, the dependent random choice, and the counting $4-$cycles in the augmented graph.

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