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Master's Programme in Mathematics and Operations Research

Weak Reverse Hölder Inequalities and Muckenhoupt Weights: Interplay and Implications

Jere Karttunen

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Author Jere Karttunen

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Abstract
Starting from the definition of a doubling measure on a metric measure spaces, the relevant theory is built to prove the equivalence of five different characterizations of the class of functions satisfying weak reverse Hölder inequality on an open subset of a metric measure space. The resulting class of functions is a relaxation of Muckenhoupt weights, which in turn arise from PDE, harmonic and Fourier analysis. Techniques used in the proofs include Vitali covering lemma, multiple key properties of Hardy-Littlewood maximal operator, and weighted inequalities. Lastly, it is shown that this class of functions, weak Muckenhoupt weights, satisfy the weak reverse Hölder inequality with all dilation factors strictly greater than one.

Keywords Metric measure spaces, doubling measure, weak Muckenhoupt weights, Hardy-Littlewood maximal operator, Vitali covering lemma

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Tiivistelmä
Alkaen tuplaavan mitan määritelmästä metrisissä mitta-avaruuksissa, johdetaan teoria, jolla näytetään viisi eri määritelmää keskenään yhtäpitäviksi heikon käänteisen Hölder-epäyhtälön toteuttavalle funktioluokalle avoimessa osajoukossa. Kyseinen funktioluokka, heikot Muckenhouptin painot, on yleistys Muckenhouptin painoista, joita esiintyy osittaisdifferentiaaliyhtälöissä, harmonisessa analyysissä sekä Fourier-analyysissä. Todistuksissa käytettyjä tekniikoita ovat muun muassa Vitalin peitelause, Hardy-Littlewood-maksimaalioperaattorin useat keskeiset ominaisuudet sekä painotetut epäyhtälöt. Lopuksi osoitetaan, että heikot Muckenhouptin painot täyttävät heikon käänteisen Hölder-epäyhtälön kaikilla aidosti yhtä suuremmilla pallon laajennuskertoimilla.

Avainsanat Metrinen mitta-avaruus, tuplaava mitta, heikot Muckenhouptin painot, Hardy-Littlewood maksimifunktio, Vitalin peitelause

Preface

I want to thank Professor Juha Kinnunen for his highly professional touch on guiding the writing of this thesis. Conversations in the biweekly meetings have been very fruitful and inspiring, whether they were about trying to find counterexamples to some of the lemmas or about general wondering about the wood preserving effects of pine tar. Every meeting gave me motivation and clarity for advancing my thesis.

Otaniemi, 12 March 2025

Jere Karttunen

Contents

1 Introduction

A metric measure space (X, d, μ) consists of a set X, a metric d, and a measure μ. This structure is posed without any additional assumptions, hence it represents the most general space where both metrics and measures can coexist harmoniously. Unlike specific cases such as Euclidean spaces or manifolds, metric measure spaces encompass a broader range of structures, allowing for much more intricate and less intuitive topologies. Thereby, metric measure spaces provide a versatile framework for various mathematical analyses exploring deep theoretical results, free from the constraints and peculiarities of more specialized spaces.

Muckenhoupt weights, widely known in harmonic analysis, arose primarily from the study of the boundedness of the Hardy–Littlewood maximal operator. These weights revolutionized the understanding of weighted inequalities found in fields such as partial differential equations, Fourier, and harmonic analysis. For instance, they provide conditions under which certain singular integral operators, crucial in Fourier analysis, remain bounded.

Being the same class of functions, Muckenhoupt weights are inseparably connected to reverse Hölder inequality. Their weak counterparts also coincide in this relation, which is well understood within Euclidean settings. Since Maurice Fréchet introduced metric spaces in 1906, many results in Euclidean spaces have been generalized to metric spaces, including the connection between weak Muckenhoupt weights and the weak reverse Hölder inequality. The principal references to the theory of Muckenhoupt weights in Euclidean spaces are [4, 6, 7, 9], and in metric measure spaces [3, 5, 8].

Assume that (X, d, μ) is a doubling metric measure space with doubling Borel regular measure μ. Let $\Omega \subset X$ be an open subset. In this thesis, a class of nonnegative, locally integrable functions — weights — satisfying a weak reverse Hölder inequality (WRH) is discussed. The WRH states that there exists a constants $C > 0$ and $p > 1$ such that

$$\left(\fint_{B(x,r)} w \, d\mu \right)^{\frac{1}{p}} \leq C \fint_{B(x,2r)} w \, d\mu, \tag{1}$$

for every open ball $B(x, r)$ with $B(x, 2r) \Subset \Omega$. This class of functions is precisely the weak Muckenhoupt A_∞ class of weights which is a generalization of A_∞ class that allows for a much broader class of weights, including nondoubling weights that can be zero on large parts of the space. The weak A_∞ class has been studied recently in [1, 10, 11, 12, 15], and indirectly through the weak reverse Hölder inequality in metric measure spaces in [1, 13, 14].

More precisely, this thesis focuses on proving the equivalence of five out of the ten characterizations for the weak A_∞ weights presented in the main source [1], with detailed proofs, supplemented with illustrative figures. This thesis aims to serve as a comprehensive resource for readers with a basic knowledge of measure and integration theory, aiding them to understand the primary research paper that forms the basis of this work. All relevant lemmas and theorems are derived from the definition of a doubling measure.

The Preliminaries section 2 includes essential definitions and results required for this thesis. Section 2.1 discusses covering theorems as well as the implications of doubling measure on a metric measure space, accompanied by proofs primarily based on "Lectures on Analysis on Metric Spaces" by Juha Heinonen [2]. Section 2.2 consists of proofs of the many important properties of the Hardy-Littlewood maximal operator. The proofs are mostly from [1].

Sections 3–3.2 contain the proof of the equivalences, and are written based on the main source of this thesis [1]. The last section 4 is dedicated to showing that one can replace the dilated ball $B(x, 2r)$ in (1) with $B(x, \sigma r)$ for any $\sigma > 1$ without altering the resulting class of functions. The equivalent characterizations are stated and four out of the five are proved. The proofs are written with unpublished notes from the authors of [1].

My contribution has been the writing of the first published version of the proof of equivalences in the general $\sigma > 1$ case, constructiong the figures as well as additional steps to the proofs to aid the reader of the thesis. Albeit being a simple matter, it is explicitly shown that the dilation factor $\sigma > 1$ is irrelevant for the resulting class of functions.

2 Preliminaries

Throughout this thesis, the underlying metric measure space (X, d, μ) is assumed to be complete, and equipped with a nontrivial Borel regular doubling measure μ. Contrary to the cubes in normed spaces, balls serve as the fundamental building blocks of covering arguments in metric spaces due to their reliance on distance functions. Therefore, all the covering arguments used are built on open or closed balls.

Definition 2.1. *Let (X, d, μ) be a metric measure space. An open ball B and a closed ball $\overline{B}$, centered at $x \in X$, with radius $r > 0$, is defined as*

$$B(x, r) = \{y \in X \ : \ d(x, y) < r\},$$
$$\overline{B}(x, r) = \{y \in X \ : \ d(x, y) \leq r\},$$

respectively.

Note that in metric spaces, above definition might not yield unique balls with respect to radius r nor center x. For bounded spaces $\Omega \subset X$, choosing R with $\operatorname{diam}(\Omega) < R < \infty$ yields $B(x, R) = B(y, R)$ and $B(x, R) = B(x, R + \epsilon)$ for every $x, y \in \Omega$ and $\epsilon > 0$. For unbounded spaces X with, e.g., a discrete metric $d(x, y) = \mathbb{1}_{x \neq y}$, one has $B(x, r) = X$ for every $x \in X$ and $r > 1$, and, $B(x, r) = \{x\}$ for every $x \in X$ and $r \leq 1$. Moreover, the balls in metric spaces might bear little to no resemblance to their Euclidean counterparts, as demonstrated on figure 1.

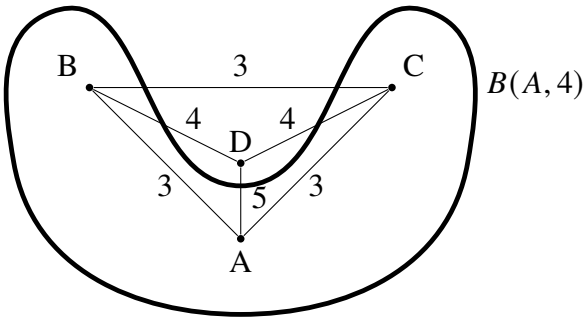

Figure 1: An open ball centered at point A with radius 4 depicted. The distances between points are given next to the connecting lines.

To enhance intuition, the remaining figures in this thesis will depict balls using the Euclidean distance metric. When obvious from the context, the following abbreviations will be used. A ball $B(x, r)$ is abbreviated as $B = B(x, r)$, and its dilation by a factor $a > 0$ as $aB = B(x, ar)$. Moreover, a radius of a ball is denoted $r(B) = r$, when $B = B(x, r)$.

The dependencies of constants are analyzed and denoted in parentheses. The measure of a μ-measurable set Ω is defined as

$$\mu(\Omega) = \int_{\Omega} d\mu < \infty.$$

An integral of a function $f \in L_1(X)$ over a measurable subset $\Omega \subset X$ is written

$$f(\Omega) = \int_\Omega f d\mu < \infty$$

for brevity. Based on the context, integral averages of functions are abbreviated as

$$\frac{1}{\mu(\Omega)} \int_\Omega f d\mu = \fint_\Omega f d\mu = \frac{f(\Omega)}{\mu(\Omega)} = f_\Omega < \infty.$$

2.1 Doubling measure

Definition 2.2. ***Doubling measure.***
A Nontrivial measure in nontrivial metric measure space (X, d, μ) is doubling, if it satisfies the following doubling condition. There exists a constant $C_d = C_d(\mu) > 1$, such that,

$$0 < \mu\left(B(x, 2r)\right) \leq C_d \mu\left(B(x, r)\right) < \infty$$

for all balls $B(x, r) \in X$.

Remark 2.3. *A measure μ is trivial if $\mu(X) = 0$. A space is trivial if X is a singleton set.*

Corollary 2.4. *Whenever μ is a nontrivial doubling measure on a nontrivial metric measure space (X, d, μ),*

$$\mu\left(B(x, R)\right) \leq C_d^{\lfloor \log_2(\frac{R}{r}) \rfloor + 1} \mu\left(B(x, r)\right),$$

for every $x \in X$ and $0 < r \leq R < \infty$.

Proof. Let $x \in X$ and $0 < r \leq R < \infty$. There exists $k = \lfloor \log_2(\frac{R}{r}) \rfloor + 1 \in \mathbb{N}$ with $2^{k-1} r < R \leq 2^k r$, and,

$$\mu\left(B(x, R)\right) \leq \mu\left(B(x, 2^k r)\right) \leq C_d^k \mu\left(B(x, r)\right).$$

□

The doubling condition for measures is like the renowed Lipschitz continuity for functions — a tool ensuring regularity and control over local behavior, imposing a structured relationship across scales. Much like Lipschitz continuity, which is not overly restrictive yet profoundly impactful analytically, the doubling condition commands significant analytical power while covering a multitude of measures. In fact, many natural measures encountered in analysis satisfy it, Lebesgue measure in Euclidean spaces being arguably one of the best known examples of such measures.

The doubling condition implies structure on the underlying space. However, it is not a causation of the measure but rather a consequence of the metric. For instance, while the Lebesgue measure is doubling for spaces equipped with Euclidean metric,

it sure is not for ones with previously presented discrete metric $d(x, y) = \mathbb{1}_{x \neq y}$, as $\mu(B(x, 1)) = 0$ and $\mu(B(x, 2)) = \mu(X) > 0$.

On the other hand, there are measures that are not doubling regardless of the underlying metric. For instance, the Dirac measure, $\delta_\omega(\Omega) = \mathbb{1}_{\omega \in \Omega}$, $\omega \in X$, cannot be doubling in nontrivial spaces, that is, spaces with more than one element. To conclude this, take a ball $B(x, r)$ with $\omega \notin B(x, r)$ and $\omega \in B(x, 2r)$. Then, the doubling condition is contradicted, as $1 \leq C_d \times 0$. If such ball does not exist, one has $\omega \in B(x, r)$ for all $r > 0$ and $x \in X$, implying $d(\omega, x) = 0$ for all $x \in X$. Hence, by the axioms of metric, the space reduces to a trivial space, formed by a singleton set $X = \{\omega\}$.

Doubling condition effectively filters out such unintuitive metrics and measures. One of the consequences of doubling measure is that every finite radius open ball has finite measure, and can be covered with finitely many arbitrary small open balls.

Lemma 2.5. *Let (X, d, μ) be a metric measure space with a doubling measure μ, $B(x, r) \subset X$ a ball, and $\sigma > 0$ constant. Then, there exists $N = N(\sigma, C_d)$ balls $\{B(x_i, \sigma r)\}_{i=1}^N$ with $\{x_i\}_{i=1}^N \subset B$ such that $B(x, r) \subset \bigcup_{i=1}^N B(x_i, \sigma r)$.*

Proof. Let $B \subset X$ be a ball, and let

$$\mathcal{F} = \left\{ B\left(z, \tfrac{1}{5}\sigma r\right) \right\}_{z \in B}$$

be an arbitrary collection of balls. By Vitali covering lemma 2.8, for any collection of open balls $\mathcal{F}$, there exist a countable subcollection of open balls $\{B_i\}_i \subset \mathcal{F}$ such that

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_i 5B_i = \bigcup_i B(z_i, \sigma r).$$

As, $B \subset \bigcup_{B \in \mathcal{F}} B$, the inclusion is proved. It remains to show that the cardinality of $\{B_i\}_i$ is finite. As each positive radii ball has positive measure for nontrivial doubling measure, this can be shown by proving that the sum of their measures is finite. This is a consequence of the inclusion

$$B_i \subset B\left(x, r + \tfrac{1}{5}\sigma r\right) \subset B(x, 2^n r),$$

for all B_i, where $n \in \mathbb{Z}^+$ depends only on σ. The sum of measures is finite as by the disjointness of the balls B_i,

$$\sum_i \mu\left(B_i\right) = \mu\left(\bigcup_i B_i\right) \leq \mu\left(B\left(x, 2^n r\right)\right) \leq C_d^n \mu\left(B\left(x, r\right)\right) < \infty.$$

□

A metric space is doubling if there exists a constant $N = N(C_d)$ such that every open ball $B(x, r)$ can be covered with at most N open balls with radii $\frac{1}{2}r$. Note that the previous definition is equivalent whether it is stated with $\frac{1}{2}r$ or with any ϵr for

$0 < \epsilon < 1$. Naturally, every metric space equipped with doubling measure is doubling by the lemma 2.5.

As was demonstrated, doubling measure is a intuitive concept. It has profound and far-reaching implications. Firstly, this fundamental property of doubling measure ensures a form of geometric regularity, thus making it amenable to various analytical techniques. Secondly, it allows the application of covering lemmas, such as the Vitali covering lemma 2.8, which in turn is pivotal in establishing differentiation theorems, (e.g. Lebesgue Differentiation Theorem 2.16) and integral estimates (e.g. (p, p)-strong type estimate for Hardy-Littlewood maximal operator 2.25).

Besides above mentioned more technical results implied by the doubling measure, it ensures more general properties of the underlying space, such as the separability of the space. In addition, if Y is a proper metric measure space, then Y is complete. The reverse implication does not hold in general. However, if Y is doubling metric measure space, then the two statements are equivalent.

Theorem 2.6. *Let Y be a doubling metric measure space. Then Y is proper if, and only if, Y is complete.*

Proof. Assume that Y is a proper metric space. To show that Y is complete, one must show that every Cauchy sequence converges, and that the limit is in Y. To this end, let $\{x_i\}_{i=1}^{\infty} \subset Y$ be a Cauchy sequence. There exists a closed ball $\overline{B}(x_1, r) \subset Y$, with sufficiently large radius $r > 0$, such that $\{x_i\}_{i=1}^{\infty} \subset \overline{B}(x_1, r)$. As Y is proper, $\overline{B}(x_1, r)$ is compact. Hence, by definition, the limit is in the ball and thus in Y.

To show the reverse implication, assume that Y is complete. Then every closed and bounded set is compact. If $\Omega \subset Y$ is closed and bounded, then, it is totally bounded, and hence also compact. □

Remark 2.7. *A set Ω in a metric space Y is totally bounded, if for every $\epsilon > 0$, there exists finite collection of open balls $\{B(x_i, \epsilon)\}_{i=1}^{N}$ with $\{x_i\}_{i=1}^{N} \subset \Omega$, such that*

$$\Omega \subset \bigcup_{i=1}^{N} B(x_i, \epsilon) .$$

As per lemma 2.5, every closed and bounded set is totally bounded in a doubling metric measure space.

A collection of sets $\{A_i\}_{i \in \mathcal{G}}$ is disjointed, if it is pairwise disjoint, i.e., $A_i \cap A_j = \emptyset$, for $i, j \in \mathcal{G}$ with $i \neq j$. This is a particularly useful property, since it holds that

$$\mu\left(\bigcup_{i \in \mathcal{G}} A_i\right) = \sum_{i \in \mathcal{G}} \mu(A_i).$$

Multiple covering theorems exist in metric spaces — some provide disjointed covers, some control the measure of the overlapping portion of the cover. Vitali covering

lemma yields disjointed cover, which comes with the tradeoff that the balls need to be dilated by a factor of 5.

Theorem 2.8. ***Vitali covering lemma***
Let $\mathcal{F}$ be a collection of open balls B in a metric space X such that

$$\operatorname{diam}\left(\bigcup_{B\in\mathcal{F}} B\right) < \infty.$$

Then $\mathcal{F}$ contains a countable subcollection of pairwise disjoint open balls $\mathcal{G} = \{B_i\}_{i=1}^{\infty}$ such that $B_i \in \mathcal{F}$, $i = 1, 2, \ldots$, such that,

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{i=1}^{\infty} 5B_i.$$

Moreover, every ball B from $\mathcal{F}$ meets a ball from $\mathcal{G}$ with radius at least $\frac{1}{2}r(B)$. That is, for every $B \in \mathcal{F}$, there exists $\tilde{B} \in \mathcal{G}$ with $r(\tilde{B}) \geq \frac{1}{2}r(B)$ such that $\tilde{B} \cap B \neq \emptyset$.

There are multiple ways of proving the Vitali covering lemma. For instance, it could be proven using Zorn's Lemma or with a so called greedy approach. The latter one uses techniques more relevant for this thesis, and thus it is proven in this thesis. It is shown that the original set of open balls $\mathcal{F}$ can be exhausted by a union of 5-balls, where the disjointed set of balls are chosen greedily among the largest remaining balls of $\mathcal{F}$ satisfying certain criterion.

Proof. Let $\mathcal{F}$ be a uniformly bounded collection of open balls B in a metric space X. It is shown, that the following process results the desired set of balls $\mathcal{G}$. At iteration i, balls $\{B(x_j, r_j)\}_{j=1}^{i-1}$ have been chosen to $\mathcal{G}$, and the next ball is selected among the largest from the remaining valid balls,

$$\mathcal{V}_i = \left\{ B \in \mathcal{F} \ : \ B \cap \bigcup_{j=1}^{i-1} B(x_j, r_j) = \emptyset \right\},$$

i.e., those that do not intersect already selected balls. This is illustrated in figure 2. As the collection $\mathcal{V}_i$ could be uncountable, there may be no ball with the largest radius, nor a ball with a supremum radius,

$$R_i = \sup\{r(B) > 0 \ : \ B \in \mathcal{V}_i\}.$$

To overcome this, a ball $B \in \mathcal{V}_i$ with $r(B) > \frac{1}{2}R_i$ is appended to the set of chosen balls $\{B_j\}_{j=1}^{i-1}$ as a ball $B_i = B(x_i, r_i)$. If there are no balls for which the above conditions are fulfilled, the process is terminated.

The chosen balls are disjoint by construction. Thus, what is left to show, is that for every ball $B \in \mathcal{F}$, there exists B_m with $r(B_m) \geq \frac{1}{2}r(B)$ such that $B_m \cap B \neq \emptyset$, and that the selected balls cover the collection $\mathcal{F}$, that is,

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{i=1}^{\infty} 5B_i.$$

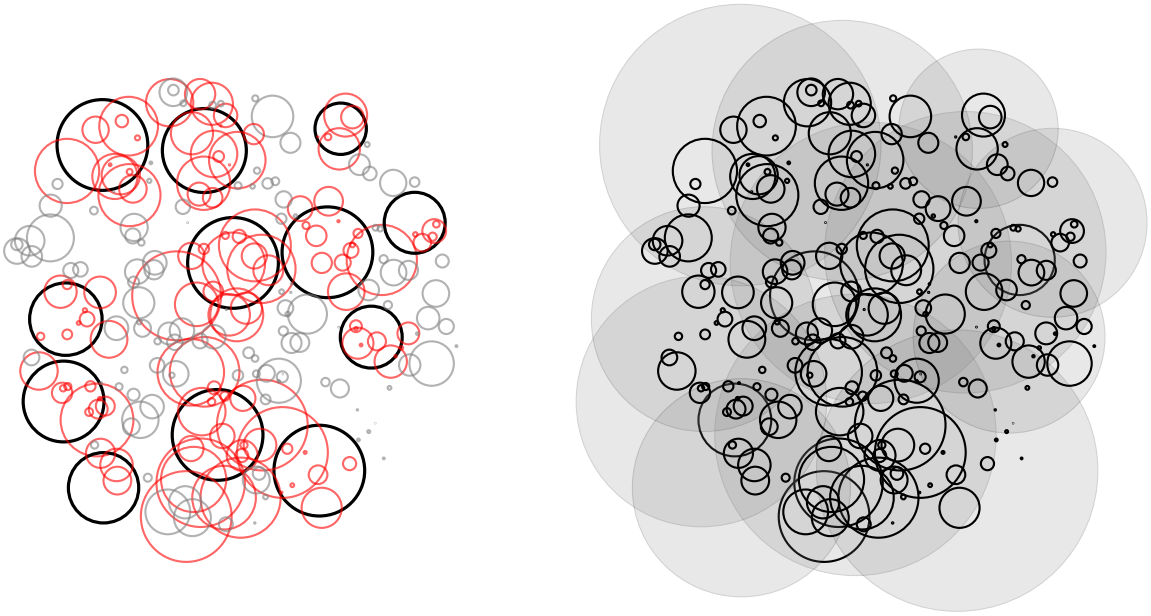

Figure 2: Vitali Covering Lemma's proof at iteration i. On the left, black balls are the chosen balls $\{B_j\}_{j=1}^{i-1} \subset \mathcal{G}$, grey balls form the collection of valid balls $\mathcal{V}_i$, and, red balls intersect with some of the chosen balls. On the right, chosen balls are dilated by a factor 3.1, which is also valid by remark 2.9. Note that the process does not terminate here, as there still exists valid balls, which implies that the latter part of the claim cannot be satisfied.

Let $B \in \mathcal{F}$ be an arbitrary ball. If it is not contained in the union of the selected balls, that is,

$$B \cap \bigcup_{i=1}^{\infty} B_i = \emptyset,$$

then at every iteration, $r(B) > 0$ would have been a valid candidate for the supremum radius. Hence, $R_i \geq r$, and thereby also $r_i > \frac{1}{2}R_i \geq \frac{1}{2}r(B) > 0$, for every $i = 1, 2, \ldots$. This contradicts the assumption of the uniform boundedness balls in $\mathcal{F}$, as the disjointness of the selected balls imply

$$\left|\bigcup_{i=1}^{\infty} B_i\right| = \sum_{i=1}^{\infty} |B_i| = \infty.$$

Hence, every ball in $\mathcal{F}$ intersects with at least one of the selected balls $\{B_i\}_{i=1}^{\infty}$. Thereby there exists smallest index m for which $B \cap B_m \neq \emptyset$. So, $B \in \mathcal{V}_{m-1}$ and hence $r_m \geq \frac{1}{2}r(B)$. The inclusion follows readily from above, as for any $z \in B \cap B_m$ and $y \in B$, it holds that,

$$|y - x_m| \leq |y - z| + |z - x_m| \leq 2r + r_m \leq 5r_m.$$

Thus, $B \subset B_m$, and consequently $\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{i=1}^{\infty} 5B_i$. □

Remark 2.9. *The factor* 5 *is not optimal in the Vitali's covering lemma 2.8. In fact, it can be shown that* $3+\epsilon$ *for any* $\epsilon > 0$ *will do, by selecting balls with* $r > \frac{1}{1+\epsilon} R_i$ *in the proof. This yields a subcollection of pairwise disjoint open balls* $\{B_i\}_{i=1}^{\infty}$ *such that for every* $i = 1, 2, \ldots$, $B_i \in \mathcal{F}$ *and*

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{i=1}^{\infty} (3+\epsilon) B_i.$$

Then, every ball in $B \in \mathcal{F}$ *meets a ball in* $\{B_i\}_{i=1}^{\infty}$ *with radius at least* $\frac{1}{1+\epsilon} r(B)$.

Remark 2.10. *The Vitali covering lemma is true also for a collection of closed balls. Figure 3 demonstrates that the optimal factor for closed balls must be at least* 3*, and that even for a finite collections of balls scaling might be required.*

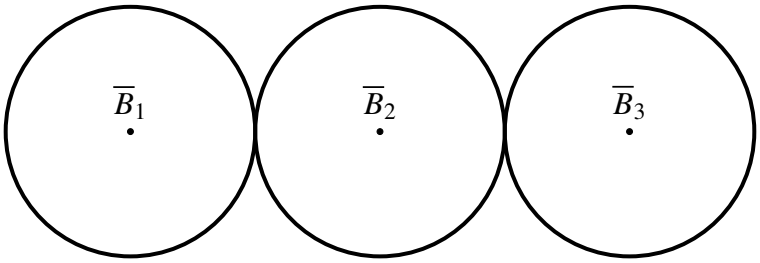

Figure 3: Three closed balls $\overline{B}_1(x_1, r)$, $\overline{B}_2(x_2, r)$, and $\overline{B}_3(x_3, r)$ with radii $r > 0$ placed such that $d(x_1, x_2) = d(x_2, x_3) = 2r$ and $d(x_1, x_3) = 4r$, so that, $\overline{B}_1 \cap \overline{B}_2 \neq \emptyset$ and $\overline{B}_2 \cap \overline{B}_3 \neq \emptyset$. The subcollections satisfying all conditions posed on them in Vitali Covering Lemma are $\{\overline{B}_2\}$ and $\{\overline{B}_1, \overline{B}_3\}$.

Remark 2.11. *Clearly, the condition that each ball* $B \in \mathcal{F}$ *meets with a ball in* $\{B_i\}_{i=1}^{\infty}$ *with radius at least* $\frac{1}{2} r(B)$*, is stronger than*

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{i=1}^{\infty} 5B_i.$$

An example where latter holds while former does not, is depicted in figure 2.

Applying Vitali covering lemma for an arbitrary set $\Omega \subset X$ can be done by setting $\mathcal{F} = \{B(x, r_x)\}_{x\in\Omega}$ for some family of balls. The Vitali covering lemma then yields disjointed open cover of the set Ω. However, this cover seldom is optimal in a sense that it does not control the overlap with the complement, i.e., often

$$\mu\left((X \setminus \Omega) \cap \bigcup_{i=1}^{\infty} 5B_i\right) \neq 0,$$

which would be desirable in analysis.

Hence, a natural question arises: how far can one optimize the countable open or closed cover of Ω, and what conditions are required from Ω? Intuitively, some form of arbitrary high granularity assumption should be required from the set of balls $\mathcal{F}$. A fine cover of a set provides provides arbitrary high granularity.

Definition 2.12. ***Fine cover***
A fine cover $\mathcal{F}$ of a subset $\Omega \subset X$ of a doubling metric measure space (X, d, μ) is a collection of closed balls, such that for every point $x \in \Omega$,

$$\inf\left\{r > 0 \ : \ \overline{B}(x, r) \in \mathcal{F}\right\} = 0.$$

A set with fine cover can be covered by a countable disjointed set of closed balls. This is called Vitali covering theorem, and its a corollary of the Vitali covering lemma. Unlike the covering lemma, which fully covers the set, the theorem covers the set up to a measure zero. This is a necessary relaxation to overcome the absence of dilation while maintaining disjointness of the cover.

Theorem 2.13. ***Vitali covering theorem***
Let $\Omega \subset X$ be a subset of a doubling metric measure space (X, d, μ) and $\mathcal{F}$ a fine cover of Ω. Then there exists a countable disjointed subcollection $\mathcal{G} \subset \mathcal{F}$ so that the closed balls in $\mathcal{G}$ cover Ω μ-a.e., namely,

$$\mu\left(\Omega \backslash \bigcup_{\mathcal{G}} \overline{B}\right) = 0.$$

Proof. First, the claim is proven for bounded sets. The proof for unbounded sets then follows as a corollary by representing unbounded sets as a countable disjoint union of bounded open sets. For each open set, the closed balls of the cover are fully contained in the set. Hence, when combining the covers, the resulting closed cover of the unbounded set remains disjointed through the union of subcovers.

Assume that $\Omega \subset X$ is bounded and has a fine cover. Let $\mathcal{F}$ be a subcollection of that fine cover with

$$\Omega \subset \bigcup_{\overline{B} \in \mathcal{F}} \overline{B} \quad \text{and} \quad \mu\left(\bigcup_{\overline{B} \in \mathcal{F}} \overline{B}\right) < \infty.$$

The Vitali covering lemma 2.8 provides a countable disjointed closed cover $\mathcal{G} = \{\overline{B}_i\}_{i=1}^{\infty} \subset \mathcal{F}$ with the following inclusions

$$\bigcup_{i=1}^{\infty} \overline{B}_i \subset \Omega \subset \bigcup_{i=1}^{\infty} 5\overline{B}_i.$$

By doubling condition, there exists a constant $C = C(C_d)$ for which

$$\sum_{i=1}^{\infty} \mu\left(5\overline{B}_i\right) \leq C \sum_{i=1}^{\infty} \mu\left(\overline{B}_i\right) \leq C\mu\left(\bigcup_{i=1}^{\infty} \overline{B}_i\right) \leq C\mu\left(\Omega\right) < \infty.$$

This implies that the union of the cover's tail diminishes to a zero measure set. More precisely, as $N \to \infty$ it holds that

$$\mu\left(\bigcup_{i>N} 5\overline{B}_i\right) \leq \sum_{i>N}^{\infty} \mu\left(5\overline{B}_i\right) \to 0.$$

Thus, it suffices to show that the part Ω which is not covered by first N chosen balls is contained in the diminishing part of Vitali cover, that is,

$$\Omega \setminus \bigcup_{i=1}^{N} \overline{B}_i \subset \bigcup_{i>N} 5\overline{B}_i, \tag{2}$$

for all $N \geq 1$. To this end, take any uncovered point $x \in \Omega \setminus \bigcup_{i=1}^{N} \overline{B}_i$. In figure 4, these points lie on the gray. There exists a closed ball $\overline{B}(x,r) \in \mathcal{F}$ that does not intersect with any of the first N closed balls $\{\overline{B}_i\}_{i=1}^{N} \subset \mathcal{G}$, that is,

$$\overline{B}(x,r) \cap \bigcup_{i=i}^{N} \overline{B}_i = \emptyset.$$

For every $\overline{B} \in \mathcal{F}$, there exists a ball $\overline{B}_j \in \mathcal{G}$ with $\overline{B} \cap \overline{B}_j \neq \emptyset$ and $\overline{B} \subset 5\overline{B}_j$. Hence, $\overline{B}(x,r) \subset 5\overline{B}_j$, for some $j > N$. As $x \in \Omega \setminus \bigcup_{i=1}^{N} \overline{B}_i$ was arbitrary, the inclusion (2) holds, and the claim is proven for bounded sets.

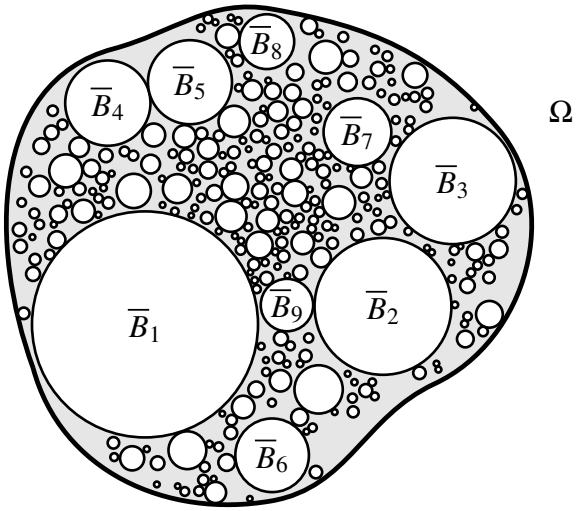

Figure 4: The covering of Ω with the first N balls $\{\overline{B}\}_{i=1}^{N}$, $\overline{B}_i \in \mathcal{G}$.

Next, assume that $\Omega \subset X$ is an unbounded set, and cover it by a countable union of bounded open sets Ω_i up to a zero measure set. That is, $\mu(\Omega \setminus \bigcup_{i=0}^{\infty} \Omega_i) = 0$. For every Ω_i, there exists a countable disjointed fine cover.

$$\mathcal{G}_i = \left\{ \overline{B} \in \mathcal{F}_i \ : \ \overline{B} \subset \Omega_i, \ \mathcal{F}_i \text{ is a fine cover of } \Omega_i \right\},$$

which is contained in the set Ω_i. Therefore, there exists a countable disjointed collection $\mathcal{G} = \bigcup_{i=0}^{\infty} \mathcal{G}_i$ of closed balls, for which

$$\mu\left(\bigcup_{i=0}^{\infty} \Omega_i \setminus \bigcup_{\overline{B} \in \mathcal{G}} \overline{B} \right) = \sum_{i=0}^{\infty} \mu\left(\Omega_i \setminus \bigcup_{\overline{B} \in \mathcal{G}} \overline{B} \right) \leq \sum_{i=0}^{\infty} \mu\left(\Omega_i \setminus \bigcup_{\overline{B} \in \mathcal{G}_i} \overline{B} \right) = 0.$$

To complete the proof, it needs to be shown that such partiotion always exists. Since any ball can be represented as a union of spheres

$$B(x,r) = \bigcup_{0\le s<r} \overline{B}(x,s) \setminus B(x,s)$$

and $\mu(B(x,r)) < \infty$ for all finite radii, at most countably many spheres can have positive measure. Hence, there exists a strictly increasing sequence of finite radii $\{r_i\}_{i=0}^{\infty}$ for which the spheres have zero measures. Thus, the respective open annuli $C_i = B_{i+1} \setminus \overline{B}_i$ cover Ω up to a zero measure set. The desired partition is obtained by defining $\Omega_i = \Omega \cap C_i$. □

Lemma 2.14. *Let (X, d, μ) be a metric space with a doubling Borel regular measure μ. Then, for every measurable set $\Omega \subset X$ with finite measure, there exists measurable sets $U, V \subset X$, where U is open, V is closed, such that $V \subset \Omega \subset U$ and*

$$\mu(U) - \epsilon \le \mu(\Omega) \le \mu(V) + \epsilon.$$

Proof. Let $\epsilon > 0$ be given. As μ is a Borel regular measure, for every measurable set $\Omega \subset X$, there exists a closed subset $V \subset \Omega$ such that

$$\mu(\Omega) - \mu(V) = \mu(\Omega \setminus V) < \frac{\epsilon}{2}.$$

Similarly, the complement of Ω, denoted by $\Omega^c = X \setminus \Omega$, can be approximated by a closed set F such that $F \subset \Omega^c$ and $\mu(\Omega^c \setminus F) \le \frac{\epsilon}{2}$. Denote $U = F^c$. Thus, $\Omega \subset U$ and

$$\begin{aligned}\mu(U \setminus \Omega) &= \mu(F^c \setminus \Omega) = \mu\left(X \setminus (F \cup \Omega)\right) = \mu\left((X \setminus \Omega) \cap (X \setminus F)\right) \\ &= \mu\left(\Omega^c \setminus F\right) < \frac{\epsilon}{2}.\end{aligned}$$

Now, $V \subset \Omega \subset U$, and

$$\mu(U) - \mu(V) = \mu(U \setminus \Omega) + \mu(\Omega \setminus V) < \epsilon,$$

which proves the claim. □

Lemma 2.15. *Let (X, d, μ) be a metric measure space. Let $\Omega \subset B$ be a measurable subset of an open ball $B \subset X$. If for some constant $\sigma, \lambda > 0$ and for all $x \in \Omega$*

$$\liminf_{r\to 0} \fint_{B(x,r)} f\,d\mu \le \sigma \quad \text{and} \quad \limsup_{r\to 0} \fint_{B(x,r)} f\,d\mu \ge \lambda$$

then

$$\lambda\mu(\Omega) \le \int_{\Omega} f\,d\mu \le \sigma\mu(\Omega).$$

Proof. Let $\Omega \subset B$ be a measurable subset of an open ball $B \subset X$. Assume that there exists constants $\sigma, \lambda > 0$ such that for all $x \in \Omega$

$$\liminf_{r\to 0} \fint_{B(x,r)} f\,d\mu \leq \sigma \quad \text{and} \quad \limsup_{r\to 0} \fint_{B(x,r)} f\,d\mu \geq \lambda.$$

By lemma 2.14, Ω can be approximated with a closed set $V \subset \Omega$ and an open set $U \supset \Omega$ by

$$\mu(U) - \epsilon < \mu(\Omega) < \mu(V) + \epsilon,$$

to an arbitrary degree of precision $\epsilon > 0$. Let $\mathcal{U}$ and $\mathcal{V}$, be the fine covers of U and V, respectively, defined by

$$\mathcal{U} = \left\{ \overline{B}(x,r) \subset U \ : \ x \in \Omega, \ \fint_{\overline{B}(x,r)} f\,d\mu \leq \sigma + \epsilon \right\},$$
$$\mathcal{V} = \left\{ \overline{B}(x,r) \subset V \ : \ x \in \Omega, \ \fint_{\overline{B}(x,r)} f\,d\mu \geq \lambda - \epsilon \right\}.$$

Moreover, let $\mathcal{F} \subset \mathcal{U}$ be the disjointed collection of closed balls from the Vitali covering theorem 2.13 for U. Then,

$$\int_\Omega f\,d\mu \leq \int_U f\,d\mu = \sum_{\overline{B}\in\mathcal{F}} \int_{\overline{B}} f\,d\mu \leq (\sigma+\epsilon) \sum_{\overline{B}\in\mathcal{F}} \mu\left(\overline{B}\right),$$

where the series can be further estimated using the approximation of Ω with U and V,

$$\sum_{\overline{B}\in\mathcal{F}} \mu\left(\overline{B}\right) = \mu\left(\bigcup_{\overline{B}\in\mathcal{F}} \overline{B}\right) = \mu(U) < \mu(\Omega) + \epsilon.$$

Similarly, if $\mathcal{G} \subset \mathcal{V}$ is the disjointed closed cover for $V \subset \Omega$, it identically follows that

$$\begin{aligned}\int_\Omega f\,d\mu \geq \int_V f\,d\mu = \sum_{\overline{B}\in\mathcal{G}} \int_{\overline{B}} f\,d\mu &\geq (\lambda-\epsilon) \sum_{\overline{B}\in\mathcal{G}} \mu\left(\overline{B}\right) \\ &\geq (\lambda-\epsilon)\left(\mu(\Omega)-\epsilon\right).\end{aligned}$$

Hereby, for any precision $\epsilon > 0$, it holds that

$$(\lambda-\epsilon)\left(\mu(\Omega)-\epsilon\right) \leq \int_\Omega f\,d\mu \leq (\sigma+\epsilon)\left(\mu(\Omega)+\epsilon\right).$$

Hence, letting $\epsilon \to 0$ proves the claim. □

Whenever $E \subset X$ is a measurable subset and the function f is integrable on every compact subset of E it is said that f is locally integrable on E, denoted $f \in L^1_{loc}(E)$.

Theorem 2.16. ***Lebesgue differentation theorem***
If $f \in L^1_{loc}(X)$ is a nonnegative locally integrable function on a metric measure space (X, d, μ), then

$$\lim_{r\to 0} \fint_{B(x,r)} f\,d\mu = f(x)$$

μ-a.e. in X. The set of points in which the above equation holds, are called the Lebesgue points of f in X.

Proof. Let $f \in L^1_{loc}(X)$ be a nonnegative locally integrable function on a metric measure space (X, d, μ). Denote the set of points in X that are not Lebesgue points by

$$E = \left\{ x \in X \ : \ \lim_{r\to 0} \fint_{B(x,r)} f\,d\mu \neq f(x) \right\}.$$

The claim now reads as $\mu(E) = 0$. By local integrability of f there exists a family of open balls $\mathcal{B}$ such that $B \subset E$ and $f \in L^1(B)$, for all $B \in \mathcal{B}$. The Vitali covering lemma 2.8 provides disjointed subcollection of open balls $\{B_i\}_{i=1}^{\infty}$ with $E \subset \bigcup_{i=1}^{\infty} 5B_i$. Let $\sigma, \lambda > 0$ with $\sigma < \lambda$ and define $A_{\sigma,\lambda} \subset E$ by

$$A_{\sigma,\lambda} = \left\{ x \in X \ : \ \liminf_{r\to 0} \fint_{B(x,r)} f\,d\mu \leq \sigma < \lambda \leq \limsup_{r\to 0} \fint_{B(x,r)} f\,d\mu \right\}.$$

As $A_{\sigma,\lambda} \cap 5B_j \subset 5B_j$ is measurable for every $B_j \in \{B_i\}_{i=1}^{\infty}$, the Lemma 2.15 states

$$\lambda\mu(A_{\sigma,\lambda} \cap 5B_j) \leq \int_{A_{\sigma,\lambda}\cap 5B_j} f\,d\mu \leq \sigma\mu(A_{\sigma,\lambda} \cap 5B_j),$$

which implies that $\mu(A_{\sigma,\lambda} \cap 5B_j) = 0$ as $\sigma < \lambda$. Consequently, for every $0 < \sigma < \lambda$,

$$\mu(A_{\sigma,\lambda}) \leq \mu\left(A_{\sigma,\lambda} \cap \bigcup_{i=1}^{\infty} 5B_i \right) \leq \sum_{i=1}^{\infty} \mu\left(A_{\sigma,\lambda} \cap 5B_i \right) = 0.$$

The density of nonnegative rational numbers $\mathbb{Q}^+$ in $\mathbb{R}^+$ implies

$$E = \bigcup_{\substack{s,t\in\mathbb{R}^+ \\ s<t}} A_{s,t} = \bigcup_{\substack{\sigma,\lambda\in\mathbb{Q}^+ \\ \sigma<\lambda}} A_{\sigma,\lambda},$$

whereas the countability guarantees the well-definedness of the series

$$\mu(E) = \sum_{\substack{\sigma,\lambda\in\mathbb{Q}^+ \\ \sigma<\lambda}} \mu(A_{\sigma,\lambda}) = 0.$$

□

2.2 Hardy-Littlewood maximal function

Definition 2.17. ***Hardy-Littlewood maximal function.*** *The Hardy-Littlewood maximal function of $f \in L^1_{loc}(X)$ is defined by*

$$Mf(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f| d\mu,$$

where the supremum is taken over all the balls $B \subset X$ containing the point x.

Lemma 2.18. ***Sublinearity of Hardy-Littlewood maximal function***
Let (X, d, μ) be a metric measure space and $f, g \in L^1_{loc}(X)$, then it holds that $M(f+g) \leq Mf + Mg$.

Proof. The sublinearity of the maximal function is a direct consequence of the triangle inequality and the sublinearity of the supremum as

$$\begin{aligned} M(f+g)(x) &= \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f+g| d\mu \\ &\leq \sup_{B \ni x} \left(\frac{1}{\mu(B)} \int_B |f| d\mu + \frac{1}{\mu(B)} \int_B |g| d\mu \right) \\ &\leq \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f| d\mu + \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |g| d\mu \\ &= Mf(x) + Mg(x). \end{aligned}$$

□

Lemma 2.19. ***Essentially boundedness of Hardy-Littlewood maximal function***
Let (X, d, μ) be a metric measure space. If $f \in L^\infty(X)$ then $Mf \in L^\infty(X)$ and $\|Mf\|_{L^\infty(X)} \leq \|f\|_{L^\infty(X)}$.

Proof. Clearly, for all open balls $B \in X$

$$\fint_B |f| d\mu \leq \frac{1}{\mu(B)} \|f\|_{L^\infty(B)} \mu(B) = \|f\|_{L^\infty(B)}.$$

Hence, $\|Mf\|_{L^\infty(X)} \leq \|f\|_{L^\infty(X)}$ which also proves the first claim in conjunction with the assumption that $f \in L^\infty(X)$. □

Lemma 2.20. ***Semicontinuity***
Hardy-Littlewood maximal function is lower semicontinuous.

Proof. To show that the Hardy-Littlewood maximal function is lower semi-continuous, it suffices to show that $E_\lambda = \{x \in \Omega \,:\, Mf(x) > \lambda\}$ is open for all $\lambda > 0$. Let $\lambda > 0$ be arbitrary and $y \in E_\lambda$ so that,

$$Mf(y) = \sup_{B \ni y} \frac{1}{\mu(B)} \int_B |f| d\mu > \lambda.$$

By the definition of supremum, there exists an open ball $B_y \ni y$ for which

$$\frac{1}{\mu(B_y)} \int_{B_y} |f| d\mu > \lambda.$$

Hence, for every point $y \in E_\lambda$, there exists an open ball containing y, making E_λ an open set by definition. As λ was arbitrary, the claim is proven. □

Lemma 2.21. ***(1,1)-weak type estimate***
Let (X, d, μ) be a metric measure space and $f \in L^1_{loc}(X)$. Then there exists a constant $C = C(C_d) > 0$ for which

$$\mu\left(\{x \in X \ : \ Mf(x) > \lambda\}\right) \leq \frac{C}{\lambda} \int_X |f| d\mu.$$

Proof. Denote $E_\lambda = \{x \in X \ : \ Mf(x) > \lambda\}$. By the supremum in the Hardy-Littlewood maximal function, it follows that for every $x \in E_\lambda$ there exists a ball $B_x \subset X$ containing x, with integral average $|f|_{B_x} > \lambda$ implying that $E_\lambda \subset \bigcup_{x \in E_\lambda} B_x$. Moreover, for every B_x, it holds that

$$1 < \frac{1}{\lambda} \fint_{B_x} |f| d\mu. \tag{3}$$

If the radii of the balls $\{B_x\}_{x \in E_\lambda}$ are uniformly bounded, Vitali covering lemma 2.8 provides disjointed subcollection of balls $\{B_i\}_{i=1}^\infty \subset \{B_x\}_{x \in E_\lambda}$ such that $E_\lambda \subset \bigcup_{i=1}^\infty 5B_i$. Hence, there exists a constant $C_0 = C_0(C_d)$ for which

$$\mu\left(E_\lambda\right) \leq \mu\left(\bigcup_{x=1}^\infty 5B_i\right) = \sum_{x=1}^\infty \mu\left(5B_i\right) \leq C_0 \sum_{x=1}^\infty \mu\left(B_i\right).$$

The series can be further estimated with (3) as

$$\sum_{x=1}^\infty \mu\left(B_i\right) \leq \frac{1}{\lambda} \sum_{x=1}^\infty \int_{B_i} |f| d\mu \leq \frac{1}{\lambda} \int_X |f| d\mu.$$

This proves the claimed (1,1)-weak type estimate for the Hardy-Littlewood maximal function

$$\mu\left(\{x \in X \ : \ Mf(x) > \lambda\}\right) \leq \frac{C}{\lambda} \int_X |f| d\mu$$

whenever the radii of the balls $\{B_x\}_{x \in E_\lambda}$ are uniformly bounded. If they are not, there exists $z \in E_\lambda$ with $r(B_z) = \infty$, and

$$\mu(E_\lambda) \leq \mu(B_z) \leq \frac{1}{\lambda} \int_{B_z} |f| d\mu = \frac{1}{\lambda} \int_X |f| d\mu.$$

Letting $C = \max\{1, C_0\}$ proves the claim. □

Lemma 2.22. ***Reverse (1,1)-weak type estimate***
Let (X, d, μ) be a metric measure space. Let $f \in L^1(X)$ be a nonnegative function, $B \subset X$ a ball, and $⨍_B f d\mu < \lambda$. Then there exists a constant $C = C(C_d) > 1$ such that

$$\int_{B \cap \{Mf > \lambda\}} f d\mu \leq C\lambda\mu\left(\{Mf > \lambda\}\right).$$

Proof. Let $f \in L^1(X)$ be a nonnegative function, and $B \subset X$ be a ball for which

$$⨍_B f d\mu < \lambda.$$

Denote $E_\lambda = \{Mf > \lambda\}$. If the gray area in figure 5, $X \setminus E_\lambda = \{Mf \leq \lambda\}$, has a zero measure, then $\mu(X) = \mu(E_\lambda)$, which directly implies the claim by

$$\int_{B \cap E_\lambda} f d\mu = \int_B f d\mu < \lambda\mu(B) \leq \lambda\mu\left(E_\lambda\right). \tag{4}$$

So, assume that $\mu(X \setminus E_\lambda) > 0$. E_λ is open by lower semicontinuity 2.20. Thus, for every $x \in E_\lambda$ the open ball $B(x, r_x)$ with $r_x = d(x, X \setminus E_\lambda) > 0$ is contained in E_λ. If $\sup_{x \in E_\lambda} r_x = \infty$, then $B \subset E_\lambda$ and consequently (4) holds. On the other hand, if $\sup_{x \in E_\lambda} r_x < \infty$, the Vitali covering lemma 2.8 provides a countable disjointed open cover $\{B_i\}_{i=1}^\infty$ with

$$\bigcup_{i=1}^\infty B_i \subset E_\lambda \subset \bigcup_{i=1}^\infty 5B_i.$$

For every $B_j \in \{B_i\}_{i=1}^\infty$, there exists $z_j \in 5B_j \cap (X \setminus E_\lambda) = 5B_j \cap \{Mf \leq \lambda\}$ implying

$$⨍_{5B_j} f d\mu \leq Mf(z_j) \leq \lambda.$$

This yields the following estimate

$$\int_{B \cap E_\lambda} f d\mu \leq \sum_{i=1}^\infty \int_{5B_i} f d\mu \leq \lambda \sum_{i=1}^\infty \mu(5B_i),$$

where the sum of the 5-ball measures can be approximated by doubling condition

$$\sum_{i=1}^\infty \mu(5B_i) \leq C(C_d) \sum_{i=1}^\infty \mu(B_i) = C(C_d)\mu\left(\bigcup_{i=1}^\infty B_i\right) \leq C(C_d)\mu(E_\lambda).$$

□

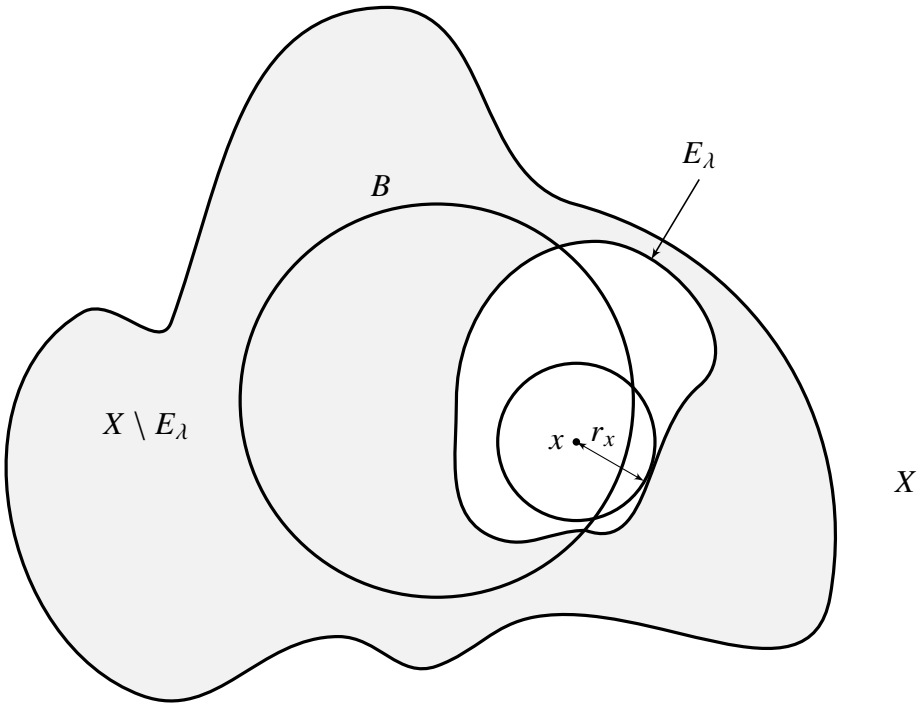

Figure 5: Spatial visualization of the Lemma 2.22.

Lemma 2.23. ***Cavalieri's principle***
Let $E \subset X$ be an arbitrary μ-measurable set and f be a μ-measurable function. Then for all $0 < p < \infty$

$$\int_E |f|^p d\mu = p \int_0^\infty \lambda^{p-1} \mu\left(\{x \in E \ : \ |f(x)| > \lambda\}\right) d\lambda.$$

Proof. Let $E \subset X$ and f be μ-measurable. Fubini's theorem readily implies

$$\begin{aligned}
\int_E |f|^p d\mu &= \int_X \mathbb{1}_E(x) |f(x)|^p d\mu(x) \\
&= \int_X \mathbb{1}_E(x) p \int_0^{|f(x)|} \lambda^{p-1} d\lambda d\mu(x) \\
&= p \int_X \int_0^\infty \mathbb{1}_E(x)\, \mathbb{1}_{\{\lambda < |f(x)|\}}\, \lambda^{p-1} d\lambda d\mu(x) \\
&= p \int_0^\infty \int_X \mathbb{1}_E(x)\, \mathbb{1}_{\{\lambda < |f(x)|\}}\, \lambda^{p-1} d\mu(x) d\lambda \\
&= p \int_0^\infty \lambda^{p-1} \int_X \mathbb{1}_E(x)\, \mathbb{1}_{\{\lambda < |f(x)|\}}\, d\mu(x) d\lambda \\
&= p \int_0^\infty \lambda^{p-1} \mu\left(\{x \in E \ : \ |f(x)| > \lambda\}\right) d\lambda.
\end{aligned}$$

□

Lemma 2.24. *Every L^p function can be expressed as a sum of L^1 and L^∞ functions.*

Proof. Assume that $f \in L^p(X)$. Let $\lambda > 0$ be given and define $f_1 = \mathbb{1}_{\{|f|>\lambda/2\}} f$ and $f_2 = \mathbb{1}_{\{|f|\leq\lambda/2\}} f$, so that $f = f_1 + f_2$. Clearly, $\|f_2\|_{L^\infty(X)} \leq \lambda/2 < \infty$. By the assumption of $f \in L^p(X)$,

$$\|f_1\|_{L^1(X)} = \int_{\{x\in X\,:\,|f|>\lambda/2\}} |f|^p |f|^{1-p} d\mu \leq \left(\frac{\lambda}{2}\right)^{1-p} \|f\|^p_{L^p(X)} < \infty,$$

i.e., $f_1 \in L^1(X)$. □

Lemma 2.25. ***(p,p)-strong type estimate***
Let (X, d, μ) be a metric measure space and $f \in L^1_{loc}(X)$. Then there exists a constant $C = C(p, d) > 0$ for which

$$\int_X |Mf|^p d\mu \leq C \int_X |f|^p d\mu.$$

Proof. Let $f \in L^1_{loc}(X)$. By the previous lemma 2.24, $f = f_1 + f_2$, where

$$f_1 = \mathbb{1}_{|f|>\lambda/2}\, f \in L^1(X),$$
$$f_2 = \mathbb{1}_{|f|\leq\lambda/2}\, f \in L^\infty(X).$$

The Hardy-Littlewood maximal function sublinear (cf. 2.18) and essentially bounded (cf. 2.19), so that μ-a.e. in X,

$$Mf(x) \leq Mf_1(x) + Mf_2(x) \leq Mf_1(x) + \frac{\lambda}{2}.$$

holds. This implies in conjunction with the $(1, 1)$-weak type estimate 2.21, that

$$\mu\left(\{Mf > \lambda\}\right) = \mu\left(\{Mf_1 > \tfrac{\lambda}{2}\}\right) \leq \frac{2C}{\lambda} \int_{\{|f|>\lambda/2\}} |f| d\mu.$$

Cavalieri's principle 2.23 and Fubini's theorem extend this to

$$\begin{aligned}
\int_X |Mf|^p d\mu(x) &= p \int_0^\infty \lambda^{p-1} \mu\left(\{Mf > \lambda\}\right) d\lambda \\
&\leq 2pC \int_0^\infty \lambda^{p-2} \int_{\{|f|>\lambda/2\}} |f| d\mu(x) d\lambda \\
&= 2pC \int_X |f| \int_0^{2|f(x)|} \lambda^{p-2} d\lambda d\mu(x) \\
&= \frac{2pC}{p-1} \int_X |f| |2f|^{p-1} d\mu(x) \\
&= \frac{2^p pC}{p-1} \int_X |f|^p d\mu(x),
\end{aligned}$$

which proves the claim. □

3 Proof of equivalence of five characterizations of the weak reverse Hölder inequality

Henceforth, the class of functions discussed is nonnegative weights $w \in L_{loc}(\Omega)$ with on an open subset $\Omega \subset X$. They ought to satisfy the weak reverse Hölder inequality with some $p > 1$ and constant $C > 0$ such that

$$\left(\fint_B w^p d\mu \right)^{\frac{1}{p}} \leq C \fint_{2B} w d\mu \tag{5}$$

for every ball B with $2B \Subset \Omega$. These functions form the weak reverse Hölder class, denoted by $WRH_p(\Omega)$, which is a relaxation of the reverse Hölder class, $RH_p(\Omega)$. The weights in $RH_p(\Omega)$ satisfy the strong version of reverse Hölder inequality, namely

$$\left(\fint_B w^p d\mu \right)^{\frac{1}{p}} \leq C \fint_B w d\mu. \tag{6}$$

$WRH_p(\Omega)$ allows nondoubling weights which can be zero on large parts of Ω, whereas doubling weights of $WR_p(\Omega)$ are either positive or identically zero μ-a.e. in Ω. A weight w satisfies reverse Hölder inequality (6) if and only if it belongs to some Muckenhoupt class $A_q(\Omega)$ with $1 < q < \infty$. Therefore, it is central in the theory of Muckenhoupt weights. By definition, w belongs to $A_q(\Omega)$ with $1 < q < \infty$, if

$$\sup_{B \subset \Omega} \left(\fint_B w d\mu \right) \left(\fint_B w^{-\frac{1}{q-1}} \right)^{q-1} < \infty.$$

Muckenhoupt classes are nested. Hence, $A_q(\Omega) \subset A_p(\Omega)$ for $1 < q < p \leq \infty$. Moreover, it can be shown that

$$A_\infty(\Omega) = \bigcup_{q>1} A_q(\Omega) = \bigcup_{p>1} RH_p(\Omega),$$

where $A_\infty(\Omega)$ can be defined with (6) or with (iv) without the dilation. Analogous results hold for weak Muckenhoupt weights derived from the weak reverse Hölder inequality (5).

By Jensen's inequality, a corollary of Hölder inequality, the weak reverse Hölder classes are nested. It states that for all open balls $B \Subset \Omega$ and $1 < q < p < \infty$,

$$\left(\fint_B w^q d\mu \right)^{\frac{1}{q}} \leq \left(\fint_B w^p d\mu \right)^{\frac{1}{p}},$$

implying $WRH_q(\Omega) \subset WRH_p(\Omega)$. Since (5) for $p = \infty$ simplifies to a strictly stronger condition, $\operatorname{ess\,sup}_B w \leq C w_{2B}$, it follows that

$$WRH_\infty(\Omega) \subset \bigcap_{p>1} WHR_p(\Omega). \tag{7}$$

WRH classes are self-improving, that is, if $w \in WRH_p(\Omega)$, then $w \in WRH_{p+\epsilon}(\Omega)$ for some $\epsilon > 0$. The self-improvement is not strong enough to imply the inclusion $WRH_p(\Omega) \subset WRH_\infty(\Omega)$ for all p. Hence, the inclusion (7) is proper. For instance, the weight

$$w(x) = \max\left\{\log\left(|x|^{-1}\right), 1\right\} \in WRH_p(\mathbb{R})$$

for all $p > 1$ but $w \notin WRH_\infty(\mathbb{R})$, as for $r < e^{-1}$, $\operatorname{ess\,sup}_{(-r,r)} w(x) = \infty$ [1]. Hence, to conclude,

$$WRH_\infty(\Omega) \subset \bigcap_{p>1} WRH_p(\Omega) \subset \bigcup_{p>1} WRH_p(\Omega) = WA_\infty(\Omega)$$

where $WA_\infty(\Omega)$ denotes the weak Muckenhoupt $A_\infty(\Omega)$ class.

$WRH_p(\Omega)$ has at least nine other equivalent formulations [1], out of which four are proved in this thesis. Out of these four, two can be extended for $p = \infty$. These strictly stronger conditions are simpler to prove, and hence demonstrated prior to $WRH_p(\Omega)$ formulations as an introduction to the topic.

Theorem 3.1. *Let (X, d, μ) be a metric measure space with a doubling measure. Assume that $\Omega \subset X$ is an open set, and let w be a weight on Ω. The following statements are equivalent.*

(i) (Weak reverse Hölder inequality for $p = \infty$) There exists a constant $C > 0$ such that for every ball B with $2B \Subset \Omega$, it holds that

$$\operatorname*{ess\,sup}_B w \leq C \fint_{2B} w d\mu.$$

(ii) (Quantitative nondoubling of weak A_∞ class) There exist constants $C > 0$ such that for every ball B with $2B \Subset \Omega$ and every measurable set $F \subset B$, it holds that

$$w(F) \leq C \left(\frac{\mu(F)}{\mu(B)}\right) w(2B).$$

(iii) (Pointwise weak Fujii-Wilson condition) There exists a constant $C > 0$ such that for every ball B with $2B \Subset \Omega$ and for every $x \in X$, it holds that

$$M\left(w \mathbb{1}_B\right)(x) \leq C \fint_{2B} w d\mu.$$

The claim is proved through equivalences (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii), which involve basic properties of doubling measure and Hardy-Littlewood maximal function in conjunction with Cavalieri's principle 2.23.

Proof. **(i)** $\implies$ **(ii)**
Assume the contents of (i). Then, there exists a constant $C > 0$ such that for every open ball B with $2B \Subset \Omega$,

$$\operatorname*{ess\,sup}_B w \leq C \fint_{2B} w d\mu \leq \frac{C}{\mu(B)} w(2B),$$

where the last inequality holds by monotonicity. For an arbitrary set $F \subset B$,

$$w(F) = \int_B w \, \mathbb{1}_F \, d\mu \leq \operatorname*{ess\,sup}_B w \mu(F),$$

holds. Substituting latter estimate to prior shows that for every ball B with $2B \Subset \Omega$ and every measurable set $F \subset B$, it holds that

$$w(F) \leq C \left(\frac{\mu(F)}{\mu(B)} \right) w(2B).$$

□

Proof. **(i)** $\impliedby$ **(ii)**
Assume, that there exist constants $C > 0$ such that for every ball B with $2B \Subset \Omega$ and every measurable set $F \subset B$, it holds that

$$w(F) \leq C \left(\frac{\mu(F)}{\mu(B)} \right) w(2B).$$

Let $t > 0$ and denote $F_t = B \cap \{w > t\}$. Then, the integral average of w over F_t is at least t, and

$$\mu(F_t)t \leq w(F_t) \leq C \left(\frac{\mu(F_t)}{\mu(B)} \right) w(2B).$$

By doubling condition, this can be further estimated to

$$\mu(F_t)t \leq \tilde{C} \mu(F_t) \fint_{2B} w d\mu,$$

where $\tilde{C} = C_d C$. This implies that $\mu(F_t) = 0$ for $t > \tilde{C} w_{2B}$. Acknowledging this in Cavalieri's principle 2.23, one obtains for every $r > 1$,

$$\int_B w^r d\mu = r \int_0^{\tilde{C} w_{2B}} t^{r-1} \mu(F_t) dt \leq \mu(B) r \int_0^{\tilde{C} w_{2B}} t^{r-1} dt,$$

which simplifies to the following estimate where the upper bound is invariant of $r > 0$

$$\left(\fint_B w^r d\mu \right)^{\frac{1}{r}} \leq \tilde{C} \fint_{2B} w d\mu.$$

Thereby, letting $r \to \infty$ gives

$$\lim_{r \to \infty} \left(\fint_B w^r d\mu \right)^{\frac{1}{r}} = \lim_{r \to \infty} \mu(B)^{-\frac{1}{r}} \|w\|_{L^r(B)} = \operatorname*{ess\,sup}_B w \leq \tilde{C} \fint_{2B} w d\mu.$$

□

Proof. **(i)** $\implies$ **(iii)**
Assume that w satisfies the weak reverse Hölder inequality for $p = \infty$. Then, for every measurable subset $F \subset B$,

$$\operatorname*{ess\,sup}_{F} w \leq \operatorname*{ess\,sup}_{B} w \leq C \fint_{2B} w d\mu.$$

Since for every $x \in \Omega$, it holds that

$$M(w\,\mathbb{1}_B)(x) = \sup_{B' \ni x} \frac{1}{\mu(B')} \int_{B' \cap B} w d\mu \leq \operatorname*{ess\,sup}_{B \cap B'} w,$$

the claim is proven by noting that $B' \cap B \subset B$ is measurable. □

Proof. **(i)** $\impliedby$ **(iii)**
Conversely, assume that there exists a constant $C > 0$ such that for every ball B with $2B \Subset \Omega$ and every $x \in X$, it holds that

$$M(w\,\mathbb{1}_B)(x) \leq C \fint_{2B} w d\mu.$$

By Lebesgue differentation theorem 2.16, for μ-a.e. in X

$$w\,\mathbb{1}_B(x) = \lim_{r \to 0} \fint_{B(x,r)} w\,\mathbb{1}_B\, d\mu \leq M(w\,\mathbb{1}_B)(x).$$

By assumption, this shows that μ-a.e. in B,

$$w(x) \leq M(w\,\mathbb{1}_B)(x) \leq C \fint_{2B} w d\mu.$$

Hence, μ-a.e. in B,

$$\operatorname*{ess\,sup}_{B} w \leq C \fint_{2B} w d\mu.$$

□

The conditions for $WRH_\infty(\Omega)$ are stronger than the first three conditions presented below. Assertion 3.1(i) is exactly 3.2(i) with $p = \infty$, similarly as assertion 3.1(ii) is 3.2(ii) with $\alpha = 1$. The weak Fujii-Wilson condition 3.1(iii) is point-wise formulation, and hence readily implies the corresponding integral condition 3.2(iii). Naturally, the converse does not hold in general.

The fundamental difference between these two sets of statements lies in the inclusions

$$WRH_\infty(\Omega) \subset \bigcap_{p>1} WRH_p(\Omega) \subset \bigcup_{p>1} WRH_p(\Omega) = WA_\infty(\Omega).$$

$WRH_\infty(\Omega)$ satisfies the statements of 3.1 and $WA_\infty(\Omega)$ class satisfies those of 3.2.

Theorem 3.2. *Let (X, d, μ) be a metric measure space with a doubling measure. Assume that $\Omega \subset X$ is an open set, and let w be a weight on Ω. The following assertations are equivalent.*

(i) (Weak reverse Hölder inequality) There exist $p > 1$ and a constant $C > 0$ such that for every ball B with $2B \Subset \Omega$, it holds that

$$\left(\fint_B w^p d\mu \right)^{\frac{1}{p}} \leq C \fint_{2B} w d\mu.$$

(ii) (Quantitative nondoubling of weak A_∞ class) There exist constants $C, \alpha > 0$ such that for every ball B with $2B \Subset \Omega$ and every measurable set $F \subset B$, it holds that

$$w(F) \leq C \left(\frac{\mu(F)}{\mu(B)} \right)^{\alpha} w(2B).$$

(iii) (Weak Fujii-Wilson condition) There exists a constant $C > 0$ such that for every ball B with $2B \Subset \Omega$ it holds that

$$\int_B M(w \mathbb{1}_B) d\mu \leq C w(2B).$$

(iv) (Qualitative weak A_∞ class characterization) For every $\eta > 0$ there exists $\epsilon > 0$ such that if B is a ball with $2B \Subset \Omega$ and $F \subset B$ is a measurable set, then $\mu(F) \leq \epsilon \mu(B)$ implies that $w(F) \leq \eta w(2B)$.

(v) There exist $\eta, \epsilon > 0$ with $\eta < C_d^{-5}$ such that if B is a ball with $2B \Subset \Omega$ and $F \subset B$ is a measurable set, then $\mu(F) \leq \epsilon \mu(B)$ implies that $w(F) \leq \eta w(2B)$.

The structure of the proof is illustrated in Figure 6. The sequence of implications is established first, beginning with the weak reverse Hölder inequality (i). The first three implications are derived from suitable choices of constants and the Hölder inequality, and are demonstrated within this chapter. The final implication, which requires extensive effort and a variety of techniques, is addressed in its own section 3.1. This includes proving and iteratively applying a complex lemma.

The equivalence between the weak Fujii-Wilson condition and other conditions is established by proving the implications of the diagonal in Figure 6. The weak Fujii-Wilson condition is directly implied by the weak reverse Hölder inequality via the (p, p)-strong type estimate for the Hardy-Littlewood maximal function 2.25. The remaining implication, from the weak Fujii-Wilson condition (iii) to the qualitative characterization of the A_∞ class (iv), is best addressed by breaking it down into multiple intermediate lemmas using the properties of the Hardy-Littlewood maximal function, as demonstrated in Section 2.2.

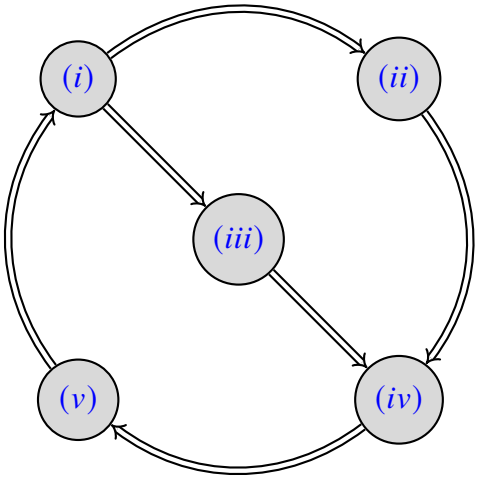

Figure 6: The structure of the proof of theorem 3.2. The outer loop of implications will be proved in Sections 3 and 3.1, and the diagonal path, $(i) \implies (iii) \implies (iv)$, in Section 3.2.

Proof. **(i) $\implies$ (ii)**
Assume the contents of the statement (i). Let $F \subset B$ be a measurable set. By applying Hölder's inequality to identity function $\mathbb{1}_F$ and weight w one obtains

$$\begin{aligned} w(F) &= \int_B \mathbb{1}_F \, w d\mu \leq \left(\int_B \mathbb{1}_F^{\frac{p}{p-1}} \, d\mu \right)^{1-\frac{1}{p}} \left(\int_B w^p d\mu \right)^{\frac{1}{p}} \\ &= \mu(F) \left(\frac{\mu(B)}{\mu(F)} \right)^{\frac{1}{p}} \left(\fint_B w^p d\mu \right)^{\frac{1}{p}}, \end{aligned}$$

where the integral average satisfies the weak reverse Hölder inequality,

$$\left(\fint_B w^p d\mu \right)^{\frac{1}{p}} \leq C \fint_{2B} w d\mu \leq \frac{C}{\mu(B)} \int_{2B} w d\mu$$

The combination of these estimates yield the claim with $\alpha = 1 - \frac{1}{p}$. □

Proof. **(ii) $\implies$ (iv)**
Assume the contents of (ii). Let $\eta > 0$ be given. Define $\epsilon = \left(\frac{\eta}{C}\right)^{\frac{1}{\alpha}} > 0$ with $C, \alpha > 0$ from previous equation. If $\mu(F) \leq \epsilon \mu(B)$, then (ii) implies

$$w(F) \leq C \left(\frac{\mu(F)}{\mu(B)} \right)^{\alpha} w(2B) \leq \eta w(2B).$$

□

Proof. **(iv) $\implies$ (v)**
Assuming (iv) gives the existence of $\epsilon > 0$ for every $\eta > 0$. Hence, in particular, it yields the existence of ϵ for $0 < \eta < C_d^{-5}$. □

3.1 Quantitative weak A_∞ condition implies weak reverse Hölder inequality

Proving the implication from (v) to the weak reverse Hölder inequality (i) relies heavily on lemma 3.3.

Lemma 3.3. *Assume that w satisfies the quantitative weak A_∞ condition as stated in (v) of Theorem 3.2. Then there exist constants $\gamma > C_d^3$, and $\beta > 0$, depending only on the parameters of (v), for which the following statement holds. Let $B = B(x_0, r_0)$ be a ball with $2B \Subset \Omega$, $0 < r < \frac{3}{2}$, and $0 < \lambda < \frac{1}{10}$. Then*

$$\int_{rB\cap\{w\geq\gamma D\}} w d\mu \leq \gamma^{-\beta} \int_{(r+\lambda)B\cap\{w\geq\gamma^{-1}D\}} w d\mu,$$

where

$$D = D(\lambda, B) = \frac{w(2B)}{\mu(2B)} C_d^{\log_2(\frac{4}{5\lambda})+1}.$$

Proof. Let $\eta, \epsilon > 0$ be constants with $\eta < C_d^{-5}$ as in (v). Choose $\gamma = \gamma(C_d, \epsilon, \eta)$ and $\beta = \beta(C_d, \epsilon, \eta)$ as

$$\gamma > \max\left\{C_d^3, \frac{C_d^2}{\epsilon}, \frac{1}{1-\eta C_D^5}\right\}, \quad \text{and,} \quad \gamma^\beta = \frac{1-\gamma^{-1}}{\eta C_d^5}.$$

Let $B = B(x_0, r_0)$ be a ball with $2B \Subset \Omega$, and, let $0 < r < \frac{3}{2}$ and $0 < \lambda < \frac{1}{10}$. For the convenience of notation, denote

$$I = \{y \in \Omega \,:\, w(y) \geq A\}, \quad \text{and,} \quad J = \{y \in \Omega \,:\, w(y) \geq a\},$$

where $A = \gamma D$ and $a = \gamma^{-1} D$. With this notation, the inequality in the claim reads as

$$w(I \cap rB) \leq \gamma^{-\beta} w\left(J \cap (r+\lambda)B\right).$$

If $\mu(I \cap rB) = 0$, there is nothing to prove, so assume that $\mu(I \cap rB) > 0$. Let the Lebesgue points of w in $I \cap rB$ be denoted as F. By definition, for every $x \in F$,

$$w(x) = \lim_{r\to 0} \fint_{B(x,r)} w d\mu > D,$$

which implies the existence of $u_x > 0$ such that $w_{B(x,r)} > D$ for all $0 < r < u_x$. Take any $y \in B(x, 5\lambda r_0)$ and $z \in B(x, 5\lambda r_0) \cap rB$, then by assumptions for r and λ, it holds that $d(y, x_0) \leq d(y, z) + d(z, x_0) \leq 5\lambda r_0 + r r_0 \leq 2r_0$. This yields the inclusions $B(x, 5\lambda r_0) \subset 2B \subset B(x, 4r_0)$ and

$$\mu(2B) \leq \mu(x, 4r_0) \leq C_d^{\lfloor \log_2(\frac{4}{5\lambda})\rfloor+1} \mu\left(B(x, 5\lambda r_0)\right),$$

where the exponent was contents of corollary 2.4. By monotonicity, this implies that

$$\fint_{B(x,5\lambda r_0)} w d\mu = \frac{w\left(B(x,5\lambda r_0)\right)}{\mu\left(B(x,5\lambda r_0)\right)} \leq \frac{w(2B)}{\mu(2B)} C_d^{\lfloor \log_2(\frac{4}{5\lambda})\rfloor+1} \leq D.$$

Hence, there exists largest lower bound $s_x > 0$ for the radii with $w_{B(x,s_x)} \leq D$ and $B(x,s_x) \Subset \Omega$, that is,

$$s_x = \inf\left\{ s > 0 \ : \ B(x,s) \Subset \Omega \text{ and } \fint_{B(x,s)} w d\mu \leq D \right\}. \tag{8}$$

Note that there exists $\tilde{s_x}$ with $s_x \leq \tilde{s_x} \leq 2s_x$ for which the conditions in (8) are met.

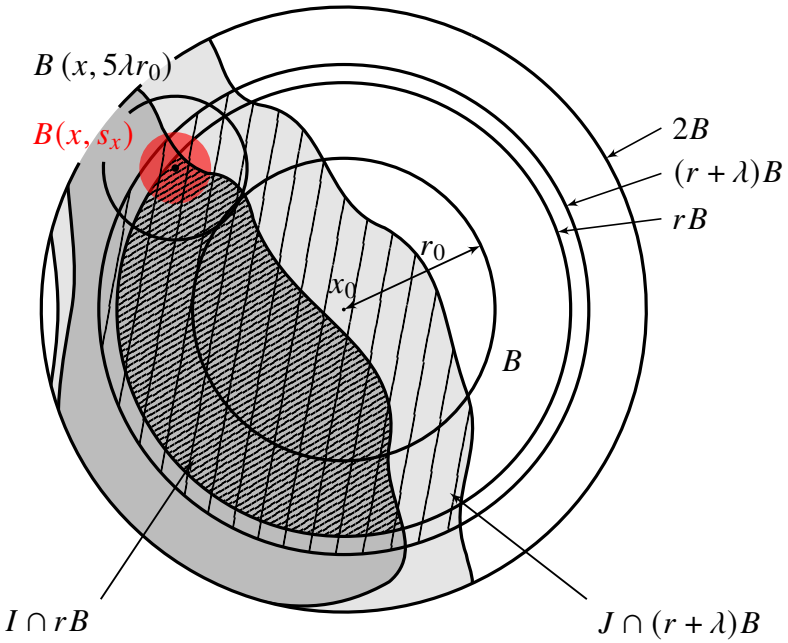

Figure 7: The distributional sets of the claim in the context of the proof.

Either $\tilde{s_x} = s_x$, or $\tilde{s_x} > s_x$ can be chosen arbitrarily close to s_x. As $1 \leq wA^{-1}$ in I,

$$\begin{aligned}
\mu\left(I \cap B(x,s_x)\right) &\leq \int_{I\cap B(x,s_x)} wA^{-1} d\mu \leq \int_{B(x,\tilde{s_x})} wA^{-1} d\mu \\
&= A^{-1}\mu\left(B(x,\tilde{s_x})\right) \fint_{B(x,\tilde{s_x})} w d\mu \\
&\leq A^{-1} D\mu\left(B(x,\tilde{s_x})\right) \leq A^{-1} D\mu(B(x,2s_x)).
\end{aligned}$$

This allows the estimation of $\mu\left(I \cap B\left(x,\frac{s_x}{2}\right)\right)$ independently of the set I by

$$\begin{aligned}
\mu\left(I \cap B(x,\tfrac{s_x}{2})\right) &\leq \mu\left(I \cap B(x,s_x)\right) \leq A^{-1} D\mu\left(B(x,2s_x)\right) \\
&\leq A^{-1} D C_d^2 \mu\left(B(x,\tfrac{s_x}{2})\right) = \gamma^{-1} C_d^2 \mu\left(B(x,\tfrac{s_x}{2})\right) \\
&< \epsilon\mu\left(B(x,\tfrac{s_x}{2})\right),
\end{aligned}$$

where the last inequality was implied by the choice of γ. As $I \cap B(x, \frac{s_x}{2})$ is a measurable subset of $B\left(x, \frac{s_x}{2}\right)$, by the assumption (v), this implies that

$$w\left(I \cap B\left(x, \tfrac{s_x}{2}\right)\right) \leq \eta w\left(B(x, s_x)\right).$$

Define $r_x = \frac{1}{10} s_x$, and further estimate $w\left(B(x, s_x)\right) \leq w\left(B(x, \tilde{s_x})\right)$ with $w_{B(x,\tilde{s_x})} \leq D$

$$w\left(B(x, s_x)\right) \leq D\mu\left(B(x, \tilde{s_x})\right) \leq D\mu\left(B(x, 2s_x)\right) \leq DC_d^5 \mu\left(B(x, r_x)\right).$$

Since $B(x, r_x) \subset B(x, s_x) \Subset \Omega$, by (8) it follows that $w_{B(x,r_x)} > D$, and consequently

$$\mu\left(B(x, r_x)\right) \leq \frac{1}{D} w\left(B(x, r_x)\right). \tag{9}$$

Combining above allows estimating $w\left(I \cap B\left(x, \frac{s_x}{2}\right)\right)$ independently of the set I as

$$w\left(I \cap B\left(x, \tfrac{s_x}{2}\right)\right) \leq \eta C_d^5 w\left(B(x, r_x)\right). \tag{10}$$

To establish the connection between the sets I and J, $w\left(B(x, r_x)\right)$ is estimated with J. Using the upper bound of $B(x, r_x) \backslash J = \{y \in B(x, r_x) \ : \ w(y) < a\}$ and estimate (9),

$$w\left(B(x, r_x) \backslash J\right) \leq a\mu\left(B(x, r_x) \backslash J\right) \leq a\mu\left(B(x, r_x)\right) \leq \frac{a}{D} w\left(B(x, r_x)\right),$$

which substituted to $w\left(B(x, r_x)\right) = w\left(B(x, r_x) \setminus J\right) + w\left(B(x, r_x) \cap J\right)$ yields

$$w\left(B(x, r_x)\right) \leq \left(1 - \gamma^{-1}\right)^{-1} w\left(J \cap B(x, r_x)\right).$$

By estimate (10), this shows that for every Lebesgue point $x \in F$, it holds that

$$w\left(I \cap B(x, 5r_x)\right) \leq \gamma^{-\beta} w\left(J \cap B(x, r_x)\right). \tag{11}$$

Since $r_x \leq \frac{\lambda}{2} r_0$, for every $z \in B(x, r_x)$ it holds that $d(x_0, z) \leq r r_0 + \frac{\lambda}{2} r_0 \leq (r + \lambda) r_0$, implying that the union of such balls is fully contained in $(r + \lambda)B$. By the density of $F \subset I \cap rB$, this establishes the inclusions

$$F \subset I \cap rB \subset \bigcup_{x \in F} B(x, r_x) \subset (r + \lambda)B.$$

Thus, the collection of balls is uniformly bounded, allowing the use of Vitali covering lemma 2.8, which provides a disjointed open cover $\{B_i\}_{i=1}^{\infty}$ with

$$I \cap rB \subset (I \cap rB) \cap \bigcup_{i=1}^{\infty} 5B_i \subset I \cap \bigcup_{i=1}^{\infty} 5B_i.$$

The claim follows by using estimate (11) and monotonicity on previous inclusion

$$w(I \cap rB) \leq \sum_{i=0}^{\infty} w(I \cap 5B_i) \leq \sum_{i=1}^{\infty} \gamma^{-\beta} w\left(J \cap B_i\right) \leq \gamma^{-\beta} w\left(J \cap (r + \lambda)B\right).$$

□

Overview of the following proof is the following. $\|w\|^p_{L^p(B)}$ is divided into a series integrals based on the values of the weight function w. The integrand in each of those disjoint sets is broken to $w^p = w^{p-1}w$, where w^{p-1} is estimated with the upper bound, and w with lemma 3.3. By choosing the constants in lemma 3.3 appropriately, the series converges and the claim is proven.

Proof. **(v)** $\implies$ **(i)**
Let w be a weight on Ω, B be an arbitrary ball with $2B \Subset \Omega$, and, $F \subset B$ a measurable set. Assume that there exist $\epsilon > 0$ and $\eta > 0$ with $\eta < C_d^{-5}$, such that $\mu(F) \leq \epsilon\mu(B)$ implies $w(F) \leq \eta w(2B)$. This renders lemma 3.3 immediately applicable. Let the constants β and γ, be as in Lemma 3.3. Choose $p > 1$ such that $2(p-1) < \beta$, and define λ_i for $i = 1, 2, \ldots$ as

$$\lambda_i = \frac{4}{5} 2^{1-2i\frac{\log(\gamma)}{\log(C_d)}}.$$

These are in accordance with the assumptions of lemma 3.3, as λ was chosen such that $\log(\gamma) > 3\log(C_d)$, and hence $\lambda_i < \frac{1}{10}$. The exponent of D_i now simplifies to

$$\log_2\left(\frac{4}{5\lambda_i}\right) + 1 = 2i\frac{\log(\gamma)}{\log(C_d)} = \frac{\log\left(\gamma^{2i}\right)}{\log(C_d)},$$

which substituted to $C_d = e^{\log(C_d)}$ yields a recursion $\gamma^{-1}D_i = \gamma D_{i-1}$, where

$$D_i = \frac{w(2B)}{\mu(2B)} C_d^{\log_2\left(\frac{4}{5\lambda_i}\right)+1} = \gamma^{2i}\frac{w(2B)}{\mu(2B)}, \tag{12}$$

for $i = 1, 2, \ldots$. Applying lemma 3.3 iteratively, one obtains the estimate

$$\begin{aligned} w(B \cap \{w \geq \gamma D_i\}) &\leq \gamma^{-\beta} w\left((1+\lambda_i)B \cap \left\{w \geq \gamma^{-1}D_i\right\}\right) \\ &= \gamma^{-\beta} w\left((1+\lambda_i)B \cap \{w \geq \gamma D_{i-1}\}\right) \\ &\leq \gamma^{-2\beta} w\left((1+\lambda_i+\lambda_{i-1})B \cap \{w \geq \gamma D_{i-2}\}\right) \\ &= \ldots \leq \gamma^{-(i-1)\beta} w\left(\left(1+\sum_{j=1}^{i}\lambda_j\right)B \cap \{w \geq \gamma D_1\}\right). \end{aligned}$$

Since $\sum_{i=1}^{\infty}\lambda_i < \frac{8}{5}\sum_{i=1}^{\infty} 2^{-6i} < \frac{1}{2}$, the dilation factors comply with the assumptions of the lemma, rendering above estimate valid. It can be further estimated to

$$w(B \cap \{w \geq \gamma D_i\}) \leq \gamma^{-(i-1)\beta} w\left(2B\right). \tag{13}$$

Integrating w^p over B in disjoint layers $B \cap \{\gamma D_i < w \leq \gamma D_{i+1}\}$, estimating w^{p-1}

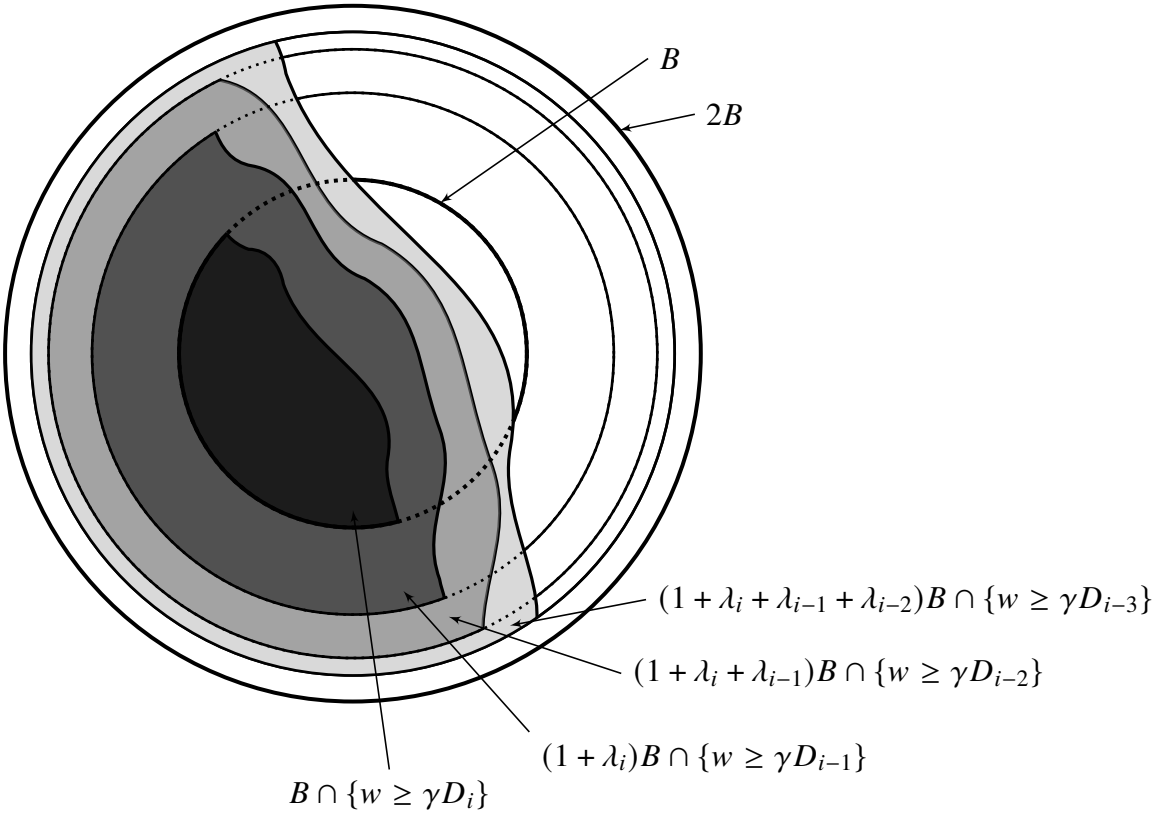

Figure 8: A spatial visualization of the sets used in the iterative application of lemma 3.3 present in the derivation of inequality (13).

with the corresponding upper bounds, and, w with (13) one obtains the estimate

$$\begin{aligned}
\fint_B w^p d\mu &= \frac{1}{\mu(B)} \left(\int_{B\cap\{w\leq\gamma D_1\}} w^p d\mu + \sum_{i=1}^{\infty} \int_{B\cap\{\gamma D_i < w \leq \gamma D_{i+1}\}} w^{p-1} w \, d\mu \right) \\
&\leq \frac{1}{\mu(B)} \left(\gamma^p D_1^p \mu(B) + \sum_{i=1}^{\infty} (\gamma D_{i+1})^{p-1} \gamma^{-(i-1)\beta} w\,(2B) \right) \\
&= \gamma^p D_1^p + \sum_{i=1}^{\infty} (\gamma D_{i+1})^{p-1} \gamma^{-(i-1)\beta} \frac{w\,(2B)}{\mu(B)} \\
&\leq \gamma^p D_1^p + C_d \sum_{i=1}^{\infty} (\gamma D_{i+1})^{p-1} \gamma^{-(i-1)\beta} \frac{w\,(2B)}{\mu(2B)}. \qquad (14)
\end{aligned}$$

Substituting D_i from (12) to (14) simplifies the terms to

$$\gamma^p D_1^p = \gamma^{3p} \left(\frac{w(2B)}{\mu(2B)} \right)^p = \gamma^{3p} \left(\fint_{2B} w d\mu \right)^p,$$

and

$$(\gamma D_{i+1})^{p-1}\gamma^{-(i-1)\beta}\frac{w\,(2B)}{\mu(2B)} = \left(\gamma^{2i+3}\frac{w(2B)}{\mu(2B)}\right)^{p-1}\gamma^{-(i-1)\beta}\frac{w\,(2B)}{\mu(2B)}$$
$$= \gamma^{(2i+3)(p-1)-(i-1)\beta}\left(\frac{w(2B)}{\mu(2B)}\right)^{p}$$
$$= \gamma^{5(p-1)}\gamma^{(i-1)(2(p-1)-\beta)}\left(\fint_{2B} w\,d\mu\right)^{p}.$$

Thus, (14) becomes

$$\fint_B w^p\,d\mu \le \left(\gamma^{3p} + C_d\gamma^{5(p-1)}\sum_{i=1}^{\infty}\gamma^{(i-1)(2(p-1)-\beta)}\right)\left(\fint_{2B} w\,d\mu\right)^{p},$$

where the geometric series converges by the choice of p, as $|\gamma^{(i-1)(2(p-1)-\beta)}| < 1$ for all i. □

3.2 Weak Fujii-Wilson condition's equivalence to weak reverse Hölder inequality

Proving the implication from weak reverse Hölder inequality (i) to weak Fujii-Wilson condition (iii) involves the (p,p)-strong type inequality for Hardy-Littlewood maximal function. The proof from (iii) to the qualitative weak A_∞ class characterization (iv) applies the remaining properties proved for maximal function in 2.2.

Proof. **(i) $\implies$ (iii)**
The claim is a direct consequence of (p,p)-strong type inequality 2.25, as

$$\int_B M(w\,\mathbb{1}_B)d\mu = \int_B (M(w\,\mathbb{1}_B)^p)^{\frac{1}{p}}\,d\mu \le \mu(B)^{\frac{p-1}{p}}\left(\int_X M(w\,\mathbb{1}_B)^p d\mu\right)^{\frac{1}{p}},$$

where, by (p,p)-strong type inequality,

$$\int_X M(w\,\mathbb{1}_B)^p d\mu \le C_{p,d}\int_X (w\,\mathbb{1}_B)^p d\mu$$
$$\le C_{p,d}\mu(B)\fint_B w^p d\mu.$$

Combining these and applying the assumption (i) yields

$$\int_B M(w\,\mathbb{1}_B)d\mu \le \mu(B)^{\frac{p-1}{p}}\left(C_{p,d}\mu(B)\fint_B w^p d\mu\right)^{\frac{1}{p}}$$
$$\le C_{p,d}^{\frac{1}{p}}C\frac{\mu(B)}{\mu(2B)}w(2B),$$

where $\frac{\mu(B)}{\mu(2B)}$ can be estimated upwards to 1, effectively removing the constant's dependency of B. Hence, the claim is proven. □

Lemma 3.4. *Let $B \subset X$ be a ball, and $w \in L^1(B)$ a nonnegative function. There exists a constant $C_1 = C_1(C_d) > 1$, such that*

$$M(w\,\mathbb{1}_B) \leq C_1 w_B,$$

everywhere in $X \backslash 2B$.

Proof. Let $B \subset X$ be a ball, and $w \in L^1(B)$ a nonnegative function. By writing out the definition of Hardy-Littlewood maximal function, the claim reads

$$\sup_{\substack{B' \ni x \\ B \cap B' \neq \emptyset}} \frac{1}{\mu(B')} \int_{B \cap B'} w \, d\mu \leq C_1 \fint_B w \, d\mu.$$

Let $x \in X \setminus 2B$, and let $B' \subset X$ be a ball such that $B \cap B \neq \emptyset$ and $x \in B'$. One candidate ball over which the supremum is taken, is depicted in figure 9.

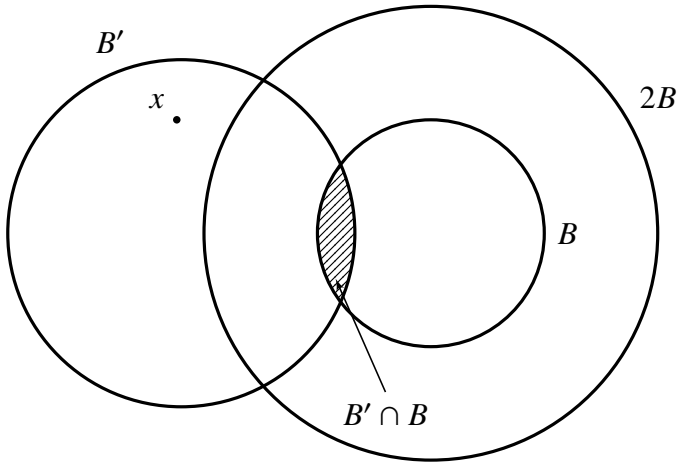

Figure 9: The Hardy-Littlewood maximal function of $w\,\mathbb{1}_B$ at a point $x \in X \setminus 2B$ is the supremum over integral averages over balls B' containing the point x. The weight function contributes to the integral only in the striped intersection of $B' \cap B$.

It is clear $B \subset 5B'$, so that $\mu(B') \geq C\mu(5B') \geq C\mu(B)$, where $C = C(C_d)$. Let $C_1 = C^{-1}$. Then by monotonicity,

$$\begin{aligned} M(w\,\mathbb{1}_B)(x) &= \sup_{\substack{B' \ni x \\ B \cap B' \neq \emptyset}} \frac{1}{\mu(B')} \int_{B \cap B'} w \, d\mu \\ &\leq \frac{1}{C\mu(B)} \sup_{B' \ni x} \int_{B \cap B'} w \, d\mu \\ &= C_1 \fint_B w \, d\mu. \end{aligned}$$

□

Lemma 3.5. *Assume that w satisfies the weak Fujii-Wilson condition (iii). There exist constants $C_2 = C_2(C_d)$ and $C_3 = C_3(C_d) > 0$, such that*

$$\int_B w \log^+\left(\frac{w}{C_2 w_{2B}}\right) d\mu \leq C_3 w(4B)$$

for every ball B with $4B \Subset \Omega$.

Proof. Let B be a ball with $4B \Subset \Omega$. By Lemma 3.4 there exists a constant $C_1 = C_1(C_d)$ such that for every $x \in X \setminus 2B$

$$M(w\,\mathbb{1}_B)(x) \leq C_1 w_B.$$

Let $\lambda, C_2 > 0$ be such that $C_1 w_B \leq C_2 w_{2B} < \lambda$. Thus, $M(w\,\mathbb{1}_B) < \lambda$ for all $x \in X \setminus 2B$, implying the inclusion $\{M(w\,\mathbb{1}_B) > \lambda\} \subset 2B$. Hence, by the reverse (1,1)-weak type estimate of Hardy-Littlewood maximal function 2.22,

$$\begin{aligned}\int_{B\cap\{w>\lambda\}} w d\mu &\leq \int_{B\cap\{M(w\,\mathbb{1}_B)>\lambda\}} w d\mu \\ &\leq C_0\lambda\mu\left(\{M(w\,\mathbb{1}_B) > \lambda\}\right) \\ &= C_0\lambda\mu\left(2B \cap \{M(w\,\mathbb{1}_B) > \lambda\}\right).\end{aligned}$$

In conjunction with Cavalieri's principle 2.23 and Fubini's theorem, this implies

$$\begin{aligned}\int_{2B} M(w\,\mathbb{1}_B) d\mu &= \int_0^\infty \mu\left(\{x \in 2B \,:\, M(w\,\mathbb{1}_B) > \lambda\}\right) d\lambda \\ &\geq \int_{C_2 w_{2B}}^\infty \mu\left(2B \cap \{M(w\,\mathbb{1}_B) > \lambda\}\right) d\lambda \\ &\geq \frac{1}{C_0}\int_{C_2 w_{2B}}^\infty \frac{1}{\lambda}\int_{B\cap\{w>\lambda\}} w d\mu d\lambda \\ &= \frac{1}{C_0}\int_{B\cap\{w>C_2 w_{2B}\}} w \int_{C_2 w_{2B}}^\infty \frac{1}{\lambda} d\lambda d\mu \\ &= \frac{1}{C_0}\int_B w \log^+\left(\frac{w}{C_2 w_{2B}}\right) d\mu.\end{aligned}$$

The claim follows by applying the assumption of the weak Fujii-Wilson condition (iii)

$$\begin{aligned}\int_B w \log^+\left(\frac{w}{C_2 w_{2B}}\right) d\mu &\leq C_0 \int_{2B} M(w\,\mathbb{1}_B) d\mu \\ &\leq C_0 \int_{2B} M(w\,\mathbb{1}_{2B}) d\mu \\ &\leq C_3 w(4B),\end{aligned}$$

where $C_3 = C_0 \times C > 0$. □

Lemma 3.6. *Assume that w satisfies the weak Fujii-Wilson condition (iii). Then, for every $\tilde{\eta} > 0$ there exists $\tilde{\epsilon} > 0$ such that for every measurable set $F \subset B$ with $4B \Subset \Omega$ and $\mu(F) \leq \tilde{\epsilon}\mu(B)$ implies $w(F) \leq \tilde{\eta} w(4B)$.*

Proof. Let $\tilde{\eta} > 0$. Choose $\gamma > 1$ and $\tilde{\epsilon} > 0$ with $C_3/\log(\gamma) \leq \tilde{\eta}/2$ and $\tilde{\epsilon} C_2 \gamma \leq \tilde{\eta}/2$. Let $F \subset B$ be measurable subset of a ball with $4B \Subset \Omega$, such that $\mu(F) \leq \tilde{\epsilon}\mu(B)$.

Divide F into a disjoint union of $F \cap \{w \leq \gamma C_2 w_{2B}\}$ and $F \cap \{w > \gamma C_2 w_{2B}\}$. The weight of the former can be approximated by

$$\begin{aligned} w\left(F \cap \{w \leq \gamma C_2 w_{2B}\}\right) &\leq \mu(F) C_2 \gamma w_{2B} \leq \tilde{\epsilon}\mu(B) C_2 \gamma w_{2B} \\ &\leq \frac{\tilde{\eta}}{2} \frac{\mu(B)}{\mu(2B)} w(2B) \leq \frac{\tilde{\eta}}{2} w(4B). \end{aligned}$$

In $F \cap \{w > \gamma C_2 w_{2B}\}$ on has $\gamma \leq \frac{w}{C_2 w_{2B}}$. Hence, by previous lemma 3.5

$$\begin{aligned} w\left(F \cap \{w > \gamma C_2 w_{2B}\}\right) &\leq \frac{1}{\log(\gamma)} \int_{F \cap \{w > \gamma C_2 w_{2B}\}} w \log\left(\frac{w}{C_2 w_{2B}}\right) d\mu \\ &\leq \frac{1}{\log(\gamma)} \int_B w \log\left(\frac{w}{C_2 w_{2B}}\right) d\mu \\ &\leq \frac{C_3 w(4B)}{\log(\gamma)} \leq \frac{\tilde{\eta}}{2} w(4B). \end{aligned}$$

Combining these estimates yields the claim, $w(F) \leq \frac{\tilde{\eta}}{2} w(4B) + \frac{\tilde{\eta}}{2} w(4B) = \tilde{\eta} w(4B)$. □

Proof. **(iii) $\Longrightarrow$ (iv):**
Let B be a ball with $2B \Subset \Omega$, and let $F \subset B$ be a measurable set. Let $\{B_i\}_{i=1}^N$ be a collection of balls from lemma 2.5 with $\sigma = 1/5$. Then $4B_i \subset 2B \Subset \Omega$, and, $\mu(B_i) \geq \tilde{c}(C_d)\mu(B)$.

Let $\eta > 0$ be given. Denote $\tilde{\eta} = N\eta$. Let $\tilde{\epsilon}$ be the parameter associated with $\tilde{\eta}$ from lemma 3.6 and $\epsilon = \tilde{c}(C_d)\tilde{\epsilon}$. If $\mu(F) \leq \epsilon\mu(B)$, then for every $i = 1, \ldots, N$

$$\frac{\mu(F \cap B_i)}{\mu(B_i)} \leq \frac{\mu(F)}{\mu(B_i)} \leq \frac{\mu(F)}{\tilde{c}(C_d)\mu(B)} = \frac{\epsilon}{\tilde{c}(C_d)} \leq \tilde{\epsilon}.$$

That is, $\mu(F \cap B_i) \leq \tilde{\epsilon}\mu(B_i)$, which, by lemma 3.6 implies

$$w(F \cap B_i) \leq \tilde{\eta} w(4B_i) \leq \tilde{\eta} w(2B)$$

for every $i = 1, \ldots, N$, and consequently,

$$w(F) \leq \sum_{i=1}^N w(F \cap B_i) \leq \tilde{\eta} \sum_{i=1}^N w(4B_i) = \tilde{\eta} N w(2B) = \eta w(2B).$$

□

4 Proof of the characterizations of the weak reverse Hölder inequality for $\sigma > 1$

As discussed in the beginning of section 3, the WRH classes are self-improving and nested, meaning that if $w \in WRH_p(\Omega)$ then $w \in WRH_q(\Omega)$ with all $1 < q \leq p + \epsilon$ for some $\epsilon > 0$. It was defined that $w \in WRH_p(\Omega)$ if it satisfied the relaxed version of the reverse Hölder inequality (5), where the integral averageon the right hand side was taken over ball $2B$.

In this section, it will be shown that $2B$ in the assertions of theorem 3.2 can be replaced with σB, where $\sigma > 1$. Only the qualitative weak A_∞ class characterization for some η requires modifications to its upper bound for η. This raises the question whether or not these assertions for different values of σ yield the same classes of functions.

It turns out they do. Unlike with $p > 1$, it is sufficient for a weight w to satisfy the assertions of 4.1 for some $\sigma_0 > 1$ to imply that it satisfies them with every $\sigma > 1$. To deduce this, let $w \in WRH_p(\Omega)$ for some $\sigma_0 > 1$. Then, by monotonicity, w satisfies the quantitative nondoubling of the weak A_∞ class (ii) with all $\sigma \geq \sigma_0$

$$w(F) \leq C \left(\frac{\mu(F)}{\mu(B)} \right)^\alpha w(\sigma_0 B) \leq C \left(\frac{\mu(F)}{\mu(B)} \right)^\alpha w(\sigma B).$$

To show that $w \in WRH_p(\Omega)$ for all $1 < \sigma < \sigma_0$, let $B \subset X$ be an arbitrary open ball, and cover it with finite number of balls $\{B_i\}_{i=1}^N$ with radii $r(B_i) < \frac{\sigma - 1}{\sigma_0} r$, cf. lemma 2.5. By construction, for every i, it holds that $\sigma_0 B_i = (\sigma - 1)B_i \subset \sigma B$. Thus, by weak reverse Hölder inequality for σ_0 one also obtains it for $\sigma < \sigma_0$ by

$$\begin{aligned}
\fint_B w^p d\mu &\leq \frac{1}{\mu(B)} \sum_{i=1}^N \mu(B_i) \fint_{B_i} w^p d\mu \\
&\leq \sum_{i=1}^N \fint_{B_i} w^p d\mu \leq C \sum_{i=1}^N \left(\fint_{\sigma_0 B_i} w d\mu \right)^p \\
&\leq C \sum_{i=1}^N \left(\frac{1}{\mu\left((\sigma - 1)B\right)} \int_{\sigma B} w d\mu \right)^p \\
&\leq C \sum_{i=1}^N \left(C_d^{\lfloor \log_2 (\frac{\sigma}{\sigma - 1}) \rfloor + 1} \frac{1}{\mu(\sigma B)} \int_{\sigma B} w d\mu \right)^p \\
&= N \tilde{C} \left(\fint_{\sigma B} w d\mu \right)^p .
\end{aligned}$$

This proves the weak reverse Hölder inequality for all $1 < \sigma < \sigma_0$. On the last inequality, it was used that $\sigma_0 B_i = (\sigma - 1)B_i \subset \sigma B$ implies $w\left((\sigma - 1)B\right) \leq w(\sigma B)$, so by corollary 2.4

$$\frac{1}{\mu\left((\sigma - 1)B\right)} \leq C_d^{\lfloor \log_2 (\frac{\sigma}{\sigma - 1}) \rfloor + 1} \frac{1}{\mu(\sigma B)}.$$

Theorem 4.1. *Let (X, d, μ) be a metric measure space with a doubling measure. Assume that $\Omega \subset X$ is an open set, and let w be a weight on Ω. The following statements are equivalent.*

(i) (Weak reverse Hölder inequality) There exist $p > 1$ and a constant $C > 0$ such that for every ball B with $\sigma B \Subset \Omega$, it holds that

$$\left(\fint_B w^p \, d\mu \right)^{\frac{1}{p}} \leq C \fint_{\sigma B} w \, d\mu.$$

(ii) (Quantitative nondoubling of weak A_∞) There exist constants $C, \alpha > 0$ such that for every ball B with $\sigma B \Subset \Omega$ and every measurable set $F \subset B$, it holds that

$$w(F) \leq C \left(\frac{\mu(F)}{\mu(B)} \right)^{\alpha} w(\sigma B).$$

(iii) (Qualitative weak A_∞ characterization) For every $\eta > 0$, there exists $\epsilon > 0$ such that if B is a ball with $\sigma B \Subset \Omega$ and $F \subset B$ is a measurable set, then $\mu(F) \leq \epsilon \mu(B)$ implies that $w(F) \leq \eta w(\sigma B)$.

(iv) There exist $\eta, \epsilon > 0$ with

$$\eta < C_d^{-\lfloor \log_2(5\sigma^2) \rfloor - 1},$$

such that if B is a ball with $\sigma B \Subset \Omega$ and $F \subset B$ is a measurable set, then $\mu(F) \leq \epsilon \mu(B)$ implies that $w(F) \leq \eta w(\sigma B)$.

Outline of the proof follows that of Sections 3 and 3.1. The constants in the following proofs are optimized in a sense that plugging $\sigma = 2$ to them yields the corresponding, previously presented proofs.

Proof. **(i) $\Longrightarrow$ (ii)**
Assume the contents of the statement (i). Let $F \subset B$ be a measurable set. By applying Hölder's inequality to identity function $\mathbb{1}_F$ and weight w one obtains

$$\begin{aligned} w(F) = \int_B \mathbb{1}_F \, w \, d\mu &\leq \left(\int_B \mathbb{1}_F^{\frac{p}{p-1}} \, d\mu \right)^{1-\frac{1}{p}} \left(\int_B w^p \, d\mu \right)^{\frac{1}{p}} \\ &= \mu(F) \left(\frac{\mu(B)}{\mu(F)} \right)^{\frac{1}{p}} \left(\fint_B w^p \, d\mu \right)^{\frac{1}{p}}, \end{aligned}$$

where the integral average satisfies the weak reverse Hölder inequality,

$$\left(\fint_B w^p \, d\mu \right)^{\frac{1}{p}} \leq C \fint_{\sigma B} w \, d\mu \leq \frac{C}{\mu(B)} \int_{\sigma B} w \, d\mu$$

The combination of these estimates yield the claim with $\alpha = 1 - \frac{1}{p}$. □

Proof. **(ii) $\implies$ (iii)**
Assume the contents of (ii). Let $\eta > 0$ be given. Define $\epsilon = \left(\frac{\eta}{C}\right)^{\frac{1}{\alpha}} > 0$ with $C, \alpha > 0$ from previous equation. If $\mu(F) \leq \epsilon\mu(B)$, then (ii) implies

$$w(F) \leq C\left(\frac{\mu(F)}{\mu(B)}\right)^{\alpha} w(\sigma B) \leq \eta w(\sigma B).$$

□

Proof. **(iii) $\implies$ (iv)**
Assuming (iii) gives the existence of ϵ for every $\eta > 0$. Hence, in particular, it yields the existence of ϵ for

$$0 < \eta < C_d^{-\lfloor \log_2(5\sigma^2)\rfloor - 1}.$$

□

Naturally, proving the implication from (iv) to the weak reverse Hölder inequality (i) requires a generalization of the lemma 3.3 that constants accommodates the general dilation factor $\sigma > 1$.

Lemma 4.2. *Assume that w satisfies (iv) of Theorem 4.1. Then there exist constants*

$$\gamma > C_d^{\frac{1}{2}\log_2\left(\frac{32\sigma}{\sigma-1}\right)} \quad \text{and} \quad \beta > 0,$$

depending only on the parameters of (iv), for which the following statement holds. Let $\sigma > 1$, and let B be a ball with $\sigma B \Subset \Omega$, $0 < r < 1 + \frac{\sigma-1}{2}$, and $\lambda < \frac{\sigma-1}{10}$. Then

$$\int_{rB\cap\{w\geq\gamma D\}} w d\mu \leq \gamma^{-\beta} \int_{(r+\lambda)B\cap\{w\geq\gamma^{-1}D\}} w d\mu,$$

where

$$D = D(\lambda, B) = \frac{w(\sigma B)}{\mu(\sigma B)} C_d^{\log_2\left(\frac{2\sigma}{5\lambda}\right)+1}.$$

Proof. Let $\eta, \epsilon > 0$ be constants with $\eta < C_d^{-\lfloor \log_2(5\sigma^2)\rfloor - 1}$ as in (iv). Choose $\gamma = \gamma(C_d, \epsilon, \eta)$ and $\beta = \beta(C_d, \epsilon, \eta)$ as

$$\gamma > \max\left\{ C_d^{\frac{1}{2}\log_2\left(\frac{32\sigma}{\sigma-1}\right)}, \frac{C_d^{\lceil \log_2(\sigma^2)\rceil}}{\epsilon}, \frac{1}{1 - \eta C_D^{\lfloor \log_2(5\sigma^2)\rfloor + 1}} \right\},$$

$$\gamma^{\beta} = \frac{1 - \gamma^{-1}}{\eta C_d^{\lfloor \log_2(5\sigma^2)\rfloor + 1}}.$$

Let $B = B(x_0, r_0)$ be a ball with $\sigma B \Subset \Omega$, and, let $0 < r < 1 + \frac{\sigma-1}{2}$ and $\lambda < \frac{\sigma-1}{10}$. For the convenience of notation, denote

$$I = \{y \in \Omega \ : \ w(y) \geq A\}, \quad \text{and,} \quad J = \{y \in \Omega \ : \ w(y) \geq a\},$$

where $A = \gamma D$ and $a = \gamma^{-1} D$. With this notation, the inequality in the claim reads as

$$w(I \cap rB) \leq \gamma^{-\beta} w\left(J \cap (r+\lambda)B\right).$$

If $\mu(I \cap rB) = 0$, there is nothing to prove, so assume that $\mu(I \cap rB) > 0$. Let the Lebesgue points of w in $I \cap rB$ be denoted as F. By definition, for every $x \in F$,

$$w(x) = \lim_{r \to 0} \fint_{B(x,r)} w d\mu > D,$$

which implies the existence of $u_x > 0$ such that $w_{B(x,r)} > D$ for all $0 < r < u_x$. Take any $y \in B(x, 5\lambda r_0)$ and $z \in B(x, 5\lambda r_0) \cap rB$, then by assumptions for r and λ, it holds that $d(y, x_0) \leq d(y, z) + d(z, x_0) \leq 5\lambda r_0 + r r_0 \leq \sigma r_0$. This yields the inclusions $B\left(x, 5\lambda r_0\right) \subset \sigma B \subset B(x, 2\sigma r_0)$ and

$$\mu(\sigma B) \leq \mu(x, 2\sigma r_0) \leq C_d^{\lfloor \log_2\left(\frac{2\sigma}{5\lambda}\right) \rfloor + 1} \mu\left(B(x, 5\lambda r_0)\right),$$

where the exponent was contents of corollary 2.4. By monotonicity, this implies that

$$\fint_{B(x,5\lambda r_0)} w d\mu = \frac{w\left(B(x, 5\lambda r_0)\right)}{\mu\left(B(x, 5\lambda r_0)\right)} \leq \frac{w(\sigma B)}{\mu(\sigma B)} C_d^{\lfloor \log_2\left(\frac{2\sigma}{5\lambda}\right) \rfloor + 1} \leq D.$$

Hence, there exists largest lower bound $s_x > 0$ for the radii with $w_{B(x,s_x)} \leq D$ and $B(x, s_x) \Subset \Omega$, that is,

$$s_x = \inf \left\{ s > 0 \ : \ B(x, s) \Subset \Omega \text{ and } \fint_{B(x,s)} w d\mu \leq D \right\}. \tag{15}$$

Note that there exists $\tilde{s_x}$ with $s_x \leq \tilde{s_x} \leq \sigma s_x$ for which the conditions in (15) are met. Either $\tilde{s_x} = s_x$, or $\tilde{s_x} > s_x$ can be chosen arbitrarily close to s_x. As $1 \leq w A^{-1}$ in I,

$$\begin{aligned}
\mu\left(I \cap B(x, s_x)\right) &\leq \int_{I \cap B(x,s_x)} w A^{-1} d\mu \leq \int_{B(x,\tilde{s_x})} w A^{-1} d\mu \\
&= A^{-1} \mu\left(B(x, \tilde{s_x})\right) \fint_{B(x,\tilde{s_x})} w d\mu \\
&\leq A^{-1} D \mu\left(B(x, \tilde{s_x})\right) \leq A^{-1} D \mu(B(x, \sigma s_x)).
\end{aligned}$$

This allows the estimation of $\mu\left(I \cap B\left(x, \frac{s_x}{2}\right)\right)$ independently of the set I by

$$\begin{aligned}
\mu\left(I \cap B(x, \tfrac{s_x}{\sigma})\right) &\leq \mu\left(I \cap B(x, s_x)\right) \leq A^{-1} D \mu\left(B(x, \sigma s_x)\right) \\
&\leq A^{-1} D C_d^{\lceil \log_2(\sigma^2) \rceil} \mu\left(B(x, \tfrac{s_x}{\sigma})\right) \\
&= \gamma^{-1} C_d^{\lceil \log_2(\sigma^2) \rceil} \mu\left(B(x, \tfrac{s_x}{\sigma})\right) \\
&< \epsilon \mu\left(B(x, \tfrac{s_x}{\sigma})\right),
\end{aligned}$$

where the last inequality was implied by the choice of γ. As $I \cap B(x, \frac{s_x}{\sigma})$ is a measurable subset of $B\left(x, \frac{s_x}{\sigma}\right)$, by the assumption (iv), this implies that

$$w\left(I \cap B\left(x, \tfrac{s_x}{\sigma}\right)\right) \leq \eta w\left(B(x, s_x)\right),$$

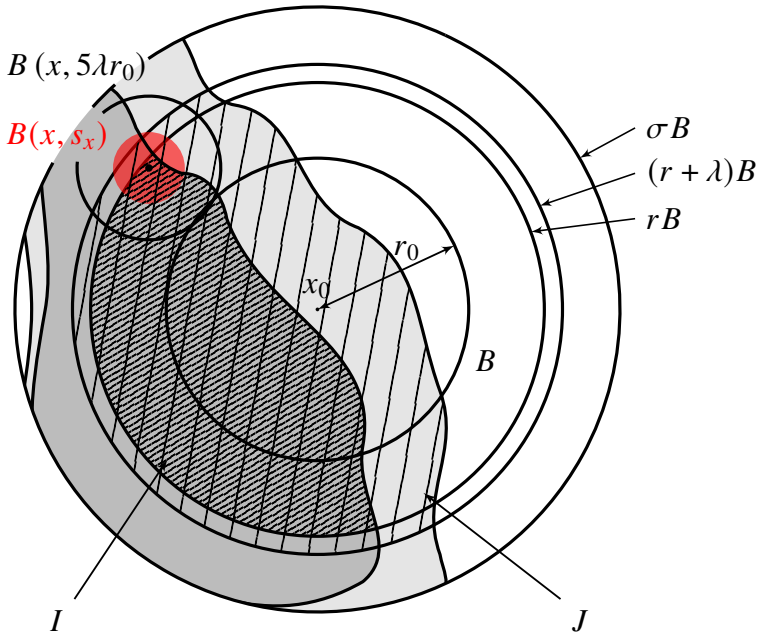

Figure 10: A spatial visualization of the arguments used in the proof.

Define $r_x = \frac{1}{5\sigma} s_x$, and further estimate $w\left(B(x, s_x)\right) \leq w\left(B(x, \tilde{s_x})\right)$ with $w_{B(x,\tilde{s_x})} \leq D$

$$\begin{aligned} w\left(B(x, s_x)\right) &\leq D\mu\left(B(x, \tilde{s_x})\right) \leq D\mu\left(B(x, \sigma s_x)\right) \\ &\leq D C_d^{\lfloor \log_2(5\sigma^2) \rfloor + 1} \mu\left(B(x, r_x)\right). \end{aligned}$$

Since $r_x < s_x$ and $B(x, r_x) \subset B(x, s_x) \Subset \Omega$, it follows by (15) that $w_{B(x,r_x)} > D$, and hence

$$\mu\left(B(x, r_x)\right) \leq \frac{1}{D} w\left(B(x, r_x)\right). \tag{16}$$

Combining above allows the estimation of $w\left(I \cap B\left(x, \frac{s_x}{\sigma}\right)\right)$ independently of the set I as

$$w\left(I \cap B\left(x, \tfrac{s_x}{\sigma}\right)\right) \leq \eta C_d^{\lfloor \log_2(5\sigma^2) \rfloor + 1} w\left(B(x, r_x)\right). \tag{17}$$

To establish the connection between the sets I and J, $w\left(B(x, r_x)\right)$ is estimated with J. Using the upper bound of $B(x, r_x) \backslash J = \{y \in B(x, r_x) \ : \ w(y) < a\}$ and estimate (16),

$$w\left(B(x, r_x) \setminus J\right) \leq a\mu\left(B(x, r_x) \setminus J\right) \leq a\mu\left(B(x, r_x)\right) \leq \frac{a}{D} w\left(B(x, r_x)\right).$$

which substituted to $w\left(B(x, r_x)\right) = w\left(B(x, r_x) \setminus J\right) + w\left(B(x, r_x) \cap J\right)$ yields

$$w\left(B(x, r_x)\right) \leq \left(1 - \gamma^{-1}\right)^{-1} w\left(J \cap B(x, r_x)\right).$$

By estimate (17), this shows that for every Lebesgue point $x \in F$, it holds that

$$w\left(I \cap B(x, 5r_x)\right) \leq \gamma^{-\beta} w\left(J \cap B(x, r_x)\right). \tag{18}$$

Since $r_x \leq \frac{\lambda}{\sigma} r_0$, for every $z \in B(x, r_x)$ it holds that $d(x_0, z) \leq r r_0 + \frac{\lambda}{\sigma} r_0 \leq (r + \lambda) r_0$, implying that the union of such balls is fully contained in $(r + \lambda)B$. By the density of $F \subset I \cap rB$, this establishes the inclusions

$$F \subset I \cap rB \subset \bigcup_{x \in F} B(x, r_x) \subset (r + \lambda)B.$$

Thus, the collection of balls is uniformly bounded, allowing the use of Vitali covering lemma 2.8, which provides a disjointed open cover $\{B_i\}_{i=1}^{\infty}$ with

$$I \cap rB \subset (I \cap rB) \cap \bigcup_{i=1}^{\infty} 5B_i \subset I \cap \bigcup_{i=1}^{\infty} 5B_i.$$

The claim follows by using estimate (18) and monotonicity on previous inclusion

$$w(I \cap rB) \leq \sum_{i=0}^{\infty} w(I \cap 5B_i) \leq \sum_{i=1}^{\infty} \gamma^{-\beta} w\left(J \cap B_i\right) \leq \gamma^{-\beta} w\left(J \cap (r + \lambda)B\right).$$

□

For convenience and brevity of the proof of *(iv)* $\implies$ *(i)*, the following lemma is stated and proved in advance.

Lemma 4.3. *Let $\sigma > 1$ and define $\alpha_\sigma(k) = \frac{1}{5} 2^{2-5k} \sigma \left(\frac{\sigma-1}{\sigma}\right)^k$. Then, for every $k = 1, 2, \ldots$ it holds that*

$$\alpha_\sigma(k) < \frac{\sigma - 1}{10}.$$

Proof. Let $\sigma > 1$ be given. Extend α_σ with respect to k from $\mathbb{Z}^+$ to $\mathbb{R}$ as

$$\begin{aligned} \alpha_\sigma &: (0, \infty) \to \mathbb{R}, \\ \alpha_\sigma(t) &:= \frac{1}{5} 2^{2-5t} \sigma \left(\frac{\sigma - 1}{\sigma}\right)^t. \end{aligned}$$

The claim can now be rephrased as $\forall t = 1, 2, \ldots : \alpha_\sigma(t) < (\sigma - 1)/10$. For $t = 1$, $\alpha_\sigma(1) = (\sigma - 1)/40 < (\sigma - 1)/10$. As α_σ is strictly decreasing function for all $t > 0$,

$$\begin{aligned} \frac{d}{dt}\alpha_\sigma(t) &= \frac{1}{5} 2^{2-5t} \left(\frac{\sigma - 1}{\sigma}\right)^t \sigma \log\left(\frac{\sigma - 1}{32\sigma}\right) \\ &< \underbrace{\frac{1}{5} 2^{2-5t} \left(\frac{\sigma - 1}{\sigma}\right)^t \sigma}_{>0} \underbrace{\log\left(\frac{1}{32}\right)}_{<0} < 0, \end{aligned}$$

the claim readily follows. □

Proof. **(iv) $\implies$ (i)**
Let w be a weight on Ω, B be an arbitrary ball with $2B \Subset \Omega$, and, $F \subset B$ a measurable set. Assume that there exist $\epsilon > 0$ and $\eta > 0$ with

$$\eta < \frac{1}{C_d^{\lfloor \log_2(5\sigma^2) \rfloor + 1}},$$

such that $\mu(F) \leq \epsilon\mu(B)$ implies $w(F) \leq \eta w(2B)$. This renders lemma 4.2 immediately applicable. Let the constants β and γ, be as in Lemma 4.2. Choose $p > 1$ with $2(p-1) < \beta$, and define λ_i for $i = 1, 2, \ldots$ as

$$\lambda_i = \frac{1}{5} 2^{2 - \frac{2i\log(\gamma)}{\log(C_d)}} \sigma.$$

By the choice of γ, the exponent in λ_i can be approximated by

$$2 - \frac{2i\log(\gamma)}{\log(C_d)} < 2 - i\log_2\left(\frac{32\sigma}{\sigma - 1}\right) = 2 + \log_2\left(\left(\frac{\sigma - 1}{2^5\sigma}\right)^i\right),$$

which in conjuction with lemma 4.3 shows that for every $i = 1, 2, \ldots,$

$$\lambda_i = \frac{1}{5} 2^{2 - \frac{2i\log(\gamma)}{\log(C_d)}} \sigma < \frac{4\sigma}{5}\left(\frac{\sigma - 1}{2^5\sigma}\right)^i = \frac{1}{5} 2^{2-5i}\sigma\left(\frac{\sigma - 1}{\sigma}\right)^i < \frac{\sigma - 1}{10}.$$

Hence, λ_i comply with the assumptions of lemma 4.2. The exponent of D_i simplifies to

$$\log_2\left(\frac{2\sigma}{5\lambda_i}\right) + 1 = \log_2\left(\frac{2\sigma}{5}\frac{5}{\sigma} 2^{\frac{2i\log(\gamma)}{\log(C_d)} - 2}\right) + 1 = \frac{\log\left(\gamma^{2i}\right)}{\log(C_d)}$$

which substituted to $C_d = e^{\log(C_d)}$ yields a recursion $\gamma^{-1} D_i = \gamma D_{i-1}$, where

$$D_i = \frac{w(\sigma B)}{\mu(\sigma B)} C_d^{\log_2\left(\frac{2\sigma}{5\lambda_i}\right) + 1} = \gamma^{2i} \frac{w(\sigma B)}{\mu(\sigma B)}, \tag{19}$$

for $i = 1, 2, \ldots$. Since $\frac{\sigma - 1}{2^5\sigma} < \frac{1}{2^5}$ holds whenever $\sigma > 1$, the sum of λ_i can be bounded by

$$\sum_{i=1}^{\infty} \lambda_i < \frac{4\sigma}{5} \sum_{i=1}^{\infty} \left(\frac{1}{2^5}\right)^i = \frac{4\sigma}{155} < \frac{\sigma - 1}{2}.$$

Consequently, the following dilation factors satisfy the assumptions lemma 4.2, which is applied iteratively to obtain

$$\begin{aligned}
w(B \cap \{w \geq \gamma D_i\}) &\leq \gamma^{-\beta} w\left((1 + \lambda_i)B \cap \left\{w \geq \gamma^{-1} D_i\right\}\right) \\
&= \gamma^{-\beta} w\left((1 + \lambda_i)B \cap \{w \geq \gamma D_{i-1}\}\right) \\
&\leq \gamma^{-2\beta} w\left((1 + \lambda_i + \lambda_{i-1})B \cap \{w \geq \gamma D_{i-2}\}\right) \\
&= \ldots \leq \gamma^{-(i-1)\beta} w\left(\left(1 + \sum_{j=1}^{i} \lambda_j\right) B \cap \{w \geq \gamma D_1\}\right),
\end{aligned}$$

By monotonicity, this can be further estimated to

$$w\left(B \cap \{w \geq \gamma D_i\}\right) \leq \gamma^{-(i-1)\beta} w(\sigma B). \tag{20}$$

Integrating w^p over B in disjoint layers $B \cap \{\gamma D_i < w \leq \gamma D_{i+1}\}$, estimating w^{p-1} with the corresponding upper bounds, and, w with (20) one obtains

$$\begin{aligned}
\fint_B w^p d\mu &= \frac{1}{\mu(B)} \left(\int_{B\cap\{w\leq\gamma D_1\}} w^p d\mu + \sum_{i=1}^{\infty} \int_{B\cap\{\gamma D_i < w \leq \gamma D_{i+1}\}} w^{p-1} w \, d\mu \right) \\
&\leq \frac{1}{\mu(B)} \left(\gamma^p D_1^p \mu(B) + \sum_{i=1}^{\infty} (\gamma D_{i+1})^{p-1} \gamma^{-(i-1)\beta} w(\sigma B) \right) \\
&= \gamma^p D_1^p + \sum_{i=1}^{\infty} (\gamma D_{i+1})^{p-1} \gamma^{-(i-1)\beta} \frac{w(\sigma B)}{\mu(B)} \\
&\leq \gamma^p D_1^p + C_d^{\lfloor \log_2(\sigma) \rfloor + 1} \sum_{i=1}^{\infty} (\gamma D_{i+1})^{p-1} \gamma^{-(i-1)\beta} \frac{w(\sigma B)}{\mu(\sigma B)}.
\end{aligned} \tag{21}$$

Last inequality was by corollary 2.4. Substituting (19) to (21), and simplifying yields

$$\gamma^p D_1^p = \gamma^{3p} \left(\frac{w(\sigma B)}{\mu(\sigma B)} \right)^p = \gamma^{3p} \left(\fint_{\sigma B} w d\mu \right)^p,$$

and

$$\begin{aligned}
(\gamma D_{i+1})^{p-1} \gamma^{-(i-1)\beta} \frac{w(\sigma B)}{\mu(\sigma B)} &= \left(\gamma^{2i+3} \frac{w(\sigma B)}{\mu(\sigma B)} \right)^{p-1} \gamma^{-(i-1)\beta} \frac{w(\sigma B)}{\mu(\sigma B)} \\
&= \gamma^{(2i+3)(p-1)-(i-1)\beta} \left(\frac{w(\sigma B)}{\mu(\sigma B)} \right)^p \\
&= \gamma^{5(p-1)} \gamma^{(i-1)(2(p-1)-\beta)} \left(\fint_{\sigma B} w d\mu \right)^p.
\end{aligned}$$

Thus, (21) becomes

$$\fint_B w^p d\mu \leq \left(\gamma^{3p} + C_d^{\lfloor \log_2(\sigma) \rfloor + 1} \gamma^{5(p-1)} \sum_{i=1}^{\infty} \gamma^{(i-1)(2(p-1)-\beta)} \right) \left(\fint_{\sigma B} w d\mu \right)^p,$$

where the geometric series converges by the choice of p, as $|\gamma^{(i-1)(2(p-1)-\beta)}| < 1$ for all i. Hence, the claim is proven. □

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