Department of Mathematics and Systems Analysis

# On Pathwise Stochastic Integration of Processes with Unbounded Power Variation

Zhe Chen

Aalto University

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# On Pathwise Stochastic Integration of Processes with Unbounded Power Variation

**Zhe Chen**

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**Aalto University**
**School of Science**
**Department of Mathematics and Systems Analysis**
**Stochastics and Statistics**

**Supervising professor**
Professor Lasse Leskelä, Aalto University, Finland

**Thesis advisor**
Dr. Lauri Viitasaari, Aalto University, Finland

**Preliminary examiners**
Professor Paavo Salminen, Åbo Akademi University, Finland

Professor Georgiy Shevchenko, Taras Shevchenko National University of Kyiv, Ukraine

**Opponent**
Professor Tommi Sottinen, University of Vaasa, Finland

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# Abstract

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**Abstract**

This dissertation concerns pathwise integrability of stochastic processes which are non-semimartingales with unbounded power variation. In this dissertation, a class of stochastic processes which can be represented as a composition of a Hölder continuous process with a nonrandom function of locally bounded variation is studied. Since the nonrandom function may contain discontinuities, stochastic processes in this class are usually of unbounded power variation. This kind of stochastic processes are of interest in many applications, for example in financial mathematics concerning option pricing. In this dissertation, new conditions are presented for the existence of generalized Lebesgue–Stieltjes integrals for the aforementioned one-dimensional stochastic processes with respect to general Hölder continuous processes. This dissertation also contains a new result on the existence of generalized Lebesgue–Stieltjes integrals for a certain class of multi-dimensional stochastic processes with respect to general Hölder continuous processes. Moreover, in this dissertation, a new proof is presented for a change of variables formula for sufficiently regular one-dimensional stochastic processes with unbounded power variation.

**Keywords** pathwise integration, Hölder process, unbounded p-variation, generalized Lebesgue-Stieltjes integration

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# Preface

I joined the stochastics and statistics research group in the beginning of 2012 to work under the supervision of Prof. Esko Valkeila. I wish to express my deep gratitude to the late Prof. Esko Valkeila for leading me into the world of research, and offering me the opportunity to spend four wonderful years in Finland.

I am very grateful to my current supervisor Prof. Lasse Leskelä, for all the comments and notes, patience and support. Lasse has guided me and inspired me with his extensive knowledge and meticulous academic attitude during the whole writing of this thesis.

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Special thanks to all my friends in Finland. Even though I'm far away from home, your friendship together with all the hot pot nights make my life full of joy and happiness.

Finally, I want to thank my parents who have always been supportive ever

since I was a kid. Without you standing behind, I couldn't have come this far. Moreover, my sincerest gratitude goes to my husband Kai Zhao. Thank you for always being there, encouraging me when I almost gave up and cheering for me when I survived.

Helsinki, February 15, 2016,

Zhe Chen

# Contents

# 1. Introduction

Stochastic calculus with respect to semimartingales has been developed over decades, and it has been successfully applied in many disciplines including physics and financial mathematics. Summaries about the theory of stochastic integration and stochastic differential equations with respect to semimartingales can be found in standard textbooks such as [20, 37, 46, 63, 64]. Applications of stochastic calculus with respect to semimartingales to financial markets can be found for example in [16, 42, 78].

As one of the fundamental examples of semimartingales, Brownian motion has been used widely in different areas. For example, it can be applied to model the stock price in financial markets. However, for some existing phenomena, Brownian motion is not an ideal model, since the increments of a Brownian motion are independent. For example, in telecommunications, asset pricing and some applications in hydrology, processes may be desired to present long-range dependence and self-similarity.

Fractional Brownian motion with stationary increments and self-similarity property was introduced by Kolmogorov in [44]. Later, the index in the correlation function of a fractional Brownian motion got the name “Hurst index” from Hurst [33] and Hurst, Black and Simaika [34]. Mandelbrot and Ness in [55] studied the process and gave the name “fractional Brownian motion” to it. Fractional Brownian motion with Hurst index $H \in (0,1)$ is not a semimartingale except when $H = \frac{1}{2}$, in which case fractional Brownian motion is a standard Brownian motion. Therefore, the theory of Itō stochastic calculus based on semimartingales cannot be applied to fractional Brownian motions and one should consider other stochastic calculus theories for fractional Brownian motions. There are many different approaches to define a stochastic integral with respect to a fractional Brownian motion. One way is to use Skorokhod integrals or divergence integrals based on Malliavin calculus [61]. For details of this approach, see [2, 3, 10, 15]. In this dissertation, Skorokhod integrals will not be considered.

Another approach is pathwise integration, i.e. integration path-by-path. It is known that the Riemann–Stieltjes integral exists if the integrand is continuous and the integrator is of bounded variation. However, by [60] we know that the $p$-variation index of almost all paths of a fractional Brownian motion $B^H$ equals $\frac{1}{H}$. For $p < \frac{1}{H}$, the $p$-variation of the path is unbounded and for $p > \frac{1}{H}$, the $p$-variation of the path is bounded. This implies that almost all paths of a fractional Brownian motion are of unbounded variation, and therefore the classical Riemann–Stieltjes integral cannot be applied here.

In 1936, Young in [89] proved that the integral $\int f \,\mathrm{d}g$ exists as a Riemann–Stieltjes integral if $f$ has bounded $p$-variation and $g$ has bounded $q$-variation for $p \geq 1$, $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} > 1$, and if $f, g$ have no common points of discontinuity. It is known that almost all paths of $B^H$ are $\alpha$-Hölder continuous for $\alpha < H$. In the case of $H \in (\frac{1}{2}, 1)$, Lin [50], Dai and Heyde [14] have defined a stochastic integral $\int_0^T \phi(t)\,\mathrm{d}B_t^H$ as a limit of Riemann sums in $L^2$ for the case that almost all paths of $\phi$ have bounded $p$-variation such that $\frac{1}{p} + \alpha > 1$.

In 1998, Zähle [90] introduced a notion of a generalized Lebesgue–Stieltjes integral by studying fractional integrals and their corresponding Weyl derivatives. In the special case where $f$ is $\lambda$-Hölder continuous and $g$ is $\mu$-Hölder continuous with $\lambda + \mu > 1$, the generalized Lebesgue–Stieltjes integral $\int f \,\mathrm{d}g$ exists and coincides with the corresponding Riemann-Stieltjes integral. Later on, in 2002 Nualart and Răşcanu [62] further studied generalized Lebesgue–Stieltjes integrals based on Zähle's results by considering fractional Sobolev-type spaces. They showed that if the integrand $f$ and integrator $g$ belong to certain fractional Sobolev-type spaces, then the generalized Lebesgue–Stieltjes integral $\int f \,\mathrm{d}g$ exists.

Consider now some Gaussian processes $X$ and $Y$. Let $f$ be a real-valued function which may contain discontinuities. In this case, the process $f(X)$ may have unbounded $p$-variation for every $p \geq 1$. One of the main goals of this dissertation is to study in which sense can we understand an integral of the form

$$\int_0^T f(X_t)\,\mathrm{d}Y_t, \tag{1.1}$$

by applying pathwise integration theory. A natural choice for us would be the generalized Lebesgue–Stieltjes integral mentioned above. Note that this has been successfully applied to the case of fractional Brownian motion, functionals of fractional Brownian motion and a class of Gaussian processes.

When Azmoodeh, Mishura and Valkeila in [6] studied a pricing model based on a geometric fractional Brownian motion with Hurst index $H > \frac{1}{2}$, they defined the integral $\int_0^T f'_-(S_t)S_t \, dB_t^H$, where $f$ is a convex function, $B_t^H$ is a fractional Brownian motion and $S_t$ is a geometric fractional Brownian motion, as a generalized Lebesgue–Stieltjes integral. The existence of the generalized Lebesgue–Stieltjes integral was proved by showing that almost all paths of the integrand and the integrator belong to certain fractional Sobolev-type spaces. Later, Tikanmäki in [84] proved the existence of the stochastic integral of some functionals of fractional Brownian motion with respect to a fractional Brownian motion in the sense of generalized Lebesgue–Stieltjes. Furthermore, Sottinen and Viitasaari in [81] generalized the theory from fractional Brownian motions into a wider class of Gaussian processes. Unfortunately, by carefully examining the proofs of theorems regarding the change of variables formula in the aforementioned articles, some gaps are present. These gaps will be explained in detail in Section 5.1. Therefore another goal of this dissertation is to fix these gaps and correct the proof of the change of variables formula.

The above pathwise stochastic integration technique should cover more stochastic processes than Gaussian processes. In this dissertation, a general class of Hölder continuous processes will be studied. Again, since we assume that $f$ in (1.1) may contain discontinuities, the integrand $f(X)$ may be of unbounded $p$-variation for $p \geq 1$. Whether the integral of the form $\int_0^T f(X_t) \, dY_t$, where $f(X)$ is a one-dimensional general unbounded $p$-variation process and $Y$ is some one-dimensional Hölder continuous process, can still be understood as a generalized Lebesgue–Stieltjes integral will be discussed in this dissertation.

Finally note that, whether the generalized Lebesgue–Stieltjes integral exists for a multidimensional process has not been studied yet, to the best of my knowledge. This dissertation will also study the existence of a stochastic integral of the form

$$\int_0^T f(X_t^1, \dots, X_t^n) dY_t,$$

where $Y$ is a Hölder continuous process and $f$ may contain discontinuities.

Naturally, from the application point of view, a change of variables formula is an interesting problem to consider. Based on the existence of the generalized Lebesgue–Stieltjes integrals, a change of variables formula of one-dimensional processes will be shown in this dissertation.

This dissertation is organized as follows. In Chapter 2, I will review several function spaces including convex functions, locally bounded variation functions, bounded $p$-variation functions and other relevant functions. One of the main tools for this research, which is the representation of convex functions with respect to some Radon measures, will be given in this chapter. Then according to the relationship between locally bounded variation functions and convex functions, such representation can also be applied to locally bounded variation functions. Moreover, fractional Sobolev-type spaces will also be discussed here to prepare for the study of generalized Lebesgue–Stieltjes integrals in the next chapters.

In Chapter 3, I will review Stieltjes integration and recall results on the existence of Stieltjes integrals for various classes of functions. After the review of classical Riemann–Stieltjes integrals and Lebesgue–Stieltjes integrals, I will move to more complicated fractional integrals and derivatives in order to introduce the generalized Lebesgue–Stieltjes integral. Finally, the generalized Lebesgue–Stieltjes integrals will be shown to exist and coincide with the Riemann–Stieltjes integrals in the special case when both the integrand and the integrator are smooth enough.

Chapter 4 will be devoted to stochastic processes and stochastic integration with respect to those processes. Firstly I will discuss several Gaussian processes, especially fractional Brownian motions. Then pathwise integration of stochastic processes with bounded $p$-variation paths for $p \geq 1$ will be discussed. Finally, results on the generalized Lebesgue–Stieltjes integrals for fractional Brownian motions, functionals of fractional Brownian motions and a class of Gaussian processes will be presented.

The main results of this dissertation will be given in Chapter 5. Firstly, limitations of previous integration techniques and gaps in the earlier literature will be explained. Then the existence of a generalized Lebesgue–Stieltjes integral of a certain class of unbounded $p$-variation processes with respect to some Hölder continuous processes will be shown. Also the existence of a generalized Lebesgue–Stieltjes integral of a class of multidimensional processes with respect to some Hölder continuous processes will be shown. Finally, a change of variables formula for one-dimensional unbounded $p$-variation processes will be given.

# 2. Analysis

In this chapter, we will review several function spaces with different properties, especially spaces of convex functions and locally bounded variation functions. A representation of a convex function plays an important role for proving the main results in Chapter 5. Moreover, fractional Sobolev-type spaces which are also crucial for the main results will be reviewed here.

## 2.1 Increasing Functions and Convex Functions

This section will start with a review of some basic and well known properties of increasing functions and convex functions. More details of these functions can be found in [17, 21, 41, 62].

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be an increasing function, i.e. $f(x) \leq f(y)$ for any $x \leq y$. Let $f(x+) := \lim_{h \to 0+} f(x+h)$ denote the right limit of $f$ at the point $x$ and $f(x-) := \lim_{h \to 0-} f(x+h)$ denote the left limit of $f$ at $x$. Moreover, denote $f_+ : x \mapsto f(x+)$ and $f_- : x \mapsto f(x-)$.

**Proposition 2.1.1.** *If $f$ is an increasing function on $\mathbb{R}$, then $f(x+)$ and $f(x-)$ exist for every point $x \in \mathbb{R}$. The function $f_+$ is increasing and right-continuous while the function $f_-$ is increasing and left-continuous. Moreover, the set of points $\{x : f(x-) \neq f(x+)\}$ is at most countable.*

This can be proved by using the definition of an increasing function, and for details, see [17].

Now consider an interval $I \subset \mathbb{R}$.

**Definition 2.1.1.** *A function $f : I \mapsto \mathbb{R}$ is called convex on $I$ if for any two points $x, y \in I$ and any $\lambda \in [0, 1]$, the following holds*

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda) f(y).$$

Denote the left-sided derivative and right-sided derivative of $f$ respectively as

$$f'_-(x) = \lim_{h \to 0-} \frac{f(x+h) - f(x)}{h}, \quad f'_+(x) = \lim_{h \to 0+} \frac{f(x+h) - f(x)}{h},$$

when the limits exist. According to the definition of a convex function, we have the following proposition.

**Proposition 2.1.2.** *If $f : I \mapsto \mathbb{R}$ is a convex function on an open interval $I$, then $f'_-$ and $f'_+$ exist for every point in $I$. The function $f'_-$ is increasing and left-continuous on $I$ while the function $f'_+$ is increasing and right-continuous on $I$. Moreover the set of points $\{x : f'_-(x) \neq f'_+(x)\}$ is at most countable. Finally, for all $[a, b] \subset I$,*

$$\int_a^b f'_+(x)\,\mathrm{d}x = f(b) - f(a) = \int_a^b f'_-(x)\,\mathrm{d}x.$$

For a proof, see [65].

**Definition 2.1.2.** *A function $f : I \mapsto \mathbb{R}$ is called Lipschitz continuous on $I$ if there exists a real constant $K \geq 0$ such that for all $x, y \in I$,*

$$|f(x) - f(y)| \leq K|x - y|.$$

**Definition 2.1.3.** *A function $f : I \mapsto \mathbb{R}$ is said to be absolutely continuous on $I$ if for every $\epsilon > 0$ there exists a number $\delta > 0$ such that*

$$\sum_{i=1}^{n} |f(x_i) - f(y_i)| < \epsilon$$

*for any $n$ and any finite collection of disjoint intervals $(x_i, y_i)$ in $I$ with*

$$\sum_{i=1}^{n} |x_i - y_i| < \delta.$$

**Definition 2.1.4.** *We say that a property of a function $f : \mathbb{R} \mapsto \mathbb{R}$ holds locally if for every point $x \in \mathbb{R}$, there is a neighbourhood $U$ such that the property holds for the restriction of $f$ into $U$.*

Let $\mathrm{Lip}^{loc}(\mathbb{R})$ denote the space of locally Lipschitz continuous functions on $\mathbb{R}$. By the definition of a convex function, we know that convex functions are locally Lipschitz continuous on $\mathbb{R}$. Let $\mathrm{AC}^{loc}(\mathbb{R})$ denote the space of locally absolutely continuous functions on $\mathbb{R}$. Since a convex function on $\mathbb{R}$ is locally Lipschitz continuous, it is absolutely continuous on any bounded closed interval of $\mathbb{R}$ (see for example [86]).

**Definition 2.1.5.** *A Radon measure on $\mathbb{R}^d$ is defined as a measure $\mu$ on the Borel $\sigma$-field of $\mathbb{R}^d$ such that $\mu(K) < \infty$ for every compact set $K$ in $\mathbb{R}^d$.*

In the following let $C(\mathbb{R})$ denote the space of continuous functions on $\mathbb{R}$, and $C_c(\mathbb{R})$ denote the space of all $f \in C_c(\mathbb{R})$ with a compact support. Let $C^1(\mathbb{R})$ denote the space of continuously differentiable functions on $\mathbb{R}$, and $C_c^1(\mathbb{R})$ denote the space of all $f \in C^1(\mathbb{R})$ with a compact support. Let $C^\infty(\mathbb{R})$ denote the space of infinitely differentiable functions on $\mathbb{R}$, and $C_c^\infty(\mathbb{R})$ denote the space of all $f \in C^\infty(\mathbb{R})$ with a compact support.

**Proposition 2.1.3.** *If $\mu$ and $\nu$ are Radon measures on $\mathbb{R}$ such that*

$$\int_{\mathbb{R}} \phi(x)\,\mu(\mathrm{d}x) = \int_{\mathbb{R}} \phi(x)\,\nu(\mathrm{d}x)$$

*for all $\phi \in C_c^\infty(\mathbb{R})$, then $\mu = \nu$.*

Before proving the proposition, we need to introduce the following lemma.

**Lemma 2.1.1.** *For any compact set $\mathcal{K}$ and open set $V$ such that $K \subset V \subset \mathbb{R}$, there exists a nonnegative function $f \in C_c^\infty(\mathbb{R})$ such that*

$$\mathbf{1}_{\mathcal{K}} \leq f \leq \mathbf{1}_V.$$

*Proof.* Such a function can be constructed as a convolution of the indicator function of a set

$$\mathcal{K}_{2\epsilon} := \{y : |x-y| \leq 2\epsilon, \text{ for some } x \in \mathcal{K}\},$$

and a smooth nonnegative function $\phi_\epsilon$ with a support in the $\epsilon$-ball centered at the origin, and choosing $\epsilon > 0$ small enough. See for example Hörmander [35, Theorem 1.4.1] for details. □

*Proof of Proposition 2.1.3.* First we will show that $\mu(\mathcal{K}) = \nu(\mathcal{K})$ for all compact sets $\mathcal{K} \subset \mathbb{R}$. Let $\mathcal{K}_{1/n}$ be the set of points $x$ such that $|x-y| < \frac{1}{n}$ for some $y \in \mathcal{K}$ and for $n \in \mathbb{N}$. By definition, $\mathcal{K}_{1/n}$ is open. According to Lemma 2.1.1, there exists $f_n \in C_c^\infty(\mathbb{R})$ such that $\mathbf{1}_{\mathcal{K}} \leq f_n \leq \mathbf{1}_{\mathcal{K}_{1/n}}$ for $n \in \mathbb{N}$. Then $f_n \to \mathbf{1}_{\mathcal{K}}$ pointwise as $n \to \infty$, and $0 \leq f_n \leq \mathbf{1}_{\mathcal{K}_1}$ for all $n \in \mathbb{N}$. Since $\mathbf{1}_{\mathcal{K}_1}$ is integrable with respect to $\mu$ and $\nu$ and $f_n \in C_c^\infty(\mathbb{R})$, by dominated convergence theorem and Lemma 2.1.1 we have that

$$\mu(\mathcal{K}) = \lim_{n\to\infty} \int f_n\,\mathrm{d}\mu = \lim_{n\to\infty} \int f_n\,\mathrm{d}\nu = \nu(\mathcal{K}).$$

Thus we have shown that $\mu(\mathcal{K}) = \nu(\mathcal{K})$ for all compact $\mathcal{K} \subset \mathbb{R}$.

Next, fix a sequence of compact sets $\mathcal{K}_1 \subset \mathcal{K}_2 \dots$ so that $\cup_{n\in\mathbb{N}}\mathcal{K}_n = \mathbb{R}$. If $C$ is a closed set, then $C \cap \mathcal{K}_n$ is compact, and $\cup_{n\in\mathbb{N}}\big(C \cap \mathcal{K}_n\big) = C$. Then we have

$$\mu(C) = \lim_{n\to\infty} \mu(C \cap \mathcal{K}_n) = \lim_{n\to\infty} \nu(C \cap \mathcal{K}_n) = \nu(C).$$

Hence $\mu(C) = \nu(C)$ for all closed $C \subset \mathbb{R}$, moreover $\mu(\mathbb{R}) = \nu(\mathbb{R})$. Since closed sets in $\mathbb{R}$ form a $\pi$-system (i.e. a collection of subsets which is closed under finite intersections) which generates the Borel sets of $\mathbb{R}$, Dynkin's identification theorem [41, Lemma 1.17] implies that the Borel measures $B \mapsto \mu(B \cap \mathcal{K}_n)$ and $B \mapsto \nu(B \cap \mathcal{K}_n)$ are equal for every $n \in \mathbb{N}$. Thus we obtain $\mu(B \cap \mathcal{K}_n) = \nu(B \cap \mathcal{K}_n)$ for every Borel set $B$ and every $n \in \mathbb{N}$. By letting $n \to \infty$, we find that

$$\mu(B) = \nu(B)$$

for every Borel set $B$. □

From [41] we know that a correspondence between Radon measures and increasing right-continuous functions can be shown through the next proposition.

**Proposition 2.1.4.** *Let $f$ be an increasing right-continuous function on $\mathbb{R}$. Then there exists a unique Radon measure $\mu_f$ on $\mathbb{R}$ such that*

$$\mu_f((a, b]) = f(b) - f(a), \quad -\infty < a < b < \infty. \tag{2.1}$$

This measure $\mu_f$ is called the Lebesgue–Stieltjes measure of $f$. Next we have the following proposition for increasing right-continuous functions.

**Proposition 2.1.5.** *If $f$ is an increasing and right-continuous real-valued function on $\mathbb{R}$, then for all $\phi \in C_c^\infty(\mathbb{R})$, the Lebesgue–Stieltjes measure $\mu_f$ of $f$ satisfies*

$$\int_{\mathbb{R}} f(x)\phi'(x)\,\mathrm{d}x = -\int_{\mathbb{R}} \phi(x)\,\mu_f(\mathrm{d}x).$$

*Proof.* Let $\phi \in C_c^\infty(\mathbb{R})$, then $\phi$ has a compact support. Choose a big enough $M$ so that the compact support of $\phi$ is strictly contained in $[-M, M]$.

From (2.1), for all $x > -M$ we have that

$$f(x) - f(-M) = \mu_f((-M, x]) = \int_{\mathbb{R}} \mathbf{1}_{(-M,x]}(u)\mu_f(\mathrm{d}u).$$

Then

$$\begin{aligned}\int_{\mathbb{R}} f(x)\phi'(x)\mathrm{d}x &= \int_{\mathbb{R}} \mathbf{1}_{(-M,M)}(x)\phi'(x)\Big(f(-M) + \int_{\mathbb{R}} \mathbf{1}_{(-M,x]}(u)\mu_f(\mathrm{d}u)\Big)\mathrm{d}x \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{(-M,x]}(u)\mathbf{1}_{(-M,M)}(x)\phi'(x)\mu_f(\mathrm{d}u)\mathrm{d}x.\end{aligned}$$

Note that the integrand is bounded in absolute value by

$$(x, u) \mapsto \mathbf{1}_{(-M,x]}(u)\mathbf{1}_{(-M,M)}(x)\|\phi'\|_\infty,$$

which is integrable with respect to $\mu_f(\mathrm{d}u)\,\mathrm{d}x$. Moreover, note that

$$\mathbf{1}_{(-M,x]}(u)\mathbf{1}_{(-M,M)}(x) = \mathbf{1}_{[u,M)}(x)\mathbf{1}_{(-M,M)}(u).$$

By Fubini's theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}} f(x)\phi'(x)\mathrm{d}x &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{(-M,x]}(u)\mathbf{1}_{(-M,M)}(x)\phi'(x)\,\mu_f(\mathrm{d}u)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} \mathbf{1}_{(-M,M)}(u)\int_{\mathbb{R}} \mathbf{1}_{[u,M)}(x)\phi'(x)\,\mathrm{d}x\,\mu_f(\mathrm{d}u) \\ &= \int_{\mathbb{R}} \mathbf{1}_{(-M,M)}(u)(\phi(M)-\phi(u))\,\mu_f(\mathrm{d}u) \\ &= -\int_{\mathbb{R}} \mathbf{1}_{(-M,M)}(u)\phi(u)\,\mu_f(\mathrm{d}u) \\ &= -\int_{\mathbb{R}} \phi(u)\,\mu_f(\mathrm{d}u).\end{aligned}$$

□

Moreover, we have the following similar proposition for increasing functions.

**Proposition 2.1.6.** *If $f$ is an increasing function on $\mathbb{R}$, then there exists a unique Radon measure $\mu$ on $\mathbb{R}$ such that*

$$\int_{\mathbb{R}} f(x)\phi'(x)\,\mathrm{d}x = -\int_{\mathbb{R}} \phi(x)\,\mu(\mathrm{d}x), \tag{2.2}$$

*for any $\phi \in C_c^\infty(\mathbb{R})$. This Radon measure $\mu$ is equal to the Lebesgue–Stieltjes measure of $f_+$ in the sense of Proposition 2.1.3.*

*Proof.* Let $f$ be an increasing function. According to Proposition 2.1.1, $f_+ : x \mapsto f(x+)$ is increasing and right-continuous. Now let $\mu_{f+}$ be the Lebesgue–Stieltjes measure of $f_+$ defined according to (2.1). By applying Proposition 2.1.5, we obtain

$$\int_{\mathbb{R}} f_+(x)\phi'(x)\,\mathrm{d}x = -\int_{\mathbb{R}} \phi(x)\,\mu_{f+}(\mathrm{d}x).$$

By Proposition 2.1.1, we know that $f = f_+$ except on a set of discontinuity points which is at most countable. Therefore, $f = f_+$ Lebesgue-almost everywhere, which implies

$$\int_{\mathbb{R}} f(x)\phi'(x)\,\mathrm{d}x = \int_{\mathbb{R}} f_+(x)\phi'(x)\,\mathrm{d}x.$$

Next we will show the uniqueness. If there exists another Radon measure $\nu$ such that

$$\int_{\mathbb{R}} f(x)\phi'(x)\,\mathrm{d}x = -\int_{\mathbb{R}} \phi(x)\,\nu(\mathrm{d}x),$$

then we have

$$\int_{\mathbb{R}} \phi(x)\,\mu(\mathrm{d}x) = \int_{\mathbb{R}} \phi(x)\,\nu(\mathrm{d}x)$$

for all $\phi \in C_c^\infty(\mathbb{R})$. By Proposition 2.1.3, we have $\mu = \nu$. □

Next we will introduce a theorem which provides a representation of a convex function with respect to a Radon measure. Before that, we need to review the mollification technique and the definition of derivative in the sense of distributions.

**Definition 2.1.6.** *Let $\eta$ be a real-valued function on $\mathbb{R}$ defined by*

$$\eta(x) = \begin{cases} C\exp\left(\frac{1}{x^2-1}\right), & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

*where $C$ is a constant chosen so that $\int_{\mathbb{R}} \eta(x)\mathrm{d}x = 1$. For every $\epsilon > 0$, let*

$$\eta_\epsilon(x) = \frac{1}{\epsilon}\eta\left(\frac{x}{\epsilon}\right).$$

*We say that $\eta_\epsilon$ is a standard mollifier.*

According to its definition, we have the following properties of a standard mollifier. For all $\epsilon > 0$,

(1) $\eta_\epsilon(x) \geq 0$ for all $x \in \mathbb{R}$,

(2) $\eta_\epsilon \in C^\infty(\mathbb{R})$,

(3) $\eta_\epsilon$ has a compact support on $\mathbb{R}$,

(4) $\int_{\mathbb{R}} \eta_\epsilon(x)\mathrm{d}x = 1$.

If a function $f : \mathbb{R} \mapsto \mathbb{R}$ is locally integrable on $\mathbb{R}$, then $f$ can be mollified by convolution with a standard mollifier as

$$f_\epsilon(x) = (f * \eta_\epsilon)(x) = \int_{\mathbb{R}} f(x-y)\eta_\epsilon(y)\mathrm{d}y = \int_{\mathbb{R}} f(y)\eta_\epsilon(x-y)\mathrm{d}y.$$

Moreover, $f_\epsilon$ satisfies the following proposition.

**Proposition 2.1.7.** *$f_\epsilon \in C^\infty(\mathbb{R})$, and $f_\epsilon \to f$ uniformly on compact sets of $\mathbb{R}$ as $\epsilon \to 0$.*

For a proof, see [40].

In the following, we will always let $\phi \in C_c^\infty(\mathbb{R})$. For a locally integrable function $f$ on $\mathbb{R}$, define

$$\langle f, \phi \rangle := \int f(x)\phi(x)\mathrm{d}x.$$

**Definition 2.1.7.** *The derivative $Df$ of a locally integrable function $f$ in the sense of distributions is defined as*

$$\langle Df, \phi \rangle := -\langle f, \phi' \rangle = -\int f(x)\phi'(x)\mathrm{d}x,$$

*and similarly the $k$-th derivative of a locally integrable function $f$ in the sense of distributions is defined as*

$$\langle D^k f, \phi \rangle := (-1)^k \langle f, \phi^{(k)} \rangle = (-1)^k \int f(x)\phi^{(k)}(x)\mathrm{d}x.$$

Therefore, if $f$ is locally integrable on $\mathbb{R}$, then $f$ has infinitely many derivatives in the sense of distributions.

If $f : \mathbb{R} \mapsto \mathbb{R}$ is an increasing function, then by Proposition 2.1.1 we know that $f_+$ and $f_-$ are both increasing, and therefore $f$, $f_+$ and $f_-$ are all locally integrable on $\mathbb{R}$. Moreover, $f = f_+ = f_-$ Lebesgue-almost everywhere, which implies that

$$\langle f, \phi \rangle = \langle f_+, \phi \rangle = \langle f_-, \phi \rangle. \tag{2.3}$$

According to Proposition 2.1.6, we have

$$-\langle f, \phi' \rangle = \langle \mu, \phi \rangle = -\langle f_+, \phi' \rangle,$$

where $\mu$ is the Lebesgue–Stieltjes measure of $f_+$. Together with equation (2.3), we obtain

$$\langle Df, \phi \rangle = \langle Df_+, \phi \rangle = \langle Df_-, \phi \rangle.$$

Hence we can conclude that the first derivative of an increasing function $f$ in the sense of distributions is the Lebesgue–Stieltjes measure $\mu$ of $f_+$. Moreover, we have $Df = Df_+ = Df_- = \mu$ in the sense of distributions.

If $f : \mathbb{R} \mapsto \mathbb{R}$ is a convex function, then $f$, $f'_+, f'_-$ are locally integrable on $\mathbb{R}$. Therefore, they have infinitely many derivatives in the sense of distributions. The first derivative $Df$ of $f$ in the sense of distributions is given by

$$\begin{aligned} \langle Df, \phi \rangle &= -\int f(x)\phi'(x)\,\mathrm{d}x \\ &= -\int f(x)\Big(\lim_{h\to 0+} \frac{\phi(x) - \phi(x-h)}{h}\Big)\,\mathrm{d}x. \end{aligned}$$

Here we can apply the Lebesgue dominated convergence theorem, since $\phi$ is smooth with a compact support and $f$ is bounded on the compact support of

$\phi$. The function $\phi'$ is bounded and Lebesgue integrable, and we have

$$\begin{aligned}
&-\int f(x)\Big(\lim_{h\to 0+}\frac{\phi(x)-\phi(x-h)}{h}\Big)\,\mathrm{d}x \\
&= -\lim_{h\to 0+}\frac{1}{h}\Big(\int f(x)\phi(x)\,\mathrm{d}x - \int f(x)\phi(x-h)\,\mathrm{d}x\Big) \\
&= -\lim_{h\to 0+}\frac{1}{h}\Big(\int f(x)\phi(x)\,\mathrm{d}x - \int f(x+h)\phi(x)\,\mathrm{d}x\Big) \\
&= \lim_{h\to 0+}\int \phi(x)\Big(\frac{f(x+h)-f(x)}{h}\Big)\,\mathrm{d}x \\
&= \int \phi(x) f'_+(x)\,\mathrm{d}x,
\end{aligned}$$

where the second equality comes from change of variables and the last equality holds because $f'_+$ is bounded on the compact support of $\phi$. This implies

$$\langle Df, \phi\rangle := -\langle f, \phi'\rangle = \langle f'_+, \phi\rangle = \langle f'_-, \phi\rangle, \tag{2.4}$$

i.e. $Df = f'_+ = f'_-$ in the sense of distributions.

Now we can state a representation theorem of convex functions.

**Theorem 2.1.1.** *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a convex function. The second derivative $D^2 f$ of $f$ exists in the sense of distributions and equals the unique Radon measure $\mu$ such that*

$$\langle D^2 f, \phi\rangle := \langle f, \phi''\rangle = \int \phi(x)\,\mu(\mathrm{d}x) \tag{2.5}$$

*for all $\phi \in C_c^\infty(\mathbb{R})$. Conversely, for any Radon measure $\mu$ on $\mathbb{R}$ there exists a unique convex function $f_\mu : \mathbb{R} \mapsto \mathbb{R}$ such that $f_\mu(0) = 0$, $f'_{\mu+}(0) = 0$ and (2.5) holds. Any convex function $g : \mathbb{R} \mapsto \mathbb{R}$ satisfying (2.5) can be represented as*

$$g(x) = f_\mu(x) + \alpha x + \beta,$$

*where $\alpha = g'_+(0)$ and $\beta = g(0)$. Moreover, for any finite Radon measure $\mu$ and any convex function $g$ satisfying (2.5),*

$$g'_+(x) - g'_+(0) = \frac{1}{2}\int \operatorname{sgn}(x-a)\,\mu(\mathrm{d}a) + C, \tag{2.6}$$

*where $C = \frac{1}{2}\mu((-\infty,\infty)) - \mu((-\infty,0])$ and*

$$\operatorname{sgn}(x) = \begin{cases} 1, & x \geq 0, \\ -1, & x < 0. \end{cases}$$

*Proof.* The first derivative $Df$ of $f$ in the sense of distributions satisfies (2.4). Let $\eta_\epsilon$ be a standard mollifier, and thus $f_\epsilon := \eta_\epsilon * f$ is a smooth convex function.

The second derivative of $f_\epsilon$ in the sense of distributions is given by

$$\begin{aligned}\langle D^2 f_\epsilon, \phi\rangle &= \int f_\epsilon(x)\phi''(x)\,\mathrm{d}x \\ &= f_\epsilon(x)\phi'(x)|_{-\infty}^{\infty} - \int f'_\epsilon(x)\phi'(x)\,\mathrm{d}x \\ &= f_\epsilon(x)\phi'(x)|_{-\infty}^{\infty} - \phi(x)f'_\epsilon(x)|_{-\infty}^{\infty} + \int \phi(x)f''_\epsilon(x)\,\mathrm{d}x \\ &= \int \phi(x)f''_\epsilon(x)\,\mathrm{d}x.\end{aligned} \tag{2.7}$$

The second and third equality in (2.7) can be obtained by integration by parts, and the first two items on the third line in (2.7) disappear because $\phi$ and its derivatives have a compact support.

Since $f_\epsilon$ is convex, we know that $f''_\epsilon \geq 0$, which implies

$$\langle D^2 f_\epsilon, \phi\rangle \geq 0$$

for all positive $\phi \in C_c^\infty(\mathbb{R})$. When $\epsilon \to 0$, by Lebesgue's dominated convergence theorem we obtain

$$L(\phi) := \langle D^2 f, \phi\rangle \geq 0$$

for all positive $\phi \in C_c^\infty(\mathbb{R})$. According to the Riesz representation theorem, a positive linear functional on $C_c(\mathbb{R})$ can be represented by a unique Radon measure. Therefore we have

$$L(\phi) = \int_{\mathbb{R}} \phi(x)\,\mu(\mathrm{d}x),$$

where $\mu$ is a Radon measure on the Borel sets of $\mathbb{R}$ (for details, see [76]).

Conversely, given a Radon measure $\mu$ on $\mathbb{R}$ we can define a function

$$h(x) = \begin{cases} \mu\big((0,x]\big), & x \geq 0, \\ -\mu\big((x,0]\big), & x < 0. \end{cases}$$

Then $h(0) = 0$, $h$ is increasing and right-continuous. We also have for $x < y$,

$$\mu\big((x,y]\big) = h(y) - h(x),$$

which implies that $\mu$ is the Lebesgue–Stieltjes measure of $h$.

Next define $f$ as

$$f(x) = \begin{cases} \int_0^x h(t)\,\mathrm{d}t, & x \geq 0, \\ -\int_x^0 h(t)\,\mathrm{d}t, & x < 0, \end{cases}$$

and $f(0) = 0$. Since $h$ is increasing, $f$ is a convex function. Moreover since $h$ is right-continuous, we have

$$f'_+(x) = h(x+) = h(x),$$

and $f'_+(0) = 0$. Then by Proposition 2.1.6 and (2.4), we have

$$\langle D^2 f, \phi \rangle = -\langle Df, \phi' \rangle = -\langle f'_+, \phi' \rangle = \int \phi(x)\, \mu(\mathrm{d}x)$$

for all positive $\phi \in C_c^\infty(\mathbb{R})$.

Next we show the uniqueness of $f$. Let $f$ and $g$ be two convex functions such that both $f$ and $g$ satisfy (2.5), $f(0) = g(0) = 0$ and $f'_+(0) = g'_+(0) = 0$. Then by the above arguments, we have

$$\int f(x)\phi''(x)\,\mathrm{d}x = \int \phi(x)\,\mu(\mathrm{d}x) = \int g(x)\phi''(x)\,\mathrm{d}x. \tag{2.8}$$

Moreover, we know that $f'_+$ and $g'_+$ are increasing and right-continuous. Let $\mu_f$ and $\mu_g$ be the Lebesgue–Stieltjes measures of $f'_+$ and $g'_+$ respectively. According to Proposition 2.1.2, we have

$$\int f'_+(x)\phi'(x)\,\mathrm{d}x = -\int \phi(x)\,\mu_f(\mathrm{d}x),$$
$$\int g'_+(x)\phi'(x)\,\mathrm{d}x = -\int \phi(x)\,\mu_g(\mathrm{d}x).$$

By equation (2.4), we have

$$\int f(x)\phi''(x)\,\mathrm{d}x = -\int f'_+(x)\phi'(x)\,\mathrm{d}x,$$
$$\int g(x)\phi''(x)\,\mathrm{d}x = -\int g'_+(x)\phi'(x)\,\mathrm{d}x.$$

Together with (2.8), we obtain $\mu_f = \mu_g = \mu$. By the definition of a Lebesgue–Stieltjes measure, we have

$$f'_+(y) - f'_+(x) = \mu\big((x, y]\big) = g'_+(y) - g'_+(x),$$

for all $x < y$. Now by the assumption $f'_+(0) = g'_+(0) = 0$, we obtain $f'_+ = g'_+$. This implies that

$$f(y) - f(x) = \int_x^y f'_+(t)\,\mathrm{d}t = \int_x^y g'_+(t)\,\mathrm{d}t = g(y) - g(x).$$

Since $f(0) = g(0) = 0$, we obtain $f = g$.

Now let $g$ be an arbitrary convex function satisfying (2.5). Let $\alpha = g'_+(0)$ and $\beta = g(0)$, thus a function

$$\tilde{g} = g(x) - \alpha x - \beta$$

is also a convex function and it satisfies (2.5) with $\tilde{g}(0) = 0$ and $\tilde{g}'_+(0) = 0$. By the above arguments we know that such a function $\tilde{g}$ is unique, therefore $\tilde{g} = f_\mu$, and

$$g(x) = f_\mu(x) + \alpha x + \beta.$$

Finally, let

$$\tilde{\mu}(x) = \frac{1}{2}\int \operatorname{sgn}(x-a)\,\mu(\mathrm{d}a).$$

Then we have

$$\tilde{\mu}(x) = \mu\big((-\infty, x]\big) - \frac{1}{2}\mu\big((-\infty,\infty)\big),$$

and $\tilde{\mu}(y) - \tilde{\mu}(x) = \mu\big((x,y]\big)$. Now $\tilde{\mu}$ is increasing and right-continuous, which implies that $\mu$ is the Lebesgue–Stieltjes measure of $\tilde{\mu}$. From preceding arguments, we know that $\mu$ is also the Lebesgue–Stieltjes measure of $g'_+$. Therefore we have

$$\begin{cases} g'_+(x) - g'_+(0) = \mu\big((0,x]\big) = \tilde{\mu}(x) - \tilde{\mu}(0), & x \geq 0 \\ g'_+(x) - g'_+(0) = -\mu\big((x,0]\big) = \tilde{\mu}(x) - \tilde{\mu}(0), & x < 0. \end{cases}$$

Hence we obtain

$$g'_+(x) - g'_+(0) = \frac{1}{2}\int \operatorname{sgn}(x-a)\,\mu(\mathrm{d}a) + C,$$

where

$$C = \frac{1}{2}\mu\big((-\infty,\infty)\big) - \mu\big((-\infty,0]\big).$$

□

**Remark 2.1.1.** *Note that since $f$ is a convex function, according to Proposition 2.1.2 we have $f'_- = f'_+$ Lebesgue-almost everywhere. We also have that $Df = f'_- = f'_+$ in the sense of distributions. The representation (2.6) also holds for the left derivative of any convex function $g$ satisfying (2.5) associated to a finite Radon measure $\mu$ by redefining* $\operatorname{sgn}(x)$ *as*

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0, \\ -1, & x \leq 0. \end{cases}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function such that the corresponding Radon measure $\mu$ has a compact support, and let $\phi$ be a positive function in $C^\infty(\mathbb{R})$ with a compact support in $[0,\infty)$ such that $\int_0^\infty \phi(y)\,\mathrm{d}y = 1$. Define for $n \in \mathbb{N}$

$$f_n(x) := n\int_0^\infty f(x+y)\phi(ny)\,\mathrm{d}y.$$

By Proposition 2.1.7, we know that $f_n$ converges to $f$ pointwise as $n \to \infty$. Moreover, $f'_n$ decreases to $f'_-$ (for details, see [65]).

According to the proof of Theorem 2.1.1, we have shown that for every $g \in C_c^\infty(\mathbb{R})$,

$$\lim_{n\to\infty}\int_{\mathbb{R}} g(x)f''_n(x)\mathrm{d}x = \int_{\mathbb{R}} g(x)\mu(\mathrm{d}x). \tag{2.9}$$

Moreover, by integration-by-parts we can show that (2.9) also holds for $g \in C_c^1(\mathbb{R})$.

## 2.2 Locally Bounded Variation Functions and Convex Functions

### 2.2.1 Functions of Locally Bounded Variation

**Definition 2.2.1.** *A partition $\pi^n$ of an interval $[a,b]$ is defined as a finite sequence of points $\pi^n = \{a = x_0^n < x_1^n < \ldots < x_{k(n)}^n = b\}$ on $[a,b]$, and the mesh of the partition is defined as*

$$|\pi^n| = \max_{1 \le i \le k(n)} |x_i^n - x_{i-1}^n|.$$

*A sequence of partitions on $[a,b]$ is denoted by $(\pi^n)$.*

**Definition 2.2.2.** *The total variation of a real-valued function $f$ on the interval $[a,b]$ is defined as*

$$V(f;[a,b]) = \sup_{(\pi^n)} \sum_{t_k \in \pi^n} |f(t_k) - f(t_{k-1})|,$$

*where the supremum runs over all partitions $\pi^n$ on $[a,b]$.*

**Definition 2.2.3.** *A function $f : [a,b] \mapsto \mathbb{R}$ is said to be of bounded variation if its total variation is bounded, i.e. $V(f;[a,b]) < \infty$. The space of functions which are of bounded variation on $[a,b]$ is denoted by $\mathrm{BV}([a,b])$.*

**Definition 2.2.4.** *A function is of locally bounded variation on $\mathbb{R}$ if it is of bounded variation over every compact set $\mathcal{K} \subset \mathbb{R}$. The space of functions which are of locally bounded variation on $\mathbb{R}$ is denoted by $\mathrm{BV}^{loc}(\mathbb{R})$.*

Next we recall the Jordan decomposition of locally bounded variation functions.

**Theorem 2.2.1.** *A function $f$ is of locally bounded variation if and only if it can be written as a difference $f = f^1 - f^2$, where $f^1$ and $f^2$ are increasing functions.*

A proof can be found in [41].

**Remark 2.2.1.** *Since $f^1$ and $f^2$ are increasing, by Proposition 2.1.1, there exist increasing, right-continuous functions $f_+^1$ and $f_+^2$ such that $f^1 = f_+^1$ Lebesgue-almost everywhere and $f^2 = f_+^2$ Lebesgue-almost everywhere. Let $\mu_1$ be the Lebesgue–Stieltjes measure of $f_+^1$ and $\mu_2$ be the Lebesgue–Stieltjes measure of $f_+^2$. By Proposition 2.1.2, we know that the right-sided derivative $g'_+$ of a convex function $g$ is increasing and right-continuous. Therefore $f_+^1$ and $f_+^2$ separately can be regarded as the right-sided derivatives of two convex functions. Moreover, according to Theorem 2.1.1, $g'_+$ together with a finite*

*Lebesgue–Stieltjes measure* $\mu_g$ *of* $g'_+$ *satisfy the representation (2.6). If* $\mu_1$ *of* $f^1_+$ *is a finite measure, then* $f^1_+$ *together with* $\mu_1$ *satisfy representation (2.6). Similarly, if* $\mu_2$ *of* $f^2_+$ *is finite, then* $f^2_+$ *together with* $\mu_2$ *satisfy representation (2.6). Note that here* $\mu_1$ *and* $\mu_2$ *are different.*

### 2.2.2 Linear Combination of Convex Functions

Recall the definition of absolutely continuous functions from Section 2.1. We have the following proposition taken from [68] for absolutely continuous functions.

**Proposition 2.2.1.** *Let* $f$ *be a real-valued function defined on an interval* $I = [a, b]$*. The function* $f$ *is absolutely continuous on* $I$ *if and only if* $f$ *is differentiable almost everywhere on* $I$*,* $f'$ *is integrable and*

$$f(x) - f(a) = \int_a^x f'(y)\,\mathrm{d}y, \quad a \leq x \leq b.$$

Now let $\mathrm{LC}_{conv} = \mathrm{LC}_{conv}(\mathbb{R})$ denote the space of functions which are linear combinations of convex functions on $\mathbb{R}$. Note that sum of two convex functions is still a convex function. Therefore for linear combinations of convex functions, it is sufficient to only consider a difference of two convex functions. We have the following proposition for linear combinations of convex functions.

**Proposition 2.2.2.** *Let* $f$ *be a real-valued function defined on* $\mathbb{R}$*. The function* $f$ *is in* $\mathrm{LC}_{conv}(\mathbb{R})$ *if and only if* $f'$ *exists almost everywhere and* $f' \in \mathrm{BV}^{loc}(\mathbb{R})$*.*

*Proof.* First, let $f'$ exist almost everywhere and be of locally bounded variation. Then for any $x \geq a$, where $a \in \mathbb{R}$ we have

$$f(x) - f(a) = \int_a^x f'(y)\,\mathrm{d}y.$$

Since $f'$ is of locally bounded variation, by Theorem 2.2.1 we have for $x \in \mathbb{R}$,

$$f'(x) = f'_1(x) - f'_2(x),$$

where $f'_1$ and $f'_2$ are increasing functions. Therefore for any $x \geq a$, we obtain

$$f(x) - f(a) = \int_a^x f'_1(y)\,\mathrm{d}y - \int_a^x f'_2(y)\,\mathrm{d}y. \tag{2.10}$$

Since the integral of an increasing function leads to a convex function, by (2.10) we see that $f$ is actually a difference of two convex functions.

Conversely, let $f \in \mathrm{LC}_{conv}(\mathbb{R})$. Then we have $f(x) = f^1(x) - f^2(x)$, where $f^1$ and $f^2$ are convex functions on $\mathbb{R}$. By Proposition 2.1.2, we know that $f'_{1+}$ and $f'_{2+}$ exist almost everywhere, and they are increasing and right-continuous.

Set $f^1(x) = f^1(a)$ when $x < a$ for any $a \in \mathbb{R}$. Let $f_n^1(x) = \frac{f^1(x+h_n)-f^1(x)}{h_n}$ with $h_n = n^{-1}$, then $f_n^1 \uparrow f'_{1+}$ as $n \to \infty$. Moreover

$$\begin{aligned}\int_a^x f_n^1(y)\mathrm{d}y &= \frac{1}{h_n}\Big(\int_a^x f^1(y+h_n)\,\mathrm{d}y - \int_a^x f^1(y)\,\mathrm{d}y\Big)\\ &= \frac{1}{h_n}\int_{a+h_n}^{x+h_n} f^1(y)\,\mathrm{d}y - \frac{1}{h_n}\int_a^x f^1(y)\,\mathrm{d}y\\ &= \frac{1}{h_n}\int_{x}^{x+h_n} f^1(y)\,\mathrm{d}y - \frac{1}{h_n}\int_a^{a+h_n} f^1(y)\,\mathrm{d}y.\end{aligned}$$

By Lebesgue differentiation theorem we find that when $n \to \infty$,

$$\tfrac{1}{h_n}\textstyle\int_x^{x+h_n} f^1(y)\,\mathrm{d}y \to f_1(x), \tag{2.11}$$

$$\tfrac{1}{h_n}\textstyle\int_a^{a+h_n} f^1(y)\,\mathrm{d}y \to f_1(a). \tag{2.12}$$

Moreover, according to monotone convergence theorem, we have

$$\int_a^x f_n^1(y)\,\mathrm{d}y \to \int_a^x f'_{1+}(y)\,\mathrm{d}y.$$

Therefore we obtain

$$\int_a^x f'_{1+}(y)\,\mathrm{d}y = f^1(x) - f^1(a), \quad a \le x.$$

Similarly, we have for $f^2$ that

$$\int_a^x f'_{2+}(y)\,\mathrm{d}y = f^2(x) - f^2(a), \quad a \le x.$$

Then

$$\begin{aligned}f(x) &= f^1(x) - f^2(x)\\ &= \Big(f^1(a) + \int_a^x f'_{1+}(y)\,\mathrm{d}y\Big) - \Big(f^2(a) + \int_a^x f'_{2+}(y)\,\mathrm{d}y\Big)\\ &= \int_a^x \big(f'_{1+}(y) - f'_{2+}(y)\big)\,\mathrm{d}y + f^1(a) - f^2(a),\end{aligned}$$

which implies that

$$f(x) - f(a) = \int_a^x \big(f'_{1+}(y) - f'_{2+}(y)\big)\,\mathrm{d}y, \quad a \le x.$$

Since $f'_{1+}$ and $f'_{2+}$ are increasing functions, by Theorem 2.2.1, a function $g$ defined as

$$g = f'_{1+} - f'_{2+}$$

is of locally bounded variation. Now we have proved that $f$ has a density which is of locally bounded variation. □

Recall that if $f$ is a real-valued function in $\mathrm{AC}^{loc}(\mathbb{R})$, then $f'$ exists almost everywhere and is locally integrable. If $f$ is a real-valued function in $\mathrm{Lip}^{loc}(\mathbb{R})$, then $f$ is almost everywhere differentiable and $f'$ is locally bounded. Hence, we conclude with the following remark.

**Remark 2.2.2.** *We can conclude the following inclusions of different function spaces as*

$$C^1(\mathbb{R}) \subset \mathrm{LC}_{conv}(\mathbb{R}) \subset \mathrm{Lip}^{loc}(\mathbb{R}) \subset \mathrm{AC}^{loc}(\mathbb{R}) \subset C(\mathbb{R}).$$

## 2.3 Other Function Spaces

In this section, we will review several other function spaces which will be considered in the following chapters. We will start with bounded $p$-variation functions.

### 2.3.1 $L^p$ Functions, Bounded $p$-variation Functions and Hölder Continuous Functions

**Definition 2.3.1.** *Let $L^p = L^p([0,T])$, $p \in [1, +\infty]$ denote the space of functions $f : [0,T] \mapsto \mathbb{R}$, with $\|f\|_{L^p} < \infty$, where*

$$\|f\|_{L^p} = \begin{cases} \left( \int_0^T |f(t)|^p \, \mathrm{d}t \right)^{1/p}, & \text{if} \quad 1 \leq p < \infty, \\ \operatorname{esssup}\{|f(t)| : t \in [0,T]\}, & \text{if} \quad p = \infty. \end{cases}$$

By Hölder's inequality, we have the following proposition for $L^p$ functions.

**Proposition 2.3.1.** *If $f \in L^p, g \in L^q$, with $p \geq 1, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

In Section 2.2 we have discussed locally bounded variation functions. Now we will review $p$-variation functions introduced by Wiener [87]. The $p$-variation of a real-valued function $f$ on $[0,T]$ along a partition $\pi^n$ is defined as

$$v_p(f; \pi^n) = \sum_{t_k \in \pi^n} |f(t_k) - f(t_{k-1})|^p \text{ for } p \geq 1.$$

**Definition 2.3.2.** *Let $f$ be a real-valued function on $[0,T]$, and $(\pi^n)$ be a sequence of partitions of $[0,T]$.*

1. *If the limit*

$$v_p^0(f; [0,T]) = \lim_{|\pi^n| \to 0} v_p(f; \pi^n) \text{ for } p \geq 1$$

*exists, then we say that $f$ has finite $p$-variation for $p \geq 1$ along the sequence of partitions $(\pi^n)$.*

2. *If*

$$v_p(f; [0,T]) = \sup_{(\pi)} v_p(f; \pi) < \infty, \text{ for } p \geq 1$$

*where the supremum is taken over all partitions* $(\pi)$ *of* $[0,T]$*, then we say that* $f$ *has bounded* $p$*-variation for* $p \geq 1$.

**Definition 2.3.3.** *Let* $\mathcal{W}_p = \mathcal{W}_p([0,T])$ *denote the space of bounded* $p$*-variation functions for* $p \geq 1$*, i.e. functions* $f : [0,T] \mapsto \mathbb{R}$ *such that*

$$v_p(f;[0,T]) < \infty.$$

If we equip this class with a norm

$$\|f\|_{[p]} := \big(v_p(f;[0,T])\big)^{\frac{1}{p}} + \|f\|_\infty,$$

then according to [18], the space $(\mathcal{W}_p, \|\cdot\|_{[p]})$ is a Banach space.

Hölder continuous functions are defined as follows.

**Definition 2.3.4.** *Let* $0 < \alpha \leq 1$*. A function* $f : [0,T] \to \mathbb{R}$ *is* $\alpha$*-Hölder continuous on* $[0,T]$ *if*

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty.$$

*Let* $C^\alpha = C^\alpha([0,T])$ *denote the space of* $\alpha$*-Hölder continuous functions on* $[0,T]$.

In order to analyse the Hölder coefficients, a result known as the Garsia-Rodemich-Rumsey inequality can be quite helpful. Let $\Psi$ denote a real-valued non-negative even function on $\mathbb{R}$ and $p$ denote a non-negative real-valued continuous even function on $[-T,T]$. Assume that $p(0) = 0$ and $\Psi(\infty) = \infty$. Moreover, assume that $\Psi(u)$ and $p(u)$ are non-decreasing for $u \geq 0$.

For $u \geq \Psi(0)$, set

$$\Psi^{-1}(u) = \sup\{v : \Psi(v) \leq u\},$$

and for $p(T) \geq u \geq 0$, set

$$p^{-1}(u) = \max\{v : p(v) \leq u\}.$$

The Garsia-Rodemich-Rumsey inequality is stated in [28], which has the following form taken from [25] .

**Lemma 2.3.1.** *Let* $f$ *be a real-valued continuous function on* $[0,T]$*. Suppose that*

$$\int_0^T \int_0^T \Psi\left(\frac{f(x) - f(y)}{p(x-y)}\right) \mathrm{d}x\, \mathrm{d}y \leq B < \infty. \tag{2.13}$$

*Then for all* $s, t \in [0,T]$ *we have*

$$|f(t) - f(s)| \leq 8 \int_0^{|t-s|} \Psi^{-1}\left(\frac{4B}{u^2}\right) \mathrm{d}p(u).$$

*Proof.* Assume without loss of generality that $T = 1$, then we will follow the same proof as in [28]. First we prove the inequality for $|f(1) - f(0)|$. Let

$$I(t) = \int_0^1 \Psi\Big(\frac{f(t) - f(s)}{p(t-s)}\Big)\,\mathrm{d}s.$$

Then by (2.13), for some $t_0 \in (0,1)$, we have $I(t_0) \leq B$. We can choose recursively a sequence $\{t_0 > t_1 > t_2 \ldots\}$ so that $t_n \to 0$ as $n \to \infty$ in the following way. Given $t_{n-1}$, define

$$d_{n-1} = p^{-1}\Big(\frac{1}{2}p(t_{n-1})\Big), \tag{2.14}$$

and choose $t_n \leq d_{n-1}$ so that

$$I(t_n) \leq \frac{2B}{d_{n-1}}. \tag{2.15}$$

Moreover,

$$\Psi\Big(\frac{f(t_n) - f(t_{n-1})}{p(t_n - t_{n-1})}\Big) \leq \frac{2I(t_{n-1})}{d_{n-1}}. \tag{2.16}$$

It is possible to find such $t_n$, because the above two inequalities can only be violated on a set of $t_n$ with a measure less than $\frac{d_{n-1}}{2}$. To see that, suppose there exist a set of $t_n$ such that

$$I(t_n) > \frac{2B}{d_{n-1}},$$

and the measure of the set of $t_n$ is greater than $\frac{d_{n-1}}{2}$. Then

$$\int_0^{d_{n-1}} I(t)\,\mathrm{d}t \geq \frac{d_{n-1}}{2} \cdot \frac{2B}{d_{n-1}} = B,$$

which is a contradiction to (2.15). Similar arguments can be applied for the second inequality. Therefore, we can always choose $t_n$ as above.

Now note that $d_n \leq d_{n-1}$ since $t_n$ is decreasing. For $n \geq 1$, (2.15) implies that

$$I(t_n) \leq \frac{2B}{d_n},$$

which is also true for $n = 0$. Therefore, for $n \geq 1$, according to the non-decreasing property of $\Psi$, (2.16) implies

$$\begin{aligned}|f(t_n) - f(t_{n-1})| &\leq p(t_{n-1} - t_n)\Psi^{-1}\Big(\frac{2I(t_{n-1})}{d_{n-1}}\Big)\\ &\leq p(t_{n-1} - t_n)\Psi^{-1}\Big(\frac{4B}{d_{n-1}^2}\Big).\end{aligned}$$

By (2.14) and $t_n \leq d_{n-1}$, we obtain

$$p(t_{n-1} - t_n) \leq 4\big(p(d_{n-1}) - p(d_n)\big).$$

Then,

$$\begin{aligned}
|f(t_0) - f(0)| &\le 4\sum_{n=1}^{\infty}\big(p(d_{n-1}) - p(d_n)\big)\Psi^{-1}\Big(\frac{4B}{d_{n-1}^2}\Big)\\
&\le 4\sum_{n=1}^{\infty}\int_{d_n}^{d_{n-1}}\Psi^{-1}\Big(\frac{4B}{u^2}\Big)\,\mathrm{d}p(u)\\
&\le 4\int_0^1 \Psi^{-1}\Big(\frac{4B}{u^2}\Big)\,\mathrm{d}p(u)
\end{aligned}$$

By using similar technique with $f(1-t)$ instead of $f(t)$, we can obtain the same result for $|f(t_0) - f(1)|$. Hence

$$\begin{aligned}
|f(1) - f(0)| &= |f(1) - f(t_0) + f(t_0) - f(0)|\\
&\le |f(t_0) - f(0)| + |f(1) - f(t_0)| \qquad (2.17)\\
&\le 8\int_0^1 \Psi^{-1}\Big(\frac{4B}{u^2}\Big)\,\mathrm{d}p(u).
\end{aligned}$$

Assume the above integral is finite. Then for general $t, s \in [-1, 1]$ set

$$\begin{aligned}
\bar{f}(t') &= f(s + t'(t-s)), \quad 0 \le t' \le 1\\
\bar{p}(u) &= p(u|s-t|).
\end{aligned}$$

Now we consider

$$\int_0^1\int_0^1 \Psi\Big(\frac{\bar{f}(t') - \bar{f}(s')}{\bar{p}(t'-s')}\Big)\mathrm{d}s'\mathrm{d}t'.$$

Let $x = s + t'(t-s)$ and $y = s + s'(t-s)$, then $t' = \frac{x-s}{t-s}$ and $s' = \frac{y-s}{t-s}$. Therefore by change of variable we get

$$\begin{aligned}
&\int_0^1\int_0^1 \Psi\Big(\frac{\bar{f}(t') - \bar{f}(s')}{\bar{p}(t'-s')}\Big)\,\mathrm{d}s'\,\mathrm{d}t'\\
=&\int_0^1\int_0^1 \Psi\Big(\frac{f(x) - f(y)}{p(x-y)}\Big)\frac{1}{|t-s|^2}\mathrm{d}x\mathrm{d}y\\
\le&\frac{B}{|t-s|^2}.
\end{aligned}$$

Therefore by (2.17), we derive

$$|f(t) - f(s)| = |\bar{f}(1) - \bar{f}(0)| \le 8\int_0^1 \Psi^{-1}\Big(\frac{4B}{u^2|t-s|^2}\Big)\,\mathrm{d}p(u|t-s|).$$

After a change of variables, we obtain the inequality. □

By choosing specific $\Psi$ and $p$, the following corollary can be derived easily. With the help of this corollary, we can study Hölder coefficients of continuous functions. For more details, see [62].

**Corollary 2.3.1.** *Let $p \ge 1$ and $\alpha > \frac{1}{p}$. There exists a constant $C = C(\alpha, p) > 0$ such that for any continuous function $f$ on $[0, T]$, and for all $0 \le s, t \le T$ we have*

$$|f(t) - f(s)|^p \le C|t-s|^{\alpha p - 1}\int_0^T\int_0^T \frac{|f(x) - f(y)|^p}{|x-y|^{\alpha p + 1}}\,\mathrm{d}x\,\mathrm{d}y. \qquad (2.18)$$

*Proof.* Let $\Psi(u) = |u|^p$, and $p(u) = |u|^{\alpha+\frac{1}{p}}$. Now assume that

$$\int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x - y|^{\alpha p+1}} \, \mathrm{d}x \, \mathrm{d}y = B.$$

Let $B < \infty$, since otherwise the inequality (2.18) is trivially true.

Then by Lemma 2.3.1, for $s, t \in [0, T]$ we have

$$\begin{aligned}
|f(t) - f(s)| &\leq 8 \int_0^{|t-s|} \Psi^{-1}\Big(\frac{4B}{u^2}\Big) \, \mathrm{d}p(u) \\
&\leq 8 \int_0^{|t-s|} \Big(\frac{4B}{u^2}\Big)^{1/p} d|u|^{\alpha+\frac{1}{p}} \\
&\leq 8(4B)^{\frac{1}{p}} \int_0^{|t-s|} u^{-\frac{2}{p}}(\alpha + \frac{1}{p})u^{\alpha+\frac{1}{p}-1} \, \mathrm{d}u \\
&= 8(4B)^{\frac{1}{p}}\Big(\alpha + \frac{1}{p}\Big) \int_0^{|t-s|} u^{\alpha-\frac{1}{p}-1} \, \mathrm{d}u \\
&\leq 32\Big(\frac{\alpha + 1/p}{\alpha - 1/p}\Big) \, |t - s|^{\alpha-\frac{1}{p}} \, B^{\frac{1}{p}}.
\end{aligned}$$

Therefore, we obtain

$$|f(t) - f(s)|^p \leq C_{\alpha,p}|t - s|^{\alpha p-1} \int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x - y|^{\alpha p+1}} \, \mathrm{d}x \, \mathrm{d}y,$$

where

$$C_{\alpha,p} = 32^p\Big(\frac{\alpha + 1/p}{\alpha - 1/p}\Big)^p.$$

□

Note that the right side of (2.18) coincides with the Gagliardo seminorm. For $p \geq 1$ and $\alpha \in (0, 1)$, the Gagliardo seminorm of a measurable function on $\Omega$ is defined as

$$[f]_{W_p^\alpha(\Omega)} := \Big(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+\alpha p}} \mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{p}}.$$

Moreover, define

$$W_p^\alpha(\Omega) := \{f \in L^p(\Omega) : [f]_{W_p^\alpha(\Omega)} < \infty\}.$$

For more details of Gagliardo seminorms, see [26].

Now consider the case when $n = 1$. If $f$ is continuous and $[f]_{W_p^\alpha(\mathbb{R})} < \infty$, then by Corollary 2.3.1, we have

$$|f(t) - f(s)|^p \leq C|t - s|^{\alpha p-1},$$

where $C$ is a constant depending on $\alpha$ and $p$. This implies that $f$ is Hölder continuous of order $\alpha - \frac{1}{p}$, if $\alpha > \frac{1}{p}$ for $p \geq 1$.

### 2.3.2 Fractional Sobolev-type Spaces

Fractional Sobolev-type spaces, or the so-called Slobodeckij-type spaces have been introduced by Aronszajn [5], Slobodeckij [79] and Gagliardo [27]. They will be considered later in the following chapters, and here I will only introduce the definition and give some properties of fractional Sobolev-type spaces.

**Definition 2.3.5.** *Fix $0 < \alpha < 1$. For any $p \in [1, \infty)$, a fractional Sobolev-type space $W^{\alpha,p}([0,T])$ is defined as*

$$W^{\alpha,p}([0,T]) := \left\{ f \in L^p([0,T]) : \frac{|f(t) - f(s)|}{|t-s|^{\frac{1}{p}+\alpha}} \in L^p([0,T] \times [0,T]) \right\},$$

*endowed with the norm*

$$\|f\|_{W^{\alpha,p}} := \Big( \int_0^T |f(t)|^p \,\mathrm{d}t + \int_0^T \int_0^T \frac{|f(t) - f(s)|^p}{|t-s|^{1+\alpha p}} \,\mathrm{d}t \,\mathrm{d}s \Big)^{\frac{1}{p}}.$$

$W^{\alpha,p}([0,T])$ is a Banach space such that

$$W^{\alpha,p}([0,T]) \subset W_p^{\alpha}([0,T]).$$

Let $\alpha > \frac{1}{p}$, for $p \geq 1$. If $f$ is continuous and $f \in W^{\alpha,p}([0,T])$, then the Gagliardo seminorm is finite, which implies $f \in C^{\alpha - \frac{1}{p}}([0,T])$, i.e.

$$W^{\alpha,p}([0,T]) \subset C^{\alpha - \frac{1}{p}}([0,T]).$$

Consider the case when $p = 1$ and the fractional Sobolev-type space $W^{\alpha,1}$. Now since $\alpha \in (0,1)$, it holds that $\alpha < \frac{1}{p}$, and we cannot conclude that $W^{\alpha,1}$ is a subspace of Hölder continuous space.

There are several other fractional Sobolev-type spaces which will be used later.

**Definition 2.3.6.** *Let $0 < \alpha < 1$.*

1. *The fractional space $W_0^{\alpha,\infty} = W_0^{\alpha,\infty}([0,T])$ is the space of measurable functions $f : [0,T] \to \mathbb{R}$ such that*

$$\|f\|_{\alpha,\infty,0} = \sup_{0 \leq t \leq T} \left( |f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{1+\alpha}} \,\mathrm{d}s \right) < \infty.$$

2. *The fractional space $W_T^{\alpha,\infty} = W_T^{\alpha,\infty}([0,T])$ is the space of measurable functions $f : [0,T] \to \mathbb{R}$ such that*

$$\|f\|_{\alpha,\infty} = \sup_{0 \leq s < t \leq T} \left( \frac{|f(t) - f(s)|}{(t-s)^{\alpha}} + \int_s^t \frac{|f(u) - f(s)|}{(u-s)^{1+\alpha}} \,\mathrm{d}u \right) < \infty.$$

*3. The fractional space* $W_0^{\alpha,1} = W_0^{\alpha,1}([0,T])$ *is the space of measurable functions* $f : [0,T] \to \mathbb{R}$ *such that*

$$\|f\|_{\alpha,1} = \int_0^T \frac{|f(t)|}{t^\alpha} \,\mathrm{d}t + \int_0^T \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{1+\alpha}} \,\mathrm{d}s \,\mathrm{d}t < \infty.$$

**Remark 2.3.1.** *For any* $0 < \epsilon < \alpha \wedge (1-\alpha)$*, it is obvious that if a function* $f \in C^{\alpha+\epsilon}([0,T])$*, then*

$$\sup_{0 \le s < t \le T} \frac{|f(t) - f(s)|}{(t-s)^\alpha} < \infty,$$

*and*

$$\sup_{0 \le s < t \le T} \int_s^t \frac{|f(u) - f(s)|}{(u-s)^{1+\alpha}} \,\mathrm{d}u < \infty.$$

*Therefore,* $f \in W_T^{\alpha,\infty}([0,T])$ *and* $f \in W_0^{\alpha,1}([0,T])$*.*

*Conversely, if* $f$ *is a real-valued function in* $W_T^{\alpha,\infty}([0,T])$*, then because the first term of the norm is finite, we obtain* $f \in C^\alpha([0,T])$*.*

*To sum up, we have the following relations for* $0 < \epsilon < \alpha \wedge (1-\alpha)$*:*

$$C^{\alpha+\epsilon}([0,T]) \subset W_T^{\alpha,\infty}([0,T]) \subset C^\alpha([0,T]), \quad C^{\alpha+\epsilon}([0,T]) \subset W_0^{\alpha,1}([0,T]).$$

# 3. Stieltjes Integration Theory

After a review of different function spaces, we are now ready to discuss Stieltjes integrals. In this chapter, we will go through Riemann–Stieltjes integrals, Lebesgue–Stieltjes integrals, and generalized Lebesgue–Stieltjes integrals which are defined in terms of fractional integrals.

## 3.1 Stieltjes Integrals

In this section, we will briefly review some well known results of Riemann–Stieltjes integrals and Lebesgue–Stieltjes integrals. More details of these two types of integrals can be found in [76, 77].

### 3.1.1 Riemann–Stieltjes Integrals

A Riemann–Stieltjes integral is a generalization of a Riemann integral, and it is defined as follows.

**Definition 3.1.1.** *Let $[a,b]$ be an interval and $\pi$ a partition of this interval, $\pi = \{a = x_0 < x_1 < \ldots < x_n = b\}$ with mesh size $|\pi| = \max_{1\le j\le n} |x_j - x_{j-1}|$. Let $f$ and $g$ be real-valued functions on $[a,b]$. A Riemann–Stieltjes sum of $f$ with respect to $g$ along the partition $\pi$ is denoted by*

$$S_\pi(f,g) = \sum_{j=1}^{n} f(x_j^*)[g(x_j) - g(x_{j-1})],$$

*where $x_j^* \in [x_{j-1}, x_j]$ for $j = 1, \ldots, n$.*

*If the limit of Riemann–Stieltjes sums exist for a sequence of partitions $(\pi)$ as $|\pi| \to 0$ and is independent of the choice of sequence and the midpoints $x_j^*$, then the limit is called the Riemann–Stieltjes integral of $f$ with respect to $g$, and we write*

$$(\text{RS}) \int_a^b f(x)\,\mathrm{d}g(x) = \lim_{|\pi|\to 0} S_\pi(f,g).$$

*In this case we say that $f$ is Riemann–Stieltjes integrable with respect to $g$.*

For the existence of Riemann–Stieltjes integrals, we have the following proposition.

**Proposition 3.1.1.** *If $f$ is continuous on $[a, b]$ and $g$ is of bounded variation on $[a, b]$, then $f$ is Riemann–Stieltjes integrable with respect to $g$.*

For a proof, see [76]. Kondurar [45] proved the existence of Riemann–Stieltjes integrals for Hölder continuous functions through the following proposition. Recall that $C^\alpha$ denotes the space of Hölder continuous functions of order $\alpha \in (0, 1)$.

**Proposition 3.1.2.** *If $f \in C^\alpha([a, b])$ for some $\alpha \in (0, 1)$ and $g \in C^\beta([a, b])$ for some $\beta \in (0, 1)$ with $\alpha + \beta > 1$, then for any $[s, t] \subset [a, b]$:*

$$(\mathrm{RS}) \int_s^t f(s)\, \mathrm{d}g(x)$$

*exists.*

For Riemann–Stieltjes integrals, we have the following proposition of integration-by-parts formula.

**Proposition 3.1.3.** *Let $f$ and $g$ be bounded functions with no common discontinuities on the interval $[a, b]$, and assume that the Riemann–Stieltjes integral of $f$ with respect to $g$ exists. Then the Riemann–Stieltjes integral of $g$ with respect to $f$ exists, and*

$$(\mathrm{RS}) \int_a^b g(x)\, \mathrm{d}f(x) = f(b-)g(b-) - f(a+)g(a+) - (\mathrm{RS}) \int_a^b f(x)\, \mathrm{d}g(x).$$

For a proof, see [30]. We also have a change of variables formula for Riemann–Stieltjes integral taken from [67].

**Proposition 3.1.4.** *Let $h$ be an increasing function on $[a, b]$ and $f$ be Riemann–Stieltjes integrable with respect to $h$ on $[a, b]$. Let $\phi : [c, d] \mapsto [a, b]$ be a strictly increasing continuous function on $[c, d]$. For any $y \in [c, d]$, let $\varphi(y) = h(\phi(y))$ and $g(y) = f(\phi(y))$. Then $g$ is Riemann–Stieltjes integrable with respect to $\varphi$ on $[c, d]$ and*

$$(\mathrm{RS}) \int_c^d g(y)\, \mathrm{d}\varphi(y) = (\mathrm{RS}) \int_a^b f(x)\, \mathrm{d}h(x).$$

***Young's Integral***

From Section 3.1.1 we know that a Riemann–Stieltjes integral exists when the integrand is continuous and the integrator is of bounded variation. Young's integral generalizes the class of Riemann–Stieltjes integrable functions to Hölder continuous functions. Recall that $C^\alpha([a, b])$ denotes the space of $\alpha$-Hölder continuous functions on $[a, b]$ with $\alpha \in (0, 1)$. If $\alpha = 1$, then the Hölder continuous functions will satisfy a Lipschitz condition, and thus they have a density. Let $\mathrm{Lip}([a,b])$ denote the space of $\alpha$-Hölder continuous functions of order $\alpha = 1$. In [70], Young's integral is constructed as follows.

**Definition 3.1.2.** *Let $f$ and $g$ be two real-valued functions in $\mathrm{Lip}([a,b])$. Young's integral of $f$ with respect to $g$ on $[s,t] \subset [a,b]$ is defined as*

$$(\mathrm{Y}) \int_s^t f(u)\,\mathrm{d}g(u) = \int_s^t f(u)g'(u)\,\mathrm{d}u.$$

*Let $\alpha, \beta > 0$ with $\alpha + \beta > 1$. The map*

$$(f,g) \in \mathrm{Lip}([a,b]) \times \mathrm{Lip}([a,b]) \mapsto \int_a^{\cdot} f\,\mathrm{d}g$$

*with values in $C^\alpha$ extends to a continuous bilinear map from $C^\alpha \times C^\beta$ to $C^\alpha$. Young's integral is defined as the value of this extension at point $(f,g) \in C^\alpha \times C^\beta$ and is denoted by*

$$(\mathrm{Y}) \int_a^t f(u)\,\mathrm{d}g(u) \quad \text{for}\, t \in [a,b].$$

Recall that $\mathcal{W}_p([a,b])$ denotes the space of functions of bounded $p$-variation for $p \geq 1$ on $[a,b]$. Actually, bounded $p$-variation functions are closely related to $\frac{1}{p}$-Hölder continuous functions. This can be shown by the following lemma taken from [18].

**Lemma 3.1.1.** *For $1 \leq p < \infty$, $f \in \mathcal{W}_p([a,b])$ if and only if $f = g \circ h$, where $h$ is a bounded nondecreasing nonnegative function on $[a,b]$ and $g$ is a Hölder continuous function of order $1/p$ defined on $[h(a), h(b)]$.*

Young's integral can also be defined slightly more generally for continuous bounded $p$-variation functions as follows.

**Definition 3.1.3.** *Let $f$ be a real-valued continuous function in $C([a,b]) \cap \mathcal{W}_p([a,b])$ and $g$ be a real-valued continuous function in $C([a,b]) \cap \mathcal{W}_q([a,b])$ for $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} > 1$. We say that $h \in C([a,b])$ is a (indefinite) Young's integral of $f$ with respect to $g$ if there exists a sequence of continuous functions $(f_n, g_n) \subset \mathcal{W}_1([a,b]) \times \mathcal{W}_1([a,b])$ which converges uniformly with uniform variation bounds in the sense*

$$\|f_n - f\|_\infty \to 0 \quad \text{and} \quad \sup_n v_p(f_n; [a,b]) < \infty,$$
$$\|g_n - g\|_\infty \to 0 \quad \text{and} \quad \sup_n v_q(g_n; [a,b]) < \infty,$$

*and*

$$\int_a^{\cdot} f_n\,\mathrm{d}g_n \to h \quad \text{uniformly on}\, [a,b] \,\text{as}\, n \to \infty.$$

*If $h$ is independent of the choice of sequence $f_n$ and $g_n$, then we write $\int_a^{\cdot} f\,\mathrm{d}g$ instead of $h$ as*

$$(\mathrm{Y})\int_a^t f(s)\,\mathrm{d}g(s) := h(t), \quad \text{for } t \in [a,b],$$

*and set*

$$(\mathrm{Y})\int_s^t f\,\mathrm{d}g := \int_a^t f\,\mathrm{d}g - \int_a^s f\,\mathrm{d}g.$$

See [25] for details. Young in [89] showed that the Riemann–Stieltjes integral can be extended to cover functions of bounded variation.

**Proposition 3.1.5.** *If a real valued function $f \in \mathcal{W}_p([a,b])$ and a real valued function $g \in \mathcal{W}_q([a,b])$ for $p \geq 1$, $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} > 1$ have no common discontinuities, then for any $[s,t] \subset [a,b]$, $f$ is Riemann–Stieltjes integrable with respect to $g$. Moreover, if $f$ and $g$ are continuous, then*

$$(\mathrm{Y})\int_s^t f\,\mathrm{d}g = (\mathrm{RS})\int_s^t f\,\mathrm{d}g.$$

*exists.*

### 3.1.2 Lebesgue–Stieltjes Integrals

The Lebesgue–Stieltjes integral is a generalization of the Riemann–Stieltjes integral in the framework of measure theory. To introduce the definition of the Lebesgue–Stieltjes integral, let $f$ be a Borel measurable function on $[a,b]$ and $g$ be a function of bounded variation on $[a,b]$. According to Theorem 2.2.1, we have $g = g^1 - g^2$, where $g^1, g^2$ are increasing functions. Moreover, recall from Proposition 2.1.6 that $g^1$ and $g^2$ are associated with Lebesgue–Stieltjes measures $\mu_1$ of $g^1_+$ and $\mu_2$ of $g^2_+$ respectively.

**Definition 3.1.4.** *The Lebesgue–Stieltjes integral of $f$ with respect to $g$ is defined as*

$$(\mathrm{LS})\int_a^b f(x)\,\mathrm{d}g(x) := \int_a^b f(x)\,\mathrm{d}\mu_1(x) - \int_a^b f(x)\,\mathrm{d}\mu_2(x)$$

*if $f$ is integrable with respect to the measures $\mu_1$ and $\mu_2$ on $[a,b]$ respectively.*

We have the following proposition for a Riemann–Stieltjes integral and a Lebesgue–Stieltjes integral.

**Proposition 3.1.6.** *If $f$ is continuous on $[a,b]$ and the Lebesgue–Stieltjes integral of $f$ with respect to $g$ exists, then the Lebesgue–Stieltjes integral coincides with the Riemann-Stieltjes integral, i.e.*

$$(\mathrm{LS})\int_a^b f\,\mathrm{d}g = (\mathrm{RS})\int_a^b f\,\mathrm{d}g.$$

A proof can be found in [77]. Moreover, we have the following remark.

**Remark 3.1.1.** *If $g$ is continuously differentiable on $(a, b)$, then the Lebesgue–Stieltjes integral becomes*

$$(\mathrm{LS}) \int_a^b f(x)\,\mathrm{d}g(x) = \int_a^b f(x)g'(x)\,\mathrm{d}x.$$

*If either $f$ or $g$ is continuously differentiable, then the integration-by-parts formula holds as follows*

$$(\mathrm{LS}) \int_a^b f(x)\,\mathrm{d}g(x) = -(\mathrm{LS}) \int_a^b g(x)\,\mathrm{d}f(x) + f(b-)g(b-) - f(a+)g(a+). \quad (3.1)$$

## 3.2 Fractional Integrals

Fractional calculus is a generalization of traditional calculus such that the exponent of traditional integral and differential operators changes from integers into fractional ones. Although the physical interpretation of fractional exponents can be difficult, fractional calculus proves to be very useful, for example in the study of a fractional Brownian motion. The main reference of fractional calculus is Samko, Kilbas, and Marichev [72].

In this section, we will discuss fractional integrals and derivatives. Then we will discuss a generalized Lebesgue–Stieltjes integral which is defined in terms of fractional integrals. The notion of generalized Lebesgue–Stieltjes integral is crucial for the main results of this thesis in Chapter 5.

### 3.2.1 Fractional Integrals and Derivatives

Let $If(x) := \int_0^x f(t)\,\mathrm{d}t$ denote the indefinite integral of a locally integrable function $f$ on $[0, T]$, and let $I^n$ be the corresponding repeated integral operator. From Cauchy's formula, we have

$$I^n f(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t)\,\mathrm{d}t. \quad (3.2)$$

Recall that the Gamma function $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1}\,\mathrm{d}u$ is a generalization of the factorial of integers to all real numbers. Therefore it is natural to replace the factorial of integers in (3.2) with a Gamma function to obtain

$$I^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t.$$

For almost all $x \in (a, b)$, the left-sided and right-sided fractional Riemann–Liouville integrals of $f$ of order $0 < \alpha < 1$ are defined as

$$\begin{aligned} I_{a+}^\alpha f(x) &:= \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)\,\mathrm{d}y, \\ I_{b-}^\alpha f(x) &:= \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y)\,\mathrm{d}y, \end{aligned}$$

where $(-1)^{-\alpha} = e^{-i\pi\alpha}$. When $f \in L^1(a,b)$, these integrals converge for almost all $x \in (a,b)$ with respect to Lebesgue measure. The left-sided and right-sided fractional Riemann–Liouville integrals of $f$ on $\mathbb{R}$ are defined as

$$\begin{aligned} I_+^\alpha f(x) &:= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-y)^{\alpha-1} f(y)\,\mathrm{d}y, \\ I_-^\alpha f(x) &:= \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_{x}^{\infty} (y-x)^{\alpha-1} f(y)\,\mathrm{d}y. \end{aligned}$$

We say that a function $f$ belongs to the domain of the integral operator $I_\pm^\alpha$, if the corresponding fractional integral converges for almost all $x \in \mathbb{R}$. For $1 \le p < \frac{1}{\alpha}$, we have $L^p(\mathbb{R}) \subset \mathcal{D}(I_\pm^\alpha)$. Moreover, we have the following Hardy-Littlewood theorem from [72].

**Proposition 3.2.1.** *Let $1 \le p < \infty$, $1 \le q < \infty$ and $0 < \alpha < 1$. The operators $I_\pm^\alpha$ are bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ if and only if $1 < p < \frac{1}{\alpha}$ and $q = p(1-\alpha p)^{-1}$, i.e. for any $1 < p < \frac{1}{\alpha}$ and $q = \frac{p}{1-\alpha p}$, there exists a constant $C_{p,q,\alpha}$ such that*

$$\left( \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(u)(x-u)^{\alpha-1}\,\mathrm{d}u \right|^q \mathrm{d}x \right)^{1/q} \le C_{p,q,\alpha} \|f\|_{L^p}.$$

If $f \in L^1(a,b)$, then the first composition formulas of fractional integration hold as

$$\begin{aligned} I_{a+}^\alpha (I_{a+}^\beta f) &= I_{a+}^{\alpha+\beta} f, \\ I_{b-}^\alpha (I_{b-}^\beta f) &= I_{a+}^{\alpha+\beta} f. \end{aligned}$$

If $\alpha + \beta \ge 1$, then the above formulas hold for any $x \in (a,b)$; otherwise the formulas hold for almost all $x \in (a,b)$.

If $f \in L^p(\mathbb{R})$, $\alpha, \beta > 0$ and $\alpha + \beta < \frac{1}{p}$, then

$$\begin{aligned} I_+^\alpha (I_+^\beta f) &= I_+^{\alpha+\beta} f, \\ I_-^\alpha (I_-^\beta f) &= I_-^{\alpha+\beta} f. \end{aligned}$$

If $f \in L^p(a,b)$, $g \in L^q(a,b)$ for $p \ge 1$, $q \ge 1$ and $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$, or $p > 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$, then the integration-by-parts formula for fractional integrals is

$$\int_a^b f(x) I_{a+}^\alpha g(x)\,\mathrm{d}x = \int_a^b g(x) I_{b-}^\alpha f(x)\,\mathrm{d}x.$$

If $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ for $p > 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$, then

$$\int_{\mathbb{R}} f(x) I_+^\alpha g(x)\,\mathrm{d}x = (-1)^\alpha \int_{\mathbb{R}} g(x) I_-^\alpha f(x)\,\mathrm{d}x.$$

Moreover, we have the following lemma which can be found in [58].

**Lemma 3.2.1.** *If $f \in L^p(\mathbb{R})$, $1 \le p < \frac{1}{\alpha}$ and $I^\alpha_\pm f = 0$ for $0 < \alpha < 1$, then $f(x) = 0$ for almost all $x \in \mathbb{R}$.*

Fractional differentiation can be viewed as an inverse operation. For $p \ge 1$, let $I^\alpha_{a+}\big(L^p(\mathbb{R})\big)$ (resp. $I^\alpha_{b-}\big(L^p(\mathbb{R})\big)$) denote the class of $f$ which is the $I^\alpha_{a+}$-integral (resp. $I^\alpha_{b-}$) of a function in $L^p(\mathbb{R})$. That is, $f \in I^\alpha_{a+}(L^p(\mathbb{R}))$ for $p \ge 1$ (resp. $f \in I^\alpha_{b-}\big(L^p(\mathbb{R})\big)$) if and only if $f = I^\alpha_{a+} g$ (resp. $f = I^\alpha_{b-} g$) for some $g \in L^p(\mathbb{R})$. Note here that Lemma 3.2.1 indicates the uniqueness of $g$. For $0 < \alpha < 1$, $g$ coincides with the fractional right-sided (left-sided) Riemann–Liouville derivative of $f$ of order $\alpha$ for $x \in \mathbb{R}$, which is defined as

$$
\begin{aligned}
D^\alpha_+ f(x) &:= \frac{1}{\Gamma(1-\alpha)} \frac{d}{\mathrm{d}x} \int_{-\infty}^{x} f(y)(x-y)^{-\alpha}\,\mathrm{d}y,\\
D^\alpha_- f(x) &:= \frac{(-1)^{\alpha+1}}{\Gamma(1-\alpha)} \frac{d}{\mathrm{d}x} \int_{x}^{\infty} f(y)(y-x)^{-\alpha}\,\mathrm{d}y.
\end{aligned}
$$

The Riemann–Liouville fractional derivatives of $f$ of order $\alpha$ for $x \in [a,b]$ are defined as

$$
\begin{aligned}
D^\alpha_{a+} f(x) &:= \frac{1}{\Gamma(1-\alpha)} \frac{d}{\mathrm{d}x} \int_{a}^{x} f(y)(x-y)^{-\alpha}\,\mathrm{d}y,\\
D^\alpha_{b-} f(x) &:= \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \frac{d}{\mathrm{d}x} \int_{x}^{b} f(y)(y-x)^{-\alpha}\,\mathrm{d}y.
\end{aligned}
$$

By putting $f = 0$ outside the interval $(a,b)$, the Riemann–Liouville fractional derivatives $D^\alpha_{a+} f$ and $D^\alpha_{b-} f$ admit the Weyl representation of fractional derivatives

$$
\begin{aligned}
D^\alpha_{a+} f(x) =& \frac{1}{\Gamma(1-\alpha)} \Big( f(x)(x-a)^{-\alpha}\\
&+ \alpha \int_a^x \big(f(x) - f(y)\big)(x-y)^{-\alpha-1}\,\mathrm{d}y \Big) \mathbf{1}_{(a,b)}(x)\\
D^\alpha_{b-} f(x) =& \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \Big( f(x)(b-a)^{-\alpha}\\
&+ \alpha \int_x^b \big(f(x) - f(y)\big)(y-x)^{-\alpha-1}\,\mathrm{d}y \Big) \mathbf{1}_{(a,b)}(x),
\end{aligned}
$$

where the integrals converge pointwise for almost all $x \in (a,b)$ for $p = 1$ and converge in $L^p$ sense for $p > 1$.

For $f \in I^\alpha_+\big(L^p(\mathbb{R})\big)$ (resp. $f \in I^\alpha_-\big(L^p(\mathbb{R})\big)$ ) with $0 < \alpha < 1$ and $p \ge 1$, we have

$$I^\alpha_+(D^\alpha_+) f = f \quad \text{and} \quad I^\alpha_-(D^\alpha_-) f = f.$$

We also have for $f \in L^1(\mathbb{R})$ that

$$D^\alpha_+(I^\alpha_+) f = f \quad \text{and} \quad D^\alpha_-(I^\alpha_-) f = f.$$

Moreover, if $f \in I^{\alpha+\beta}_{a+}\big(L^1(\mathbb{R})\big)$ with $\alpha \ge 0$, $\beta \ge 0$ and $\alpha + \beta \le 1$, then we have the following composition formula for fractional derivatives

$$D^\alpha_{a+}(D^\beta_{a+} f) = D^{\alpha+\beta}_{a+} f. \tag{3.3}$$

If $f \in I^{\alpha}_{a+}\big(L^p[a,b]\big)$, $g \in I^{\alpha}_{b-}\big(L^q[a,b]\big)$ for $0 < \alpha < 1$, where $p \geq 1$, $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$, then the integration-by-parts formula for fractional derivatives is

$$\int_a^b f(x) D^{\alpha}_{b-} g(x) \,\mathrm{d}x = (-1)^{\alpha} \int_a^b g(x) D^{\alpha}_{a+} f(x) \,\mathrm{d}x. \tag{3.4}$$

Finally, according to [62] we know that the linear space $I^{\alpha}_{a+}\big(L^p[a,b]\big)$ is a Banach space with respect to the norm

$$\|f\|_{I^{\alpha}_{a+}(L^p[a,b])} = \|f\|_{L^p[a,b]} + \|D^{\alpha}_{a+} f\|_{L^p[a,b]},$$

and a similar conclusion also holds for $I^{\alpha}_{b-}(L^p[a,b])$.

### 3.2.2 Generalized Lebesgue–Stieltjes Integrals

In this section we will discuss the theory of the generalized Lebesgue–Stieltjes integration, which was introduced by Zähle in [90–92]. Later it was developed in fractional Sobolev-type spaces by Nualart and Răşcanu in [62].

***Generalized Lebesgue–Stieltjes Integrals***

Consider two real-valued functions $f$ and $g$ defined on $[a,b] \subset \mathbb{R}$. We denote

$$\begin{aligned} f_{a+}(x) &:= \mathbf{1}_{(a,b)}(x)\big(f(x) - f(a+)\big), \\ g_{b-}(x) &:= \mathbf{1}_{(a,b)}(x)\big(g(x) - g(b-)\big). \end{aligned}$$

**Definition 3.2.1.** *If $f_{a+} \in I^{\alpha}_{a+}\big(L^p[a,b]\big)$, $g_{b-} \in I^{1-\alpha}_{b-}\big(L^q[a,b]\big)$ for $p \geq 1$, $q \geq 1$, $\frac{1}{p} + \frac{1}{q} \leq 1$ and $0 \leq \alpha \leq 1$, then the generalized Lebesgue–Stieltjes integral of $f$ with respect to $g$ is defined as*

$$\begin{aligned} \text{(gLS)} \int_a^b f(x) \,\mathrm{d}g(x) =& (-1)^{\alpha} \int_a^b D^{\alpha}_{a+} f_{a+}(x) D^{1-\alpha}_{b-} g_{b-}(x) \,\mathrm{d}x \\ &+ f(a+)\big(g(b-) - g(a+)\big). \end{aligned} \tag{3.5}$$

Note that the integral $\int_a^b D^{\alpha}_{a+} f_{a+}(x) D^{1-\alpha}_{b-} g_{b-}(x) \,\mathrm{d}x$ is well defined by Proposition 2.3.1. Moreover, we have the following proposition from Proposition 2.1 in [90].

**Proposition 3.2.2.** *In Definition 3.2.1, the right-hand-side of (3.5) is independent of the choice of $\alpha$.*

*Proof.* Let $(\alpha', p', q')$ be another numbers which fulfil all the conditions of

Definition 3.2.1. Moreover, let $\alpha' = \alpha + \gamma$. Then we have

$$
\begin{aligned}
&(-1)^{\alpha'} \int_a^b D_{a+}^{\alpha'} f(x) D_{b-}^{1-\alpha'} g_{b-}(x)\,\mathrm{d}x \\
=&(-1)^{\alpha+\gamma} \int_a^b D_{a+}^{(\alpha+\gamma)} f(x) D_{b-}^{1-(\alpha+\gamma)} g_{b-}(x)\,\mathrm{d}x \\
=&(-1)^{\alpha+\gamma} \int_a^b D_{a+}^{\alpha} (D_{a+}^{\gamma} f)(x) D_{b-}^{1-(\alpha+\gamma)} g_{b-}(x)\,\mathrm{d}x \\
=&(-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f(x) D_{b-}^{\gamma} (D_{b-}^{1-(\alpha+\gamma)} g_{b-})(x)\,\mathrm{d}x \\
=&(-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f(x) D_{b-}^{1-\alpha} g_{b-}(x)\,\mathrm{d}x
\end{aligned}
$$

where the third equality comes from the second integration-by-parts formula (3.4) and the composition formula (3.3). □

**Remark 3.2.1.** *For $\alpha p < 1$, we have $f_{a+} \in I_{a+}^{\alpha}\big(L^p[a,b]\big)$ if and only if $f \in I_{a+}^{\alpha}\big(L^p[a,b]\big)$ and $f(a+)$ exists. In this case we have*

$$
\begin{aligned}
D_{a+}^{\alpha} f_{a+}(x) &= D_{a+}^{\alpha} \mathbf{1}_{(a,b)}(x)\big(f(x) - f(a+)\big) \\
&= D_{a+}^{\alpha} f(x) - \frac{1}{\Gamma(1-\alpha)} \frac{f(a+)}{(x-a)^{\alpha}} \mathbf{1}_{(a,b)}(x).
\end{aligned}
$$

According to Remark 3.2.1, we can rewrite (3.5) as

$$
\begin{aligned}
(\mathrm{gLS}) \int_a^b f(x)\,\mathrm{d}g(x) = \int_a^b (-1)^{\alpha} \Big( D_{a+}^{\alpha} f(x) - \frac{1}{\Gamma(1-\alpha)} \frac{f(a+)}{(x-a)^{\alpha}} \Big) D_{b-}^{1-\alpha} g_{b-}(x)\,\mathrm{d}x& \\
+ f(a+)\big(g(b-) - g(a+)\big)& \\
=(-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f(x) D_{b-}^{1-\alpha} g_{b-}(x)\,\mathrm{d}x& \\
- f(a+) I_{b-}^{1-\alpha}(D_{b-}^{1-\alpha} g_{b-})(a) + f(a+)\big(g(b-) - g(a+)\big)& \\
=(-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f(x) D_{b-}^{1-\alpha} g_{b-}(x)\,\mathrm{d}x.&
\end{aligned}
\tag{3.6}
$$

For $\alpha = 0$, (3.6) can be written as

$$
(\mathrm{gLS}) \int_a^b f(x)\,\mathrm{d}g(x) = \int_a^b f(x) g'(x)\,\mathrm{d}x.
$$

For $\alpha = 1$, (3.6) can be written as

$$
(\mathrm{gLS}) \int_a^b f(x)\,\mathrm{d}g(x) = -\int_a^b f'(x) g(x)\,\mathrm{d}x + f(b-)g(b-) - f(a+)g(a+),
$$

which coincides with the corresponding integration-by-parts formula (3.1) for Lebesgue–Stieltjes integrals in Section 3.1.2.

According to [90], we have the following lemma which shows that under certain conditions, the generalized Lebesgue–Stieltjes integral agrees with the Lebesgue–Stieltjes integral.

**Lemma 3.2.2.** *If $f_{a+} \in I^{\alpha}_{a+}\big(L^p[a,b]\big)$, $g_{b-} \in I^{1-\alpha}_{b-}\big(L^q[a,b]\big) \cap BV([a,b])$, $p, q \geq 1$, $1/p + 1/q \leq 1$, $0 < \alpha < 1$ and*

$$\int_a^b I^{\alpha}_{a+}(|D^{\alpha}_{a+} f_{a+}|)(x)|g|\,(\mathrm{d}x) < \infty,$$

*then*

$$(\mathrm{gLS}) \int_a^b f(x)\,\mathrm{d}g(x) = (\mathrm{LS}) \int_a^b f(x)\,\mathrm{d}g(x).$$

For a proof, see [90].

**Remark 3.2.2.** *If $f$ is continuous and all the conditions of $f$, $g$ in Lemma 3.2.2 are fulfilled, then*

$$(\mathrm{gLS}) \int_a^b f(x)\,\mathrm{d}g(x) = (\mathrm{RS}) \int_a^b f(x)\,\mathrm{d}g(x).$$

Next, let us review some properties of the generalized Lebesgue–Stieltjes integral. For more details, see [90].

**Proposition 3.2.3.**

*1) Let $s, t \in [a,b]$, $s < t$ and assume that functions $f, g$ satisfy the following two assumptions:*

*(i) $(f \cdot \mathbf{1}_{(s,t)}) \in I^{\alpha}_{+}\big(L^p[a,b]\big)$, $g_{b-} \in I^{1-\alpha}_{-}\big(L^q[a,b]\big)$ for some $0 < \alpha < 1$, $p \geq 1$, $q \geq 1$ and $1/p + 1/q \leq 1$,*

*(ii) $f_{s+} \in I^{\alpha'}_{+}\big(L^{p'}[s,t]\big)$, $g_{t-} \in I^{1-\alpha'}_{-}\big(L^{q'}[s,t]\big)$ for some $0 < \alpha' < 1$, $p' \geq 1$, $q' \geq 1$ and $1/p' + 1/q' \leq 1$.*

*Then we have*

$$(\mathrm{gLS}) \int_s^t f\,\mathrm{d}g = (\mathrm{gLS}) \int_a^b \mathbf{1}_{(s,t)} f\,\mathrm{d}g.$$

*2) For $a \leq s < t < u \leq b$, we have*

$$(\mathrm{gLS}) \int_s^t f\,\mathrm{d}g + (\mathrm{gLS}) \int_t^u f\,\mathrm{d}g = (\mathrm{gLS}) \int_s^u f\,\mathrm{d}g - f(t)\big(g(t+) - g(t-)\big),$$

*if all the integrals exist in the sense of (3.5).*

*3)*

$$(\mathrm{gLS}) \int_a^b f_1\,\mathrm{d}g = (\mathrm{gLS}) \int_a^b f_2\,\mathrm{d}g,$$

*if $f_1 = f_2$ Lebesgue-almost everywhere and both integrals exist in the sense of (3.5).*

**Definition 3.2.2.** *If $f_{b-} \in I_{b-}^{\tilde{\alpha}}(L^{\tilde{p}})$, $g_{a+} \in I_{a+}^{1-\tilde{\alpha}}(L^{\tilde{q}})$ for some $\frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} \leq 1$ and $0 \leq \tilde{\alpha} \leq 1$. Define the backward generalized Lebesgue–Stieltjes integral as*

$$\begin{aligned}(\mathrm{gLS}) \int_a^b \mathrm{d}g(x)\, f(x) =&(-1)^{\tilde{\alpha}} \int_a^b D_{b-}^{\tilde{\alpha}} f_{b-}(x) D_{a+}^{1-\tilde{\alpha}} g_{a+}(x)\, \mathrm{d}x \\ &+ f(b-)\big(g(b-) - g(a+)\big).\end{aligned} \tag{3.7}$$

Then we obtain the following integration-by-parts formula.

**Proposition 3.2.4.** *If $f$ and $g$ satisfy the conditions of Definition 3.2.1 and Definition 3.2.2, then*

*1)*

$$(\mathrm{gLS}) \int_a^b f\, \mathrm{d}g = (\mathrm{gLS}) \int_a^b \mathrm{d}g\, f.$$

*2)*

$$(\mathrm{gLS}) \int_a^b f\, \mathrm{d}g = f(b-)g(b-) - f(a+)g(a+) - (\mathrm{gLS}) \int_a^b g\, \mathrm{d}f.$$

Zähle in [90] proved the existence of the generalized Lebesgue–Stieltjes integral for Hölder continuous functions, and showed that the integral coincides with the corresponding Riemann–Stieltjes integral.

**Proposition 3.2.5.** *If $f \in C^{\alpha}([a,b]), g \in C^{\beta}([a,b])$ for some $\alpha + \beta > 1$, then the integral $(\mathrm{gLS}) \int_a^b f\, \mathrm{d}g$ exists in the sense of (3.5) and coincides with the corresponding Riemann–Stieltjes integral, i.e.*

$$(\mathrm{gLS}) \int_a^b f(x)\, \mathrm{d}g(x) = (\mathrm{RS}) \int_a^b f(x)\, \mathrm{d}g(x).$$

For a proof, see [90].

***Generalized Lebesgue–Stieltjes Integrals in Fractional Sobolev-type Spaces***

Recall that we have introduced fractional Sobolev-type spaces $W_0^{\alpha,1}$ and $W_T^{1-\alpha,\infty}$ in Section 2.3.2. Nualart and Răşcanu in [62] showed the existence of the generalized Lebesgue–Stieltjes integral for functions that belong to certain fractional Sobolev-type spaces.

**Proposition 3.2.6.** *If $f \in W_0^{\alpha,1}([0,T])$ and $g \in W_T^{1-\alpha,\infty}([0,T])$, then $f$ is generalized Lebesgue–Stieltjes integrable with respect to $g$ over $[0,t]$ for all $t \leq T$ in the sense of (3.5).*

If $g \in W_T^{1-\alpha,\infty}([0,T])$, then its restriction to $(0,t) \subset [0,T]$ belongs to

$I_{t-}^{1-\alpha}(L^{\infty}(0,t))$ for all $t$. Moreover, we have

$$\begin{aligned}\Lambda_\alpha(g) &:= \sup_{0<s<t<T} |D_{t-}^{1-\alpha} g_{t-}(s)| \\ &\le \frac{1}{\Gamma(\alpha)} \|g\|_{1-\alpha,\infty} \\ &< \infty.\end{aligned}$$

If $f \in W_0^{\alpha,1}([0,T])$, then the restriction of $f$ to $(0,t) \subset [0,T]$ belongs to $I_{0+}^{\alpha}\big(L^1(0,t)\big)$ for all $t$.

For any $t \in [0,T]$, according to Proposition 3.2.3, we have

$$(\mathrm{gLS}) \int_0^t f \,\mathrm{d}g = (\mathrm{gLS}) \int_0^T \mathbf{1}_{(0,t)} f \,\mathrm{d}g,$$

if certain conditions are fulfilled. Moreover, we have the following estimates taken from [62].

**Proposition 3.2.7.** *If $f \in W_0^{\alpha,1}([0,T])$ and $g \in W_T^{1-\alpha,\infty}([0,T])$, then*

$$\begin{aligned}\left|(\mathrm{gLS}) \int_0^t f \,\mathrm{d}g\right| &\le \int_0^t |(D_{0+}^{\alpha} f)(s)(D_{t-}^{1-\alpha} g_{t-})(s)| \,\mathrm{d}s \\ &\le \Lambda_\alpha(g) \|f\|_{\alpha,1} \\ &\le \frac{1}{\Gamma(\alpha)} \|f\|_{\alpha,1} \|g\|_{1-\alpha,\infty}.\end{aligned}$$

**Corollary 3.2.1.** *Fix a parameter $0 < \alpha < 1$. Suppose we have a sequence of functions $f_n \in W_0^{\alpha,1}([0,T])$ and a function $f \in W_0^{\alpha,1}([0,T])$, so that $\|f_n - f\|_{\alpha,1} \to 0$ as $n \to \infty$. If $g \in W_T^{1-\alpha,\infty}([0,T])$, then*

$$(\mathrm{gLS}) \int_0^t f_n \,\mathrm{d}g \to (\mathrm{gLS}) \int_0^t f \,\mathrm{d}g.$$

## 3.3 Relationships between Different Integrals

Now we have introduced different kinds of Stieltjes integrals. I end up this chapter by summarizing implications and equivalences of these integrals under different conditions.

1) If $f \in C([a,b]) \cap \mathcal{W}_p([a,b])$ and $g \in C([a,b]) \cap \mathcal{W}_q([a,b])$ for $p \ge 1$, $q \ge 1$, $\frac{1}{p} + \frac{1}{q} > 1$, then (Y) $\int_a^b f \,\mathrm{d}g$ exists, and

$$(\mathrm{Y}) \int_a^b f \,\mathrm{d}g = (\mathrm{RS}) \int_a^b f \,\mathrm{d}g.$$

2) If $f \in C([a,b])$ and $g$ is increasing, then the existence of (LS) $\int_a^b f \,\mathrm{d}g$ implies the existence of (RS) $\int_a^b f \mathrm{d}g$, and

$$(\mathrm{LS}) \int_a^b f \,\mathrm{d}g = (\mathrm{RS}) \int_a^b f \,\mathrm{d}g.$$

3) If $f \in C([a,b])$ and $g \in BV([a,b])$, then (RS) $\int_a^b f \, \mathrm{d}g$, (LS) $\int_a^b f \, \mathrm{d}g$ and (gLS) $\int_a^b f \, \mathrm{d}g$ exist. Moreover, we have

$$\text{(LS)} \int_a^b f \, \mathrm{d}g = \text{(RS)} \int_a^b f \, \mathrm{d}g = \text{(gLS)} \int_a^b f \, \mathrm{d}g.$$

4) If $f$ and $g$ satisfy the assumptions of Lemma 3.2.2, then (gLS) $\int_a^b f \, \mathrm{d}g$ exists. Moreover, we have

$$\text{(gLS)} \int_a^b f \, \mathrm{d}g = \text{(LS)} \int_a^b f \, \mathrm{d}g.$$

5) If $f \in C^\alpha([a,b])$ for $\alpha \in (0,1)$ and $g \in C^\beta([a,b])$ for $\beta \in (0,1)$ with $\alpha+\beta > 1$, then (gLS) $\int f \, \mathrm{d}g$ exists. Moreover, we have

$$\text{(gLS)} \int f \, \mathrm{d}g = \text{(RS)} \int f \, \mathrm{d}g.$$

7) If $f \in W_0^{\alpha,1}, g \in W_T^{1-\alpha,\infty}$, then (gLS) $\int_a^b f \, \mathrm{d}g$ exists.

# 4. Stochastic Integration Theory

After the review of integration with respect to different classes of functions, now we move to a review of integration with respect to stochastic processes. Firstly, several stochastic processes which we are interested in will be introduced. Secondly, stochastic integration methods with respect to these processes will be discussed, with focus on pathwise integration theory.

## 4.1 Stochastic Processes

In this section, I will review several stochastic processes. I will begin with the well known semimartingales, and then go beyond semimartingales to more general Gaussian processes including fractional Brownian motions.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, where the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions, i.e. it is complete and right-continuous.

Let $X = (X_t)_{t\geq 0}$ be a real-valued stochastic process defined on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. For each $\omega \in \Omega$, the mapping $t \mapsto X_t(\omega)$ is called a sample path of $X$. For each $t \geq 0$, $X_t$ is a random variable. We say that two stochastic processes $X$ and $Y$ are equivalent in law if they have the same finite dimensional distributions, denoted by $X \stackrel{d}{=} Y$.

A centered stochastic process is a process with $\mathbb{E}[X_t] = 0$ for every $t \geq 0$. In the following, we assume that the process is centered.

**Definition 4.1.1.** *The covariance function of a centered stochastic process $X$ is a function $R : [0,\infty) \times [0,\infty) \mapsto \mathbb{R}$ defined by*

$$R(s,t) = \mathbb{E}[X_t X_s], \tag{4.1}$$

*provided the expectation on the right exists for all $s \geq 0$ and $t \geq 0$.*

**Definition 4.1.2.** *A stochastic process $X$ is called stationary if for every $h \geq 0$, we have*

$$(X_{t+h} : t \geq 0) \stackrel{d}{=} (X_t : t \geq 0).$$

**Definition 4.1.3.** *A stochastic process $X$ such that $X_0 = 0$ a.s. has stationary increments if for any $s \geq 0$,*

$$(X_{t+s} - X_s : t \geq 0) \stackrel{d}{=} (X_t : t \geq 0).$$

**Definition 4.1.4.** *A stochastic process $X$ is called self-similar with index $\gamma > 0$, if for any $a > 0$,*

$$(X_{at} : t \geq 0) \stackrel{d}{=} (a^\gamma X_t : t \geq 0).$$

**Definition 4.1.5.** *A stationary sequence $(X_n)_{n \in \mathbb{N}}$ with finite variance is said to exhibit long-range dependence if the autocorrelation function $\rho(n) = \mathbb{E}[X_0 X_n]$ satisfies*

$$\sum_{n=1}^{\infty} \rho(n) = \infty.$$

*If $\sum_{n=1}^{\infty} \rho(n) < \infty$, then the stationary sequence $(X_n)_{n \in \mathbb{N}}$ is said to exhibit short-range dependence.*

For different definitions of long-range dependence, see also [8] and [29].

**Definition 4.1.6.** *A stochastic process $X$ is said to be a.s. continuous if for almost all $\omega \in \Omega$, the function $t \to X_t(\omega)$ is continuous.*

**Definition 4.1.7.** *Let $X$ and $Y$ be two stochastic processes defined on the same probability space. $X$ and $Y$ are said to be the modifications of each other if for each $t \geq 0$,*

$$X_t = Y_t \quad a.s.$$

In this dissertation, we mainly consider stochastic processes on a compact interval $[0, T]$.

**Definition 4.1.8.** *A stochastic process $X$ is $\alpha$-Hölder continuous if there exists a finite random variable $C(\omega)$ such that for every $s, t \in [0, T]$ we have*

$$\sup_{s,t \in [0,T], s \neq t} \frac{|X_s - X_t|}{|t - s|^\alpha} \leq C(\omega) \quad a.s.$$

The following Kolmogorov–Chentsov continuity theorem taken from [63] gives a criterion for the continuity of a stochastic process.

**Theorem 4.1.1.** *A stochastic process $X = (X_t)_{t \in [0,T]}$ has an a.s. continuous modification $\tilde{X}$, if there exist constants $\alpha, \beta, c > 0$ such that*

$$\mathbb{E}|X_t - X_s|^\alpha \leq c|t - s|^{1+\beta},$$

*for $s, t \in [0, T]$.*

For a proof, see [82].

**Remark 4.1.1.** *Almost all paths of the modification* $\tilde{X}$ *are locally Hölder continuous of any order* $\lambda \in (0, \frac{\beta}{\alpha})$*, i.e. there exists an a.s. finite and positive random variable* $C = C(\omega)$ *such that*

$$|\tilde{X}_t(\omega) - \tilde{X}_s(\omega)| \leq C(\omega)|t-s|^\lambda,$$

*for* $s, t \in [0, T]$ *and almost all* $\omega \in \Omega$*. Note here that the Hölder coefficients can be computed by using Corollary 2.3.1.*

The quadratic variation of a process is defined as follows.

**Definition 4.1.9.** *Let* $(\pi^n)$ *be a sequence of partitions* $\pi^n = \{0 = t_0^n < \ldots < t_{k(n)}^n = T\}$ *such that* $|\pi^n| = \max_{j=1,\ldots,k(n)} |t_j^n - t_{j-1}^n| \to 0$ *as* $n \to \infty$*. Let* $X$ *be a continuous stochastic process. The quadratic variation process of* $X$ *along the sequence* $(\pi^n)$ *is defined as*

$$\langle X, X \rangle_t = \lim_{n\to\infty} \sum_{t_j^n \in \pi^n \cap (0,t]} \left( X_{t_j^n} - X_{t_{j-1}^n} \right)^2,$$

*if the limit exists in the convergence of probability.*

Next let us consider semimartingales which have well-defined quadratic variations. We will review some definitions first.

**Definition 4.1.10.** *A family of random variables* $(X_i)_{i\in I}$*, where* $I$ *is any set in* $\mathbb{R}$*, is said to be uniformly integrable if*

$$\lim_{n\to\infty} \sup_{i\in I} \mathbb{E}[|X_i| \mathbf{1}_{\{|X_i|\geq n\}}] = 0.$$

**Definition 4.1.11.** *A stochastic process* $X$ *is said to be càdlàg if it has sample paths which are right-continuous with left limits a.s.*

**Definition 4.1.12.** *We say a stochastic process* $X$ *is adapted to the filtration* $(\mathcal{F}_t)_{t\in[0,T]}$ *if* $X_t$ *is* $\mathcal{F}_t$*-measurable for each* $t \in [0, T]$*.*

**Definition 4.1.13.** *A real-valued, adapted process* $X$ *is called a martingale with respect to the filtration* $(\mathcal{F}_t)_{t\in[0,T]}$ *if*

*1)* $\mathbb{E}[|X_t|] < \infty$ *for all* $t \in [0, T]$*;*

*2)* $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ *a.s. for any* $0 \leq s \leq t \leq T$*.*

**Definition 4.1.14.** *A random variable* $\tau : \Omega \mapsto [0, \infty]$ *is called a stopping time with respect to the filtration* $(\mathcal{F}_t)_{t\geq 0}$ *if* $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$*, for every* $t \geq 0$*.*

If $\tau$ is a stopping time, let $X^\tau$ denote the stopped process as

$$X_t^\tau = X_{t\wedge\tau},$$

for $t \in [0, T]$.

A local martingale is defined as follows.

**Definition 4.1.15.** *An adapted, càdlàg process $X = (X_t)_{t\geq 0}$ is a local martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if there exists a sequence of increasing stopping times $T_n$ with $\lim_{n\to\infty} T_n = +\infty$ a.s. such that for every $n \geq 1$, $X^{T_n}\mathbf{1}_{\{T_n>0\}}$ is a uniformly integrable martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$.*

Finally, we have the definition of a semimartingale as follows.

**Definition 4.1.16.** *An adapted, càdlàg process $X = (X_t)_{t\geq 0}$ is called a semimartingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, if it can be written as $X_t = X_0 + M_t + A_t$ with $M_0 = A_0 = 0$ for $t \geq 0$. Here $M$ is a local martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and $A$ is an adapted process of a.s. locally bounded variation sample paths.*

Note that the quadratic variation of a semimartingale has finite variation paths. For details, see [64]. The most important example of a semimartingale is Brownian motion.

### 4.1.1 Lévy Processes

Next we will consider Lévy processes. For a detailed study of Lévy processes, see [73], and for applications of Lévy processes in mathematical finance, see [74].

**Definition 4.1.17.** *A stochastic process $L = (L_t)_{t\in[0,T]}$ is called a Lévy process if it satisfies the following conditions*

*1) $L_t - L_s \overset{d}{=} L_{t-s}$, for $0 \leq s \leq t$,*

*2) $L_t - L_s$ is independent of $L_s$, for $0 \leq s \leq t$,*

*3) $\mathbb{P}(L_0 = 0) = 1$,*

*4) $L$ is stochastically continuous (also called continuous in probability or $\mathbb{P}$-continuous), i.e. for $s \geq 0$,*

$$X_{t+s} - X_s \xrightarrow{P} 0, \quad as \quad t \to 0.$$

*5) paths of $L$ are a.s. càdlàg.*

As an example of Lévy process, Brownian motion with drift is the only (non deterministic) Lévy process which is a.s. continuous. Another example of Lévy process is Poisson process, whose paths contain jumps. Moreover, a compensated Poisson process is a martingale.

**Definition 4.1.18.** *A compound Poisson process $Y = (Y_t)_{t\in[0,T]}$ is defined as*

$$Y_t = \sum_{i=1}^{N_t} X_i,$$

*where $N = (N_t)_{t\in[0,T]}$ is a Poisson process with parameter $\lambda$, and $X_i$ are any sequence of identically distributed and independent (i.i.d.) random variables which are also independent of $N$.*

A compound Poisson process is also a Lévy process, and it satisfies the following proposition.

**Proposition 4.1.1.** *If $\mathbb{E}[X_i] < \infty$, then a compensated compound Poisson process defined as*

$$M_t := Y_t - \lambda t \mathbb{E}[X_1], \quad t \in [0,T]$$

*is a martingale.*

In the following, we will discuss some properties of Lévy processes and review some well known results of Lévy processes. The infinitely divisible distributions and Lévy processes are closely related. Due to the infinitely divisible distribution property, one can characterize Lévy processes via a characteristic triplet.

**Definition 4.1.19.** *A real-valued random variable $X$ is said to have an infinitely divisible distribution if for all $n \in \mathbb{N}$, there exists a sequence of i.i.d. random variables $X_1^n, \dots, X_n^n$ such that*

$$X \overset{d}{=} X_1^n + \dots, +X_n^n.$$

*Alternatively, we can say that the probability law $\mu$ of $X$ is infinitely divisible if for all $n \in \mathbb{N}$, there exists another probability law $\mu_n$ of $X_n^n$ such that*

$$\mu = \mu_n^{*n},$$

*where $\mu_n^{*n}$ is the n-fold convolution of $\mu_n$.*

The celebrated Lévy–Khinchin formula presents the characterization of random variables with infinitely divisible distributions via their characteristic functions.

**Theorem 4.1.2** (Lévy–Khinchin). *The probability law $\mu$ of a real-valued random variable $X$ is infinitely divisible if and only if there exists a unique characteristic triplet $(a, \sigma, \nu)$, where $a \in \mathbb{R}, \sigma \geq 0$ and $\nu$ is a measure on $\mathbb{R}$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge x^2)\nu(\mathrm{d}x) < \infty$, such that the characteristic function of $X$ is*

$$\mathbb{E}(e^{iuX}) = \exp\Big(iau - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \big(e^{ixu} - 1 - ixu\mathbf{1}_{|x|<1}(x)\big)\nu(\mathrm{d}x)\Big). \quad (4.2)$$

For the proof of the theorem, see [47] or [73]. Here $(a, \sigma, \nu)$ is called the Lévy triplet or the characteristic triplet, and $\nu$ is called the Lévy measure.

Now consider a Lévy process $L$. For any $n \in \mathbb{N}$ and any $t \in [0, T]$, we have

$$L_t = L_{\frac{t}{n}} + (L_{\frac{2t}{n}} - L_{\frac{t}{n}}) + \ldots + (L_{\frac{nt}{n}} - L_{\frac{(n-1)t}{n}}).$$

Due to the property of stationary and independent increments of Lévy processes, $(L_{\frac{kt}{n}} - L_{\frac{(k-1)t}{n}})_{k=1,\ldots,n}$ is an i.i.d. sequence of random variables. Therefore the random variable $L_t$ has infinitely divisible distribution for every $t \in [0, T]$.

### 4.1.2 Gaussian Processes

Gaussian processes have many applications in different fields due to the central limit theorem and many properties they posses. There are many references for Gaussian processes, including for example [1, 31, 36, 49, 56]. Among many Gaussian processes, I will only discuss fractional Brownian motions and consider some properties of fractional Brownian motions in this section.

**Definition 4.1.20.** *A stochastic process $X = (X_t)_{t\in[0,T]}$ is called Gaussian if for any finite collection of time points $t_1, \ldots, t_n \in [0, T]$, the random vector $(X_{t_1}, \ldots, X_{t_n})$ is a multivariate Gaussian random variable.*

From [7], we know that for Gaussian processes, the Kolmogorov–Chentsov condition is also necessary for Hölder continuity.

**Proposition 4.1.2.** *A Gaussian process $X$ has an a.s. Hölder continuous modification $\tilde{X}$ of any order $\alpha < H$, i.e.*

$$|\tilde{X}_t - \tilde{X}_s| \leq C_\epsilon |t - s|^{H-\epsilon}, \epsilon > 0, \quad (4.3)$$

*if and only if there exist constants $c_\epsilon$ such that*

$$d_X(t, s) \leq c_\epsilon |t - s|^{H-\epsilon}, \epsilon > 0,$$

*where $d_X(t, s) := \big(\mathbb{E}(X_t - X_s)^2\big)^{1/2}$ for $s, t \in [0, T]$. Moreover, the random variables $C_\epsilon$ in (4.3) satisfy*

$$\mathbb{E}[\exp(aC_\epsilon^k)] < \infty,$$

*for any constants $a \in \mathbb{R}$ and $k < 2$; also for $k = 2$ and small enough positive $\alpha$. In particular, the moments of all orders of $C_\epsilon$ are finite.*

***Fractional Brownian motion***

Fractional Brownian motions have been widely applied to many areas including mathematical finance and physics models. The study of fractional Brownian motion goes back to Kolmogorov and Yaglom, and for more details, see [44, 55, 88].

**Definition 4.1.21.** *A fractional Brownian motion with Hurst index $H \in (0,1)$ is a Gaussian process $B^H = (B^H_t)_{t\in[0,T]}$ satisfying the following properties*

*1) $B^H_0 = 0$,*

*2) $\mathbb{E}[B^H_t] = 0$, $t \in [0,T]$,*

*3) $\mathbb{E}[B^H_t B^H_s] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$, $s,t \in [0,T]$.*

It is evident from the definition that the covariance of a fractional Brownian motion is homogeneous of order $2H$, and hence it follows that $B^H$ is $H$-self similar. Moreover, $B^H$ has stationary increments since $\mathbb{E}|B^H_t - B^H_s|^2 = |t-s|^{2H}$ (for a detailed proof, see [48]).

**Definition 4.1.22.** *A stationary sequence $(Z_n)_{n\in\mathbb{N}}$ with*

$$Z_n := B^H_{n+1} - B^H_n,$$

*where $B^H$ is a fractional Brownian motion with Hurst index $H$, is called a fractional Gaussian noise with Hurst index $H$.*

**Proposition 4.1.3.** *For a fractional Gaussian noise $(Z_n)_{n\in\mathbb{N}}$, the covariance function is*

$$\rho(n) := \mathbb{E}[Z_{n+k}Z_k] = \frac{1}{2}\big((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\big), \quad n \in \mathbb{N}.$$

*For $H \neq \frac{1}{2}$, we have*

$$\rho(n) \sim H(2H-1)n^{2H-2},$$

*as $n \to \infty$. Therefore, for $n > 0$, we have*

$$\rho(n) \quad \begin{cases} < 0, & H \in (0, \frac{1}{2}), \\ = 0, & H = \frac{1}{2}, \\ > 0, & H \in (\frac{1}{2}, 1). \end{cases}$$

**Proposition 4.1.4.** *For fractional Gaussian noise* $(Z_n)_{n\in\mathbb{N}}$ *with covariance function* $\rho(n)$*, we have*

$$\begin{cases} \displaystyle\sum_{n=1}^{\infty}\rho(n) = \infty, & \textit{if } H \in (\frac{1}{2}, 1), \quad (\textit{long-range dependence}) \\ \displaystyle\sum_{n=1}^{\infty}|\rho(n)| < \infty, & \textit{if } H \in (0, \frac{1}{2}). \quad (\textit{short-range dependence}) \end{cases}$$

The preceding propositions show that when $H \in (\frac{1}{2}, 1)$, $B^H$ exhibits long-range dependence; while for $H \in (0, \frac{1}{2})$, $B^H$ exhibits short-range dependence.

Next, let us look at the path property of a fractional Brownian motion.

**Proposition 4.1.5.** *Fractional Brownian motion* $B^H$ *has a continuous modification whose paths are locally* $\lambda$*-Hölder continuous for any* $\lambda < H$*. Moreover for any* $\lambda \geq H$*, the trajectories of fractional Brownian motion are almost surely nowhere* $\lambda$*-Hölder continuous on any interval.*

*Proof.* For any $\alpha > 0$, by the self-similarity and stationary increments properties of $B^H$, we have

$$\mathbb{E}[|B_t^H - B_s^H|^\alpha] = \mathbb{E}\big[\big||t-s|^H B_1^H\big|^\alpha\big] = |t-s|^{\alpha H}\mathbb{E}[|B_1^H|^\alpha].$$

Therefore by Kolmogrov continuity theorem 4.1.1, $B^H$ has a continuous modification whose paths are of locally $\lambda$-Hölder continuous for $\lambda < H$.

According to [4], a fractional Brownian motion satisfies the following law of the iterated logarithm

$$\mathbb{P}\Big(\lim_{t\downarrow 0}\frac{B_t^H}{t^H\sqrt{\ln\ln(1/t)}} = 1\Big) = 1.$$

Therefore it follows that the trajectories of $B^H$ cannot be $\lambda$-Hölder continuous with $\lambda \geq H$. □

**Proposition 4.1.6.** *If* $H > \frac{1}{2}$*, then the quadratic variation of a fractional Brownian motion along any sequence of partitions with mesh converging to* 0 *is* $\langle B^H, B^H\rangle_t = 0$*; if* $H < \frac{1}{2}$*, then* $\langle B^H, B^H\rangle_t$ *does not exist. Moreover,* $B^H$ *is of unbounded variation almost surely.*

*Proof.* When $H > 1/2$, take $\alpha \in (1/2, H)$. According to the Hölder continuity of $B^H$, we know that $B^H$ is $\alpha$-Hölder continuous for $\alpha \in (1/2, H)$. Therefore for any sequence $(\pi_n)$ of partitions on $[0, T]$ such that $|\pi_n| \to 0$, we obtain

$$\begin{aligned} \langle B^H, B^H\rangle_T &= \lim_{|\pi_n|\to 0}\sum_{t_k\in\pi_n}(B_{t_k}^H - B_{t_{k-1}}^H)^2 \\ &\leq C^2(\omega)\lim_{|\pi_n|\to 0}\sum_{t_k\in\pi_n}(t_k - t_{k-1})^{2\alpha} \\ &\leq C^2(\omega)\lim_{|\pi_n|\to 0}|\pi_n|^{2\alpha-1}\sum_{t_k\in\pi_n}(t_k - t_{k-1}) \\ &= 0, \end{aligned}$$

as $n \to \infty$ almost surely, where $C$ is a random Hölder coefficient of $B^H$.

Consider a partition of $[0,T]$ denoted as $\pi = \{0 = t_0^n < t_1^n < \ldots < t_{k(n)}^n = T\}$, where $t_k^n = \frac{kT}{n}, 0 \le k \le n$. Then by the self-similarity property of $B^H$, we have

$$|B^H_{t_k^n} - B^H_{t_{k-1}^n}| \stackrel{d}{=} \left(\frac{T}{n}\right)^H |B^H_k - B^H_{k-1}|.$$

Therefore, the $p$-variation of $B^H$ along the partition $\pi$ is

$$\begin{aligned} v_p(B^H,\pi) &= \sum_{k=1}^n |B^H_{t_k^n} - B^H_{t_{k-1}^n}|^p \stackrel{d}{=} (\frac{T}{n})^{Hp} \sum_{k=1}^n |B^H_k - B^H_{k-1}|^p \\ &= T^{pH} n^{1-pH} \frac{1}{n} \sum_{k=1}^n |B^H_k - B^H_{k-1}|^p \\ &\to \begin{cases} \infty, & p < \frac{1}{H}, \\ T\mathbb{E}|B_1^H|^{1/H}, & p = \frac{1}{H}, \\ 0, & p > \frac{1}{H}, \end{cases} \end{aligned}$$

as $n \to \infty$. The convergence can be shown to hold almost surely and in $L^2(\Omega, \mathbb{P})$ by a result from ergodic theory. □

Finally we will mention the well known non-semimartingale property of a fractional Brownian motion with $H \neq \frac{1}{2}$.

**Proposition 4.1.7.** *A fractional Brownian motion $B^H$ is a semimartingale if and only if $H = \frac{1}{2}$.*

For a proof, see [51] or [66].

## 4.2 Pathwise Stochastic Integration

For integration with respect to semimartingales, we can simply apply the well-known Itō integration theory. However, for other processes such as a fractional Brownian motion with Hurst index $H \neq \frac{1}{2}$, or other processes which are not semimartingales, Itō integration theory cannot be applied. Thus, we need other approaches to define integrals with respect to nonsemimartingale stochastic processes. One possible way is to define the integral path by path ($\omega$ by $\omega$). Therefore, properties of sample paths become one of the most important issues to study for pathwise stochastic integration.

Next, I will discuss three different approaches to pathwise integration of stochastic processes. The first is a forward type integral introduced by Föllmer in 1981 [22]. The second is based on the bounded $p$-variation of paths which is introduced by Young in 1936 [89]. The third is the generalized Lebesgue–Stieltjes integral introduced by Zähle in 1998 [90] and further developed by

Nualart and Răşcanu in 2002 [62]. Rough path theory introduced by Lyons [52] can be applied to integration of processes with bounded $p$- and $q$-variation paths beyond the case $\frac{1}{p} + \frac{1}{q} > 1$. However, rough path theory cannot be applied to processes of unbounded $p$-variation for $p \geq 1$, which is the main focus of this dissertation.

### 4.2.1 Föllmer Integrals

Föllmer showed in [22] (see also Sondermann [80]) that stochastic calculus for quadratic variation processes can be developed path-by-path without probability.

Recall that for a fixed $\omega \in \Omega$, the path of stochastic process $X$ is just a real-valued function of $t$. In this section, let $x(t)$ be a real-valued function on $[0, T]$ which is right-continuous with left limits, and denote

$$x_t = x(t), \quad \Delta x_t = x_t - x_{t-}, \quad \Delta x_t^2 = (\Delta x_t)^2.$$

**Definition 4.2.1.** *Let $(\pi^n)$ be a sequence of partitions $\pi^n = \{0 \leq t_0^n < \ldots < t_{k(n)}^n \leq T\}$ such that $|\pi^n| = \max_{j=1,\ldots,k(n)} |t_j^n - t_{j-1}^n| \to 0$ as $n \to \infty$. Let $\delta_t$ be the Dirac measure at $t$ for $t \in [0, T]$. We say that $x$ is of quadratic variation along $(\pi^n)$ if the measures*

$$\xi_n = \sum_{t_i^n \in \pi^n} (x_{t_i^n} - x_{t_{i-1}^n})^2 \delta_{t_{i-1}}$$

*converge weakly to a finite Radon measure $\xi$ on $[0, T]$, where the atomic part of $\xi$ is given by the quadratic jumps of $x$ along $(\pi^n)$:*

$$[x, x]_t = [x, x]_t^c + \sum_{s \leq t} \Delta x_s^2.$$

*Here $[x, x]$ denotes the distribution function of $\xi$ and $[x, x]_t^c$ denotes the continuous part of $\xi$.*

The Föllmer integral is defined as follows.

**Definition 4.2.2.** *Let $(\pi^n)$ be a sequence of partitions $\pi^n = \{0 \leq t_0^n < \ldots < t_{k(n)}^n \leq T\}$ such that $|\pi^n| = \max_{j=1,\ldots,k(n)} |t_j^n - t_{j-1}^n| \to 0$ as $n \to \infty$. Let $f$ be a real-valued function and $x$ be a real-valued function which is right-continuous with left limits on $[0, T]$. The Föllmer integral of $f$ with respect to $x$ over an interval $[0, t]$ along the sequence of partitions $(\pi^n)$ for $t \in [0, T]$ is defined as*

$$(\mathrm{F}) \int_0^t f(x_s)\, \mathrm{d}x_s = \lim_{n \to \infty} \sum_{t_j^n \in \pi^n \cap (0, t]} f(x_{t_{j-1}^n})(x_{t_j^n} - x_{t_{j-1}^n}), \tag{4.4}$$

*if the limit exists.*

For details about the Föllmer integral, see [80].

**Lemma 4.2.1.** *Let $x$ be right-continuous with left limits and of quadratic variation along a sequence of partitions $(\pi^n)$, and let $f$ be a function in $C^2(\mathbb{R})$. Then a pathwise Itō formula can be written as*

$$f(x_t) - f(x_0) = (\mathrm{F}) \int_0^t f'(x_{s-})\,\mathrm{d}x_s + \frac{1}{2}\int_0^t f''(x_s)\,\mathrm{d}[x,x]_s + \sum_{s\leq t}\big(f(x_s) - f(x_{s-}) - f'(x_{s-})\Delta x_s - \frac{1}{2}f''(x_{s-})\Delta x_s^2\big).$$

*Here the second integral on the right side is a Lebesgue integral with respect to the finite Radon measure $\xi$ defined in Definition 4.2.1. In particular, the Föllmer integral exists along the sequence of partitions $(\pi^n)$.*

The Föllmer integral is a forward-type Riemann–Stieltjes integral, therefore the existence of a Riemann–Stieltjes integral implies the existence of the corresponding Föllmer integral. In general, the existence of Föllmer integral is hard to prove. In some special cases such as in the case of processes with finite quadratic variation, the existence can be proved.

In the following, let us consider stochastic processes instead of real-valued functions. The existence of Föllmer integrals can be shown for quadratic variation processes via the following lemma.

**Lemma 4.2.2.** *Let $(\pi^n)$ be a sequence of partitions $\pi^n = \{0 \leq t_0^n < \ldots < t_{k(n)}^n \leq T\}$ such that $|\pi^n| = \max_{j=1,\ldots,k(n)} |t_j^n - t_{j-1}^n| \to 0$ as $n \to \infty$. Let $X$ be a continuous quadratic variation process along the sequence of partitions $(\pi^n)$ and let $f \in C^{1,2}([0,T]\times\mathbb{R})$. For $0 \leq s < t \leq T$,*

$$f(t, X_t) = f(s, X_s) + \int_s^t \frac{\partial f}{\partial t}(u, X_u)\,\mathrm{d}u + (\mathrm{F})\int_s^t \frac{\partial f}{\partial x}(u, X_u)\,\mathrm{d}X_u + \frac{1}{2}\int_s^t \frac{\partial^2 f}{\partial x^2}(u, X_u)\,\mathrm{d}\langle X\rangle_u. \quad a.s.$$

*Note that the last integral on the right-hand side of the above equation is a Lebesgue integral. In particular, the Föllmer integral exists along the sequence of partitions $(\pi^n)$ and has a continuous modification.*

For a proof and details, see [80].

**Remark 4.2.1.** *Note that in the above result, the existence of the Föllmer integral is a consequence of the existence of other terms. Hence the existence of the integral is not proved directly but it is rather a consequence of the pathwise Itō formula.*

An interesting example where the Föllmer integral exists and the Itō integral does not exist is a bifractional Brownian motion. In [32], the authors introduced bifractional Brownian motions using the following definition.

**Definition 4.2.3.** *A bifractional Brownian motion* $B^{H,K} = (B_t^{H,K})_{t\in[0,T]}$ *is a centered Gaussian process starting from zero with covariance function*

$$R_{H,K}(t,s) = \frac{1}{2^K}\Big(\big(t^{2H} + s^{2H}\big)^K - |t-s|^{2HK}\Big),$$

*where* $H \in (0,1)$ *and* $K \in (0,1]$*.*

If $K = 1$, then $B^{H,1}$ is just a fractional Brownian motion with Hurst index $H \in (0,1)$. If $2HK = 1$ and $K = 1$, then $H = \frac{1}{2}$ and $B^{1/2,1}$ is a Brownian motion.

A bifractional Brownian motion is not a semimartingale in the case when $\frac{1}{2} < HK < 1$ and in the case when $2HK = 1$ with $K \neq 1$. The quadratic variation of a bifractional Brownian motion along any sequence of partitions $(\pi^n)$ such that $|\pi^n| \to 0$ is zero for $\frac{1}{2} < HK < 1$. In [69], the authors showed that in the case when $2HK = 1$ with $K \neq 1$, the bifractional Brownian motion has finite non-trivial quadratic variation along any sequence of partitions $(\pi^n)$ such that $|\pi^n| \to 0$. Therefore Föllmer integral exists for $\frac{1}{2} < HK < 1$ and for $2HK = 1$ with $K \neq 1$.

### 4.2.2 Young Integrals

Recall from Section 3.1.1 that Young's integral can be understood as a Riemann–Stieltjes integral for the case when the paths of a stochastic process are Hölder continuous or, slightly more generally, of bounded $p$-variation for $p \geq 1$ with certain restrictions. In the following, we will give several examples of Young's integration applied to different stochastic processes with bounded $p$-variation paths.

It is well known that almost all sample paths of a Brownian motion are of unbounded variation. However, they are of bounded $p$-variation on any bounded interval for $p > 2$ (for details, see [83]). In fact, from [18], we know that almost all paths of a right-continuous martingale are of locally bounded $p$-variation for $p > 2$, and therefore almost all paths of a semimartingale are of locally bounded $p$-variation for $p > 2$.

Recall that Young proved the existence of Riemann–Stieltjes integrals for bounded $p$-variation and bounded $q$-variation functions with no common discontinuities for $p \geq 1$, $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} > 1$. Since almost all paths of a semimartingale are of unbounded $p$-variation for $p \leq 2$, Young's integration cannot be applied in this case. However, standard Itō integration can be applied in this case.

Now consider a general Gaussian process $X = (X_t)_{t\in[0,T]}$, and denote

$\sigma_X(s,t) := \mathbb{E}|X_s - X_t|$ for $s,t \in [0,T]$. Define

$$G(\sigma_X; p) := \sup_{\pi^n}\{\sum_{t_i\in\pi^n} \sigma_X(t_{i-1},t_i)^p\},$$

where the supremum is taken over all finite partitions $\pi^n$ on $[0,T]$. For a real-valued function $f$ on $[a,b]$, define the index of $p$-variation of $f$ as

$$v(f;[a,b]) := \inf\{p \geq 1 : v_p(f;[a,b]) < \infty\},$$

and if the set is empty, then let $v(f;[a,b]) = \infty$.

For the $p$-variation index of the sample paths of an centered Gaussian process $X$, we have the following proposition taken from [39].

**Proposition 4.2.1.** *If $q > p^*$, where $p^* = \inf\{p \geq 1 : G(\sigma_X; p) < \infty\}$, then $X$ has bounded $q$-variation on $[0,T]$ almost surely. Conversely, if $q < p^*$, then $X$ has unbounded $q$-variation on $[0,T]$ almost surely.*

Many processes do satisfy this path boundedness, and I will present some examples. Consider a fractional Brownian motion $B^H$ with index $H \in (0,1)$. Almost all paths of $B^H$ are of bounded $p$-variation for any $p > \frac{1}{H}$, and for $p < \frac{1}{H}$, almost all paths of $B^H$ are of unbounded $p$-variation. Moreover, the $p$-variation index of $B^H$ with Hurst index $H$ is

$$v(B^H) = \frac{1}{H}$$

with probability one. Therefore, Young's integration can be applied for a stochastic integral with respect to $B^H$ with Hurst index $H \in (\frac{1}{2},1)$ (see [57]). However, if the integrand fails to be smooth enough or it is of unbounded $p$-variation, Young's integration will no longer be appropriate.

Next, let us consider paths of Lévy processes. The $p$-variation of the paths of a Lévy process has been studied in [9]. For a Lévy process with the characteristic function (4.2) with $\sigma = 0$, the $p$-variation with $1 < p < 2$ is bounded with probability one, if and only if the integral

$$\int_{\mathbb{R}\setminus\{0\}} (1 \wedge |x|^p)\, \nu(\mathrm{d}x)$$

is finite, where $\nu$ is a Lévy measure. If the above integral is infinite, then the $p$-variation of the corresponding Lévy process is unbounded almost surely.

**Definition 4.2.4.** *A Lévy process $L$ is called an $\alpha$-stable Lévy motion of index $\alpha$ if $L$ has the characteristic function (4.2) with $\sigma = 0$, and the Lévy measure $\nu$ satisfies*

$$\nu(\mathrm{d}x) = \begin{cases} rx^{-1-\alpha}\,\mathrm{d}x, & x > 0, \\ q(-x)^{-1-\alpha}\,\mathrm{d}x, & x < 0, \end{cases}$$

*for $\alpha \in (0,2)$ and $r, q \geq 0$ with $r + q > 0$.*

The $p$-variation of an $\alpha$-stable Lévy motion was studied in [24], and the authors obtained the following result.

**Proposition 4.2.2.** *Let $L$ be an $\alpha$-stable Lévy motion of index $\alpha \in (0, 2)$ which has no drift for $\alpha < 1$ and the Lévy measure is symmetric for $\alpha = 1$. Then $L$ has bounded $p$-variation for $1 < p < 2$ with probability 1 if $p > \alpha$, and unbounded $p$-variation for $p \leq \alpha$.*

Integration with respect to this kind of processes has been discussed in [57] for $p > \alpha$. However, when $p \leq \alpha$, $L$ has unbounded $p$-variation, and hence Young's integration cannot be applied.

The above examples show that the properties of paths can be very useful for Young's integration. However, there is a limitation of Young's integration since processes need to be of bounded $p$-variation for $p \geq 1$. For processes with unbounded $p$-variation, Young's integration is not applicable. Moreover, in Young's integration theory, integrands are also limited to be of bounded $p$-variation which should be relaxed to more general class of processes.

**Remark 4.2.2.** *Note that rough path theory introduced by Lyons [52] can be applied to the integration of processes with bounded $p$- and $q$-variation paths beyond the case $\frac{1}{p} + \frac{1}{q} > 1$. Especially, the theory has been successfully applied to study differential equations with smooth coefficients and a non-regular driving process, i.e. a process which is $\alpha$-Hölder continuous with some $\alpha < \frac{1}{2}$ (although a restriction $\alpha > \frac{1}{4}$ appears in many of the cases). There exists an abundance of literature on rough path theory, see [25, 53, 54]. However, rough path theory cannot be applied to processes which have unbounded $p$-variation paths for any $p \geq 1$, which is the main focus of this dissertation.*

### 4.2.3 Generalized Lebesgue–Stieltjes Integrals

Recall that we have discussed generalized Lebesgue–Stieltjes integrals in Chapter 3.2.2. Next we will review some results regarding to the existence of pathwise generalized Lebesgue–Stieltjes integrals for stochastic processes with unbounded $p$-variation. Azmoodeh, Mishura and Valkeila in 2010 [6] proved the existence of the generalized Lebesgue–Stieltjes integral for geometric fractional Brownian motions. Later Tikanmäki in 2012 [84] showed the existence of the generalized Lebesgue–Stieltjes integral for functionals of fractional Brownian motions. Sottinen and Viitasaari in 2014 [81] presented the existence of the generalized Lebesgue–Stieltjes integral for a general class of Gaussian processes. These appear to be the only known results for processes with unbounded

$p$-variation, however, they all contain some gaps in the proof of theorems of change of variables formulas and Riemann–Stieltjes integrals, which will be explained in Chapter 5. However, in Chapter 5, we will show that although the proofs contain some gaps, the results they obtained are still valid.

#### Existence of gL–S Integrals for Fractional Brownian Motions

Recall from Section 4.1.2 that fractional Brownian motion $B^H$ is not a semimartingale when the Hurst index $H \neq \frac{1}{2}$. Therefore standard Itō integration theory cannot be applied in the case of a fractional Brownian motion.

In the paper [6], the authors showed that the integral can be understood in the sense of a generalized Lebesgue–Stieltjes integral for geometric fractional Brownian motions. I will present their main result here.

**Theorem 4.2.1.** *[6] Let $S_t = e^{B_t^H}$ be a geometric fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$ for $t \in [0, T]$. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a convex function. Then the pathwise integral*

$$(\text{gLS}) \int_0^T f'_-(S_t) S_t \, \mathrm{d}B_t^H$$

*exists almost surely.*

#### Existence of gL–S Integrals for Functionals of Fractional Brownian Motions

Functional Itō calculus was studied by Dupire in 2010 [19], and later developed by Cont and Fornié in [12, 13, 23]. With the help of a functional change of variables formula introduced in [12], Tikanmäki in 2012 [84] showed the existence of a pathwise integral for functionals of fractional Brownian motions with respect to fractional Brownian motions in the generalized Lebesgue–Stieltjes sense.

Let $S_t = e^{B_t^H}$ be a geometric fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$ for $t \in [0, T]$, and denote

$$G_t = \exp\Big(\frac{1}{T} \int_0^t \log S_s \, \mathrm{d}s\Big) S_t^{\frac{T-t}{T}}.$$

Then we have the following theorem taken from [84].

**Theorem 4.2.2.** *[84] Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a convex function. Then for any $t \in [0, T]$, the integral*

$$(\text{gLS}) \int_0^t \frac{T-s}{T} f'_-(G_s) G_s \, \mathrm{d}B_s^H$$

*exists almost surely.*

Moreover, we have an analogous theorem for arithmetic averages.

**Theorem 4.2.3.** *[84] Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a convex function. Then for any $t \in [0, T]$, the integral*

$$(\text{gLS}) \int_0^t f'_-\Big(\frac{T-s}{T} S_s + \frac{1}{T}\int_0^s S_u \, \mathrm{d}u\Big)\frac{T-s}{T} S_s \, \mathrm{d}B_s^H$$

*exists almost surely.*

The author also proved a change of variables formulas for functionals of geometric averages and arithmetic averages separately.

**Existence of gL–S Integrals for Gaussian Processes**

In the paper of Sottinen and Viitasaari in 2014 [81], the authors proved the existence of generalized Lebesgue–Stieltjes integrals for a general class of Gaussian processes.

Let $X = (X_t)_{t\in[0,T]}$ and $Y = (Y_t)_{t\in[0,T]}$ be two stochastic processes and consider the following notations

$$\begin{aligned} R(t,s) &= \mathbb{E}[X_t X_s], \\ W(t,s) &= \mathbb{E}[(X_t - X_s)^2], \\ V(t) &= \mathbb{E}[X_t^2], \\ w^*(t) &= \sup_{0\le s\le T-t} W(t+s, s). \end{aligned}$$

The authors in [81] consider a general class of Gaussian processes defined as follows.

**Definition 4.2.5.** *For $0 < \alpha < 1$, we say that a centered continuous Gaussian process $X$ belongs to the class $\mathcal{X}^\alpha$ if*

*1) $R(t,s) > 0$ for every $s, t > 0$,*

*2) the incremental variance as $t \to 0$ satisfies*

$$w^*(t) = Ct^{2\alpha} + o(t^{2\alpha}),$$

*where $C > 0$ is a constant,*

*3) there exist a $\delta > 0$ such that when $s \le \delta$,*

$$V(s) \ge cs^2,$$

*where $c > 0$ is a constant,*

*4) there exists a* $\delta > 0$ *such that*

$$\sup_{0<t<2\delta} \sup_{t/2\leq s\leq t} \frac{R(s,s)}{R(t,s)} < \infty.$$

Here are some examples of processes that belong to the class $\mathcal{X}^\alpha$.

**Example 4.2.1.** *Note that for processes with stationary increments,*

$$\begin{aligned} R(t,s) &= \frac{1}{2}[V(t)+V(s)-V(t-s)], \\ W(t,s) &= V(t-s), \\ w^*(t) &= V(t). \end{aligned}$$

*A zero mean continuous Gaussian process with stationary increments belongs to class* $\mathcal{X}^\alpha$ *if and only if*

$$V(t) > 0,$$

*for all* $t > 0$, *and*

$$V(t) = Ct^{2\alpha} + o(t^{2\alpha}),$$

*as* $t \to 0$.

*Fractional Brownian motions with index* $\alpha \in (0,1)$ *belong to the class* $\mathcal{X}^\alpha$.

**Example 4.2.2.** *For stationary processes*

$$\begin{aligned} R(t,s) &= r(t-s), \\ W(t,s) &= 2[r(0)-r(t-s)], \\ V(t) &= r(0), \\ w^*(t) &= 2[r(0)-r(t)]. \end{aligned}$$

*Thus, a stationary process belongs to the class* $\mathcal{X}^\alpha$ *if and only if*

$$r(t) > 0,$$

*for all* $t$, *and*

$$r(0) - r(t) = Ct^{2\alpha} + o(t^{2\alpha}),$$

*as* $t \to 0$.

*Fractional Ornstein-Uhlenbeck processes with index* $\alpha \in (0,1)$ *belong to the class* $\mathcal{X}^\alpha$ *( for fractional Ornstein-Uhlenbeck processes, see [11]).*

The most important requirement of class $\mathcal{X}^\alpha$ is that the Gaussian process $X$ has a version whose trajectories are Hölder continuous of order $\lambda$ for any $\lambda < \alpha$ on $[0,T]$. Moreover, this class of Gaussian processes should not be too smooth with some mild assumptions on covariance and variance.

For the class $\mathcal{X}^\alpha$, the authors in [81] proved the following main result.

**Theorem 4.2.4.** *Let $f$ be a linear combination of real-valued convex functions on $\mathbb{R}$. Let $X$, $Y$ be two stochastic processes such that $Y \in W_T^{1-\beta,\infty}$ and $X \in \mathcal{X}^\alpha$ for $\alpha > \frac{1}{3}$. If $\beta < \alpha \wedge (3\alpha - 1)$, then the integral*

$$(\mathrm{gLS}) \int_0^T f'_-(X_t)\, \mathrm{d}Y_t$$

*exists almost surely.*

They also showed that the generalized Lebesgue–Stieltjes integral in this case can be approximated by Riemann–Stieltjes sums. Moreover, they derived a change of variables formula. For more details, see [81].

# 5. Integration of Stochastic Processes of Unbounded Power Variation

In this chapter, we are going to study how to define a stochastic integral of the form $\int_0^T F(X_t)\,\mathrm{d}Y_t$, where $X, Y$ are some Hölder continuous processes and $F$ is a real-valued function of locally bounded variation. If $Y$ is not a semimartigale, then Itō integration theory cannot be applied. Moreover, since $F$ may contain discontinuities, according to Proposition 5.0.3 and Remark 5.0.3 below, paths of the process $F(X)$ may be of unbounded $p$-variation for $p \geq 1$. Therefore we need to determine in which sense the above integral exists.

Let $F$ be a function of locally bounded variation on $\mathbb{R}$, i.e. $F \in BV^{loc}(\mathbb{R})$. According to Theorem 2.2.1, we have $F = f^1 - f^2$, where $f^1$ and $f^2$ are increasing functions. By Proposition 2.1.1, for increasing functions $f^1$ and $f^2$ there exist $f^1_+$ and $f^2_+$, which are right-continuous and increasing so that $f^1 = f^1_+$ Lebesgue-almost everywhere and $f^2 = f^2_+$ Lebesgue-almost everywhere. Note that by Proposition 2.1.4, $f^1_+$ (resp. $f^2_+$) defines a unique Lebesgue–Stieltjes measure $\mu_1$ (resp. $\mu_2$). Since for the main results we will consider generalized Lebesgue–Stieltjes integrals related to $F$, by linearity of the generalized Lebesgue–Stieltjes integral, instead of $F$ it is sufficient to consider $f^1$. Moreover, according to 3) in Proposition 3.2.3 and the fact that $f^1 = f^1_+$ Lebesgue-almost everywhere, instead of $f^1$ it is sufficient to consider $f^1_+$. In the following, we will let $f$ denote such an increasing, right-continuous function, and let $\mu$ denote the Lebesgue–Stieltjes measure of $f$.

If $\mu$ of $f$ has a compact support, then by Remark 2.2.1, $f$ together with $\mu$ satisfy the representation (2.6). Now assume that the support of $\mu$ is not compact. Define a function $\tilde{f}_n$ for any $n \in \mathbb{N}$ by

$$\tilde{f}_n(x) = \begin{cases} f(-n), & \text{if} \quad x < -n, \\ f(x), & \text{if} \quad -n \leq x \leq n, \\ f(n), & \text{if} \quad x > n. \end{cases}$$

Then $\tilde{f}_n$ is increasing, right-continuous and bounded for every $n \in \mathbb{N}$. Hence the Lebesgue–Stieltjes measure $\mu_n$ of $\tilde{f}_n$ has a compact support in $[-n,n]$ for every $n \in \mathbb{N}$. Note that $\mathbb{R} = \cup_{n\in\mathbb{N}}[-n,n]$, and for every $n \in \mathbb{N}$, we have $\tilde{f}_n = f$ on $[-n,n]$. In the following, we will consider a function $f \circ h$, where $h$ is a continuous function on $[0,T]$. Since $h$ is a bounded function on a compact set, one can always find some $n$ so that there exists a compact set $\mathcal{K}_n$ which satisfies $[\inf_{t\in[0,T]} h(t), \sup_{t\in[0,T]} h(t)] \subset \mathcal{K}_n$. Moreover, we have $\tilde{f}_n \circ h = f \circ h$ on $[0,T]$. Therefore, we can always assume that the Lebesgue–Stieltjes measure $\mu$ of $f$ has a compact support.

According to the above arguments, in order to prove the main results, instead of a locally bounded variation function, it is enough for us to consider an increasing, right-continuous function for which the Lebesgue–Stieltjes measure has a compact support. In the following, we denote such a function as $f$, and assume that the Lebesgue–Stieltjes measure $\mu$ of $f$ has a compact support $\mathcal{K}$. Therefore by representation (2.6), we have

$$f(x) = \frac{1}{2}\int_{\mathcal{K}} \operatorname{sgn}(x-a)\,\mu(\mathrm{d}a) + C, \tag{5.1}$$

where $C = \frac{1}{2}\mu((-\infty,\infty)) - \mu((-\infty,0]) + f(0)$.

For a continuous function $h$ defined on $[0,T]$, let us consider $g(x) := \mathbf{1}_{\{h(x)\geq c\}}$, where $c$ is a point in $\mathbb{R}$. The following proposition shows that $g$ may be of unbounded $p$-variation for $p \geq 1$.

**Proposition 5.0.3.** *Let $c \in \mathbb{R}$ be fixed. A function $g(x) := \mathbf{1}_{\{h(x)\geq c\}}$ defined on $[0,T]$ is of bounded $p$-variation for all $1 \leq p < \infty$ if and only if $h$ crosses level $c$ finitely many times.*

*Proof.* For $0 \leq s < t \leq T$, we have

$$\begin{aligned} g(t) - g(s) &= \mathbf{1}_{\{h(t)\geq c\}} - \mathbf{1}_{\{h(s)\geq c\}} \\ &= \mathbf{1}_{\{h(s)<c\leq h(t)\}} - \mathbf{1}_{\{h(t)<c\leq h(s)\}}. \end{aligned} \tag{5.2}$$

Let $\pi^n$ be any finite partition on $[0,T]$ such that

$$\pi^n = \{0 = t_0^n < t_1^n < \ldots < t_{k(n)}^n = T\}.$$

Let $(\pi^n)$ denote a sequence of partitions $\pi^n$ on $[0,T]$. Recall from Chapter 2.3 that the $p$-variation of $g$ on $[0,T]$ for $p \geq 1$ is defined as

$$v_p(g;[0,T]) = \sup_{(\pi^n)} v_p(g;\pi^n) = \sup_{(\pi^n)} \sum_{t_i\in\pi^n} \left|g(t_i) - g(t_{i-1})\right|^p,$$

where the supremum is taken over all finite partitions on $[0,T]$. For $g(x) = \mathbf{1}_{\{h(x)\geq c\}}$, by (5.2) we have

$$v_p(g;[0,T]) = \sup_{(\pi^n)} \sum_{t_i\in\pi^n} \left|\mathbf{1}_{\{h(t_{i-1})<c\leq h(t_i)\}} - \mathbf{1}_{\{h(t_i)<c\leq h(t_{i-1})\}}\right|^p. \tag{5.3}$$

At first, let $g$ be of bounded $p$-variation for $p \geq 1$ on $[0,T]$, i.e.

$$v_p(g;[0,T]) < \infty.$$

According to (5.3), we have

$$\sup_{(\pi^n)} \sum_{t_i \in \pi^n} \left| \mathbf{1}_{\{h(t_{i-1})<c\leq h(t_i)\}} - \mathbf{1}_{\{h(t_i)<c\leq h(t_{i-1})\}} \right|^p < \infty. \tag{5.4}$$

Now on the interval $[t_{i-1}, t_i]$, if either $h(t_{i-1}) < c \leq h(t_i)$ or $h(t_i) < c \leq h(t_{i-1})$ holds, then

$$\left| \mathbf{1}_{\{h(t_{i-1})<c\leq h(t_i)\}} - \mathbf{1}_{\{h(t_i)<c\leq h(t_{i-1})\}} \right|^p = 1.$$

Otherwise,

$$\left| \mathbf{1}_{\{h(t_{i-1})<c\leq h(t_i)\}} - \mathbf{1}_{\{h(t_i)<c\leq h(t_{i-1})\}} \right|^p = 0.$$

The finiteness of (5.4) implies that either $h$ never crosses level $c$ on $[0,T]$, or on finitely many time intervals $h$ crosses level $c$; otherwise (5.4) will become infinity. Hence if $g$ is of bounded $p$-variation, $h$ can only cross level $c$ finitely many times.

Conversely, let $h$ cross level $c$ finitely many times. If we have $\sup_{(\pi^n)} v_p(g;\pi^n) < \infty$, then $g$ is of bounded $p$-variation. Now assume that $\sup_{(\pi^n)} v_p(g;\pi^n) = \infty$ for $p \geq 1$, i.e.

$$\sup_{(\pi^n)} \sum_{t_i \in \pi^n} \left| \mathbf{1}_{\{h(t_{i-1})<c\leq h(t_i)\}} - \mathbf{1}_{\{h(t_i)<c\leq h(t_{i-1})\}} \right|^p = \infty. \tag{5.5}$$

Since

$$\left| \mathbf{1}_{\{h(t_{i-1})<c\leq h(t_i)\}} - \mathbf{1}_{\{h(t_i)<c\leq h(t_{i-1})\}} \right| = \begin{cases} 1, & h \text{ crosses level } c \text{ on } [t_{i-1}, t_i], \\ 0, & h \text{ does not cross level } c \text{ on } [t_{i-1}, t_i], \end{cases}$$

(5.5) implies that there are infinitely many intervals, on which $h$ crosses level $c$ either from above or from below, and this is a contradiction. Therefore, we obtain

$$v_p(g;[0,T]) < \infty.$$

□

Proposition 5.0.3 implies that if $h$ crosses level $c$ infinitely many times, then $g$ is of unbounded $p$-variation for every $p \geq 1$.

**Remark 5.0.3.** *Now consider a function $g(x) := f\big(h(x)\big)$, where $f$ is an increasing, right-continuous function on $\mathbb{R}$ and $h$ is a continuous function on*

$[0, T]$. *Note that the set of discontinuity points of $f$ is at most countable. If $h$ crosses a discontinuity point of $f$ infinitely many times, then by following similar steps in Proposition 5.0.3, we can see that $g$ is of unbounded p-variation for all $p \geq 1$.*

When the integrand is of unbounded $p$-variation for all $p \geq 1$, Young's integration cannot be applied. However, In Chapter 4, we have shown that generalized Lebesgue–Stieltjes integral can be applied for fractional Brownian motions and a general class of Gaussian processes. Now let us take a step forward to apply the generalized Lebesgue–Stieltjes integral for a more general class of processes with unbounded $p$-variation paths for $p \geq 1$.

## 5.1 Earlier Literature

In [6,81,84], integration of fractional Brownian motions, functionals of fractional Brownian motions and a class of Gaussian processes were discussed. To the best of my knowledge, they are the only existing results regarding the pathwise stochastic integration of unbounded $p$-variation processes. Results of integration of fractional Brownian motions and a general class of Gaussian processes have been applied in the papers [59, 75, 85].

Let $F$ be a function in $BV^{loc}(\mathbb{R})$. According to the statements in the beginning of this chapter, instead of $F$ it is sufficient for us to only consider an increasing and right-continuous function $f$, for which the Lebesgue–Stieltjes measure $\mu$ has a compact support $\mathcal{K}$.

Let $\phi$ be a positive function in $C^\infty(\mathbb{R})$ with a compact support in $[0, \infty)$ such that $\int_{\mathbb{R}} \phi(x) \mathrm{d}x = 1$. According to the mollification technique summarized in Chapter 2, one can define a sequence of functions

$$f_n(x) = n \int_0^\infty f(x + y)\phi(ny)\,\mathrm{d}y, \quad n \in \mathbb{N}. \tag{5.6}$$

Here $f_n \in C^\infty(\mathbb{R})$ and $f_n \to f$ pointwise as $n \to \infty$. Moreover, by (2.9) we have for every $g \in C_c^\infty(\mathbb{R})$

$$\lim_{n\to\infty} \int_{\mathbb{R}} g(x) f_n'(x)\,\mathrm{d}x = \int_{\mathbb{R}} g(x)\,\mu(\mathrm{d}x). \tag{5.7}$$

As a direct consequence of equation (5.7), we obtain the following lemma.

**Lemma 5.1.1.** *Let $g : [0, T] \to \mathbb{R}$ be any function, and let $f : \mathbb{R} \to \mathbb{R}$ be an increasing, right-continuous function such that the Lebesgue–Stieltjes measure $\mu$ of $f$ has a compact support $\mathcal{K}$. Let $f_n$ be defined as in (5.6). Then*

$$\int \mathbf{1}_{\{g(s)<a<g(t)\}}\,\mu(\mathrm{d}a) \leq \liminf_n \int \mathbf{1}_{\{g(s)<a<g(t)\}} f_n'(a)\,\mathrm{d}a.$$

*Proof.* The proof follows similar ideas as in the proof of the portmanteau theorem concerning the convergence of probability measures (see e.g. [43]).

Since $f$ is increasing on $\mathbb{R}$, $\mu$ is a nonnegative measure on $\mathbb{R}$. Moreover, note that $\mu$ has a compact support $\mathcal{K}$, therefore $f$ is a bounded function on $\mathbb{R}$ and $\mu$ is a finite measure. Since $f$ is increasing and bounded on $\mathbb{R}$, $f_n$ is also increasing and bounded on $\mathbb{R}$ for every $n \in \mathbb{N}$, which implies that the Lebesgue–Stieltjes measure $\mu_n$ of $f_n$ is a nonnegative finite measure on $\mathbb{R}$ for every $n \in \mathbb{N}$. Moreover, $\mu_n$ satisfies

$$\mu_n(\mathrm{d}x) = f_n'(x)\,\mathrm{d}x.$$

By the fact that $f_n \to f$ as $n \to \infty$, we obtain $\mu_n(-\infty,\infty) \to \mu(-\infty,\infty)$ as $n \to \infty$. Therefore, by normalizing, we can regard $\mu$ and $\mu_n$ as probability measures assuming they are nonzero.

According to Remark 18.1 in [38], if for probability measures $\mu$ and $\mu_n$,

$$\int g(x)\,\mu_n(\mathrm{d}x) \to \int g(x)\,\mu(\mathrm{d}x), \tag{5.8}$$

holds for all functions $g$ in $C_c^\infty(\mathbb{R})$, then

$$\mu_n \Rightarrow \mu,$$

where $\Rightarrow$ means converges weakly. Therefore, according to portmanteau theorem,

$$\int g(x)\,\mu_n(\mathrm{d}x) \to \int g(x)\,\mu(\mathrm{d}x),$$

holds for all bounded continuous functions $g$ on $\mathbb{R}$. Note that since equation (5.7) holds for all $g$ in $C_c^\infty(\mathbb{R})$, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}} g(x) f_n'(x)\,\mathrm{d}x = \int_{\mathbb{R}} g(x)\,\mu(\mathrm{d}x), \tag{5.9}$$

for all bounded continuous functions $g$ on $\mathbb{R}$.

For any open bounded set $G \subset \mathbb{R}$, define $l_N(x) = 1 \wedge (N \cdot d(x, G^c))$, where $d(x, G^c)$ is the distance from $x$ to the set $G^c$. For every $N$, $l_N$ is a continuous function with a compact support, and $0 \le l_N \le \mathbf{1}_G$, $l_N \uparrow \mathbf{1}_G$ as $N \to \infty$.

Now for an open set $G = (a, b)$, we can rewrite $l_N$ as

$$l_N(x) = N(x-a)\mathbf{1}_{\{a<x\le \frac{1}{N}+a\}} + \mathbf{1}_{\{\frac{1}{N}+a<x\le b-\frac{1}{N}\}} + N(b-x)\mathbf{1}_{\{b-\frac{1}{N}<x<b\}}.$$

Note that $l_N$ is a continuous bounded function on $\mathbb{R}$. Thus we can use equation (5.9) for $g = l_N$ to obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}} l_N(x) f_n'(x)\,\mathrm{d}x = \int_{\mathbb{R}} l_N(x)\,\mu(\mathrm{d}x).$$

Consequently we get

$$\int_{\mathbb{R}} l_N(x)\,\mu(\mathrm{d}x) = \liminf_n \int_{\mathbb{R}} l_N(x) f_n'(x)\,\mathrm{d}x \le \liminf_n \int_{\mathbb{R}} \mathbf{1}_G(x) f_n'(x)\,\mathrm{d}x.$$

Let now $N \to \infty$, we obtain

$$\mu(G) \le \liminf_n \int_{\mathbb{R}} \mathbf{1}_G(x) f_n'(x)\,\mathrm{d}x.$$

By choosing $G = \big(g(s), g(t)\big)$ where $g(s) < g(t)$, we obtain

$$\int_{\mathbb{R}} \mathbf{1}_{\{g(s)<a<g(t)\}}\,\mu(\mathrm{d}a) \le \liminf_n \int_{\mathbb{R}} \mathbf{1}_{\{g(s)<a<g(t)\}} f_n'(a)\,\mathrm{d}a.$$

□

**Remark 5.1.1.** *Note that by following a similar proof and choosing* $G = (g(t), g(s))$ *when* $g(t) < g(s)$, *it can be shown that Lemma 5.1.1 also holds for the case* $\mathbf{1}_{\{g(t)<a<g(s)\}}$.

Let now $B^H$ be a fractional Brownian motion with $H > \frac{1}{2}$, and $f$ be a real-valued convex function on $\mathbb{R}$. In [6], the authors proved the existence of the generalized Lebesgue–Stieltjes integral (gLS) $\int_0^T f_-'(B_t^H)\mathrm{d}B_t^H$, and they proved a change of variables formula

$$f(B_T^H) - f(B_0^H) = (\mathrm{gLS}) \int_0^T f_-'(B_t^H)\,\mathrm{d}B_t^H \quad a.s. \tag{5.10}$$

To obtain such a result, the authors applied a change of variables formula to a smooth approximation $f_n$ of the convex function $f$ defined as (5.6). Then by using fractional Sobolev-type space techniques, they showed the convergence of integrals

$$(\mathrm{gLS}) \int_0^T f_n'(B_t^H)\,\mathrm{d}B_t^H \to (\mathrm{gLS}) \int_0^T f_-'(B_t^H)\,\mathrm{d}B_t^H \quad a.s.$$

To prove such a result, one needs to find an integrable dominant for the difference $f_n' - f_-'$ in terms of the norm $||\cdot||_{1-\beta,1}$, where $\beta \in (1-H, \frac{1}{2})$ (see also the proof of Theorem 5.4.2). In [6] it was argued that, starting from equation (2.9), one can take any sequence of functions $\phi_\epsilon$ in $C_c^\infty(\mathbb{R})$ converging to the Dirac delta $\delta_a$ at point $a$ in the following sense

$$\lim_{\epsilon\to 0} \int \phi_\epsilon(x) g(x)\,\mathrm{d}x = g(a),$$

for any continuous function $g$ with a compact support. Therefore one obtains that

$$\lim_{\epsilon\to 0} \int \phi_\epsilon(x) f_n''(x)\,\mathrm{d}x = f_n''(a).$$

On the other hand, by (5.7), one has

$$\lim_{n\to\infty} \int \phi_\epsilon(x) f_n''(x)\,\mathrm{d}x = \int \phi_\epsilon(x)\,\mu(\mathrm{d}x).$$

Finally it leads to the conclusion that $\sup_n f_n''(a) < \infty$ uniformly in $n$. However, the statement is false in general. Since if $\mu$ has an atom at point $a$, we obtain

$$\int \phi_\epsilon(x)\,\mu(\mathrm{d}x) \to \infty.$$

Actually, by Lebesgue decomposition theorem (see [68] for a reference), the measure $\mu$ can be decomposed as

$$\mu = \mu_{AC} + \mu_{SC} + \mu_{SD},$$

where $\mu_{AC}$ is absolutely continuous with respect to Lebesgue measure, $\mu_{SC}$ is singular continuous and $\mu_{SD}$ is singular discontinuous, i.e. $\mu_{SD}$ corresponds to the atoms of the measure $\mu$. Now the statement $\sup_n f_n''(a) < \infty$ is clearly true for $\mu_{AC}$ and false for $\mu_{SD}$. It is unclear if the statement is whether true or not for $\mu_{SC}$.

However, if $\sup_n f_n''(a) < \infty$ does indeed hold, then by applying Lemma 5.1.1 to a stochastic process $X$, we obtain

$$\int \mathbf{1}_{X_s<a<X_t}\,\mu(\mathrm{d}a) \leq C|X_t - X_s|.$$

Consequently, for every $\alpha$-Hölder continuous process $X$, we would derive that

$$f_-'(X) \in W_0^{\alpha-\epsilon,1} \subset C^{\alpha-2\epsilon}.$$

In other words, any function of locally bounded variation applied to Hölder continuous process $X$ would still be Hölder continuous. Clearly, such a result is true only if $f_-'$ is sufficiently smooth, in which case the integration would reduce back to Young integration theory for Hölder continuous processes.

The aforementioned flawed argument was also applied in the proofs of the theorems to generalize the results of [6] to a general class of Gaussian processes. Although by examining the proof in [81], it is clear that the proof of the existence of the generalized Lebesgue–Stieltjes integral is correct provided that $\mu_{SC}([-\epsilon,\epsilon]) = 0$ for small enough $\epsilon$. We also note that similar techniques were applied in [84], where the author studied the average of geometric fractional Brownian motion and proved a change of variables formula in that case. To obtain the result, the author in [84] proved that for a given functional $X_t$ and the approximating sequence $f_n(X_t)$, one has

$$\mathbb{E}||f_n||_{\beta,1} \to \mathbb{E}||f||_{\beta,1},$$

for $\beta \in (0,1)$. Then the author applied dominated convergence theorem to obtain the result. However, this is not sufficient to apply dominated convergence theorem. Moreover, it is not even clear whether it holds that $\mathbb{E}||f||_{\beta,1} < \infty$ (see also Remark 5.2.3).

The new results presented below not only generalize the pathwise stochastic integrals for more general processes, but also fix the aforementioned gaps. In the following, we will show that although the proofs of the change of variables formula in [6, 81, 84] contain some flaws, their results are still valid.

## 5.2 Integration of One-dimensional Unbounded $p$-variation Processes

In the following section, the existence of the generalized Lebesgue–Stieltjes integrals of a certain class of unbounded $p$-variation processes for $p \geq 1$ with respect to general Hölder continuous processes will be shown. For simplicity, we will start from deterministic functions first.

### 5.2.1 Existence of Generalized Lebesgue–Stieltjes Integral for Unbounded $p$-variation Functions

Recall that $C^{\alpha}([0,T])$ denotes the space of $\alpha$-Hölder continuous functions on $[0,T]$ and $BV^{loc}(\mathbb{R})$ denotes the space of locally bounded variation functions on $\mathbb{R}$.

According to the arguments in the beginning of Chapter 5, in order to prove Theorem 5.2.1 and Theorem 5.2.2, it is sufficient to consider an increasing, right-continuous function $f$ such that the Lebesgue–Stieltjes measure of $f$ has a compact support, instead of a general locally bounded variation function. Let $h$ be an $\alpha$-Hölder continuous function on $[0,T]$ with $\alpha \in (0,1)$. Note that if $h$ has infinitely many crossings of a discontinuity point of $f$, then according to Remark 5.0.3, $f \circ h$ may be of unbounded $p$-variation for any $p \geq 1$. Now let us make the following assumption for $h$.

**Assumption 5.2.1.** *Let $h$ be a continuous function on $[0,T]$. Assume that for any $\delta \in (0,1)$, there exists a constant $M$ such that*

$$\int_0^T |h(t) - a|^{-\delta} \,\mathrm{d}t \leq M. \tag{5.11}$$

*for all $a \in \mathbb{R}$.*

Then we have the existence theorem of the generalized Lebesgue–Stieltjes integral.

**Theorem 5.2.1.** *Let $F$ be a function in $BV^{loc}(\mathbb{R})$. Let $h$ be a function in $C^{\alpha}([0,T])$ with $\alpha \in (0,1)$ satisfying Assumption 5.2.1, and $g$ be a function in $C^{\gamma}([0,T])$ with $\gamma$ in $(0,1)$ such that $\alpha + \gamma > 1$. Then the integral*

$$(\mathrm{gLS}) \int_0^T F\big(h(t)\big) \,\mathrm{d}g(t)$$

*exists.*

*Proof.* The proof follows the ideas applied in [6, 84]. If we choose some $\beta \in (1-\gamma, \alpha)$, then $g \in W_T^{1-\beta,\infty}$ by Remark 2.3.1. Hence by Proposition 3.2.6, the integral is well-defined if $F\big(h(t)\big) \in W_0^{\beta,1}$ for all $t \in [0,T]$.

Note that by the arguments in the beginning of Chapter 5, instead of a locally bounded variation function $F$, we consider an increasing, right-continuous function $f$ such that the Lebesgue–Stieltjes measure $\mu$ of $f$ has a compact support $\mathcal{K}$. By Remark 2.2.1, $f$ together with $\mu$ satisfy the representation (2.6).

In order to show that $f\big(h(\cdot)\big) \in W_0^{\beta,1}$, what we we need to prove is

$$\|f\big(h(\cdot)\big)\|_{\beta,1} < \infty.$$

For the first term in the norm, by the fact that $f$ is bounded on compact sets, we have

$$\int_0^T \frac{\big|f\big(h(t)\big)\big|}{t^\beta}\,\mathrm{d}t \le \sup_{0\le t\le T}\big|f\big(h(t)\big)\big| \int_0^T \frac{1}{t^\beta}\,\mathrm{d}t < \infty.$$

For the second term in the norm, according to representation (2.6), we have

$$\begin{aligned}
&\int_0^T\int_0^t \frac{\big|f\big(h(t)\big)-f\big(h(s)\big)\big|}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t \\
=&\int_0^T\int_0^t \frac{\big|\frac{1}{2}\int_{\mathcal{K}}\operatorname{sgn}\big(h(t)-a\big)\,\mu(\mathrm{d}a)-\frac{1}{2}\int_{\mathcal{K}}\operatorname{sgn}\big(h(s)-a\big)\,\mu(\mathrm{d}a)\big|}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t \qquad (5.12)\\
=&\int_0^T\int_0^t \frac{\int_{\mathcal{K}}\big(\mathbf{1}_{\{h(s)<a\le h(t)\}}+\mathbf{1}_{\{h(t)<a\le h(s)\}}\big)\,\mu(\mathrm{d}a)}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t.
\end{aligned}$$

We will consider the term $\mathbf{1}_{\{h(s)<a\le h(t)\}}$ first. By Tonelli's theorem we obtain

$$\int_0^T\int_0^t \frac{\int_{\mathcal{K}}\big(\mathbf{1}_{\{h(s)<a\le h(t)\}}\big)\,\mu(\mathrm{d}a)}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t = \int_{\mathcal{K}}\int_0^T\int_0^t \frac{\mathbf{1}_{\{h(s)<a\le h(t)\}}}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t\,\mu(\mathrm{d}a). \qquad (5.13)$$

According to Assumption 5.2.1, for any $a \in \mathbb{R}$, the set of points $x$ such that $h(x) = a$ has Lebesgue measure 0. This implies that, for any $s$ and $a$

$$\mathbf{1}_{\{h(s)<a\le h(t)\}} = \mathbf{1}_{\{h(s)<a<h(t)\}} \qquad (5.14)$$

for Lebesgue-almost every $t$ in $[0,T]$. Hence it suffices to consider $\mathbf{1}_{\{h(s)<a<h(t)\}}$ instead of $\mathbf{1}_{\{h(s)<a\le h(t)\}}$.

Denote

$$\begin{aligned}
J &:= \int_0^T\int_0^t \frac{\int_{\mathcal{K}}\big(\mathbf{1}_{\{h(s)<a<h(t)\}}\big)\,\mu(\mathrm{d}a)}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t,\\
J_1(a) &:= \int_0^T\int_0^t \frac{\mathbf{1}_{\{h(s)<a<h(t)\}}}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t,
\end{aligned}$$

then (5.13) is equivalent to

$$J = \int_{\mathcal{K}} J_1(a)\,\mu(\mathrm{d}a).$$

The last hitting time of function $h$ into level $a$ during the time interval $[0,t]$ for $t \in [0,T]$ is defined as

$$T_t(a) := \sup\{u \in [0,t] : h(u) = a\}.$$

If $h$ never hits $a$ on $[0,t]$, then let $T_t(a) = 0$. In the case of $T_t(a) = 0$ for some $t \in [0,T]$, the integral $J$ is finite due to the fact that $\mathbf{1}_{\{h(s)<a<h(t)\}} = 0$. In the case of $T_t(a) > 0$ for some $t \in [0,T]$, if $h(t) > a$, one obtains $T_t(a) < t$ for these $t \in [0,T]$.

Therefore we have

$$\begin{aligned}
J_1(a) &= \int_0^T \int_0^t \frac{\mathbf{1}_{\{h(s)<a<h(t)\}}}{(t-s)^{1+\beta}} \, \mathrm{d}s\mathrm{d}t \\
&\leq \int_0^T \int_0^{T_t(a)} \frac{\mathbf{1}_{\{a<h(t)\}}}{(t-s)^{1+\beta}} \, \mathrm{d}s\mathrm{d}t \\
&= \int_0^T \mathbf{1}_{\{a<h(t)\}} \frac{\big(t-T_t(a)\big)^{-\beta}}{\beta} \, \mathrm{d}t - \int_0^T \mathbf{1}_{\{a<h(t)\}} \frac{t^{-\beta}}{\beta} \, \mathrm{d}t \\
&\leq \int_0^T \mathbf{1}_{\{a<h(t)\}} \frac{\big(t-T_t(a)\big)^{-\beta}}{\beta} \, \mathrm{d}t.
\end{aligned}$$

Now

$$J_1(a) \leq \int_0^T \mathbf{1}_{\{a<h(t)\}} \frac{\big(t-T_t(a)\big)^{-\beta}}{\beta} \mathrm{d}t. \tag{5.15}$$

Since $h \in C^\alpha([0,T])$, there exists a constant $H_\alpha(h) := \sup_{0\leq s<t\leq T} \frac{|h(t)-h(s)|}{|t-s|^\alpha}$ such that

$$|h(t)-h(s)| \leq H_\alpha(h)\, |t-s|^\alpha. \tag{5.16}$$

Now let $s = T_t(a)$, we obtain

$$|h(t)-a| \leq H_\alpha(h)\, |t-T_t(a)|^\alpha.$$

Note that when $a < h(t)$ holds for some $t$, $|h(t)-a|$ and $|t-T_t(a)|$ cannot be zero. Therefore, in this case we derive that

$$|t-T_t(a)|^{-\beta} \leq H_\alpha^\delta(h)\, |h(t)-a|^{-\delta},$$

where $\delta = \frac{\beta}{\alpha}$.

Now (5.15) implies

$$J_1(a) \leq \frac{H_\alpha^\delta(h)}{\beta} \int_0^T |h(t)-a|^{-\delta} \, \mathrm{d}t.$$

Since $\delta \in (0,1)$, by Assumption 5.2.1 we obtain

$$J_1(a) \leq \frac{H_\alpha^\delta(h)}{\beta} M = C,$$

where $C$ is a constant independent of $a$.

Therefore we have

$$J = \int_{\mathcal{K}} J_1(a)\, \mu(\mathrm{d}a) \leq C\, \mu(\mathcal{K}) < \infty.$$

The other term $\mathbf{1}_{\{h(t)<a\leq h(s)\}}$ in (5.12) can be treated similarly by changing order of the integrals, and considering the first hitting time of function $h$ into level $a$ during the time interval $[s,T]$ for $s \in [0,T]$, which is defined as

$$\tilde{T}_s(a) := \sup\{u \in [s,T] : h(u) = a\}.$$

If $h$ never hits $a$ on $[s,T]$, then let $\tilde{T}_s(a) = T$. In the case of $\tilde{T}_s(a) = T$ for some $s \in [0,T]$, the integral $J$ is finite due to the fact that $\mathbf{1}_{\{h(t)<a<h(s)\}} = 0$. In the case of $\tilde{T}_s(a) < T$ for some $s \in [0,T]$, if $h(s) > a$, one obtains $\tilde{T}_s(a) > s$ for these $s \in [0,T]$. Then following similar arguments as above, we derive the conclusion. $\square$

**Remark 5.2.1.** *Let $G$ be a function on $[0,T]$ defined as*

$$G(x) = F\big(h(x)\big),$$

*where $h$ is in $C^{\alpha+\epsilon}([0,T])$ for $\alpha \in (0,1)$, $\epsilon \in (0,1-\alpha)$ and $F$ is in $BV^{loc}(\mathbb{R})$. By the proof of Theorem 5.2.1, we notice that the space of all functions $G$ on $[0,T]$ is a subset of the space $W_0^{\alpha,1}([0,T])$.*

In above, $h$ may have uncountably many crossings of some level $a$. However, if $h$ has countably many crossings of level $a$ for any $a \in \mathbb{R}$ over $[0,T]$, and the set $\{t \in [0,T] : h(t) = a\} \cap [0,c]$ is finite for every $c < T$, then we can define a sequence of hitting points of $h$ for level $a$ on $[0,T]$ as

$$\begin{aligned} \tau_k^a &= 0, \text{ for } k = 0, \\ \tau_k^a &= \inf\{u \in (\tau_{k-1}^a, T] : h(u) = a\}, \text{ for } k = 1, 2, \ldots \end{aligned}$$

In this case, instead of Assumption 5.2.1, we have the following assumption.

**Assumption 5.2.2.** *Let $h$ be a continuous function on $[0,T]$. Assume that $h$ has countably many crossings of any level $a$, and the set $\{t \in [0,T] : h(t) = a\} \cap [0,c]$ is finite for every $c < T$. Moreover, assume that for some $\delta \in (0,1)$, there exists a constant $M$ such that*

$$\Big|\sum_{k=1}^{\infty} (\tau_k^a - \tau_{k-1}^a)^{1-\delta}\Big| \leq M,$$

*for all $a \in \mathbb{R}$.*

Then we have the following theorem.

**Theorem 5.2.2.** *Let $F$ be a function in $BV^{loc}(\mathbb{R})$. Let $g$ be a function in $C^\gamma([0,T])$ with some $\gamma \in (0,1)$. Let $h$ be a function in $C^\alpha([0,T])$ with $\alpha \in (0,1)$ such that $\alpha+\gamma > 1$. Moreover, let $h$ satisfy Assumption 5.2.2 with $\delta \in (1-\gamma, \alpha)$. Then the integral*

$$(\text{gLS}) \int_0^T F\big(h(t)\big)\,\mathrm{d}g(t)$$

*exists.*

*Proof.* By the arguments in the beginning of Chapter 5, it is enough to consider an increasing, right-continuous function $f$ such that the Lebesgue–Stieltjes measure $\mu$ of $f$ has a compact support $\mathcal{K}$.

Choosing $\beta = \delta$ and following the same steps as in the proof of Theorem 5.2.1, we obtain (5.15). Now for (5.15), we can split the integral into a sum of countably many pieces as

$$\begin{aligned}\int_0^T \mathbf{1}_{\{a<h(t)\}}\big(t - T_t(a)\big)^{-\beta}\,\mathrm{d}t &= \sum_{k=1}^{\infty} \int_{\tau_{k-1}^a}^{\tau_k^a} \mathbf{1}_{\{a<h(t)\}}\big(t - T_t(a)\big)^{-\beta}\,\mathrm{d}t \\ &\leq \sum_{k=1}^{\infty} \frac{\big(\tau_k^a - \tau_{k-1}^a\big)^{1-\beta}}{1-\beta}.\end{aligned}$$

Since $\beta = \delta \in (1-\gamma, \alpha)$, according to Assumption 5.2.2 we have

$$\sum_{k=1}^{\infty} \big(\tau_k^a - \tau_{k-1}^a\big)^{1-\beta} \leq M,$$

where $M$ is independent of $a$. Therefore we obtain

$$\sup_{a\in\mathcal{K}} J_1(a) < \infty,$$

which implies

$$J = \int_{\mathcal{K}} J_1(a)\,\mu(\mathrm{d}a) < \infty.$$

□

Here is an example of continuous functions $h$ with countably infinite crossings of level $a = 0$.

**Example 5.2.1.** *Consider a function*

$$h(t) = t\sin\Big(\frac{T}{t}\Big),$$

*for $t \in [0,T]$. We can see that $h$ is continuous and has countably infinite crossings of level 0. Moreover, we can find a sequence of hitting points for level 0 on $[0,T]$ as*

$$\tau_k^0 = \frac{T}{k\pi}, \quad k = 1, 2, \ldots.$$

*Note that now the last hitting point of $h$ for level $0$ is*

$$T_t(0) = \frac{T}{\pi}.$$

*Therefore, according to (5.15) we have*

$$\begin{aligned} J_1(0) &\le \int_0^T \left(t - \frac{T}{\pi}\right)^{-\beta} \mathrm{d}t \\ &\le C, \end{aligned}$$

*where $C$ is a constant.*

### 5.2.2 Existence of Generalized Lebesgue–Stieltjes Integral for Processes of Unbounded $p$-variation

Now we have shown the existence of the generalized Lebesgue–Stieltjes integral for deterministic functions. In this section, we will consider a continuous stochastic process $X = (X_t)_{t\in[0,T]}$. We say a process $X \in C^\alpha([0,T])$ with $\alpha \in (0,1)$, if almost all paths of $X$ belong to the space $C^\alpha([0,T])$.

**Theorem 5.2.3.** *Let $F$ be a real-valued function in $BV^{loc}(\mathbb{R})$. Let $X \in C^\alpha([0,T])$ with $\alpha \in (0,1)$ be a process such that for almost every path, the Assumption 5.2.1 is satisfied with a random constant $M = M(\omega)$. Let $Y$ be a process in $C^\gamma([0,T])$ with $\gamma \in (0,1)$ such that $\alpha + \gamma > 1$. Then*

$$(\mathrm{gLS}) \int_0^T F(X_t)\, \mathrm{d}Y_t$$

*exists a.s.*

*Proof.* Note that for different path of $X$, the random constant $M$ in Assumption 5.2.1 will be a different constant. Again, we can only consider an increasing, right-continuous function $f$. Assume first that the Lebesgue–Stieltjes measure of $f$ has a compact support. For every $\omega \in \Omega$, processes $X$ and $Y$ will become Hölder continuous functions on $[0,T]$ and $M$ will become a constant. Then by following similar steps as in the proof of Theorem 5.2.1, we can prove the theorem.

Next, assume that the Lebesgue–Stieltjes measure of $f$ does not have a compact support. For any $n \in \mathbb{N}$, define

$$\Omega_n := \{\omega \in \Omega : \sup_{t\in[0,T]} |X_t| \in [0,n]\},$$

and a function $\tilde{f}_n$ by

$$\tilde{f}_n(x) = \begin{cases} f(-n), & \text{if} \quad x < -n, \\ f(x), & \text{if} \quad -n \le x \le n, \\ f(n), & \text{if} \quad x > n. \end{cases}$$

Now $f = \tilde{f}_n$ on $[-n, n]$ and the Lebesgue–Stieltjes measure of $\tilde{f}_n$ has a compact support. Now the integral

$$(\text{gLS}) \int_0^T \tilde{f}_n(X_t)\,\mathrm{d}Y_t$$

is well defined a.s. on $\Omega_n$ for all $n \in \mathbb{N}$. Moreover, we have

$$(\text{gLS}) \int_0^T \tilde{f}_n(X_t)\,\mathrm{d}Y_t = (\text{gLS}) \int_0^T f(X_t)\,\mathrm{d}Y_t$$

almost surely on $\Omega_n$. Since $\Omega = \cup_{n\in\mathbf{N}}\Omega_n$, we obtain

$$(\text{gLS}) \int_0^T \tilde{f}_n(X_t)\,\mathrm{d}Y_t = (\text{gLS}) \int_0^T f(X_t)\,\mathrm{d}Y_t$$

almost surely for every $\omega \in \Omega$. This implies that we only need to consider some $\tilde{f}_n$ such that the Lebesgue–Stieltjes measure of $\tilde{f}_n$ has a compact support. $\square$

Similar to the deterministic case, if $X$ has countably crossings of any level $a \in \mathbb{R}$, then define a sequence of hitting times of level $a$ for $X$ as

$$\begin{aligned}
\tau_a(k) &= 0, \text{ for } k = 0, \\
\tau_a(k) &= \inf\{u \in (\tau_a(k-1), T] : X_u = a\}, \text{ for } k = 1, 2, \ldots
\end{aligned}$$

Then we obtain a similar theorem of Theorem 5.2.2.

**Theorem 5.2.4.** *Let $F$ be a real-valued function in $BV^{loc}(\mathbb{R})$. Let $X$ be a process in $C^\alpha([0,T])$ with $\alpha \in (0,1)$, such that Assumption 5.2.2 is satisfied with a random constant $M = M(\omega)$. Let $Y$ be a process in $C^\gamma([0,T])$ with $\gamma \in (0,1)$ such that $\alpha + \gamma > 1$. Then*

$$(\text{gLS}) \int_0^T F(X_t)\,\mathrm{d}Y_t$$

*exists a.s.*

The proof follows the same arguments as in Theorem 5.2.3 and Theorem 5.2.2.

In the following section, we will discuss the existence of the generalized Lebesgue–Stieltjes integral for processes with densities. However, note that Theorem 5.2.3 and Theorem 5.2.4 can be applied to processes without densities if certain assumptions are satisfied.

***Existence of Generalized Lebesgue–Stieltjes Integral for Processes with Densities***

Instead of pathwise assumption for a process $X$, now let us consider a process $X$ which has a density. In this section, I will show that if $X$ satisfies some density assumption, the existence of the generalized Lebesgue–Stieltjes integral can also be shown.

**Assumption 5.2.3.** *Let $X$ be a stochastic process on $[0,T]$ with values in $\mathbb{R}$. We assume that for almost every $t \in [0,T]$, $X_t$ has a density $p_t(y)$ and there exists a function $g_t \in L^1([0,T])$ such that $\sup_y p_t(y) \leq g_t$.*

The following examples should convince the readers that the assumption is not very restrictive.

**Example 5.2.2.** *Let $X$ be a stationary stochastic process such that $X_0$ has a bounded density $p(y)$. Then $\sup_y p(y) = C < \infty$, and consequently we can take $g(t) = C$. Therefore $X$ satisfies Assumption 5.2.3.*

**Example 5.2.3.** *Let $X$ be a Hölder continuous Gaussian process with variance function $V(t)$. Then we have*

$$\sup_y p_t(y) = \frac{1}{\sqrt{2\pi V(t)}}.$$

*Consequently, $X$ satisfies Assumption 5.2.3 provided that $V(t) \geq ct^{2\beta}$ for some $\beta < 1$. Especially, this is usually satisfied for every interesting Gaussian process $X$. Indeed, a natural assumption is that $V(t) > 0$ for every $t > 0$, i.e. there is some randomness involved. On the other hand, for many interesting cases we have $X_0 = 0$, and thus one only has to study the behaviour of $V(t)$ at zero. Now if $V(t) \leq ct^{2\beta}$ with some $\beta > 1$, then the process is Hölder continuous of order $\beta$ at the origin. Hence if $\beta > 1$, the process would be constant which is hardly interesting. Similarly, in the limiting case $V(t) \sim t^2$ the process is differentiable in the mean square sense, and consequently one can apply classical integration techniques.*

**Example 5.2.4.** *Assume that a process $X$ satisfies Assumption 5.2.3. If we add a deterministic drift $f(t)$ and consider a process $Y_t = X_t - f(t)$, it is again clear that then the process $Y$ satisfies Assumption 5.2.3. Hence the results are valid if one adds a suitably regular drift term into the model.*

**Example 5.2.5.** *Assume that a process $X_1$ with density $\tilde{p}_t$ satisfies Assumption 5.2.3 and a process $X_2$ with density $\hat{p}_t$ is independent of $X_1$. Then the density of a process $Y = X_1 + X_2$ satisfies*

$$p_t^Y = \int \tilde{p}_t(z-y)\hat{p}_t(y)\,\mathrm{d}y < g(t)\int \hat{p}_t(y)\,\mathrm{d}y = g(t),$$

*where $g \in L^1([0,T])$. Consequently, the process $Y$ also satisfies Assumption 5.2.3.*

*Now consider a process $Z = X_1 + X_2 + \ldots + X_n$, where each $X_i$ has a density and is independent with each other. Moreover, assume that $X_1$ satisfies Assumption 5.2.3. Define $Y_2 = X_1 + X_2$, then by the above statements, it is* *true that $Y_2$ satisfies Assumption 5.2.3. Repeat this procedure for $n > 2$ and in the end define $Z = Y_n = Y_{n-1} + X_n$. We have that $Z$ satisfies Assumption 5.2.3. To sum up, a finite sum of independent processes where only one process satisfies Assumption 5.2.3 also satisfies Assumption 5.2.3.*

**Remark 5.2.2.** *In general, Malliavin calculus is a powerful tool to study the existence and smoothness of the density (see, e.g. [61]). Moreover, it is known that any random variable lying in some fixed Wiener-chaos admits a density. In particular, a finite sum of such variables admits a density.*

The following lemma is a version of a similar lemma taken from [84], and it is one of our key ingredients to prove the main theorems.

**Lemma 5.2.1.** *Let $X$ be a stochastic process such that Assumption 5.2.3 is satisfied. Then for every function $f : [0,T] \to \mathbb{R}$ and every $\alpha \in (0,1)$ it holds*

*1)*

$$\mathbb{E}|X_t - f(t)|^{-\alpha} \leq \frac{2}{1-\alpha} g(t) + 1.$$

*2)*

$$\mathbb{E}\int_0^T |X_t - f(t)|^{-\alpha} \,\mathrm{d}t \leq \frac{2}{1-\alpha} \|g\|_{L^1([0,T])} + T.$$

*Proof.* 1) According to Assumption 5.2.3, we obtain

$$\begin{aligned}
\mathbb{E}|X_t - f(t)|^{-\alpha} &= \int_{\mathbb{R}} |y - f(t)|^{-\alpha} p_t(y) \,\mathrm{d}y \\
&= \int_{f(t)-1}^{f(t)+1} |y - f(t)|^{-\alpha} p_t(y) \,\mathrm{d}y + \int_{\mathbb{R}\setminus[f(t)-1, f(t)+1]} |y - f(t)|^{-\alpha} p_t(y) \,\mathrm{d}y \\
&\leq g(t) \int_{f(t)-1}^{f(t)+1} |y - f(t)|^{-\alpha} \,\mathrm{d}y + 1 \\
&= \frac{2}{1-\alpha} g(t) + 1.
\end{aligned}$$

2) We have

$$\mathbb{E}\int_0^T |X_t - f(t)|^{-\alpha} \,\mathrm{d}t = \int_0^T \mathbb{E}|X_t - f(t)|^{-\alpha} \,\mathrm{d}t.$$

Thus from 1) we obtain

$$\mathbb{E}\int_0^T |X_t - f(t)|^{-\alpha} \,\mathrm{d}t \leq \int_0^T \frac{2}{1-\alpha} g(t) \mathrm{d}t + T \leq \frac{2}{1-\alpha} \|g\|_{L^1([0,T])} + T.$$

□

The main theorem is stated as follows.

**Theorem 5.2.5.** *Let $F$ be a real-valued function in $BV^{loc}(\mathbb{R})$. Let $X \in C^{\alpha}([0,T])$ with $\alpha \in (0,1)$ satisfy the Assumption 5.2.3, and $Y \in C^{\gamma}([0,T])$ with $\gamma \in (0,1)$ such that $\alpha + \gamma > 1$. Then the integral*

$$(\mathrm{gLS}) \int_0^T F(X_t)\, \mathrm{d}Y_t$$

*exists a.s.*

*Proof.* Choose some $\beta \in (1-\gamma, \alpha)$, and by Remark 2.3.1, we know that $Y \in W_T^{1-\beta,\infty}$ a.s. Hence according to Proposition 3.2.6, the integral is well-defined if $F(X) \in W_0^{\beta,1}$ a.s.

By the arguments in the beginning of Chapter 5 and the proof of Theorem 5.2.3, instead of $F$, we only need to consider an increasing and right-continuous function $f$, such that the Lebesgue–Stieltjes measure $\mu$ of $f$ has a compact support $\mathcal{K}$.

Following similar arguments as in the proof of Theorem 5.2.1, what we need to show is that

$$\tilde{J} := \int_{\mathcal{K}} \int_0^T \int_0^t \frac{\mathbf{1}_{\{X_s < a \le X_t\}}}{|t-s|^{1+\beta}}\, \mathrm{d}s\mathrm{d}t\, \mu(\mathrm{d}a) < \infty, \quad a.s.$$

where $\mu$ is the Lebesgue–Stieltjes measure of $f$.

Note here that since $X_t$ has a density for almost every $t \in [0,T]$, for every fixed $a$, every $s$ and almost every $t$ in $[0,T]$ we have

$$\mathbf{1}_{\{X_s < a \le X_t\}} = \mathbf{1}_{\{X_s < a < X_t\}} \quad a.s.$$

Denote

$$J := \int_{\mathcal{K}} \int_0^T \int_0^t \frac{\mathbf{1}_{\{X_s < a < X_t\}}}{|t-s|^{1+\beta}}\, \mathrm{d}s\mathrm{d}t\, \mu(\mathrm{d}a).$$

Now $\tilde{J} - J$ is a non-negative random variable. By applying Tonelli's theorem, we obtain that $\mathbb{E}(\tilde{J} - J) = 0$. Therefore,

$$\tilde{J} = J \quad a.s.$$

Denote

$$J_1(a) := \int_0^T \int_0^t \frac{\mathbf{1}_{\{X_s < a < X_t\}}}{|t-s|^{1+\beta}}\, \mathrm{d}s\mathrm{d}t,$$

and we have

$$J = \int_{\mathcal{K}} J_1(a)\, \mu(\mathrm{d}a).$$

The last hitting time of $X$ into level $a$ during the time interval $[0,t]$ is defined as

$$T_t(a) := \sup\{u \in [0,t] : X_u = a\}.$$

When $T_t(a) = 0$ for some $t \in [0,T]$, one has $\mathbf{1}_{\{X_s < a < X_t\}} = 0$, and the above integral $J$ becomes 0. Therefore, we only need to consider the case when

$T_t(a) > 0$ for some $t \in [0, T]$. Note that on the set $\{\omega \in \Omega : a < X_t\}$ we have $T_t(a) < t$.

Following similar arguments in the proof of Theorem 5.2.1, we obtain

$$J_1(a) \leq \int_0^T \mathbf{1}_{\{a<X_t\}} \frac{\big(t - T_t(a)\big)^{-\beta}}{\beta} \, \mathrm{d}t.$$

Since $X \in C^\alpha([0,T])$, there exists a random constant $H_\alpha(X) := \sup_{0 \leq s < t \leq T} \frac{|X_t - X_s|}{|t-s|^\alpha}$ such that

$$|X_t - X_s| \leq H_\alpha(X)\, |t-s|^\alpha. \tag{5.17}$$

Now if we let $s = T_t(a)$, then on the event $A_{a,t} := \{\omega \in \Omega : X_s = a, \text{ for some } s \in [0,t)\}$, we obtain

$$|X_t - a| \leq H_\alpha(X)\, |t - T_t(a)|^\alpha.$$

For each $t > 0$, on the event $A_{a,t} \cap \{a < X_t\}$, it holds that $|X_t - a|$ and $|t - T_t(a)|$ are nonzero. Therefore, on $A_{a,t} \cap \{a < X_t\}$ we have

$$|t - T_t(a)|^{-\beta} \leq H_\alpha^\delta(X)\, |X_t - a|^{-\delta},$$

where $\delta = \frac{\beta}{\alpha}$.

Therefore we obtain

$$\int_0^T \mathbf{1}_{\{a<X_t\}} \frac{\big(t - T_t(a)\big)^{-\beta}}{\beta} \, \mathrm{d}t \leq \beta^{-1} H_\alpha^\delta(X) \int_0^T |X_t - a|^{-\delta} \, \mathrm{d}t. \tag{5.18}$$

Next denote $J_2(a) := \int_0^T |X_t - a|^{-\delta} \, \mathrm{d}t$. Now (5.18) becomes

$$J_1(a) \leq \beta^{-1} H_\alpha^\delta(X)\, J_2(a).$$

Since $\delta \in (0,1)$, according to Assumption 5.2.3 and Lemma 5.2.1, we get

$$\mathbb{E} J_2(a) \leq C_1,$$

where $C_1$ is a finite constant independent of $a$.

Now

$$\begin{aligned} J &= \int_{\mathcal{K}} J_1(a)\, \mu(\mathrm{d}a) \\ &\leq \beta^{-1} H_\alpha^\delta(X) \int_{\mathcal{K}} J_2(a)\, \mu(\mathrm{d}a). \end{aligned} \tag{5.19}$$

Since

$$\mathbb{E} \int_{\mathcal{K}} J_2(a)\, \mu(\mathrm{d}a) \leq \int_{\mathcal{K}} \mathbb{E} J_2(a)\, \mu(\mathrm{d}a) \leq \mu(\mathcal{K})\, C_1,$$

together with the fact that $H_\alpha^\delta(X)$ is finite for almost every path of $X$, we obtain

$$J = \int_{\mathcal{K}} J_1(a)\, \mu(\mathrm{d}a) < \infty \quad a.s.$$

□

**Remark 5.2.3.** *Note that I am not claiming that*

$$\mathbb{E}\int_0^T\int_0^t \frac{\int_{\mathcal{K}} \mathbf{1}_{\{X_s<a<X_t\}}\,\mu(\mathrm{d}a)}{|t-s|^{1+\alpha}}\,\mathrm{d}s\mathrm{d}t<\infty$$

*in general. Indeed, the upper bound depends on the random variable representing the Hölder constant of the process $X$ which may or may not have a finite expectation.*

**Remark 5.2.4.** *In [84], the author showed the existence of the generalized Lebesgue–Stieltjes integral of some functionals of fractional Brownian motion. For Hölder continuous processes satisfying Assumption 5.2.3, the existence of generalized Lebesgue–Stieltjes integral of those functionals can also be shown by similar arguments in [84]. Moreover, in order to find the Hölder coefficients in (5.17), Garsia-Rodemich-Rumsey inequality reviewed in Chapter 2 can be applied as a tool.*

Consider a process defined as

$$Z_t = F(X_t), \tag{5.20}$$

where $F$ is a real-valued function in $BV^{loc}(\mathbb{R})$ and $X$ is a stochastic process in $C^{\alpha+\epsilon}([0,T])$ for $\alpha\in(0,1)$ and $\epsilon\in(0,1-\alpha)$ satisfying Assumption 5.2.3. We are not assuming that a density of $Z$ itself exists. Indeed, this is rarely the case as can be seen by considering the function $F(x)=\mathbf{1}_{x\geq 0}$.

**Remark 5.2.5.** *Theorem 5.2.5 implies that the sample paths of the above process $Z$ lie between the space of $(\alpha+\epsilon)$-Hölder continuous functions $C^{\alpha+\epsilon}([0,T])$ and the fractional Besov space $W_0^{\alpha,1}([0,T])$.*

Let $\tau$ be an a.s. finite random variable. Then the following theorem is a straightforward consequence of Theorem 5.2.5, and the proof goes analogously to the proof of similar theorem in [81].

**Theorem 5.2.6.** *Let $F$ be a real-valued function in $BV^{loc}(\mathbb{R})$. Let $X\in C^{\alpha}([0,T])$ with $\alpha\in(0,1)$ satisfy Assumption 5.2.3, and $Y\in C^{\gamma}([0,T])$ with $\gamma\in(0,1)$ such that $\alpha+\gamma>1$. Let $\tau\leq T$ be a bounded random time. Then the integral*

$$(\mathrm{gLS})\int_0^\tau F(X_t)\,\mathrm{d}Y_t$$

*exists a.s.*

*Proof.* By the same arguments as in the beginning of Chapter 5 and the proof of Theorem 5.2.3, it is enough to consider an increasing, right-continuous $f$, such that the Lebesgue–Stieltjes measure of $f$ has a compact support $\mathcal{K}$.

Choose some $\beta \in (1-\gamma, \alpha)$, then $Y_t \mathbf{1}_{\{t \le \tau\}} \in W_T^{1-\beta,\infty}$ a.s. by Remark 2.3.1. Thus by Proposition 3.2.6, we only need to show that $f(X_t)\mathbf{1}_{\{t\le\tau\}} \in W_0^{\beta,1}$ a.s.

For the first term in the norm $\|f(X_t)\mathbf{1}_{\{t\le\tau\}}\|_{\beta,1}$, we can make an upper bound by

$$\mathbf{1}_{\{t\le\tau\}} \le 1,$$

and proceed as in the proof of Theorem 5.2.5. For the second term in the norm $\|f(X_t)\mathbf{1}_{\{t\le\tau\}}\|_{\beta,1}$, we have

$$\int_0^T \int_0^t \frac{|f(X_t)\mathbf{1}_{\{t\le\tau\}} - f(X_s)\mathbf{1}_{\{s\le\tau\}}|}{|t-s|^{1+\beta}} \,\mathrm{d}s\mathrm{d}t.$$

If $s \le t \le \tau$, then we can proceed as in the proof of Theorem 5.2.5. If $\tau < s \le t$, then the integral becomes 0. If $\tau < t$ and $s \le \tau$, now by the fact that $f$ is bounded on compact sets, we have

$$\begin{aligned}
&\int_\tau^T \int_0^\tau \frac{|f(X_s)|}{|t-s|^{1+\beta}} \,\mathrm{d}s\mathrm{d}t \\
\le& C_1(\beta) \sup_{s\in[0,T]} |f(X_s)| \int_\tau^T (t-\tau)^{-\beta} \,\mathrm{d}t \\
\le& C(\beta) \sup_{s\in[0,T]} |f(X_s)| \, T^{1-\beta} \\
<& \infty,
\end{aligned}$$

where $C(\beta)$ and $C_1(\beta)$ are constants depend on the index $\beta$. This completes the proof. □

## 5.3 Integration of Multidimensional Processes

Having studied the pathwise integration of one-dimensional unbounded $p$-variation processes for $p \ge 1$, now we want to apply the generalized Lebesgue–Stieltjes integral for multidimensional unbounded $p$-variation processes with $p \ge 1$. In the following, we denote $\mathbf{x} = (x_1, \ldots, x_d)$ and $\mathbf{a} = (a_1, \ldots, a_d)$ for $d$-dimensional vector.

**Definition 5.3.1.** *$\mathcal{R}(\mathbb{R}^d)$ is defined to be the class of functions $f : \mathbb{R}^d \mapsto \mathbb{R}$ and functions $g = f$ Lebesgue-almost everywhere so that*

$$f(\mathbf{x}) := \sum_{k=1}^n f_k\big(\Pi_k(\mathbf{x})\big),$$

*where $n \in \mathbb{N}$, $\Pi_k : \mathbb{R}^d \mapsto \mathbb{R}^{\sigma(k)}$ is a coordinate projection, and $f_k : \mathbb{R}^{\sigma(k)} \mapsto \mathbb{R}$ with some integer $\sigma(k) \le d$ is a function satisfying*

$$f_k(\mathbf{x}) := c_k \int_{\mathbb{R}^{\sigma(k)}} \prod_{i=1}^{\sigma(k)} \mathrm{sgn}(x_i - a_i)\, \mu(\mathrm{d}\mathbf{a}) + d_k,$$

*where* $\operatorname{sgn}(x) = \mathbf{1}_{\{x \geq 0\}} - \mathbf{1}_{\{x<0\}}$, $c_k$, $d_k$ *are some real numbers for every* $k$ *and* $\mu$ *is a finite Radon measure on* $\mathbb{R}^{\sigma(k)}$.

Through the following examples, we can see that the class $\mathcal{R}(\mathbb{R}^d)$ includes many interesting functions.

**Example 5.3.1.** *Let* $f(x_1, \ldots, x_d) := \sum_{k=1}^{d} f_k(x_k)$, *where* $f_k \in BV(\mathbb{R})$ *for every* $k = 1, \ldots, d$. *By Jordan decomposition,* $f_k = f_k^1 - f_k^2$, *where* $f_k^1$ *and* $f_k^2$ *are increasing functions. For* $f_k^1$ *and* $f_k^2$, *there exist right-continuous and increasing* $f_{k+}^1$ *and* $f_{k+}^2$ *such that* $f_k^1 = f_{k+}^1$ *and* $f_k^2 = f_{k+}^2$ *Lebesgue-almost everywhere. According to Remark 2.2.1,* $f_{k+}^1$ *(resp.* $f_{k+}^2$*) together with the finite Lebesgue–Stieltjes measure* $\mu_k^1$ *(resp.* $\mu_k^2$*) of* $f_{k+}^1$ *(resp.* $f_{k+}^2$*) satisfy the representation (2.6).*

*For every* $k = 1, \ldots, d$,

$$\begin{aligned} f_{k+}(x_k) &= \frac{1}{2}\int_{\mathbb{R}} \operatorname{sgn}(x_k - a)\, \mu_k^1(da) + C_k^1 - \frac{1}{2}\int_{\mathbb{R}} \operatorname{sgn}(x_k - a)\, \mu_k^2(da) - C_k^2 \\ &= \frac{1}{2}\int_{\mathbb{R}} \operatorname{sgn}(x_k - a)\, (\mu_k^1 - \mu_k^2)(\mathrm{d}a) + C_k^1 - C_k^2 \\ &= \frac{1}{2}\int_{\mathbb{R}} \operatorname{sgn}(x_k - a)\, \mu_k(\mathrm{d}a) + C_k, \end{aligned}$$

*where* $\mu_k = \mu_k^1 - \mu_k^2$ *is a finite signed measure, and* $C_k = C_k^1 - C_k^2$ *with some constants* $C_k^1$ *and* $C_k^2$. *Moreover* $f_{k+} = f_k$ *Lebesgue-almost everywhere.*

*Therefore,*

$$f(x_1, \ldots, x_d) = \frac{1}{2}\sum_{k=1}^{d}\int_{\mathbb{R}} \operatorname{sgn}(x_k - a)\, \mu_k(da) + \sum_{k=1}^{d} C_k.$$

*Clearly,* $f \in \mathcal{R}(\mathbb{R}^d)$.

**Example 5.3.2.** *Let* $f(x_1, \ldots, x_d) := \prod_{k=1}^{d} f_k(x_k)$, *where* $f_k \in BV(\mathbb{R})$ *for every* $k = 1, \ldots, d$. *Follow the same arguments as in Example 5.3.1, we obtain*

$$f_{k+}(x_k) = \frac{1}{2}\int_{\mathbb{R}} \operatorname{sgn}(x_k - a)\, \mu_k(\mathrm{d}a) + C_k,$$

*where* $\mu_k$ *is a finite signed measure,* $C_k$ *is some constant and* $C_k$ *may be different for every* $k$. *Moreover* $f_{k+} = f_k$ *Lebesgue-almost everywhere.*

*Now consider $d = 2$ for simplicity,*

$$
\begin{aligned}
&\prod_{k=1}^{2} f_{k+}(x_k)\\
=&\prod_{k=1}^{2}\Big(\frac{1}{2}\int_{\mathbb{R}} \operatorname{sgn}(x_k - a)\,\mu_k(\mathrm{d}a) + C_k\Big)\\
=&\frac{1}{4}\int_{\mathbb{R}} \operatorname{sgn}(x_1 - a)\,\mu_1(\mathrm{d}a)\int_{\mathbb{R}} \operatorname{sgn}(x_2 - a)\,\mu_2(\mathrm{d}a)\\
&+\frac{C_1}{2}\int_{\mathbb{R}} \operatorname{sgn}(x_2 - a)\,\mu_2(\mathrm{d}a) + \frac{C_2}{2}\int_{\mathbb{R}} \operatorname{sgn}(x_1 - a)\,\mu_1(\mathrm{d}a) + C_1C_2\\
=&\frac{1}{4}\int_{\mathbb{R}^2}\prod_{k=1}^{2} \operatorname{sgn}(x_k - a_k)\,\mu(\mathrm{d}\mathbf{a}) + \frac{C_1}{2}\int_{\mathbb{R}} \operatorname{sgn}(x_2 - a)\,\mu_2(\mathrm{d}a)\\
&+\frac{C_2}{2}\int_{\mathbb{R}} \operatorname{sgn}(x_1 - a)\,\mu_1(\mathrm{d}a) + C_1C_2,
\end{aligned}
$$

*where $\mu := \mu_1 \otimes \mu_2$ is the product measure of the signed measures $\mu_1$ and $\mu_2$.*

*Therefore, $f(x_1, x_2) \in \mathcal{R}(\mathbb{R}^2)$. For $d > 2$, we can follow the same arguments to show that $f \in \mathcal{R}(\mathbb{R}^d)$.*

### 5.3.1 Existence of Generalized Lebesgue–Stieltjes Integral for Multivariable Functions

Similar to one-dimensional case, we will start with deterministic functions first. Consider a $d$-dimensional vector $\alpha = (\alpha_1, \ldots, \alpha_d)$ where every $\alpha_k \in (0, 1)$ for $k = 1, \ldots, d$.

**Definition 5.3.2.** *Let $\mathbf{h} = (h^1, \ldots, h^d)$ be a function from $[0, T]$ into $\mathbb{R}^d$. We denote $h \in C^{\alpha}([0, T])$ if every function $h^k \in C^{\alpha_k}([0, T])$ for $k = 1, \ldots, d$.*

**Theorem 5.3.1.** *Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ be a function in the class $\mathcal{R}(\mathbb{R}^d)$, $\mathbf{h}$ be a function from $[0, T]$ into $\mathbb{R}^d$ such that $\mathbf{h} \in C^{\bar{\alpha}}([0, T])$ and $g$ be a function in $C^{\gamma}([0, T])$ with $\gamma \in (0, 1)$. Suppose that every $h^k$ satisfies Assumption 5.2.1 with a constant $M_k$ for $k = 1, \ldots, d$. Moreover, if $\alpha_k + \gamma > 1$ for every $k = 1, \ldots, d$, then the integral*

$$
(\mathrm{gLS})\int_0^T f\big(\mathbf{h}(t)\big)\,\mathrm{d}g(t)
$$

*exists.*

*Proof. Step 1.* Take $f = f(x_1, \ldots, x_d)$ such that

$$
f(x_1, \ldots, x_d) = \int_{\mathbb{R}^d}\prod_{i=1}^{d} \operatorname{sgn}(x_i - a_i)\,\mu(\mathrm{d}\mathbf{a}),
$$

where $\mu$ is a finite Radon measure on $\mathbb{R}^d$.

Let $\mathbf{x} = (x_1, \dots, x_d)$ and $\mathbf{y} = (y_1, \dots, y_d)$. Observe that

$$\begin{aligned} f(\mathbf{y}) - f(\mathbf{x}) &= \int_{\mathbb{R}^d} \prod_{i=1}^{d} \operatorname{sgn}(y_i - a_i)\,\mu(\mathrm{d}\mathbf{a}) - \int_{\mathbb{R}^d} \prod_{i=1}^{d} \operatorname{sgn}(x_i - a_i)\,\mu(\mathrm{d}\mathbf{a}) \\ &= \int_{\mathbb{R}^d} \Big( \prod_{i=1}^{d} \operatorname{sgn}(y_i - a_i) - \prod_{i=1}^{d} \operatorname{sgn}(x_i - a_i) \Big)\,\mu(\mathrm{d}\mathbf{a}). \end{aligned}$$

Note that

$$\operatorname{sgn}(y_i - a_i) - \operatorname{sgn}(x_i - a_i) = 2\mathbf{1}_{\{x_i < a_i \le y_i\}} - 2\mathbf{1}_{\{y_i < a_i \le x_i\}}.$$

Moreover,

$$|\operatorname{sgn}(y_i - a_i) - \operatorname{sgn}(x_i - a_i)| \le 2.$$

Moreover, we obtain

$$\Big| \prod_{i=1}^{d} \operatorname{sgn}(y_i - a_i) - \prod_{i=1}^{d} \operatorname{sgn}(x_i - a_i) \Big| \le 2^d \sum_{i=1}^{d} \big( \mathbf{1}_{\{x_i < a_i \le y_i\}} + \mathbf{1}_{\{y_i < a_i \le x_i\}} \big).$$

Therefore we can conclude

$$|f(y) - f(x)| \le 2^d \int_{\mathbb{R}^d} \sum_{i=1}^{d} \Big( \mathbf{1}_{\{x_i < a_i \le y_i\}} + \mathbf{1}_{\{y_i < a_i \le x_i\}} \Big)\,\mu(\mathrm{d}\mathbf{a}). \tag{5.21}$$

Choose some $\beta \in \big(1 - \gamma, \min(\alpha_1, \dots, \alpha_d)\big)$, then $g \in W_T^{1-\beta,\infty}$. According to Proposition 3.2.6, the generalized Lebesgue–Stieltjes integral exists if

$$\|f\big(\mathbf{h}(t)\big)\|_{\beta,1} < \infty.$$

For the first term in the norm $\|\cdot\|_{\beta,1}$, according to (5.21), $|f|$ is bounded, therefore

$$\int_0^T \frac{\big|f\big(h^1(t), \dots, h^d(t)\big)\big|}{t^\beta}\,\mathrm{d}t \le \sup_{0 \le t \le T} \big|f\big(h^1(t), \dots, h^d(t)\big)\big| \int_0^T \frac{1}{t^\beta}\,\mathrm{d}t < \infty.$$

For the second term in the norm $\|\cdot\|_{\beta,1}$, according to (5.21), we obtain

$$\begin{aligned} &\big|f\big(h^1(t), \dots, h^d(t)\big) - f\big(h^1(s), \dots, h^d(s)\big)\big| \\ \le& 2^d \int_{\mathbb{R}^d} \Big( \sum_{i=1}^{d} \mathbf{1}_{\{h^i(s) < a_i \le h^i(t)\}} + \sum_{i=1}^{d} \mathbf{1}_{\{h^i(t) < a_i \le h^i(s)\}} \Big)\,\mu(\mathrm{d}\mathbf{a}) \\ =& 2^d \sum_{i=1}^{d} \int_{\mathbb{R}^d} \big( \mathbf{1}_{\{h^i(s) < a_i \le h^i(t)\}} + \mathbf{1}_{\{h^i(t) < a_i \le h^i(s)\}} \big)\,\mu(\mathrm{d}\mathbf{a}). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &\int_0^T \int_0^t \frac{\big|f\big(h^1(t), \dots, h^d(t)\big) - f\big(h^1(s), \dots, h^d(s)\big)\big|}{|t - s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t \\ \le& 2^d \sum_{i=1}^{d} \int_0^T \int_0^t \frac{\int_{\mathbb{R}^d} \big( \mathbf{1}_{\{h^i(s) < a_i \le h^i(t)\}} + \mathbf{1}_{\{h^i(t) < a_i \le h^i(s)\}} \big)\,\mu(\mathrm{d}\mathbf{a})}{|t - s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t. \end{aligned}$$

By symmetry we only consider the case $\mathbf{1}_{\{h^i(s)<a_i\le h^i(t)\}}$. According to Tonelli's theorem, we obtain

$$\begin{aligned}&\int_0^T\int_0^t\int_{\mathbb{R}^d}\frac{\mathbf{1}_{\{h^i(s)<a_i\le h^i(t)\}}}{|t-s|^{1+\beta}}\,\mu(\mathrm{d}\mathbf{a})\,\mathrm{d}s\mathrm{d}t\\&=\int_{\mathbb{R}^d}\int_0^T\int_0^t\frac{\mathbf{1}_{\{h^i(s)<a_i\le h^i(t)\}}}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t\,\mu(\mathrm{d}\mathbf{a}).\end{aligned}$$

According to Assumption 5.2.1 and similar arguments which derive (5.14) in the proof of Theorem 5.2.1, we obtain

$$\begin{aligned}J^i:=&\int_{\mathbb{R}^d}\int_0^T\int_0^t\frac{\mathbf{1}_{\{h^i(s)<a_i\le h^i(t)\}}}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t\,\mu(\mathrm{d}\mathbf{a})\\=&\int_{\mathbb{R}^d}\int_0^T\int_0^t\frac{\mathbf{1}_{\{h^i(s)<a_i<h^i(t)\}}}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t\,\mu(\mathrm{d}\mathbf{a}),\end{aligned}\tag{5.22}$$

and

$$J_1(a_i):=\int_0^T\int_0^t\frac{\mathbf{1}_{\{h^i(s)<a_i<h^i(t)\}}}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t.$$

Therefore we have

$$J^i=\int_{\mathbb{R}^d}J_1(a_i)\,\mu(\mathrm{d}\mathbf{a}).$$

Recall that for every function $h^i$ with $i=1,\dots,d$, the last hitting time of $h^i$ into level $a_i$ during the time interval $[0,t]$ for $t\in[0,T]$ is defined as

$$T_t^i(a_i):=\sup\{u\in[0,t]:h^i(u)=a_i\}.$$

If $h^i$ never hits $a_i$ on $[0,t]$, let $T_t^i(a_i)=0$. If $T_t^i(a_i)=0$ for some $t$, then $\mathbf{1}_{\{h^i(s)<a_i<h^i(t)\}}$ is 0 for these $t$. Therefore we only need to consider the case when $T_t^i(a_i)>0$. If $h^i(t)>a_i$ for some $t$, then $T_t^i(a_i)<t$ for these $t$. Thus by following the same arguments as in the proof of Theorem 5.2.1, we derive

$$\begin{aligned}J_1(a_i)&\le\int_0^T\int_0^t\frac{\mathbf{1}_{\{h^i(s)<a_i<h^i(t)\}}}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t\\&\le\int_0^T\mathbf{1}_{\{a_i<h^i(t)\}}\frac{\big(t-T_t^i(a_i)\big)^{-\beta}}{\beta}\,\mathrm{d}t.\end{aligned}$$

Since $h^i\in C^{\alpha_i}([0,T])$, by following similar arguments in the proof of Theorem 5.2.1 we obtain

$$J_1(a_i)\le\frac{H_{\alpha_i}^{\delta_i}(h^i)}{\beta}\int_0^T|h^i(t)-a_i|^{-\delta_i}\,\mathrm{d}t,$$

where $H_{\alpha_i}(h^i)$ is the Hölder coefficient of $h^i$ and $\delta_i=\frac{\beta}{\alpha_i}$. Since $\delta_i\in(0,1)$, according to Assumption 5.2.1, it follows

$$J_1(a_i)\le\frac{H_{\alpha_i}^{\delta_i}(h^i)}{\beta}M_i=C_i,$$

where $C_i$ is a constant independent of $a$. This implies

$$J^i=\int_{\mathbb{R}^d}J_1(a_i)\,\mu(\mathrm{d}\mathbf{a})<\infty.$$

Therefore, we conclude

$$\begin{aligned}&\int_0^T\int_0^t \frac{|f\big(h^1(t),\ldots,h^d(t)\big)-f\big(h^1(s),\ldots,h^d(s)\big)|}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t\\ \le& 2^d\sum_{i=1}^d\int_0^T\int_0^t \frac{\int_{\mathbb{R}^d}\big(\mathbf{1}_{\{h^i(s)<a_i\le h^i(t)\}}+\mathbf{1}_{\{h^i(t)<a_i\le h^i(s)\}}\big)\,\mu(\mathrm{d}\mathbf{a})}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t\\ <&\infty.\end{aligned}$$

*Step 2.* For general $f\in\mathcal{R}(\mathbb{R}^d)$, by linearity of the generalized Lebesgue–Stieltjes integral, we can obtain the result.

□

**Remark 5.3.1.** *Let $f:\mathbb{R}^d\mapsto\mathbb{R}$ be a function in $C^2(\mathbb{R}^d)$ and let $g:\mathbb{R}\mapsto\mathbb{R}$ be a function in $BV(\mathbb{R})$. We have already known that $g=g^1-g^2$ for increasing $g_1$ and $g_2$, and $g^1=g^1_+$ (resp. $g^2=g^2_+$) Lebesgue-almost surely for right-continuous increasing $g^1_+$ (resp. $g^2_+$). Moreover, $g^1_+$ (resp. $g^2_+$) together with the finite Lebesgue–Stieltjes measure $\mu_1$ (resp. $\mu_2$) of $g^1_+$ (resp. $g^2_+$) admit the representation (2.6). Let $\mathbf{x}=(x_1,\ldots,x_d)$. For $g_+=g^1_+-g^2_+$ we get*

$$\begin{aligned}g_+\big(f(\mathbf{x})\big)&=\frac{1}{2}\int_{\mathbb{R}}\operatorname{sgn}\big(f(\mathbf{x})-a\big)\mu(\mathrm{d}a)+C\\&=\frac{1}{2}\int_{\mathbb{R}}\big(\mathbf{1}_{\{a\le f(\mathbf{x})\}}-\mathbf{1}_{\{f(\mathbf{x})<a\}}\big)\mu(\mathrm{d}a)+C,\end{aligned}$$

*where $\mu=\mu_1-\mu_2$ is a finite signed measure, and $C$ is some constant. Since $f$ is in $C^2(\mathbb{R}^d)$, by Taylor expansion, we obtain*

$$f(\mathbf{x})=f(0)+\sum_{k=1}^d\partial_{x_k}f(0)x_k+\sum_{|r|=2}h_r(\mathbf{x})^2, \tag{5.23}$$

*where $\lim_{\mathbf{x}\to\mathbf{0}}h_r(\mathbf{x})=0$. Now let $x_k=u^k(t)$, where $u^k(t)$ is a Hölder continuous function of order $\alpha_k\in(0,1)$ for every $k=1,\ldots,d$ on $[0,T]$. Let $\mathbf{x}=\mathbf{u}(t)=\big(u^1(t),\ldots,u^d(t)\big)$, and (5.23) becomes*

$$\begin{aligned}f(\mathbf{x})=&f\big(\mathbf{u}(t)\big)=f\big(u^1(t),\ldots,u^d(t)\big)\\=&f(0)+\sum_{k=1}^d\partial_{x_k}f(0)u^k(t)+o\Big(\big(\mathbf{u}(t)\big)^2\Big).\end{aligned}$$

*It is clear that $f$ is Hölder continuous of order $\min(\alpha_1,\ldots,\alpha_d)$. Now we have*

$$g_+\Big(f\big(\mathbf{u}(t)\big)\Big)=\frac{1}{2}\int_{\mathbb{R}}\big(\mathbf{1}_{\{a\le f(\mathbf{u}(t))\}}-\mathbf{1}_{\{f(\mathbf{u}(t))<a\}}\big)\mu(\mathrm{d}a)+C,$$

*Lebesgue-almost everywhere, which reduces back to one-dimensional case. Therefore for $v\in C^\gamma([0,T])$ with $\gamma\in(0,1)$ such that $\alpha_k+\gamma>1$ for every $k=1,\ldots,d$, if Assumption 5.2.1 is satisfied for every $u^k$, then*

$$(\mathrm{gLS})\int_0^T g\circ f\circ\mathbf{u}(t)\,\mathrm{d}v(t)$$

*exists.*

### 5.3.2 Existence of Generalized Lebesgue–Stieltjes Integral for Multidimensional Processes

Consider a $d$-dimensional vector $\alpha = (\alpha_1, \dots, \alpha_d)$ where every $\alpha_k \in (0,1)$ for $k = 1, \dots, d$.

**Definition 5.3.3.** *Let $X = (X^1, \dots, X^d)$ be a multidimensional stochastic process on $[0,T]$. We denote $X \in C^{\alpha}([0,T])$ if for every $k = 1, \dots, d$ we have $X^k \in C^{\alpha_k}([0,T])$.*

**Theorem 5.3.2.** *Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ be a function in $\mathcal{R}(\mathbb{R}^d)$. Let $X$ be a process in $C^{\alpha}([0,T])$ such that every $X^k$ for $k = 1, \dots, d$ satisfies Assumption 5.2.3 and $Y$ be a process in $C^{\gamma}([0,T])$ with $\gamma \in (0,1)$. If $\alpha_k + \gamma > 1$ for every $k = 1, \dots, d$, then the integral*

$$\text{(gLS)} \int_0^T f(X_t)\,\mathrm{d}Y_t$$

*exists a.s.*

*Proof.* Choose some $\beta \in (1-\gamma, \min(\alpha_1, \dots, \alpha_d))$, then the paths of $Y$ belong to the space $W_T^{1-\beta,\infty}$ a.s. According to Proposition 3.2.6, the generalized Lebesgue–Stieltjes integral exists if

$$\|f(X_t^1, \dots, X_t^d)\|_{\beta,1} < \infty \quad a.s.$$

Following similar arguments as in the proof of Theorem 5.3.1, the first term in the norm $\|\cdot\|_{\beta,1}$ is finite a.s. For the second term in the norm $\|\cdot\|_{\beta,1}$, we can write as follows

$$\begin{aligned} &|f(X_t^1, \dots, X_t^d) - f(X_s^1, \dots, X_s^d)| \\ \le& 2^d \int_{\mathbb{R}^d} \Big( \sum_{i=1}^d \mathbf{1}_{\{X_s^i < a_i \le X_t^i\}} + \sum_{i=1}^d \mathbf{1}_{\{X_t^i < a_i \le X_s^i\}} \Big)\,\mu(\mathrm{d}\mathbf{a}) \\ =& 2^d \sum_{i=1}^d \int_{\mathbb{R}^d} \big( \mathbf{1}_{\{X_s^i < a_i \le X_t^i\}} + \mathbf{1}_{\{X_t^i < a_i \le X_s^i\}} \big)\,\mu(\mathrm{d}\mathbf{a}). \end{aligned} \tag{5.24}$$

We will only consider the case $\mathbf{1}_{\{X_s^1 < a_1 \le X_t^1\}}$ since the other case can be treated similarly. Note that $X_t^k$, for $k = 1, \dots, d$ has a density for almost every $t$, therefore for every $s$ and almost every $t \in [0,T]$ we have

$$\mathbf{1}_{\{X_s^1 < a_1 \le X_t^1\}} = \mathbf{1}_{\{X_s^1 < a_1 < X_t^1\}} \quad a.s.$$

The rest of the proof goes similarly as the proof of Theorem 5.2.5 and 5.3.1. □

## 5.4 Change of Variables Formula

In this section, we will prove a change of variables formula and we will fix the gaps in literature as explained in section 5.1. First, we will recall a change of variables formula for smooth functions taken from [71].

**Theorem 5.4.1.** *Let $F$ be a real-valued function in $C^2(\mathbb{R})$.*

*(i) If $h$ is a function in $C^\alpha([0,T])$ with $\alpha > \frac{1}{2}$, then*

$$F\big(h(T)\big) - F\big(h(0)\big) = (\mathrm{RS}) \int_0^T F'\big(h(t)\big) \,\mathrm{d}h(t).$$

*(ii) If $X$ is a Hölder continuous process of order $\alpha > \frac{1}{2}$ on $[0,T]$, then*

$$F(X_T) - F(X_0) = (\mathrm{RS}) \int_0^T F'(X_t) \,\mathrm{d}X_t.$$

*Proof.* It is well known that by Taylor's theorem and the fact that the quadratic variation of $\alpha$-Hölder continuous function with $\alpha > \frac{1}{2}$ is 0, one can derive

$$F\big(h(t)\big) = F\big(h(0)\big) + (\mathrm{RS}) \int_0^t F'\big(h(s)\big) \,\mathrm{d}h(s), \quad t \in [0,T].$$

If $X$ is a Hölder continuous process of order $\alpha > \frac{1}{2}$, then by Taylor's theorem and the fact that the quadratic variation of $X$ is zero, we obtain

$$F(X_t) = F(X_0) + (\mathrm{RS}) \int_0^t F'(X_t) \,\mathrm{d}X_t, \quad t \in [0,T].$$

□

Theorem 5.4.1 is quite clear since a smooth function of a Hölder continuous function is still Hölder continuous. Therefore, by Young's integration theory, Riemann–Stieltjes integral exists.

Next we will consider the general case when $F$ may not be smooth enough. Recall from Section 2.2.2 that we define $LC_{conv}(\mathbb{R})$ to be the space of functions which are linear combinations of convex functions on $\mathbb{R}$.

**Theorem 5.4.2.** *Let $F$ be a real-valued function in $LC_{conv}(\mathbb{R})$.*

*(i) If $h$ is a function in $C^\alpha([0,T])$ with $\alpha > \frac{1}{2}$ such that Assumption 5.2.1 is satisfied, then*

$$F\big(h(T)\big) - F\big(h(0)\big) = (\mathrm{gLS}) \int_0^T F'_+\big(h(t)\big) \,\mathrm{d}h(t). \tag{5.25}$$

*(ii) If $X$ is a Hölder continuous process of order $\alpha > \frac{1}{2}$ on $[0,T]$ such that Assumption 5.2.3 is satisfied, then*

$$F(X_T) - F(X_0) = (\mathrm{gLS}) \int_0^T F'_+(X_t)\, \mathrm{d}X_t. \tag{5.26}$$

Before giving the proof, recall the following fundamental result first. Consider a space $S = [0,T]^2$, equipped with a measure $\eta$ given by

$$\eta(A) = \int_0^T \int_0^T \mathbf{1}_A \,\mathrm{d}s\mathrm{d}t,$$

where $A \subset S$. Clearly, $\eta$ is the Lebesgue measure on $\mathbb{R}^2$ restricted to $S$, thus $\eta$ is a finite measure on $S$. Consequently, it is straightforward to obtain the following theorem.

**Theorem 5.4.3.** *Let $f_n$ be a sequence of positive and integrable functions on $(S,\eta)$. Let $f$ be a positive and integrable function on $(S,\eta)$ such that $f_n(s,t) \to f(s,t)$ for almost every $(s,t) \in S$ as $n \to \infty$. Then the following statements are equivalent:*

*1) the sequence $f_n$ is $\eta$-uniformly integrable,*

*2) $f_n \to f$ in $L^1(S,\eta)$ as $n \to \infty$,*

*3) $\int f_n \mathrm{d}\eta \to \int f\,\mathrm{d}\eta$ as $n \to \infty$.*

**Remark 5.4.1.** *Usually, the above theorem can be found for probability measures. Note that the only key feature is the fact that the measure $\eta$ needs to be finite.*

The following lemma is needed for proving Theorem 5.4.2.

**Lemma 5.4.1.** *Let $h$ be a function in $C^\alpha([0,T])$ with order $\alpha \in (0,1)$ such that Assumption 5.2.1 is satisfied. Let $\beta$ be a constant in $(0,\alpha)$, then*

$$J_1(a) := \int_0^T \int_0^t \frac{\mathbf{1}_{\{h(s)<a<h(t)\}}}{|t-s|^{1+\beta}} \,\mathrm{d}s\mathrm{d}t \tag{5.27}$$

*is a continuous function with a compact support.*

*Proof.* Since $h$ is a continuous function on $[0,T]$, $h$ is bounded over $[0,T]$. For $a \in \mathbb{R}$ which lies outside of the range of $h$, $J_1(a)$ will be 0. This implies that $J_1(a)$ has a compact support.

Let $\epsilon > 0$ and first consider

$$
\begin{aligned}
J_1(a+\epsilon) - J_1(a) &= \int_0^T \int_0^t \frac{\mathbf{1}_{\{h(s)<a+\epsilon<h(t)\}} - \mathbf{1}_{\{h(s)<a<h(t)\}}}{|t-s|^{1+\beta}} \mathrm{d}s\mathrm{d}t \\
&= \int_0^T \int_0^t \frac{\mathbf{1}_{\{a\le h(s)<a+\epsilon<h(t)\}} - \mathbf{1}_{\{h(s)<a<h(t)\le a+\epsilon\}}}{|t-s|^{1+\beta}} \mathrm{d}s\mathrm{d}t \\
&= \int_0^T \int_0^t \frac{\mathbf{1}_{\{a\le h(s)<a+\epsilon<h(t)\}}}{|t-s|^{1+\beta}} \mathrm{d}s\mathrm{d}t - \int_0^T \int_0^t \frac{\mathbf{1}_{\{h(s)<a<h(t)\le a+\epsilon\}}}{|t-s|^{1+\beta}} \mathrm{d}s\mathrm{d}t \\
:=& I_1(a,\epsilon) - I_2(a,\epsilon).
\end{aligned}
$$

For $I_2$, we have

$$
\begin{aligned}
I_2(a,\epsilon) &= \int_0^T \int_0^t \frac{\mathbf{1}_{\{h(s)<a<h(t)\le a+\epsilon\}}}{|t-s|^{1+\beta}} \mathrm{d}s\mathrm{d}t \\
&\le \int_0^T \int_0^t \frac{\mathbf{1}_{\{h(s)<a<h(t)\}}}{|t-s|^{1+\beta}} \mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

Therefore, according to Theorem 5.2.1, $\frac{\mathbf{1}_{\{h(s)<a<h(t)\le a+\epsilon\}}}{|t-s|^{1+\beta}}$ has an integrable upper bound with respect to the Lebesgue measure $\eta$.

Recall that the last hitting time of $h$ into level $a$ over the time $[0,t]$ is defined as

$$
T_t(a) := \sup\{u \in [0,t] : h(u) = a\}.
$$

If $h$ never hits $a$ on $[0,t]$, then let $T_t(a) = 0$. If $T_t(a) > 0$ for some $t \in [0,T]$ and $a < h(t)$ holds, then $T_t(a) < t$ for those $t$. Then we have

$$
\begin{aligned}
I_2(a,\epsilon) &= \int_0^T \int_0^t \frac{\mathbf{1}_{\{h(s)<a<h(t)\le a+\epsilon\}}}{|t-s|^{1+\beta}} \mathrm{d}s\mathrm{d}t \\
&= \int_0^T \int_0^{T_t(a)} \frac{\mathbf{1}_{\{h(s)<a<h(t)\le a+\epsilon\}}}{|t-s|^{1+\beta}} \mathrm{d}s\mathrm{d}t \\
&= \int_0^T \mathbf{1}_{\{a<h(t)\le a+\epsilon\}} \int_0^{T_t(a)} \frac{\mathbf{1}_{\{h(s)<a\}}}{|t-s|^{1+\beta}} \mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

Note that

$$
\mathbf{1}_{\{a<h(t)\le a+\epsilon\}} \to 0,
$$

as $\epsilon \to 0$ from above. Therefore, by applying dominated convergence theorem to $I_2(a,\epsilon)$, we obtain

$$
\begin{aligned}
&\lim_{\epsilon\to 0} I_2(a,\epsilon) \\
=& \int_0^T \lim_{\epsilon\to 0} \mathbf{1}_{\{a<h(t)\le a+\epsilon\}} \int_0^{T_t(a)} \frac{\mathbf{1}_{\{h(s)<a\}}}{|t-s|^{1+\beta}} \mathrm{d}s\mathrm{d}t \\
=& 0,
\end{aligned}
$$

as $\epsilon \to 0$ from above.

Now let us turn to $I_1$, which is

$$
\begin{aligned}
I_1(a,\epsilon) &= \int_0^T \int_0^t \frac{\mathbf{1}_{\{a\le h(s)<a+\epsilon<h(t)\}}}{|t-s|^{1+\beta}} \mathrm{d}s\mathrm{d}t \\
&\le \int_0^T \int_0^t \frac{\mathbf{1}_{\{h(s)<a+\epsilon<h(t)\}}}{|t-s|^{1+\beta}} \mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

Therefore, according to Theorem 5.2.1, $\frac{\mathbf{1}_{\{a\le h(s)<a+\epsilon<h(t)\}}}{|t-s|^{1+\beta}}$ has an integrable upper bound with respect to the Lebesgue measure $\eta$.

Now we can rewrite

$$\begin{aligned} I_1(a,\epsilon) &= \int_0^T \int_0^t \frac{\mathbf{1}_{\{a\le h(s)<a+\epsilon<h(t)\}}}{|t-s|^{1+\beta}} \,\mathrm{d}s\mathrm{d}t \\ &= \int_0^T \int_s^T \frac{\mathbf{1}_{\{a\le h(s)<a+\epsilon<h(t)\}}}{|t-s|^{1+\beta}} \,\mathrm{d}t\mathrm{d}s. \end{aligned}$$

Define $\tilde{T}_s(a+\epsilon)$ to be the first hitting time of $h$ into level $a+\epsilon$ over the time interval $[s,T]$, i.e.

$$\tilde{T}_s(a+\epsilon) := \inf\{u \in [s,T] : h(u) = a+\epsilon\}.$$

If $h$ never hits $a+\epsilon$ on $[s,T]$, then let $\tilde{T}_s(a+\epsilon) = T$. If $\tilde{T}_s(a+\epsilon) < T$ and $h(s) < a+\epsilon$ holds for some $s$, then $\tilde{T}_s(a+\epsilon) > s$. Then

$$\begin{aligned} I_1(a,\epsilon) &= \int_0^T \int_s^T \frac{\mathbf{1}_{\{a\le h(s)<a+\epsilon<h(t)\}}}{|t-s|^{1+\beta}} \,\mathrm{d}t\mathrm{d}s \\ &= \int_0^T \int_{\tilde{T}_s(a+\epsilon)}^T \frac{\mathbf{1}_{\{a\le h(s)<a+\epsilon<h(t)\}}}{|t-s|^{1+\beta}} \,\mathrm{d}t\mathrm{d}s \\ &= \int_0^T \mathbf{1}_{\{a\le h(s)<a+\epsilon\}} \int_{\tilde{T}_s(a+\epsilon)}^T \frac{\mathbf{1}_{\{a+\epsilon<h(t)\}}}{|t-s|^{1+\beta}} \,\mathrm{d}t\mathrm{d}s \\ &\le \int_0^T \mathbf{1}_{\{a\le h(s)<a+\epsilon\}} \int_{\tilde{T}_s(a+\epsilon)}^T \frac{1}{|t-s|^{1+\beta}} \,\mathrm{d}t\mathrm{d}s \\ &\le \int_0^T \mathbf{1}_{\{a\le h(s)<a+\epsilon\}} \beta^{-1} \big(\tilde{T}_s(a+\epsilon)-s\big)^{-\beta} \,\mathrm{d}s. \end{aligned}$$

According to the proof of Theorem 5.2.1, we have

$$\big(\tilde{T}_s(a+\epsilon)-s\big)^{-\beta} \le H_\alpha^\delta(h)|a+\epsilon-h(s)|^{-\delta},$$

where $H_\alpha(h)$ is the Hölder constant of $h$ and $\delta = \frac{\beta}{\alpha}$. Therefore we obtain

$$I_1(a,\epsilon) \le \frac{H_\alpha^\delta(h)}{\beta} \int_0^T \mathbf{1}_{\{a\le h(s)<a+\epsilon\}} |a+\epsilon-h(s)|^{-\delta} \,\mathrm{d}s.$$

By Hölder inequality, for $p>1$, $q>\frac{1}{\delta}$ such that $\frac{1}{p}+\frac{1}{q}=1$, we obtain

$$\begin{aligned} &\int_0^T \mathbf{1}_{\{a\le h(s)<a+\epsilon\}} |a+\epsilon-h(s)|^{-\delta} \,\mathrm{d}s \\ \le&\Big(\int_0^T \big(\mathbf{1}_{\{a\le h(s)<a+\epsilon\}}\big)^p \,\mathrm{d}s\Big)^{1/p} \Big(\int_0^T |a+\epsilon-h(s)|^{-\delta q} \,\mathrm{d}s\Big)^{1/q} \qquad (5.28) \\ \le&\Big(\int_0^T \mathbf{1}_{\{a\le h(s)<a+\epsilon\}} \,\mathrm{d}s\Big)^{1/p} M^{1/q}, \end{aligned}$$

since by Assumption 5.2.1, for $\delta q \in (0,1)$,

$$\int_0^T |a+\epsilon-h(s)|^{-\delta q} \,\mathrm{d}s \le M.$$

Now

$$\int_0^T \mathbf{1}_{\{a\le h(s)<a+\epsilon\}}\,\mathrm{d}s = \lambda(\{s \le t : h(s) \in [a, a+\epsilon)\}),$$

where $\lambda$ is the Lebesgue measure. Since $\lambda(\{s \le T : h(s) \in [a, a+\epsilon)\}) \to \lambda(\{s \le T : h(s) = a\})$ as $\epsilon \to 0$ from above, according to Assumption 5.2.1 we have $\lambda(\{s \le T : h(s) = a\}) = 0$. Therefore, by applying dominated convergence theorem to $I_1(a,\epsilon)$, we conclude $I_1(a,\epsilon) \to 0$ as $\epsilon \to 0$ from above.

For $J_1(a) - J_1(a-\epsilon)$, by following similar arguments as above, we can also derive that

$$J_1(a) - J_1(a-\epsilon) \to 0,$$

as $\epsilon \to 0$ from above. Thus we complete the proof. □

**Remark 5.4.2.** *If all the conditions in Lemma 5.4.1 are fulfilled, then by following similar arguments in the proof of Lemma 5.4.1, it can be shown that*

$$\tilde{J}_1(a) := \int_0^T \int_0^t \frac{\mathbf{1}_{\{h(t)<a<h(s)\}}}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t \tag{5.29}$$

*is also a continuous function with a compact support.*

*Proof of Theorem 5.4.2.* (i) Let $F_n$ be the smooth approximation of $F$ defined as in equation (5.6). Then $F_n$ converges to $F$ pointwise, and $F_n'$ increases to $F_+'$. Since $F_n$ is smooth, by Theorem 5.4.1, for $t \in [0,T]$ we have

$$F_n\big(h(t)\big) - F_n\big(h(0)\big) = (\mathrm{RS})\int_0^t F_n'\big(h(s)\big)\,\mathrm{d}h(s).$$

Note that $F_n\big(h(t)\big) \to F\big(h(t)\big)$ and $F_n\big(h(0)\big) \to F\big(h(0)\big)$ Lebesgue-almost everywhere. Hence we only need to prove the convergence of integrals

$$(\mathrm{RS})\int_0^t F_n'\big(h(s)\big)\,\mathrm{d}h(s) \to (\mathrm{gLS})\int_0^t F_+'\big(h(s)\big)\,\mathrm{d}h(s).$$

Since $F \in LC_{conv}(\mathrm{I\!R})$, by Proposition 2.2.2, we have $F_+' \in BV^{loc}(\mathrm{I\!R})$. By linearity of differentiation, instead of $F$ together with $F_+'$, it is sufficient to consider a convex function $f$ together with its right-sided derivative $f_+'$. Assume that the Lebesgue–Stieltjes measure $\mu$ of $f_+'$ has a compact support $\mathcal{K}$. Otherwise, by similar arguments in the beginning of Chapter 5, we can always construct a sequence of $\tilde{f}_n$ which equals $f$ on $[-n,n]$ and the Radon measure corresponding to $\tilde{f}_n$ has a compact support. Therefore, by Theorem 2.1.1, $f_+'$ admits a representation (2.6) with respect to $\mu$.

Note that, by linearity of the generalized Lebesgue–Stieltjes integral, instead of $F_n$ it is sufficient to consider smooth approximation $f_n$ of $f$ defined as equation (5.6). Since $f_n$ is smooth, for $t \in [0,T]$ we get

$$f_n\big(h(t)\big) = f_n\big(h(0)\big) + (\mathrm{gLS})\int_0^t f_n'\big(h(s)\big)\,\mathrm{d}h(s).$$

This can be seen by Proposition 3.2.5. Moreover, we can assume that the support of $f_n''$ equals the support of $\mu$. Indeed, as above we can define auxiliary function $\tilde{f}_{n,k}$ for which $\tilde{f}_{n,k}''$ has a compact support and $\tilde{f}_{n,k}$ is equal to $f_n$ on the set $[-n,n]$. According to (5.9), if $g$ is a real-valued continuous function with a compact support, then

$$\lim_{n\to\infty}\int_{\mathbb{R}} g(x)f_n''(x)\,\mathrm{d}x = \int_{\mathbb{R}} g(x)\,\mu(\mathrm{d}x).$$

In order to prove the theorem, it is enough to prove the convergence

$$\text{(gLS)}\int_0^t f_n'\big(h(s)\big)\,\mathrm{d}h(s) \to \text{(gLS)}\int_0^t f_+'\big(h(s)\big)\,\mathrm{d}h(s).$$

Choose some $\beta \in (1-\alpha,\alpha)$. Since $h \in C^\alpha([0,T])$, one obtains $h \in W_T^{1-\beta,\infty}([0,T])$. Therefore, by Proposition 3.2.7 and Corollary 3.2.1, to prove the convergence of the Lebesgue–Stieltjes integrals, it is sufficient to show

$$\|f_n'\big(h(t)\big) - f_+'\big(h(t)\big)\|_{\beta,1} \to 0.$$

For the first term in the norm $\|\cdot\|_{\beta,1}$, by the boundedness of $f_+'$ on compact sets and the fact that $f_n' \downarrow f_+'$, we have

$$\begin{aligned}
&\frac{|f_n'\big(h(t)\big) - f_+'\big(h(t)\big)|}{t^\beta}\\
\leq&\frac{\sup_{t\in[0,T]}|f_n'\big(h(t)\big)| + \sup_{t\in[0,T]}|f_+'\big(h(t)\big)|}{t^\beta}\\
\leq&\frac{\sup_{t\in[0,T]}|f_1'\big(h(t)\big)| + \sup_{t\in[0,T]}|f_+'\big(h(t)\big)|}{t^\beta}\\
\in& L^1([0,T],\mathrm{d}t),
\end{aligned}$$

which is an integrable upper bound. Thus by Lebesgue dominated convergence theorem, we obtain

$$\int_0^T \frac{|f_n'\big(h(t)\big) - f_+'\big(h(t)\big)|}{t^\beta}\,\mathrm{d}t \to 0.$$

For the second term in the norm $\|\cdot\|_{\beta,1}$, we have

$$\begin{aligned}
&\frac{|f_n'\big(h(t)\big) - f_+'\big(h(t)\big) - f_n'\big(h(s)\big) + f_+'\big(h(s)\big)|}{|t-s|^{1+\beta}}\\
\leq&\frac{|f_n'\big(h(t)\big) - f_n'\big(h(s)\big)|}{|t-s|^{1+\beta}} + \frac{|f_+'\big(h(t)\big) - f_+'\big(h(s)\big)|}{|t-s|^{1+\beta}}.
\end{aligned}$$

Denote now

$$g_n(s,t) = \frac{|f_n'\big(h(t)\big) - f_n'\big(h(s)\big)|}{|t-s|^{1+\beta}},$$

and

$$g(s,t) = \frac{|f_+'\big(h(t)\big) - f_+'\big(h(s)\big)|}{|t-s|^{1+\beta}}.$$

Then we obtain $g_n \to g$ Lebesgue-almost everywhere. Since $f_n$ is smooth and $h \in C^\alpha([0,T])$, we have

$$|f_n'(h(t)) - f_n'(h(s))| = \big|f_n''(\xi)|h(t) - h(s)|\big| \leq |H_\alpha(h) f_n''(\xi)|\,|t-s|^\alpha,$$

where $\xi$ is between $h(s)$ and $h(t)$, and $H_\alpha(h)$ is the Hölder coefficient of $h$. Therefore the sequence $g_n$ is integrable with respect to $\eta$ for every $n$.

Moreover, function $g$ is positive and integrable according to the proof of Theorem 5.2.1. Furthermore, since $f_n$ is a smooth convex function and we assume that the support of $f_n''$ equals the support of $\mu$, it follows that $\mu_n(dx) = f_n''(x)dx$ and $\mu_n$ has a compact support $\mathcal{K}$. Therefore we can apply representation (2.6) for $f_n'$ with $\mu_n$ to get

$$\frac{|f_n'(h(t)) - f_n'(h(s))|}{|t-s|^{1+\beta}} = \frac{|\int_{\mathcal{K}}(\mathbf{1}_{\{h(s)<a\leq h(t)\}} + \mathbf{1}_{\{h(t)<a\leq h(s)\}})f_n''(a)\mathrm{d}a|}{|t-s|^{1+\beta}}.$$

Hence, by applying Tonelli's theorem and the fact that (5.14) holds Lebesgue-almost everywhere, we derive

$$\begin{aligned}
&\int_0^T \int_0^t \frac{|f_n'(h(t)) - f_n'(h(s))|}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t \\
&= \int_0^T \int_0^t \frac{|\int_{\mathcal{K}}(\mathbf{1}_{\{h(s)<a<h(t)\}} + \mathbf{1}_{\{h(t)<a<h(s)\}})f_n''(a)\,\mathrm{d}a|}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t \\
&= \int_{\mathcal{K}} \int_0^T \int_0^t \frac{(\mathbf{1}_{\{h(s)<a<h(t)\}} + \mathbf{1}_{\{h(t)<a<h(s)\}})}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t\, f_n''(a)\,\mathrm{d}a \\
&\to \int_{\mathcal{K}} \int_0^T \int_0^t \frac{(\mathbf{1}_{\{h(s)<a<h(t)\}} + \mathbf{1}_{\{h(t)<a<h(s)\}})}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t\, \mu(\mathrm{d}a) \\
&= \int g\,\mathrm{d}\eta,
\end{aligned}$$

where the convergence takes place according to (5.9). Indeed, according to Lemma 5.4.1 and Remark 5.4.2, a function

$$a \to \int_0^T \int_0^t \frac{(\mathbf{1}_{\{h(s)<a<h(t)\}} + \mathbf{1}_{\{h(t)<a<h(s)\}})}{|t-s|^{1+\beta}}\,\mathrm{d}s\mathrm{d}t$$

is a continuous and positive function with a compact support.

Consequently, we obtain

$$\int g_n(s,t)\,\mathrm{d}\eta \to \int g(s,t)\,\mathrm{d}\eta,$$

which together with Theorem 5.4.3 implies

$$\int |g_n - g|\,\mathrm{d}\eta \to 0.$$

In other words, we have

$$\int_0^T \int_0^t \frac{\big||f_n'(h(t)) - f_n'(h(s))| - |f_+'(h(t)) - f_+'(h(s))|\big|}{(t-s)^{\beta+1}}\,\mathrm{d}s\mathrm{d}t \to 0.$$

To conclude, it remains to note that

$$\begin{aligned}&\big||f_n'\big(h(t)\big) - f_n'\big(h(s)\big)| - |f_+'\big(h(t)\big) - f_+'\big(h(s)\big)|\big|\\ =&|f_n'\big(h(t)\big) - f_n'\big(h(s)\big) - f_+'\big(h(t)\big) + f_+'\big(h(s)\big)|,\end{aligned}$$

since $f_n'$ and $f_+'$ are increasing functions. This shows that

$$\|f_n'\big(h(t)\big) - f_+'\big(h(t)\big)\|_{\beta,1} \to 0,$$

which concludes the proof.

(ii) For process $X$, we can do the above proof path-by-path, and the only thing we need is the existence of the generalized Lebesgue–Stieltjes integral $\int_0^t f_+'(X_s)\mathrm{d}X_s$, which can be shown through Theorem 5.2.5.

□

For the class $\mathcal{R}(\mathbb{R}^d)$, it is unclear if the change of variables formula holds for multidimensional functions and processes. This is because for a multidimensional function $f$ such that every partial derivative belongs to the class $\mathcal{R}(\mathbb{R}^d)$, we cannot conclude that the corresponding partial derivative of the mollified $f_n$ also belongs to the class $\mathcal{R}(\mathbb{R}^d)$.

# 6. Conclusion

Since Itō integration cannot be applied beyond semimartingales, other integration theories are needed for that case. In order to study the stochastic integral of unbounded $p$-variation processes for $p \geq 1$ with respect to general processes beyond semimartingales, this dissertation applies pathwise integration theory based on the generalized Lebesgue–Stieltjes integral.

In this dissertation, a certain class of unbounded $p$-variation processes for $p \geq 1$ has been considered. This class of processes is a composition of a Hölder continuous process $X$ with a nonrandom function $f$ which is of locally bounded variation. The existence of a generalized Lebesgue–Stieltjes integral of $f(X)$ which satisfies Assumption 5.2.1 with respect to a general Hölder continuous process has been shown almost surely in Theorem 5.2.3. Moreover, if $X$ satisfies a not very restrictive density Assumption 5.2.3, the generalized Lebesgue–Stieltjes integral also exists almost surely according to Theorem 5.2.5.

In addition to one-dimensional case, this dissertation also considered a multi-dimensional process $X = (X^1, \ldots, X^d)$, where each $X^i$ satisfies Assumption 5.2.3. The existence of a generalized Lebesgue–Stieltjes integral of a certain multidimensional process with respect to a general Hölder continuous process was shown almost surely in Theorem 5.3.2.

Based on the existence of the generalized Lebesgue–Stieltjes integrals, this dissertation has proved a change of variables formula for the aforementioned one-dimensional unbounded $p$-variation processes in Theorem 5.4.2. Moreover, the gaps that had been discovered in the proofs of the change of variables formula in the literature have also been fixed in this dissertation.

The pathwise stochastic integration technique of the generalized Lebesgue–Stieltjes integral has been applied to fractional Brownian motions, functionals of fractional Brownian motions and a class of Gaussian processes. Now this dissertation has confirmed the validity of the pathwise integration theory to a general class of unbounded $p$-variation processes with respect to a general class of Hölder continuous processes. However, since we still need the density Assumption 5.2.3 for the process, one of the possible future studies related to this research could be the analysis of processes without densities.

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