Department of Mathematics and Systems Analysis

Finite Element Methods for Flow in Porous Media

Juho Könnö

Aalto University

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Juho Könnö

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Abstract

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Abstract
This thesis studies the application of finite element methods to porous flow problems. Particular attention is paid to locally mass conserving methods, which are very well suited for typical multiphase flow applications in porous media. The focus is on the Brinkman model, which is a parameter dependent extension of the classical Darcy model for porous flow taking the viscous phenomena into account. The thesis introduces a mass conserving finite element method for the Brinkman flow, with complete mathematical analysis of the method. In addition, stochastic material parameters are considered for the Brinkman flow, and parameter dependent Robin boundary conditions for the underlying Darcy flow. All of the theoretical results in the thesis are also verified with extensive numerical testing. Furthermore, many implementational aspects are discussed in the thesis, and computational viability of the methods introduced, both in terms of usefulness and computational complexity, is taken into account.

Keywords Finite element methods, porous media, Brinkman model, a posteriori error estimates

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Tiivistelmä

Tekijä
Juho Könnö

Väitöskirjan nimi
Elementtimenetelmän sovelluksia huokoisen aineen virtaukseen

Julkaisija Perustieteiden korkeakoulu

Yksikkö Matematiikan ja systeemianalyysin laitos

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Tiivistelmä
Väitöskirja käsittelee elementtimenetelmän soveltamista huokoisen aineen virtaustehtäviin. Erityishuomion kohteena ovat lokaalisti massan säilyttävät elementtimenetelmät, joiden tärkeys korostuu erityisesti käytännön sovelluksissa tyypillisissä monifaasivirtauksissa. Huomion keskipisteenä on Brinkmanin malli, joka laajentaa huokoiselle virtaukselle usein käytettyä Darcyn mallia ottamalla huomioon myös viskoottiset efektit. Mallille esitellään massan säilyttävä elementtimenetelmä, sekä menetelmän kattava matemaattinen analyysi. Lisäksi väitöskirjassa tutkitaan stokastisten materiaaliparametrien mallintamista Brinkmanin tehtävän yhteydessä, sekä parametririippuvan Robin-tyyppisen reunaehdon asettamista Darcyn tehtävälle. Kaikki teoreettiset tulokset on vahvistettu kattavin numeerisin kokein. Väitöskirjassa kiinnitetään myös huomiota menetelmien käytännön toteutukseen ja laskennalliseen vaativuuteen, sekä niiden soveltuvuuteen käytännön ongelmiin.

Avainsanat Elementtimenetelmä, virtaus huokoisessa aineessa, Brinkmanin tehtävä, a posteriori analyysi

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Priimaa tuloo vaikka tavallista yrittääs.

Preface

Little did I know at the age of two upon insisting on helping my father dig a sewer ditch in the rain and breeze with a shovel larger than the tiny man himself that almost exactly 24 years later I would be still dealing with the same subject in the form of a dissertation. Well, maybe somewhat taller and dressed bit more posh for the occasion than back in the day but in essence the problem is still that of figuring out how the keep the fluids in the soil under control.

This thesis consists of three publications and one research report and was written at the Department of Mathematics and Systems Analysis at Aalto University during the period 2008 – 2011. Writing the thesis would not have been possible without the financial support from the Finnish Cultural Foundation, the Finnish Graduate School in Engineering Mechanics, Finnish Foundation for Technology Promotion, and the Emil Aaltonen Foundation. In addition I would like to recognize the financial support from the Finnish Research Programme on Nuclear Waste Management (KYT2010) project.

I am extremely grateful to my advisor professor Rolf Stenberg for his guidance, sharing of knowledge and continuous support on this journey into the realms of soil mechanics, not forgetting many nice moments of conversation on topics outside of mathematics, too. I have also had the privilege of collaborating with Dr. Dominik Schötzau from the University of British Columbia and with professor Christoph Schawb and Dr. Claude Gittelson from ETH Zürich, and I would like to thank all of them for many inspirational discussions and new aspects in the field of finite element methods. In particular I am indebted to my colleagues Antti Hannukainen and Mika Juntunen for their valuable comments, support, and collaboration. Many moments of both success and failure have been shared either facing a blackboard or a keyboard.

It has been an honour having professor Erik Burman and Dr. Martin Vohralik, both highly regarded experts in the field, as the pre-examiners of the thesis.

I would also like to thank all of my colleagues for an unforgettable half a decade in the righteous end of the thrid floor corridor. The triple Anttis, Mika, Harri, Jarkko, Helena and my five-year cellmate Santtu deserve a special thank you for creating a unique atmosphere for work, very few aspects of life were left uncovered in our discussions – either in the coffee room, swimming pool, sauna or around a barbeque. There has not been a single day without laughter, no matter how deep the scientific abyss.

Finally I would like to thank my parents for their continuous support and encouragement in my studies.

Vaasa, October 14, 2011,

Juho Könnö

Contents

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I Juho Könnö, Dominik Schötzau and Rolf Stenberg. Mixed Finite Element Methods for Problems with Robin Boundary Conditions. *SIAM Journal on Numerical Analysis*, 49(11), pp. 285-308, 2011.

II Juho Könnö and Rolf Stenberg. Analysis of H(div)-conforming Finite Elements for the Brinkman Problem. Accepted for publication in *Mathematical Models and Methods in Applies Sciences*, doi:10.1142/S0218202511005726, 2011.

III Juho Könnö and Rolf Stenberg. Numerical Computations with H(div)-Finite Elements for the Brinkman Problem. Accepted for publication in *Computational Geosciences*, doi:10.1007/s10596-011-9259-x, Preprint: arXiv:1103.5338v1 2011.

IV Claude Gittelson, Juho Könnö, Christoph Schwab and Rolf Stenberg. The Multi-Level Monte Carlo Finite Element Method for the Stochastic Brinkman Problem. Submitted to *Numerische Mathematik*, Preprint: ETH Zürich, Seminar für Angewandte Mathematik, Research Report 2011-31, 2011.

Author’s Contribution

Publication I: “Mixed Finite Element Methods for Problems with Robin Boundary Conditions”

Major parts of the analysis and writing, as well as all of the numerical experiments, are due to the author.

Publication II: “Analysis of H(div)-conforming Finite Elements for the Brinkman Problem”

The author is responsible for the writing and a major part of the analysis.

Publication III: “Numerical Computations with H(div)-Finite Elements for the Brinkman Problem”

The author is responsible for the writing and all of the numerical examples in the paper. The hybridization technique in Section 4 and the extension to non-constant permeability are due to the author.

Publication IV: “The Multi-Level Monte Carlo Finite Element Method for the Stochastic Brinkman Problem”

The author is responsible for writing Sections 5 and 7, as well as for adapting the finite element techniques and the analysis thereof to the stochastic framework. All of the numerical computations are performed by the author.

1. Introduction

In recent years a growing demand for efficient, accurate and reliable simulation methods has emerged in the field of geomechanics. In particular, the modelling of fluid flow in porous media is a central problem within the field with various applications in hydrogeology, soil contamination modelling, and petroleum engineering, to name a few. Most subsurface flows take place in different rock and soil types with varying porosities, thus rendering problems in geomechanics very challenging numerically due to highly irregular physical data, uncertainty in both the geometry and the parameter values, and last but not least the sheer size of the problems at hand. Another problematic aspect are the extremely long time scales, with the longest simulated intervals ranging typically from tens of years in petroleum engineering to extreme time intervals of tens of thousands of years in nuclear waste disposal applications.

Applications in hydrogeology encompass e.g. groundwater modelling, soil drainage, tracking the distribution of pollutants, and recently also nuclear waste disposal. The growing need for advanced simulations is to a great extent due to constantly tightening environmental regulations of industrial installations requiring careful risk assessment. For example, in undergound nuclear waste disposal it is of great importance to accurately model the water breakthough time to the capsules containing the radioactive waste with a timescale of tens of years, as well as the transport of different chemical agents in the groundwater undermining the structural integrity of the bentonite buffer during a period of thousands of years. Naturally, in such a volatile application the reliability of the computational results is a key issue.

Another important major application of subsurface flow models is petroleum engineering. Although the first signs of the use of petroleum date back to 4000 BC, it is only recently that the high demand for oil has induced a massive need for efficient extraction techniques, and thus for advanced simulation methods for enhanced oil recovery. The computational models in petroleum engineering are characterized by very heteregenous and possibly stochastic material data and the massive physical size of the problems. Consequently, many of the numerical methods in subsurface flow modelling stem from the need to utilize the scarce computational resources with utmost efficiency in massive reservoir simulations whilst still retaining some essential properties such as local mass conservation in the numerical methods employed.

Apart from geomechanical engineering, porous flow problems emerge in a variety of industrial applications, ranging from e.g. filtration technology and composite resin infusion to biomedical modelling of permeable cell walls. For example, in resin infusion molding of composite laminates one models the fiberglass or carbon fiber matrix as a porous medium. This results in a two-phase flow problem with air and resin flowing both inside the porous fibres as well as the void space left between the fibres.

This thesis addresses two porous flow models – namely the Darcy model and the more complicated Brinkman model [19, 1, 2, 3]. In the following we shall first introduce both of the models, and discuss the applicability of the two to different physical problems. The thesis focuses on three distinct problems related to the aforementioned flow models.

First, a parameter dependent boundary condition for the Darcy flow model is analyzed. This Robin type boundary condition allows one to move continuously between a pressure and a normal velocity boundary condition. A similar boundary condition was analyzed in [20], but the robustness with respect to the parameter ε was not studied. Both a priori and residual based a posteriori estimates are presented for the problem. It is also shown, that by using hybridization for the velocity field, the resulting system matrix is not ill-conditioned in the normal velocity boundary condition limit.

Next, a locally mass conserving finite element discretization of the Brinkman flow model is analyzed. The approach taken in the thesis employs $H(\mathrm{div})$-conforming finite elements to acertain the local conservation of mass discussed later in detail in Chapters 2 and 3. The tangential continuity of the velocity field required by the Brinkman model is then weakly enforced using a symmetric interior penalty Galerkin method. Similar techniques have been analyzed for the Stokes flow in [11, 15, 23, 22], whereas an approach based on H^1-conforming finite elements for the Brinkman problem has been widely analyzed e.g. in references [14, 4, 12]. A complete a priori and a residual based a posteriori analysis is presented, and all of the results are verified by extensive numerical testing.

The third and final focal point of the thesis is the simulation of stochastic material parameters for the Brinkman flow. In the rapidly growing field of stochastic finite element methods, problems in soil mechanics play an important role, since oftentimes the data for the permeability field is naturally of stochastic nature. Here, the multi level Monte Carlo technique [5, 13] is applied to the Brikman problem with a log normal stochastic permeability field. A stabilized conforming Stokes-based finite element approach presented in [14] is adapted to meet the demands of the multi level Monte Carlo method, and extensive numerical tests verify the results.

2. Mathematical models for porous flow

The quantities of interest in porous flow models are the *pore pressure* p and the *velocity* $\boldsymbol{u}$ of the fluid. In the following we present phenomenologically the Darcy and Brinkman models, for a detailed and rigorous derivation, cf. [19, 1, 2] and the references therein.

Let μ denote the *dynamic viscosity* of the fluid. Roughly speaking, viscosity describes the thickness of the fluid. For example, water is often described as a thin and honey as a thick fluid. In engineering applications the viscosities of the co-flowing fluids ofter vary by several orders of magnitude. In resin infusion the epoxy resin is very thick with a viscosity of several hundreds of centipoise (cP) compared to the air present in the matrix. Similarly, water is often used as the driving fluid in enhanced oil recovery, which is very thin with a viscosity of approximately one centipoise when compared to heavy crude oils having viscosities of hundreds or even thousands of centipoise.

The *permeability* is denoted by $\boldsymbol{K}$. In general, permeability is a symmetric tensor quantity. In numerous practical situations in geomechanics the permeability tensor is of the diagonal form. However, when using e.g. upscaling methods [18] for the permeability field, the resulting effective permeability tensor is often highly anisotropic. The unit for permeability is Darcy, $1\ \mathrm{D} = 9.869233 \times 10^{-13}\ \mathrm{m}^2$, commonly permeabilities are given in mD. Typically the permeability is a highly heterogeneous quantity, and the magnitude of variations might be extremely large. In Table 2.1 some typical permeabilities for different types of soil and rock are presented [7].

To clarify the heterogeneity of the permeability field, the logarithm of the permeability field for one layer of the the SPE10 benchmark dataset [10] describing a typical highly heterogenous oil reservoir is plotted in Figure 2.1. Evidently, the jumps in the material parameters in realistic reservoir applications are often of several orders of magnitude. Further-

Permeability mD	Property	Examples
$10^8 - 10^6$	Pervious	Clean gravel
$10^6 - 10^4$	Pervious	Clean sand, gravel and sand
$10^4 - 10^1$	Semipervious	Oil rocks, peat, fine sand
$10^1 - 10^{-1}$	Semipervious	Sandstone, stratified clay
$10^{-1} - 10^{-3}$	Impervious	Limestone, dolomite, clay
$10^{-3} - 10^{-5}$	Impervious	Granite, breccia

Table 2.1. Permeabilities for different soil and rock types.

more, Figure 2.1 also shows how the permeability fields in certain types of reservoirs are very chanellized localizing the flow to certain regions of the computational domain and thus underlining the need for adaptive methods in the numerical simulation of subsurface flows. Similarly, in nuclear waste disposal one is interested in the flow of groundwater in the extremely narrow void channels between the bentonite blocks.

In addition, it should be kept in mind that the derived quantities of interest, such as the well pressures and the production rates in petroleum engineering, as well as the saturation distribution depend both on the pore pressure p and the fluid velocity $\boldsymbol{u}$. Similarly, in industrial applications one wishes to keep the hydraulic pressures on a safe level while simulatenously e.g. maximizing the flow through an oil filter. Thus it is essential to design finite elements methods that perform equally well for both of the aforementioned variables.

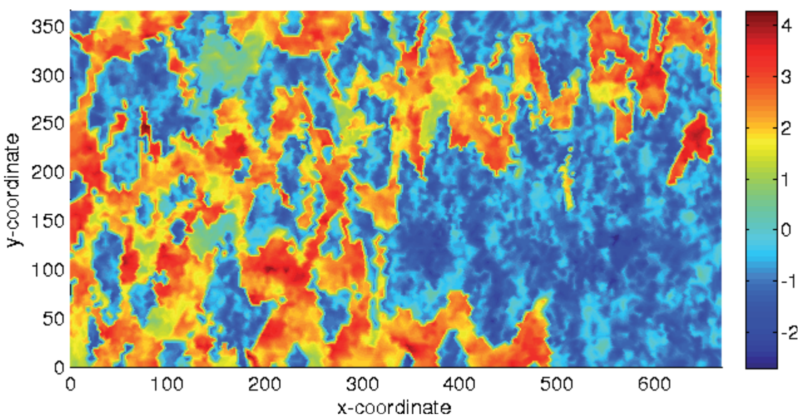

Figure 2.1. Logarithm of the permeability field in layer 68 of the SPE10 dataset in mD.

2.1 The Darcy model

The Darcy flow model is the simplest and by far the most widely used porous flow model. In the Darcy model the flow is directly proportional to the pressure gradient via the relation [19, 7]

$$\boldsymbol{u} = -\frac{1}{\mu}\boldsymbol{K}\nabla p. \tag{2.1}$$

Assuming the fluid to be incompressible, the Darcy equations read

$$\mu\boldsymbol{K}^{-1}\boldsymbol{u} + \nabla p = \boldsymbol{f}, \tag{2.2}$$

$$\operatorname{div}\,\boldsymbol{u} = g. \tag{2.3}$$

Here, the loading $\boldsymbol{f}$ comprises of body loadings to the fluid, most commonly gravity effects. The function g is a source term, describing e.g. injection and production wells in a groundwater or oil reservoir.

Normally one enforces either the pressure or the normal velocity on the boundary. In a nuclear waste management application, for example, one might prescribe the groundwater pressure on the boundary between the bentonite buffer and the borehole wall in the bedrock, and a no-flow condition on the boundary between the bentonite and the waste capsule. In article I we analyse the following Robin type boundary condition for the Darcy problem,

$$\varepsilon\boldsymbol{u}\cdot\boldsymbol{n} + p = \varepsilon u_{n,0} + p_0. \tag{2.4}$$

Here, $\varepsilon \geq 0$ is a parameter which allows one to move between the limiting pressure boundary condition $p = p_0$ as $\varepsilon = 0$ and the normal flow boundary condition $\boldsymbol{u}\cdot\boldsymbol{n} = u_{n,0}$ as $\varepsilon \to \infty$.

2.2 The Brinkman model

In the Brinkman model, one adds an effective viscosity term to the Darcy model. Thus the model constitutes a parameter dependent combination of the porous Darcy flow and the viscous Stokes flow. The Brinkman model is best suited for modelling very porous materials and domains with cracks or flow channels. The main advantage of the Brinkman model is the ability to move from the Darcy regime to the Stokes regime and back by altering the material parameters only. With $\tilde{\mu}$ denoting the *effective viscosity* of the fluid, the Brinkman equations for an incompressible fluid read [19, 18]

$$-\tilde{\mu}\Delta \boldsymbol{u} + \mu \boldsymbol{K}^{-1}\boldsymbol{u} + \nabla p = \boldsymbol{f}, \tag{2.5}$$

$$\operatorname{div} \boldsymbol{u} = g. \tag{2.6}$$

A common choice for $\tilde{\mu}$ is to take the effective viscosity equal to the dynamic viscosity, i.e. $\tilde{\mu} = \mu$, however more refined models depending on e.g. the *porosity* ϕ of the porous medium exist, see e.g. [18].

Mathematically the nature of the problem changes radically depending on the ratio of the coefficients of the two velocity terms in equation (2.5). For very large permeabilities the flow takes place in almost void space, and the viscous part dominates. In this situation the flow is essentially of the Stokes type, whereas for more impermeable materials the Darcy part is the dominant term. Therefore the numerical method for solving the Brinkman equation must be chosen carefully to assure stability and accuracy of the method for all possible parameter values. For example in reservoir simulation a large portion of the domain is typically in the Darcy regime, but on the other hand in e.g. filter applications the void space governed by the Stokes limit constitutes a major part of the domain. This motivates the design of numerical methods that perform well in both regimes and simultaneously allow for a seamless transition between the two limiting models.

An approach based on finite elements originally designed for the Darcy problem is covered in this thesis in articles II and III. Advantages of the chosen approach include the intrisic local mass conservation property of the finite element space and the ability to enhance the pressure approximation afterwards by a postprocessing scheme presented in paper II. However, these elements are more complex to implement and computationally more demanding than discretizations based on Stokes-type elements analyzed in e.g. [14, 4].

2.3 Local mass conservation - why?

A central part of the thesis deals with finding a locally mass conserving finite element method for the Brikman problem. But what makes this property so important and desirable? To shed light on the issue, let us recall that in practice almost all applications of porous flow models are multiphase problems. That is, two or more fluids such as oil and water or air and epoxy resin co-exist in the porous matrix. For simplicity, let us demonstrate the importance of the local mass conservation property in the simplest possible framework by considering a two phase incompressible Darcy flow of oil and water with no capillary or gravity effects.

Let $\boldsymbol{u} = \boldsymbol{u}_o + \boldsymbol{u}_w$ be the total flow, in which $\boldsymbol{u}_o$ and $\boldsymbol{u}_w$ are the velocities for the oil and water components, respectively. Since the flow is assumed incompressible, we have

$$\text{div } \boldsymbol{u} = 0 \tag{2.7}$$

in the absence of sources or sinks. The *water saturation* S describes the fraction of water of the total pore volume inside the porous matrix. The saturation evolves in time as [9]

$$\frac{\partial S}{\partial t} + \text{div}(f_w(S)\boldsymbol{u}) = g_w, \tag{2.8}$$

in which $f_w(S)$ is the saturation dependent flow fraction of water and g_w the source loading for the water component. Using the product rule for divergence yields

$$\frac{\partial S}{\partial t} + f_w'(S)\nabla S\cdot \boldsymbol{u} + f_w(S)\text{div } \boldsymbol{u} = g_w. \tag{2.9}$$

Clearly, the last term on the left hand side should vanish for an incompressible flow. However, it is insufficient for the divergence to vanish globally in the weak sense, since this could lead to spurious modes that create artificial sources or sinks in individual elements. Thus we try to find a method that satisfies the *equilibrium property*

$$\text{div } \boldsymbol{V}_h \subset Q_h, \tag{2.10}$$

and the *commutative diagram property*

$$\text{div } \boldsymbol{R}_h = P_h\text{div}. \tag{2.11}$$

Here the finite dimensional spaces $\boldsymbol{V}_h$ and Q_h are the approximation spaces for the velocity and pressure, respectively. $\boldsymbol{R}_h$ is a special interpolation operator for $H(\text{div}, \Omega)$ functions to $\boldsymbol{V}_h$, and P_h is the L^2-projection to Q_h. For details on the properties of the interpolation operator $\boldsymbol{R}_h$, cf. article II and the references therein. These properties quarantee that the aforementioned spurious modes cannot occur. Since the time intervals simulated in geomechanics are typically very long, from days to years, it is of utmost importance that accumulation of unphysical saturation does not occur during the computations.

2.4 Stochastic permeability fields

As mentioned, in soil mechanics one often encounters permeabily fields for which only some statistical quantities are known. The aim is to simulate such flow fields based on data such as the covariance and mean value of the permeability field numerically. One of the most common models for the permeability field is the *log normal* model. That is, the logarithm of the permeability field is normally distributed. Thus the permeability field is of the form

$$\boldsymbol{K} = \boldsymbol{K}_0 \exp(\boldsymbol{G}), \tag{2.12}$$

in which $\boldsymbol{G}$ is an $\mathbb{R}^d$-valued, symmetric Gaussian field and $\boldsymbol{K}_0$ is a symmetric, positive definite $d \times d$ matrix. The random field $\boldsymbol{G}$ has the Karhunen-Loève expansion

$$\boldsymbol{G} = \sum_{n=1}^{\infty} Y_n \sqrt{\lambda_n} \boldsymbol{\Phi}_n, \tag{2.13}$$

in which $(\lambda_n, \boldsymbol{\Phi}_n)$ are the eigenpairs of the covariance operator corresponding to the random field $\boldsymbol{G}$, and Y_n are standard normal random variables. For details, see e.g. [5] and article IV. In simple cases, the eigenpairs for the covariance operator can be computed explicitly in some simple domains, such as in a square or a circle. However, in a more general setting one has to solve the eigenpairs numerically using e.g. finite elements.

For computations, the infinite Karhunen-Loève series (2.13) must be truncated. Thus the permeability field $\boldsymbol{K}$ is approximated with a truncated field $\boldsymbol{K}_N$ as

$$\boldsymbol{K}_N := \boldsymbol{K}_0 \exp\left(\sum_{n=1}^{N} Y_n \sqrt{\lambda_n} \boldsymbol{\Phi}_n \right). \tag{2.14}$$

In article IV a multi level Monte Carlo method is considered for such a permeability field. In a multi level Monte Carlo method the key ingredient is to compute the samples on multiple nested meshes balancing the error between the discretization error and the stochastic truncation error. The analysis can be easily adapted to other models with log normal random fields, too.

3. Numerical methods

In this section the main numerical methods deployed in the articles are covered. The details of applying these techniques to each of the individual problems in the thesis are presented in the articles, thus the main focus here is to shed light on the ideas behind each of the different numerical techniques and the underlying reasons for using a specific method.

3.1 Discretizations of the $H(\mathrm{div})$ space

The space $H(\mathrm{div},\Omega)$ is composed of those functions $\boldsymbol{u}$ for which it holds $\boldsymbol{u} \in L^2(\Omega)$ and $\mathrm{div}\ \boldsymbol{u} \in L^2(\Omega)$. For the discretized space $\boldsymbol{V}_h$ the condition $\boldsymbol{V}_h \subset H(\mathrm{div},\Omega)$ translates into a continuity condition over the interelement boundaries $E \in \mathcal{E}_h$ of the mesh $\mathcal{K}_h$. More exactly, one requires that the normal component $\boldsymbol{u}\cdot \boldsymbol{n}$ is continuous across the interelement boundaries.

Typically $H(\mathrm{div})$-conforming finite element spaces appear in the context of mixed methods, for example we seek for the velocity of the fluid in $\boldsymbol{V}_h$ and the pressure in Q_h. In what follows, the spaces $\boldsymbol{V}_h$ and Q_h are chosen such that the method is stable, and that the equilibrium property

$$\mathrm{div}\boldsymbol{V}_h \subset Q_h \tag{3.1}$$

and the commutative diagram property (2.11) hold. Consequently, the weak divergence condition

$$(\mathrm{div}\ \boldsymbol{u}, q) = (g, q), \quad \forall q \in Q_h \tag{3.2}$$

yields $\mathrm{div}\ \boldsymbol{u} = P_h g$, in which $P_h : L^2(\Omega) \to Q_h$ is the L^2-projection to the pressure space. Thus one immediately recognizes that for example the incompressibility condition

$$\mathrm{div}\ \boldsymbol{u} = 0 \tag{3.3}$$

is satisfied exactly for $H(\mathrm{div},\Omega)$-conforming elements satisfying (3.1). This is the main motivation for such an approach to the Brinkman problem in papers II and III. Oftentimes this property is referred to as *local mass conservation*. As an example we consider in the following the simple first order Brezzi-Douglas-Marini (BDM) element [8] for which

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in H(\mathrm{div},\Omega) \mid \boldsymbol{v}|_K \in [P_1(K)]^2\}, \tag{3.4}$$

and the corresponding pressure space is

$$Q_h = \{q \in L^2(\Omega) \mid q|_K \in P_0(K)\}. \tag{3.5}$$

The degrees of freedom for this element are the average and the first moment over the element edges, cf. Figure 3.1. The pressure space is discontinuous over the interelement edges.

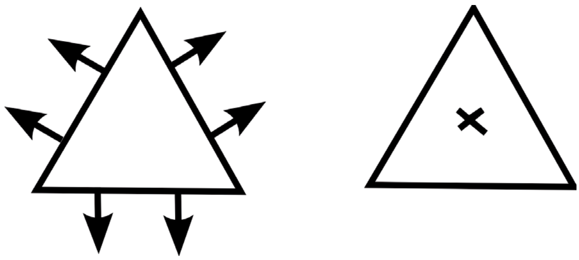

Figure 3.1. Degrees of freedom for the lowest-order BDM element

3.2 Enforcing continuity via penalization

It is often beneficial to relax the continuity requirements to some extent, however in return some extra work has to be done in order to stabilize the method. As mentioned earlier, only the normal component of the velocity is required to be continuous in the case of $H(\mathrm{div})$-conforming elements. In order to approximate the second order term describing the viscous effects in the Brinkman model, the continuity of the tangential component is weakly enforced akin to traditional discontinuous Galerkin (DG) methods. This matter is discussed in detail in article II.

To fix ideas, consider the scalar Poisson problem

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega \end{aligned} \tag{3.6}$$

discretized with elementwise discontinuous finite elements from the space $V_h = \{v \in L^2(\Omega) \mid v|_K \in P_k(K)\}$. Due to the discontinuity multiplication by an arbitrary test function $v \in V_h$ and partial integration of the first equation yields

$$\sum_{K\in\mathcal{K}_h} (\nabla u, \nabla v)_K - \langle \frac{\partial u}{\partial \boldsymbol{n}}, v\rangle_{\partial K} = (f, v). \tag{3.7}$$

To stabilize the method, we modify the weak formulation as follows:

$$(\nabla u, \nabla v) + \sum_{E\in\mathcal{E}_h} \left(\frac{\alpha}{h_E} \langle [\![u]\!], [\![v]\!]\rangle_E - \langle \{\frac{\partial u}{\partial \boldsymbol{n}}\}, [\![v]\!]\rangle_E - \langle [\![u]\!], \{\frac{\partial v}{\partial \boldsymbol{n}}\}\rangle_E \right) = (f, v). \tag{3.8}$$

Here $[\![\cdot]\!]$ and $\{\cdot\}$ denote the jump and average on the edge E, respectively. The above symmetric interior penalty Galerkin (SIPG) formulation (see e.g. [21]) guarantees that for a suitably chosen stabilization parameter α the formulation is stable and an optimal convergence rate with respect to the polynomial degree of the space V_h is attained. In the context of setting Dirichlet boundary conditions the above formulation is often referred to as Nitsche's method [17].

In articles II and III the SIPG formulation is employed to stabilize certain families of $H(div)$-conforming elements for the Brinkman problem, as well as to enforce the boundary conditions weakly. The resulting finite element approximation is thus intrinsically locally mass conserving and stable for all parameter values of the Brinkman model. In addition, weakly enforcing the boundary conditions alleviates the numerical problems related to handling boundary layers stemming from no-flow boundary conditions when approaching the Darcy limit.

3.3 Postprocessing for the pressure

As a model problem, the Darcy problem with the material parameters set to unity is considered. In the discretized form we seek a velocity-pressure pair $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h \subset H(\mathrm{div}, \Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} (\boldsymbol{u}, \boldsymbol{v}) - (\mathrm{div}\, \boldsymbol{v}, p) &= (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\ -(\mathrm{div}\, \boldsymbol{u}, q) &= -(g, q), \quad \forall q \in Q_h, \end{aligned} \tag{3.9}$$

in which $\boldsymbol{f}$ and g are given sufficiently smooth loading functions.

To analyze the convergence of the finite element discretization, the following mesh dependent norm is used for the pressure

$$\|p\|_h^2 = \sum_{K\in\mathcal{K}_h} \|\nabla p\|_{0,K}^2 + \sum_{E\in\mathcal{E}_h} \frac{1}{h_E} \|[\![p]\!]\|_{0,E}^2, \tag{3.10}$$

whilst for the velocity the L^2-norm is employed. Note, that due to the equilibrium property (3.1) we need not separetely estimate the error in the divergence, since $\operatorname{div} \boldsymbol{u}_h = P_h g$. This yields the following suboptimal convergence result for the pressure

$$\|P_h p - p_h\|_h \leq Ch \tag{3.11}$$

when the lowest order Brezzi-Douglas-Marini elements are employed. The fact that the pressure solution p_h only converges to the L^2-projection $P_h p$ of the exact solution onto the finite element space Q_h is simply due to the lack of approximation properties of the pressure space, which in this case is that of elementwise constant functions. However, a simple postprocessing procedure can be shown to remedy this by seeking the postprocessed pressure p_h^* in an augmented space [16]. For example, for the first order BDM element we choose

$$Q_h^* = \{q \in L^2(\Omega) \mid q|_K \in P_2(K)\} \tag{3.12}$$

and compute the postprocessed pressure $p_h^* \in Q_h^*$ through

$$P_h p_h^* = p_h, \tag{3.13}$$

$$(\nabla p_h^*, \nabla q)_K = (\boldsymbol{u}_h, \nabla q)_K \qquad \forall q \in (I - P_h) Q_h^*|_K. \tag{3.14}$$

It can then be shown [16], that full convergence rate is recovered for the pressure, that is

$$\|p - p_h\|_h \leq Ch. \tag{3.15}$$

Note, that the postprocessed pressure is still discontinuous across the interelement boundaries.

The postprocessing method can be applied to a wide variety of different families of $H(div)$-conforming elements. In articles I, II and III this technique is applied to more complicated problems to recover the optimal convergence rate for the pressure variable. It is noteworthy that the procedure is performed elementwise thus being computationally inexpensive compared to solving the original linear system, and also allowing for efficient parallelization due to the localized nature.

3.4 A posteriori estimators

In the analysis of finite element methods the error estimates are divided into two categories - namely *a priori* and *a posteriori* estimates. The former are asymptotic error estimates of the form

$$\|u - u_h\|_1 \leq Ch, \tag{3.16}$$

which for example for the Poisson problem (3.6) tells that the error in the $H^1(\Omega)$-norm is directly dependent on the mesh size h. However, the constant C depends on some higher Sobolev norm of the exact solution u, and thus cannot be computed in practice since the exact solution u is not known.

On the other hand, in a posteriori estimates one seeks for an *estimator* η which is a function of the discrete solution u_h and the loading and boundary condition functions. The aim is to find an estimator satisfying e.g. for the model Poisson problem

$$c\eta \leq \|u - u_h\|_1 \leq C\eta. \tag{3.17}$$

For this simple problem, such an estimator is

$$\eta^2 = \sum_{K \in \mathcal{K}_h} h_K^2 \|\Delta u_h + f\|_{0,K}^2 + \sum_{E \in \mathcal{E}_h} h_E \|[\![\frac{\partial u_h}{\partial \boldsymbol{n}}]\!]\|_{0,E}^2, \tag{3.18}$$

in which $[\![\cdot]\!]$ denotes the jump of a function and $\boldsymbol{n}$ is the normal vector on a face $E \in \mathcal{E}_h$.

The constants c and C should not depend on the solution or the computational mesh. However, sometimes these constants are unknown and might depend e.g. on the shape of the domain, but they are nevertheless known to be bounded. For parameter dependent problems, such as the Robin-type boundary conditions in I and the Brinkman problem in II and III, it is crucial that the constants are also independent of the parameters. Deriving such parameter independent a posteriori bounds is one of the key ingredients in this thesis.

3.5 Numerical challenges in the transition regime

In the Brinkman model, the nature of the problem changes dramatically as one passes numerically from the Darcy to the Stokes regime. This manifests itself in numerical anomalies in the transition zone, which are clearly seen in the numerical results.

More exactly, the balancing of the error components changes from a velocity dominated error to a pressure dominated error as the interior penalty term becomes more dominant. As shown in the numerical results in III, the postprocessing technique is only efficient in removing the pressure jumps in the Darcy regime to give the optimal convergence rate proven in II, and the error levels are of different magnitude for the two limiting cases. Thus, it is natural for the convergence to exhibit a drop in the transition regime since the relative error is considerably higher for equivalent mesh densities in the non-conforming Stokes regime.

The numerical results in III neglect the error terms from the external loading and the constants in the a posteriori bounds. Naturally, the constants in the efficiency and reliability estimates differ if we consider the limiting cases only, and accordingly the unscaled value of estimator in the numerical examples overshoots the estimator in the Stokes regime, and undershoots in the Darcy regime.

The discrepancy between the estimator and the exact error in the transition zone $t \approx h$ is most likely due to the saturation assumption not holding in this narrow regime, as the error might grow as a result of boundary layer type effects that are discovered on a finer mesh. During subsequent refinements the convergence is however regained, since one passes numerically into the Stokes regime in which the saturation assumption holds true.

3.6 Hybridization techniques

Sometimes it is desirable to break the continuity of the finite element space on all or a certain subset of the interelement boundaries, and enforce the continuity on these edges via Lagrange multipliers. Such techniques are known as *hybridized methods*.

The model mixed finite element problem (3.9) can be hybridized on all internal edges as follows [6, 8]: Find $(\boldsymbol{u}_h, p_h, m_h) \in \widetilde{\boldsymbol{V}}_h \times Q_h \times M_h$ such that

$$(\boldsymbol{u}_h, \boldsymbol{v}) - \sum_{K \in \mathcal{K}_h} (\operatorname{div} \boldsymbol{v}, p_h)_K + \sum_{K \in \mathcal{K}_h} \langle \boldsymbol{v} \cdot \boldsymbol{n}_{\partial K}, m_h \rangle_{\partial K} = (\boldsymbol{f}, \boldsymbol{v}), \tag{3.19}$$

$$- \sum_{K \in \mathcal{K}_h} (\operatorname{div} \boldsymbol{u}_h, q)_K = (g, q), \tag{3.20}$$

$$\sum_{K \in \mathcal{K}_h} \langle \boldsymbol{u}_h \cdot \boldsymbol{n}_{\partial K}, r \rangle_{\partial K} = 0 \tag{3.21}$$

for all $(\boldsymbol{v}, q, r) \in \widetilde{\boldsymbol{V}}_h \times Q_h \times M_h$, in which $\widetilde{\boldsymbol{V}_h}$ corresponds to the space $\boldsymbol{V}_h$ with no continuity restrictions across interelement boundaries and $\boldsymbol{n}_{\partial K}$ is the outer normal of the element K. M_h is a suitably chosen space of Lagrange multipliers on the hybridized edges, e.g. for the lowest-order BDM elements M_h is composed of first-order polynomials on the edges

$E \in \mathcal{E}_h$.

The algebraic system corresponding to the hybridized equations is of the form

$$\begin{aligned} \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}p + \boldsymbol{C}m &= f \\ \boldsymbol{B}^T\boldsymbol{u} &= g \\ \boldsymbol{C}^T\boldsymbol{u} &= 0, \end{aligned}$$

in which $\boldsymbol{A}$ is a block diagonal matrix and $(\boldsymbol{u}, p, m)$ are now the coefficient vectors associated with the finite element solution. One can now eliminate the velocity and pressure variables ending up with a system for the Lagrange multipliers only. For example for the lowest order BDM elements the blocksize of the matrix $\boldsymbol{A}$ is only 6×6, thus inverting $\boldsymbol{A}$ is computationally very cheap. The resulting system matrix for the Lagrange multipliers is of the form

$$\boldsymbol{C}^T(\boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{A}^{-1} - \boldsymbol{A}^{-1})\boldsymbol{C}. \tag{3.22}$$

This matrix is symmetric and positive definite [8] in contrast to the original saddle point system, and hence well-suited for standard linear solvers. Hybridization can also be easily adapted to domain decomposition by hybridizing the finite element spaces only on the skeleton of the domain decomposition mesh, and using subdomain solvers for inverting the matrix $\boldsymbol{A}$ simultaneously on several computational nodes. Hybridization techniques are considered in detail for both the Darcy problem and the Brinkman problem in articles I and III, respectively.

3.7 The multi-level Monte Carlo method

As previously mentioned, the permeability $\boldsymbol{K}$ is often known only as a statistical quantity. That is, one has a stochastic model or uncertain measurement data for the expected value and covariance of the permeability field, thus underlining the importance of finding efficient simulation methods for stochastic porous flow models. Traditional Monte Carlo methods rely on randomizing several realizations of the stochastic field and computing a corresponding finite element solution for the quantities of interest, which are then averaged to get quantities such as the expected value of the velocity and pressure fields. A major drawback of traditional Monte Carlo methods is that they are computationally very expensive.

As a remedy, multi level Monte Carlo methods have been proposed and analyzed in e.g. [5, 13]. They are based on a hierarchy of finite element discretizations and a varying level of approximation for the stochastic parameter. The number of Monte Carlo samples per mesh level is varied based on the convergence properties of the Karhunen-Loève expansion (2.13) of the stochastic parameter. In paper IV the multi level Monte Carlo method is applied to the Brinkman equations with a stochastic permeability field, and combined with a robust stabilized mixed finite element method based on [14].

From the finite element point of view, a major challenge is to find a stable finite element method, such that the finite element spaces are nested on a hierarchy of uniformly refined meshes to keep the workload low in the multi level method. In addition, for stabilized methods, the dependence of the stabilization parameter on the stochastic quantities must be carefully studied. Due to the high number of samples computed and the fact that virtually no internode communication is required, the method is very well suited for massively parallel computations.

4. Concluding remarks

The main findings in this thesis can be summarized as follows.

I In this article the Darcy problem with a parameter dependent boundary condition is studied. We introduce a weak formulation for enforcing the boundary condition, along with a rigorous a priori and a posteriori analysis. The postprocessing method of [16] for the scalar variable is shown to be applicable for this type of a problem, thus yielding optimal convergence rates for the proposed method. It is shown that all the a priori estimates and the reliability estimate for the a posteriori indicator are independent of the parameter ε in the boundary condition. All of the theoretical results are verified with numerical tests, which also suggest that the efficiency estimate is independent of the parameter, even though the proof is not complete.

II The article presents a complete and rigorous analysis of applying $H(\mathrm{div})$-conforming finite elements for the Brinkman problem. A suitable mesh dependent norm for the problem is presented, in which we prove optimal convergence estimates robust in the effective viscosity parameter t. Thus the proposed method is applicable for the whole range of problems from the Darcy flow to a viscous Stokes flow covered by the Brinkman model. We also extend the aforementioned postprocessing method to the Brinkman equations to achieve optimal convergence rate for the pressure. The residual based a posteriori indicator introduced is shown to be both reliable and efficient for all values of the parameter $t \geq 0$.

III This paper is a continuation of paper II. The estimates are extended to cover a non-constant permeability field, and a hybridization technique is presented for the SIPG formulation of the problem. We also address applying the hybridization method to domain decomposition. A major part of the paper deals with numerically verifying both the results in paper II, as well as the new results presented in this paper. In addition, the applicability of the a posteriori indicator to adaptive mesh refinement is demonstrated employing realistic material data.

IV In this work the stochastic Brinkman problem with a log normal permeability field is studied. Rigorous error estimates are derived both for the stochastic and the spatial discretization errors. A Stokes-based stabilized finite element method proposed in [14] is modified to fulfill the requirements of the multi level Monte Carlo method. In particular, great attention is given to analyzing the computational complexity of the method. Finally, all of the results are verified with extensive numerical tests, verifying both the predicted convergence behaviour, as well as the work load estimates.

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Errata

Publication I

The proof of Proposition 5.5 is incorrect, and thus the proof of Proposition 5.6 is flawed as regards the parts based on using Proposition 5.5. Accordingly, the proof for the efficiency estimate of Theorem 5.8 is incomplete for boundary edges with a non-vanishing ε. However, there is strong numerical evidence that the estimator proposed is also efficient as shown in the numerical results in Section 7. We can show the following suboptimal estimate for the boundary edges with $\varepsilon \neq 0$,

$$\eta_E^2 \leq (\varepsilon + h_E)\|(\sigma_h - \sigma)\cdot \boldsymbol{n}\|_{0,E}^2 + \frac{1}{\varepsilon + h_E}\|u_h^* - u\|_{0,E}^2 + (\varepsilon + h_E)\|g - g_h\|_{0,E}^2.$$

The above can be shown by directly inserting the exact boundary condition into the boundary edge estimator η_E yielding

$$\begin{aligned}\varepsilon(\sigma_h\cdot \boldsymbol{n} - g_h) + u_h^* - u_0 &= \varepsilon(\sigma_h\cdot \boldsymbol{n} - g_h) + u_h^* - \varepsilon(\sigma\cdot \boldsymbol{n} - g) - u \\ &= \varepsilon(\sigma_h - \sigma)\cdot \boldsymbol{n} + (u_h^* - u) + \varepsilon(g - g_h).\end{aligned}$$

Using the triangle inequality and the relation $\varepsilon/\sqrt{\varepsilon + h_E} \leq \sqrt{\varepsilon} \leq \sqrt{\varepsilon + h_E}$ gives the desired result. Note, that the above estimate is suboptimal in the sense that given an irregular boundary load g the contribution from the boundary load error can be substantial and grows as the root of the ε parameter. Furthermore, the flux estimate is in the $\|\cdot\|_{\varepsilon,h}$ norm in contrast to the reliability and convergence estimates given in the L^2 norm. However, assuming a certain degree of regularity for the solution the estimator can be shown to converge with the same ratio as the error.

The authors are working towards presenting a corrected proof for the original results.

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