Department of Real Estate, Planning and Geoinformatics

Aspects of Modelling Geographic Information in Geodesy and Geodynamics

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Karin Kollo

Aalto University

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Aspects of Modelling Geographic Information in Geodesy and Geodynamics

Karin Kollo

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Abstract

We studied aspects of modelling geographical information from the geodetic and geodynamic viewpoints. The data for our studies were acquired by a variety of methods: laser scanning, levelling and satellite positioning.

The major subject of this dissertation is quality and its measures. We study various modelling approaches on geodetic data aimed at use cases in the fields of geodesy, geodynamics and geographic information science. The dissertation discusses internal and external quality aspects of the modelling and the data used. The main objectives of the research are related to data modelling aspects and fitness for use within the fields of study as expressed quantitatively in various quality measures. The novelty of the dissertation is in the application of appropriate quality measures, like precision or accuracy, in these fields of research from the viewpoint of fitness for use, which for the various models depend on input data precision as well as on the envisaged applications of the model product.

This dissertation has three main topics. Firstly, the modelling of gravity based heights was studied. Three different modelling methods were used: kriging, fuzzy modelling and bilinear affine transformation. We studied the use of geostatistical and geodetic methods for the construction of digital elevation models and height transformation surfaces. Within these methods also quality measures were formulated showing their applicability in height modelling. We found that the quality of the results is dependent on the data point distribution and the availability of precise geoid heights. Secondly, precision measures for gravimetric geoid determination were derived for two test areas. For this study three error sources were investigated: the error of omission, the aliasing error and the out-of-area error. We showed that error sources were dependent on the spatial extent and accuracy of the gravimetric measurements. Thirdly, the modelling of post-glacial land uplift was investigated, using two methods: land uplift prediction by least-squares collocation, and Glacial Isostatic Adjustment (GIA) modelling using two different ice models and fitting the Earth model parameter values. In the land uplift recovery study, possibilities were investigated for projecting the land uplift forward in time. From this study a statistical model for predicting land uplift rate from point velocity rates using a relatively simple formulation was derived by the least squares collocation technique, with error propagation into the predicted land uplift. The GIA modelling study gave us experience in building land uplift models. We found that the two methods, land uplift rate prediction and physical GIA modelling, though being very different, are giving similar accuracy measures for the derived land uplift values.

Keywords bilinear affine transformation, digital elevation model, fuzzy modelling, geoid precision, glacial isostatic adjustment, kriging, least squares collocation

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Helsinki, October 2015

Karin Kollo

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Abbreviations

BIFROST	Baseline Interferences for Fennoscandian Rebound Observations, Sea Level and Tectonics
CORS	Continuously Operating Reference Stations
DEM	Digital Elevation Model
ECEF	Earth Centered Earth Fixed
ETRF	European Terrestrial Reference Frame
ETRS	European Terrestrial Reference System
GIA	Glacial Isostatic Adjustment
GIS	Geographical Information System
GNSS	Global Navigation Satellite System
GOCE	Gravity field and steady-state Ocean Circulation Explorer
GRACE	Gravity Recovery and Climate Experiment
GRS80	Geodetic Reference System 1980
IERS	International Earth Rotation and Reference System Service
ITRF	International Terrrestrial Reference Frame
ITRS	International Terrestrial Reference System
LGM	Last Glacial Maximum
MSL	Mean Sea Level
NNR	No Net Rotation
PGR	Post Glacial Rebound

PREM	Preliminary Earth Model
RMS	Root Mean Square
SLE	Sea Level Equation
SNARF	Stable North American Reference Frame
TIN	Triangulated Irregular Network

List of Publications

The dissertation is based on five articles, four of which are published in peer-reviewed journals and one is a conference full paper. Reference to these publications in the text is made using capital letters from A to E as follows:

Publication A: Kollo, Karin (2008). Two alternative methods for height transformation. Taylor & Francis. Geodesy and Cartography, 34(1), pp. 5-11. ISSN 2029-6991 (Print), 2029-7009 (Online). DOI: 10.3846/1392-1541.2008.34.5-11.

Publication B: Sunila, Rangsima and Kollo, Karin (2009). Kriging and Fuzzy Approaches for DEM. In: Ed: Stein, A., Shi, W. and Bijker, W. Quality Aspects in Spatial Data Mining. Ch. 9, pp. 101-114. Taylor & Francis Group. ISBN: 978-1-4200-6926-6. DOI: 10.1201/9781420069273.ch9.

Publication C: Vermeer, Martin and Kollo, Karin (2007). Geoid precision from limited-area gravimetric surveys. Taylor & Francis. Geodesy and Cartography, 33(1), pp. 3-8. ISSN 2029-6991 (Print), 2029-7009 (Online). DOI: 10.1080/13921541.2007.9636708.

Publication D: Kollo, Karin and Vermeer, Martin (2011). Modelling land uplift rates and their error propagation. Taylor and Francis. Geodesy and Cartography, 37(1), pp. 33-40. ISSN 2029-6991 (Print), 2029-7009 (Online). DOI: 10.3846/13921541.2011.559941.

Publication E: Kollo, Karin, Spada, Giorgio and Vermeer, Martin (2015). Studying Earth rheology using GNSS permanent stations and GIA modelling tools. Geophysical Society of Finland. Geophysica, 51(1), in print.

Author’s contribution

Publication A: The author was responsible for the entirety of the work described in the paper.

Publication B: The second author was responsible for computations of fuzzy model construction for DEM, analysing and summarising corresponding parts of the paper and the paper was written by both authors together. The research idea originated in discussions between the authors.

Publication C: The second author was responsible for acquiring and processing Estonian gravity survey data and wrote the corresponding chapter in the paper. The original idea came from the first author.

Publication D: The first author was partly responsible for designing and writing the code and analyzing and summarizing the results and drawing of figures; she was fully responsible for calculations, both authors together wrote the paper. The research idea was developed jointly by the authors.

Publication E: The first author had the main responsibility for writing the article with minor assistance from second and third authors, all computations and figures. The research idea was proposed by the first author.

Summary of Publications

Publication A: This article introduces two different methods for height transformation: the bilinear affine transformation approach and fuzzy modelling, and applies them to the territory of Estonia. We showed that using these methods, a transformation surface can be determined and its accuracy in computing the heights of arbitrary points can be evaluated. Precision estimates for the individual case studies are presented.

Publication B: This article discusses the construction of digital elevation models (DEM) using the kriging and fuzzy modelling approaches. As input data height data from a laser scanning survey for a sub-urban area of Helsinki were used. We demonstrated the construction of digital elevation models with their quality measures. Additionally, a comparison with an independently constructed TIN model was made.

Publication C: In the article rough theoretical estimates for the precision of a gravimetric geoid model and the structure of the uncertainty budget computed from discrete data points were presented. We presented calculations for two case studies, Estonia and Finland.

Publication D: In the article a statistical model for predicting the uplift rates from the existing point uplift rates was derived with its empirical signal covariance function for the Fennoscandian land uplift area. We showed how the model uplift rate can be calculated for an arbitrary point. If one knows uplift rates, heights of points in the terrain, and GNSS positioning together with a precise geoid model, then one may project geodetic heights forward in time.

Publication E: In the article we found best fitting Earth model parameters with their uncertainties for ice models ICE-5G and KL05 in the North-American and Fennoscandian uplift areas and compared them with results from other published studies.

1. Introduction

Nowadays geographical information plays a central role in society. Almost everything is based on geographical information, the geographical information itself is based on position information acquired by, or tied in with, for example, though not only, geodetic measurements.

Geographical information stands for "geographical knowledge from investigation of geographical features", whatever kind of geographical features are studied (Peuquet, 2002). More often terrain based features are described – spatial information, boundaries, heights, etc. Within the spatial information, one also needs *spatial framework* information, i.e., geodetic datum and reference systems. These are based on precise geodetic control points, which help to make the link to a *real physical* location.

The dissertation interconnects geodesy, geodynamics and geographical modelling. Geodesy is the science of measuring the Earth and its size and shape; geodynamics is the discipline for studying the Earth's size, shape and orientation change over time. Geographical modelling studies phenomena on, above or below the surface of the Earth. A central question in geographic information is: how geographic information can be presented and used (Peuquet, 2002).

Within the fields of geographic information presentation and usage, the term *quality* is often used. Data quality is an overall term for various concepts, like precision, accuracy, consistency, completeness, validity, uncertainty, timeliness, etc. As it is not possible to make a perfect representation of real world phenomena in GIS, some deficiency in the quality of the final products is inevitable (Longley et al., 2001).

Generally, different geographic models are used to represent reality. The model is always simpler than the reality it represents and, therefore, different types and magnitudes of errors are always present. It should be realised that there is no such thing as the correct geographic data model

(Longley et al., 2001).

For modelling purposes, the geographical information can be collected by measurements as objects (e.g., in geodesy, in photogrammetry, or obtained by digitizing from pre-existing sources) or as fields (e.g., remote sensing) (Devillers and Jeansoulin, 2006). Furthermore, from geodetic measurements, the results are given in the form of discrete objects. From geodynamical studies, it is possible to obtain the change in coordinates, i.e., the rates or velocities, which are given as continuous objects, i.e., time series. With the help of geographic information and its modelling tools, we may gain a better understanding of the Earth system. Also, although this is not an objective of this study, it is possible to make these results visually attractive and useful to public understanding.

In connection with geographic information systems we used two terms: data and information, which do not mean the same thing. Data means just numerical values, but information, on the other hand, means data plus metadata, i.e., *interpreted* data.

The practice of modelling itself includes the acquisition of input data, methods for analysing the input data, choice of study area, modelling techniques, computational methods and programs, etc. Each of these aspects needs to be considered in designing geographical information systems (GIS); the most crucial one is the precision and spatial distribution of input data. From the dissertation, additional value in geographical modelling was acquired and its results are applicable to further studies and may be used by cartographers, geodesists and other geoscientists.

Quality can be expressed by internal or external measures. Internal quality measures are, e.g., precision and completeness. An external quality measure is, e.g., accuracy. Accuracy shows the measurement error against a reference dataset, e.g., the difference from ground truth. Precision describes the consistency of repeated measurements, i.e., how close repeated measurements are to each other. Collectively, accuracy and precision can be referred to as *uncertainty* (JCGM/WG 1, 2008).

Applications of relevance to this dissertation include determination of height models, geoid models and land uplift rate models. The quality of these models will vary, e.g., height models used in, e.g., large scale urban mapping, have to be centimetre precision, whereas for mapping of remote areas on small scale they may be decimetre or even metre precision. Similar considerations apply to the other model types.

As input data, many different datasets can be used for modelling. For example, in a Digital Elevation Model (DEM) point height data from levelling or laser scanning can be used. In this case, the model is composed of crisp data and the result can be visualised as a field or a surface. Another possibility is to incorporate into the model different physical parameters together with the evaluation data sets. In this case, one can see the model as an object (Devillers and Jeansoulin, 2006). Besides input data, there is usually some kind of computation strategy involved in model computation. The computation strategy should be in accordance with the model's suitability for purpose and intended use (Zhang and Goodchild, 2002). For example, for DEM, geostatistical methods can be used. In geodynamics, various types of advanced mathematical and physical approaches are used, e.g., the sea-level equation used in Glacial Isostatic Adjustment (GIA) modelling.

In building models, proper reference systems have to be used: the spatial framework includes the reference systems tied to the Earth. It must be considered that the Earth moves – both around its own axis and around the Sun, movements that have to be taken into account. There are two basic aspects of reference systems – the geometrical aspect and the physical aspect. The geometrical aspect refers to the International Terrestrial Reference System (ITRS) and its realisations. Coordinates are geocentric and may be presented either in rectangular form or as geodetic coordinates on the reference ellipsoid. The physical aspect includes the height system and the figure of the Earth – the geoid. The geoid corresponds, not to the actual surface of the Earth, but to a surface defined by the gravity field of the Earth, which is known as the physical figure of the Earth.

1.1 Motivation and aim of the dissertation

The motivation for the dissertation is to study various modelling approaches on geodetic data, to be used in the fields of geodesy, geodynamics and geographic information science. The dissertation discusses issues of internal and external quality aspects of both the modelling techniques and the data used. The models in geodesy, geodynamics and geoinformatics can be thought of as approximations to reality, in which the key element is uncertainty. Modelling can be based on treating reality either as objects or as fields. Both modelling approaches have been used within the dissertation. Geodesy deals with point measurements on the millimetre accuracy level, i.e., the accuracy characteristics are well measured and well known.

However, users sometimes do not need as accurate results, their applications can accept precisions on the centimetre or decimetre level. Also, techniques from geographic information science and geostatistics can be used to generalise results obtained into lower resolution, lower precision results that better match actual user requirements.

The novelty of the dissertation is in the application of appropriate precision measures in the various fields of research. The fitness for use of the different models depends on the input data precision as well as on the envisaged applications of the model product. E.g., in geodesy an appropriate precision can be from millimetres to centimetres, but in geographic information science the precision may vary within larger bounds.

1.2 Structure of the dissertation

The summary part of the dissertation comprises five chapters. The first chapter introduces the subject and presents the motivation and research questions addressed. The second chapter gives an overview of the theoretical background and related research. An overview of the research methods, input data and other materials used are presented in the third chapter. Chapter four presents our main findings and discusses them, while chapter five presents conclusions.

1.3 Key concepts

The key concept of the dissertation is finding criteria for judging the fitness for use of models in geodesy, geodynamics and geoinformatics. Within the different modelling approaches, different quality measures are considered. These quality measures are usually stated in the negative; they include imprecision, uncertainty, standard error, standard deviation, variance and the root mean square (RMS) error.

1.4 Objectives and research topics

The main objectives of the research are the data modelling aspects related to the fitness for use as expressed quantitatively in various quality measures within the fields of geodesy, geodynamics and geoinformatics. In the dissertation, different models for these disciplines with their quality measures are developed. From this the main research question can be formulated:

> *Which measures for model quality (such as accuracy or precision) can be used to judge the fitness for use in various use cases?*

The research topics for this dissertation are the following (see Table 1.1):

1. constructing height transformation surfaces, i.e., models, using various techniques

2. deriving empirical land uplift models for GIA affected areas

3. identifying quality measures for modelling appropriate to each model's use case.

Table 1.1. Research topics addressed by each publication

Publication	Research topic
Two alternative methods for height transformation	(1), (3)
Kriging and Fuzzy Approaches for DEM	(1), (3)
Geoid precision from limited-area gravimetric surveys	(3)
Modelling land uplift rates and their error propagation	(2), (3)
Studying Earth rheology using GNSS permanent stations and GIA modelling tools	(2), (3)

The practical study of these research topics was done as follows:

I. The main characteristics for building empirical models for use in geodesy, geodynamics and geoinformatics. The dissertation discusses model construction for DEM, for transformation approaches and for geodynamic studies. For DEM construction, the issues with height modelling together with DEM compilation are addressed. The main idea was to present alternative ways to model heights in a geographical area. Two problems were addressed in this context: one related to DEM modelling and the other to height transformation techniques. Three approaches were used: fuzzy modelling, kriging and the bilinear transformation approach. By using these methods, different surfaces were constructed with their precision measures. Also, more complex models were used, e.g., for GIA modelling. As the GIA has an impact both on vertical and horizontal crustal movements, various Earth model parameters have to be incorporated into the models. To study the land uplift, two methods were used. Firstly, a functional land uplift model was derived with its signal covariance function. For this, point coordinates and their rates of change were used, obtained from GNSS and high-precision levelling. Secondly, physical GIA models were constructed using the sea-level equation (SLE) and SELEN software (Spada and Stocchi, 2007), and validated with the crustal motion values for the land uplift areas (Fennoscandia and North-America) obtained from GNSS time series.

II. The main characteristics defining quality measures for models. Quality, e.g., accuracy or precision, or more generally, *uncertainty*, plays an important role in model construction. Numerically expressed uncertainty is the first measure for users – does this model with this accuracy or precision, meet user needs? There are different quality concepts in use – internal quality, e.g., precision, and external quality, e.g., accuracy. Also, there exist other qualitative and quantitative measures of quality. In the dissertation, various quality measures, associated with different models are used to describe the uncertainty or quality of the models.

2. Theoretical background

Theoretical quality estimates allow us to judge in a numerical fashion whether the quality of a model is sufficient for a particular application. Specifying what quality requirement any particular application has, belongs to the domain of study of that application. For example, the quality, e.g., the accuracy or precision, required for a digital elevation model will depend on which kind of area will be mapped and on which scale. From this requirement one can infer which modelling techniques and input datasets will be fit for purpose.

There are many other theoretical concepts of direct relevance for this dissertation, which will be discussed below.

2.1 Spatial framework

2.1.1 Reference frames

One of the main tasks of modern geodesy is to define and maintain a *global terrestrial reference frame* in order to measure and map the Earth's surface. A geodetic reference frame may be considered as a consistent set of geodetic stations with assigned coordinate values, velocities, and an epoch of validity (Altamimi et al., 2007). The reference frame connects observations in space and time, and also defines the framework for global and regional observations (Blewitt et al., 2010).

In geodesy, a *datum* is a reference point or surface to refer geodetic coordinates or heights to. A horizontal datum is used to describe the location of a point on the Earth's surface in a certain coordinate system, a vertical datum is used to describe an elevation in a certain height system. Nowadays datums are three-dimensional, having, e.g., rectangular coordinates (X, Y, Z), or geographic coordinates (φ,λ,h) based on a reference ellipsoid, e.g., GRS80.

An Earth-fixed (i.e. co-rotating with the Earth) and Earth-centered (ECEF) system of spatial coordinates is used as the fundamental terrestrial coordinate system (Torge, 2001). The International Terrestrial Reference System (ITRS) is realised by the International Earth Rotation and Reference System Service (IERS) through a global set of space geodetic observing sites. The geocentric Cartesian coordinates and velocities of the observing sites comprise the International Terrestrial Reference Frame (ITRF) (Torge, 2001).

The European Terrestrial Reference System (ETRS) is a regional coordinate reference system for Europe, based on the ITRS, but is fixed to (co-moves with) the stable part of Eurasian tectonic plate, coinciding with ITRS for the epoch 1989.0, and called ETRS89 (Torge, 2001).

The Stable North American Reference Frame (SNARF) is a regional coordinate reference system used in North America. SNARF defines a reference frame that represents the stable interior of North America in order to interpret intra-plate (relative) motions, which are affected by Glacial Isostatic Adjustment (GIA) (Blewitt et al., 2006).

In geodesy, the three-dimensional motion of the reference frame is constrained by assuming that sites do not move in the radial direction, i.e., they move laterally only, as a rotation around the geocentre with a specific "pole" and rotation rate. One commonly used model to describe plate tectonic motion is the NNR-NUVEL 1A (DeMets et al., 1994). It takes into account the general drift of a plate, but in the areas of intra-plate deformation the model cannot be applied, as it accounts only for the horizontal plate motion (Koivula et al., 2006).

If one wishes to use Global Navigation Satellite System (GNSS) positioning techniques to provide the height information, one has to have detailed information on the form of the Earth's gravity field (Vermeer, 1988). The reason for this is that the GNSS satellites provide coordinates in a system which is near-geocentric and has no direct connection with the Earth's figure (Vermeer, 1988; Ekman, 1995).

Networks of continuously operating reference stations (CORS) provide an accurate method for determining present-day crustal 3-D deformations. Both horizontal and vertical motions can be measured simultaneously. From time series of decadal length, the horizontal velocities can be measured at the mm/a level and vertical rates about 2 times less accurately (Koivula et al., 2006). Nowadays GNSS measurements are used for providing constraints to GIA modelling (Steffen et al., 2006).

2.1.2 Height systems

In common usage, elevations are often cited as heights above mean sea level, also called *gravity based heights*. Mean Sea Level (MSL) is a tidal datum which is described as the arithmetic mean of the hourly water elevation taken over an 18.6 years Lunar cycle (Ekman, 1995).

The geoid, introduced by Gauss in 1828, is usually defined as the equipotential surface (level surface) of the Earth‘s gravity field which is closest to the mean sea level of the oceans (Ekman, 1995). Postglacial rebound has an impact on heights through the uplift of the crust, but due to the associated inflow of mantle material below the crust it also affects the gravity field, and therefore, one has to specify an epoch for which the geoid is valid (Ekman, 1995).

A vertical datum is used for measuring the elevations of points on the Earth’s surface. Vertical datums are either tidal (based on sea levels), gravimetric (based on a geoid) or geodetic (based on reference ellipsoid models). The heights are needed within the local gravity field, therefore one has to have information about the precise shape of the geoid (Vermeer, 1988).

2.1.3 Transformation methods for heights

The height information is always connected to geoid modelling and its error propagation. Height data are usually obtained by levelling, a technique which is very labour intensive and costly. Nowadays GNSS measurements can be used, which are much faster and cheaper, but in order to use GNSS measurements for height determination, one needs a precise geoid model to transform GNSS heights to heights above sea level.

For height transformation, locally (e.g., within one triangle of a Delaunay triangulation) a bilinear or affine transformation can be used (Publication A):

$$H = h + c_0 + c_1 x + c_2 y, \tag{2.1}$$

where H is the orthometric or normal height, h is the ellipsoidal height, x and y are map projection coordinates and c_0, c_1, c_2 are transformation parameters. Note that in Publication A all height units (in tables and histograms) are cm.

In this one-dimensional case, the model reduces to a TIN (Triangulated Irregular Network) representation.

Another way to look at this is to use barycentric coordinates as a way to describe the bilinear transformation as follows (Publication A):

$$H_i = h_i + p_i^A \left(H_A - h_A\right) + p_i^B \left(H_B - h_B\right) + p_i^C \left(H_C - h_C\right), \tag{2.2}$$

where p_i^A, p_i^B and p_i^C are the barycentric coordinates of point i relative to the triangle corners A, B and C. (Note that this formula differs from that given in Publication A, which has the sign of the height differences reversed. Eq. 2.2 is the one actually used in the computations.)

2.2 Geostatistical methods

Modelling procedures are usually accompanied by some kind of statistical measures. For example, geostatistical methods are by nature mathematical methods to produce interpolative data (Longley et al., 2001).

2.2.1 Fuzzy modelling

Fuzzy set theory was invented by Lofti Zadeh in the 1960's. It provides an rational basis for handling imprecise entities (Niskanen, 2003).

Fuzzy logic can be understood as a many-valued logic. In traditional logic theory a two-valued logic (true or false) is used, in fuzzy logic the concept of partial truth can be used, where the logic value may range between completely true and completely false (Niskanen, 2003). Fuzziness is an approximation and it is derived from characteristics of the real world and human knowledge (Peuquet, 2002). In the real world, objects are not always defined by sharp boundaries, they are changing in space and in time (Stein et al., 2009).

In fuzzy logic, objects can have partial degrees to belong to the classes (Longley et al., 2001). Fuzzy modelling can be thought as approximate rather than accurate. One of the most important characteristics of fuzzy sets is that the approach can be used for datasets where the boundaries are not defined precisely, and it is impossible to establish clearly the membership function (Longley et al., 2001). By using given input values, the fuzzy approach is fitted by using neural network programming to the chosen function for the given values.

One of the most used fuzzy systems is the Takagi-Sugeno-Kang model, where the consequent part uses a linear combination of the input variables and the antecedent part uses linguistic variables (Niskanen, 2003). For linear fuzzy sets, the formula can be presented as (The MathWorks, Inc., 2015):

$$\text{IF } x \text{ is } A \text{ and } y \text{ is } B \text{ THEN } z = f\left(x, y\right), \tag{2.3}$$

where A and B are fuzzy sets in the antecendent part and $z = f\left(x, y\right)$ is a crisp function in the consequent part. Usually $f\left(x, y\right)$ is the function containing input variables x and y.

A typical rule for the Takagi-Sugeno-Kang model is expressed as (The MathWorks, Inc., 2015):

$$\text{IF input } 1 = x \text{ and input } 2 = y, \text{ THEN output is } z = ax + by + c, \tag{2.4}$$

where x and y are the input values and a, b, and c are constants.

The algorithm for using fuzzy modelling includes the following steps (The MathWorks, Inc., 2015):

1. Fuzzification, i.e, choosing the membership functions to be used and mapping input values to membership values.

2. Weighting of the each rule (using, e.g., the multiplying or minimising techniques).

3. Generating consequents (either fuzzy or crisp) for each rule.

4. Defuzzification, i.e. computing crisp values from fuzzy sets.

2.2.2 Kriging

Geostatistical methods for interpolation are often based on the fact that the spatial variation of continuous attributes is too irregular to be modelled by a simple mathematical function, additionally they provide quality measures for the interpolation (Burrough and McDonnell, 1998). One interpolation method using geostatistics is known as kriging, after D. G. Krige (Burrough and McDonnell, 1998). It uses a statistical model for spatial continuity in the interpolation of unknown values, based on values at neighbouring points (Sunila et al., 2004).

The variogram is an important tool to determine the optimal weight for interpolation (Burrough and McDonnell, 1998). In kringing, the variogram is used to investigate the spatial continuity and how this continuity changes as a function of distance and direction.

Ordinary kriging is a variation of the weighted interpolation technique, the basic equation used is (Burrough and McDonnell, 1998):

$$\hat{Z}\left(x_0\right) = \sum_{i=1}^{n} \lambda_i \underline{z}\left(x_i\right), \tag{2.5}$$

where n is the number of sample points; λ_i is the weight of each sample point and $\underline{z}\left(x_i\right)$ is the value at each of the points.

Ordinary kriging can be thought of as a true interpolator, e.g., the surface for interpolation coincides with the values at the data points (Burrough and McDonnell, 1998).

2.3 Glacial isostatic adjustment

2.3.1 Overview

Land uplift, also called post-glacial rebound (PGR) or glacial isostatic adjustment (GIA), is caused by changes in the continental ice sheet loading in high-latitude areas. GIA has an impact on horizontal and vertical coordinates, specifically in the regions where the phenomenon is still ongoing.

GIA is described as the rise of land masses that were depressed by the weight of glaciers during the last glacial period (Peltier, 1990; Fjeldskaar, 1991). During the last glacial maximum, the ice was up to two-three kilometres thick in Fennoscandia and North-America, and water from the oceans was tied up in these large ice sheets (Sella et al., 2007). The weight of the ice caused a depression of the Earth surface. At the end of the ice age, when the glaciers melted, the removal of the weight of the ice caused land uplift, and the meltwater caused global sea level rise. Because of the high mantle viscosity, thousands of years will pass for the Earth to regain its equilibrium state (Steffen and Wu, 2011)

The Earth is influenced by the time-dependent behaviour of ice sheets. With the main periodicity of the 100 ka period, the ice caps have regularly grown and decayed (Le Meur, 1996; Johansson et al., 2002). This influence is described as a “memory effect”, meaning that ice loads in the past continue to have an effect today, i.e., the Earth has not yet recovered the equilibrium state corresponding to the present-day load (Le Meur, 1996; Johansson et al., 2002).

Ice loading generates both elastic and viscous deformation. After unloading, the elastic deformation is recovered instantaneously while viscous deformation recovers according to the relaxation time of the specific deformation mode. Practically, this relaxation is stopped by the start of another ice age and the deformation that is observable today is the result of a series of glacial cycles (Whitehouse, 2009).

GIA is a slow process which decays exponentially and is determined by the mantle viscosity (Douglas and Peltier, 2002). Glacial isostatic adjustment is affected by the global history of the deglaciations, constraints for it are obtained from geomorphology and sea level data (Whitehouse, 2009).

To study the GIA process, several data sources are in use, including time series of sea level data and tide gauges, high-precision levelling and gravity measurements (Johansson et al., 2002). High precision measurements of crustal deformations in three dimensions were not possible before the appearance of space geodetic techniques (Koivula et al., 2006).

2.3.2 Regions of land uplift

The uplift of the crust has influenced the Earth mostly, but not only, on the northern hemisphere. For GIA studies the Fennoscandian uplift area is very well investigated and may be considered a best of breed. But the same changes can be seen also in, e.g., North America, Antarctica and Australia. In the following, the land uplift in two areas – Fennoscandia and North America will be described.

The land uplift regions in Fennoscandia and North America (Laurentide) are similar in many ways (Walcott, 1973):

- In both regions, the shield merges with a continental platform one side and with heavily glaciated mountains that border the adjacent ocean at the other side.

- Both regions have an elliptical shape, the major axis of the Laurentide uplift region is directed north-west, for the Fennoscandian uplift region it is directed north-east.

- To the north of the major uplift region, there is a smaller region of rebound.

The difference between the Fennoscandian and North American uplift regions is in their horizontal extent, in which they differ by a factor of about two (Walcott, 1973).

2.3.2.1 Fennoscandia

In recent decades, the Fennoscandian region has offered a great potential for post-glacial rebound studies (Le Meur, 1996). The main reason is the fact that the Earth still experiences the influence of the past ice-loading events (Le Meur, 1996). A small contribution also comes from the reloading of sea water due to the land uplift itself (Ekman, 1988).

In Fennoscandia, the uplift is well determined by geodetic observations. In 1992, the project called BIFROST (Baseline Inferences for Fennoscandian Rebound Observations, Sea Level and Tectonics) was created (Johansson et al., 2002). One of the primary goals of BIFROST was to use the three-dimensional measurements from the GNSS network to constrain models of the GIA process in Fennoscandia (Johansson et al., 2002; Lidberg et al., 2010).

For Fennoscandia, the maximum absolute land uplift is about 1 cm/a, in very good agreement with results from the BIFROST project, which gives the absolute land uplift value as 11 mm/a (Ekman, 2009; Mäkinen, 2000; Scherneck et al., 2001; Lidberg, 2007).

The vertical motion is usually accompanied with a horizontal motion. The horizontal motion is relatively slow where the radial motion is large (as in the uplift centre). The horizontal motion increases within distances from the uplift centre and can reach about 1 to 2 mm/a (Milne et al., 2001). The motion is everywhere radially away from the centre of the uplift area (Ekman, 2009; Mäkinen, 2000; Scherneck et al., 2001; Lidberg, 2007; Milne et al., 2001). In Fennoscandia the horizontal rebound component has a unique property – the deformation is dominated by surface extension throughout the uplift area (Scherneck et al., 2001).

For Fennoscandia, several land uplift models have been obtained over the last decades, e.g., Ekman (1996), Lambeck et al. (1998a) and Vestøl (2006). These models are based on different data types: sea level records, lake level records, repeated high-precision levelling, and time series from continuous GNSS stations. In these models different modelling techniques were used, nevertheless, they all agree about the maximum uplift rate in

Fennoscandia (about 10 mm/a) (Ekman, 1996; Staudt et al., 2004; Lambeck et al., 1998a; Vestøl, 2006; Müller et al., 2005).

2.3.2.2 North America

Besides the uplift in Fennoscandia, similar patterns are visible in North America. The North American uplift area is two-three times larger than the Fennoscandian one, but the Fennoscandian uplift is well determined by geodetic observations. The reason is that the distribution of GNSS stations in North America is not as good as in Fennoscandia, mostly due to the inaccessibility of large parts of Canada and North America.

In North America, the present-day uplift rates are about 1 cm/a near Hudson Bay (Latychev et al., 2005; Sella et al., 2007). To the south of the Great Lakes, the predicted subsidence is about 1-2 mm/a, higher rates (3 mm/a) are found to the northwest of Laurentia, in locations on the periphery of more than one glaciation centre (Latychev et al., 2005).

The horizontal velocities are scattered and they show a spoke-like pattern of motions directed outward and increasing in amplitude away from the area of maximum uplift (Latychev et al., 2005; Sella et al., 2007). This trend turns around in the far-field area of the deglaciation, where the motions are directed towards Hudson Bay (Latychev et al., 2005; Sella et al., 2007). There is no radial pattern visible as it is in Fennoscandia (Sella et al., 2007). The predicted horizontal motions are about 1-2 mm/a in the near field and close to 1 mm/a in the far field of Hudson Bay (Latychev et al., 2005). Some of the horizontal motion is a combination of local site effects and intraplate tectonic signal (Sella et al., 2007).

2.3.3 GIA models

The deformation of the solid Earth is a key process of GIA. Nowadays it may be observed using GNSS technology to measure horizontal and vertical deformation rates relative to the centre of the Earth (Scherneck et al., 2001; Johansson et al., 2002; Lidberg et al., 2010). Accompanying this deformation is sea level change. Present-day rates of relative sea level change are measured using tide gauges. A third observable relates to changes in the gravity field. This signal is the observable in the rate of change of the present-day gravity field measured by the Gravity Recovery and Climate Experiment (GRACE) satellite mission or terrestrial gravity surveys. (Whitehouse, 2009)

The glacial and post-glacial readjustment process of the Earth depends on the space-time history of the large ice sheets and the rheology of the Earth's lithosphere and mantle (Wu et al., 1998). The first input to the GIA model is the ice loading history, it determines the ocean loading history via the sea-level equation (Lambeck et al., 1998b). Thereafter the combined loading (ice and water) is applied to the chosen Earth model, which is the second input to the GIA model. Once the solid Earth deformation is calculated, the resulting change in relative sea level can be determined. (Whitehouse, 2009)

With the help of GIA an overview of three major Earth processes can be obtained (Sella et al., 2007): first, the delayed response to deglaciation helps to constrain the viscosity structure of the mantle; second, GIA signals provide constraints on the distribution and thickness of ice; third, GIA causes a deformation of continental plates and possibly causes seismic events.

And vice versa, GIA produces the following measurable effects (Ekman, 2009): vertical crustal motion, global sea level change, horizontal crustal motion, gravity field change, the Earth's rotational motion change, and a state of stress leading to multiple small earthquakes.

In GIA modelling, a description of the Earth structure consists of parameters for lithosphere thickness and mantle viscosity. The method for solving the sea-level equation depends on the choice of Earth structure. (Whitehouse, 2009)

Earth models used in GIA studies apply spherical geometry to represent the whole Earth. These models consist of an elastic lithosphere of constant thickness and viscoelastic mantle layers. The mantle is divided usually into the upper and lower mantle or a multi-layer structure is used, each layer has usually a single viscosity value. (Whitehouse, 2009)

Space geodetic techniques have the sensitivity to recover horizontal deformation due to GIA, they add important information on the viscosity structure of the Earth mantle and lithosphere thickness, thus helping to place tighter bounds on ice shield parameters (Scherneck et al., 2001). As the horizontal movements are generally smaller than the vertical movements, their detection is a more difficult (Vaníček and Krakiwsky, 1986). Moreover, the horizontal motions are sensitive to the gradient of the radial motions, i.e., they are small in regions where the radial velocities are at a maximum (Mitrovica et al., 2001).

2.3.4 Earth models

The solid Earth deforms due to the variable loads on its surface. The deformation can be divided into an instantaneous and a time-dependent component. The instantaneous component is modelled by the Hooke or elastic deformation model, which describes reversible deformations that will revert instantly when the load vanishes. The time-dependent component is modelled by the Newton or plastic deformation model, which will not revert in this way.

In all models which use spherical geometry, the spherically layered Earth model (Preliminary Reference Earth Mode or PREM) from Dziewonski and Anderson (1981) is used to determine the Earth's radial elastic and density structure (Whitehouse, 2009). Viscosity values can be obtained by inversion or can be estimated from independent geophysical studies. In different studies different Earth model parameters are used, but the range of mantle layer viscosities is for upper mantle $2 \times 10^{20} < \eta_{um} < 10^{21}$ Pa s and lower mantle $2 \times 10^{21} < \eta_{lm} < 10^{23}$ Pa s respectively (see as well Lambeck et al. 1998b; Steffen and Kaufmann 2006; Milne et al. 2001; Lambeck et al. 1998a; Tushingham and Peltier 1991; Thatcher and Pollitz 2008; Moisio and Mäkinen 2006; Wieczerkowski et al. 1999; Lidberg et al. 2010; Kaufmann and Lambeck 2000; Milne et al. 2004; Whitehouse 2009 and references therein). For the lower mantle the value of 8×10^{22} Pa s was given already by Walcott (1973). The viscous structure is represented by a simple three-layer model defined by an elastic lithosphere and uniform upper and lower mantle viscosities, where the boundary between the viscous layers coincides with the seismic discontinuity at a depth of 670 km (PREM). The uplift data require continental lithospheric thickness to be about 70-200 km (Whitehouse, 2009). There is a viscosity difference between the upper and lower mantle of approximately 1-2 orders of magnitude (Walcott, 1973).

Different sets of viscosity values are used throughout different studies, an overview is given in Table 2.1.

2.3.5 Ice models

For the ice model three types of data can be used: ice margin data, ice loading data and global sea level data. There are two ways of constraining ice models: the first is using the relative sea-level history. The second uses

Table 2.1. Earth model parameters

upper mantle viscosity	lower mantle viscosity	references
$(3-4) \times 10^{20}$ Pa s	5×10^{21} Pa s	(Lambeck et al., 1998b) (Lambeck et al., 1998a)
4×10^{20} Pa s	2×10^{22} Pa s	(Wieczerkowski et al., 1999) (Milne et al., 2004) (Steffen et al., 2006)
5×10^{20} Pa s	5×10^{21} Pa s	(Scherneck et al., 2001) (Latychev et al., 2005) (Lidberg et al., 2010)
5×10^{20} Pa s	2×10^{22} Pa s	(Ekman, 2009)
7×10^{20} Pa s	1×10^{22} Pa s	(Steffen and Kaufmann, 2006)
8×10^{20} Pa s	10^{22} Pa s	(Milne et al., 2001)
10^{21} Pa s	2×10^{21} Pa s	(Peltier, 1998a)

an iterative method to solve the sea-level equation. (Whitehouse, 2009)

In order to tune glacial history, adjustments to the Earth model are carried out in parallel. If a specific ice model is used, it is important to use the proper radial viscosity profile developed in parallel with the Earth model, otherwise the result will not match to the observational data (Whitehouse, 2009).

There are several ice models published over the years by different authors, namely Peltier, Lambeck, Mitrovica and Milne, Wu, Kaufmann, Zhong, Paulson and Wahr, Sabadini and Spada, Vermeersen, Fjeldskaar, and others. All of the authors have published local ice models, but some of them have worked with the global ice models too.

Global ice models ICE-(3, 4, 5, 6)G were developed by Richard Peltier and his co-workers, these models were fitted to the observational data using radial viscosity profiles. Although the most recent model is the ICE-6G ice model (Argus et al., 2014; Peltier et al., 2015), also ICE-5G (Peltier, 2004), ICE-4G (Peltier, 2002) and ICE-3G (Tushingham and Peltier, 1991) are widely used, often in combination with alternative local ice models. The ice and Earth models are tuned to fit relative sea level, geomorphological and geoid data. (Whitehouse, 2009)

ICE-5G is an updated version of ICE-4G, for the ice model new observational data were added, e.g., historical sea level data, ice margin data, GIA data as well as geodetic and gravimetric measurements from different regions (Whitehouse, 2009). ICE-5G ice model should be used together with the VM2 Earth Model (Peltier, 2004).

Global ice models KL05 (or ANU05) were developed by Kurt Lambeck and his co-workers, these models are based on observational data from ice sheet history, Earth structure, and the records of climate, glacial cycles, and sea-level change (Whitehouse, 2009). The model KL05 has been assembled from several regional ice models (Whitehouse, 2009): the Fennoscandian part, which covers also the Barents Sea, FBK8 from Lambeck et al. (1998b), Laurentide and Greenland parts of ICE-1 from (Peltier and Andrews, 1976), the British Isles ice model from (Lambeck, 1993) and the ANT3 Antarctic model from (Nakada and Lambeck, 1998).

Models ICE-5G and KL05 differ in several aspects, including the mantle viscosity profile, the ice distribution and the history of equivalent sea-level measurements (Spada and Galassi, 2012). But the spatial scale of these models at the Last Glacial Maximum is similar, although ice thickness is counted differently in the areas of deglaciation (Whitehouse, 2009). Because of the mentioned differences, both ice models will give different output when used in conjunction with a GIA model (Whitehouse, 2009).

2.3.6 Sea level

The GIA component of sea-level change is evaluated solving the sea-level equation (SLE), all terms of the SLE are dependent on the history of ice thickness variation (Spada and Galassi, 2012). During the LGM global sea level was reduced due to the large volume of water retained in continental ice sheets (Rittenour, 2015). During deglaciation the meltwater returns to the oceans and sea level rises. Sea level has varied by more than 120 m during glacial/interglacial cycles (Church et al., 2008). Geological records of sea level changes show that the redistribution of the meltwater is not the same everywhere in the oceans, this is due to the gravitational attraction change and the change in centrifugal potential due to the Earth's variable rotation (Whitehouse, 2009).

The uplift of the crust relative to mean sea level can be detected from long time measurements of sea and lake levels. The geocentric uplift (denoted by $\dot{h}$) can be obtained from (Ekman, 1988):

$$\dot{h} = \dot{H}_a + \dot{H}_e + \dot{H}_g, \tag{2.6}$$

where $\dot{H}_a$ is the apparent land uplift (i.e., the relative motion of land and sea surface), $\dot{H}_e$ is the eustatic sea-level rise and $\dot{H}_g$ is the geoid uplift.

The absolute, i.e., geocentric land uplift is needed when dealing with gravity decrease due to the land uplift as well as determining the uplift component from GNSS time series. The latter is used to study apparent and absolute land uplift differences and gives the possibility to study sea-level changes (Ekman, 1988).

The prediction of relative sea-level variations is a complicated process: the ocean redistribution is directed by the gravitational field and deformations of the solid Earth, the gravitational field itself is perturbed by the direct gravitational effect of the ocean redistribution and the solid Earth deformation (Mitrovica et al., 2010). This circularity is solved by the sea-level equation (Farrell and Clark, 1976).

2.3.7 The sea-level equation

The theory of glacial isostatic adjustment can predict the history of relative sea level variations, given as function of $S(\varphi, \lambda, t)$, which is known as the *sea-level equation* (Peltier, 1998b). The sea-level equation shows the spatial and temporal change in ocean bathymetry, where the gravity potential over the sea surface shall be spatially constant for a specific deglaciation chronology and viscoelastic Earth model (Spada and Stocchi, 2005).

The sea-level equation has been discussed already in many publications (see Whitehouse (2009) and references therein). One can write (Lambeck et al., 1998b):

$$\Delta\zeta(\varphi, t) = \Delta\zeta_e(t) + \Delta\zeta_I(\varphi, t) + \Delta\zeta_T(\varphi, t), \tag{2.7}$$

where $\Delta\zeta(\varphi, t)$ is the mean sea level at location φ and time t, measured with respect to present sea level; $\Delta\zeta_e(t)$ is the eustatic sea-level change, which is defined as: $\Delta\zeta_e(t)$ = change in ocean volume / ocean surface area; $\Delta\zeta_I(\varphi, t)$ is the additional change that results from the isostatic adjustment of the crust to the changing ice-water surface load; $\Delta\zeta_T(\varphi, t)$ is any additional tectonic contribution resulting from geophysical factors (Lambeck et al., 1998b).

To solve the sea-level equation, the solution shall include the whole

Earth. This ensures that water produced by the melting of ice sheets is consistently redistributed throughout the oceans. (Whitehouse, 2009)

A simple algorithm to solve the sea-level equation can be described as follows (Whitehouse, 2009):

- Using initial predictions for the global distribution of the change in ocean height for a given time step ($\Delta\zeta\left(\varphi, t\right)$), the resulting global distribution of the change in sea level in the spectral domain is calculated;

- This solution is transformed to the spatial domain, and projected to the ocean function (a function that has the value of 1 on the oceans and 0 on land);

- The solution is transformed back to the spectral domain for a next estimate for ocean height change ($\Delta\zeta\left(\varphi, t\right)$). The process is repeated until convergence is achieved for the ocean height change for that time step.

3. Research methods and materials

The main research idea was to investigate a variety of modelling approaches, their various precision measures and their suitability for different use cases. The choices made in modelling included modelling strategies and precision estimation methods. The main research question was:

> *Which measures for model quality (such as accuracy or precision) can be used to judge the fitness for use in various use cases?*

In this research, the common elements of modelling approaches in the field of geodesy, geodynamics and geoinformatics were discussed from this viewpoint.

3.1 Publication A

Publication A introduces two different methods for height transformation: the bilinear affine transformation approach and fuzzy modelling.

In geodesy, in general, transformation methods are used to find missing information by means of various mathematical approaches. By its nature, the height transformation can be thought of as a linear approach, but, because it is dependent on the geoid, this would require a more complicated approximating function. The main assumption of this research was the geoid piecewise linearity over the study area in Estonia. For input data, data from the Estonian geodetic network were used.

Firstly, the affine bilinear transformation technique was applied to a triangulated network covering the study area. Within every triangle, barycentric coordinates were used in order to calculate normal heights for points. In the triangle nodes known rectangular map projection coordinates (x, y) as well as ellipsoidal (h) and normal heights (H^*) were used.

Secondly, the fuzzy method was used, taking advantage of multi-valued reasoning. The fuzzy membership functions were fitted to the input data and the transformation surface was formed. In order to find a suitable fuzzy algorithm, different models with different membership functions were created. The most suitable for the elevation surface construction were triangular and Gaussian models with different numbers of membership functions. Thereafter height values for data points were derived from the transformation surface.

For both the bilinear affine transformation and fuzzy modelling approaches, the transformation surfaces were determined and the heights of the points were computed with their error measures.

3.2 Publication B

Publication B discusses the possibility for DEM construction and quality measures when using kriging and fuzzy approaches.

A Digital Elevation Model (DEM) was used to present topographic information. For DEM construction, several different methods can be used. In this article two of these were studied: fuzzy modelling and kriging. The input data were height data from a laser scanning survey, altogether 2000 laser-scanning points, which were situated in the area of about $2\,\mathrm{km}^2$ in the Rastila area in Helsinki. The data used were rectangular point coordinates in the map projection plane and heights above sea level in the range of 0 to 18 m.

For constructing a DEM by the fuzzy modelling method, the Matlab Fuzzy Toolbox was used. Two methods of fuzzy modelling were chosen for the study – grid partition and subclustering. Altogether 20 models were computed, from which three models were chosen by the smallest RMS value.

Kriging is a geostatistical method based on least-squares interpolation producing optimal field predictions from discrete data points. For the construction of the kriging DEM, first the candidate variograms were computed. For the selection the RMS value was used, which shows how well the model is fitted to the empirical variogram. From the RMS analysis the exponential variogram was chosen as the most suitable one. Thereafter the ordinary kriging method was implemented to estimate and interpolate the data and the kriging DEM map was produced.

As the result, two DEM models with their quality measures were computed.

3.3 Publication C

Publication C discusses theoretical estimates for the gravimetric geoid precision as well as the structure of the uncertainty – i.e., the uncertainty budget. In this context the sources of uncertainty and their relative contributions as well as data coverage (i.e., how lacking data outside the border and the limited resolution of global models affect the precision) were studied. The example calculations are given for two case study areas – Finland and Estonia.

In the study three geoid error sources were considered:

- The error of omission. This error represents geoid error directly caused by the finite spatial density of the gravity survey.

- The aliasing error. This error represents geoid error due to finite spatial density of the gravity survey, the part of the field above the truncation degree. For this error two approaches were given, one of them used the concept of white noise and the other used the Stokes integral. For the aliasing error the accuracy was computed for the different grid spacings. These calculations assumed an infinite extent of the gravitational survey data.

- The out-of-area error. This error acknowledges the fact that gravimetric data may not be available for neighbouring areas.

The input data needed for the calculation of the geoid errors were the mean separation of gravimetric measurement points and the average "error of prediction" of the gravimetric survey for two test regions (see Table 3.1).

Table 3.1. Input data for geoid precision study (Publication C)

Indicator	Finland	Estonia
Average "error of prediction" [mGal]	± 2	± 3
Mean separation of gravimetric points [km]	4	5

3.4 Publication D

Publication D introduces a statistical model for predicting the uplift rates from the existing point uplift rates with its empirical signal covariance function.

In this study we investigated, given the precision of the land uplift values obtained from GNSS time series, how precise the land uplift value predicted at an arbitrary point would be. In order to find the solution, firstly, one should know the functional behaviour of the land uplift model, and secondly, the general stochastic behaviour of local uplift deviations from this functional model. These deviations can be characterised by a signal covariance function estimated empirically by least-squares collocation.

The derived model allows the prediction of point height values above sea level if the following are given:

- point coordinates allowing to extract the uplift rates from the model;
- current point height as measured by GNSS;
- a geoid model for extracting the point geoid height using point coordinates.

Two different datasets for obtaining uplift values for Fennoscandia were used: data from the BIFROST project (Johansson et al., 2002) and data from the last Finnish precise levellings, jointly adjusted with the previous levelling campaigns. From the BIFROST project, uplift values for the whole Fennoscandian uplift area and uplift values for the Fennoscandian central area were used. One has to be aware that the BIFROST project provides geocentric land uplift values; the dataset from the Finnish precise levelling, on the other hand, provides land uplift values relative to mean sea level.

Firstly, a functional model for uplift rate prediction was derived based on a 2D elliptical geometry. Thereafter plausible initial values for the model parameters were chosen. The computation iteratively improved the model parameter values. It showed the quality of the functional model for predicting land uplift at an arbitrary point. After parameter estimation, an uncertainty model over the Fennoscandian area was derived by using the least-squares collocation method. As a result, an empirical covariance function for residuals relative to the functional model from the previous stage was derived for quality assessment.

3.5 Publication E

Publication E describes the modelling of GIA in the North American and Fennoscandian uplift areas, deriving fitted Earth model parameters and their uncertainties, using the ice models ICE-5G and KL05 as input.

In this article the focus was on GIA processes in North America and Fennoscandia. For these areas GIA modelling was carried out using the free software SELEN for the visco-elastic modelling (Spada and Stocchi, 2007). For the reference dataset GNSS data from CORS were used. For North America the dataset from Sella et al. (2007) and for Fennoscandia the BIFROST dataset from Lidberg et al. (2010) were used.

The study was performed in different stages. Firstly, the sensitivity of the results to the maximum harmonic degree included in the model was tested. As a result, a maximum harmonic degree of 72 was chosen, as including higher degree numbers did not significantly change the results obtained. Secondly, the GIA computation was carried out in order to find the Earth model parameters yielding the best fit with the GNSS-based velocity field. With both ice models the following Earth model parameters were included in the estimation process: upper mantle viscosity, lower mantle viscosity and lithosphere thickness. In addition, an alternative, two-step method (the “2D+1D approach”) was tested against the more exact 3D approach. For this 2D+1D approach we considered the estimation of Earth model parameters in two steps: firstly, mantle viscosity values were fitted and thereafter the lithosphere thickness was estimated.

For optimal fitting the χ^2 goodness of fit measure was used (Milne et al.,

2001) to test the GIA induced velocity against the velocity values from the GNSS time series. In the computations the Fennoscandian dataset was used for testing the methodology, and afterwards the same computations were performed for the North American uplift area. As the result, the optimum Earth model parameters were found for both ice models having the smallest χ^2 misfit with the GNSS data.

4. Discussion of research results

4.1 Transformation surface and DEM

Height modelling, as discussed in this dissertation, covers transformation surface modelling and DEM modelling methods. Transformation methods are always used within the context of some reference frame, relations between coordinate systems are described by coordinate transformations.

4.1.1 Obtaining height values by means of a transformation surface (Publication A)

Among coordinate transformation methods, the height transformation is the easiest one, as it has only one dimension. In this study different interpolation methods for finding correct gravity based, i.e., orthometric or normal heights (in the absence of a precise geoid model) were tested by using plane coordinates and ellipsoidal heights from GNSS as input data. The purpose was to construct a height transformation surface covering the Estonian territory by using two different methods.

Table 4.1 presents an overview of the used methods, using standard deviation as a quality measure. All three models were based on the triangulation of the area of study. The difference between the three models was in triangle size, i.e., the distance between triangle nodes.

Table 4.1. Comparison of quality measures for bilinear and fuzzy approaches

	Standard deviation (cm)		
	Model 1	Model 2	Model 3
triangle size (km), approx.	60	85	150
Bilinear transformation	9.9	15.0	31.0
Fuzzy Gaussian	40.0	23.0	27.0
Fuzzy, triangular	34.0	22.0	27.0

The bilinear transformation approach has a very simple and understandable mathematical structure. Table 4.1 shows that Model 1 has the best quality measures. This conclusion is somewhat expected for the model having the smallest triangle size. The weakness of this method is in its piece-wise linear nature, as of course the geoid surface is not linear at all.

The fuzzy modelling approach, especially with neural network programming (i.e., neuro-fuzzy), is complex by its nature. In our research, we found that the triangular membership function was more suitable for use with the input data at our disposal. As the triangular membership function is linear, it gives a good fit for the geoid in the study area, which is rather smooth. On the other hand, the Gaussian membership function should theoretically give a better fitting transformation surface, being smooth like the geoid, rather than angular. The fuzzy approach produced rather poor results (see Table 4.1) with large standard deviations, especially for the smallest triangles. Fuzzy modelling is meant to be used with large datasets, in case of which it performs well, as neural network programming has good abilities to extract useful information from these datasets.

Within this publication the weaknesses of the fuzzy modelling approach were the sparse dataset used and long distances between the data points, which did not show the method's ability at its best.

Both presented approaches can be used in a height prediction process when a precise geoid model is not available. The fitness for use of these algorithms is dependent on the given input dataset, which should be of sufficient size. The achieved precision of the bilinear approach could make this a useful method in some geodetic applications. The results from fuzzy modelling are less useful for the given input dataset. We observe that limitations are posed by the density and homogeneity of the input dataset, which may, as seen by these case studies, affect the overall quality of the results.

4.1.2 Obtaining DEM by means of a transformation surface (Publication B)

Digital elevation modelling shows the possibilities of using various modelling concepts to refer to different topographic features. Two methods were used within the study – fuzzy modelling and ordinary kriging, both widely used in production settings for DEM construction. The aim of the study was to investigate these methods and their suitability for DEM modelling as well as to obtain quality measures for output products.

The statistics of the computed models is presented in Table 4.2. Note that the statistical information for the TIN model is given only for comparison.

Table 4.2. RMS statistics for the fuzzy and kriging approaches

Model/statistical quantity	Fuzzy	Kriging	TIN
RMS (cm)	4.21	3.27	4.60

Comparing the two methods used in this case study, one could conclude that the kriging approach gives a better error measure and the graphical representation of the height surface looks good (see Publication B for details). When the number of observations is adequate, the kriging technique provides a better fit in elevation surface modelling than the fuzzy approach. When using these algorithms, one has to be aware that obtaining reliable output is critically dependent on the input data and their spatial distribution.

4.1.3 Quality measures for height transformation surface models (Publications A and B)

Transformation methods and geostatistical analysis use probabilistic quality measures while modelling geospatial data. Transformation methods have basically two precision measures: the precision of transformed coordinates and the overall transformation precision. In this study the latter was used for comparison. For geostatistical analysis continuous variables are usually tested against reference data using RMS or similar accuracy measures.

In Publications A and B, the quality measures were computed for DEM models and transformation surfaces. For Publication A the achieved quality measures are presented in Table 4.1. The input data density was also very sparse, being about 1 point for $2000\,\mathrm{km}^2$. For the given data density, the methods used can not be applied for high-precision geodetic applications, but may be sufficient for cartographic or GIS purposes. For Publication B two different methods were used for DEM construction and error estimation. The quality measures for DEM models are presented in Table 4.2. The input point density for this study was about 2000 points for the area of $2\,\mathrm{km}^2$. These quality measures suggest that these methods can also be used in some geodetic applications.

These two studies indicated (Table 4.1 and Table 4.2) that the quality measures improved when the data point density increased. This suggests that the results of the first study would have been better if a denser dataset could have been used (see Publication A). Unfortunately, this was not available at the time.

The results for height modelling showed that both the transformation (traditional application) and DEM (non-traditional application) methods were suitable for geodetic or GIS applications. The most important factors to affect the quality measures were the precision and spatial distribution of input data.

4.2 Quality measures for the geoid model (Publication C)

For geoid models different quality measures as well as the structure of the uncertainty were investigated and evaluated for the two test areas, Finland and Estonia, see paragraph 3.3 and Table 3.1.

Table 4.3. Geoid quality measures: error of omission and aliasing error

Indicator	Finland	Estonia
Error of omission [mm]	±3	±3.6
Aliasing error (point separation 20 km) [mm]	±4.8	±6.2
Aliasing error (point separation 50 km) [mm]	±6.2	±7.8

The comparison of quality measures showed them to be similar for the test areas, considering the difference in the area size and the measurement precision of input gravity data (see Table 4.3). For one error source (see Table 4.4), the results were different, but for this error source other phenomena, like data availability in border areas and neighbouring countries, were in play.

Table 4.4. Geoid quality measures: out-of-area error [mm]

study area	Finland			Estonia		
$\ell\downarrow$ max $(\lambda, \delta) \rightarrow$	200	200	200	200	200	200
max $(\ell, \delta) \rightarrow$	20	50	100	20	50	100
10	13.74	10.66	7.54	24.92	19.34	13.67
20	50.67	39.32	37.80	94.40	73.24	51.79

In the Table 4.4 ℓ is the correlation length of the gravity anomalies, δ is the distance of the evaluation point from the border, and λ is the semi-wavelength of the global reference model used. Nowadays this will always be at least as good as GOCE, i.e., 200 km.

This study showed that in order to achieve a good geoid precision, a good coverage and high quality of input data are needed.

4.3 Land uplift rate recovery and GIA

Geodynamic studies are mainly based on so-called *physical* GIA modelling approaches. GIA is a complex of problems to which applies the approach of *visco-elastic Earth modelling* with its parameters, like radial viscosity profile, lithosphere thickness and ice load model. However, the tricky part is that all these parameters are themselves variables in the GIA process.

4.3.1 Land uplift rate model (Publication D)

In this study, a method was derived to predict the uplift rate in an arbitrary point for which the position coordinates were given in the Fennoscandian uplift region, if the uplift rates in a set of discrete data points were known. This, together with a geoid uplift model, allows prediction of gravity based heights for the future.

This analysis yields the precision of the uplift rate of an arbitrary predicted point anywhere in the terrain, which can be height-connected to levelling benchmarks using GNSS and a precise geoid model. The RMS of the residuals of fit is presented in Table 4.5.

Table 4.5. RMS of the residuals of fit for a land uplift model to the BIFROST and Finnish precise levelling datasets

Model	RMS [mm a^{-1}]
BIFROST, whole area	± 1.685
BIFROST, central area	± 0.852
Finnish precise levelling	± 0.314

This study showed the possibilities of uplift modelling using the least squares collocation method. In the research simple functional models, the estimates of the signal covariance functions of land uplift residuals and a standard deviation describing their estimation precision were derived for two input datasets. The method agrees well with quality measures found in independent studies, i.e., in the Finnish precise levelling and BIFROST.

4.3.2 GIA modelling (Publication E)

In this study physical GIA modelling was carried out in two test regions in Fennoscandia and North America. Optimal Earth model parameters specific for the ice models ICE-5G and KL05 were estimated. The optimality criterion used was χ^2 goodness of fit with GNSS crustal-motion observations.

In the study different tests were performed, as already described in paragraph 3.5. The sensitivity to the maximum harmonic degree included was tested for the both ice models. This test showed that the choice of maximum harmonic degree had an important effect on computation time, but from the specific value of maximum harmonic degree, the results did not change significantly. For example, for a maximum harmonic degree of 72 the difference with the results for the maximum harmonic degree of 128 remained below 3% for the both ice models (See Publication E for details). Also a comparison between the 2D+1D and 3D computational approaches was performed, see paragraph 3.5. Results showed that although the 3D approach did give a slightly smaller χ^2 misfit, results were not significantly different.

As a result, optimal Earth model parameters were obtained for two ice models in two test regions. For Fennoscandia, the results agree with those from other published studies. Moreover, the results for upper mantle viscosity and lithosphere thickness showed good agreement with the nominal Earth model parameters which were used to calculate the ice models, but the values for lower mantle viscosity were different, especially for the ice model KL05. For North America, similar computations were performed only for the ICE-5G ice model, and the results of optimal Earth parameter fit were all different from the nominal ice model parameters.

4.3.3 Quality measures for the land uplift models (Publications D and E)

In Publications D and E the post-glacial land uplift modelling errors were investigated. As computed GIA models are often taken as a representation of the physical Earth, these models are usually tested on reference datasets, in our case the GNSS time-series or land uplift rates obtained from precise levelling. Within the GIA studies, precision measures like RMS and methods like χ^2-fitting were used to validate the models.

In Publication D promising results were achieved. For the high-precision levelling data an RMS of ± 0.314 mm/a was obtained. For the GNSS dataset a value of ± 0.852 mm/a for the Fennoscandian central area and of ± 1.685 mm/a for the whole area including the forebulge were obtained. The differences between these two results indicate that the chosen simple functional model may not be sufficient to model the land uplift when using BIFROST data, especially for the whole uplift area.

In Publication E physical GIA modelling was carried out. This implements the search of the optimal Earth model parameters using the predefined ice models. For the Earth model parameter search the criterion of χ^2 goodness of fit to the reference dataset was used. As reference, the velocity field from GNSS permanent station time series was used both for North America and Fennoscandia. The RMS values obtained from the study are presented in Table 4.6.

Table 4.6. RMS quality measure for the GIA models (Publication E)

Model	RMS [mm a^{-1}]
Fennoscandia, KL05	±1.772
Fennoscandia, ICE-5G	±1.253
North America, KL05	±1.535
North America, ICE-5G	±1.627

The results from both studies showed (Tables 4.5 and 4.6) that for the Fennoscandian uplift area, the both methods gave similar precision measures for the same reference datasets, i.e., land uplift values can be predicted using an empirical model with the same precision as physical GIA modelling.

4.4 Outline

One of the most important considerations in model building is the quality and fitness for use of the final output product. From the user's viewpoint the methods used in modelling shall give an understanding about their usefulness, as well as quality estimates. Another consideration is the input dataset and its spatial distribution and homogeneity. With a sparse dataset coverage, a good method may still give poor results. It is the user's decision, what are the uncertainty limits for the final output product. From these limits, suitable modelling methods for the specific input dataset can be chosen. From various use cases discussed in this dissertation, one can conclude that the internal quality measure, i.e., precision, may be used to judge a model's fitness for use.

We have shown that not all modelling methods can cope with all input datasets, e.g., as shown in the Publication A, where the input data were very sparse and the used method's abilities were not shown at their best. On the other hand, within dense datasets of good coverage (e.g., Publication B), the method's behaviour was indeed impressive. The same was shown in Publication C, i.e., input data density and coverage has an effect on the theoretical uncertainty estimates, as we compared there the various geoid uncertainty contributions for two case study areas. Although the one area was about seven times larger then the other, the data density and coverage for it was much better, which has an influence on the computed uncertainties. In the Publications D and E we showed that two different independent modelling methods can give similar precision estimates for land uplift rates for the same study area using these two approaches.

5. Conclusions

In this dissertation a variety of modelling approaches of a spatial nature were investigated. These were demonstrated to be able to model geographical information from different perspectives. All methods used were formulated in mathematical terms and could offer added value to geodetic and geoinformatics studies.

In the geosciences different features are often modelled as spatially correlated random variables and they usually are location dependent (Zhang and Goodchild, 2002). Within modelling one has the possibility to choose the precision measures. Besides the different quality concepts, the precision of modelling depends also on the input data and their spatial distribution.

In the dissertation, modelling aspects in geodesy, geoinformatics and geodynamics were investigated. In relation to geodesy, transformation methods as well as quality measures for geoid models were discussed. In relation to geoinformatics, fuzzy modelling and kriging methods to be used in height transformation surface modelling and DEM were investigated. In relation to geodynamics, land uplift rate prediction computation and GIA modelling were performed.

The following conclusions can be drawn according to the research topics:

1. Several different mathematical and geostatistical methods were used in height modelling. These include the affine bilinear method, and the fuzzy and kriging modelling methods. The study showed that the transformation approach using either traditional or non-traditional methods can be used in various geodetic and GIS applications. In both studies, quality measures were evaluated for DEM models and transformation surfaces.

    (a) In the first study on transformation surfaces, the achieved quality in terms of standard deviations was about 10 cm for the traditional (bilinear affine) transformation approach, and 20 cm for non-traditional fuzzy modelling.

(b) In the second study on digital elevation modelling, the achieved accuracy was about 3 to 5 cm. Although the size of the area and the number of input data points as well as the point distribution were very different from the earlier case study, the results showed the suitability of these modelling approaches to be used in this kind of research.

One can conclude that the quality measures improve when the data point density increases, thus, results can even be better while using a still denser input dataset, especially when using the traditional transformation method. As a result, we showed that these modelling techniques could be used in constructing terrain height models (DEM) to be used in GIS, but to some extent also in geodesy, e.g., for the prediction of heights when no geoid model is available.

2. In the geoid uncertainty study we showed that the uncertainty was dependent both on gravity data coverage and on the precision of gravity measurements, as well as on the availability of gravity data from neighbouring countries in border areas. Also the underlying global geopotential model that was used had an effect.

3. In GIA modelling, we investigated two methods for predicting land uplift rates. These include land uplift modelling by least squares collocation and GIA model computations by visco-elastic modelling.

   (a) Firstly, an empirical model for predicting the land uplift rate was derived. Achieved quality measures were following: for the high-precision levelling data an RMS of ± 0.314 mm/a; for the GNSS dataset an RMS of ± 0.852 mm/a for the Fennoscandian central area and of ± 1.685 mm/a for the whole area including the forebulge were obtained.

   (b) Secondly, physical GIA modelling was performed. The quality measures were very promising, for the Fennoscandian uplift area an RMS of ± 1.772 mm/a and of ± 1.253 mm/a were obtained for ice models KL05 and ICE-5G respectively. For the North American uplift area an RMS of ± 1.535 mm/a and of ± 1.627 mm/a were obtained for ice models KL05 and ICE-5G respectively.

For both studies, reference datasets from GNSS time series were used to validate the constructed models. As a result, we showed that land uplift values can be predicted empirically with similar accuracy as the physics based GIA modelling, as RMS was about 1.2 to 1.5 mm/a for both studies as shown in the case study of the Fennoscandian region.

The dissertation showed different use cases in the field of geodesy, geodynamics and geoinformatics. We constructed various models using various modelling methodologies and input datasets. The models were studied and discussed from the viewpont of fitness for use, i.e., suitable quality measures for these models were evaluated. The quality measures achieved for the various models showed their applicability in the fields of geodesy, geodynamics and geoinformation science.

In this dissertation a large amount of numerical results were obtained and some conclusions drawn. The purpose of such research shall always be insight, not only numerical results. Indeed, this provided an opportunity to compare the chosen methods and to understand the theories behind them. The obtained understanding pertains not only to the modelling itself and its use in different applications, but also to the choice of parameters used in the modelling. They all together serve as the basis for decision making on different levels.

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We studied various aspects of modelling geographical information from the geodetic and geodynamic points of view. The data for our studies were acquired by a variety of methods: laser scanning, levelling and satellite positioning.

The major subject of this dissertation is quality and its measures. We study various modelling approaches on geodetic data aimed at use cases in the fields of geodesy, geodynamics and geographic information science. The dissertation discusses internal and external quality aspects of the modelling and the data used.

The main objectives of the research are related to the data modelling aspects and fitness for use within the fields of study as expressed quantitatively in various quality measures. The novelty of the dissertation is in the application of appropriate quality measures, like precision or accuracy from the viewpoint of fitness for use, which for the various models depends on the input data precision as well as on the envisaged applications of the model product.

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