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BV capacity and Hausdorff content

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Abstract

In this thesis, we consider a complete and doubling metric measure space which supports a weak Poincaré inequality. We construct a set function called BV capacity, which is defined in terms of functions of bounded variation. When restricted to compact sets, this capacity is equivalent to a variational capacity which is defined in terms of functions in the first order Sobolev space with integrable gradients. Hence BV capacity is useful to characterise not only the space of bounded variation functions, but also this Sobolev space. We study the basic properties of BV capacity and we recognise its connection to the perimeter outer measure. The main purpose of this work is to understand how, for compact sets, the BV capacity and the Hausdorff content of codimension one are equivalent. The key tools to achieve this are the coarea formula, the relative isoperimetric inequality and a modified version of the boxing inequality for metric spaces.

Keywords BV capacity, BV functions, Hausdorff content, boxing inequality, coarea formula, weak Poincaré inequalities, doubling metric measure space

Contents

1 Introduction

In the Euclidean space $\mathbb{R}^n$, the Sobolev space $W^{1,p}(\mathbb{R}^n)$, where $1 \leq p < \infty$, consists of functions which are integrable to power p and have a weak derivative which is also integrable to power p. An alternative definition in a metric measure space X is the following *Newtonian space*

$$N^{1,p}(X) := \left\{ u \in L^p(X) \mid \|u\|_{N^{1,p}(X)} < \infty \right\} /\sim,$$

where

$$\|u\|^p_{N^{1,p}(X)} := \int_X |u| \, d\mu + \inf_g \int_X g \, d\mu$$

and the infimum is taken over all p-weak upper gradients g of u. Two functions $u, v \in L^p(X)$ are equal in $N^{1,p}(X)$, that is, $u \sim v$, if and only if $\|u - v\|_{N^{1,p}(X)} = 0$. Exceptional sets for these Sobolev type functions are measured by the so-called Sobolev p-capacity, which under stronger assumptions on the metric space, is coarser than the variational p-capacity, see Theorem 7.7. in [17, p. 428]. The *variational p-capacity* is defined as

$$\operatorname{cap}_p(E) := \inf_g \|g\|^p_{L^p(X)}$$

for all sets $E \subset X$, where the infimum is taken over all p-weak upper gradients g of functions $u \in N^{1,p}(X)$ such that $u = 1$ in E. This definition agrees with Definition 2.4. in [17, p. 404].

It turns out that the spaces $N^{1,p}(X)$ for $1 < p < \infty$ behave quite differently than the space $N^{1,1}(X)$. In particular, the Hausdorff content of codimension p is only equivalent (for compact sets) to the variational p-capacity when $p = 1$. This is mentioned in [17] and explored in [8, 7]. The main goal of this thesis is to understand this equivalence for $p = 1$.

Other capacities possible to consider are BV-capacities, where BV stands for bounded variation. As the name suggests, they are defined in terms of functions of bounded variation. The definition we use in this thesis is the following. For all sets $E \subset X$,

$$\operatorname{cap}_{\mathrm{BV}}(E) := \inf \|Du\|(X), \tag{1}$$

where the infimum is taken over all functions $u \in L^1(X)$ such that $\|Du\|(X) < \infty$, $0 \leq u \leq 1$ and $u = 1$ in a neighbourhood U of E. By a *neighbourhood* of E, we mean an open subset of X containing E. Here $\|Du\|(X)$ stands for the total variation of u. In Euclidean spaces, the total variation of a BV-function is well-known and is covered, for instance, in [2, 10, 11]. A generalisation of this concept to the metric setting was carried out by Miranda in [20].

It is true that, for compact sets, cap_1 is equivalent to $\operatorname{cap}_{\mathrm{BV}}$. This can be deduced by Theorems 6.2. and 6.4. in [17, pp. 419–420]. BV-capacity has the advantage that functions of bounded variation form a more flexible class than the admissible functions for the variational 1-capacity do. Thus in this thesis we turn our focus to BV-capacity and aim to study its equivalence to the Hausdorff content of codimension 1. This equivalence implies that all estimates that hold for Hausdorff content hold for BV-capacity, and vice-versa. In particular, we get measure-theoretical tools to study functions of bounded variation.

In section 2, we cover the preliminary notions and results that we will use throughout the thesis. We consider a complete and doubling metric measure space and assume it supports a 1-weak Poincaré inequality. We briefly discuss the theory of p-weak upper gradients, since 1-weak upper gradients are the main ingredient in the construction of the total variation of an integrable function. Still in this section, we study weak (q, p)-Poincaré inequalities and consider some estimates for the relative measure of balls. We mention [4] as one of the main references.

In section 3, we analyse the basic properties of the BV-capacity proposed in (1), such as finite subadditivity and outer regularity. This analysis is mostly based on the article [13, pp. 57-64]. Moreover, we recognise the connection between BV-capacity and the perimeter outer measure. We also construct a modified BV-capacity for which outer measurability and the Choquet property are satisfied. The main tool to show those two properties is a global Gagliardo-Nirenberg type inequality.

The main goal in section 4 is to obtain a modified version of the boxing inequality for our metric measure space setting. To obtain that, we need a relative isoperimetric inequality, which can be interpreted as a geometric version of the 1-weak Poincaré inequality. The boxing inequality was introduced by Gustin in [12] and our version is based on [17, pp. 406–408].

Finally, in section 5, we address the equivalence between BV-capacity and Hausdorff content of codimension 1 for compact sets. We start by defining Hausdorff content in the metric setting. Then we note that, as a consequence of the boxing inequality, for bounded open sets of finite perimeter, the perimeter gives an upper bound for the Hausdorff content of codimension 1. Together with the coarea formula, this is the main tool to show the pretended equivalence. Most of the writing of this section is based on [17, pp. 408–411].

2 Preliminaries

In this section, we start by introducing a complete and doubling metric measure space. With such a space at hands, we have a notion of dimension and also some control over the relative measure of balls. Next, we discuss p-weak upper gradients, which are a generalisation of the absolute value of the gradient in Euclidean spaces. These will allow us to define and study functions of bounded variation in the metric setting. Lastly, we examine weak (q, p)-Poincaré inequalities and list some basic but important results. Proofs for most of the mentioned results can be found in [4].

2.1 Metric measure space setting

Throughout this thesis, we consider a complete metric measure space (X, d, μ) where d is a norm and μ is a Borel regular outer measure. When referring to this space, we might simply write X.

A *ball* in X with a center $x \in X$ and a radius $0 < r < \infty$ is

$$B(x, r) := \{y \in X \mid d(x, y) < r\}.$$

We assume that

$$0 < \mu(B) < \infty$$

for every ball B in X. This guarantees that X is separable, see Proposition 1.6. in [4, p. 5]. We also impose that $\mu(X) = \infty$, so that there exist compact sets of non-trivial BV-capacity.

Remark 2.1. *If $K \subset X$ is a compact set, then $\mu(K) < \infty$.*

Reason. It is true that

$$K \subset \bigcup_{x \in K} B(x, 1)$$

and so, by compactness of K, there exists $n \in \mathbb{N}$ and $x_i \in K$ with $i = 1, \dots, n$ such that

$$K \subset \bigcup_{i=1}^{n} B(x_i, 1).$$

By the assumption that $\mu(B) < \infty$ for every ball B in X and by finite subadditivity of μ, we get

$$\mu(K) \leq \sum_{i=1}^{n} \mu(B(x_i, 1)) < \infty.$$

■

We note that Remark 2.1 implies that μ is a Radon outer measure.

For every $x \in X$ and $0 < r, \tau < \infty$, we denote $B(x, \tau r)$ by $\tau B(x, r)$. Moreover, for any subset E of X, the *diameter* of E is

$$\operatorname{diam}(E) := \sup_{x,y \in E} d(x, y).$$

The *distance between a point $x \in X$ and a set $E \subset X$* is

$$d(x, E) := \inf_{y \in E} d(x, y).$$

Similarly, the *distance between two sets $E_1, E_2 \subset X$* is

$$d(E_1, E_2) := \inf_{x \in E_1} d(x, E_2).$$

We assume there exists a constant $c_D \geq 1$ such that

$$\mu(2B) \leq c_D \mu(B)$$

for every ball B in X. This means that μ is a *doubling measure*. The constant c_D is the *doubling constant* associated to μ. One consequence of the doubling condition is that X is a *proper space*, meaning that closed and bounded sets are compact.

The next lemma corresponds to Lemma 3.3. in [4, p. 66].

Lemma 2.2. *Let (X, d, μ) be a metric measure space such that μ is a doubling measure with doubling constant c_D. For all $x \in X$, $0 < r < \infty$, $x' \in B(x, r)$ and $0 < r' \leq r$, we have*

$$\frac{\mu(B(x', r'))}{\mu(B(x, r))} \geq C \left(\frac{r'}{r}\right)^s,$$

where $s = \log_2 c_D$ and $C = c_D^{-2}$.

Besides providing a density estimate for the measure of balls, the previous Lemma reflects that the doubling condition introduces a notion of dimension to the space X. Indeed, note that if $X = \mathbb{R}^n$ with the usual Euclidean metric and the n-dimensional Lebesgue measure, then

$$\mu(2B(x, r)) = \omega_n (2r)^n = 2^n \omega_n r^n = 2^n \mu(B(x, r))$$

for every $x \in X$ and $0 < r < \infty$. Here ω_n denotes the n-dimensional volume of the unitary ball $B(0, 1)$. In particular, $c_D = 2^n$ serves as a doubling constant and, in that case, $s = \log_2 c_D = n$. Hence, in the Euclidean case, s corresponds to the dimension of the space. We can generalise that idea to the metric space setting and interpret the constant $s = \log_2 c_D$ as a dimensional parameter inherent to the space X.

The last assumption we make on the space X is that it ought to satisfy a weak 1-Poincaré inequality. To make sense of this inequality, we need to first define p-weak upper gradients.

2.2 *p*-weak upper gradients

The main ingredients to construct the variation of a function $u \in L^1(X)$ are 1-weak upper gradients. These also appear in the weak 1-Poincaré inequality mentioned in the previous section 2.1.

In order to discuss the theory of p-weak upper gradients, we need to present some basic notions on paths. A *path* in X is a continuous function $\gamma : [a, b] \to X$, where $[a, b]$ is a closed interval in $\mathbb{R}$. If $\gamma(a) = x$ and $\gamma(b) = y$, then γ is a path joining x and y, where x and y are called the *initial* and *end* points of γ, respectively. The *length of a path* $\gamma : [a, b] \to X$ is

$$\ell(\gamma) := \sup \sum_{i=1}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all subdivisions of the form

$$a = t_1 \leq t_2 \ldots \leq t_n = b.$$

A *rectifiable path* is a path γ such that $\ell(\gamma) < \infty$. The *length function* associated to a path $\gamma : [a, b] \to X$ is the function

$$\begin{aligned} s_\gamma : [a, b] &\to [0, \ell(\gamma)] \\ t &\mapsto \ell\left(\gamma_{|[a,t]}\right). \end{aligned}$$

For every rectifiable path γ, there exists a unique path $\tilde{\gamma} : [0, \ell(\gamma)] \to X$ such that $\tilde{\gamma}$ is Lipschitz and $\gamma = \tilde{\gamma} \circ s_\gamma$, see Theorem 4.2.1. in [3, p. 59]. The function $\tilde{\gamma}$ is called the parametrization of γ by the *arc length*.

Let γ be a rectifiable path and g be a nonnegative Borel function. The *line integral* of g over γ is

$$\int_\gamma g \, ds := \int_0^{\ell(\gamma)} g(\tilde{\gamma}(t)) \, dt.$$

We proceed to briefly discuss the theory of p-weak upper gradients for $1 \leq p < \infty$. We consider the general case for $1 \leq p < \infty$, even though in the next sections we only deal with the case corresponding to $p = 1$. This part of our work is mostly based on sections 1.3, 1.5, 1.6 and 2.2–2.4 of [4].

Before stating what a p-weak upper gradient is, we need to define the p-modulus of a path family.

Definition 2.3 (p-modulus of a path family)**.** *Let $1 \leq p < \infty$ and let Γ be a family of paths in X. The p-modulus of Γ is*

$$\mathrm{Mod}_p(\Gamma) = \inf \int_X g^p \, d\mu,$$

where the infimum is taken over all nonnegative Borel functions g satisfying

$$\int_\gamma g \, ds \geq 1$$

for every locally rectifiable path $\gamma \in \Gamma$.

A family of paths Γ is *p-exceptional* if $\mathrm{Mod}_p(\Gamma) = 0$. We say that a property holds for *p-almost every path* if it holds for every path except for a p-exceptional family of paths.

We are ready to announce what are p-weak upper gradients.

Definition 2.4 (p-weak upper gradient)**.** *Let* $1 \leq p < \infty$ *and* $u : X \to [-\infty, \infty]$. *We say that a nonnegative Borel function* g *is a* p*-weak upper gradient of* u *if*

$$|u(x) - u(y)| \leq \int_\gamma g\, ds$$

for all $x, y \in X$ *and for* p*-almost every rectifiable path* γ *joining* x *and* y *in* X.

Absolutely continuous functions on paths are extremely useful in the characterisation of p-weak upper gradients. We say a function $f : [a, b] \to \mathbb{R}$ is *absolutely continuous* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon$$

for any $n \in \mathbb{N}$ and $a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n \leq b$ such that

$$\sum_{i=1}^{n} (b_i - a_i) < \delta.$$

A natural notion of absolute continuity for functions in metric spaces follows. A function $u : X \to \mathbb{R}$ is *absolutely continuous on a path* γ if $u \circ \gamma$ is absolutely continuous. We denote the class of functions which are absolutely continuous on p-almost every rectifiable path by $\mathrm{ACC}_p(X)$.

Moreover, consider $D^p(X)$ to be the set of μ-measurable functions which have a p-weak upper gradient in $L^p(X)$. Then it is true that $D^p(X) \subset \mathrm{ACC}_p(X)$.

Let us now list some important properties of p-weak upper gradients. First, we state the existence of a minimal p-weak upper gradient.

Lemma 2.5. *If* $u \in D^p(X)$, *there exists a minimal* p*-weak upper gradient* $g_u \in L^p(X)$ *in the sense that*

$$g_u \leq g$$

μ*-almost everywhere for all* p*-weak upper gradients* g *of* u. *This minimal element is unique up to sets of measure* μ *zero.*

Until the end of this section, we will use the notation g_u for the minimal p-weak upper gradient of u, whenever $u \in D^p(X)$.

We now make some remarks about p-weak upper gradients of Lipschitz functions. These will be important for future arguments.

Remark 2.6. *Let $1 \leq p < \infty$. If $u \in \mathrm{Lip}_{loc}(X)$, then u has a p-weak upper gradient. Moreover, if u is L-Lipschitz in a compact set $K \subset X$, where L is the minimal Lipschitz constant for u in K, then $L\chi_K$ is the minimal p-weak upper gradient of $u|_K$.*

Reason. *The pointwise lower dilation of a function $u \in \mathrm{Lip}_{loc}(X)$, which is defined as*

$$\mathrm{lip}\, u(x) = \liminf_{r \to 0} \sup_{y \in B(x,r)} \frac{|u(y) - u(x)|}{r}$$

for all $x \in X$, is a p-weak upper gradient of u for any $1 \leq p < \infty$. For a reference, see Proposition 1.14. in [4, p. 9].

Now suppose u is L-Lipschitz in a compact set $K \subset X$, where L is the minimal Lipschitz constant for u in K. Then for every $x, y \in K$ and every rectifiable path γ in K joining x and y, it holds that

$$|u(x) - u(y)| \leq Ld(x, y) \leq L\ell(\gamma) = \int_\gamma L\chi_K \, ds.$$

This implies that $L\chi_K$ is a p-weak upper gradient of $u|_K$. It is also true that $u|_K \in D^p(X)$ since

$$\int_X |L\chi_K|^p \, d\mu = L^p \mu(K) < \infty.$$

It is easy to check that if there would exist a distinct p-weak upper gradient g of $u|_K$ such that $g < L\chi_K$ μ-almost everywhere, then L would not be the minimal Lipschitz constant for $u|_K$. Hence $L\chi_K$ is the minimal p-weak upper gradient of $u|_K$.

∎

Remark 2.7. *If $X = \mathbb{R}^n$ and $u \in C^1_{loc}(X)$, then $|\nabla u|$ is an upper gradient of u, see Corollary 1.15. in [4, p. 9].*

We state a powerful result for the construction of a p-weak upper gradient for the affine combination of two functions $u, v \in D^p(X)$. For a proof, see Lemma 2.18. in [4, p. 47].

Lemma 2.8. *Let $u, v \in D^p(X)$ and $\eta \in \mathrm{Lip}(X)$ be such that $0 \leq \eta \leq 1$. Then $w = (1 - \eta)u + \eta v \in D^p(X)$ and*

$$g = (1 - \eta)g_u + \eta g_v + |v - u|g_\eta$$

is a p-weak upper gradient of w.

The following glueing lemma, which corresponds to Lemma 2.19. in [4, p. 48], is a simple but useful result in the construction of p-weak upper gradients.

Lemma 2.9. *Let $E \subset X$ be a μ-measurable set and $f \in \mathrm{ACC}_p(X)$. Suppose there exist $u_1, u_2 \in D^p(X)$ such that*

$$f|_E = u_1 \quad \text{and} \quad f|_{X \setminus E} = u_2.$$

If g_1 and g_2 are p-weak upper gradients of u_1 and u_2, respectively, then

$$g = g_1\chi_E + g_2\chi_{X\setminus E}$$

is a p-weak upper gradient of f. Moreover, if g_1 and g_2 are the minimal p-weak upper gradients of u_1 and u_2, respectively, then g is the minimal p-weak upper gradient of f.

In the following example, we compute a 1-weak upper gradient for a type of cutoff function which is commonly used to obtain estimates for the variation of functions.

Example 2.10. *Let $x \in X$, $0 < r < \infty$, $\varepsilon > 0$ and define*

$$u(y) = \max\left\{0, 1 - \frac{d\left(y, B\left(x,r\right)\right)}{\varepsilon}\right\}$$

for all $y \in X$. Then,

$$g = \frac{1}{\varepsilon}\chi_{B(x,r+\varepsilon)}$$

is a 1-weak upper gradient of u.

Reason. *We start by noting that $u = 1$ in $B\left(x,r\right)$ since $d\left(y, B\left(x,r\right)\right) = 0$ for every $y \in B\left(x,r\right)$. Moreover $u = 0$ in $X \setminus B\left(x,r+\varepsilon\right)$ since $d\left(y, B\left(x,r\right)\right) \geq \varepsilon$ for every $y \in X \setminus B\left(x,r+\varepsilon\right)$. Hence we can write u in the following manner:*

$$\begin{aligned} u &= \left(1 - \frac{d\left(y, B\left(x,r\right)\right)}{\varepsilon}\right)\chi_{B(x,r+\varepsilon)} \\ &= \left(1 - \frac{d\left(y, B\left(x,r\right)\right)}{\varepsilon}\right)\chi_{B(x,r+\varepsilon)} + 0\chi_{X\setminus B(x,r+\varepsilon)} \\ &= u_1 + u_2 \end{aligned}$$

The function $g_2 = 0$ is a 1-weak upper gradient of u_2. On the other hand, u_1 is $\frac{1}{\varepsilon}$-Lipschitz. Indeed, let $y, z \in B\left(x, r+\varepsilon\right)$ and assume, without loss of generality, that $d\left(y, B\left(x,r\right)\right) \geq d\left(z, B\left(x,r\right)\right)$.

$$\begin{aligned} |u(y) - u(z)| &= \frac{d\left(y, B\left(x,r\right)\right) - d\left(z, B\left(x,r\right)\right)}{\varepsilon} \\ &\leq \frac{d\left(y,z\right) + d\left(z, B\left(x,r\right)\right) - d\left(z, B\left(x,r\right)\right)}{\varepsilon} \\ &= \frac{1}{\varepsilon}d\left(y,z\right). \end{aligned}$$

Hence $g_1 = \frac{1}{\varepsilon}\chi_{B(x,r+\varepsilon)}$ is a 1-weak upper gradient of u_1. Now, $B\left(x, r+\varepsilon\right)$ is a Borel set and hence μ-measurable. Function u is $\frac{1}{\varepsilon}$-Lipschitz and hence in $\mathrm{ACC}_1\left(X\right)$ and both g_1 and g_2 are integrable functions in X. Thus by the glueing Lemma, see Lemma 2.9, we conclude that

$$g = g_1 + g_2 = \frac{1}{\varepsilon}\chi_{B(x,r+\varepsilon)}$$

is a 1-weak upper gradient of u.

2.3 Basic results

In this section, we formally introduce weak (p, q)-Poincaré and discuss some consequences of the doubling condition and the weak 1-Poincaré inequality. We also state Vitali's 5-covering theorem.

Definition 2.11 (Weak (q, p)-Poincaré inequality)**.** *Let $1 \leq q, p < \infty$. A metric measure space X supports a weak (q, p)-Poincaré inequality if there exist constants $c_P > 0$ and $\tau \geq 1$ such that for all balls $B = B(x, r) \subset X$, with $x \in X$ and $0 < r < \infty$, all functions $u \in L^1_{\text{loc}}(X)$ and all p-weak upper gradients g of u,*

$$\left(\fint_B |u - u_B|^q \ d\mu \right)^{1/q} \leq c_P r \left(\fint_{\tau B} g^p \ d\mu \right)^{1/p}.$$

If $q = 1$, then we say X supports a weak p-Poincaré inequality.

Here,

$$u_B := \fint_B u \ d\mu = \frac{1}{\mu(B)} \int_B u \ d\mu.$$

As mentioned in section 2.1, we impose X to support a 1-Poincaré inequality. In terms of definition 2.11, this means there exist constants $c_P > 0$ and $\tau \geq 1$, which we call the *Poincaré constant* and the *dilation constant*, respectively, such that for all balls $B = B(x, r) \subset X$, with $x \in X$ and $0 < r < \infty$, all functions $u \in L^1_{\text{loc}}(X)$ and all 1-weak upper gradients g of u,

$$\fint_B |u - u_B| \ d\mu \leq c_P r \fint_{\tau B} g \ d\mu. \tag{2}$$

We make the choice $q = p = 1$ because in the next sections we focus on the study of functions of bounded variation, which are related to the first-order Sobolev space $N^{1,1}(X)$. The weak 1-Poincaré inequality provides a relation between the inherent measure and metric of X and 1-weak upper gradients. Moreover, it insures that, in every ball, the mean oscillation of u is not much greater than the integral average of its 1-weak upper gradients. A consequence of the 1-Poincaré inequality is that X is connected, see Proposition 4.2. in [4, p. 85].

The next two results are useful in section 3.3, namely, in obtaining a Gagliardo-Nirenberg type inequality. The first one, which corresponds to Theorem 4.21. in [4, p. 92] translates that, under certain conditions, we have a self-improving weak Poincaré inequality.

Proposition 2.12. *Assume X satisfies a p-Poincaré inequality with dilation constant τ and that there exist constants $C > 0$ and $s > p$ such that*

$$\frac{\mu(B(x', r'))}{\mu(B(x, r))} \geq C \left(\frac{r'}{r} \right)^s$$

for all $x \in X$, $r > 0$, $x' \in B(x, r)$ and $0 < r' \leq r$. Then X supports a (p^*, p)−Poincaré inequality with $p^* = sp/(s - p)$ and dilation constant 2τ.

The next Lemma and respective Corollary can be found as Lemma 3.7. and Corollary 3.8., respectively, in [4, p. 67].

Lemma 2.13. *Let (X, d, μ) be a metric measure space such that μ is a doubling measure with doubling constant c_D and X is connected. Then, for every $0 < \alpha < 1$, there exists a constant $\xi = \xi(\alpha, c_D)$ such that $0 < \xi < 1$ and*

$$\mu(\alpha B) \leq \xi \mu(B)$$

for all balls $B \subset X$.

Corollary 2.14. *Let (X, d, μ) be a metric measure space such that μ is a doubling measure with doubling constant c_D and X is connected. Then there exist constants $C > 0$ and $\sigma > 0$ such that*

$$\frac{\mu(B(x', r'))}{\mu(B(x, r))} \leq C \left(\frac{r'}{r}\right)^{\sigma}$$

for all $x \in X$, $r > 0$, $x' \in B(x, r)$ and $0 < r' \leq r$.

Lemmas 2.2 and 2.13 provide lower and upper bounds for the density of balls with respect to the ratio of their radii.

We state Vitali's 5-covering theorem. A proof for it can be found, for instance, in [14, p. 2]

Theorem 2.15. *Let $\mathcal{B}$ be a family of balls in X with uniformly bounded radii. Then there exists a countable subfamily $\mathcal{B}' \subset \mathcal{B}$ of pairwise disjoint balls such that*

$$\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{B \in \mathcal{B}'} 5B.$$

3 BV-capacity

The central piece of this section is BV-capacity, defined in terms of functions of bounded variation.

We first introduce the notions of variation and perimeter and announce the coarea formula. We omit most of the proofs related to variation and perimeter, but they can all be found in [13, pp. 54–57]. Secondly, we characterise BV-capacity. Some properties satisfied by this capacity are finite subadditivity, outer regularity and downwards monotone convergence for compact sets. Next we obtain a global Gagliardo-Nirenberg type inequality. This is done so that we can construct an alternative BV-capacity satisfying, for example, countable subaddtivity.

3.1 Variation and perimeter

Functions of bounded variation are a fundamental piece in the construction of BV-capacity. We need to make sense of variation of locally integrable functions in X. That is the purpose of this section.

Definition 3.1 (Variation)**.** *Let $u \in L^1_{\mathrm{loc}}(X)$. For every open set $U \subset X$, we define the variation of u in U as*

$$\|Du\|(U) := \inf \left\{ \liminf_{i\to\infty} \int_U g_{u_i}\, d\mu \right\},$$

where the infimum is taken over all possible sequences of 1-weak upper gradients g_{u_i} of functions $u_i \in \mathrm{Lip}_{loc}(X)$, with $i \in \mathbb{N}$, such that $u_i \to u$ in $L^1_{loc}(U)$ as $i \to \infty$.

The total variation of u is given by $\|Du\|(X)$.

This construction of variation is (a slightly modified version of) the one introduced by Miranda in [20, p. 984] and it is a generalisation of the variation of functions in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, with $n \in \mathbb{N}$. The idea behind it is, after fixing an open set $U \subset X$, to relax the domain of definition of the functional $u \mapsto \int_U g_u$, which is well-defined for functions $u \in \mathrm{Lip}_{\mathrm{loc}}(U)$. It is possible to do that in a proper way since $\mathrm{Lip}_{\mathrm{loc}}(X)$ is dense in $L^1_{\mathrm{loc}}(X)$. The set of functions of bounded variation is taken to be the set for which the relaxed functional corresponding to X is finite.

Definition 3.2 (Function of bounded variation)**.** *A function $u \in L^1(X)$ is said to be of bounded variation, or a* BV*-function, if the total variation of u is finite, that is, if $\|Du\|(X) < \infty$. We denote the space of bounded variation functions by* $\mathrm{BV}(X)$.

For a fixed $u \in \mathrm{BV}(X)$, we can extend the definition of variation to every set $A \subset X$ by defining

$$\|Du\|(A) := \inf \{\|Du\|(U) \mid A \subset U, U \subset X \text{ is open}\}.$$

Then Theorem 3.4. in [20, p. 988] implies that $\|Du\|(\cdot)$ is a finite Borel outer measure.

The next Lemma, which can found as Lemma 2.6. in [13, p. 55], states there exists a minimising sequence of compactly supported Lipschitz functions for the total variation.

Lemma 3.3. *Let $u \in \mathrm{BV}(X)$. Then there exist $u_i \in \mathrm{Lip}_c(X)$ and upper gradients g_{u_i} of u_i, with $i \in \mathbb{N}$, such that $u_i \to u$ in $L^1(X)$ as $i \to \infty$ and*

$$\|Du\|(X) = \lim_{i\to\infty} \int_X g_{u_i}\, d\mu.$$

We state some basic properties of variation. These are present in Theorem 2.7. in [13, p. 56].

Proposition 3.4. *Let $u, v \in L^1_{\mathrm{loc}}(X)$ and $U \subset X$ be an open set. Then,*

1. $\|D(\alpha u)\|(U) = |\alpha|\,\|Du\|(U)$ *for all* $\alpha \in \mathbb{R}$*;*
2. $\|D(u+v)\|(U) \le \|Du\|(U) + \|Dv\|(U)$*;*
3. $\|D\max\{u,v\}\|(U) + \|D\min\{u,v\}\|(U) \le \|Du\|(U) + \|Dv\|(U)$.

Note that properties 1. and 2. in the previous Proposition reflect the sublinearity of variation.

Remark 3.5. *As a consequence of property 3. in the Proposition above, we get that if $u_i \in L^1_{loc}(X)$, for $i = 1, \ldots, n$, and $U \subset X$ is an open set, then*

$$\left\|D \max_{1\le i\le n} u_i\right\|(U) \le \sum_{i=1}^n \|Du_i\|(U) \tag{3}$$

and

$$\left\|D \min_{1\le i\le n} u_i\right\|(U) \le \sum_{i=1}^n \|Du_i\|(U)\,. \tag{4}$$

Reason. *We show only (3) since the argument for (4) is very similar. We use an induction argument. If $n = 1$ the inequality is trivial and if $n = 2$, it agrees with part 3. of the previous Proposition. Thus let $n > 2$ and suppose the result holds for $n-1$. Let $v = \max_{1\le i\le n-1} u_i$. Note that $\max_{1\le i\le n} u_i = \max\{v, u_n\}$. Using part 3. of Proposition 3.4 and the induction hypothesis, we obtain*

$$\begin{aligned}\left\|D \max_{1\le i\le n} u_i\right\|(U) &= \|D\max\{v,u_n\}\|(U)\\ &\le \|D\max\{v,u_n\}\|(U) + \|D\min\{v,u_n\}\|(U)\\ &\le \|Dv\|(U) + \|Du_n\|(U)\\ &= \left\|D \max_{1\le i\le n-1} u_i\right\|(U) + \|Du_n\|(U)\\ &\le \sum_{i=1}^n \|Du_i\|(U)\,.\end{aligned}$$

■

It is also true that variation is lower semicontinuous with respect to the L^1-norm. This property is mentioned in Proposition 3.6 in [20, p. 993].

Proposition 3.6 (Lower semicontinuity)**.** *Let $U \subset X$ be an open set and $u_i \in L^1_{loc}(X)$, with $i \in \mathbb{N}$, be such that $\|Du_i\|(U) < \infty$ for all $i \in \mathbb{N}$ and $u_i \to u$ in $L^1_{loc}(U)$ as $i \to \infty$. Then*

$$\|Du\|(U) \leq \liminf_{i\to\infty} \|Du_i\|(U).$$

We now focus on a very particular class of L^1-functions, characteristic functions of μ-measurable sets. When considering the variation of these functions, we get the new concept of perimeter.

Definition 3.7 (Perimeter)**.** *Let $E \subset X$ be a μ-measurable set and $A \subset X$. The perimeter of E in A is*

$$P(E, A) := \|D\chi_E\|(A).$$

If $A = X$, then we obtain the perimeter of E, $P(E, X)$. E is said to have finite perimeter if $P(E, X) < \infty$.

Note that by construction of $\mathrm{BV}(X)$, a μ-measurable set $E \subset X$ has finite perimeter if and only if $\chi_E \in \mathrm{BV}(X)$. Moreover, for every μ-measurable set $E \subset X$ such that $\chi_E \in \mathrm{BV}(X)$, $P(E, \cdot) = \|D\chi_E\|(\cdot)$ is a finite Borel outer measure.

Remark 3.8. *The empty set and the whole space X have perimeter zero.*

Reason. *Let g be the zero function, let $x, y \in X$ and let γ be a rectifiable path joining x and y. Then,*

$$\int_\gamma g\, ds = 0.$$

Since $\chi_\emptyset$ is identically zero, g is clearly a 1-weak upper gradient of $\chi_\emptyset$. Therefore, by definition of perimeter and variation,

$$0 \leq P(\emptyset, X) = \|D\chi_\emptyset\|(X) \leq \int_X g\, d\mu = 0,$$

which implies that $P(\emptyset, X) = 0$. Moreover, since χ_X is identically one, it holds that

$$|\chi_X(x) - \chi_X(y)| = 0 \leq \int_\gamma g\, d\mu.$$

Hence, g is also a 1-weak upper gradient of χ_X and so

$$0 \leq P(X, X) = \|D\chi_X\|(X) \leq \int_X g\, d\mu = 0.$$

We conclude that $P(X, X) = 0$. ■

The next coarea formula is derived in Remark 4.3. in [20, p. 997]. It is a way to compute the variation of a function by integrating over the perimeter of its upper level sets.

Theorem 3.9 (Coarea formula)**.** *Let $u \in \mathrm{BV}(X)$ and $U \subset X$ be an open set. Then*

$$\|Du\|(U) = \int_{-\infty}^{\infty} P(\{u > \lambda\}, U)\ d\lambda;$$

The integration on the right-hand side is taken with respect to the one-dimensional Lebesgue measure.

The Corollary below is crucial to show the main result of this thesis.

Corollary 3.10. *Let $E \subset X$ and $u \in \mathrm{BV}(X)$ be such that $0 \leq u \leq 1$. Then,*

1. $\|Du\|(X) = \int_0^1 P(\{u > \lambda\}, X)\ d\lambda.$
2. *There exists λ_0 with $0 < \lambda_0 < 1$ such that*

$$P(\{u > \lambda_0\}, X) \leq \|Du\|(X).$$

Proof. Let $E \subset X$ and $u \in \mathrm{BV}(X)$ be such that $0 \leq u \leq 1$.

1. By the coarea formula, Theorem 3.9, we have

$$\|Du\|(X) = \int_{-\infty}^{\infty} P(\{u > \lambda\}, X)\ d\lambda.$$

Note that since $0 \leq u \leq 1$, if $\lambda \geq 1$, then

$$P(\{u > \lambda\}, X) = P(\emptyset, X) = 0.$$

Moreover, if $\lambda \leq 0$, then

$$P(\{u > \lambda\}, X) = P(X, X) = 0.$$

Hence,

$$\|Du\|(X) = \int_0^1 P(\{u > \lambda\}, X)\ d\lambda.$$

2. Suppose that for all λ with $0 < \lambda < 1$,

$$P(\{u > \lambda_0\}, X) > \|Du\|(X).$$

Then, by part 1., we have

$$\|Du\|(X) = \int_0^1 P(\{u > \lambda\}, X)\ d\lambda > \|Du\|(X),$$

which is impossible. Hence there exists λ_0 with $0 < \lambda_0 < 1$ such that

$$P(\{u > \lambda_0\}, X) \leq \|Du\|(X).$$

□

3.2 Basic properties of BV-capacity

We present a rigorous definition of BV-capacity.

Definition 3.11 (BV-capacity). *Let $E \subset X$. We say that $u \in \mathrm{BV}(X)$ is an admissible function for the* BV*-capacity of E if $0 \leq u \leq 1$ and $u = 1$ in a neighbourhood of E. The set of admissible functions for the capacity of E is denoted by $\mathcal{A}_{\mathrm{BV}}(E)$. We then define the* BV*-capacity of E as*

$$\mathrm{cap}_{\mathrm{BV}}(E) := \inf_{u \in \mathcal{A}_{\mathrm{BV}}(E)} \|Du\|(X).$$

Here we take the infimum of only the variation of u, while in Definition 3.1. in [13, p. 57], the authors infimise $\int_X |u|\, d\mu + \|Du\|(X)$. Leaving the term corresponding to the L^1-norm of u out makes certain estimates and computations easier to handle. It is true that some important properties of the capacity, such as countable subadditivity, are lost. However, finite subadditivity still holds and that is enough to obtain the result on the equivalence between BV-capacity and Hausdorff content for compact sets.

Remark 3.12. *If $K \subset X$ is a compact set, then $\mathrm{cap}_{\mathrm{BV}}(K) < \infty$.*

Reason. Let $K \subset X$ be a compact set and $B = B(x, r) \subset X$ be a ball containing K, with $x \in X$ and $0 < r < \infty$. Consider the function

$$u(y) = \max\left\{0, 1 - \frac{1}{r} d\left(y, B(x, r)\right)\right\}$$

for all $y \in X$. We show that $u \in \mathcal{A}_{\mathrm{BV}}(K)$. By Example 2.10 with $\varepsilon = r$, the function

$$g = \frac{1}{r}\chi_{B(x,2r)}$$

is a 1*-weak upper gradient of u. Then*

$$\begin{aligned} \|Du\|(X) &\leq \int_X g\, d\mu \\ &= \int_X \frac{1}{r}\chi_{B(x,2r)}\, d\mu \\ &= \frac{\mu\left(B(x, 2r)\right)}{r} < \infty \end{aligned}$$

and so $u \in \mathrm{BV}(X)$. Besides that, by construction of u,

$$0 \leq u \leq 1.$$

Now, for all $y \in B(x, r)$, it holds that

$$d(y, B(x, r)) = 0,$$

which implies that $u = 1$ in $B(x, r)$. Note that $B(x, r)$ is a neighbourhood of K. This suffices to conclude that $u \in \mathcal{A}_{\mathrm{BV}}(K)$. Finally, we get

$$\mathrm{cap}_{\mathrm{BV}}(K) \leq \|Du\|(X) < \infty.$$

■

For compact sets, the admissible functions for the BV-capacity can be assumed to be compactly supported and Lipschitz continuous. For the proof of this result, we use the techniques employed in the proof of Theorem 3.9. in [13, pp. 63–64].

Lemma 3.13. *Let $K \subset X$ be a compact set. Then*

$$\mathrm{cap}_{\mathrm{BV}}(K) = \inf \|Du\|(X),$$

where the infimum is taken over all functions $u \in \mathcal{A}_{\mathrm{BV}}(K) \cap \mathrm{Lip}_c(X)$.

Proof. Let $K \subset X$ be a compact set. It is clear that

$$\mathrm{cap}_{\mathrm{BV}}(K) \leq \inf \|Du\|(X)$$

where the infimum is taken over all functions $u \in \mathcal{A}_{\mathrm{BV}}(K) \cap \mathrm{Lip}_c(X)$ since $\mathcal{A}_{\mathrm{BV}}(K) \cap \mathrm{Lip}_c(X) \subset \mathcal{A}_{\mathrm{BV}}(X)$. To show the reverse inequality, let $\varepsilon > 0$ and consider $u \in \mathcal{A}_{\mathrm{BV}}(K)$ such that

$$\|Du\|(X) < \mathrm{cap}_{\mathrm{BV}}(K) + \varepsilon. \tag{5}$$

Since $u \in \mathcal{A}_{\mathrm{BV}}(X)$ and K is compact, there exists a neighbourhood U of K such that $u = 1$ in U and $\mu(U) < \infty$. Moreover, again by compactness of K and since $X \setminus U$ is closed, there exists an open set $U' \subset U$ such that $K \subset U' \Subset U$ and $\mathrm{dist}(U', X \setminus U) > 0$. Let $(u_i)_{i\in\mathbb{N}} \subset \mathrm{Lip}_c(X)$ be the minimising sequence associated to u from Lemma 3.3. Then $u_i \to u$ in $L^1(X)$ as $i \to \infty$, $0 \leq u_i \leq 0$ for all $i \in \mathbb{N}$ and there exist 1-weak upper gradients g_{u_i} of u_i, with $i \in \mathbb{N}$, such that

$$\|Du\|(X) = \lim_{i\to\infty} \int_X g_{u_i}\, d\mu.$$

Consider a cutoff function $\eta \in \mathrm{Lip}_c(X)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in U' and $\eta = 0$ in $X \setminus U$. Define

$$v_i = (1-\eta)u_i + \eta$$

for all $i \in \mathbb{N}$ and

$$v = (1-\eta)u + \eta.$$

Observe that $v_i \to v$ in $L^1(X)$ as $i \to \infty$. We aim to show that $v \in \mathcal{A}_{\mathrm{BV}}(K) \cap \mathrm{Lip}_c(X)$ and $\|Dv\|(X) \leq \|Du\|(X)$.

It is straightforward that $v \in \mathrm{Lip}_c(X)$ since u and η are in $\mathrm{Lip}_c(X)$ and this class of functions is closed under addition and multiplication. Moreover since both u and η are bounded above by 1 and below by 0, the same thing holds for v. Now, for every $x \in U$, since $u(x) = 1$, we have

$$v(x) = (1-\eta(x))u(x) + \eta(x) = 1 - \eta(x) + \eta(x) = 1.$$

This shows that $v = 1$ in U, which is a neighbourhood of K. The only thing that remains to check in order to conclude that $v \in \mathcal{A}_{\mathrm{BV}}(K) \cap \mathrm{Lip}_c(X)$ is that $v \in \mathrm{BV}(X)$.

We do that by constructing 1-weak upper gradients g_{v_i} for v_i, with $i \in \mathbb{N}$, and using lower semicontinuity of variation.

We start by noting that η has a minimal 1-weak upper gradient g_η which is bounded and satisfies $g_\eta = 0$ in $X \setminus U$.

To see this, denote by K' the support of η and conclude that $\eta = \eta\chi_{K'}$. This is due the fact that $\eta|_{X\setminus K'} = 0$. Let $L \geq 0$ be the minimal constant such that $\eta|_{K'}$ is L-Lipschitz. Then, by Remark 2.6, the function $L\chi_{K'}$ is the minimal 1-weak upper gradient of $\eta|_{K'} \in D^1(X)$. On the other hand, the constant function zero is the minimal 1-weak upper gradient of $\eta|_{X\setminus K'} = 0$, which is obviously an integrable function in X. Thus $\eta|_{X\setminus K'} \in D^1(X)$. Note that K' is a Borel set and hence μ-measurable. On the other hand, $\eta \in \mathrm{ACC}_1(X)$ since $\eta \in \mathrm{Lip}_c(X)$. A trivial application of the glueing Lemma for 1-weak upper gradients, see Lemma 2.9, gives us the minimal 1-weak upper gradient for η,

$$g_\eta = L\chi_{K'}.$$

We notice that g_η is bounded by L and it is zero in $X \setminus K'$. Since $K' \Subset U$, $g_\eta = 0$ in $X \setminus U$.

Now, let $i \in \mathbb{N}$. Since $u_i \in \mathrm{Lip}_c(X)$, we can apply an identical argument as the one used for η, to conclude the minimal 1-weak upper gradient g_{u_i} of u_i is bounded and integrable in X.

With these tools at hand, we employ Lemma 2.8 to u_i and the constant function 1 to conclude

$$g_{v_i} = (1-\eta)g_{u_i} + |1-u_i|g_\eta = (1-\eta)g_{u_i} + (1-u_i)g_\eta$$

is a 1-weak upper gradient of v_i. We used the fact that $u_i \leq 1$ and the zero function is the minimal 1-weak upper gradient of the constant function 1.

Recall that $u_i, \eta \geq 0$ and $\eta = 0$ in $X \setminus U$. Then we observe that

$$\begin{aligned}
\|Dv_i\|(X) &\leq \int_X g_{v_i}\, d\mu \\
&= \int_X (1-u_i)g_\eta\, d\mu + \int_X (1-\eta)g_{u_i}\, d\mu \\
&\leq \int_X g_\eta\, d\mu + \int_X g_{u_i}\, d\mu \\
&\leq \|g_\eta\|_{L^\infty(X)}\, \mu(U) + \int_X g_{u_i}\, d\mu.
\end{aligned}$$

Since g_η is bounded and $\mu(U) < \infty$, the first term in the right-hand side of the inequality above is finite. Because $\int_X g_{u_j}\, d\mu \to \|Du\|(X)$ as $j \to \infty$, the second term is also finite, since otherwise we would have $\|Du\|(X) = \infty$, which contradicts u being a function of bounded variation. Thus $\|Dv_i\|(X) < \infty$ for all $i \in \mathbb{N}$ and by lower semicontinuity of variation, see Proposition 3.6, we get

$$\begin{aligned}\|Dv\|(X) &\leq \liminf_{i\to\infty} \|Dv_i\|(X)\\ &\leq \liminf_{i\to\infty} \int_X (1-u_i) g_\eta \, d\mu + \liminf_{i\to\infty} \int_X (1-\eta) g_{u_i} \, d\mu \\ &\leq \left\| g_\eta \right\|_{L^\infty(X)} \liminf_{i\to\infty} \int_U |u-u_i| \, d\mu + \liminf_{i\to\infty} \int_X g_{u_i} \, d\mu \\ &\leq \liminf_{i\to\infty} \int_X g_{u_i} \, d\mu \\ &= \|Du\|(X) < \infty.\end{aligned}$$

Here we used the fact that $g_\eta = 0$ in $X \setminus U$, $u = 1$ in U and $u_i \to u$ in $L^1(X)$ as $i \to \infty$. This shows $v \in \mathrm{BV}(X)$ and consequently $v \in \mathcal{A}_{\mathrm{BV}}(K) \cap \mathrm{Lip}_c(X)$. Furthermore, the inequality immediately above and (5) yield

$$\|Dv\|(X) \leq \|Du\|(X) < \mathrm{cap}_{\mathrm{BV}}(K) + \varepsilon.$$

By taking the limit when $\varepsilon \to 0$, we obtain

$$\|Dv\|(X) \leq \mathrm{cap}_{\mathrm{BV}}(K).$$

This implies

$$\inf \|Du\|(X) \leq \mathrm{cap}_{\mathrm{BV}}(K),$$

where the infimum is taken over all functions $u \in \mathcal{A}_{\mathrm{BV}}(K) \cap \mathrm{Lip}_c(X)$. □

We now present some basic properties of BV-capacity. This section follows [13, pp. 57–62] very closely.

Theorem 3.14. BV*-capacity satisfies*

1. $\mathrm{cap}_{\mathrm{BV}}(\emptyset) = 0$*;*
2. *(monotonicity)* $\mathrm{cap}_{\mathrm{BV}}(E_1) \leq \mathrm{cap}_{\mathrm{BV}}(E_2)$ *for all* $E_1, E_2 \subset X$ *such that* $E_1 \subset E_2$*.*

Proof. We prove both items separately.

1. The function $u = 0$ is an obvious choice as an admissible function for the BV-capacity of the empty set. Moreover, it is a 1-weak upper gradient of itself. Hence

$$0 \leq \mathrm{cap}_{\mathrm{BV}}(\emptyset) \leq \|Du\|(\emptyset) \leq \int_\emptyset 0 \, d\mu = 0$$

and so

$$\mathrm{cap}(\emptyset) = 0.$$

2. Let E_1, E_2 be subsets of X such that $E_1 \subset E_2$. Let $u \in \mathcal{A}_{\mathrm{BV}}(E_2)$. Note that $u = 1$ on a neighbourhood of E_1 since $u = 1$ on a neighbourhood of E_2 and every neighbourhood of E_2 is also a neighbourhood of E_1. Thus $u \in \mathcal{A}_{\mathrm{BV}}(E_1)$, which implies that $\mathcal{A}_{\mathrm{BV}}(E_2) \subset \mathcal{A}_{\mathrm{BV}}(E_1)$. By definition of $\mathrm{cap}_{\mathrm{BV}}(E_i)$ as an infimum over $\mathcal{A}_{\mathrm{BV}}(E_i)$, for $i = 1, 2$, we observe

$$\mathrm{cap}_{\mathrm{BV}}(E_1) \leq \mathrm{cap}_{\mathrm{BV}}(E_2).$$

□

Corollary 3.15. BV-*capacity is a finite subadditive function.*

Proof. We show by induction that

$$\mathrm{cap}_{\mathrm{BV}}\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \mathrm{cap}_{\mathrm{BV}}(E_i)$$

for every $E_i \subset X$, with $i = 1, \ldots, n$ and $n \in \mathbb{N}$. The statement is obvious for $n = 1$. Let $n > 1$ and suppose

$$\mathrm{cap}\left(\bigcup_{i=1}^{n-1} E_i\right) \leq \sum_{i=1}^{n-1} \mathrm{cap}(E_i).$$

Then, by monotonicity of BV-capacity and the induction hypothesis, one gets

$$\mathrm{cap}\left(\bigcup_{i=1}^{n} E_i\right) \leq \mathrm{cap}_{\mathrm{BV}}\left(\bigcup_{i=1}^{n-1} E_i\right) + \mathrm{cap}_{\mathrm{BV}}(E_n) \leq \sum_{i=1}^{n} \mathrm{cap}(E_i).$$

□

BV-capacity satisfies strong subadditivity.

Proposition 3.16. *Let $E_1, E_2 \subset X$. Then*

$$\mathrm{cap}_{\mathrm{BV}}(E_1 \cup E_2) + \mathrm{cap}_{\mathrm{BV}}(E_1 \cap E_2) \leq \mathrm{cap}_{\mathrm{BV}}(E_1) + \mathrm{cap}_{\mathrm{BV}}(E_2).$$

Proof. Let $E_1, E_2 \subset X$. If $\mathrm{cap}_{\mathrm{BV}}(E_1) + \mathrm{cap}_{\mathrm{BV}}(E_2) = \infty$, the result follows trivially. So assume $\mathrm{cap}_{\mathrm{BV}}(E_1) + \mathrm{cap}_{\mathrm{BV}}(E_2) < \infty$. In particular, both $\mathrm{cap}_{\mathrm{BV}}(E_1)$ and $\mathrm{cap}_{\mathrm{BV}}(E_2)$ are finite. Let $\varepsilon > 0$ and, for $i = 1, 2$, consider $u_i \in \mathcal{A}_{\mathrm{BV}}(E_i)$ such that

$$\|Du_i\|(X) < \mathrm{cap}_{\mathrm{BV}}(E_i) + \frac{\varepsilon}{2}.$$

We start by showing that $\max\{u_1, u_2\} \in \mathcal{A}_{\mathrm{BV}}(E_1 \cup E_2)$ and $\min\{u_1, u_2\} \in \mathcal{A}_{\mathrm{BV}}(E_1 \cap E_2)$. Indeed, by Proposition 3.4,

$$\begin{aligned}\|D \max\{u_1, u_2\}\|(X) &\leq \|D \max\{u_1, u_2\}\|(X) + \|D \min\{u_1, u_2\}\|(X) \\ &\leq \|Du_1\|(X) + \|Du_2\|(X) < \infty\end{aligned}$$

since $u_1, u_2 \in \mathrm{BV}(X)$. Similarly we can show that

$$\|D \min\{u_1, u_2\}\|(X) < \infty.$$

Hence

$$\max\{u_1, u_2\}, \min\{u_1, u_2\} \in \mathrm{BV}(X).$$

Now, since $0 \le u_i \le 1$ for $i = 1, 2$, we get

$$0 \le u_1 \le \max\{u_1, u_2\} \le 1$$

and

$$0 \le \min\{u_1, u_2\} \le u_1 \le 1.$$

Finally, let U_i be a neighbourhood of E_i such that $u_i = 1$ on U_i, for $i = 1, 2$. Let $x \in U_1 \cup U_2$. If $x \in U_i$, then $u_i(x) = 1$ and so

$$1 = u_i(x) \le \max\{u_1, u_2\}(x) \le 1,$$

which implies

$$\max\{u_1, u_2\}(x) = 1.$$

Hence, $\max\{u_1, u_2\} = 1$ in $U_1 \cup U_2$, where $U_1 \cup U_2$ is a neighbourhood of $E_1 \cup E_2$. Now let $x \in U_1 \cap U_2$. Then, $u_1(x) = 1$ and $u_2(x) = 1$, which implies

$$\min\{u_1, u_2\}(x) = 1.$$

Thus $U_1 \cap U_2$ is a neighbourhood of $E_1 \cap E_2$ where $\min\{u_1, u_2\} = 1$. We conclude that $\max\{u_1, u_2\} \in \mathcal{A}_{\mathrm{BV}}(E_1 \cup E_2)$ and $\min\{u_1, u_2\} \in \mathcal{A}_{\mathrm{BV}}(E_1 \cap E_2)$. Again by Proposition 3.4,

$$\begin{aligned} \mathrm{cap}_{\mathrm{BV}}(E_1 \cup E_2) + \mathrm{cap}_{\mathrm{BV}}(E_1 \cap E_2) &\le \|D \max\{u_1, u_2\}\|(X) + \|D \min\{u_1, u_2\}\|(X) \\ &\le \|Du_1\|(X) + \|Du_2\|(X) \\ &< \mathrm{cap}_{\mathrm{BV}}(E_1) + \mathrm{cap}_{\mathrm{BV}}(E_2) + \varepsilon. \end{aligned}$$

By taking the limit when $\varepsilon \to 0$, we obtain the pretended result. □

BV-capacity is outer regular, meaning that any set $E \subset X$ can be approximated by open sets from the outside in terms of BV-capacity.

Proposition 3.17. *For every $E \subset X$,*

$$\mathrm{cap}_{\mathrm{BV}}(E) = \inf\left\{\mathrm{cap}_{\mathrm{BV}}(U) \mid E \subset U,\ U \textit{ is open}\right\}.$$

Proof. Let $E \subset X$. We prove both inequalities separately. By monotonicity of BV-capacity,

$$\mathrm{cap}_{\mathrm{BV}}(E) \leq \mathrm{cap}_{\mathrm{BV}}(U),$$

for every open set $U \subset X$ such that $E \subset U$. By taking the infimum over all such sets U, we get

$$\mathrm{cap}_{\mathrm{BV}}(E) \leq \inf\left\{\mathrm{cap}_{\mathrm{BV}}(U) \mid E \subset U,\, U \text{ is open}\right\}.$$

If $\mathrm{cap}_{\mathrm{BV}}(E) = \infty$,

$$\inf\left\{\mathrm{cap}_{\mathrm{BV}}(U) \mid E \subset U,\, U \text{ is open}\right\} \leq \mathrm{cap}_{\mathrm{BV}}(E)$$

follows trivially. Otherwise, let $\varepsilon > 0$ and $u \in \mathcal{A}_{\mathrm{BV}}(E)$ such that

$$\|Du\|(X) < \mathrm{cap}_{\mathrm{BV}}(E) + \varepsilon.$$

Let U be a neighbourhood of E such that $u = 1$ in U. Then it is obvious that $u \in \mathcal{A}_{\mathrm{BV}}(U)$ and so

$$\begin{aligned}\inf\left\{\mathrm{cap}_{\mathrm{BV}}(U) \mid E \subset U,\, U \text{ is open}\right\} &\leq \mathrm{cap}_{\mathrm{BV}}(U)\\ &\leq \|Du\|(X)\\ &< \mathrm{cap}_{\mathrm{BV}}(E) + \varepsilon.\end{aligned}$$

By considering the limit when $\varepsilon \to 0$, we finish the proof. □

With our definition of BV-capacity, it is not possible to apply the same techniques as in [13] to show the Choquet property and, in particular, upwards monotone convergence. However, downwards monotone convergence for compact sets still holds.

Proposition 3.18. *Let $(K_i)_{i\in\mathbb{N}}$ be a decreasing sequence of compact subsets of X. Then*

$$\mathrm{cap}_{\mathrm{BV}}\left(\bigcap_{i=1}^{\infty} K_i\right) = \lim_{i\to\infty} \mathrm{cap}_{\mathrm{BV}}(K_i).$$

Proof. Let $(K_i)_{i\in\mathbb{N}}$ be a decreasing sequence of compact subsets of X and consider

$$K = \bigcap_{i=1}^{\infty} K_i.$$

If there exists $i \in \mathbb{N}$ such that $K_i = \emptyset$, then for all $j \geq i$, $K_j = \emptyset$. Additionally, $K = \emptyset$. In that case,

$$\mathrm{cap}_{\mathrm{BV}}(K) = 0 = \lim_{i\to\infty} \mathrm{cap}_{\mathrm{BV}}(K_i).$$

So we can assume $K_i \neq \emptyset$ for all $i \in \mathbb{N}$ and, in that case, $K \neq \emptyset$. By monotonicity of BV-capacity,

$$\operatorname{cap}_{\mathrm{BV}}\left(\bigcap_{i=1}^{\infty} K_i\right) \leq \operatorname{cap}_{\mathrm{BV}}(K_i)$$

for all $i \in \mathbb{N}$. Therefore,

$$\operatorname{cap}_{\mathrm{BV}}\left(\bigcap_{i=1}^{\infty} K_i\right) \leq \lim_{i\to\infty} \operatorname{cap}_{\mathrm{BV}}(K_i)\,.$$

To prove the reverse inequality, let $U \subset X$ be an open set containing K. We note there exists $i \in \mathbb{N}$ such that $K_j \subset U$ for all $j \geq i$. Indeed, suppose by contradiction that for all $i \in \mathbb{N}$, there exists $j \geq i$ such that $K_j \cap (X \setminus U) \neq \emptyset$. Since $\bigcap_{i=1}^{\infty} K_i = K \neq \emptyset$, this means

$$\bigcap_{i=1}^{\infty}\bigcup_{j=i}^{\infty} K_j \cap (X \setminus U) \neq \emptyset.$$

But $(K_i)_{i\in\mathbb{N}}$ is a decreasing sequence, so

$$\bigcap_{i=1}^{\infty}\bigcup_{j=i}^{\infty} K_j \cap (X \setminus U) = \bigcap_{i=1}^{\infty} K_i \cap (X \setminus U) = K \cap (X \setminus U)\,.$$

This shows that $K \cap (X \setminus U) \neq \emptyset$, which contradicts the fact that $K \subset U$. Therefore, there exists $i \in \mathbb{N}$ such that $K_j \subset U$ for all $j \geq i$. In particular, by monotonicity of BV-capacity, there exists $i \in \mathbb{N}$ such that for all $j \geq i$,

$$\operatorname{cap}_{\mathrm{BV}}(K_j) \leq \operatorname{cap}_{\mathrm{BV}}(U)\,.$$

Thus

$$\lim_{i\to\infty} \operatorname{cap}_{\mathrm{BV}}(K_i) \leq \operatorname{cap}_{\mathrm{BV}}(U)\,.$$

Since this holds for all open sets $U \subset X$ containing K, we get

$$\lim_{i\to\infty} \operatorname{cap}_{\mathrm{BV}}(K_i) \leq \inf\left\{\operatorname{cap}_{\mathrm{BV}}(U) \mid K \subset U,\ U \text{ is open}\right\}.$$

Finally, we apply Proposition 3.17 to K and obtain

$$\lim_{i\to\infty} \operatorname{cap}_{\mathrm{BV}}(K_i) \leq \operatorname{cap}_{\mathrm{BV}}(K)\,.$$

□

The coarea formula gives rise to an equivalent definition of BV-capacity in terms of perimeter.

Proposition 3.19. *Let $E \subset X$. Then*

$$\operatorname{cap}_{\mathrm{BV}}(E) = \inf\{P(A,X) \mid E \subset \operatorname{int} A,\ A \subset X\}.$$

Proof. Let $E \subset X$. We show both inequalities in the statement above separately. Consider $A \subset X$ such that $E \subset \operatorname{int} A$. If $P(A,X) = \infty$, then $\operatorname{cap}_{\mathrm{BV}}(E) \leq P(A,X)$ follows trivially. Otherwise, by definition of perimeter, $\chi_A \in \mathrm{BV}(X)$. It is also true that $0 \leq \chi_A \leq 1$ and $\chi_A = 1$. Note that $\operatorname{int} A$ is a neighbourhood of E. Hence $\chi_A \in \mathcal{A}_{\mathrm{BV}}(E)$ and

$$\operatorname{cap}_{\mathrm{BV}}(E) \leq \|D\chi_A\|(X) = P(A,X).$$

Since this holds for all sets $A \subset X$ for which $E \subset \operatorname{int} A$, by taking the infimum over that class of sets, we get

$$\operatorname{cap}_{\mathrm{BV}}(E) \leq \inf\{P(A,X) \mid E \subset \operatorname{int} A,\ A \subset X\}.$$

If $\operatorname{cap}_{\mathrm{BV}}(E) = \infty$, the opposite inequality follows trivially. Otherwise, let $\varepsilon > 0$ and $u \in \mathcal{A}_{\mathrm{BV}}(E)$ such that

$$\|Du\|(X) < \operatorname{cap}_{\mathrm{BV}}(E) + \varepsilon.$$

By Corollary 3.10, there exists λ_0 such that $0 < \lambda_0 < 1$ and

$$P(\{u > \lambda_0\}, X) \leq \|Du\|(X) < \operatorname{cap}_{\mathrm{BV}}(E) + \varepsilon.$$

Let U be a neighbourhood of E such that $u = 1$ in U. Since $\lambda_0 < 1$, $U \subset \{u > \lambda_0\}$. Consequently, $U = \operatorname{int} U \subset \operatorname{int}\{u > \lambda_0\}$ and $E \subset \operatorname{int}\{u > \lambda_0\}$. Finally,

$$\begin{aligned}\inf\{P(A,X) \mid E \subset \operatorname{int} A,\ A \subset X\} &\leq P(\{u > \lambda_0\}, X) \\ &< \operatorname{cap}_{\mathrm{BV}}(E) + \varepsilon\end{aligned}$$

and by taking the limit when $\varepsilon \to 0$, the pretended result follows. □

3.3 Gagliardo-Nirenberg inequality

Throughout the next two sections, we make the extra assumption that $c_D > 2$ and consider $1^* = \frac{s}{s-1}$, where $s = \log_2 c_D$. Note that by imposing $c_D > 2$, we get $s > 1$. The assumptions in Proposition 2.12 are then satisfied for $p = 1$ and, as a consequence, we get a weak $(1^*, 1)$-Poincaré inequality, that is,

$$\left(\fint_B |u - u_B|^{1^*}\, d\mu\right)^{1/1^*} \leq c_P r \fint_{2\tau B} g\, d\mu$$

for all balls $B = B(x,r) \subset X$, with $x \in X$ and $0 < r < \infty$, all locally integrable functions u and all 1-weak upper gradients g_u of u.

We furthermore assume there exist a constant $C = C(c_D)$, a point $x_0 \in X$ and a sequence $0 < r_i < \infty$, with $i \in \mathbb{N}$, such that $\lim_{i\to\infty} r_i = \infty$ and $\mu(B(x_0, r_i)) \geq C r_i^s$

for all $i \in \mathbb{N}$. This condition will be sufficient to show a global Gagliardo-Nirenberg inequality for L^{1^*}-functions which have bounded variation. Many of the techniques used in this section can be recognised in section 5.5 in [4]. In the next arguments, we deal with many constants that depend at most in the underlying constants c_D, c_P and τ. To make the reading lighter, we abuse notation and denote most of those constants by the same letter c.

We first present an estimate for the L^{1^*}-norm of a compactly supported function in terms of its 1-weak upper gradients.

Lemma 3.20. *There exists a constant $c = c(c_D, c_P)$ such that*

$$\|u\|_{L^{1^*}(X)} \le c \int_X g \, d\mu$$

for all $u \in L_c^{1^*}(X)$ and all 1-weak upper gradients g of u.

Proof. Let $u \in L_c^{1^*}(X)$ and consider $C = C(c_D)$, $x_0 \in X$ and $0 < r_i < \infty$, with $i \in \mathbb{N}$, such that $\lim_{i\to\infty} r_i = \infty$ and

$$\mu\left(B\left(x_0, r_i\right)\right) \ge C r_i^s \tag{6}$$

for all $i \in \mathbb{N}$. Consider $i \in \mathbb{N}$ such that $r_i \ge 1$ and $\operatorname{supp} u \subset B_i$. Denote $B\left(x_0, r_i\right)$ by B_i. Since X satisfies a weak $(1^*, 1)$-Poincaré inequality, we get

$$\begin{aligned}
&\left(\int_{2B_i} |u|^{1^*} \, d\mu\right)^{1/1^*} \\
&\le \mu\left(2B_i\right)^{1/1^*} \left(\fint_{2B_i} \left|u - u_{2B_i}\right|^{1^*} \, d\mu\right)^{1/1^*} + \mu\left(2B_i\right)^{1/1^*} \left|u_{2B_i}\right| \\
&\le c r_i \mu\left(2B_i\right)^{1/1^*-1} \int_{4\tau B_i} g \, d\mu + \mu\left(2B_i\right)^{1/1^*} \left|u_{2B_i}\right|,
\end{aligned} \tag{7}$$

for some constant c depending on c_P and c_D.

By Lemma 2.13, there exists $\xi = \xi(c_D)$ such that $0 < \xi < 1$ and

$$\frac{\mu\left(B_i\right)}{\mu\left(2B_i\right)} \le \xi.$$

Since $\operatorname{supp} u \subset B_i$ and by Hölder's inequality we obtain

$$\begin{aligned}
\left|u_{2B_i}\right| &= \left|\fint_{2B_i} u \, d\mu\right| \le \fint_{2B_i} |u| \, \chi_{B_i} \, d\mu \\
&= \frac{1}{\mu\left(2B_i\right)} \left(\int_{2B_i} |u|^{1^*}\right)^{1/1^*} \mu\left(B_i\right)^{1-1/1^*} \\
&= \frac{\mu\left(2B_i\right)^{1/1^*}}{\mu\left(2B_i\right)} \left(\fint_{2B_i} |u|^{1^*}\right)^{1/1^*} \mu\left(B_i\right)^{1-1/1^*} \\
&= \left(\frac{\mu\left(B_i\right)}{\mu\left(2B_i\right)}\right)^{1-1/1^*} \left(\fint_{2B_i} |u|^{1^*}\right)^{1/1^*} \\
&\le \xi^{1-1/1^*} \left(\fint_{2B_i} |u|^{1^*}\right)^{1/1^*}.
\end{aligned}$$

Thus we get the following estimate for the second term in the right-hand side of (7).

$$\mu\left(2B_i\right)^{1/1^*}\left|u_{2B_i}\right| \le \xi^{1-1/1^*}\left(\int_{2B_i}|u|^{1^*}\right)^{1/1^*}.$$

So from (7) we get

$$(1-\xi^{1-1/1^*})\left(\int_{2B_i}|u|^{1^*}\,d\mu\right)^{1/1^*} \le c r_i \mu\left(2B_i\right)^{1/1^*-1}\int_{4\tau B_i} g\,d\mu.$$

By (6), we have

$$\mu\left(2B_i\right)^{1/1^*-1} = \mu\left(2B_i\right)^{\frac{s-1}{s}-1} = \mu\left(2B_i\right)^{-1/s} \le \left(Cr_i^s\right)^{-1/s} = cr_i^{-1}$$

and consequently

$$\begin{aligned}(1-\xi^{1-1/1^*})\left(\int_{2B_i}|u|^{1^*}\,d\mu\right)^{1/1^*} &\le c r_i r_i^{-1}\int_{4\tau B_i} g\,d\mu \\ &\le c\int_X g\,d\mu,\end{aligned}$$

where c is dependent on c_D and c_P. After dividing both sides of the inequality above by $(1-\xi^{1-1/1^*})$, we get

$$\left(\int_{2B_i}|u|^{1^*}\,d\mu\right)^{1/1^*} \le c\int_X g\,d\mu.$$

Since $\operatorname{supp} u \subset B_i$, we conclude that

$$\left(\int_X |u|^{p^*}\,d\mu\right)^{1/1^*} \le c\int_X g\,d\mu.$$

□

We can extend the previous result and obtain a global Gagliardo-Nirenberg type inequality for functions of bounded variation.

Theorem 3.21 (Gagliardo-Nirenberg inequality)**.** *There exists a constant $c = c(c_D, c_P)$ such that*

$$\|u\|_{L^{1^*}(X)} \le c\,\|Du\|\,(X)$$

for all $u \in \mathrm{BV}\,(X) \cap L^{1^*}\,(X)$.

Proof. Let $u \in \mathrm{BV}(X) \cap L^{1^*}(X)$ and consider a sequence $u_i \in \mathrm{Lip}_c(X)$, with $i \in \mathbb{N}$, as in Lemma 3.3 such that $u_i \to u$ in $L^1(X)$ as $i \to \infty$ and

$$\lim_{i\to\infty} \int_X g_{u_i}\, d\mu = \|Du\|(X) < \infty,$$

where g_{u_i} is a 1-weak upper gradient of u_i for all $i \in \mathbb{N}$. Fix $i \in \mathbb{N}$ and let $x \in X$ and $0 < r < \infty$ be such that $\operatorname{supp} u_i \subset B(x,r)$. Using the weak $(1^*, 1)$-Poincaré inequality yields

$$\begin{aligned}
\left(\int_X |u_i|^{1^*}\, d\mu\right)^{1/1*} &\le \left(\int_{B(x,r)} |u_i|^{1^*}\, d\mu\right)^{1/1*} \\
&\le \mu(B(x,r))^{1/1^*} \left(\fint_{B(x,r)} \left|u_i - (u_i)_{B(x,r)}\right|^{1^*} d\mu\right)^{1/1^*} \\
&\quad + \mu(B(x,r))^{1/1^*} \left|(u_i)_{B(x,r)}\right| \\
&\le c_P r \mu(B(x,r))^{1/1^*-1} \int_{2\tau B(x,r)} g_{u_i}\, d\mu \\
&\quad + \mu(B(x,r))^{1/1^*-1} \|u_i\|_{L^1(X)} \\
&\le c_P r \mu(B(x,r))^{1/1^*-1} \int_X g_{u_i}\, d\mu \\
&\quad + \mu(B(x,r))^{1/1^*-1} \|u_i\|_{L^1(X)} \\
&< \infty.
\end{aligned}$$

The finiteness of the previous expression stands on the fact that $u \in \mathrm{BV}(X)$, $\int_X g_{u_j} \to \|Du\|(X)$ as $j \to \infty$ and $u_i \in L^1(X)$. This shows that $u_i \in L_c^{1^*}(X)$ and so by Lemma 3.20, there exists $c = c(c_D, c_P)$ such that

$$\|u_i\|_{L^{1^*}(X)} \le c \int_X g_{u_i}\, d\mu$$

for all $i \in \mathbb{N}$. Since $u_i \to u \in L^1(X)$ as $i \to \infty$, there exists a subsequence $(u_{i_k})_{k\in\mathbb{N}}$ of $(u_i)_{i\in\mathbb{N}}$ for which $u_{i_k} \to u$ μ-almost everywhere as $k \to \infty$. In that case, $|u_{i_k}|^{1^*} \to |u|^{1^*}$ μ-almost everywhere as $k \to \infty$ and by Fatou's Lemma we obtain

$$\|u\|_{L^{1^*}(X)} \le \liminf_{k\to\infty} \left\|u_{i_k}\right\|_{L^{1^*}(X)} \le c \liminf_{k\to\infty} \int_X g_{u_{i_k}}\, d\mu = c\,\|Du\|(X).$$

□

3.4 Modified BV-capacity

In this section, we consider a modified version of BV-capacity where the class of admissible functions is restricted to $L^{1^*}(X)$. The main reason to do this is to obtain a version of capacity which is countable subadditive. We denote this new capacity by BV^*-capacity or $\mathrm{cap}_{\mathrm{BV}^*}$, in short. As in section 3.2, many of our results and techniques are based in [13, pp. 58–63].

Definition 3.22 (Modified BV-capacity)**.** *Let $E \subset X$. We define the modified* BV*-capacity of E, or* BV^**-capacity of E, as*

$$\mathrm{cap}_{\mathrm{BV}^*}(E) := \inf_{u \in \mathcal{A}_{\mathrm{BV}^*}(E)} \|Du\|(X),$$

where $\mathcal{A}_{\mathrm{BV}^*}(E) = \mathcal{A}_{\mathrm{BV}}(E) \cap L^{1^*}(X)$.

Remark 3.23. *For all sets $E \subset X$,*

$$\mathrm{cap}_{\mathrm{BV}}(E) \leq \mathrm{cap}_{\mathrm{BV}^*}(E)$$

since $\mathcal{A}_{\mathrm{BV}^*}(E) \subset \mathcal{A}_{\mathrm{BV}}(E)$.

We can show that, when restricted to compact sets, BV-capacity and the modified BV-capacity coincide.

Lemma 3.24. *If $K \subset X$ is compact, then*

$$\mathrm{cap}_{\mathrm{BV}}(K) = \mathrm{cap}_{\mathrm{BV}^*}(K).$$

Proof. The inequality

$$\mathrm{cap}_{\mathrm{BV}}(K) \leq \mathrm{cap}_{\mathrm{BV}^*}(K)$$

follows by the previous Remark 3.23. To show the inequality in the opposite direction, let $\varepsilon > 0$ and consider $u \in \mathcal{A}_{\mathrm{BV}}(X) \cap \mathrm{Lip}_c(X)$ such that

$$\|Du\|(X) < \mathrm{cap}_{\mathrm{BV}}(K) + \varepsilon.$$

Recall such a function exists by Lemma 3.13. Since $u \in \mathcal{A}_{\mathrm{BV}}(X)$, we have $0 \leq u \leq 1$ and so

$$\int_X |u|^{1^*} \, d\mu \leq \int_{\mathrm{supp}\, u} 1 \, d\mu = \mu(\mathrm{supp}\, u) < \infty$$

This implies that $u \in L^{1^*}(X)$ and, as a consequence, $u \in \mathcal{A}_{\mathrm{BV}^*}(X)$. Therefore

$$\mathrm{cap}^*_{\mathrm{BV}}(K) \leq \|Du\|(X) < \mathrm{cap}_{\mathrm{BV}}(K) + \varepsilon.$$

By taking the limit when $\varepsilon \longrightarrow 0$, we get

$$\mathrm{cap}^*_{\mathrm{BV}}(K) \leq \mathrm{cap}_{\mathrm{BV}}(K).$$

□

The previous result allows us to conclude that BV-capacity and the modified BV-capacity satisfy the same properties in the class of compact subsets of X.

It also happens that all the basic properties of BV-capacity listed in section 3.2 are inherited by the modified BV-capacity since all the admissible functions used in the arguments are in $L^{1^*}(X)$. Namely, the BV^*-capacity satisfies $\mathrm{cap}_{\mathrm{BV}^*}(\emptyset) = 0$, it is monotone, see Theorem 3.14, it is outer regular, see Proposition 3.17, and it satisfies downwards monotone convergence for compact sets, see Proposition 3.18.

The global Gagliardo-Nirenberg inequality provides a tool to show that BV^*-capacity is countable subadditive and therefore an outer measure.

Theorem 3.25. BV**-capacity is an outer measure.*

Proof. As previously stated, BV*-capacity satisfies $\mathrm{cap}_{\mathrm{BV}^*}(\emptyset) = 0$ and monotonicity. It remains to show countable subadditivity. Let $E_i \subset X$, with $i \in \mathbb{N}$, and consider $E = \bigcup_{i=1}^{\infty} E_i$. If

$$\sum_{i=1}^{\infty} \mathrm{cap}_{\mathrm{BV}^*}(E_i) = \infty,$$

it is trivial that

$$\mathrm{cap}_{\mathrm{BV}^*}(E) \leq \sum_{i=1}^{\infty} \mathrm{cap}_{\mathrm{BV}^*}(E_i).$$

Hence we may assume that

$$\sum_{i=1}^{\infty} \mathrm{cap}_{\mathrm{BV}^*}(E_i) < \infty.$$

In particular,

$$\mathrm{cap}_{\mathrm{BV}^*}(E_i) < \infty$$

for all $i \in \mathbb{N}$. Let $\varepsilon > 0$ and, for all $i \in \mathbb{N}$, take $u_i \in \mathcal{A}_{\mathrm{BV}^*}(E_i)$ such that

$$\|Du_i\|(X) < \mathrm{cap}_{\mathrm{BV}^*}(E_i) + 2^{-i}\varepsilon. \tag{8}$$

Consider the function defined by $u(x) = \sup_{i\in\mathbb{N}} u_i(x)$, for all $x \in X$. By the Gagliardo-Nirenberg inequality, see Theorem 3.21, it hols that

$$\begin{aligned}
\|u\|_{L^{1^*}(X)} \leq \sum_{i=1}^{\infty} \|u_i\|_{L^{1^*}(X)} &\leq c \sum_{i=1}^{\infty} \|Du_i\|(X) \\
&\leq c \sum_{i=1}^{\infty} \left(\mathrm{cap}_{\mathrm{BV}^*}(E_i) + 2^{-i}\varepsilon\right) \\
&= c \left(\sum_{i=1}^{\infty} \mathrm{cap}_{\mathrm{BV}^*}(E_i) + \varepsilon\right) < \infty.
\end{aligned}$$

This implies that $u \in L^{1^*}(X)$. Now, for all $i \in \mathbb{N}$, define

$$v_i = \max_{1\leq j\leq i} u_j.$$

Recall that $u_i \in L^1(X)$ for all $i \in \mathbb{N}$. Then

$$\|v_i\|_{L^1(X)} \leq \sum_{j=1}^{i} \|u_j\|_{L^1(X)} < \infty$$

and so $v_i \in L^1(X)$ for all $i \in \mathbb{N}$. Moreover, $v_i \to u$ in $L^1(X)$ as $i \to \infty$. Hence, by (8), lower semicontinuity of variation, see Proposition 3.6, and Remark 3.5, we get

$$\begin{aligned}\|Du\|(X) &\leq \liminf_{i\to\infty} \|Dv_i\|(X) = \liminf_{i\to\infty} \left\| D \max_{1\leq j\leq i} u_j \right\| (X) \\ &\leq \liminf_{i\to\infty} \sum_{j=1}^{i} \|Du_j\|(X) = \sum_{i=1}^{\infty} \|Du_i\|(X) \\ &\leq \sum_{i=1}^{\infty} \left(\mathrm{cap}_{\mathrm{BV}^*}(E_i) + 2^{-i}\varepsilon\right) \\ &= \sum_{i=1}^{\infty} \mathrm{cap}_{\mathrm{BV}^*}(E_i) + \varepsilon < \infty. \end{aligned} \tag{9}$$

We have shown that $u \in L^{1^*}(X)$ and $u \in \mathrm{BV}(X)$. Moreover, $0 \leq u_i \leq 1$ for all $i \in \mathbb{N}$, so by construction of u as the supremum of u_i for all $i \in \mathbb{N}$, we have that $0 \leq u \leq 1$. Now, for all $i \in \mathbb{N}$, let U_i be a neighbourhood of E_i such that $u_i = 1$ in U_i. Then it is trivial to check that $U = \bigcup_{i=1}^{\infty} U_i$ is a neighbourhood of E such that $u = 1$ on U. With this, we can conclude that $u \in \mathcal{A}_{\mathrm{BV}^*}(E)$. Therefore, by (9),

$$\mathrm{cap}_{\mathrm{BV}^*}(E) \leq \|Du\|(X) \leq \sum_{i=1}^{\infty} \mathrm{cap}_{\mathrm{BV}^*}(E_i) + \varepsilon.$$

By letting $\varepsilon \to 0$, we obtain

$$\mathrm{cap}_{\mathrm{BV}^*}(E) \leq \sum_{i=1}^{\infty} \mathrm{cap}_{\mathrm{BV}^*}(E_i).$$

□

We introduce the notion of Choquet capacity.

Definition 3.26 (Choquet capacity)**.** *A function $\nu : \mathcal{P}(X) \to [0, \infty]$ is a Choquet capacity if*

1. *(monotonicity) $\nu(E_1) \leq \nu(E_2)$ for all $E_1, E_2 \subset X$ such that $E_1 \subset E_2$;*
2. *(upwards monotone convergence) for all increasing sequences $(E_i)_{i\in\mathbb{N}}$ of sets in X,*

$$\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{i\to\infty} \nu(E_i); \tag{10}$$

3. *(downwards monotone convergence for compact sets) for all decreasing sequences $(K_i)_{i\in\mathbb{N}}$ of compact sets in X,*

$$\nu\left(\bigcap_{i=1}^{\infty} K_i\right) = \lim_{i\to\infty} \nu(K_i). \tag{11}$$

All Borel sets are capacitable by a Choquet capacity, see [6, p. 84]. In practice, this means that Borel sets can be approximated from the outside by open sets and from the outside by compact sets in terms of the capacity.

Theorem 3.27 (Choquet's capacitability theorem)**.** *If a set function $\nu : \mathcal{P}(X) \to [0, \infty]$ is a Choquet capacity, then all Borel sets $E \subset X$ are capacitable by ν, that is,*

$$\begin{aligned}\nu(E) &= \inf\{\nu(U) \mid E \subset U,\ U \text{ is open}\} \\ &= \sup\{\nu(K) \mid K \subset E,\ K \text{ is compact}\}.\end{aligned}$$

It turns out that BV^*-capacity is a Choquet capacity.

Theorem 3.28. BV^**-capacity is a Choquet capacity.*

Proof. Monotonicity and downwards monotone convergence of BV^*-capacity hold just like they hold for BV-capacity. We show upwards monotone convergence. Let $(E_i)_{i\in\mathbb{N}}$ be an increasing sequence of sets in X. By monotonicity, it is clear that

$$\mathrm{cap}_{\mathrm{BV}^*}(E_i) \le \mathrm{cap}_{\mathrm{BV}^*}\left(\bigcup_{i=1}^{\infty} E_i\right)$$

for all $i \in \mathbb{N}$ and so by taking the limit when $i \to \infty$, one gets

$$\lim_{i\to\infty} \mathrm{cap}_{\mathrm{BV}^*}(E_i) \le \mathrm{cap}_{\mathrm{BV}^*}\left(\bigcup_{i=1}^{\infty} E_i\right).$$

If $\lim_{i\to\infty} \mathrm{cap}_{\mathrm{BV}^*}(E_i) = \infty$, then

$$\mathrm{cap}_{\mathrm{BV}^*}\left(\bigcup_{i=1}^{\infty} E_i\right) \le \lim_{i\to\infty} \mathrm{cap}_{\mathrm{BV}^*}(E_i)$$

follows trivially and we are done. Assume then that $\lim_{i\to\infty} \mathrm{cap}_{\mathrm{BV}^*}(E_i) < \infty$. In particular, $\mathrm{cap}_{\mathrm{BV}^*}(E_i) < \infty$ for all $i \in \mathbb{N}$. So let $\varepsilon > 0$ and for all $i \in \mathbb{N}$, take $u_i \in \mathcal{A}_{\mathrm{BV}^*}(E_i)$ such that

$$\|Du_i\|(X) < \mathrm{cap}_{\mathrm{BV}^*}(E_i) + 2^{-i}\varepsilon. \tag{12}$$

Define

$$u = \sup_{i\in\mathbb{N}} u_i$$

and, for all $i \in \mathbb{N}$,

$$v_i = \max_{1\le j\le i} u_j.$$

As in the proof of Theorem 3.25, it holds that $u \in \mathcal{A}_{\mathrm{BV}^*}\left(\bigcup_{i=1}^{\infty} E_i\right)$, $v_i \in L^1(X)$ for all $i \in \mathbb{N}$ and $v_i \to u$ in $L^1(X)$ as $i \to \infty$. Moreover, by Remark 3.5,

$$\|Dv_i\|(X) \le \sum_{j=1}^{i} \|Du_j\|(X) < \infty,$$

for all $i \in \mathbb{N}$. In turn, lower semicontinuity of variation, see Proposition 3.6, yields

$$\|Du\|(X) \le \liminf_{i\to\infty} \|Dv_i\|(X). \tag{13}$$

Now, set $v_0 = 0$, so that $\|Dv_0\|(X) = 0$, and $E_0 = \emptyset$, so that $\mathrm{cap}_{\mathrm{BV}^*}(E_0) = 0$. Furthermore, define

$$w_i = \min\{v_{i-1}, u_i\}$$

for all $i \in \mathbb{N}$. Note that by definition of v_i,

$$v_i = \max\{v_{i-1}, u_i\}$$

for all $i \in \mathbb{N}$.

Let $i \in \mathbb{N}$. We aim to show that $w_i \in \mathcal{A}_{\mathrm{BV}^*}(E_{i-1})$. Since both v_{i-1} and u_i are in $\mathrm{BV}(X)$, part 3. of Proposition 3.4 implies that $w_i \in \mathrm{BV}(X)$. On the other hand, $w_i \in L^{1^*}(X)$ since $u_j \in L^{1^*}(X)$ for all $1 \le j \le i$. Furthermore, since $0 \le u_j \le 1$ for all $1 \le j \le i$, it is true that

$$0 \le w_i \le 1.$$

Note that if $u_j = 1$ in a neighbourhood U_j of E_j, with $1 \le j \le i$, then $v_{i-1} = 1$ in $\bigcup_{j=1}^{i-1} U_j$. Hence $w_i = 1$ in

$$\left(\bigcup_{j=1}^{i-1} U_j\right) \cap U_i.$$

Note that the set above is a neighbourhood of $E_{i-1} \cap E_i$. Since the sequence $(E_i)_{i\in\mathbb{N}}$ is increasing, $E_{i-1} \cap E_i = E_{i-1}$ and so we can assert that $w_i = 1$ in a neighbourhood of E_{i-1}. This shows that $w_i \in \mathcal{A}_{\mathrm{BV}^*}(E_{i-1})$.

By (12) and part 3. of Proposition 3.4, we get

$$\begin{aligned}\left\|Dv_j\right\|(X) + \mathrm{cap}_{\mathrm{BV}^*}\left(E_{j-1}\right) &\le \left\|Dv_j\right\|(X) + \left\|Dw_j\right\|(X)\\ &= \left\|\max\left\{v_{j-1}, u_j\right\}\right\|(X) + \left\|\min\left\{v_{j-1}, u_j\right\}\right\|(X)\\ &\le \left\|Dv_{j-1}\right\|(X) + \left\|Du_j\right\|(X)\\ &\le \left\|Dv_{j-1}\right\|(X) + \mathrm{cap}_{\mathrm{BV}^*}\left(E_j\right) + 2^{-j}\varepsilon\end{aligned}$$

for all $1 \le j \le i$. By rearranging the terms and summing over j, we get

$$\begin{aligned}&\sum_{j=1}^{i}(\left\|Dv_j\right\|(X) - \left\|Dv_{j-1}\right\|(X))\\ &\le \sum_{j=1}^{i}(\mathrm{cap}_{\mathrm{BV}^*}\left(E_j\right) - \mathrm{cap}_{\mathrm{BV}^*}\left(E_{j-1}\right)) + \sum_{j=1}^{i} 2^{-j}\varepsilon.\end{aligned}$$

By the formula for telescopic sums and recalling that $\|Dv_0\|(X) = 0$ and $\mathrm{cap}_{\mathrm{BV}^*}(E_0) = 0$, we obtain

$$\|Dv_i\|(X) \le \mathrm{cap}_{\mathrm{BV}^*}(E_i) + \sum_{j=1}^{i} 2^{-j}\varepsilon. \tag{14}$$

Finally, since $u \in \mathcal{A}_{\mathrm{BV}^*}\left(\bigcup_{i=1}^{\infty} E_i\right)$ and by considering (13) and (14) we conclude

$$\begin{aligned}\operatorname{cap}_{\mathrm{BV}^*}\left(\bigcup_{i=1}^{\infty} E_i\right) &\le \|Du\|(X) \le \liminf_{i\to\infty} \|Dv_i\|(X)\\ &\le \liminf_{i\to\infty}(\operatorname{cap}_{\mathrm{BV}^*}(E_i) + \sum_{j=1}^{i} 2^{-j}\varepsilon)\\ &= \liminf_{i\to\infty} \operatorname{cap}_{\mathrm{BV}^*}(E_i) + \varepsilon.\end{aligned}$$

Since $\lim_{i\to\infty} \operatorname{cap}_{\mathrm{BV}^*}(E_i)$ is finite, it coincides with $\liminf_{i\to\infty} \operatorname{cap}_{\mathrm{BV}^*}(E_i)$. Hence, by taking the limit when $\varepsilon \to 0$ in the last inequality, one gets

$$\operatorname{cap}_{\mathrm{BV}^*}\left(\bigcup_{i=1}^{\infty} E_i\right) \le \lim_{i\to\infty} \operatorname{cap}_{\mathrm{BV}^*}(E_i).$$

□

Corollary 3.29. *All Borel sets $E \subset X$ are capacitable by* BV**-capacity, that is,*

$$\begin{aligned}\operatorname{cap}_{\mathrm{BV}^*}(E) &= \inf\left\{\operatorname{cap}_{\mathrm{BV}^*}(U) \mid E \subset U,\ U \text{ is open}\right\}\\ &= \sup\left\{\operatorname{cap}_{\mathrm{BV}^*}(K) \mid K \subset E,\ K \text{ is compact}\right\}.\end{aligned}$$

Proof. This follows directly from Theorem 3.27 and Theorem 3.28. □

The modified BV-capacity is finer than the measure μ.

Proposition 3.30. *Let $E \subset X$. Then*

$$\mu(E) \le c \operatorname{cap}_{\mathrm{BV}^*}(E)^{1^*},$$

where $c = c(c_D, c_P)$.

Proof. Let $E \subset X$. If $\operatorname{cap}_{\mathrm{BV}^*}(E) = \infty$, we are done. Otherwise, let $\varepsilon > 0$ and take $u \in \mathcal{A}_{\mathrm{BV}^*}(E)$ such that

$$\|Du\|(X) < \operatorname{cap}_{\mathrm{BV}^*}(E) + \varepsilon.$$

Consider a neighbourhood $U \subset X$ of E such that $u = 1$ in U. We observe that by the Gagliardo-Nirenberg inequality, see Theorem 3.21,

$$\begin{aligned}\mu(E) \le \mu(U) &= \int_U |u|^{1^*}\, d\mu\\ &= \|u\|_{L^{1^*}(X)}^{1^*} \le (c\,\|Du\|(X))^{1^*}\\ &< \left(c \operatorname{cap}_{\mathrm{BV}^*}(E) + \varepsilon\right)^{1^*},\end{aligned}$$

where $c = c(c_D, c_P)$. By taking the limit when $\varepsilon \to 0$, we get

$$\mu(E) \le c \operatorname{cap}_{\mathrm{BV}^*}(E)^{1^*}.$$

□

4 Boxing inequality

The main goal in this section is to obtain a modified version of the boxing inequality for our metric measure space setting. Note that we return to our basic assumptions on the space X described in section 2, that is, we drop the stronger assumptions introduced in section 3.3.

We start by introducing a relative isoperimetric inequality, which is a geometric version of the 1-weak Poincaré inequality for μ-measurable sets. Once we have this important geometric tool, we can proceed to study the boxing inequality.

4.1 Relative isoperimetric inequality

The result that follows states that, locally, the measure of a set or of its complement cannot surpass a scaling of the perimeter of the set. This is a geometric version of the weak 1-Poincaré inequality that can be applied to sets of finite perimeter. An appopriate reference for this inequality is Theorem 4.5 in [20, p. 998].

Lemma 4.1 (Relative isoperimetric inequality)**.** *Let $E \subset X$ be a μ-measurable set of finite perimeter. Then,*

$$\min \{\mu (B \cap E), \mu (B \setminus E)\} \leq c_P r P (E, \tau B),$$

for every ball $B = B(x,r) \subset X$, where $x \in X$ and $0 < r < \infty$.

Proof. Let $E \subset X$ be a μ-measurable set of finite perimeter and consider a ball $B = B(x,r) \subset X$ with $x \in X$ and $0 < r < \infty$. Take $u = \chi_E$. Since E is a set of finite perimeter, $u \in \mathrm{BV}(X)$ and so by Lemma 3.3, there exist $u_i \in \mathrm{Lip}_c(X)$ and upper gradients g_{u_i} of u_i, with $i \in \mathbb{N}$, such that $u_i \to u$ in $L^1(X)$ as $i \to \infty$ and

$$\lim_{i\to\infty} \int_{\tau B} g_{u_i}\, d\mu = \|D\chi_E\|(\tau B) = P(E, \tau B).$$

Note that $(u_i)_B \to u_B$ in $L^1(X)$ as $i \to \infty$. Using Fatou's Lemma and the weak 1-Poincaré inequality leads to

$$\begin{aligned}
&\fint_B |u - u_B|\, d\mu \\
&\leq \fint_B \liminf_{i\to\infty} \left(|u - u_i| + |u_i - (u_i)_B| + |u_B - (u_i)_B|\right)\, d\mu \\
&= \fint_B \liminf_{i\to\infty} |u_i - (u_i)_B|\, d\mu \leq \liminf_{i\to\infty} \fint_B |u_i - (u_i)_B|\, d\mu \\
&\leq c_P r \liminf_{i\to\infty} \fint_{\tau B} g_{u_i}\, d\mu \\
&= \frac{c_P r}{\mu(\tau B)} \liminf_{i\to\infty} \int_{\tau B} g_{u_i}\, d\mu \\
&= \frac{c_P r}{\mu(\tau B)} P(E, \tau B).
\end{aligned} \tag{15}$$

Now by construction of u,

$$u_B = \frac{1}{\mu(B)} \int_B \chi_E \, d\mu = \frac{\mu(B \cap E)}{\mu(B)}$$

and so

$$|u - u_B| = \frac{\mu(B \cap E)}{\mu(B)} \text{ on } B \setminus E$$

and

$$|u - u_B| = \frac{\mu(B \setminus E)}{\mu(B)} \text{ on } B \cap E.$$

Moreover, since μ is a regular outer meausure and E is a μ-measurable set, we get

$$\begin{aligned}
\mu(B) &= \mu(B \cap E) + \mu(B \setminus E) \\
&= \max\{\mu(B \cap E), \mu(B \setminus E)\} + \min\{\mu(B \cap E), \mu(B \setminus E)\} \\
&\leq 2 \max\{\mu(B \cap E), \mu(B \setminus E)\}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\fint_B |u - u_B| \, d\mu \\
&= \frac{1}{\mu(B)} \left(\int_{B \cap E} |u - u_B| \, d\mu + \int_{B \setminus E} |u - u_B| \, d\mu \right) \\
&= \frac{1}{\mu(B)} \left(\frac{\mu(B \cap E)\, \mu(B \setminus E)}{\mu(B)} + \frac{\mu(B \setminus E)\, \mu(B \cap E)}{\mu(B)} \right) \\
&= \frac{2}{\mu(B)} \frac{\mu(B \cap E)\, \mu(B \setminus E)}{\mu(B)} \\
&\geq \frac{2}{\mu(\tau B)} \frac{\max\{\mu(B \cap E), \mu(B \setminus E)\} \min\{\mu(B \cap E), \mu(B \setminus E)\}}{2 \max\{\mu(B \cap E), \mu(B \setminus E)\}} \\
&= \frac{\min\{\mu(B \cap E), \mu(B \setminus E)\}}{\mu(\tau B)}.
\end{aligned} \tag{16}$$

Finally, (15) and (16) together imply

$$\begin{aligned}
\frac{\min\{\mu(B \cap E), \mu(B \setminus E)\}}{\mu(\tau B)} &\leq \fint_B |u - u_B| \, d\mu \\
&\leq \frac{c_P r}{\mu(\tau B)} P(E, \tau B),
\end{aligned}$$

that is,

$$\min\{\mu(B \cap E), \mu(B \setminus E)\} \leq c_P r P(E, \tau B).$$

□

4.2 Boxing inequality

After having introduced the relative isoperimetric inequality, we are ready to announce and prove the Boxing inequality. This result provides a covering for open bounded sets of finite perimeter U such that, after proper scaling, the total measure of the covering is not much larger than the perimeter of U. We base our study in Theorem 3.1 in [17, pp. 406–408].

Theorem 4.2. *Let $U \subset X$ be an open bounded set of finite perimeter. Then there exists a constant $c = c(c_D, c_P, \tau)$ and a collection $\{B(x_i, \tau r_i)\}_{i \in \mathbb{N}}$ of pairwise disjoint balls satisfying the following properties:*

1. $U \subset \bigcup_{i=1}^{\infty} B(x_i, 5\tau r_i)$;
2. $\frac{1}{2c_D} < \frac{\mu(B(x_i, r_i) \cap U)}{\mu(B(x_i, r_i))} \leq \frac{1}{2}$ *for all* $i \in \mathbb{N}$;
3. *(Boxing inequality)* $\sum_{i=1}^{\infty} \frac{\mu(B(x_i, 5\tau r_i))}{5\tau r_i} \leq cP(U, X)$.

Proof. Let $U \subset X$ be an open bounded set such that $P(U, X) < \infty$. Since U is bounded, there exists a ball $B \subset X$ such that $U \subset B$. In particular, $\mu(U) \leq \mu(B) < \infty$.

We proceed to construct a covering of U satisfying the properties announced in the statement of the theorem. Take $x \in U$. Since U is an open set, there exists $0 < \rho_x < \infty$ such that $B(x, \rho_x) \subset U$ and hence

$$\frac{\mu(B(x, \rho_x) \cap U)}{\mu(B(x, \rho_x))} = 1. \tag{17}$$

Moreover, since $\mu(U) < \infty$ and $\mu(X) = \infty$,

$$\lim_{r \to \infty} \frac{\mu(B(x, r) \cap U)}{\mu(B(x, r))} \leq \lim_{r \to \infty} \frac{\mu(U)}{\mu(B(x, r))} = \frac{\mu(U)}{\mu(X)} = 0. \tag{18}$$

This shows that the density of U in $B(x, \rho_x)$ is 1 and, as $r \to \infty$, the density of U in $B(x, r)$ tends to 0. We now apply a stopping time argument. By (17) and (18), we observe there exists $k_x \in \mathbb{N}$ such that

$$\frac{\mu(B(x, 2^m \rho_x) \cap U)}{\mu(B(x, 2^m \rho_x))} > \frac{1}{2} \tag{19}$$

for all $m \in \{0, \ldots, k_x - 1\}$ and

$$\frac{\mu\left(B\left(x, 2^{k_x} \rho_x\right) \cap U\right)}{\mu\left(B\left(x, 2^{k_x} \rho_x\right)\right)} \leq \frac{1}{2}. \tag{20}$$

By (19) and the doubling condition, we then get

$$\frac{\mu\left(B\left(x, 2^{k_x} \rho_x\right) \cap U\right)}{\mu\left(B\left(x, 2^{k_x} \rho_x\right)\right)} \geq \frac{\mu\left(B\left(x, 2^{k_x - 1} \rho_x\right) \cap U\right)}{c_D \mu\left(B\left(x, 2^{k_x - 1} \rho_x\right)\right)} > \frac{1}{2c_D}. \tag{21}$$

Recall that x was chosen arbitrarily from U. Then, for all $x \in U$, denote $r_x = 2^{k_x}\rho_x$ and consider the collection $\{B(x,\tau r_x)\}_{x\in U}$. With Vitali's 5-covering theorem in mind, we show this collection has uniformly bounded radii.

Indeed, take $x_0 \in U$ and let $0 < R_0 < \infty$ be such that $U \subset B(x_0, R_0)$ and

$$\frac{\mu(U)}{\mu(B(x_0,R_0))} \leq \frac{1}{2}.$$

Such R_0 exists since U is bounded and

$$\lim_{r\to\infty} \frac{\mu(U)}{\mu(B(x_0,r))} = 0.$$

Consider now $R = \operatorname{diam}(U) + R_0 < \infty$. For every $x \in U$, $B(x,R) \supset B(x_0,R_0)$. Indeed, by triangle's inequality,

$$\begin{aligned} d(y,x) &\leq d(y,x_0) + d(x_0,x) \\ &< R_0 + \operatorname{diam}(U) = R \end{aligned}$$

for every $y \in B(x_0,R_0)$. Hence we get

$$\frac{\mu(B(x,R)\cap U)}{\mu(B(x,R))} \leq \frac{\mu(U)}{\mu(B(x_0,R_0))} \leq \frac{1}{2}.$$

Recall that (19) holds for $m = k_x - 1$. Thus $2^{k_x-1}\rho_x < R$, which implies that $r_x = 2^{k_x}\rho_x < 2R$. Thus we get

$$\tau r_x < 2R\tau < \infty$$

for all $x \in U$, that is, $2R\tau$ is a uniform bound for the radii τr_x, with $x \in U$.

By Vitali's 5-covering Theorem, see Theorem 2.15, there exists a countable subcollection $\{B(x_i,\tau r_i)\}_{i\in\mathbb{N}} \subset \{B(x,\tau r_x)\}_{x\in U}$ of pairwise disjoint balls such that

$$U \subset \bigcup_{i=1}^{\infty} B(x_i, 5\tau r_i).$$

This proves the first statement of the theorem. The second statement follows then immediately from (20) and (21). That is, for all $i \in \mathbb{N}$,

$$\frac{1}{2c_D} < \frac{\mu(B(x_i,r_i)\cap U)}{\mu(B(x_i,r_i))} \leq \frac{1}{2}. \tag{22}$$

It remains to show the last statement of the theorem, which corresponds to the Boxing inequality. Let $i \in \mathbb{N}$. Since μ is a Borel outer measure, the rightmost inequality in (22) implies

$$\begin{aligned} \mu(B(x_i,r_i)\cap U) &= 2\mu(B(x_i,r_i)\cap U) - \mu(B(x_i,r_i)\cap U) \\ &\leq \mu(B(x_i,r_i)) - \mu(U \cap B(x_i,r_i)) \\ &= \mu(B(x_i,r_i)\setminus U). \end{aligned}$$

In particular,

$$\min \{\mu (B (x_i, r_i) \cap U), \mu (B (x_i, r_i) \setminus U)\} = \mu (B (x_i, r_i) \cap U).$$

Then the relative isoperimetric inequality, see Lemma 4.1, applied to U and $B(x_i, r_i)$ yields

$$\mu (B (x_i, r_i) \cap U) \leq c_P r_i P (U, B (x_i, \tau r_i)).$$

Using that and the leftmost inequality in (22) leads to

$$\begin{aligned} \frac{\mu (B (x_i, r_i))}{r_i} &< \frac{2c_D \mu (B (x_i, r_i) \cap U)}{r_i} \\ &\leq \frac{2c_D c_P r_i P (U, B (x_i, \tau r_i))}{r_i} \\ &= cP (U, B (x_i, \tau r_i)), \end{aligned} \tag{23}$$

where c is a constant which depends only on c_D and c_P.

We now note that by the doubling condition, we have

$$\frac{\mu (B (x_i, 5\tau r_i))}{5\tau r_i} \leq c \frac{\mu (B (x_i, r_i))}{r_i} \tag{24}$$

for some constant c only depending on c_D and τ. Let us show this fact. Consider $\ell \in \mathbb{N}$ such that $5\tau 2^{-\ell} \leq 1$. Then

$$\mu (B (x_i, 5\tau r_i)) \leq c_D^{\ell} \mu \left(B \left(x_i, 5\tau 2^{-\ell} r_i\right)\right) \leq c_D^{\ell} \mu (B (x_i, r_i))$$

and so

$$\frac{\mu (B (x_i, 5\tau r_i))}{5\tau r_i} \leq \frac{c_D^{\ell} \mu (B (x_i, r_i))}{r_i}.$$

Here we used the fact that $\tau \geq 1$. Finally, by (24), (23) and the fact that $P(U, \cdot)$ is a Borel outer measure, we have

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{\mu (B (x_i, 5\tau r_i))}{5\tau r_i} &\leq c \sum_{i=1}^{\infty} \frac{c_D^{\ell} \mu (B (x_i, r_i))}{r_i} \\ &\leq c \sum_{i=1}^{\infty} P (U, B (x_i, \tau r_i)) \\ &= cP \left(U, \cup_{i=1}^{\infty} B (x_i, \tau r_i)\right) \\ &\leq cP (U, X), \end{aligned}$$

where c only depends on c_D, c_P and τ. Note that to use countable additivity of $P(U, \cdot)$ it is crucial that the collection $\{B (x_i, \tau r_i)\}_{i \in \mathbb{N}}$ is pairwise disjoint. □

5 BV-capacity and Hausdorff Content

In this chapter, we focus on the main result of this thesis, the equivalence between BV-capacity and Hausdorff content of codimension 1 for compact sets.

First, we define Hausdorff content of codimension 1 in the metric measure space setting. We obtain an important upper bound for the mentioned Hausdorff content in terms of perimeter. This is due to the boxing inequality. After that we work on a two-sided estimate which shows the pretended equivalence. Most of the results obtained here are inspired by [17, pp. 409–411].

5.1 Hausdorff content of codimension 1

For every $x \in X$ and $0 \leq r < \infty$, let

$$h\left(B\left(x,r\right)\right) := \frac{\mu\left(B\left(x,r\right)\right)}{r} \quad \text{if } r > 0$$

and $h\left(\emptyset\right) := 0$. We take the convention that $B(x,0) = \emptyset$ for all $x \in X$.

The way to define Hausdorff content of codimension 1 is by applying a Carathéodory construction to h. We do it as in [17, p. 408].

Definition 5.1 (Hausdorff content of codimension 1)**.** *Let $E \subset X$. We define the Hausdorff content of codimension 1 of E as*

$$\mathcal{H}^h_\infty\left(E\right) := \inf \sum_{i=1}^{\infty} h\left(B\left(x_i,r_i\right)\right),$$

where the infimum is taken over all collections of balls $\{B(x_i,r_i)\}_{i\in\mathbb{N}}$ such that

$$E \subset \bigcup_{i=1}^{\infty} B\left(x_i,r_i\right),$$

with $x_i \in X$ and $0 \leq r_i < \infty$, for $i \in \mathbb{N}$. Since we only discuss codimension 1, we will also refer to Hausdorff content of codimension 1 by simply Hausdorff content.

Remark 5.2. $\mathcal{H}^h_\infty\left(K\right) < \infty$ *for all compact sets $K \subset X$.*

Reason. For every compact subset K of X, there exist $x \in X$ and $0 < r < \infty$ such that $K \subset B\left(x,r\right)$. Then,

$$\mathcal{H}^h_\infty\left(K\right) \leq h\left(B\left(x,r\right)\right) = \frac{\mu\left(B\left(x,r\right)\right)}{r} < \infty.$$

■

Hausdorff content is an outer measure by Theorem 2.33 in [5, pp. 92–93]. In this work, since we deal solely with Hausdorff content of compact sets, we only need the first two properties of outer measurability.

Lemma 5.3. *Hausdorff content satisfies*

1. $\mathcal{H}^h_\infty(\emptyset) = 0$;

2. *(monotonicity)* $\mathcal{H}^h_\infty(E_1) \leq \mathcal{H}^h_\infty(E_2)$ *for all* $E_1, E_2 \subset X$ *such that* $E_1 \subset E_2$.

Proof. It is clear that $\mathcal{H}^h_\infty(\emptyset) = 0$ by definition of h. Monotonicity also comes trivially. Indeed, every covering by balls of E_2 is also a covering by balls of E_1 whenever $E_1 \subset E_2 \subset X$. Then, $\mathcal{H}^h_\infty(E_1) \leq \mathcal{H}^h_\infty(E_2)$ follows from the definition of Hausdorff content as an infimum over coverings by balls. □

We present a relation between the measure-theoretical notion of Hausdorff content and the geometric notion of perimeter. This is due to the boxing inequality presented in the previous section 4.2.

Proposition 5.4. *There exists a constant* $c = c(c_D, c_P, \tau)$ *such that*

$$\mathcal{H}^h_\infty(U) \leq cP(U, X),$$

for all bounded open sets $U \subset X$ *of finite perimeter.*

Proof. Let $U \subset X$ be a bounded open set of finite perimeter. The boxing inequality, see Theorem 4.2, provides a covering for U consisting of a countable collection of balls $\{B(x_i, 5\tau r_i)\}_{i\in\mathbb{N}}$, with $x_i \in X$ and $0 < r_i < \infty$, such that

$$\sum_{i=1}^{\infty} \frac{\mu(B(x_i, 5\tau r_i))}{5\tau r_i} \leq cP(U, X),$$

for a constant $c = c(c_D, c_P, \tau)$. Then, by definition of $\mathcal{H}^h_\infty(U)$,

$$\mathcal{H}^h_\infty(U) \leq \sum_{i=1}^{\infty} \frac{\mu(B(x_i, 5\tau r_i))}{5\tau r_i} \leq cP(U, X).$$

□

We can extend the previous result to all sets $E \subset X$.

Corollary 5.5. *There exists a constant* $c = c(c_D, c_P, \tau)$ *such that*

$$\mathcal{H}^h_\infty(E) \leq c \inf\{P(U, X) \mid E \subset U,\ U \subset X \text{ is bounded and open}\} \tag{25}$$

for every set $E \subset X$.

Proof. Let $E \subset X$. If the right-hand of (25) is infinite, the result follows trivially. Otherwise, let $\varepsilon > 0$ and $U \subset X$ be a bounded open set such that $E \subset U$ and

$$P(U, X) < \inf\{P(U, X) \mid E \subset U,\ U \subset X \text{ is bounded and open}\} + \varepsilon.$$

In particular, $P(U, X) < \infty$ and so Proposition 5.4 implies

$$\mathcal{H}^h_\infty(U) \leq cP(U, X),$$

where c only depends on c_D, c_P and τ. By monotonicity of Hausdorff content, see Lemma 5.3, we have

$$\begin{aligned}\mathcal{H}^h_\infty(E) &\leq \mathcal{H}^h_\infty(U) \leq cP(U, X)\\ &\leq c(\inf\{P(U, X) \mid E \subset U,\ U \subset X \text{ is bounded and open}\} + \varepsilon).\end{aligned}$$

By taking the limit when $\varepsilon \to 0$, we obtain the pretended result. □

5.2 Equivalence between BV-capacity and Hausdorff content

A way to show that BV-capacity and Hausdorff are equivalent is to provide an upper bound and a lower bound for the Hausdorff content in terms of BV-capacity.

We start by considering a lower bound.

Lemma 5.6. *Let $K \subset X$ be a compact set. Then,*

$$\mathrm{cap}_{\mathrm{BV}}(K) \leq c_D \mathcal{H}^h_\infty (K) .$$

Proof. Let $K \subset X$ be a compact set. Let $\varepsilon > 0$ and consider $x_i \in X$, $0 \leq r_i < \infty$, with $i \in \mathbb{N}$, such that

$$K \subset \bigcup_{i=1}^{\infty} B(x_i, r_i)$$

and

$$\sum_{i=1}^{\infty} h(B(x_i, r_i)) < \mathcal{H}^h_\infty (K) + \varepsilon. \tag{26}$$

Since K is compact, there exists $n \in \mathbb{N}$ such that

$$K \subset \bigcup_{i=1}^{n} B(x_i, r_i) .$$

By finite subadditivity of BV-capacity, see Corollary 3.15, we get

$$\mathrm{cap}_{\mathrm{BV}}(K) \leq \mathrm{cap}_{\mathrm{BV}}\left(\bigcup_{i=1}^{n} B(x_i, r_i) \right) \leq \sum_{i=1}^{n} \mathrm{cap}_{\mathrm{BV}}(B(x_i, r_i)) .$$

Let $i \in \mathbb{N}$ and define

$$u_i(x) = \max\left\{ 0, 1 - \frac{1}{r_i} d(x, B(x_i, r_i)) \right\}$$

for all $x \in X$. We proceed to show that $u_i \in \mathcal{A}_{\mathrm{BV}}(B(x_i, r_i))$. By construction,

$$0 \leq u_i \leq 1.$$

For all $x \in B(x_i, r_i)$, it holds that

$$d(x, B(x_i, r_i)) = 0,$$

which implies that $u_i(x) = 1$. Hence $u_1 = 1$ in $B(x_i, r_i)$, which is a trivial neighbourhood of $B(x_i, r_i)$. By Example 2.10 with $\varepsilon = r_i$,

$$g_i = \frac{1}{r_i} \chi_{B(x_i, 2r_i)}$$

is a 1-weak upper gradient of u_i. Hence

$$\|Du_i\|(X) \le \int_X g_i \, d\mu < \infty$$

and so $u_i \in \mathrm{BV}(X)$. This shows that $u_i \in \mathcal{A}_{\mathrm{BV}}(B(x_i, r_i))$. Thus using the doubling condition and () yields

$$\begin{aligned}
\mathrm{cap}_{\mathrm{BV}}(K) &\le \sum_{i=1}^{n} \mathrm{cap}_{\mathrm{BV}}(B(x_i, r_i)) \le \sum_{i=1}^{\infty} \mathrm{cap}_{\mathrm{BV}}(B(x_i, r_i)) \\
&\le \sum_{i=1}^{\infty} \|Du_i\|(X) \le \sum_{i=1}^{\infty} \int_X g_i \, d\mu \\
&= \sum_{i=1}^{\infty} \frac{\mu(B(x_i, 2r_i))}{r_i} \le c_D \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} \\
&= c_D \sum_{i=1}^{\infty} h(B(x_i, r_i)) < c_D \left(\mathcal{H}_\infty^h(K) + \varepsilon \right).
\end{aligned}$$

By taking the limit when $\varepsilon \to 0$, we get

$$\mathrm{cap}_{\mathrm{BV}}(K) \le c_D \mathcal{H}_\infty^h(K).$$

□

Remark 5.7. *In the previous lemma, if instead of* BV*-capacity we consider the modified* BV*-capacity defined in section 3.4, the previous lemma can be extended to every subset $E \subset X$.*

Reason. First, if $\mathcal{H}_\infty^h(E) = \infty$, the inequality

$$\mathrm{cap}_{\mathrm{BV}}^*(E) \le c_D \mathcal{H}_\infty^h(E)$$

follows trivially. Otherwise, we can consider the same admissible functions u_i, $i \in \mathbb{N}$, since they are bounded and compactly supported and hence in $L^{1^*}(X)$. Note that the only reason why compactness of K is important in Lemma 5.6 is to insure we obtain

$$\mathrm{cap}_{\mathrm{BV}}(K) \le \sum_{i=1}^{\infty} \mathrm{cap}_{\mathrm{BV}}(B(x_i, r_i))$$

using only finite subadditivity of BV*-capacity. Recall that, in opposition to* BV*-capacity, the modified* BV*-capacity is countable subadditive. Thus we get*

$$\mathrm{cap}_{\mathrm{BV}}^*(E) \le \sum_{i=1}^{\infty} \mathrm{cap}_{\mathrm{BV}}^*(B(x_i, r_i)).$$

The rest of the proof follows similarly. ■

We state and present a proof of our final result.

Theorem 5.8. *There exists a constant $c = c(c_D, c_P, \tau)$ such that*

$$\frac{1}{c}\,\mathrm{cap}_{\mathrm{BV}}(K) \le \mathcal{H}^h_\infty(K) \le c\,\mathrm{cap}_{\mathrm{BV}}(K), \tag{27}$$

for all compact sets $K \subset X$.

Proof. Let $K \subset X$ be a compact set. By Lemma 5.6,

$$\mathrm{cap}_{\mathrm{BV}}(K) \le c_D \mathcal{H}^h_\infty(K),$$

that is,

$$\frac{1}{c_D}\,\mathrm{cap}_{\mathrm{BV}}(K) \le \mathcal{H}^h_\infty(K). \tag{28}$$

We proceed to show the rightmost inequality of (27). Let $\varepsilon > 0$ and $u \in \mathcal{A}_{\mathrm{BV}}(K) \cap \mathrm{Lip}_c(X)$ such that

$$\|Du\|(X) < \mathrm{cap}_{\mathrm{BV}}(K) + \varepsilon. \tag{29}$$

Recall such a function can be taken by Lemma 3.13. By Corollary 3.10,

$$\|Du\|(X) = \int_0^1 P(\{u > \lambda\}, X)\, d\lambda$$

and there exists λ_0 with $0 < \lambda_0 < 1$ such that

$$P(\{u > \lambda_0\}, X) \le \|Du\|(X).$$

Hence, by (29), we get

$$P(\{u > \lambda_0\}, X) < \mathrm{cap}_{\mathrm{BV}}(K) + \varepsilon.$$

Denote $\{u > \lambda_0\}$ by U. Note that U is an open set since it is the pre-image of the open interval (λ_0, ∞) by the Lipschitz continuous function u. Moreover, $U \subset \operatorname{supp} u$ and so U is bounded. Set U is also a neighbourhood of K since for all $x \in K$, $u(x) = 1 > \lambda_0$. Thus, Corollary 5.5 yields

$$\mathcal{H}^h_\infty(K) \le cP(U, X) < c\left(\mathrm{cap}_{\mathrm{BV}}(K) + \varepsilon\right),$$

where c only depends on c_D, c_P and τ. By letting $\varepsilon \longrightarrow 0$, one gets

$$\mathcal{H}^h_\infty(K) \le c\,\mathrm{cap}_{\mathrm{BV}}(K).$$

□

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