Quantum To Classical Transition

Joel Järnefelt

School of Science

Bachelor's thesis
Espoo 11.11.2024

Supervisor

Jani-Petri Martikainen

Advisor

Jani-Petri Martikainen

Aalto University
School of Science

Aalto University
School of Science

Aalto University, P.O. BOX 11000, 00076 AALTO
www.aalto.fi

Abstract of the bachelor's thesis

Author Joel Järnefelt

Title Quantum to Classical transition

Degree programme Bachelor's Programme in Science and Technology

Major Engineering physics **Code of major** SCI3028

Teacher in charge Jani-Petri Martikainen

Advisor Jani-Petri Martikainen

Date 11.11.2024 **Number of pages** 36 **Language** English

Abstract

In this bachelor's thesis we study the effect of decoherence on quantum properties. In this thesis, the nature of decoherence is explored along with its mathematical representation. With this knowledge, implications of decoherence are pointed out, namely the disappearance of certain quantum properties on entangled systems. This is shown in a simple way first and then a more general framework in the form of the master equation is presented.

As the effect of decoherence in quantum mechanics is noticeable, a section on interpretations of quantum mechanics has also been added. In this section we discuss the relative state interpretation, or more colloquially the many words interpretation, and its attempt of addressing some of the foundational issues in quantum mechanics, such as the measurement postulate. To this end, we discuss about a particular scientific paper on a relative state interpretation concept called envariance and how this relates to decoherence.

In the final section we discuss the experimental aspect of decoherence research. Two experiments are presented: interferometry on C_{70} molecules and decoherence in SQUIDs.

C_{70} interferometry is very similar to double slit experiment as it seeks to find interference patterns between non-decohered C_{70} plane waves. Since the wavelength of these particles is very small due to their large mass, ordinary double slit experiments do not work and more sophisticated experimental setups are shown. From results of this experiment we can see that the interactions between air molecules and C_{70} particles as well as the temperature of C_{70} molecule are directly correlated with the intensity of the interference pattern and thus the quantum nature of these particles.

The final experiment uses SQUIDs, which are essentially very small superconducting current loops. Small is a relative term though since a clockwise or a counterclockwise current rotation corresponds to a movement of 10^9 electrons. In this experiment the current is put in a quantum superposition, where the current goes clockwise and counterclockwise at the same time and then its decoherence is studied with Ramsey interferometry. With this technique we can track the decay of the superposition at every step in the course of ~ 50 nanoseconds.

Keywords Quantum physics, Decoherence, Open quantum systems

Contents

1 Introduction

Quantum and classical mechanics seem incompatible. In our classical world, we know that physical systems have objective, unchanging and measurement-independent qualities. Quantum mechanics presents a model which is almost completely in opposition of this idea. The most fundamental objects in quantum mechanics are completely abstract in their nature and the connection they have to our world can be very complicated to understand. Therefore, it would be very easy just to call classical and quantum mechanics two completely different theories of nature with no connection whatsoever. However, this should not be the case. As quantum physics determines the mechanics of the building blocks of reality, we should expect that every single phenomenon in nature is ultimately quantum mechanical. There should not be anything intrinsically classical about the universe since all macroscopic structures are made up from fundamental constituents, governed by quantum mechanics.

This thesis will present a bridge between these two views on nature. More precisely, classical properties of nature will be derived from laws of quantum mechanics that do not assume something inherently classical that exists outside of quantum mechanical description.

The main problem this thesis seeks to present a solution for is why any inherently quantum mechanical effects, such as interference or superpositions, do not seem to exist beyond a certain scale by only using normal quantum mechanics. When these results from quantum mechanics are understood, the thesis will briefly explore of how these results can inspire a particular interpretation the fundamental assumptions of basic quantum mechanics. After that, the thesis will switch subjects and present a derivation for an approximate model that accounts for the disappearance of quantum effects. Then, the final sections show two different experiments that utilize the understanding that is obtained in the preceding sections to create phenomena that seem to blur the line between classical and quantum worlds.

The discussion of quantum phenomena in this thesis is by no means exhaustive for several reasons. First, to completely understand the conjunction of quantum and classical worlds, one would need to have a model that would predict quantum mechanics and general relativity. Currently, this theory does not exist. However, there are some advanced theories that can quantize special relativity and almost all force fields of nature. For simplicity, we will not cover these either. Taking into consideration all of these points, we see that this thesis only shows the union of non relativistic unitary quantum mechanics and classical mechanics without gravity. This union can loosely be called decoherence.

2 Theoretical Background

Decoherence is a process in which a quantum system gets entangled with its environment and loses certain quantum properties. To understand what this means, it is necessary to understand what separates classical and quantum properties, what entanglement means and how to describe systems which have been entangled.

Entanglement means that the system can no be longer described with a state vector $|\psi\rangle$. Instead, only the system-environment pair is now describable with a state vector $|\Psi\rangle$. To see this, consider a pair of two state systems, described with symbols $|i\rangle_j$, where $i, j \in \{0, 1\}$. The whole system can now be found in four states $\{|0\rangle_0|0\rangle_1, |1\rangle_0|0\rangle_1, |0\rangle_0|1\rangle_1, |1\rangle_0|1\rangle_1\}$. If these two subsystems are not entangled, it means that, by definition, a state vector can be associated with each of them. If it is possible to associate a state vector with each subsystem, it means that the probability of measuring any state in any of the subsystems is independent of the of the other subsystems. This places some restrictions on the four states. Without the loss of generality, if the state vectors are $|\psi_0\rangle = \alpha|0\rangle_0 + \beta|1\rangle_0$ and $|\psi_1\rangle = \gamma|0\rangle_1 + \delta|1\rangle_1$, then the probability amplitudes for any non entangled state has to be of the form $\{\alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta\}$ for any $\{\alpha, \beta, \gamma, \delta\} \in \mathbb{C}^4$. These numbers, however, are not the only possible amplitudes one can give. Omitting normalization, consider the set $\{1, 0, 0, 1\}$. This is not a possible combination since $\alpha\gamma = \beta\delta = 1$, so none of the coefficients cannot be zero. Some of them have to be zero though, since there are the middle coefficients which are zero. The only explanation is that there can be states in the basis $\{|0\rangle_0|0\rangle_1, |1\rangle_0|0\rangle_1, |0\rangle_0|1\rangle_1, |1\rangle_0|1\rangle_1\}$ which cannot be expressed using two independent state vectors which describe the subsystems alone.

These kind of entangled systems can exist in nature. For example consider the electron orbital. It is the quantum state of an electron that is bound to an atomic nucleus. Electrons are fermions. In many cases, two electrons share almost exactly the same quantum state, the orbital, to spare energy. The Pauli exclusion principle states that no two fermions can have the same quantum state. This means that the spins of those two electrons must be opposed to each other, to make the quantum states orthogonal. Spin of an electron is a two state system so the formalism discussed in previous paragraph applies to this case. In this opposite spin situation the pair is described by a state vector with $\{0, 1, 1, 0\}$, which is an entangled state.

2.1 Density matrices

The information of this section is based on [1]. Although a state vector cannot be associated to a subsystem of an entangled state, it is possible to extract statistics from these subsystems using the density matrix/operator formalism. A density matrix of a quantum system is given by

$$\hat{\rho} = |\Psi\rangle\langle\Psi| \tag{1}$$

From this it is possible to attain expected values for observables $\hat{O}$ by using matrix product and the trace operation. The trace operation for an hermitian operator, i.e. any observable, $\hat{A}$, is defined as follows. First define an orthonormal basis $\{|i\rangle\}$ using the eigenvectors of $\hat{A}$ and calculate $\sum_i \langle i|\hat{A}|i\rangle$. These eigenvectors correspond to eigenstates with a given measurement result r_i. Now we can see that

$$\mathrm{Tr}(\hat{\rho}\hat{O}) = \sum_i \langle i|\hat{\rho}\hat{O}|i\rangle = \sum_i \langle i|\psi\rangle\langle\psi|\hat{O}|i\rangle = \sum_i r_i \langle i|\psi\rangle\langle\psi|i\rangle = \langle\hat{O}\rangle \tag{2}$$

What is useful with the density matrix formalism is when $\hat{\rho}$ does not describe a state vector. A density matrix which does this is called a mixed density matrix. This matrix is defined with a classical ensemble of quantum states given below

$$\hat{\rho}_m = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{3}$$

Here p_i denotes the classical probability of recovering a given quantum state $|\psi_i\rangle$. The matrix has this form since the operation given in Eq. (2) works for mixed density matrices as well. To show this, it is necessary to remember that the expected value for the ensemble is given by the weighted sum of all the expected values for the individual states. Below is a mathematical derivation which leans on steps recovered in Eq. (2). More precisely, the following identity was used $\langle\hat{O}\rangle_i = \sum_j r_j \langle j|\psi_i\rangle\langle\psi_i|j\rangle$.

$$\begin{aligned}\langle\hat{O}\rangle_{ensemble} &= \sum_i p_i \langle\hat{O}\rangle_i = \sum_{ij} p_i r_j \langle j|\psi_i\rangle\langle\psi_i|j\rangle = \sum_{ij} p_i \langle j|\psi_i\rangle\langle\psi_i|\hat{O}|j\rangle \\ &= \sum_j \langle j|\hat{\rho}_m\hat{O}|j\rangle = \mathrm{Tr}(\hat{\rho}_m\hat{O})\end{aligned} \tag{4}$$

One caveat is necessary here. Although one can get a mixed density matrix given by the procedure in Eq. (3), the same matrix can be obtained from other sources. This means that if one is confronted with a mixed density matrix, it cannot be ascertained from what ensemble it came from.

2.1.1 Metrics for density matrices

There are mathematical procedures of distinguishing between pure and mixed density matrices. One algorithm is known as the purity-metric. In this process the matrix is squared and the trace of the result is computed. The idea comes from the fact that a pure matrix is its own square $\hat{\rho}^2 = \hat{\rho}\hat{\rho} = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \hat{\rho}$ but for a mixed state this is not the case $\hat{\rho}^2 = \sum_{ij} p_i p_j |\psi_i\rangle\langle\psi_j|\langle\psi_i|\psi_j\rangle$. Since the equation Eq. (2) works for both mixed and pure matrices, by selecting $\hat{O} = \hat{I}$, we can see that the trace of the non squared matrix is always 1. However, the trace of the squared mixed matrix is then $\sum_i p_i^2$.

One other metric for measuring the purity of a density operator is to measure its entropy. Entropy in this context refers to information, meaning Shannon entropy. It is a measure of uncertainty of a distribution, which a mixed density matrix is. More precisely, it is the expected measure of information gain on measurement. Note that here measurement refers to recovering a particular quantum state from the mixed state distribution. The form of this information gain is in this context $g(p_i) = -\ln(p_i)$. Looking at this function we can see that this function goes to 0 when $p_i = 1$, which means that recovering a quantum state which had a probability of occurrence of 1 yields us no new information. If $p_i \to 0$ then $g \to \infty$, so more unlikely events give out much more information, in this case base-e bits or nats. The reason why this $g(p_i)$ is called information gain has its reasons in information theory, but the connection to normal physical entropy is clear via the statistical mechanics formula via Gibbs entropy formula $S = -\sum_i p_i \ln p_i$ [2].

To put this Shannon entropy in a quantum context, we recover the Von-Neumann entropy or the quantum entropy [1]. It is a function of the density operator and is defined as

$$S(\hat{\rho}) = -\mathrm{Tr}(\hat{\rho}\ln\hat{\rho}) \tag{5}$$

If the density operator is composed of orthogonal quantum states, then the matrix can be expressed in a basis where it is diagonal so we get that

$$-\mathrm{Tr}(\hat{\rho}\ln\hat{\rho}) = -\mathrm{Tr}\left(\begin{bmatrix} p_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & p_n \end{bmatrix}\begin{bmatrix} \ln(p_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ln(p_n) \end{bmatrix}\right) = \sum_i p_i \ln p_i = H(X) \tag{6}$$

Here $H(X)$ is the Shannon entropy. It can be shown that for a mixture of pure states, the quantum entropy obeys the following inequality [1].

$$S(\hat{\rho}) \leq H(X) \tag{7}$$

This can be understood because if the different quantum states are non orthogonal, they start to have overlap. This means that they resemble each other and this starts to "narrow down" the different possibilities thus making the entropy smaller.

2.1.2 Superpositions

Mixed density matrices are useful since with them one can obtain measurement statistics from quantum systems which cannot be expressed as state vectors. For example, a density matrix with a following expression, $\hat{\rho} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$ describes a quantum system that does not have quantum coherences between the states $|0\rangle$ and $|1\rangle$. It is classically in either one of the systems, with a probability $\frac{1}{2}$ of either one. This has observable consequences, when comparing a similar state which is in a superposition of $|0\rangle$ and $|1\rangle$. To see this, if the system was described with a state vector of $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, the probability of finding the state in either one is the the same in both expressions. However, this is only true if time evolution is not taken into account. The equation of motion for quantum systems described with a state vector is given by the Schrödinger's equation.

$$\hat{H}|\psi\rangle = i\frac{\partial}{\partial t}|\psi\rangle \tag{8}$$

As can bee seen from this equation, the time evolution of the state vector, and thus the system, depends on the Hamiltonian and crucially the system's state vector itself. This means that the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ in general evolves in time differently than either $|0\rangle$ or $|1\rangle$ since they are different vectors in state space. To elaborate this fact, consider a time independent Hamiltonian of the form $\hat{H} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. This kind of Hamiltonian can encountered, if one places an electron, or some other spin-$\frac{1}{2}$

particle in a homogeneous magnetic field that points in the y-direction. For time independent Hamiltonians, the solution to Eq. (8) has the following form

$$|\psi(t)\rangle = e^{i\hat{H}t}|\psi(0)\rangle \tag{9}$$

The given Hamiltonian has a property that $\hat{H}^2 = \hat{I}$. Thus, the exponential in Eq. (9) has the form

$$\begin{aligned} e^{i\hat{H}t} &= \hat{I} + i\hat{H}t + \frac{1}{2!}i^2\hat{I}t^2 + \frac{1}{3!}i^3\hat{H}t^3 ... \\ &= \begin{pmatrix} 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 ... + & -t + \frac{1}{3!}t^3 ... \\ t - \frac{1}{3!}t^3 ... & 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 ... \end{pmatrix} \\ &= \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \end{aligned} \tag{10}$$

If the system is described as the state vector given above and time is chosen such that $\cos(t) = \sin(t)$, the state vector will evolve to $|\psi(t)\rangle = |1\rangle$. However, if the system is in an ensemble of $|0\rangle$ and $|1\rangle$, so it is described by $\hat{\rho}$, a classical distribution, both of these possibilities will evolve into a superposition states, namely $|\psi_0(t)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|\psi_1(t)\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle)$, respectively. As both of these state have 50% chance of measuring either $|0\rangle$ or $|1\rangle$ and the the ensemble contains only these states, at the time $\cos(t) = \sin(t)$, the classical distribution has evolved such that one will find 50% $|0\rangle$ and 50% $|1\rangle$ states. The state vector, however, is only found to be in the state $|1\rangle$ at that time.

The above example illustrates that superpositions in quantum mechanics cannot be interpreted as mere classical ensembles. There are other phenomena as well which show this. Probably the most famous one is the double slit experiment, where in some parts of the distribution of the electron the probability density goes to zero where it couldn't go if the distribution was described by a sum of two single slit distributions. This means that there are different kinds of ways interference effects of superpositions manifest themselves. The example above is a specimen of a temporal interference effects whereas double slit demonstrates spatial ones.

2.2 Reduced density matrices

We will now discuss how to use these mixed density matrices to describe systems that are entangled. The source of this section is again [1]. The solution is to use so called reduced density matrices. These matrices work in a similar way compared to other density matrices. Namely, these reduced density matrices obey the following identity.

$$\langle \hat{O}_{\mathcal{A}} \rangle = \mathrm{Tr}_{\mathcal{A}}(\hat{\rho}_{\mathcal{A}}\hat{O}_{\mathcal{A}}) \tag{11}$$

In the equation Eq. (11), $\hat{\rho}_{\mathcal{A}}$ describes the reduced density matrix for system described by a Hilbert subspace $\mathcal{A}$. $\hat{O}_{\mathcal{A}}$ is the observable operator responsible for the same subspace. $\mathrm{Tr}_{\mathcal{A}}$ denotes the partial trace for $\mathcal{A}$. This section will go over these terms one by one.

First object for inspection is $\hat{O}_{\mathcal{A}}$. This is simply the mathematical symbol for a measurement done in one of the subsystems. Even though entangled systems cannot be described as two distinct subsystems in the state vector formalism, in practice separate measurements must be able to be done to them. Otherwise there could not be any subsystems to which one could partition the total system. As an example we could use the entangled electron positron pair. It is certainly feasible to make an experiment which only measures the spin of the positron and not the electron. This measurement might indeed influence the spin properties of the electron but the measurement event can exclude that data in principle. In mathematical formalism this physical principle emerges in the form of an abstract operation called the tensor product. The full meaning of this tensor product operation is not essential to understand. We will instead cover couple of its relevant properties. A Hilbert space which contains a state vector that completely describes a physical system can be decomposed to subspaces given that the physical system itself can be decomposed to physical subsystems. These Hilbert subspaces then combine together to form the original Hilbert space via the tensor product. Below is an example with three subsystems.

$$\mathcal{H}_{total} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}} \otimes \mathcal{H}_{\mathcal{C}} \tag{12}$$

Above, the three vector subspaces now describe three different physical subsystems. These subspaces work in the same way as in elementary quantum mechanics.

The state vector that now occupies $\mathcal{H}_{total}$ can be expressed in a basis that is also composed of tensor products

$$|\psi_{total}\rangle = \sum_{i\in\mathcal{A}, j\in\mathcal{B}, k\in\mathcal{C}} c_{ijk} |i\rangle \otimes |j\rangle \otimes |k\rangle \tag{13}$$

In Eq. (13) all of the state vectors that are in the tensor product are basis vectors in the subspaces they belong into. If c_{ijk} can be expressed in terms of a product of complex amplitudes from each subspace, then $|\psi_{total}\rangle$ is not entangled. Otherwise it is.

Operators work intuitively in this tensor product formalism, i.e. one can decompose a total operator to ones only acting in some chosen subspaces and then combine these together using the tensor product very similarly as in Eq. (12).

These properties can explain $\hat{O}_{\mathcal{A}}$. We can mathematically express it as

$$\hat{O}_{total} = \hat{O}_{\mathcal{A}} \otimes \hat{O}_{\mathcal{E}} \tag{14}$$

Here $\hat{O}_{\mathcal{E}}$ is the operator in with Hilbert space that is not associated with the subsystem of interest. When exploring quantum decoherence, the split is usually done in terms of subsystem of interest, denoted by $\mathcal{A}$, and the environment, denoted by $\mathcal{E}$.

Using the properties discussed in equations above, we can see how $\hat{O}_{total}$ acts on $|\psi_{total}\rangle$.

$$\hat{O}_{total}|\psi_{total}\rangle = (\hat{O}_{\mathcal{A}} \otimes \hat{O}_{\mathcal{E}}) \sum_{i\in\mathcal{A},j\in\mathcal{E}} c_{ij}|i\rangle \otimes |j\rangle = \sum_{i\in\mathcal{A},j\in\mathcal{E}} c_{ij}\hat{O}_{\mathcal{A}}|i\rangle \otimes \hat{O}_{\mathcal{E}}|j\rangle \tag{15}$$

Having understood what $\hat{O}_{\mathcal{A}}$ is, we will now cover the partial trace and reduced density matrix. The explanation of the partial trace and reduced density matrix can be most easily accomplished by discovering them by decomposing a general state vector in the same way as equation Eq. (13) does, and then calculating an observable-average for an observable given in equation Eq. (14), where $\hat{O}_{\mathcal{E}} = \hat{I}_{\mathcal{E}}$. This is a mathematical way of marking an event in which a measurement is done to a subsystem but not to the environment. Below is a state vector with a desired decomposition.

$$|\Psi\rangle = \sum_i c_i|\psi_i\rangle|\phi_i\rangle \tag{16}$$

Here $|\psi_i\rangle$ refers to a particular quantum state in subsystem and $|\phi_i\rangle$ environment. For this example, the subsystem basis $\{|\psi_i\rangle\}$ is orthonormal in its own subspace. The tensor product symbol is omitted in Eq. (16). In this inspection we will form a density matrix from the state vector from Eq. (16) and calculate an average from an observable that is in the form that was stated above, i.e $\hat{O} = \hat{O}_{\mathcal{A}} \otimes \hat{I}_{\mathcal{E}}$. The eigenstates of this operator will be $|i\rangle|j\rangle$ in this notation. For increased clarity we remind that this mathematical operation signifies an experiment where one calculates average of some observable from a subsystem which may be entangled with its environment.

$$\begin{aligned}
\langle\hat{O}\rangle = \mathrm{Tr}(\hat{\rho}\hat{O}) &= \sum_{ij}\langle j|\langle i|\hat{\rho}\hat{O}|i\rangle|j\rangle \\
&= \sum_{ij}\langle j|\langle i|\hat{\rho}(\hat{O}_{\mathcal{A}} \otimes \hat{I}_{\mathcal{E}})|i\rangle|j\rangle \\
&= \sum_{ij}\langle j|\langle i|\hat{\rho}\hat{O}_{\mathcal{A}}|i\rangle\hat{I}_{\mathcal{E}}|j\rangle \\
&= \sum_{i}\langle i|\sum_{j}\langle j|\hat{\rho}|j\rangle\hat{O}_{\mathcal{A}}|i\rangle \\
&= \sum_{i}\langle i|(\mathrm{Tr}_{\mathcal{E}}\hat{\rho})\hat{O}_{\mathcal{A}}|i\rangle \\
&= \mathrm{Tr}_{\mathcal{A}}(\hat{\rho}_{\mathcal{A}}\hat{O}_{\mathcal{A}})
\end{aligned} \tag{17}$$

As can be seen, the partial trace is a normal trace operation but restricted to a particular subspace. The reduced density matrix then is the density matrix which has been partially traced over the environment, e.g. by all of the degrees of freedom not in the subsystem of interest. As can be seen from Eq. (17), one can extract measurement statistics from an entangled subsystem if partial trace and reduced matrices are used.

2.2.1 Form of reduced density matrices

A natural continuation of this study is to ascertain the mathematical expression of $\mathrm{Tr}_{\mathcal{E}}\rho = \hat{\rho}_{\mathcal{A}}$. It will be derived from Eq. (16). From it we can see that the total density matrix has a form of

$$\hat{\rho} = |\Psi\rangle\langle\Psi| = \sum_{ij} c_i(c_j)^* |\psi_i\rangle\langle\psi_j| \otimes |\phi_i\rangle\langle\phi_j| \tag{18}$$

As equation Eq. (17) yields us a formula for obtaining reduced density matrices, we will now use it to obtain $\hat{\rho}_{\mathcal{A}}$ with the assumption that for all k, $|\phi_k\rangle = \sum_m d_k^m |m_k\rangle$.

$$\begin{aligned}
\hat{\rho}_{\mathcal{A}} &= \sum_j \langle j|(\sum_{ik} c_i(c_k)^* |\psi_i\rangle\langle\psi_k| \otimes |\phi_i\rangle\langle\phi_k|)|j\rangle \\
&= \sum_j \langle j|(\sum_{ik} c_i(c_k)^* |\psi_i\rangle\langle\psi_k| \otimes \sum_{ml} d_i^{(m)} d_k^{*(l)} |m_i\rangle\langle m_k|)|j\rangle \\
&= \sum_{ik} c_i(c_k)^* |\psi_i\rangle\langle\psi_k| \sum_{jml} d_i^{(m)} d_k^{*(l)} \delta_{jm}\delta_{jl} \\
&= \sum_{ik} c_i(c_k)^* |\psi_i\rangle\langle\psi_k| \sum_j d_i^{(j)} d_k^{*(j)} \\
&= \sum_{ik} c_i(c_k)^* |\psi_i\rangle\langle\psi_k| \langle\phi_i|\phi_k\rangle
\end{aligned} \tag{19}$$

As can be seen, the degree of presence of off diagonal elements in the $\{|\psi_i\rangle\}$ basis is determined by the overlap of $\langle\phi_i|\phi_k\rangle$. If $\langle\phi_i|\phi_k\rangle = \delta_{ik}$, the reduced density matrix is diagonal. This means that it does not describe a quantum state that can be represented as a state vector. This is because, like in the pure and mixed density matrices, the reduced density matrix trace is one, which can be seen by substituting $\hat{O}_{\mathcal{A}} = \hat{I}_{\mathcal{A}}$ in Eq. (17) or Eq. (11). If that is the case, and there are at least two distinct ψ states, then the values for those states in the diagonal have to be less than one. That means that the sum of squares, or the purity metric, is always less than one. Same applies to entropy as the probability of occurrence of different states is less than one so $g_i = -\ln(\hat{\rho}_{ii}) > 0$ and therefore $S > 0$.

2.3 Reduced density matrices and decoherence

Having the proper mathematical machinery in place, we can understand how classical mechanics and quantum mechanics join together. In general, there are three different ways that quantum and classical mechanics separate from each other.

1. The absence of superpositions in classical systems.
2. The preferred basis problem. In classical physics, eigenstates of position, not of momentum or energy, are always measured.
3. The stochastic and non-stochastic nature of quantum and classical physics, respectively.

2.3.1 Ideal measurement

To understand quantum mechanics itself can explain these seeming differences, let us first understand what an ideal measurement would mean in quantum mechanical context. This is necessary since the measurement event seems to be the inflection point between these two worlds. After all, superpositions and nonstochasticity are never directly observable.

The way quantum mechanics is brought to the measurement event in a way that includes the observer is to use so called pointer states [3]. These states represent the state vector of the environment, or macroscopic measurement device in the tensor product formalism. These pointer states will then evolve to encode some information about subsystem onto themselves, since that is what measurement means. More formal explanation is as follows. Given a subsystem in a quantum state $|s_i\rangle$ and the pointer state in a "ready" state $|p_r\rangle$ prior to measurement, then in an ideal measurement, the pointer state evolves in time to a quantum state $|p_i\rangle$ without changing the state $|s_i\rangle$. This means that the information regarding the subsystem has now been coded in the $|p\rangle$-states. If the quantum system is in a superposition of different state, then the whole quantum system will after measurement go in a entangled state given below.

$$|\Psi\rangle = \sum_i c_i |s_i\rangle |p_i\rangle \tag{20}$$

This is because the Schrödinger equation given in Eq. (8) is linear. Thus, if all of the time evolutions $|s_i\rangle|p_r\rangle \to |s_i\rangle|p_i\rangle$ are valid, then their linear combinations work as well. It can now be seen that entanglement has been created just by observing superposition states.

The argument below will now show how decoherence influences interference effects between different $|s_i\rangle$ states and ultimately why classical systems do not exhibit superpositions. From the definition of density matrices we can see that for a generic basis vector $|b\rangle$ for $|\Psi\rangle$ it holds that,

$$\langle b|\hat{\rho}|b\rangle = \langle b|\Psi\rangle\langle\Psi|b\rangle = \Pr(|b\rangle) \tag{21}$$

This vector $|b\rangle$ is the same Hilbert space as the state vector itself. However, if one is interested in the probability of occurrence of a state $|\alpha\rangle$ which exists in a particular subspace of the total Hilbert space, such that for any $|b\rangle$ we can say that $|b\rangle = |\alpha\rangle|\beta\rangle$, then all of the possible $|\beta\rangle$ states must be summed over.

$$\Pr(|\alpha\rangle) = \sum_\beta \langle\beta|\langle\alpha|\hat{\rho}|\alpha\rangle|\beta\rangle = \langle\alpha|\sum_\beta\langle\beta|\hat{\rho}|\beta\rangle|\alpha\rangle = \langle\alpha|\hat{\rho}_\beta|\alpha\rangle \tag{22}$$

By using a reduced density matrix, the probability of occurrence of a state can be obtained in the same way as in a normal density matrix, see Eq. (21). This works as long as the correct reduced density matrix, as there are many ways to reduce a density matrix.

To put this knowledge in use, a reduced density matrix for the basis $\{|s_i\rangle\}$ will be formed, using the equation Eq. (20). Then, the probability of occurrence of a general state $|e\rangle$ calculate will be calculated in the subspace spanned by $\{|s_i\rangle\}$.

$$\Pr(|e\rangle) = \langle e|\hat{\rho}_{s_i}|e\rangle = \sum_i |c_i|^2 \langle e|s_i\rangle\langle s_i|e\rangle + \sum_{i\neq j} c_i c_j^* \langle e|s_i\rangle\langle s_j|e\rangle\langle p_i|p_j\rangle \tag{23}$$

We can see from Eq. (23) that the first sum is the classical probability of recovering the state $|e\rangle$. This is because it is the sum of probabilities of recovering the state $|e\rangle$ in a given $|s_i\rangle$ weighted with the probability of recovering $|s_i\rangle$. On the other hand, the second term is a function of different states mixed together and therefore must be the contribution of quantum interference. As can be seen, it is modulated with the inner product term between different pointer states. If different environment measurement outcomes coincide with state vectors that have little overlap, meaning that the environment or measurement device is able to distinguish what state it is observing, the interference dies off.

This the reason why classically behaving objects are not found in superpositions. They are always interacting with their environment in such a way that some of the environmental degrees of freedom that interact with them "escape" i.e. leave the system of interest. This escaping means, that to model them, a reduced density operator is needed for which equation Eq. (23) holds. For these macroscopic systems, the leaving different leaving environment states $|p_i\rangle$ are intuitively clearly orthogonal. The reader can think of how the light scatters off of their hand when it is rotated in some way. It changes very dramatically even for extremely slight modifications in macroscopic or mesoscopic range.

This derivation is only partially complete, and the problem here lies with the point 3 made in the beginning of this subsection. The problem is due to the fact that in order to use equations Eq. (21) and onward, Born's rule, and the collapse postulate need to be observed. Therefore these two phenomena are not explained within the decoherence framework. This problem is called the measurement problem and it has to be rectified at an interpretational level. This will be discussed in the next subsection that discusses decoherence and the relative state interpretation.

2.3.2 Preferred bases

Before discussing the interpretational problems of quantum mechanics we will illustrate one important property of these pointer states. By using them, we will have an answer to second important problem with classical and quantum mechanics, that being the preferred basis problem, meaning we can know why state vectors lose their coherences in some bases but not in others [4]. Some examples in nature of this are why electrons have well defined energy levels but not momentum and position or why macroscopic objects are precisely position eigenstates. The reason arises from the particular interaction the quantum system has with its environment. For example, consider a quantum state with the following form

$$|\phi\rangle = c_0|s_0\rangle - \sum_{i\neq 0} c_i|s_i\rangle \tag{24}$$

Using this quantum state we create a new superposition of the form

$$
\begin{aligned}
|\Phi\rangle = \xi(\bar{c})(|\psi\rangle + |\phi\rangle) &= \xi(\bar{c})(\sum_i c_i|s_i\rangle + |c_0|s_0\rangle - \sum_{i\neq 0} c_i|s_i\rangle) \\
&= 2\xi(\bar{c})c_0|s_0\rangle = |s_0\rangle
\end{aligned}
\tag{25}
$$

Above, $\xi(\bar{c})$ is a normalization constant that depends on the list of c-values, $\bar{c}$. Using the interactions from before we get

$$
|\Phi\rangle|p_r\rangle = |s_0\rangle|p_r\rangle \rightarrow |s_0\rangle|p_0\rangle = |\Phi\rangle|p_0\rangle \tag{26}
$$

As can be seen, this equation is separable, which means that in the context of equation Eq. (23) that all $\langle p_i|p_j\rangle$ terms are one since all vectors $|p_i\rangle$ are the same, namely $|p_0\rangle$. This means that all of the interference effects between $|\psi\rangle$ and $|\phi\rangle$ are fully present. So in the presence of interactions in which the environment is not able to differentiate between different quantum states, in this case $|\psi\rangle$ and $|\phi\rangle$, superpositions of different states do not get decohered.

This means that if the time evolutions which the composite system undergoes are only of form $|s_0\rangle|p_r\rangle \rightarrow |s_0\rangle|p_0\rangle$, then quantum states that can form superpositions that produce different $|s_i\rangle$ states are able to persist in those superpositions. This means that the environment is not able to directly detect the superposition.

In practice, however, one is not given the different evolutions used in, for example, in Eq. (26). Instead, what is known is the total Hamiltonian $\hat{H}$. It is decomposable to three parts, such that $\hat{H} = \hat{H}_S + \hat{H}_E + \hat{H}_I$, where the subscripts refer to to the system, environment and the interaction between them, respectively. This Hamiltonian is always possible to express as $\hat{H} = \hat{S} \otimes \hat{E}$ where $\hat{S}$ and $\hat{E}$ again differentiate between system and the environment. The time evolution operator generated by $\hat{H}$ changes as well to accommodate the system environment split as $\exp(i\hat{S} \otimes \hat{E}t) = \exp(i\hat{S}t) \otimes \exp(i\hat{E}t)$. In this context, the pointer states are those which do not become entangled with the environment under the evolution governed by $\hat{H}$. This statement is expressed mathematically in equation Eq. (27).

$$
\exp(i\hat{S}t)|s_i\rangle\exp(i\hat{E}t)|E_0\rangle = |s_i\rangle\exp(iS_it)\exp(i\hat{E}t)|E_0\rangle = |s_i\rangle|E_i(t)\rangle \tag{27}
$$

What can be seen from equation Eq. (27) is that $|s_i\rangle$ has to be an eigenstate of $\hat{S}$. If this is not the case, $|s_i\rangle$ will have to be decomposed to vectors that are eigenstates of $\hat{S}$. In general this decomposition produces a linear combination of vectors with different eigenvalues, which combine with $|E_i\rangle$ to create different environment vectors for each eigenvector of $\hat{S}$ in the linear combination. This causes the system get entangled with its environment and thereby destroying the coherences.

Another way to put the eigenvalue demand on pointer states is to use the so called commutability criterion. In this criterion, one defines pointer observables which have the form

$$
\hat{O}_p = \sum_i o_i|s_i\rangle\langle s_i|. \tag{28}
$$

These pointer observables refer to any measurements that could be done to the pointer states and these o_i are the measurement results. If all $|s_i\rangle$ are eigenstates of $\hat{H}$, then $[\hat{O}, \hat{H}] = 0$, which is the commutability criterion.

In many macroscopic applications we can approximate that $\hat{H} \approx \hat{H}_{int}$ [5]. This means that the eigenstates of the interaction are the eigenstates of the system. Basically all interactions larger than the atomic nucleus are either electromagnetic or gravitational, which have potentials of $\hat{V} \propto \frac{1}{\hat{x}}$. These commute with the position operator so that means that we only see position eigenstates in normal life.

In quantum mechanical applications $\hat{H}$ is largely described by $\hat{H}_{\mathcal{S}}$. The eigenstates of this operator are the usual energy eigenstates found in atomic orbitals for example.

Quantum Brownian motion is an example in which both of these operators play an important role [5]. Then the quantum system will be compromise of normal particles and energy eigenstates.

2.4 Relative states

In the previous section, we mentioned that decoherence does not solve the famous measurement problem or the differences between the stochastic and non-stochastic natures of quantum and classical descriptions of reality. In this section, we will show how the so called relative state interpretation can potentially shed some light on this issue. We note here that this is just one potential interpretation of non-relativistic quantum mechanics and therefore it is not proven that this is the correct way to view quantum mechanics. However, this interpretation does couple well with the decoherence based framework and hence a chapter will be devoted to it.

The relative state interpretation is sometimes called the many world's interpretation or Everett's model [6]. In this model there is no state vector collapse, only a so-called branching. This means that the observer, and the entire rest of the universe get tangled up in the superposition. The state vector always evolves unitarily and measurement results of a subsystem are to be interpreted relative to other subsystems in the whole state vector, hence the name relative states. To illustrate this point, suppose we have a two state system in a superposition $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, and an observer $|O\rangle$ which evolves to $|O_0\rangle$ or $|O_1\rangle$ depending on the result, then the resultant state vector will be just $|\Psi\rangle = \alpha|0\rangle|O_0\rangle + \beta|1\rangle|O_1\rangle$. As can be seen, there are two observers now, each corresponding to a different measurement outcome. Also each outcome can see just one result, which matches our experience.

A common problem attributed to this approach is the preferred basis problem. If for each possible outcome we get a separate observer/universe, then the amount of terms in the expansion of the state vectors becomes really important. From elementary quantum mechanics we know that some observables do not commute, so an eigenstate of a particular observable is a superposition of a different observable. This means then, that if we have a case where a state vector is in an eigenstate of one observable, and the universe measures that, there is only one outcome. If on the other hand the universe measures a noncommutable observable, there is at least two outcomes. So how many outcomes there are then, one or many? The results obtained from the previous section help to correct this issue. If we use a more complete form of the state vector given in this chapter, we should include the environment. This means that the state vector is actually of the form $|\Psi\rangle = \alpha|0\rangle|O_0\rangle|E_0\rangle + \beta|1\rangle|O_1\rangle|E_1\rangle$. Now, if $|E_0\rangle$ and $|E_1\rangle$ have little to no overlap, the quantum coherences only manifest themselves at this subsystem-observer-environment level. At the local subsystem-observer level, where there is no state vector description if $\langle E_0|E_1\rangle \approx 0$, quantum coherence does not reveal itself in the measurement statistics given by the reduced density matrix. Since all actual observers are macroscopic and therefore strongly coupled to the environment, this orthogonality condition holds in our experience. Even though the overlap between different environment states is never actually zero and therefore it is always possible to see some interference in principle, the overlap is so close to it that it does not make difference in any practical application.

2.4.1 Probabilities

However, an important but subtle point was overlooked with this fix to the preferred basis problem. The fix obviously relies on decoherence. Decoherence uses reduced density matrices to obtain the measurement statistics via the formula of $\langle \hat{O}_{\mathcal{A}} \rangle = \mathrm{Tr}_{\mathcal{E}}(\hat{\rho}_{\mathcal{A}} \hat{O}_{\mathcal{A}})$. This formula is obtained by using the Born's rule which is a property of the collapse postulate. In the beginning we stated that the relative state interpretation is a no-collapse model, so decoherence cannot be used directly. In this light, we have not solved the preferred basis problem unless there is a way to derive the Born's rule without resorting to a collapse postulate.

There have been many attempts to derive the Born probabilities straight from unitary quantum mechanics. Many of them have been shown to be circular or insufficient, see papers [7; 4; 8]. However, there is one proof made by Zurek [9] which has shown promise. This derivation uses envariance, or environment assisted invariance.

In order to understand how this envariance works, we first need to understand how this paper bu Zurek defines probabilities. Probabilities on a general level are not directly defined, but a scenario with equal probabilities is. This can then be used to define general case. The idea is that if a system has distinct degrees of freedoms that can be interchanged without changing the description of the system, that description describes perfect ignorance and therefore each of the degrees of freedom are equally likely, therefore equiprobable. An example from a classical world is called for. Suppose there are two playing cards, one being hearts and one spades. The red card gets designated with letter A and black with B. Suppose, that the observer for this system does not know to which of colour A and B refers to. From this observer's ignorant point of view, A and B designation can be interchanged without changing anything. The observer can therefore say that the probability that A is paired with the red card is the same as it with the black card. It turns out that certain entangled quantum systems exhibit a symmetry that very much resembles this case, which is then used to motivate the association of probabilities with the square modulus of the coefficients of the state vector.

2.4.2 Envariance

Now that we understand probabilities, we can move on to the envariance part. Envariance is defined to be a class of symmetries. Precisely, if one has a tensor product state vector

$$|\psi\rangle = \sum_i a_i |s_i\rangle |e_i\rangle \tag{29}$$

The $\{|s_i\rangle\}$ and $\{|e_i\rangle\}$ span their own Hilbert subspaces. This vector is defined to be envariant with respect to an unitary operation $\hat{U} = \hat{u} \otimes \hat{I}$ if and only if there exists another operation $\hat{U}' = \hat{I} \otimes \hat{u}'$ such that $\hat{U}\hat{U}' = \hat{I}$. Physically this means that one does an operation in a system A which is entangled to system B, that operation can completely be undone just by operating on system B.

When our mathematical framework is in order, we need to connect the mathematical object $|\Psi\rangle$ to a physical system or systems that it describes. We do not need to fully specify the connection between state vectors and the physical systems they represent, only a few properties that are relevant in deriving Born's rule. In the paper presented in this thesis, the connection is made with three assumptions. In the list below, the state vector is assumed to have the form of Eq. (29), with $|s_i\rangle$ describing subsystem and $|e_i\rangle$ environment.

1. All actions that change the state of the subsystem are represented by unitary transformations that are done in its subspace. This means that transformations of the form $\hat{u} = \hat{I}_s \otimes \hat{u}_e$ are incapable of changing the subsystem.

2. The state of the subsystem is completely determined by a state vector in its Hilbert space.

3. If there is a larger entangled state vector which also describes the environment, then this state vector completely describes the subsystem.

With these axioms, Born's rule can be obtained. As was stated earlier, we will first determine the equiprobability case, meaning we prove that a state vector of the form of Eq. (30) has equally probable outcomes.

$$|\psi\rangle = a \sum_i |s_i\rangle |e_i\rangle \tag{30}$$

The rough proof is very simple. This kind of state vector is envariant with respect to any transformation with the form given below

$$\hat{u}_{ij} = |s_i\rangle\langle s_j| + |s_j\rangle\langle s_i| \tag{31}$$

This is true because a countertransformation with a following form undoes the transformation.

$$\hat{u}'_{ij} = |e_i\rangle\langle e_j| + |e_j\rangle\langle e_i| \tag{32}$$

There could have been arbitrary phase factors in Eq. (31) and Eq. (32), but for simplicity they were left out. We can see that these transformations correspond to swapping operations identical to those presented in the probability subsection. Applying both of these swapping transformations left the total state vector to its original state, and thus did not affect the state of the subsystem according to fact 3. However, this return transformation was done with an operation that, by fact 1, did not affect the subsystem at all. This must mean that the first swapping transformation could not have affected the subsystem either. Consequently, swapping states in maximally entangled states that have equal coefficients does not result in any measurable change. This was the ignorance definition of equiprobability that was given in the probability subsection.

This equiprobability proof can be generalized quite easily to rational but unequal probabilities. This is good enough since rational numbers are a dense set, so there is no measurable need to go any further. The proof of this claim does not provide any new insights to this issue at hand and will therefore be omitted from this thesis.

Envariance is actually such a powerful tool that it can be used to bypass decoherence entirely and be used to derive pointer states completely all on its own. As we have already shown how decoherence explains pointer states, it will be redundant to show it using envariance only. What is essential to know is that pointer states can be known straight from unitary quantum mechanics and necessarily do not need decoherence.

2.4.3 Conclusion

Having derived probabilities from unitary quantum mechanics, one could say that we have a complete description of classicality from the state vector level without resorting to odd postulates such as the collapse postulate. One must recognize however, that it is not proven that the relative states interpretational framework is the correct way to interpret quantum mechanics. This is currently an open issue. Still, using decoherence and envariance one could claim that this is the interpretation which still works and offers the least amount of postulates. We also remind the reader that the proof was done in context of nonrelativistic quantum theory which is not a complete theory of nature.

2.5 The master equation

Very useful way of modeling decoherence is to use the so called master equations. They are a way of modeling the evolution of the subsystem in a way that does not need the evolution of the environment to work. This is useful since in the study of open quantum systems, the environment not known. This is so because if one would want fully describe the environment, then the model would have to include the entire universe. As this equation is so useful in decoherence research, in this chapter we will derive this equation of motion. A detailed analysis of master equations can be found at Ref. [10] We will also briefly cover one decoherence model with the master equation and show how it can be used to extract useful results.

The first step for deriving master equation is to make a Schrödinger-like equation for the reduced density matrices of the form

$$\frac{d}{dt}\hat{\rho}_{\mathcal{S}}(t) = \hat{\mathcal{L}}[\hat{\rho}_{\mathcal{S}}(t)]. \tag{33}$$

As can be seen from equation Eq. (33), the equation has a few nice properties. First of all, it satisfies the property of not depending on the time evolution of the environment. Another nice property is that it is local in time, so on the right side, there are no time arguments that differ from the left hand side. Obtaining this equation using the principles of quantum mechanics is not possible exactly but one can arrive at it using certain approximations that will become apparent as we derive the form of Eq. (33).

To begin, we need to familiarize ourselves with the interaction picture. In this model we decompose the entire Hamiltonian to two parts as

$$\hat{H} = \hat{H}_0 + \hat{V}. \tag{34}$$

Here the term $\hat{V}$ is the so called interaction term and will correspond in our case to $\hat{H}_{int}$. Using Eq. (2) we know that the following formula works.

$$\begin{aligned} \langle \hat{A}(t) \rangle &= \mathrm{Tr}(\hat{\rho}(t)\hat{A}(t)) \\ &= \mathrm{Tr}(\exp(i\hat{H}_0 t)\hat{\rho}(t)\exp(-i\hat{H}_0 t)\exp(i\hat{H}_0 t)\hat{A}(t)\exp(-i\hat{H}_0 t)) \\ &= \mathrm{Tr}(\exp(-i\hat{H}_0 t)\exp(i\hat{H}_0 t)\hat{\rho}(t)\hat{A}(t)) \\ &= \mathrm{Tr}(\hat{\rho}(t)\hat{A}(t)) \end{aligned} \tag{35}$$

In the equation above, we used the trace cyclic invariance. From this equation we can see that the following unitary transformations do not change the expected value formula.

$$\begin{aligned} \hat{\rho}(t) \to \hat{\rho}^{(I)}(t) &= \exp(i\hat{H}_0 t)\hat{\rho}(t)\exp(-i\hat{H}_0 t) \\ &= \exp(i\hat{H}_0 t)\exp(-i\hat{H} t)\hat{\rho}(0)\exp(i\hat{H} t)\exp(-i\hat{H}_0 t) \end{aligned} \tag{36}$$

$$\hat{A}(t) \to \hat{A}^{(I)}(t) = \exp(i\hat{H}_0 t)\hat{A}(t)\exp(-i\hat{H}_0 t) \tag{37}$$

These transformations move us into the interaction picture in which the derivation for Eq. (33) is mostly done in.

Now we need to find the equation of motion for the interaction density operator. Using the product rule for derivatives and Schrödinger equation we know that the equation of motion for normal density matrices is

$$\frac{d}{dt}\hat{\rho}(t) = -i[\hat{H}, \hat{\rho}(t)]. \tag{38}$$

Now, using the product rule again with the definition of $\hat{\rho}^{I}$, equation Eq. (35), and properties of matrix exponentials we get that

$$\begin{aligned} \frac{d}{dt}\hat{\rho}^{(I)}(t) &= i[\hat{H}_0, \hat{\rho}^{(I)}(t)] - i\exp(i\hat{H}_0 t)[\hat{H}, \hat{\rho}(t)]\exp(-i\hat{H}_0 t) \\ &= i[\hat{H}_0, \hat{\rho}^{(I)}(t)] - i[\hat{H}_0, \hat{\rho}^{(I)}(t)] - i[\hat{V}^{(I)}(t), \hat{\rho}^{(I)}(t)] \\ &= -i[\hat{V}^{(I)}(t), \hat{\rho}^{(I)}(t)]. \end{aligned} \tag{39}$$

Now we can see the reason for the the unitary transformations Eq. (36) and Eq. (37) are useful. Their evolution is solely generated by the interaction.

The thesis will now use the equation Eq. (39) to get an equation that looks like Eq. (33), by first integrating it.

$$\hat{\rho}^{(I)}(t) = \hat{\rho}^{(I)}(0) - i\int_0^t dt'[\hat{H}_{int}(t'), \hat{\rho}^{(I)}(t')] \tag{40}$$

Then we put this new equation back into Eq. (39)

$$\begin{aligned} \frac{d}{dt}\hat{\rho}^{(I)}(t) &= -i[\hat{H}_{int}(t), \hat{\rho}^{(I)}(0) - \int_0^t dt'[\hat{H}_{int}(t'), \hat{\rho}^{(I)}(t')]] \\ &= -i[\hat{H}_{int}(t), \hat{\rho}^{(I)}(0)] - \int_0^t dt'[\hat{H}_{int}(t), [\hat{H}_{int}(t'), \hat{\rho}^{(I)}(t')]] \end{aligned} \tag{41}$$

As we are trying to derive an expression for $\hat{\rho}_{\mathcal{S}}$, we need to define a new quantity, the reduced interaction density matrix, $\hat{\rho}_{\mathcal{S}}^{(I)} = \mathrm{Tr}_{\mathcal{E}}(\hat{\rho}^{(I)})$.

$$\frac{d}{dt}\hat{\rho}_{\mathcal{S}}^{(I)}(t) = -\int_0^t dt'\mathrm{Tr}_{\mathcal{E}}([\hat{H}_{int}(t), [\hat{H}_{int}(t'), \hat{\rho}^{(I)}(t')]]) \tag{42}$$

Here the first term in Eq. (41) was put away due to the fact we are working in a thermal equilibrium.

This paragraph will perform the approximations in this derivation. In the first scheme, the Born approximation, we assume that the environment is much larger than the subsystem and in thermal equilibrium. What this means is that the environment stays time invariant. What we will also assume is that the subsystem is minimally entangled to the environment and stays that way. That is to say the subsystem is maximally decohered at all times. These statements mathematically mean that $\hat{\rho}(t) = \hat{\rho}_{\mathcal{S}}(t) \otimes \hat{\rho}_{\mathcal{E}}$. This non-entanglement condition is observed at all times in the interaction picture as well. This is so, since the unitary transformations for the interaction picture, defined in Eq. (36) and Eq. (37), are only functions of $\hat{H}_0$. These do not, by definition, contain any interaction between the environment and subsystem that could in principle create system-environment entanglements. As the environment does not evolve this means that in the interaction picture, we have that $\hat{\rho}^I(t) = \hat{\rho}_{\mathcal{S}}^{(I)}(t) \otimes \hat{\rho}_{\mathcal{E}}$.

The second approximation, called the Markov approximation, is one in which we introduce the interaction Hamiltonian fully, namely $\hat{H}_{int} = \sum_\alpha \hat{S}_\alpha \otimes \hat{E}_\alpha$, and open up the formula Eq. (42) with this information and the Born condition.

$$\frac{d}{dt}\hat{\rho}_{\mathcal{S}}^{(I)}(t) = -\int_0^t dt'\mathrm{Tr}_{\mathcal{E}}\sum_{\alpha\beta}([\hat{S}(t)_\alpha \otimes \hat{E}(t)_\alpha, [\hat{S}(t')_\beta \otimes \hat{E}(t')_\beta, \hat{\rho}_{\mathcal{S}}^{(I)}(t') \otimes \hat{\rho}_{\mathcal{E}}]]) \tag{43}$$

In this equation the terms $\hat{S}$ and $\hat{E}$ operators are the interaction picture versions. Explicitly they are of the form

$$\begin{aligned}
\hat{H}_{int} &= \exp(i\hat{H}_0 t)(\sum_\alpha \hat{S}_\alpha \otimes \hat{E}_\alpha)\exp(-i\hat{H}_0 t) \\
&= \exp(i\hat{H}_{\mathcal{S}} t)\exp(i\hat{H}_{\mathcal{E}} t)(\sum_\alpha \hat{S}_\alpha \otimes \hat{E}_\alpha)\exp(-i\hat{H}_{\mathcal{S}} t)\exp(-i\hat{H}_{\mathcal{E}} t) \\
&= \sum_\alpha \exp(i\hat{H}_{\mathcal{S}} t)\hat{S}_\alpha \exp(-i\hat{H}_{\mathcal{S}} t) \otimes \exp(i\hat{H}_{\mathcal{E}} t)\hat{E}_\alpha \exp(i\hat{H}_{\mathcal{E}} t) \\
&= \sum_\alpha \hat{S}_\alpha(t) \otimes \hat{E}_\alpha(t)
\end{aligned} \tag{44}$$

The decomposition on the second line does not violate the Baker-Campbell-Hausdorff theorem, since $[\hat{H}_{\mathcal{S}}, \hat{H}_{\mathcal{E}}] = 0$.

If one opens up the commutators in Eq. (44), and distributes the trace operators and has them only act on environment operators, the following expression is obtained.

$$\mathrm{Tr}(\hat{\mathfrak{g}}_{\alpha\beta})(\hat{\mathfrak{S}}_{\alpha\beta} - \hat{\mathfrak{T}}_{\alpha\beta}) - \mathrm{Tr}(\hat{\mathfrak{q}}_{\alpha\beta})(\hat{\mathfrak{Q}}_{\alpha\beta} - \hat{\mathfrak{K}}_{\alpha\beta}) \tag{45}$$

Below is a list of all of the Fraktur operators used in Eq. (45). They all have arguments of (t, t') so they are omitted in Eq. (45).

1. $\hat{\mathfrak{g}}_{\alpha\beta} = \hat{E}_\alpha(t)\hat{E}_\beta(t')\hat{\rho}_{\mathcal{E}}$
2. $\hat{\mathfrak{S}}_{\alpha\beta} = \hat{S}_\alpha(t)\hat{S}_\beta(t')\hat{\rho}_{\mathcal{S}}^{(I)}(t')$
3. $\hat{\mathfrak{T}}_{\alpha\beta} = \hat{S}_\beta(t')\hat{\rho}_{\mathcal{S}}^{(I)}(t')\hat{S}_\alpha(t)$
4. $\hat{\mathfrak{q}}_{\alpha\beta} = \hat{E}_\alpha(t)\hat{\rho}_{\mathcal{E}}\hat{E}_\beta(t')$
5. $\hat{\mathfrak{Q}}_{\alpha\beta} = \hat{\rho}_{\mathcal{S}}^{(I)}(t')\hat{S}_\beta(t')\hat{S}_\alpha(t)$
6. $\hat{\mathfrak{K}}_{\alpha\beta} = \hat{S}_\alpha(t)\hat{\rho}_{\mathcal{S}}^{(I)}(t')\hat{S}_\beta(t')$

As was supposed in the Born approximation, the environment is at thermal equilibrium, so $[\hat{H}_{\mathcal{E}}, \hat{\rho}_{\mathcal{E}}] = 0$. This means that

$$\begin{aligned}
\mathcal{C}_{\alpha\beta}(t, t') &= \mathrm{Tr}(\hat{\mathfrak{g}}_{\alpha\beta}) \\
&= \mathrm{Tr}(\exp(i\hat{H}_0 t)\hat{E}_\alpha \exp(-i\hat{H}_0 t)\exp(i\hat{H}_0 t')\hat{E}_\beta(t')\exp(-i\hat{H}_0 t')\hat{\rho}_{\mathcal{E}}) \\
&= \mathrm{Tr}(\exp(i\hat{H}_0(t - t'))\hat{E}_\alpha \exp(-i\hat{H}_0(t - t'))\hat{E}_\beta(t')\hat{\rho}_{\mathcal{E}}) \\
&= \mathcal{C}_{\alpha\beta}(t - t')
\end{aligned} \tag{46}$$

Using the result from Eq. (46) can see that $\mathrm{Tr}(\hat{\mathfrak{q}}_{\alpha\beta}) = \mathcal{C}_{\alpha\beta}(t' - t)$, because as an argument for the trace operation, which is permutation invariant, $\hat{\mathfrak{g}}_{\alpha\beta}(t, t') = \hat{\mathfrak{q}}_{\alpha\beta}(t', t)$.

Looking at the definition of $\mathcal{C}_{\alpha\beta}$, it can be seen that it is the autocorrelation term for the environment. Autocorrelation is a measure of how correlated two observables

$\hat{E}_\alpha$ and $\hat{E}_\beta$ are in different times, meaning how much information one gains of $\hat{E}_\beta$ at t' from knowing $\hat{E}_\alpha$ at t. In the Markov approximation, we approximate that the environment is very noisy, meaning that the correlation times decay at a much smaller timescales than any meaningful change in $\hat{\rho}_\mathcal{S}$. This means two things:

1. The argument t' in $\hat{\rho}_\mathcal{S}^{(I)}(t')$ can be replaced with t
2. t' can be integrated to ∞.

This means that we end up with an integral like this

$$\frac{d}{dt}\hat{\rho}_\mathcal{S}^{(I)}(t) = -\int_0^\infty d\tau \sum_{\alpha\beta} \mathcal{C}_{\alpha\beta}(\tau)(\hat{\mathfrak{S}}_{\alpha\beta} - \hat{\mathfrak{T}}_{\alpha\beta}) - \mathcal{C}_{\alpha\beta}(-\tau)(\hat{\mathfrak{Q}}_{\alpha\beta} - \hat{\mathfrak{K}}_{\alpha\beta}) \tag{47}$$

In this equation substitution $\tau = t - t'$ was used. This form of is now independent of t' as an argument for $\hat{\rho}_\mathcal{S}^{(I)}$, which makes this differential equation local in time, which was a desired property. Next, we transform the equation back to Schrödinger picture and do some simplifying notation.

Knowing the unitary transformations that define the interaction picture, we can do a countertransformation of the form

$$\hat{\rho}_\mathcal{S} = \exp(-i\hat{H}_\mathcal{S}t)\hat{\rho}_\mathcal{S}^{(I)}\exp(i\hat{H}_\mathcal{S}t). \tag{48}$$

If this equation is differentiated with respect to time we can get an equation of motion for the reduced density matrix using the interaction picture.

$$\frac{d}{dt}\hat{\rho}_\mathcal{S} = -i[\hat{H}_\mathcal{S}, \hat{\rho}_\mathcal{S}] + \exp(-i\hat{H}_\mathcal{S}t)\frac{d}{dt}\hat{\rho}_\mathcal{S}^{(I)}\exp(i\hat{H}_\mathcal{S}t) \tag{49}$$

To this expression one can insert the result for $\frac{d}{dt}\hat{\rho}_\mathcal{S}^{(I)}$. Then the matrix exponentials will travel into the differences, such as $(\hat{\mathfrak{S}}_{\alpha\beta} - \hat{\mathfrak{T}}_{\alpha\beta})$. Doing this, and expanding the definition of the Fraktur operators we get that

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_\mathcal{S} &= -i[\hat{H}_\mathcal{S}, \hat{\rho}_\mathcal{S}] + \exp(-i\hat{H}_\mathcal{S}t)\frac{d}{dt}\hat{\rho}_\mathcal{S}^{(I)}\exp(i\hat{H}_\mathcal{S}t) \\ &= -i[\hat{H}_\mathcal{S}, \hat{\rho}_\mathcal{S}] - \int_0^\infty d\tau \sum_{\alpha\beta} \mathcal{C}_{\alpha\beta}(\tau)[\hat{S}_\alpha, \hat{S}_\beta(-\tau)\hat{\rho}_\mathcal{S}] - \mathcal{C}_{\alpha\beta}(-\tau)[\hat{\rho}_\mathcal{S}\hat{S}_\beta(-\tau), \hat{S}_\alpha]\end{aligned} \tag{50}$$

As the integral runs from $\tau = 0$ to $\tau = \infty$, the time dependent operators can be integrated out to yield

$$\frac{d}{dt}\hat{\rho}_\mathcal{S} = -i[\hat{H}_\mathcal{S}, \hat{\rho}_\mathcal{S}] - \sum_\alpha [\hat{S}_\alpha, \hat{B}_\alpha\hat{\rho}_\mathcal{S}] - [\hat{\rho}_\mathcal{S}\hat{C}_\alpha, \hat{S}_\alpha] \tag{51}$$

Here the operators $\hat{B}$ and $\hat{C}$ are defined as

$$\hat{O}_\alpha = \int_0^\infty d\tau \sum_\beta \mathcal{C}_{\alpha\beta}(\pm\tau)\hat{S}_\beta(\tau). \tag{52}$$

The plus-case is operator $\hat{B}$. These results yield us the desired result for the time evolution of $\hat{\rho}_\mathcal{S}$. One could now build decoherence models using this equation. This will be done in the following subsection.

2.5.1 Master equation and decoherence models

Having derived the master equation, we can now use it to illustrate the effect on decoherence on open quantum systems. This section intends to briefly show the effect of interaction with the environment to the coherences of the system. More precisely, a model will be presented in which we can see the disappearance of spatial superpositions and other environmental effects. This section is more focused on the results of using master equation formalism. This means that the derivation will be kept brief and not all steps will be shown. A good source for the derivation of the master equation of quantum Brownian motion can be found at Ref. [11]

When modeling decoherence, one must make a model of the environment. Surprisingly, depending on the context, we can use only two such models to describe a wide range of physical situations. We can either model the environment as being a collection of non-interacting harmonic oscillators or two state systems. For more detail on this issue, especially for oscillator models, will refer the reader to Ref. [12].

In this section, we will choose the environment to be made up from harmonic oscillators and we will choose the subsystem to be a harmonic oscillator as well. This harmonic oscillator will then be linearly coupled to all of the environment oscillators.

In the literature, this model is usually called the quantum Brownian motion. One can understand this name, if one considers atoms bound to a lattice. When the particles interact with each other, they experience Brownian motion. Atoms near their equilibrium positions can be approximated to exist in a harmonic oscillators. This a very common approximation in many areas of physics. When near a potential minimum, one can very commonly make a quadratic approximation of the potential function. This justifies the choice of harmonic oscillators for both the subsystem and the environment. Linear coupling is done because it is the most simple choice to model.

With this knowledge, we will now write down the total Hamiltonian of the system.

$$\hat{H} = \hat{H}_{\mathcal{S}} + \hat{H}_{\mathcal{E}} + \hat{H}_{int} = \frac{\hat{P}^2}{2M} + \frac{M\Omega^2\hat{X}^2}{2} + \sum_i \left(\frac{\hat{p}_i^2}{2m_i} + \frac{m_i\omega_i^2\hat{x}_i^2}{2} \right) + \hat{X} \otimes \sum_i c_i\hat{x}_i \quad (53)$$

When we look at the master equation, we see that we first need to compute $\mathcal{C}(\tau)$. We will denote it as

$$\mathcal{C}(\tau) = \langle \hat{E}(\tau)\hat{E}\hat{\rho}_{\mathcal{E}} \rangle = \sum_{ij} c_i c_j \langle \hat{x}_i(\tau)\hat{x}_j\hat{\rho}_{\mathcal{E}} \rangle = \sum_i c_i^2 \langle \hat{x}_i(\tau)\hat{x}_i\hat{\rho}_{\mathcal{E}} \rangle \quad (54)$$

The final addition to Eq. (54) was motivated via the fact that the environmental oscillators are uncorrelated and the expected value of $\hat{x}$ of an oscillator is zero.

Next steps will be skipped due to brevity. In these steps one uses the creation and annihilation operators, the definition of interaction operators and the fact that the oscillators are in thermal equilibrium at temperature T. When these steps are done, we get that

$$\mathcal{C}(\tau)=\sum_i \frac{c_i^2}{2m_i\omega_i}\left(\coth\left(\frac{\omega_i}{2k_BT}\right)\cos(\omega_i\tau)-i\sin(\omega_i\tau)\right)=\nu(\tau)-i\eta(\tau) \tag{55}$$

Here $\nu(\tau)$ and $\eta(\tau)$ are called the noise and dissipation kernels, respectively. However, understanding the exact form $\mathcal{C}(\tau),\nu(\tau)$ and $\eta(\tau)$ is not important here. Essential to understand is that these quantities can be obtained. Then, using these quantities, we can obtain $\hat{B}$ and $\hat{C}$ from the equation Eq. (52), and from there the master equation of the system. When this is done, we will get a result shown below.

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_{\mathcal{S}}(t)=-i\left[\frac{\hat{P}^2}{2M}+\frac{M\Omega^2\hat{X}^2}{2}+\frac{M\tilde{\Omega}^2\hat{X}^2}{2},\hat{\rho}_{\mathcal{S}}(t)\right]-i\gamma[\hat{X},\{\hat{P},\hat{\rho}_{\mathcal{S}}(t)\}]\\ -D[\hat{X},[\hat{X},\hat{\rho}_{\mathcal{S}}(t)]]-f[\hat{X},[\hat{P},\hat{\rho}_{\mathcal{S}}(t)]]\end{aligned} \tag{56}$$

The $\{\}$ sign above denotes the anticommutator $\{\hat{A},\hat{B}\}=\hat{A}\hat{B}+\hat{B}\hat{A}$. The coefficients $\tilde{\Omega}^2$, γ, D and f are all derived from the kernels defined from Eq. (55). Furthermore, they all have a physical interpretation. In this chapter, we will focus on a few of them.

We will examine the coefficient D first. This will be responsible for the disappearance of spatial coherences. In order to see this effect, we need to transform the equation Eq. (56) to position representation. What this means is that we multiply the whole equation by $\langle X|$ from the left and $|X'\rangle$ from the right and expanding the operators in the equation in position basis. This will create the equation of motion for the coefficients of $\hat{\rho}$ in position space. While doing this, we will only consider the effect $D[\hat{X},[\hat{X},\hat{\rho}_{\mathcal{S}}(t)]]$ term. The reason for this is that this is the only term that does not contain the momentum operator. This means that for this term, no spatial derivatives will exist.

$$\begin{aligned}\frac{d}{dt}\rho(X,X',t)=-D\langle X|(\hat{X}\hat{X}\hat{\rho}-2\hat{X}\hat{\rho}\hat{X}+\hat{\rho}\hat{X}\hat{X})|X'\rangle\\ =-D(X^2\rho-2XX'\rho\ +\rho X'^2)=-D(X-X')\rho(X,X',t)\end{aligned} \tag{57}$$

This is a simple equation with an exponentially decaying solution of

$$\rho(X,X',t)=\exp(-D(X-X')t) \tag{58}$$

Here the master equation on harmonic oscillators has demonstrated the decay of spatial coherences, with the parameter D. As can be seen, the decay speed increases with larger spatial differences. Of course, this equation is not entirely the same as Eq. (56) in position representation. However, we can see that the effect the parameter D brings to this equation is that which dampens spatial coherences. Other terms have more complicated dynamics on ρ due to the presence of $\hat{P}$.

One can calculate other interesting results too, which we will cover very briefly. If one obtains the equation of motion for the expected value of momentum via the trace rule of density matrices, one gets that

$$\frac{d}{dt}\langle\hat{P}\rangle(t) = -M(\Omega^2 - \tilde{\Omega})^2)\langle\hat{X}\rangle(t) - 2\gamma\langle\hat{P}\rangle(t) \tag{59}$$

Examining only the effect of the second term, we see that the momentum scales down exponentially due to γ. This is why it is called the dissipation term.

A connection to normal Brownian motion can be made by calculating the long time limit of the variance of $\hat{X}$. As with classical physics [13], the quantum version scales too with linear time.

$$\Delta X^2 = \frac{D}{2M^2\gamma^2}t \tag{60}$$

As the results in this section have shown, models of decoherence can be made quite easily with the master equation. Furthermore, the environmental systems can usually be easily approximated to be simple systems which can be handled with pure quantum mechanical formalism.

Having derived some theoretical results from standard quantum mechanics, this thesis will now move to practical scenarios. In these scenarios, we will see the effect of decoherence in nature.

3 Experiments on decoherence

In this section, two experiments of decoherence will be covered. The first one part discusses a generalization of the double slit experiment which uses C_{70} molecules as test particles. In the second part, an experiment that blurs the line between microscopic and macroscopic world will be shown. In that experiment a current loop with billions of traveling electrons will be brought to superposition and its decoherence will be measured.

3.1 C_{70} interferometry

In 2002 a group led by Anton Zeilinger [14] in the university of Vienna performed an interferometry experiment using C_{70} molecules instead of electrons. The experiment showed that quantum mechanics is not contained only to the world of subatomic particles but in principle applies to everything. It also showed clearly how the quantum effects disappear on larger particles and why we do not usually see these weird effects on larger scales.

In practice, to perform the famous double slit experiment that was done to electrons is impractical with such massive particles. This is because the wavelength of the plane-wave of a massive particle is the de-Broglie wavelength

$$\lambda = \frac{h}{mv} \tag{61}$$

If one inserts the values for C_{70} molecule, the spatial delocalization is of the order of a few picometers. To have a slit that could detect the interferences, it should be around as slim as the delocalization scale of C_{70} molecule.

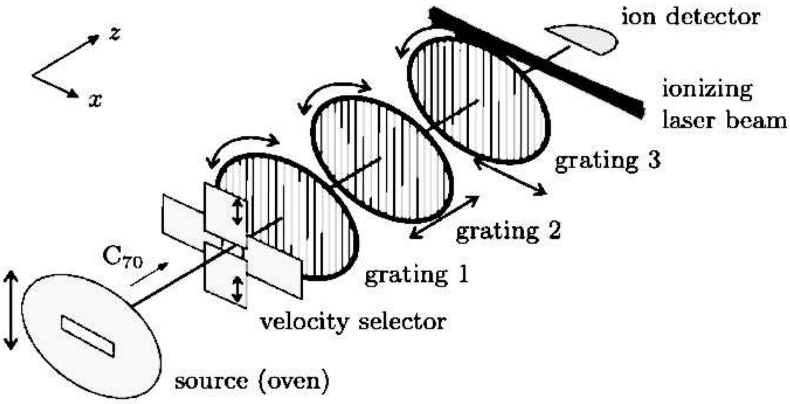

Figure 1: The experimental setup of C_{70} interferometry. Three gratings were used instead of one for practical reasons. First one was to induce spatial coherences in the C_{70}, second one was the main slit and final one masked for the Talbot pattern. A slightly modified source picture is available at [14].

The experiment used the so called Talbot-Lau effect. The experimental setup is shown in Figure 1. Consider a plane wave that is sent to a grating which has slits of width d. At an integer multiple of a so called Talbot length $L = \frac{d^2}{\lambda}$, the diffracted wave will mimic the shape of the grating, meaning the wave will have maxima where the slits were. This formula is very handy since the length has a d^2-term in it. This means that to have a Talbot length of the order of 1 meter, the slit only needs to have a width $\sqrt{\lambda}$ meters. As λ is a few picometers, we can see that this corresponds to a d-value of a micrometer. This is very doable, and figure 1 shows how experimenters implemented this setup. Figure 2 shows the Talbot pattern on the detector. When measuring the diffraction pattern, the experiments verified the interference pattern to be a Talbot pattern by varying λ by varying the particle speed. When λ was varied, that varied the Talbot length, so the place where the maxima were placed differed as a function of the C_{70} particle speed.

Decoherence comes into the picture when the pressure in the test chamber was varied as can be seen in Figure 3. What the experimenters found was that the visibility, which is a metric which measures the strength of the interference patterns, decreased exponentially with increasing air pressure. This is because the scattering events of C_{70} with air molecules are mediated by the electric field which measures position. This causes decoherence in position basis.

There was also another source of spatial decoherence, which is the emission of thermal photons, seen in Figure 4. This could be seen by heating the molecules to different temperatures. The higher the temperature, the larger the decoherence effects.

What this experiment shows in practice is that term "wave-particle duality" from quantum mechanics does apply to anything and macroscopic objects could just be called open quantum systems. The quantum effects are just easier to see with very small particles because they interact weakly with their environment and weakly emit thermal radiation. Researches have wanted to show this and there

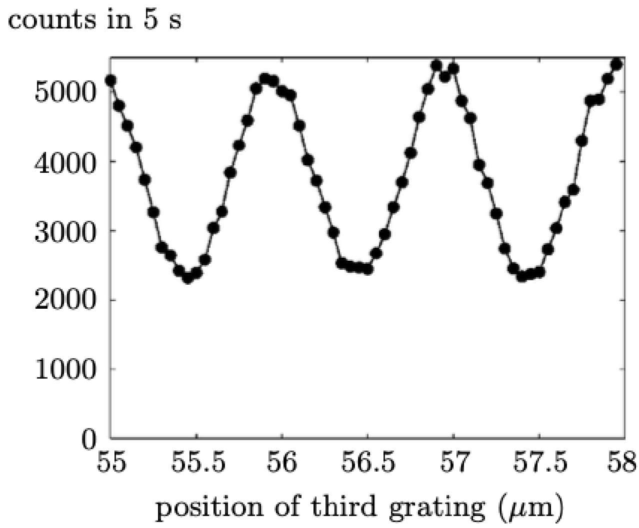

Figure 2: Results of the experiment and the Talbot pattern at the Talbot length L. This particular graph is a typical result for a 5 second run. The continuous graph is the theoretical prediction. Source: [14].

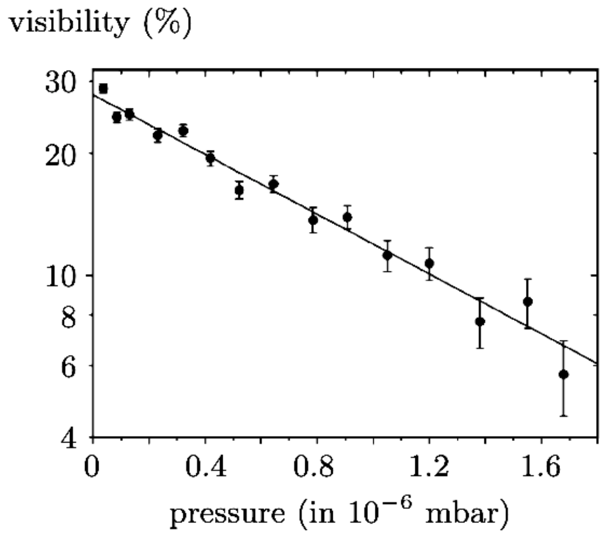

Figure 3: Dependence of air pressure in terms of localization or classicality of the C_{70} molecule, expressed in terms of visibility. Higher visibility means lower classicality. Visibility in this experiment is defined as the relative difference between the low and high points of the sinusoid seen in the previous figure. Source: [14]

have been considerations of making these interferometry experiments for even larger particles, even for small viruses [16]. For reasons that have been stated above, these experiments are hard to conduct in practice and thus haven't been done yet.

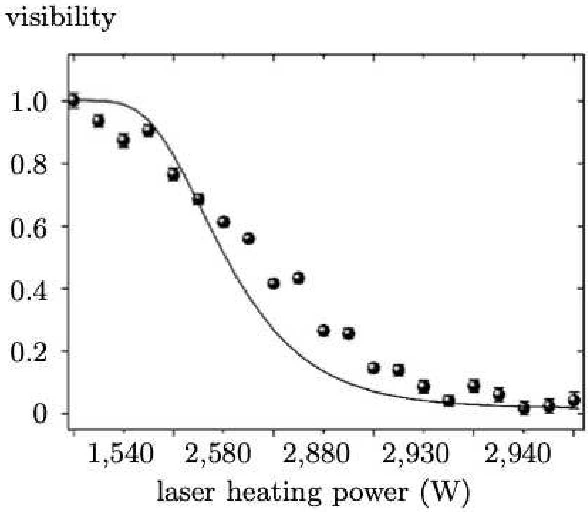

Figure 4: Same visibility metric being used in different temperature C_{70} molecules as defined via the laser heating power. Source: [15]

3.2 SQUIDs

SQUID is an abbreviation of the words superconducting quantum interference device. These objects are very interesting as they exhibit quantum coherence on macroscopic scales and thus provide a good ground to test decoherence. A through analysis on the physics of SQUIDs is found at Ref. [17].

The physics behind SQUIDs is that of Bose-Einstein condensation and superconductivity. In these effects, there are two phenomena at play. In these phenomena it is assumed that the medium is a metal with ions and an electron gas around it.

1. In a medium, the particles are normally confined to being particles by scattering events as was established in the C_{70} interferometry experiment. The spatial distribution length of the wave packets from scattering decoherence can be shown [18] to be of the size

$$\lambda_{th} \propto \frac{1}{\sqrt{T}} \tag{62}$$

This distribution size is also called the thermal wavelength. As can be seen from the from the equation above if one starts to cool the medium, the scattering events starts to decrease thus creating more spatial coherences. At some point the length will be of the order of the container.

2. When the container, that can superconduct in low temperatures, is cool enough, the free electrons in it start to form so called Cooper pairs. Cooper pairs resemble the Debye shielding in plasma physics but are a low-temperature quantum equivalent. The idea is that one electron attracts positive ions. On larger distances this conglomeration of positive ions then attract electrons near it. What happens is that one electron gets correlated with the electron that caused the disturbance. This correlation strength is very weak and this is the reason for the low temperature

requirement.

The electrons in Cooper pairs interact with each other and other Cooper pairs, if there are any. These interactions cause a gap in the allowed energy spectrum of the electrons, meaning that the exited states become quantized much like in atomic orbitals. This gap gets bigger with lower temperatures. What this gap does, is that it prohibits the electrons from feeling the thermal noise coming from the phonons, which are the quantized lattice vibrations. This allows the electrons to flow with no electrical resistance.

The final remark is that the Cooper pair is a pair consisting of two electrons it is a boson with integer spin. Bosons are described with state vectors that are symmetric with respect to particle exchange, meaning that they can share quantum states. This means that if the container is very cool, all of the Cooper pairs will be at the same ground quantum state meaning a single particle ground state will describe the entire macroscopic electron gas environment adequately.

Now we will turn to the actual SQUID. It is a superconducting loop with a small slit of resistor, called the Josephson junction, in it. A schematic picture of it is shown below in Figure 5. It has a current running around it and it is immersed in an external magnetic field that can be controlled.

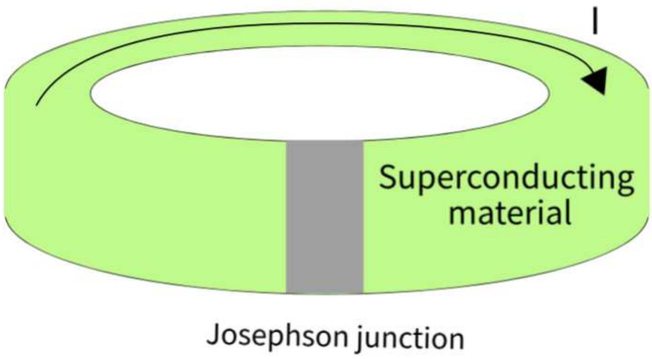

Figure 5: Schematic picture of SQUID. Source: Author

As the current is superconducting, the current density, which completely determines the actual current, is described with a complex function, a quantum mechanical state vector, Ψ. Any complex function on a loop can be described to be

$$\Psi(\theta) = \Psi_0(\theta)\exp(i\varphi(\theta)) \tag{63}$$

Here θ is real number and Ψ_0, φ are real functions. For simplicity of notation it is assumed that the current density on the loop does not have any r-dependence.

When one goes around the loop, Ψ has to stay the same. When this information is added to the facts on how Ψ gives the description of the current density and that $\nabla \times \vec{B} \propto \vec{J}$, then we have that the magnetic flux that penetrates the interior of the loop has to be $\Phi = n\Phi_0$, where $n \in \mathbb{N}$ and $\Phi_0 = h/2e$ is called the flux quantum. This allows $\varphi(\theta = 2\pi) = 2\pi n$. If one introduces the Josephson junction to this setup, it introduces a phase difference in φ, denoted $\Delta\gamma$, and we have a new equation

$$2\pi n = \Phi/\Phi_0 + \Delta\gamma \tag{64}$$

The flux quantum and the induced phase differences are fixed, so the quantum mechanical description of this whole current loop system is determined by Φ. One can derive it's equation of motion for $\Psi(\Phi, t)$, in natural units, to be

$$i\frac{\partial}{\partial t}\Psi = (-\frac{1}{2C}\frac{\partial^2}{\partial\Phi^2} + \frac{1}{2L}(\Phi - \Phi_{ext})^2 - \frac{1}{2\pi}I_c\Phi_0\mathrm{Cos}(2\pi\frac{\Phi}{\Phi_0}))\Psi \tag{65}$$

Here the C:s, L:s and I:s refer to the physical properties of the loop. Φ_{ext} is the external magnetic flux. Derivation for this equation can be seen in [11]. Equation (65) can be rearranged in a more instructive way

$$i\frac{\partial}{\partial t}\Psi = (\frac{\hat{P}_\Phi^2}{2C} + U(\Phi))\Psi \tag{66}$$

As can be seen, this looks very much like the usual Schrödinger equation for a particle with mass $2C$ in one dimension, which is denoted Φ. The wave function one recovers from solving this "particle" equation gives the current of the entire loop which can determine the motion of billions of Cooper pairs.

The potential, which ultimately determines the whole motion can be controlled with Φ_{ext}. Experimentally relevant situation is one where Φ_{ext} is chosen in such a way that $U(\Phi)$ takes on the form of double well potential. The lowest energy eigenstates of this system are localized near the minimums of the potential. This means that they are approximate eigenstates of Φ, meaning the magnetic flux and therefore the current is approximately well defined.

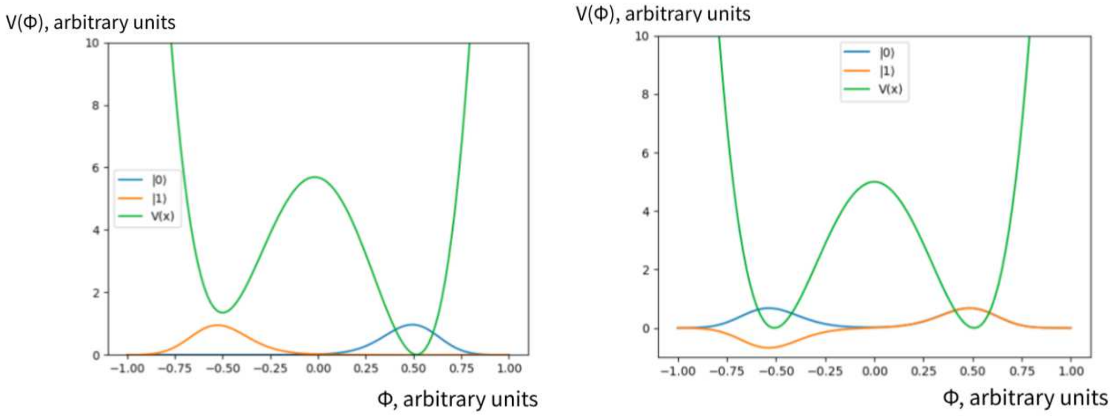

Figure 6: Numerical solutions to the first and second energy eigenstates with tilted and symmetrical double wells. Source: Author

This system already has some interesting quantum properties. Looking at the Figure 6 above, which reveals the two lowest eigenstates of the problem, one can determine that classically it is impossible to traverse between the two minimas if the system energy is low enough. Classically, of course, one would have a system which is always an eigenstate of Φ, so the magnetic field and current are well defined. Interesting things happen if one excites the quantum state localized near minimum 2 with a microwave pulse to an energy state which has energy that is higher than the lowest energy state in minimum 1 but which is still localized in minimum 2. Then it can happen that the microwave exited energy state can decay to a lower energy state in minimum 1, essentially creating quantum tunneling. The effect of this tunneling is the reversal of the polarity of the small magnet.

As one can see from the plots from above, the energy eigenstate solutions become more delocalized to the separate wells as Φ_{ext} is chosen such that the well becomes more symmetric. These symmetric states then represent a superposition of currents representing the motion of billions of Cooper pairs.

Next, we will discuss how to measure these symmetric states. This was done indirectly using so called Rabi oscillations [19]. The idea behind them is that if one applies a microwave pulse to the SQUID, which is in one of the energy eigenstates, it will start to oscillate sinusoidally between the ground and first exited states, with a characteristic frequency, called the Rabi frequency. As the state oscillates, it will also oscillate between the localized well states. This means that the current and the induced magnetic field from the current varies sinusoidally while the pulse is active. To be precise the average current $\langle I \rangle$ varies as the state evolves in time. This average current can be calculated by setting the state vector in the same point in Hilbert space and performing a projective measurement on the current basis thousands of times. This projective measurement was made when the pulse stopped. In the experiment the current was measured by a magnetometer which was in fact another SQUID. Researchers proved that the current oscillations were the product of Rabi oscillations by noticing that the frequency of the oscillations was linearly proportional to the amplitude of the microwave pulse, which is a property of the Rabi oscillation. In the next page, figure 7 shows the results when this was done.

These SQUIDs are useful for making ever larger quantum experiments as the properties of U and therefore the energy eigenstates is only dependent on the properties of the Josephson junction. Therefore the size of the are enclosed by the loop can be increased more easy than to scale the C_{70} interferometry experiment on more massive particles.

The final part we will see about these SQUIDs is their decoherence mechanics. The way decoherence was seen in SQUIDs was by using so called Ramsey interferometry. In Ramsey interferometry, the SQUID is initialized to state $|\psi\rangle = |0\rangle$. Then a microwave pulse is applied to this state, creating Rabi oscillations. The pulse is timed in such a way that it transforms $|\psi\rangle$ to $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$. Then this state is allowed to evolve freely. Without loss of generality and omitting global phases, we can say that $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\exp(i\phi[t])|1\rangle)$. Now we apply the same microwave pulse as the first to the state. This changes the state to $|\psi\rangle = \sin(\frac{\phi[t]}{2})|0\rangle - \cos(\frac{\phi[t]}{2})|1\rangle$. As we can see, the time the state evolved freely determines gives a trigonometric dependence to the probabilities of finding $|\psi\rangle$ in $|0\rangle$ or $|1\rangle$ and also to the classical current states, which are just superpositions of $|0\rangle$ and $|1\rangle$. Figure 7 shows this

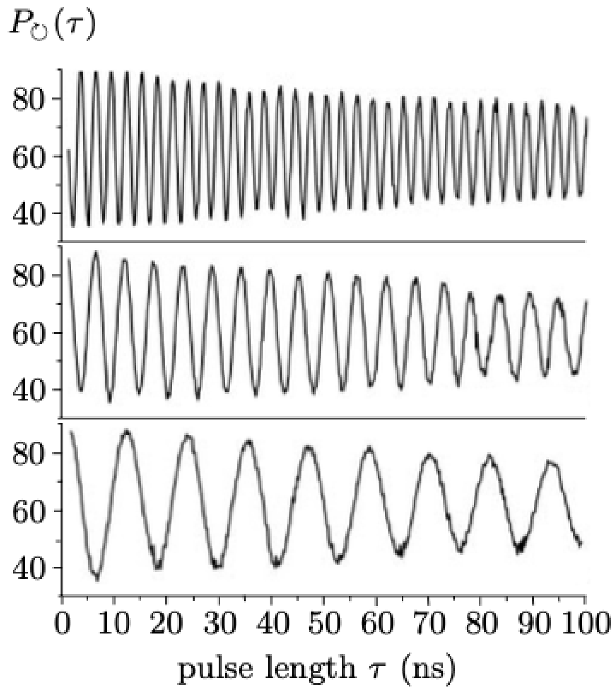

Figure 7: Rabi frequencies with three different microwave pulse amplitudes. Here the probability refers to the probability of the state vector to be in clockwise current in the end of pulse, whose length is given on the x axis. Source: [19]

sinusoidal dependence for different microwave pulses. For a closer look at Ramsey interferometry in general, see the original paper [20].

The Ramsey interferometry assumes that ρ_ψ stays as a pure state. If this is not the case, the sinusoidal dependence starts to dampen. If the state is completely decohered, the local phase relations between $|0\rangle$ or $|1\rangle$ are destroyed. Then the time spent freely evolving does not influence the recovery probabilities anymore. This effect can be seen in Figure 8 below.

Chiorescu et al. [19] measured these recovery probabilities and obtained a plot for them in free evolution time τ. The experiment showed that the probabilities initially obeyed very well the Ramsey theory predictions, and on the timescales of few tens of nanoseconds the sinusoid dependence decohered off.

While Ramsey interferometry experiments provide good experiments to measure the quantity of decoherence in SQUIDs, it does not explain the decoherence mechanism. This has been studied and the mechanism seems to be defects within the SQUID itself, not its coupling to an external circuit. These defects can be modeled as a 2 state system. Since the SQUID is a qubit itself, the decoherence model is a spin-$\frac{1}{2}$ particle in a spin-$\frac{1}{2}$. These results were established in [21], [22].

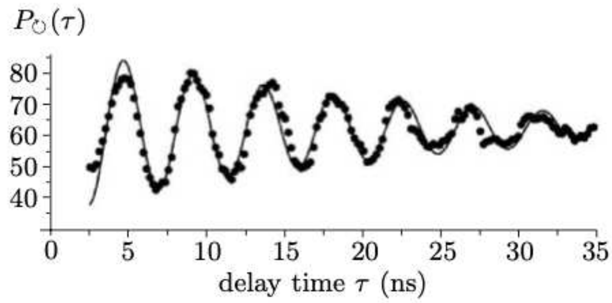

Figure 8: The gradual disappearance of trigonometric phase relations due to decoherence. Source: [19]

4 Conclusions

The interplay between quantum and classical physics is a complex and interesting area of study. While decoherence provides a powerful framework for understanding the emergence of classical behavior from the quantum world, it does not fully resolve all the fundamental questions. In particular, the measurement problem and the stochastic nature of quantum measurements remain open challenges. The relative state interpretation offers a potential solution by incorporating the observer and environment into the quantum description, but its ultimate validity remains a topic of ongoing debate. However, the practical advantages of this approach are clear demonstrated by the master equation formalism.

Experiments, such as those involving C_{70} molecules and SQUIDs, have demonstrated decoherence in action, showing how interactions with the environment lead to the suppression of quantum interference effects. These experiments also highlight the delicate balance between quantum and classical behavior, as even macroscopic systems like SQUIDs can exhibit quantum coherence under carefully controlled conditions. Further research in this area will undoubtedly deepen our understanding of the quantum-classical divide and may even lead to the development of new technologies that exploit the unique properties of quantum systems.

References

[1] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information*, 10th ed. Cambridge: Cambridge university press, 2010.

[2] e. Bl, B. Stephen, and K. M. ell, “Concepts in Thermal Physics.”

[3] C. A. Brasil and L. A. d. Castro, “Understanding the Pointer States,” *European Journal of Physics*, vol. 36, no. 6, p. 065024, Nov. 2015.

[4] W. H. Zurek, “Decoherence, einselection, and the quantum origins of the classical,” *Reviews of Modern Physics*, vol. 75, no. 3, pp. 715–775, May 2003.

[5] M. Schlosshauer, “Quantum Decoherence,” *Physics Reports*, vol. 831, pp. 1–57, Oct. 2019.

[6] H. Everett, ““Relative State” Formulation of Quantum Mechanics.”

[7] R. D. Gill, “On an Argument of David Deutsch,” May 2004.

[8] H. Barnum, C. M. Caves, J. Finkelstein, C. A. Fuchs, and R. Schack, “Quantum probability from decision theory?” *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, vol. 456, no. 1997, pp. 1175–1182, May 2000.

[9] W. H. Zurek, “Probabilities from Entanglement, Born’s Rule from Envariance,” *Physical Review A*, vol. 71, no. 5, p. 052105, May 2005.

[10] H. J. Carmichael, “Dissipation in Quantum Mechanics: The Master Equation Approach,” in *Statistical Methods in Quantum Optics 1: Master Equations and Fokker-Planck Equations*, H. J. Carmichael, Ed. Berlin, Heidelberg: Springer, 1999, pp. 1–28.

[11] U. Weiss, *Quantum dissipative systems*, 4th ed. New Jersey: World Scientific, 2012.

[12] R. P. Feynman and F. L. Vernon, “The theory of a general quantum system interacting with a linear dissipative system,” *Annals of Physics*, vol. 24, pp. 118–173, Oct. 1963.

[13] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, ser. Graduate Texts in Mathematics. New York, NY: Springer, 1998, vol. 113.

[14] B. Brezger, L. Hackermüller, S. Uttenthaler, J. Petschinka, M. Arndt, and A. Zeilinger, “Matter-Wave Interferometer for Large Molecules,” *Physical Review Letters*, vol. 88, no. 10, p. 100404, Feb. 2002.

[15] L. Hackermüller, K. Hornberger, B. Brezger, A. Zeilinger, and M. Arndt, “Decoherence of matter waves by thermal emission of radiation,” *Nature*, vol. 427, no. 6976, pp. 711–714, Feb. 2004.

[16] M. Arndt, O. Nairz, and A. Zeilinger, “Interferometry with Macromolecules: Quantum Paradigms Tested in the Mesoscopic World,” in *Quantum [Un]speakables: From Bell to Quantum Information*, R. A. Bertlmann and A. Zeilinger, Eds. Berlin, Heidelberg: Springer, 2002, pp. 333–350.

[17] J. Clarke, “SQUIDS: Theory and Practice,” in *The New Superconducting Electronics*, H. Weinstock and R. W. Ralston, Eds. Dordrecht: Springer Netherlands, 1993, pp. 123–180.

[18] E. Joos and H. D. Zeh, “The emergence of classical properties through interaction with the environment,” *Zeitschrift für Physik B Condensed Matter*, vol. 59, no. 2, pp. 223–243, Jun. 1985.

[19] I. Chiorescu, Y. Nakamura, C. J. P. M. Harmans, and J. E. Mooij, “Coherent quantum dynamics of a superconducting flux qubit,” *Science (New York, N.Y.)*, vol. 299, no. 5614, pp. 1869–1871, Mar. 2003.

[20] N. F. Ramsey, “A Molecular Beam Resonance Method with Separated Oscillating Fields,” *Physical Review*, vol. 78, no. 6, pp. 695–699, Jun. 1950.

[21] J. M. Martinis, K. B. Cooper, R. McDermott, M. Steffen, M. Ansmann, K. D. Osborn, K. Cicak, S. Oh, D. P. Pappas, R. W. Simmonds, and C. C. Yu, “Decoherence in Josephson Qubits from Dielectric Loss,” *Physical Review Letters*, vol. 95, no. 21, p. 210503, Nov. 2005.

[22] J. M. Martinis, S. Nam, J. Aumentado, and C. Urbina, “Rabi Oscillations in a Large Josephson-Junction Qubit,” *Physical Review Letters*, vol. 89, no. 11, p. 117901, Aug. 2002.