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TRADING VARIANCE

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Optiohinnoittelussa yleinen oletus on, että on mahdollista omistaa optio ja deltasuojata pois riski kohde-etuuden hinnan muutosten suunnasta. Teorian mukaan tällaisen position tuotto riippuu ainoastaan toteutuneesta varianssista, eli heilahteluista kohde-etuuden hinnassa.

Käytännössä tulokseen vaikuttaa muun muassa polku, jota rahoitusinstrumentin hinta seuraa periodin aikana. Tässä diplomityössä kuvataan tämä polkuriippuvuus ja muut tärkeät tekijät, jotka vaikuttavat tulokseen, kun deltasuojataan optiota. Teoriaosan lopussa kuvataan, miten tulosta voidaan yrittää parantaa käytännössä deltasuojaamalla yhden option sijaan tiettyä optioportfoliota.

Empiirisessä osassa muodostetaan teorian mukainen optioportfolio Matlabissa ja tarkastellaan tämän ominaisuuksia. Simuloinnin avulla verrataan tuloksia yhden option ja optioportfolion deltasuojaamisesta.

Avainsanat: deltasuojaus, varianssiswap, optiohinnoittelu

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<p>A common assumption in option pricing theory is that it is possible to own an option and hedge away the risk from the direction of the underlying asset's price. The return from such a strategy will in theory only depend on the realized variance of the underlying asset.</p> <p>In practice, the return is affected by other factors as well, for example the path that the asset's price takes. The objective of this thesis is to describe the factors that should be taken into consideration when delta hedging an option. Delta hedging a certain option portfolio is suggested as a means to get better delta hedging results in practice.</p> <p>In the empirical part, the suggested option portfolio is built in Matlab. Its features are examined. The results from delta hedging a single option and delta hedging an option portfolio are examined by simulation.</p>		
Keywords: delta hedging, variance swap, option pricing		

Preface

For me, writing a thesis about option pricing was an opportunity to combine skills and interests from both school and work. I would like to thank my supervisor professor Kai Zenger and instructor D.Sc. Jorma Selkänaho for their positive attitude towards this challenge.

I would also like to thank my girlfriend, family, friends and colleagues for the support and patience during this process.

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Symbols and abbreviations

$R_t, R(t)$	return of financial asset
$P, P_t,$ and $P(t)$	price of financial asset
μ	drift of financial asset
σ	volatility of financial asset
σ^2	variance of financial asset
$dW(t)$	Brownian motion or Wiener process
K	strike price of option
C_T	payoff of call option at time T
F	price of forward contract
H	expression for forward contract
w, V	option value
w_S	option value's first partial derivative with respect to the stock price S
w_{SS}	option value's second partial derivative with respect to the stock price S
r	interest rate
Γ	gamma, second partial derivate of option price
M	stock price value
c	call option price
\mathbf{c}	vector of call option payoffs for different outcomes
κ	option strike which separates liquid puts and calls, often forward level
w_c, w_p	weights of call and put options in portfolio
Π_{CP}	portfolio of call and put options
K_{var}	market expectations of fair variance
J	variable that indicates magnitude of jump in market price

1. Introduction

This thesis deals with variance and volatility of financial assets and how these can be traded.

Volatility is a synonym for standard deviation and a measure for dispersion (Poon & Granger, 2003). Together with an assumption about the probability distribution of a financial asset, volatility tells us about the probability of different investment outcomes. In 1952, Harry Markowitz suggested that volatility would be used as a single measure for investment risk (Markowitz, 1952). In modern risk assessment, volatility has become the most common risk measure (Djupsjöbacka, 2006).

There are several reasons for trading volatility. The simplest is that one might have a view on the magnitude of moves in financial markets, but not on their direction. Volatility tends to go up when prices of financial assets go down. Hence, trading volatility can be used for portfolio insurance purposes. (Allen et al., 2006; Black, 1976; Szado, 2009)

This work presents a way to trade so that the payoff is directly proportional a period's realized variance. The starting point is the Black-Scholes option pricing model from 1973. This shows the basic idea of delta hedging an option in order to trade variance. This method has some pitfalls. Among other things, the profit not only depends on realized variance, but also on the path of the stock price. The theory from the 1990's, of how a variance swap can be replicated, is presented as a solution to this problem.

In the empirical part we use simulation to test how delta hedging a single option performs in comparison to the more advanced method, which tries to replicate a variance swap.

2. Mathematical models in finance

It is worthwhile to say a few words about the difference between modeling physical and economical systems. In physics and natural sciences, one should distinguish between theories and models. Theories can be thought of as facts. They precisely describe how something works. Maxwell's equations are an example. They accurately describe light, and will continue to do so in the future. Models, on the other hand, describe what something resembles. They are used to describe something we don't yet fully understand by expressing it in terms of better understood phenomena. For example, Brownian motion accurately describes the random movement of small particles in physics. In finance, we use of the same theory to model stock prices, even though it is widely known that this is a vast simplification.

It is worth noting that in finance there are almost only models. This is because economical systems have memory. Stock markets might react differently to the same input in the future, because they learnt something the last time. Perhaps the only true theory in finance is the law of one price. It says that if two assets give exactly the same payoffs, then their prices should be the same. The law of one price can be used to derive the price of complicated payoffs. This is done by replicating the complicated payoff using elementary building blocks such as stocks and bonds. The price of the complicated payoff will be the price of the replicating portfolio with equal payoff. In 1973, for example, Black, Scholes and Merton derived the price of an option by showing how it can be replicated using stocks and bonds. (Derman, 2010)

In this work, we are interested in finding a replicating portfolio, whose payoff is an asset's accrued variance over a certain time period.

3. History and literature review

This chapter is divided into two sections. The first section gives an introduction to the history of options and option pricing. The second section deals with the much shorter history of volatility derivatives.

3.1. Stock and option price modeling

You might get the picture that option pricing dates back to 1973 when the Black-Scholes-Merton option pricing model was published. However, option pricing of at least some form took place in the Netherlands already in the 17th century. In the late 19th and early 20th century there were active option markets in at least New York, London and Paris. Texts from 1902-1904 by Nelson and Higgins mention put-call parity, which is central in modern option pricing. In literature, Stoll (Stoll, 1969) is often credited for discovering the put-call parity in 1969. World War 2 is suggested as an explanation for information being lost and then re-discovered by academics in the 1960's and 70's. (Haug & Taleb, 2011)

In 1900 Bachelier laid the ground for theoretical option pricing models in his doctoral dissertation. His starting point was that stock prices followed a random walk and were hence normally distributed. In 1961 Sprenkle (Sprenkle, 1961) improved the idea by assuming that prices follow a lognormal distribution whereby, among other things, prices cannot become negative. When prices are lognormal, returns will be normal (Gaussian). Sprenkle discovered most of what we today call the Black-Scholes-Merton option pricing formula. (Haug & Taleb, 2011)

During the 60's, several authors derived option or warrant pricing formulas of the same general form as presented in Chapter 7. In addition to Sprenkle, these include Ayres (Ayres, 1963), Boness (Boness, 1964), Samuelson (Samuelson, 1965), Baumol, Malkiel and Quandt (Baumol et al., 1966) and Chen (Chen, 1970). However, at that time, no one succeeded in determining all the appropriate parameters. (Black & Scholes, 1973)

The general idea in the models was to compute the expected payoff from the option at maturity, and discount it to present. This method had two problems. First, in addition to the assumption of lognormal prices, a stock's expected return had to be known in order to calculate the option's expected payoff. Secondly, a discount rate had to be assigned to discount the payoff to present. The risk of an option depends on both the price of the underlying asset and time to maturity, which makes it hard to find the correct discount rate.

In the late 60's, Black started applying the Capital Asset Pricing Model (CAPM) to derive a differential equation for the price of a warrant. He came up with an equation indicating that the value of a warrant did not depend on how the stock's risk was split between risk that could be diversified away and risk that couldn't. In other words, the value seemed to depend only on the total risk, as measured for example with the stock's standard deviation. Further, a warrant's value seemed not to depend on the stock's expected return. At the time, Black did not manage to solve his equation.

Later, Black and Scholes started working together on the option pricing problem. Their starting point was to assume that the price of an option depended on the underlying stock's volatility and not on its expected return. By this assumption, they could pick any expected return for the stock. They chose the expected return to be a constant interest rate. In CAPM terms, this meant that the stock's beta was zero. It came to them, that if the stock had zero beta, then so would the option. If the option always had an expected return equal to the interest rate, then this interest rate would be the appropriate discount rate. Sprenkle had derived the expression for an option's expected payoff given a constant return on the stock. Black and Scholes took Sprenkle's formula and used the risk-free interest rate for discounting. The resulting option price satisfied their differential equation. They were confident of having found the right formula.

Merton was simultaneously working on his own version of the option pricing formula. He pointed out to Black and Scholes that by continuously trading stock or options, a hedged position which was riskless arose. This was considered the most general derivation. In the final version of Black and Scholes' paper, the equation was derived this way. (Black, 1989)

In 1997, Black had passed away, but Merton and Scholes were awarded the Nobel Prize in economics for their achievement. The associated press release (1997) states among other things the following:

“Black, Merton and Scholes made a vital contribution by showing that it is in fact not necessary to use any risk premium when valuing an option. This does not mean that the risk premium disappears; instead it is already included in the stock price.”

The Black-Scholes formula relies on strict assumptions that don't apply in practice. Haug and Taleb point not only to their experience as traders but also to market dynamics as proof that the Black-Scholes model is not used in practice for pricing options. If the model was used in practice, option prices would not react to rising demand of an option of some strike, since traders could simply

manufacture more options using the Black-Scholes model. In reality, supply and demand drive prices. (Haug & Taleb, 2011)

However, the Black-Scholes model has become the standard way of quoting options. This is done by taking an observed option price and running it through the option pricing formula to see what volatility level the price corresponds to. This can be compared to bond yields, which are a convenient way of quoting bond prices. (Derman et al., 1998)

3.2. Volatility derivatives

While option prices depend on volatility, their payoffs only depend on the price of the underlying asset at maturity. In the mid-1990's derivatives whose payoffs depended on the volatility during the entire lifetime of the derivative emerged. These are called volatility derivatives. (Carr & Lee, 2009)

A variance swap is a good example of such a derivative. At maturity, the buyer of a variance swap pays the swap strike and receives the period's realized variance. Buying a variance swap is the same as being long variance (Allen et al., 2006). How the fair strike of a variance swap is calculated, and how a variance swap can be replicated, is described in Chapter 8.

The first variance swap is claimed to have been traded by UBS in 1993. At that time, the method used was to simply quote the at-the-money volatility with a safety margin. A more accurate method for calculating the fair strike price emerged starting with Neuberger's idea of a log contract in 1990 and 1994 (Neuberger, 1994). The precise theory was laid out at least by Peter Carr and Dilip Madan in 1998 (Carr & Madan, 1998). As the theory became known, trading variance swaps became popular. At the time, hedge funds found it attractive to sell volatility. Banks used the newly developed theory to replicate variance swaps and hedge their risk. Loosely speaking, selling volatility was a profitable strategy until the financial crisis of end 2008. At that time, volatility levels exploded.

In the beginning, variance swaps were available only for indices, but later also for individual stocks. Because natural logarithms of prices are used for calculating the floating part of a variance swap, the variance would become infinite if the stock's value became zero. Hence a cap was introduced limiting the payoff to 2,5 times the variance swap strike for the floating part. After the crisis of 2008, this convention was introduced also for indices. (Carr & Lee, 2009)

The Chicago Board Options Exchange (CBOE) reacted to the development of volatility derivatives by introducing their first version of the renowned VIX volatility index in 1993. In the beginning, it was calculated from short term near-the-money options on the S&P 100 index. In 2003, it was

revised to its current form, whereby the underlying was changed to the S&P 500 index, the rate was determined according to the theory presented in the late 90's and the annualization was changed from calendar days to business days. This improved the popularity of the index. Options on the VIX index were introduced later and became the CBOE's second most liquid option contract after those of the S&P 500 index itself. The popularity stems from the VIX index's negative correlation with the equity index. (Carr & Lee, 2009)

4. Mathematical tools for stock and option price modeling

This section briefly summarizes the main mathematical building blocks used in the paper, starting with the return on a stock.

4.1. Price and return of a stock

Notation often differs between continuous and discrete time finance, since the theories were developed separately. In the continuous case, prices are commonly denoted S , S_t or $S(t)$. It is common to drop the time dependence term for simplicity. In the discrete case P , P_t , and $P(t)$ are used to denote price.

In both the continuous and discrete case, R is used for returns. In the discrete case it is defined

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \quad (1)$$

In the continuous case this becomes

$$R(t) = \frac{(S(t + \Delta t) - S(t))}{S(t)} \quad (2)$$

where Δt is a very short time period. Note that the expression can be written.

$$1 + R(t) = \frac{S(t + \Delta t)}{S(t)} \quad (3)$$

Logarithmic returns are defined

$$\ln \frac{S(t + \Delta t)}{S(t)} \quad (4)$$

Logarithmic returns are often used in practice due to the convenient additive feature; a year's logarithmic return equals the sum of the year's daily logarithmic returns. Note that when R is small we can write

$$R(t) \approx \ln(1 + R(t)) = \ln S(t + \Delta t) - \ln S(t) \quad (5)$$

(Alexander, 2008)

We take a closer look at the returns in Section 4.4.

4.2. The price process

It is common to model stocks by Geometric Brownian motion. In this model, stock prices are assumed lognormally distributed whereby returns are normally distributed. The distributions and the

motivation for using them are presented in the next section. Here is the general expression for the price process:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t) \quad (6)$$

where μ is drift, σ is volatility or standard deviation and $dW(t)$ is a stochastic differential term called Brownian motion or Wiener process. In the basic case, the drift and volatility are constants. (Alexander, 2008)

A stochastic process is a Wiener process if the following conditions hold:

$$W(0) = 0$$

Set $r < s \leq t < u$. Then $W(u) - W(t)$ and $W(s) - W(r)$ are independent random variables.

Set $s < t$. Then $W(t) - W(s)$ is normally distributed with mean 0 and standard deviation $\sqrt{t - s}$.

$W(t)$ is continuous.

Put in words, the price process is such that a small relative price change consists of drift, which is linear in time, and a normally distributed increment with mean 0 and volatility linearly related to the square root of time. (Djupsjöbacka, 2006)

The Brownian motion property, that increments are independent random variables, indicates that profits cannot be made by analyzing past prices. (Sigman, 2006)

4.3. Probability distributions

Prices and returns of financial assets are random variables. Often returns are modeled with a normal distribution, whereby prices are lognormally distributed. The most important reason for using the normal distribution is ease of use. Only two parameters are needed to exactly specify the normal distribution; the expectation μ and standard deviation (volatility) σ , which is a measure for dispersion around the expectation. Secondly, the normal distribution is a stable distribution. Due to the stable property, the sum of two normally distributed random variables also has a normal distribution, whose expectation and variance are easy to determine. A property of the lognormal distribution, on the other hand, is that it gives zero probability for negative outcomes. This makes it well suited for modeling stock prices, which cannot become negative. (Alexander, 2008)

Next, the properties of the normal and lognormal distribution are defined.

4.3.1. Normal distribution

The normal distribution has the following probability density function:

$$\varphi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad (7)$$

for $-\infty < x < \infty$. This is a symmetric bell shaped curve centered at the expectation μ . It has dispersion about its mean measured by the standard deviation σ . The notation

$$X \sim N(\mu, \sigma^2) \quad (8)$$

means that X has a normal distribution with mean μ and variance σ^2 . $N(0, 1)$ is a special case and called the standard normal distribution. The letter Z is often used to denote a standard normal variable.

$$Z \sim N(0, 1) \quad (9)$$

Any normally distributed random variable $X \sim N(\mu, \sigma^2)$ can be transformed into a standard normal random variable $Z \sim N(0, 1)$ by the following transformation:

$$Z = \frac{X - \mu}{\sigma} \quad (10)$$

(Alexander, 2008)

The standard normal probability density function is got by setting $\mu = 0$ and $\sigma^2 = 1$:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (11)$$

By the expression $N(x)$ is meant the integral of the standard normal density function $\varphi(x)$ over $-\infty$ to x :

$$N(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{\tau^2}{2}} d\tau \quad (12)$$

It is the probability that a random variable $x \sim N(0, 1)$ is equal to or smaller than x . The probability that x is larger than x will be $1 - N(x) = N(-x)$. See any basic probability book for reference.

4.3.2. Lognormal distribution

If a random variable has a lognormal distribution, then its logarithm has a normal distribution. A lognormal distribution has the following density function:

$$f(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ln(y) - \mu)^2}{2\sigma^2}\right) \quad (13)$$

for $-\infty < x < \infty$, where μ and σ^2 are the expectation and variance of $\ln(Y)$. In other words, they are the expectation and variance of the associated normal density function. The expectation and variance of Y are the following:

$$E(Y) = \exp\left(\mu + \frac{1}{2}\sigma^2\right) \quad (14)$$

$$V(Y) = \exp(2\mu + \sigma^2) (\exp(\sigma^2) - 1) \quad (15)$$

(Alexander, 2008)

4.4. A closer look at μ , σ and returns

In later sections, probabilities for specific stock price outcomes are needed. These are calculated using the assumed probability density functions presented above. To be able to calculate probabilities from the density functions, the appropriate drifts and volatilities have to be known. One might easily lose track of how μ and σ are defined when dealing with both normal and lognormal distributions. Hence, it is useful to take a second look at the definitions.

As stated before, prices are often assumed lognormal, whereby their logarithms have a normal distribution. Because of the normal distribution's stable property, the sum of two normally distributed variables is normally distributed as well. The difference between the logarithms of a certain price outcome and the price in the beginning $\ln S_t - \ln S = \ln\left(\frac{S_t}{S}\right)$ is called the period's logarithmic return. It has the following normal distribution:

$$\ln\left(\frac{S_t}{S}\right) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right) \quad (16)$$

The mean or expectation is $\left(\mu - \frac{\sigma^2}{2}\right)t$ and variance is $\sigma^2 t$. The period's continuously compounded return is defined as follows:

$$\frac{\ln S_t - \ln S}{t} \sim N\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{t}\right) \quad (17)$$

From the above equations (16) and (17), an exact definition for the term σ^2 is "the variance of the continuously compounded rate of return over a time interval of length one".

Next we will find an exact definition for the expectation μ . The return relative over the time interval $[0, t]$ is

$$\frac{S_t}{S} \quad (18)$$

where S is the stock price at time 0. Its logarithm has a normal distribution, which is defined above.

The expression for its expectation becomes

$$E\left(\frac{S_T}{S}\right) = \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) + \frac{1}{2}\sigma^2\right\} = \exp(\mu t) \quad (19)$$

So μ is “the logarithm of the expected return relative over a period of length one”.(Nielsen, 1992)

4.5. Expected value and variance

The expected value of the random variable X is defined as follows

$$E(X) = \mu = \int_{-\infty}^{\infty} xf(x) dx \quad (20)$$

where $f(x)$ is the density function. (Alexander, 2008)

4.6. Ito’s lemma

To differentiate a Wiener process, Ito’s lemma has to be applied. Assume that the variable to be differentiated is of the form

$$dx = a(x, t) + b(x, t)dW \quad (21)$$

where dW is a Wiener process and a and b are functions of x and t . Ito’s lemma says that a function $G(x, t)$ follows the process

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}bdW \quad (22)$$

This is needed, among other things, for computing the derivatives of an option’s price, which is a function of stock price and time.(Hull, 2000)

4.7. Empirical findings of asset returns

In the above sections, we have presented tools often used for modeling financial asset prices and returns. It is important to bear in mind that these are only models of how asset prices and returns behave in reality.

In reality, returns are not normally distributed. Compared to the normal distribution, real returns have fat tails. If returns followed a normal distribution, 68% of the returns would fall within \pm one standard deviation from the expectation. In the real world, 80-99% of the observations tend to fall inside the interval \pm one standard deviation around the mean. (Taleb, 2009)

Djupsjöbacka uses daily returns on the Dow Jones Industrial Average index from 1928 to 2003. He uses a Jarque-Bera test, which strongly rejects the hypothesis of normally distributed returns. He for example finds that 40,56% of the returns are within $\left[\mu - \frac{\hat{\sigma}}{3}, \mu + \frac{\hat{\sigma}}{3}\right]$, while it should be 26,11% for a normal distribution. The proportion of returns more than five standard deviations from the expectation is 0,34%, which resembles about one return in a year. In a normal distribution, such a return is almost impossible. (Djupsjöbacka, 2006)

Mandelbrot recognized that empirical research rejected the assumption about normally distributed returns. He suggested that one should use power laws instead. (Mandelbrot, 1963)

5. Volatility

This chapter presents some features of volatility.

5.1. Three types of volatility

There are three types of volatility to distinguish between. These are

- statistical volatility
- implied volatility and
- process volatility.

Process volatility is the true volatility, the sigma which scales the Brownian motion. It cannot be observed, but has to be estimated using statistical methods. The estimate that is got this way is called statistical volatility. Implied volatility refers to the volatility level that options prices point to, when all other inputs of the option pricing models are known. (Djupsjöbacka, 2006)

For example, let us assume that stock prices follow a geometric Brownian motion with volatility 20% and no drift. Figure 1 shows the estimated volatilities for three-month-periods, measured from daily closing prices.

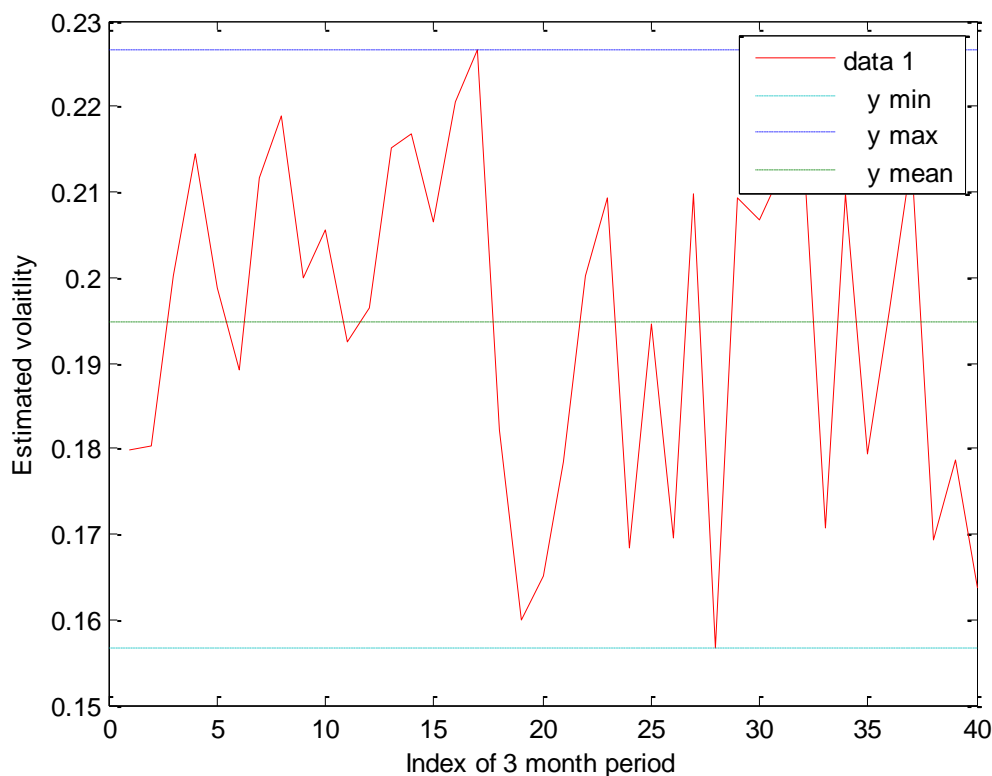


Figure 1 Estimated volatility in 40 periods, with process volatility 20% (0.2)

We see that the estimate of volatility ranges from about 16% to 23%, with a mean close to 20%.

5.2. What is volatility

In finance, volatility, abbreviated σ , usually means the standard deviation of returns, calculated from observations over a certain time period. The formula for calculating variance is as follows:

$$\hat{\sigma}^2 = \frac{1}{N-1} \sum_{i=1}^t (R_i - \bar{R})^2 \quad (23)$$

where R_t is the return at time t and \bar{R} is the mean return. The formula is an unbiased estimate of the data set's variance. This means that the expected value of the estimator equals the process' true variance (Alexander, 2008). Note that due to Jensen's inequality, the square root of the unbiased estimate of variance is a biased estimate of volatility. In finance, the mean is often assumed to be zero, whereby estimate noise decreases.

As showed in Chapter 4, volatility scales the increments of the Brownian motion process.

Often volatility is used as a measure of risk. For example, the Sharpe ratio measures the performance of an investment in relation to risk and is calculated by dividing the return in excess of the risk-free return by the standard deviation. However, risk is often associated with small or negative returns while the standard deviation makes no difference between positive and negative returns.

In Chapter 4 we said that volatility and mean exactly define an entire normal distribution. Note that the correct measure of dispersion depends on the probability distribution, and standard deviation is the correct measure of dispersion only for some distributions.

Researchers have found that using sampling intervals of for example five or fifteen minutes produces an accurate estimate of volatility. The estimate becomes noisy for shorter sampling intervals. (Poon & Granger, 2003)

5.3. Volatility smile and skew

One characteristic feature of implied volatility is the volatility smile. If implied volatilities are calculated from options of different strikes, different volatilities are got. Volatilities will be higher for options with strikes further away from the at-the-money level. This is called volatility smile. If the smile isn't symmetrical, we talk about skew. Before the stock market crash of 1987, there was little smile and skew; the volatility of an option was about the same, regardless of strike. (Derman, 1999)

One reason for the smile is the assumption of normally distributed returns that the Black-Scholes formula makes. The normal distribution will give a small probability for very large and small returns. In reality, the probability of a return does not decrease as fast as the normal distribution implies when the magnitude of the return increases. A higher probability of bigger moves means that the options are more likely to end up in-the-money and be exercised at the expense of the option seller. Hence, the option sellers demand a higher price for them. When the higher option price is put in the Black-Scholes option pricing formula, it will imply a higher volatility.

A second reason is that the Black-Scholes model assumes constant volatility which does not hold true in the real world. Stochastic volatility alone causes smile. (Poon & Granger, 2003)

5.4. Volatility in relation to market moves

Fisher Black was one of the first to examine the relationship between returns and volatility. He had noticed that as stock prices move up, volatility usually goes down and vice versa. The inverse relationship is illustrated in Figure 2. He was also amazed at the magnitude of the change in volatility. In his tests, he found that a 1% change in stock price led to change in volatility of more than 1%. When stock went up, volatility would fall even in dollar terms. For example, a stock might be worth 20 dollar and have typical daily moves of 50 cents. Then it can rise to 40 dollars and start having daily moves of 38 cents.

Black identified leverage as one reason for the moves in volatility in relation to return. If the value of a leveraged company goes down, the company will become more leveraged which increases volatility. On the other hand if the company has no debt, it still has operating leverage, as income varies more than expenses. During bad times income will fall more than expenses and small moves in income will lead to higher percentage changes in profits. Black stated that leverage alone was unlikely to account for the large moves in volatility. Another interesting feature that Black found was that volatility tends to stay at elevated levels for some time after it has gone up. (Black, 1976)

Three-Month Implied Volatilities of SPX Options

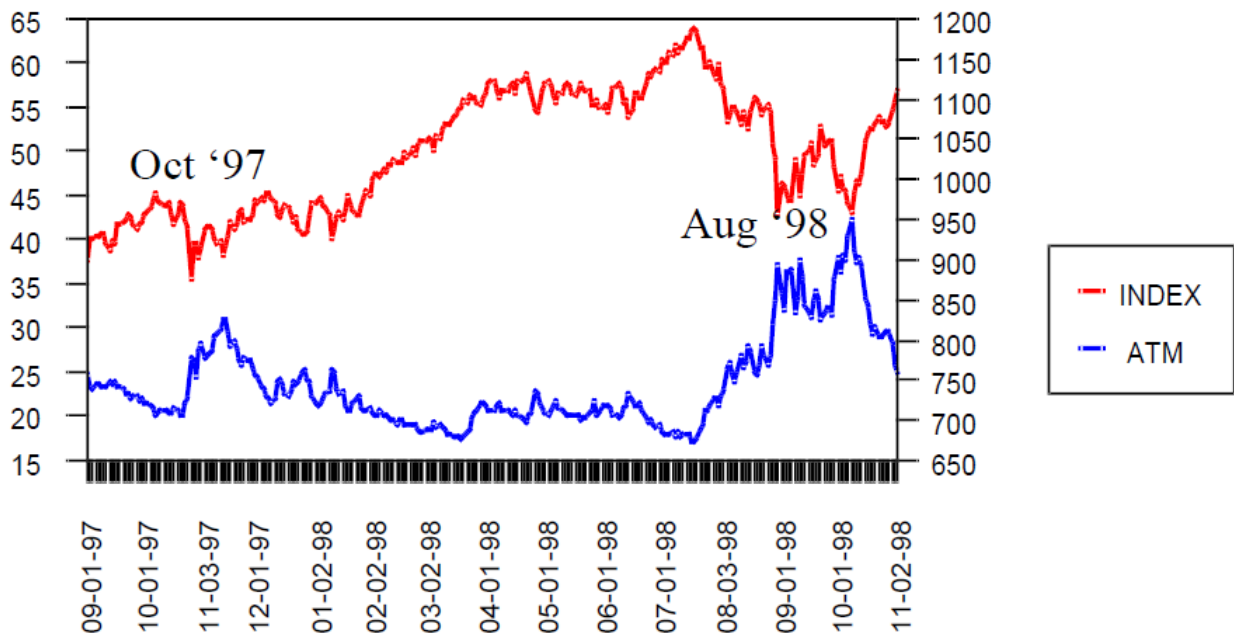


Figure 2 Volatility and index level have an inverse relationship. ATM stands for at-the-money volatility. Figure taken from (Derman, 1999)

6. Options

There are two kinds of basic options; calls and puts. Call options give its owner the right, but not the obligation, to buy a certain asset for a specified price at a specified time. Put options give the owner the right to sell the underlying asset for a certain price at a certain time. If the option can only be exercised (use the right) at maturity, it is called a european option while options that can be exercised at any time during their lifetime are called american. (Hull, 2000)

Next, the theory of option prices is presented. The theory is relevant since variance or volatility is one of the central variables affecting the option price. The option pricing formulas give a relationship between volatility and price. In Chapter 7, the derivation of the Black-Scholes option pricing model suggests a way to trade variance. In order to grasp the idea of options and their value, we will first derive the result more heuristically. Then we move on to show how Black, Scholes and Merton derived their model.

6.1. Where does the option price come from

To start off, we examine a European call option on a stock and explain where its price comes from. The price of the option can be thought of as the discounted expected payoff that the owner gets from the option. The option has a strike price K , which is the level at which stock can be bought at time T . Let us denote the stock price S . On the future time T , when the option expires, S can be either higher or lower than the option strike price. If S is lower than K , it is cheaper to buy the stock directly from the market and leave the option unused. If S is higher than K , then the owner of the option can buy the stock for a lower price than what it is worth (and for example sell it right away for a profit). The payoff of the call option C at time T can be expressed in the following way:

$$C_T = \max\{0, S_T - K\} = \begin{cases} S_T - K, & S_T > K \\ 0, & \text{otherwise} \end{cases} \quad (24)$$

This can further be split into two parts; paying the strike price K and receiving the stock worth S_T .

Let the former be C_T^1 and the latter C_T^2 .

$$C_T^1 = \begin{cases} -K, & S_T > K \\ 0, & \text{otherwise} \end{cases} \quad (25)$$

$$C_T^2 = \begin{cases} S_T, & S_T > K \\ 0, & \text{otherwise} \end{cases} \quad (26)$$

The expected future cash flow from paying the exercise price is

$$E[C_T^1] = -KP\{S_T > K\} \quad (27)$$

where $P\{S_T > K\}$ is the probability that the stock price is higher than the exercise price at maturity T . The expected (negative) payoff is the strike price to be paid times the probability that it will be paid. The second payoff C_T^2 is slightly more complicated, and can be written as follows:

$$E[C_T^2] = E[S_T | S_T > K] P\{S_T > K\} \quad (28)$$

In other words, this is the expected price of the stock, given that the price is higher than K , times the probability that the price is higher than K .

We assume that prices are lognormally distributed. Then $\ln S_t$ is normally distributed and has mean $(\ln S_0 + (\mu - \sigma^2)t)$ and variance $\sigma^2 t$:

$$\ln S_t \sim N(\ln S_0 + (\mu - \sigma^2)t, \sigma^2 t) \quad (29)$$

(Nielsen, 1992)

Under this assumption, the probabilities can be calculated from the known probability distribution. What remains to do in order to determine the price of the option is to discount the expected payoff. Then the formula for the present value or price of the option looks like this:

$$C_t = e^{-r(T-t)} (E[S_T - K]_+) = e^{-r(T-t)} \{S e^{\mu(T-t)} N(d_1) - KN(d_2)\} \quad (30)$$

where r is the appropriate discount rate and μ is the stock's drift rate. $S e^{\mu(T-t)} N(d_1)$ and $-KN(d_2)$ are the expected payoffs from receiving the stock and paying the strike price. $N(d_1)$ and $N(d_2)$ are probabilities calculated using the normal distribution's cumulative distribution functions. These depend on the stock's drift, the volatility and the stock's price in the beginning. $N(d_1)$ and $N(d_2)$ are thoroughly explained in the next section.

Formulas equivalent to equation (30) were derived in the 60's by several researchers. At the time, no one found a way to determine or estimate the correct risk premium, which would have given the appropriate discount rate. This was later discovered by Black, Scholes and Merton. (Black & Scholes, 1973)

Next, we present one way in which the appropriate discounting rate could have been found. The price of a put option can be written:

$$P_t = e^{-r(T-t)} (E[K - S_T]_+) = e^{-r(T-t)} \{KN(-d_2) - S e^{\mu(T-t)} N(-d_1)\} \quad (31)$$

Put-call parity was (re)discovered by Stoll in the year 1969. It gives a relationship between put and call prices, and says that a long position in a call and short position in a put gives exactly the same

payoff as a forward contract F on the stock. By using this relationship and the formulas for put and call prices, the following expression for a forward's price can be derived:

$$F = C_t - P_t = e^{-r(T-t)}\{S e^{\mu(T-t)} - K\} \quad (32)$$

The value of the forward contract, on the other hand, is:

$$H = S - K e^{R(T-t)} \quad (33)$$

where R is the zero-coupon riskless discount rate for the time horizon. Requiring equations (32) and (33) to be equal gives that both the discount rate r and the drift μ are equal to the zero-coupon rate R . Then the formula becomes exactly the Black-Scholes option pricing formula published in 1973. (Derman & Taleb, 2005)

The Black-Scholes option pricing formula is as follows:

$$C = S_0 N(d_1) - K e^{-rT} N(d_2) \quad (34)$$

$$P = K e^{-rT} N(-d_2) - S_0 N(-d_1) \quad (35)$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad (36)$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \quad (37)$$

(Hull, 2000)

Next, the terms $N(d_1)$ and $N(d_2)$ are explained in order to fully understand the above formula. After this, we move on to present the derivation of the formula as it appeared in the 1973 paper by Black and Scholes.

6.2. Understanding $N(d_1)$ and $N(d_2)$

Often little attention is paid to the probability calculations $N(d_1)$ and $N(d_2)$. Here a thorough explanation is provided for those interested. Let us start off with the easier expression of the expected payoff from paying the strike price at maturity, which was defined in the following way:

$$E[C_T^1] = -K P\{S_T > K\} \quad (38)$$

We can calculate the probability $P\{S_T > K\}$ by remembering that

$$\ln\left(\frac{S_t}{S}\right) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right) \quad (39)$$

We transform this to a standard normal distribution as presented in Subsection 4.3.1:

$$Z = \frac{\ln\left(\frac{K}{S}\right) - \left(\mu - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \quad (40)$$

$$Z \sim N(0,1) \quad (41)$$

By using the appropriate μ , derived in the previous section, we get

$$Z = \frac{\ln\left(\frac{K}{S}\right) - \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \quad (42)$$

Then

$$\{S_T > K\} = 1 - N(Z) = N(-Z) = N(d_2) \quad (43)$$

In other words, we have the same expression for d_2 as in the Black-Scholes formula:

$$d_2 = -\frac{\ln\left(\frac{K}{S}\right) - \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \quad (44)$$

The expected (negative) payoff from paying the exercise price K is hence $KN(d_2)$. If we use $\mu = r$, the appropriate discounting rate will be the risk free rate r . The present value then becomes $e^{-r\tau}KN(d_2)$.

Next, we move on to the more complicated expected payoff from receiving the stock if the stock finishes in the money, that is if the option is exercised. This payoff was expressed in the following way:

$$C_T^2 = \begin{cases} S_T, & S_T > K \\ 0, & \text{otherwise} \end{cases} \quad (45)$$

The expected value of S_T conditional on $S_T > K$ is got from Section 4.5:

$$E[C_T^2] = \int_x^\infty S f(S) dS \quad (46)$$

Recall that the logarithm of S_T has the following normal distribution:

$$\log S_t \sim N(\log S + (\mu - \sigma^2)t, \sigma^2 t) \quad (47)$$

For convenience, let us denote this $\ln S_t \sim N(m, s^2)$ in the following calculations, and switch back to the actual values for m and s^2 once the calculations are ready. Here, upper case S stands for stock price and lower case s^2 for variance. The associated standard normal transformation becomes

$$v = \frac{\ln S - m}{s} \quad (48)$$

Denote the standardized transformation for the strike price K by D :

$$D = \frac{\ln K - m}{s} \quad (49)$$

We get:

$$\begin{aligned} E[C_T^2] &= \int_D^\infty S(v) \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv = \int_D^\infty e^{sv+m} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv \\ &= \frac{1}{\sqrt{2\pi}} \int_D^\infty e^{sv+m} e^{-\frac{v^2}{2}} dv \end{aligned} \quad (50)$$

where the following equality has been used:

$$e^{sv+m} = \exp\left(s * \frac{\ln S - m}{s} + m\right) = e^{\ln S} = S \quad (51)$$

$$e^{sv+m} e^{-\frac{v^2}{2}} = \exp\left(sv + m - \frac{v^2}{2}\right) \quad (52)$$

$$sv + m - \frac{v^2}{2} = -\frac{(v-s)^2}{2} + m + \frac{s^2}{2} \quad (53)$$

We get

$$E[C_T^2] = \frac{\exp\left(m + \frac{s^2}{2}\right)}{\sqrt{2\pi}} \int_D^\infty \exp\left(-\frac{(v-s)^2}{2}\right) dv \quad (54)$$

By changing integrating variable to $y = v - s$ we get the following result:

$$\begin{aligned} \frac{\exp\left(m + \frac{s^2}{2}\right)}{\sqrt{2\pi}} \int_{D-s}^\infty \exp\left(-\frac{y^2}{2}\right) dy &= \exp\left(m + \frac{s^2}{2}\right) (1 - N(D-s)) \\ &= \exp\left(m + \frac{s^2}{2}\right) N(s-D) \end{aligned} \quad (55)$$

The correct values for m and s^2 were the following:

$$m = \ln S + (\mu - \sigma^2)t \quad (56)$$

$$s^2 = \sigma^2 t \quad (57)$$

Once again, using $\mu = r$

$$m = \ln S + (r - \sigma^2)t \quad (58)$$

Then we get:

$$m + \frac{s^2}{2} = \ln S + r\tau \quad (59)$$

$$D = \frac{\ln K - \ln S - (r - \sigma^2)\tau}{\sigma\sqrt{\tau}} = -d_2 \quad (60)$$

We define d_1 :

$$s - D = \sigma\sqrt{\tau} + d_2 = d_1 \quad (61)$$

The sought expected payoff from receiving the stock conditional on $S_T > K$ becomes

$$C_T^2 = \exp\left(m + \frac{s^2}{2}\right) N(s - D) = \exp(\ln S + r\tau) N(d_1) = Se^{r\tau} N(d_1) \quad (62)$$

The appropriate discount rate is the risk free rate r . The present value simplifies to $SN(d_1)$.

(Nielsen, 1992)

7. The Black-Scholes model

As stated earlier, no one had succeeded in determining the correct parameters for the option pricing model before Black, Scholes and Merton. In this section we introduce the assumptions behind and the derivation for their differential equation that describes the dynamics of option prices. One of the assumptions is that volatility is known and constant (Derman & Taleb, 2005). After the equation has been derived, we move on to consider the real life case, where volatility is not a known constant. We show how the ideas from the Black-Scholes derivation can be used to trade volatility.

7.1. The assumptions

These are the assumptions made about stock price and market dynamics as stated in the Black-Scholes paper from 1973:

1. The short-term interest rate is known and is constant through time.
2. The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is lognormal. The variance rate of the return on the stock is constant.
3. The stock pays no dividends or other distributions.
4. The option is “European”, that is, it can only be exercised at maturity.
5. There are no transaction costs in buying or selling the stock or the option.
6. It is possible to borrow any fraction of the price of the security to buy it or to hold it, at the short-term interest rate.
7. There are no penalties to short selling. A seller who does not own a security will simply accept the price of the security from a buyer on some future date by paying him an amount equal to the price the security on that date.

(Black & Scholes, 1973)

7.2. Derivation of the Black-Scholes differential equation

In this section, the derivation of the Black-Scholes differential equation is presented. The derivation is important, since it suggests a way to trade, if one wants to have a profit proportional to the period's realized variance.

The basic idea is the following: Suppose the value of an option increases by about 50 cent when the stock price rises by one dollar. Then, one can go short (sell) two options and long (buy) one stock, to create a portfolio where the price changes from the options and the stock cancel each other out. As the stock price changes and time passes, either the amount of options, the amount of stock or

both have to be adjusted in order to keep the price changes equal. As explained below, this kind of portfolio should earn the risk free interest rate. From this basic idea, the Black-Scholes differential equation can be derived. (Black 1989)

Consider a portfolio consisting of a long position in stock and a short position in options. We denote the stock price S and the option price w . The correct amount of options to go short in order to create a portfolio whose value does not depend on the stock price over the next time step is the following:

$$\frac{1}{w_S(S, t)} \quad (63)$$

where w_S is the option value's first partial derivative with respect to the underlying stock's price S . The equity value of this portfolio is then

$$S - \frac{w}{w_S} \quad (64)$$

Over the following short time step, the price change of this is

$$\Delta S - \frac{\Delta w}{w_S} \quad (65)$$

(Black & Scholes, 1973)

By stochastic calculus, an expression for Δw is

$$\Delta w = w_S \Delta S + w_t \Delta t + \frac{1}{2} w_{SS} \sigma^2 S^2 \Delta t \quad (66)$$

A simplified explanation for the equation (66) is given in the next section.

The expression for the price change of the portfolio becomes

$$\begin{aligned} \Delta S - \frac{\Delta w}{w_S} &= \Delta S - \frac{w_S \Delta S + w_t \Delta t + \frac{1}{2} w_{SS} \sigma^2 S^2 \Delta t}{w_S} = - \frac{w_t \Delta t + \frac{1}{2} w_{SS} \sigma^2 S^2 \Delta t}{w_S} \\ &= - \frac{\left(w_t + \frac{1}{2} w_{SS} \sigma^2 S^2 \right) \Delta t}{w_S} \end{aligned} \quad (67)$$

Due to the constant volatility assumption, the change in value of the portfolio becomes entirely deterministic. Since the return has no uncertainty, the change in value of the portfolio must equal the risk-free interest rate times the equity value of the portfolio:

$$-\frac{\left(w_t + \frac{1}{2}w_{SS}\sigma^2S^2\right)\Delta t}{w_S} = \left(S - \frac{w}{w_S}\right)r\Delta t \quad (68)$$

, which can be rearranged to

$$w_t + \frac{1}{2}w_{SS}\sigma^2S^2 = r(w - Sw_S) \quad (69)$$

Equation (69) is the famous Black-Scholes differential equation. (Black & Scholes, 1973)

The differential equation can be solved to give the same call and put price values as derived earlier.

This is done by rearranging it to a diffusion equation. See for example

<http://www.francoiscoppex.com/blackscholes.pdf>.

7.3. Change in option value Δw

Here is a brief explanation for where the expression for Δw in equation (66) comes from.

The formal derivation of the expression is got by using Ito's lemma. One can use a Taylor expansion to understand the result:

$$\Delta w = w_S\Delta S + w_t\Delta t + \frac{1}{2}[w_{SS}(\Delta S)^2 + w_{tt}(\Delta t)^2 + 2w_{St}\Delta S\Delta t] \quad (70)$$

where w_{SS} is the option price's second partial derivative with respect to the stock price and w_{tt} the same with respect to time. Black and Scholes assume that the position is adjusted continuously, whereby the terms $w_{tt}(\Delta t)^2$ and $2w_{St}\Delta S\Delta t$ go to zero and can be neglected. An expression for $(\Delta S)^2$ is

$$(\Delta S)^2 = \sigma^2S^2\Delta t \quad (71)$$

where higher order terms of Δt have been neglected. By combining the equations we get the following expression for the change in option price:

$$\Delta w = w_S\Delta S + w_t\Delta t + \frac{1}{2}w_{SS}\sigma^2S^2\Delta t \quad (72)$$

7.4. The Greeks

Once we have the equation for the price of an option, we can compute derivatives of the equation with respect to the underlying stock price, time, volatility level etc. These can be used to make a portfolio independent of for example small moves in stock price. The name "Greeks" comes from Greek letters often used for denoting the sensitivities. The first and second partial derivatives with

respect to the underlying stock's price, delta and gamma, are used extensively throughout this paper.

7.4.1. Delta

The option price's first partial derivative with respect to the underlying stock's price is called delta. In the previous section, it was denoted w_S and was used to create a portfolio whose value does not depend on small moves in the stock price. Such a portfolio is called delta neutral. Delta is often denoted Δ . Below are the deltas for call and put options:

$$\Delta(\text{call}) = N(d_1) \quad (73)$$

$$\Delta(\text{put}) = N(d_1) - 1 \quad (74)$$

(Hull, 2000)

7.4.2. Gamma

The option price's second partial derivative with respect to the underlying stock's price is called gamma. In the Black-Scholes derivation, we denoted gamma w_{SS} . Gamma is the same for put and call options:

$$\Gamma = \frac{N'(d_1)}{S_0} \quad (75)$$

where N' is the standard normal probability density function:

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (76)$$

Note that $N'(x)$ is not the same as $N(x)$, which is the cumulative probability distribution for a standardized normal distribution. (Hull, 2000)

7.4.3. Theta

Theta, denoted θ , is the option price's sensitivity to passing time. The a bit more complicated equations are

$$\theta(\text{call}) = -\frac{S_0 N'(d_1) \sigma}{2\sqrt{T}} - rK e^{-rT} N(d_2) \quad (77)$$

$$\theta(\text{put}) = -\frac{S_0 N'(d_1) \sigma}{2\sqrt{T}} + rK e^{-rT} N(-d_2) \quad (78)$$

where $N'(x)$ is defined as in equation (76). (Hull, 2000)

7.5. Delta hedging

The Black-Scholes differential equation is

$$w_t + \frac{1}{2} w_{SS} \sigma^2 S^2 = r(w - Sw_S) \quad (79)$$

For simplicity, let us assume zero interest rates. Interest rates are in fact quite close to zero at the time of writing (3M Euribor about 0,2%).

The equation tells us that under the Black-Scholes assumptions, when delta hedging an option, the gamma profit $\frac{1}{2} w_{SS} \sigma^2 S^2$ equals the time value decay of the option. Peter Carr derives the following expression for the delta hedging profit, if one sells an option and continuously delta hedges it:

$$P\&L = [V(S_0, o, \sigma_i) - V(S_0, o, \sigma_h)]e^{rt} + \int_0^T e^{r(T-t)} (\sigma_h^2 - \sigma_t^2) \frac{S_t^2}{2} \frac{\partial^2}{\partial S^2} V(S_t; t; \sigma_h) dt \quad (80)$$

where V stands for option value. σ with sub indexes i , h and t refer to implied, hedging and true volatility. The same assumptions are made as in the Black-Scholes model except for known volatility.

From the expression we see that by selling an option for an implied volatility lower than true volatility, and hedging at the true volatility, you make the following profit:

$$[V(S_0, o, \sigma_i) - V(S_0, o, \sigma_h)]e^{rt} \quad (81)$$

, which is the price difference between the current option price (priced at implied volatility) and the correct option price (priced at true volatility). The profit has no uncertainty.

In reality you of course do not know the true volatility, so a relevant question is whether one should hedge at implied volatility or an estimate of true volatility.

Hedging at implied volatility gives us the integral as profit

$$\int_0^T e^{r(T-t)} (\sigma_h^2 - \sigma_t^2) \frac{S_t^2}{2} \frac{\partial^2}{\partial S^2} V(S_t; t; \sigma_h) dt \quad (82)$$

This profit is path dependent due to the term $\frac{S_t^2}{2} \frac{\partial^2}{\partial S^2} V(S_t; t; \sigma_h)$. You will get profit as long as the true volatility is lower than the implied volatility, which is used for hedging. The amount of profit you get depends on the path that the stock price takes. (Carr, 1999; Ahmad & Wilmott, 2005)

To sum up, the two main problems of delta hedging in a continuous world are that

- 1) You do not know the realized volatility at which it would be attractive to hedge
- 2) The profit is path dependent due to the so called dollar-gamma $\frac{S_t^2}{2} \frac{\partial^2}{\partial S^2} V(S_t; t; \sigma_h)$

In the next chapter we present a solution to these two problems. Even after this you are left with problems arising from discrete rather than continuous delta hedging.

8. The variance swap

In the previous chapter we showed how buying options and delta hedging them gives a payoff that depends both on realized variance and the path that the stock price takes. It would be desirable to have a profit which only depends on realized variance. This chapter presents the theory behind variance swaps as a solution for this problem. A variance swap is a contract where one either pays or receives the difference between realized and expected variance over a time period. Such contracts can be entered with banks. Banks can hedge their risk by delta hedging a certain options portfolio. In this section we describe how this portfolio is built.

8.1. The log contract

In the beginning of the 1990's Neuberger pointed out that there was no proper instrument for hedging or trading volatility. He noted that you had to know the correct volatility, in order to hedge away an option's risk from the underlying asset's price direction. As a solution he presented the log contract. It is a contract that pays the natural logarithm of the underlying asset's price at maturity. He derived the price for such a contract, under Black-Scholes assumptions, to be

$$\ln(F(t)) - \frac{1}{2} \sigma^2 (T - t) \quad (83)$$

where $F(t)$ is the futures price at time T . He noted that the first partial derivative with respect to the underlying's price, the delta

$$\Delta = \frac{1}{F} \quad (84)$$

did not depend on any forecast of volatility. He confirmed in tests, that by delta hedging a log contract, one got accurate exposure to the realized variance. If the delta hedge was adjusted daily, then one got exposure to the daily estimated variance. If the hedge was adjusted every Friday at noon, then one got exposure to realized variance estimated every Friday at noon. (Neuberger, 1994)

The second partial derivative of the log contract, the gamma, is

$$\Gamma = -\frac{1}{F^2} \quad (85)$$

Hence the "dollar gamma" will be constant. This suggests that the log contract is a solution to the path dependency problem. Another way to derive the useful features of a log contract is this:

Consider the following stock price process

$$\frac{dS(t)}{S(t)} = \mu(t, \dots)dt + \sigma(t, \dots)dW(t) \quad (86)$$

Here, both the drift and volatility are functions of for example time and other parameters, and are hence not assumed constant. Here we assume continuous trading. Knowing the process for $S(t)$, the process for $\ln S(t)$ is got from Ito's lemma:

$$d \ln(S_t) = \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dW \quad (87)$$

Subtracting equation (87) from equation (86), we get the following expression:

$$\frac{dS(t)}{S(t)} - d \ln(S_t) = \frac{1}{2} \sigma^2 dt \quad (88)$$

, which can be rearranged to

$$2 \left(\frac{dS(t)}{S(t)} - d \ln(S_t) \right) = \sigma^2 dt \quad (89)$$

Integrating from 0 to T and dividing by T gives the following:

$$\frac{2}{T} \int_0^T \frac{dS(t)}{S(t)} - d \ln(S_t) = \frac{1}{T} \int_0^T \sigma^2 dt \quad (90)$$

The right hand side of equation (90) is the definition of the period's continuously-sampled variance. The left hand side becomes

$$\frac{2}{T} \left[\int_0^T \frac{dS(t)}{S(t)} - \ln \frac{S_t}{S_0} \right] \quad (91)$$

Equation (91) describes the position needed to capture realized variance. $\int_0^T \frac{dS(t)}{S(t)}$ is the payoff from a stock position that is continuously adjusted to consist of $\frac{1}{S_t}$ shares. In other words, the value of the stock position is kept constant. $\ln \frac{S_t}{S_0}$ is a static position in a contract that pays the natural logarithm of the total return on expiry. Alternatively, it can be written $\ln S_t - \ln S_0$, whereby it is a combination of contract that pays the logarithm of the stock price at maturity and a bond. (Demeterfi et al., 1999)

8.2. Replication of a log contract

There are no log contracts available, but they can be replicated using call options. In this section we show the general result that any payoff at time T, which is a twice differentiable function of an asset's market value at time T, is possible to replicate using call options.

We start by defining an “elementary claim” on a stock as a security that pays one dollar payoff at a given time T , only if the stock is worth exactly M . If the stock takes any other value, then the elementary claim will expire worthless. Assume that the stock price at time T has a discrete probability distribution, where M can take the following values:

$$M = \{1, 2, 3, \dots, N\} \quad (92)$$

Denote the price of call options $c(X, T)$, where X is the strike price and T time to maturity. Denote the associated payoff vector for different stock price outcomes $\mathbf{c}(X, T)$ (bold). Figure 3 gives the payoffs for call options with strikes 0, 1 and 2 for different values of the stock M at maturity T :

	$\mathbf{c}(0, T)$	$\mathbf{c}(1, T)$	$\mathbf{c}(2, T)$
$M(T) = 1$	$\begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ N \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ N - 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ N - 2 \end{pmatrix}$
$M(T) = 2$			
$M(T) = 3$			
\vdots			
$M(T) = N$			

Figure 3 The payoffs of call options with strikes 0, 1 and 2 for different stock prices M . The table has been copied from (Breedon & Litzenberger, 1978).

Note from Figure 3 what happens to a call option’s payoff vector, when the strike price is increased by one; all the non-zero payoffs are decreased by one. In other words, when the strike price is increased from X to $X + 1$, the following happens: The payoff from market price $M = X + 1$ goes to zero. The payoffs from market prices $M > X + 1$ are decreased by the increase in strike price. From this follows that in our example $\mathbf{c}(X, T) - \mathbf{c}(X + 1, T)$ gives a payoff of 1 for all $M \geq X + 1$. Equivalently, $\mathbf{c}(X + 1, T) - \mathbf{c}(X + 2, T)$ gives a payoff of 1 for all $M \geq X + 2$.

If you want a payoff of 1 dollar only if $M(T) = 1$, this is done with the following portfolio of options:

$$[\mathbf{c}(0, T) - \mathbf{c}(1, T)] - [\mathbf{c}(1, T) - \mathbf{c}(2, T)] \quad (93)$$

In the same way, one can create a payoff that pays one dollar for any $M(T)$. More generally, the portfolio that gives one dollar as payoff only for M looks like this:

$$\mathbf{c}(M - 1, T) + \mathbf{c}(M + 1, T) - 2\mathbf{c}(M, T) \quad (94)$$

To generalize further, let the step size between possible values of M be ΔM . The portfolio that pays one dollar for a certain M will then be

$$\frac{1}{\Delta M} \{c(M - \Delta M, T) + c(M + \Delta M, T) - 2c(M, T)\} \quad (95)$$

Denote the price of this portfolio P

$$P(M, T, \Delta M) = \frac{1}{\Delta M} \{c(M - \Delta M, T) + c(M + \Delta M, T) - 2c(M, T)\} \quad (96)$$

The expression for the price of the portfolio divided by the step size is

$$\frac{P(M, T, \Delta M)}{\Delta M} = \frac{c(M - \Delta M, T) + c(M + \Delta M, T) - 2c(M, T)}{(\Delta M)^2} \quad (97)$$

Note that when ΔM goes to zero, the above expression becomes the definition of the call option price's second derivative with respect to the stock price M :

$$\lim_{\Delta M \rightarrow 0} \frac{P(M, T, \Delta M)}{\Delta M} = \frac{\partial^2 c(X, T)}{\partial X^2} \Big|_{X=M} \quad (98)$$

P is the price of getting one dollar as payment if a certain M happens. If the price is 0.5, it implies that there is a 0.5 probability that the certain M will happen. In the continuous case, P is analogous to a probability density function. The probability density function is the option's gamma.

To get a portfolio whose payoff is a function q of the market price of the stock, we can hold separate portfolios each paying one dollar for all the possible values of M . If these portfolios are weighted by the function $q(M)$, the result will be a portfolio that pays $q(M)$ for every M . The price of this payoff becomes

$$\int_{M_T} q(M) \frac{\partial^2 C(X = M, T)}{\partial X^2} dM \quad (99)$$

(Breedon & Litzenberger, 1978)

Using a Taylor expansion, the following expression can be derived from equation (99)

$$\begin{aligned} f(F_T) = & f(\kappa) + f'(\kappa)[(F_T - \kappa)^+ - (\kappa - F_T)^+] + \int_0^\kappa f''(K)(K - F_T)^+ dK \\ & + \int_\kappa^\infty f''(K)(F_T - K)^+ dK \end{aligned} \quad (100)$$

The expression explains how the payoff $f(F_T)$ at maturity can be created. The entire expression is static. κ is the option strike which separates liquid puts and calls, and is often assumed to be the

forward level. The first term on the right hand side resembles a position in bonds that pays $f(\kappa)$ at time T, the second term resembles a futures contract with strike κ in the amount of $f'(\kappa)$, the third a continuous strip of puts with strikes up to κ and the fourth a strip of calls with strikes from κ . (Carr & Madan, 1998)

Remember that the following describes how to capture a period's realized variance.

$$\frac{2}{T} \left[\int_0^T \frac{dS(t)}{S(t)} - \ln \frac{S_T}{S_0} \right] \quad (101)$$

The first term comes from continuous balancing. The second term is the function that we wish to replicate using the Breeden-Litzenberger method:

$$f(S_T) = \frac{2}{T} \ln \frac{S_T}{S_0} \quad (102)$$

Let us first compute the derivatives:

$$f' = \frac{2}{T} \frac{1}{S_T} \quad (103)$$

$$f' = \frac{2}{T} \frac{1}{S_T} \quad (104)$$

$$f'' = -\frac{2}{T} \frac{1}{S_T^2} \quad (105)$$

Inserting these into equation (100) gives:

$$\begin{aligned} \frac{2}{T} \left[\ln \frac{S_T}{S_0} \right] &= \frac{2}{T} \left[\ln \frac{\kappa}{S_0} + \frac{1}{\kappa} [(S_T - \kappa)^+ - (\kappa - S_T)^+] + \int_0^\kappa -\frac{1}{K^2} (K - S_T)^+ dK \right. \\ &\quad \left. + \int_\kappa^\infty -\frac{1}{K^2} (S_T - K)^+ dK \right] \end{aligned} \quad (106)$$

Actually, we wanted the negative of equation (106). Accounting for this and rearranging a bit gives

$$\begin{aligned} -\frac{2}{T} \left[\ln \frac{S_T}{S_0} \right] &= \frac{2}{T} \left[\ln \left(\frac{S_0}{\kappa} \right) - \frac{1}{\kappa} [(S_T - \kappa)^+ - (\kappa - S_T)^+] \right] \\ &\quad + \int_0^\kappa \frac{2}{TK^2} (K - S_T)^+ dK + \int_\kappa^\infty \frac{2}{TK^2} (S_T - K)^+ dK \end{aligned} \quad (107)$$

Equation (107) tells us that the payoff $-\frac{2}{T} \left[\ln \frac{S_T}{S_0} \right]$ at maturity is replicated with a bond (short or long depends on κ and F), short position in a futures contract and long position in options ($\frac{2}{TK^2}$ is a positive number). Rearranging this for the payoff of the option portfolio, we get the following expression

$$\begin{aligned} \Pi_{CP}(S_T) &= \int_0^\kappa \frac{2}{TK^2} (K - S_T)^+ dK + \int_\kappa^\infty \frac{2}{TK^2} (S_T - K)^+ dK \\ &= \frac{2}{T} \left[\frac{1}{\kappa} [(S_T - \kappa)^+ - (\kappa - S_T)^+] - \ln \left(\frac{S_0}{\kappa} \right) - \ln \frac{S_T}{S_0} \right] \\ &= \frac{2}{T} \left[\frac{S_T - \kappa}{\kappa} - \ln \left(\frac{S_T}{\kappa} \right) \right] \end{aligned} \quad (108)$$

In reality, there is not an infinite strip of options available to compute the payoff. The payoff from the continuous strip of options will have to be approximated using the options that are available. The convex payoff function $f = \Pi_{CP}(S_T)$ will be approximated by a piece-wise linear function.

The weights of the available options are got by iteration. The weight of the first call option (with strike K_0) is

$$w(K_0) = \frac{f(K_{1c}) - f(K_0)}{K_{1c} - K_0} \quad (109)$$

The weight of the second call option is equivalent, but we have to consider the contribution of the first call option. Because of this contribution, we will have to add less of the second call option.

$$w_c(K_1) = \frac{f(K_{2c}) - f(K_{1c})}{K_{2c} - K_{1c}} - w(K_0) \quad (110)$$

The general formula for the weight of call options w_c and put options w_p is

$$w_c(K_{n,c}) = \frac{f(K_{n+1,c}) - f(K_{n,c})}{K_{n+1,c} - K_{n,c}} - \sum_{i=0}^{n-1} w_c(K_{i,c}) \quad (111)$$

$$w_p(K_{n,p}) = \frac{f(K_{n+1,p}) - f(K_{n,p})}{K_{n+1,p} - K_{n,p}} - \sum_{i=0}^{n-1} w_p(K_{i,p}) \quad (112)$$

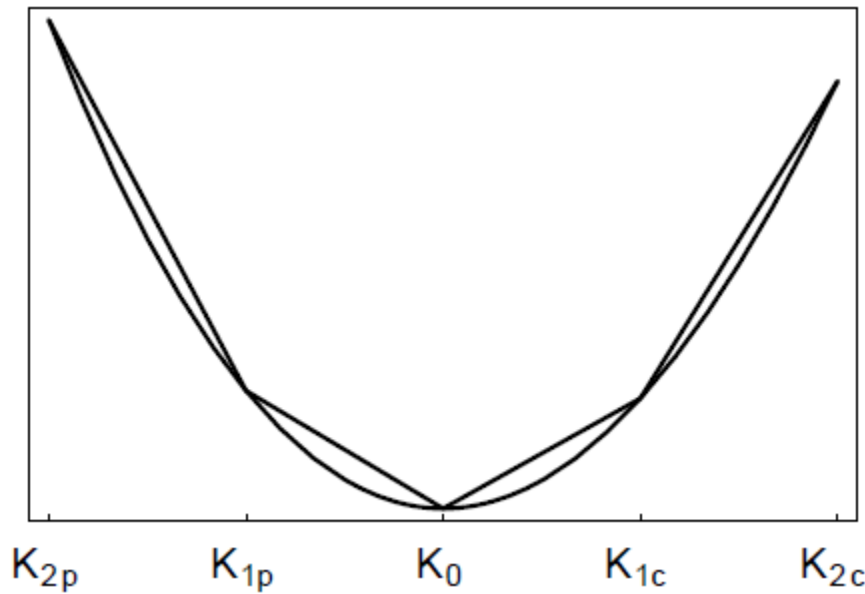


Figure 4 Illustration of how the convex payoff function is replicated by a piecewise linear function. Note that the piecewise linear approximation will be equal to or larger than the convex function for all values. The picture is taken from (Demeterfi et al., 1999)

As can be seen from Figure 4, the value of the approximation will always be higher than or equal to the intended function value. Hence, the strategy will give more payoff than the true variance in ideal conditions. In reality trading costs might of course cancel the extra payoff and more. How many options to use for the replication is an interesting question which will be explored in the empirical part.

8.3. Connection to market forecast of volatility

Market expectations of future realized volatility are often used as a barometer for the condition and vulnerability of the economy and financial markets. Even monetary policy can be affected by these expectations. (Poon & Granger, 2003) In this section we present how the correct market expectation is formed. The VIX-index is formed in a similar way.

8.3.1. Computing the market expectation

In Section 8.1 we derived the following expression:

$$\frac{1}{T} \int_0^T \sigma^2 dt = \frac{2}{T} \left[\int_0^T \frac{dS(t)}{S(t)} - \ln \frac{S_t}{S_0} \right] \quad (113)$$

A continuously balanced position in the stock has a forward price which grows at the risk-free rate r :

$$\frac{dS(t)}{S(t)} = rdt + \sigma dW(t) \quad (114)$$

$$E \left[\int_0^T \frac{dS(t)}{S(t)} \right] = rT \quad (115)$$

The remaining part is

$$-\frac{2}{T} \left[\ln \frac{S_T}{S_0} \right] = \frac{2}{T} \left[\ln \left(\frac{S_0}{\kappa} \right) - \frac{S_T - \kappa}{\kappa} \right] + \Pi_{CP} \quad (116)$$

Next we take the expectation of this expression at time T. The strike κ of the futures contract is chosen so that the expected profit from it is zero. If the strike is chosen differently, the expected payoff is given by

$$E \left[\frac{S_T - \kappa}{\kappa} \right] = \frac{S_0}{\kappa} e^{rT} - 1 \quad (117)$$

The market price or present value of the options portfolio can be thought of as the discounted value of the payoffs at time T. The expected payoff is the market price adjusted for the discount factor.

$$E[\Pi_{CP}] = e^{rT} \Pi_{CP, \text{present value}} \quad (118)$$

Combining the above equations and using the market price for the options portfolio, we get the market expectations of fair variance K_{var} during the coming time period as:

$$K_{var} = \frac{2}{T} \left[rT - \left[\frac{S_0}{\kappa} e^{rT} - 1 \right] + \ln \left(\frac{S_0}{\kappa} \right) \right] + e^{rT} \Pi_{CP, \text{present value}} \quad (119)$$

(Demeterfi et al., 1999)

8.4. Sensitivity to jumps in the stock price

Stock price jumps affect the replicating portfolio of a variance swap in two ways. First, the price might jump so that it exceeds the limited range of option strikes in the portfolio (think for example of bio-tech companies in a situation of a new successful drug for cancer). Secondly, a jump causes the replicating strategy to capture a different amount than the true realized variance. In this section we look at how jumps affect the payoff in theory. In the empirical part we will analyze the effect that jumps might have in reality.

The starting point to assess the effect of jumps is the additive feature of variance. Assuming here that a jump occurs only over one time step, the expression for variance can be written as follows:

$$V = \frac{1}{T} \left[\sum_{i=1}^N \left(\frac{\Delta S_i}{S_{i-1}} \right)^2 \right] = \frac{1}{T} \sum_{no\ jumps} \left(\frac{\Delta S_i}{S_{i-1}} \right)^2 + \frac{1}{T} \left(\frac{\Delta S}{S} \right)^2_{jump} \quad (120)$$

Here, the root mean square is used as an approximation of variance. Let J be a variable for the magnitude of the jump so that $J = 0,1$ corresponds to a 10% jump downwards in the asset price. A jump upwards has a negative J . The contribution of the jump can be written using J as follows:

$$\frac{1}{T} \left(\frac{\Delta S}{S} \right)^2_{jump} = \frac{J^2}{T} \quad (121)$$

Recall the expression for the variance-replicating strategy:

$$V \equiv \frac{1}{T} \int_0^T \sigma^2 dt = \frac{2}{T} \int_0^T \frac{dS(t)}{S(t)} - d \ln(S_t) = \frac{2}{T} \left[\int_0^T \frac{dS(t)}{S(t)} - \ln \frac{S_t}{S_0} \right] \quad (122)$$

Relaxing the assumption of continuity, the equation (122) becomes

$$V = \frac{2}{T} \left[\left(\sum_{i=1}^N \frac{\Delta S_i}{S_{i-1}} \right) - \ln \frac{S_t}{S_0} \right] = \frac{2}{T} \left[\sum_{i=1}^N \frac{\Delta S_i}{S_{i-1}} - \ln \frac{S_i}{S_{i-1}} \right] \quad (123)$$

The variance captured by this replicating strategy over one time step in case of a jump can be written in the following way:

$$\frac{2}{T} \left(\frac{\Delta S_i}{S_{i-1}} - \ln \frac{S_i}{S_{i-1}} \right)_{jump} = \frac{2}{T} [-J - \ln(1 - J)] \quad (124)$$

To see the error in the replicating strategy when a jump occurs, we subtract the effect that the jump has on variance from the effect it has on the replicating strategy. This can be thought of as the P&L of a replicating strategy due to a jump.

$$P\&L\ due\ to\ a\ jump = \frac{2}{T} [-J - \ln(1 - J)] - \frac{J^2}{T} \quad (125)$$

The Taylor expansion of the natural logarithm gives the following:

$$-\ln(1 - J) = J + \frac{J^2}{2} + \frac{J^3}{3} + \dots \quad (126)$$

Combining the equations (125) and (126) gives

$$P\&L\ due\ to\ a\ jump = \frac{2}{T} \left[-J + J + \frac{J^2}{2} + \frac{J^3}{3} + \dots \right] - \frac{J^2}{T} = \frac{2J^3}{3T} + \dots \quad (127)$$

The linear and quadratic terms are the same for the variance and the replicating strategy, so they cancel out. The leading error is cubic. A large move downward ($J > 0$) will lead to a profit if you are long the replicating strategy or short the variance swap and vice versa.

Figure 5 illustrates the effect of a single jump on a short position in a variance swap (long position in replicating strategy) when $T=1$ year.

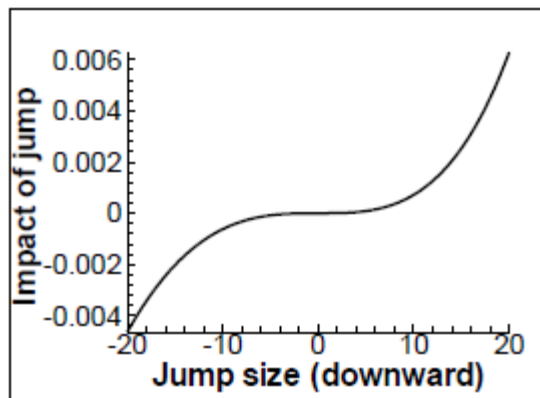


Figure 5 Illustration of how a jump in stock price affects the variance swap replicating strategy.

Picture is taken from (Derman, 1999).

9. The empirical part

The aim of the empirical part is to build an understanding of how well the theory might work in practice and what the main problems are. We compare two main methods for trading variance; buying and delta hedging a single option and doing the same for a variance swap replicating portfolio.

9.1. Research questions

The research questions include:

- Can a replicated variance swap replace a real variance swap entered with a bank?
- Does a variance swap replicating strategy give better results than simply delta hedging a single option? We will refer to the variance swap replicating strategy as the “sophisticated” strategy and hedging a single option as the “simple” strategy.

The first question is interesting for example if variance swaps aren’t available to you. What should you do in such a situation? The second question is important, since the traditional do-it-yourself solution is to buy an option and delta hedge it.

9.2. Research methods

We use simulation to answer the above questions. To conduct the research, a Matlab script is written. These are the inputs for the script:

- an arbitrary amount of daily closing prices
- length of time period, in which the closing prices are split
- distance between option strikes as percentage of index level on first day of period
- amount of options used (same for calls and puts)
- transaction cost as percentage of deal

The script first splits the closing prices to periods of desired length. For all periods, the option portfolio for the variance swap replicating strategy is created using the given specifications.

Transaction costs are not examined at this time, since the focus is on understanding the dynamics of delta-hedging. Transaction costs are also very individual. For small investors, trading costs might make delta hedging unprofitable regardless of how liquid the underlying asset is. For financial institutions this is a smaller problem, unless the underlying is less liquid and hence has a large spread.

9.3. Source of input data

Two sources of price data were considered:

- real closing prices from the S&P 500 index and
- randomly generated lognormal price data

We chose to focus on the latter. The first reason for this is that by knowing the exact features of our price data, we know what kind of results to expect and have a better chance of noticing the relevant differences between theory and practice. If we used real price data, it would be more difficult to tell whether discrepancies between theory and practice depend on the input or on the model.

A second reason is that we want to compare the performance between the simple and the sophisticated strategy. For the simple strategy we need to know what volatility to use for hedging. A simple way to do this is to create prices whose return volatilities are known, and use this as hedge volatility. Note that this is different from taking a period with a certain realized volatility and hedging with this, since the realized volatility will be a random variable. The magnitude of the error from not knowing the correct delta is assessed by running simulations with different hedging volatilities.

9.4. How to choose the options for the replicating strategy

The key decisions for the strategy are

- how many options to use i.e. how far from the current level do we buy options and
- what is the spacing between the bought options.

To decide how far from the initial level we should buy options, we take a look at the price history of the S&P 500 index from January 1950 to April 2012. The period is divided into periods of 3 months starting from the first day. Three months as time horizon was chosen randomly. Below is a histogram of the three-month-returns:

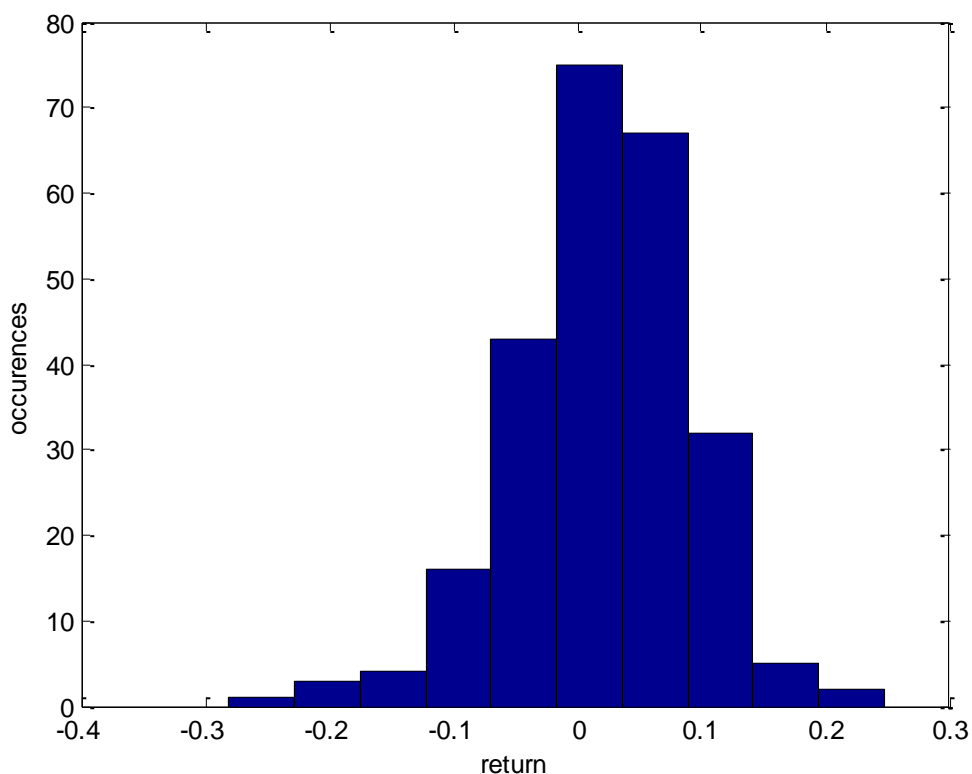


Figure 6 Histogram of three-month returns of the S&P 500 index from the 1950's. Closing prices taken from Yahoo Finance.

From Figure 6, we see what was discussed in Section 4.7; returns do not seem normally distributed.

It's good to keep in mind that the above analysis depends on how the three-month-periods happen to be chosen. By choosing the periods differently, periods with even larger absolute returns might have been found.

As variance swaps are often done for hedging purposes, it is desirable that the strategy works also in rare circumstances. Looking at the histogram, we found that holding options until 25% above and below the initial index level might be reasonable. Even if the asset price moved outside the covered area, chances are that the area was covered for most of the time. If available, even more options naturally give more security.

The other question is what distance between option strikes gives sufficiently constant gamma. To start off, we use 5% from the initial index level as distance between strikes.

9.5. Test setting

Two ways are used to analyze the performance of the variance swap replicating strategy:

- First we take a look at what kind of deltas and vegas are achieved
- Then we look at actual payoffs, comparing the performance of a variance swap replicating strategy to delta hedging a single call option

We chose to examine three-month-periods. For the complicated strategy five call and five put options are used. The distance between the strikes is 5%. So if the initial stock price level is 100, our portfolio will consist of

- call options of strikes 100, 105, 110, 115 and 120
- put options of strikes 100, 95, 90, 85 and 80

We created 10 years of lognormal closing prices with drift zero and standard deviation 20%.

9.6. The achieved deltas and gammas

Knowing the correct delta is important in order to know how much stock to hold so that the profit does not depend on whether prices go up or down. The delta of a log-contract is $1/S$. Figure 7 illustrates the delta of a portfolio which is built using 10 options. The initial stock price level is 100. Call options are bought with strikes 100, 105, 110, 115 and 120. Put options are bought with strikes 80, 85, 90, 95 and 100. The red line in the plot is the desired delta. This is $-1/S*2/T$. T is a quarter of a year in this example. The blue, green and yellow lines are the deltas of the replicating portfolio at the beginning of the period, at the middle of the period and one week before expiration respectively. We see that the replication is best at the middle of the option range. The replication becomes more exact as time passes and quickly becomes inexact when the stock price moves outside of the area of bought options. Based on the plot, the approximation becomes inexact faster when the asset prices fall than when they rise.

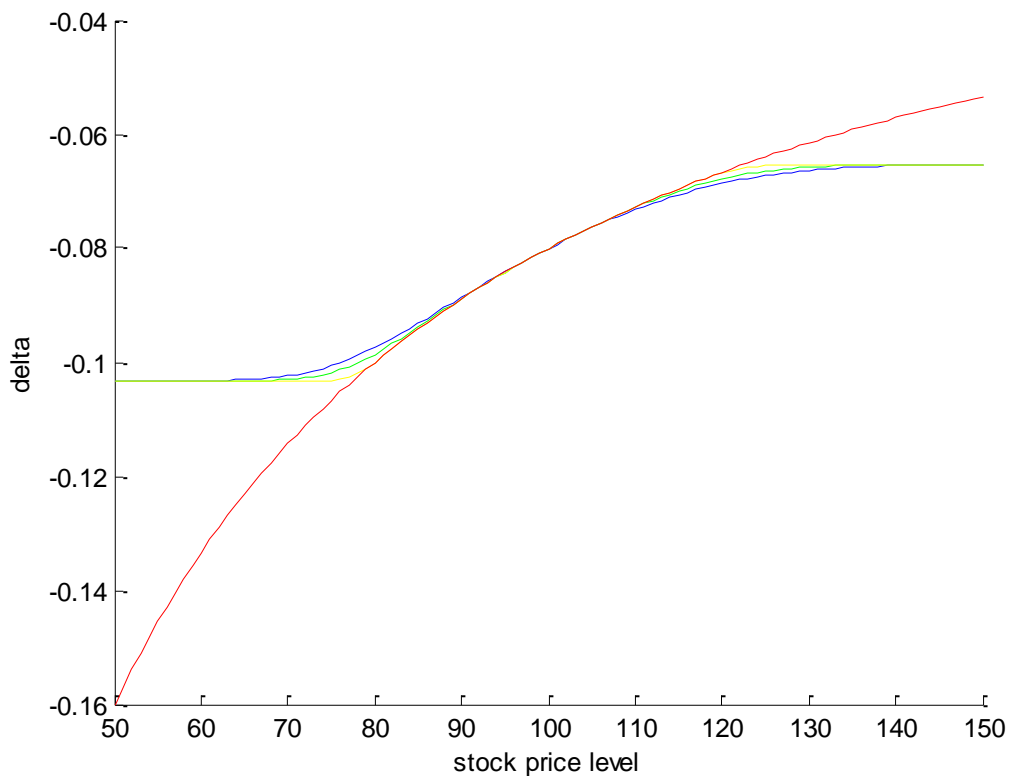


Figure 7 Plot of ideal delta in red. In blue, green and yellow we have the achieved deltas at the beginning, halfway in the period and one week before expiration. Three-month-periods were analyzed.

Holding a position in $2/T$ times $1/S$ stocks is a good delta hedge if the stock price lies in the option strike interval. After this, the hedge becomes inexact. In this case either the portfolio of options has to be reweighted or then the delta of the portfolio has to be calculated with the Black-Scholes formula.

As explained in the theory part, gamma plays a big role if one wants to trade variance. In order to capture variance as profit, gamma times S^2 (so called dollar gamma) should be a constant. In other words, gamma should be proportional to $1/S^2$.

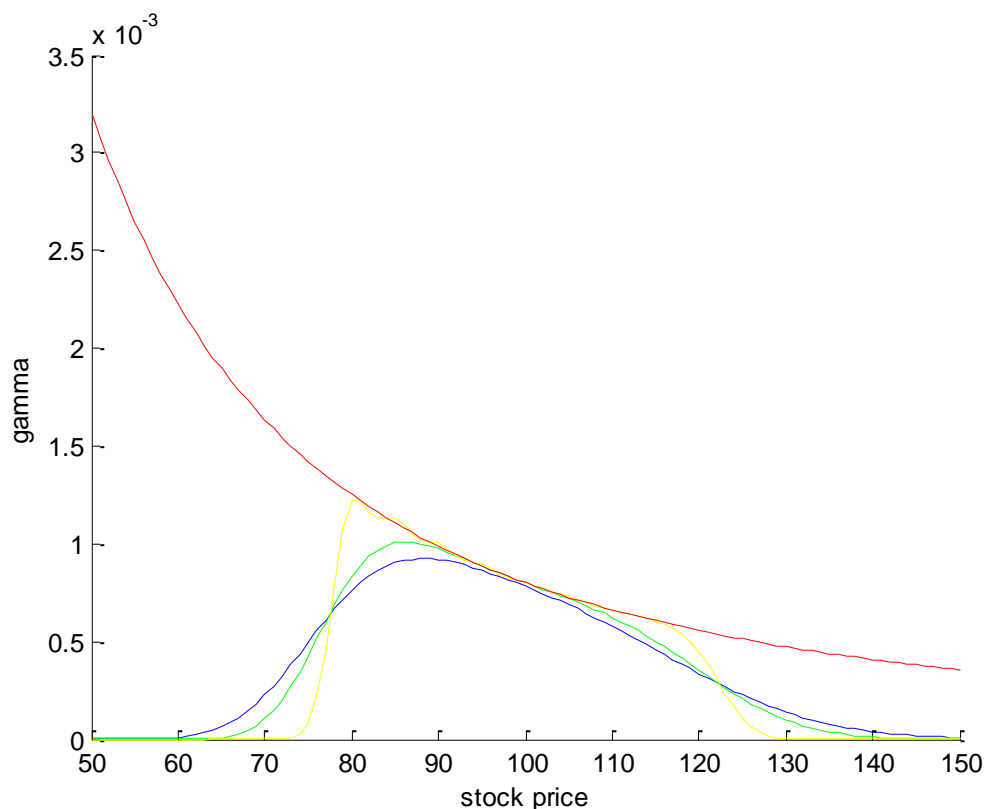


Figure 8 Plot of ideal gamma in red. In blue, green and yellow we have the achieved gammas at the beginning, halfway in the period and one week before expiration. Three-month-periods were analyzed.

Figure 8 shows the same for gamma as Figure 7 for delta. The red line is the gamma of an ideal log-contract. The blue, green and yellow lines are the gammas of the replicating portfolio at the beginning, half-way in the period and one week before expiry. We see that the replication is exact only close to the middle of the covered range. The replication noticeably improves closer to maturity. Closer to maturity the replication is more exact in the covered area, while it is more sensitive to the stock price moving outside of the covered range.

The gammas are calculated using a volatility level of 20%.

In practice, this means that the strategy is more vulnerable to the underlying drifting away from the initial level in the beginning of the period than towards the end of the period. As with delta, the gamma replication seems to deteriorate faster when the underlying price drifts below the covered strikes than when it moves above them.

We conclude that we can be fairly satisfied with the achieved deltas and gammas under normal circumstances, but have to be careful if the underlying moves outside the covered strikes.

9.7. Comparing payoffs

A natural question is how much there is to gain from the more sophisticated method compared to just buying a single option and delta hedging it with the Black-Scholes delta.

To test this, 10 years of ideally distributed daily closing prices with volatility 20% are generated. Drift is zero. The daily closing prices are divided into three-month-periods. The more sophisticated portfolio is generated using 5 call and 5 put options, starting from at-the-money. The distance between strikes is 5% of initial stock price level. These numbers were chosen arbitrarily.

The price of all the option portfolios was 0.0415. This is because all the option portfolios were built using the same assumptions. The option portfolios were built in such a way, that their price is the market forecast of variance. The reason for the price being higher than 0.04 is the way in which the log contract is replicated; the payoff function will be slightly higher than it should be when the function is approximated using few options (see Figure 4).

For comparison, one at the money call option with the same price 0.0415 is bought and delta hedged according to the Black-Scholes delta. In a way, this is an unfair test setting, since we now know the correct delta to hedge with. In the real world, one of the benefits of the more sophisticated method is that you do not need to know the realized volatility.

Figure 9 shows the results. In blue are the payoffs from delta hedging a single call option. In red are the corresponding payoffs from the more advanced method. In green is the number produced by a day-to-day estimator. Remember that the initial price is 0.0415, so a payoff below that level will mean a loss.

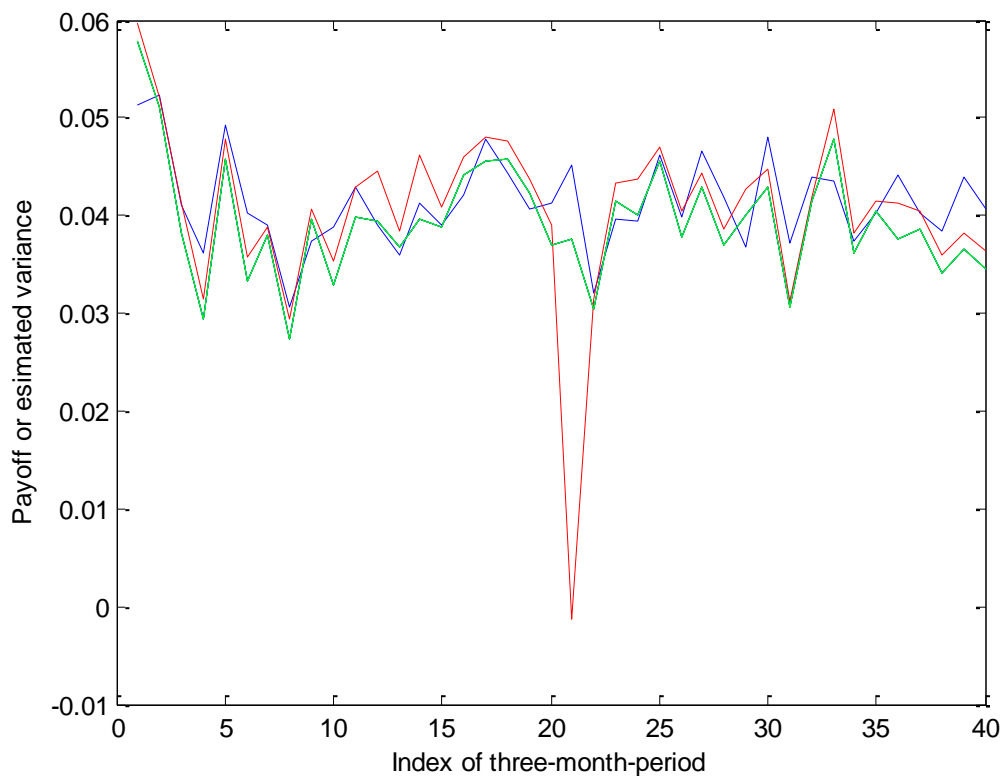


Figure 9 Payoff from delta hedging a call option is depicted blue. In red we have the payoff from a variance swap replication strategy. In green is the close-to-close estimate of variance.

The first reaction is that there does not seem to be a huge difference in performance between the two strategies, except in period 21. The poor performance in period 21 is explained by a rare price development, where the price drifts upwards during the entire period. Because of this, the stock price ends up far outside the area covered by options. Consequently the delta used for balancing ($1/S$) is no longer applicable. The traditional method does a far better job, since the delta is more correct, since it is calculated with the Black-Scholes model. In Figure 10 are depicted the price development in period 21 in blue and the development in period 22 in green, for comparison.

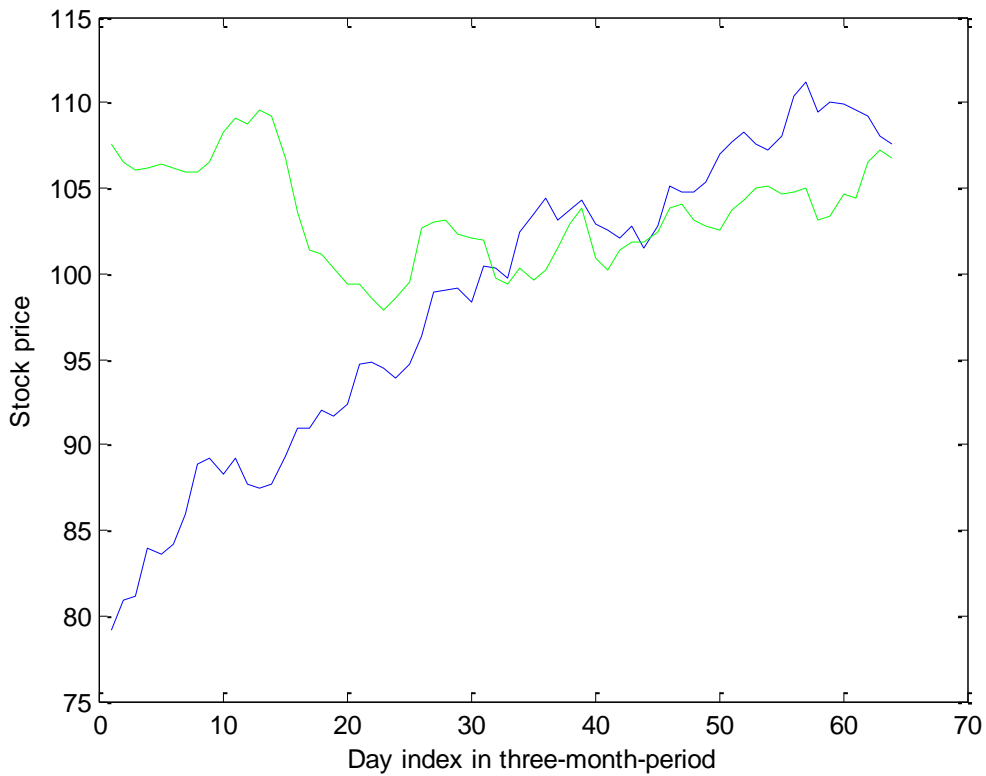


Figure 10 The price development in period 21 in blue. For comparison, in green we have a more normal price development from period 22.

The next task is to assess the effect of not knowing the correct hedging volatility has. In Figure 11 we have added have the result from using a hedge volatility of 24% in cyan and in green the result from using 16% as hedge volatility.

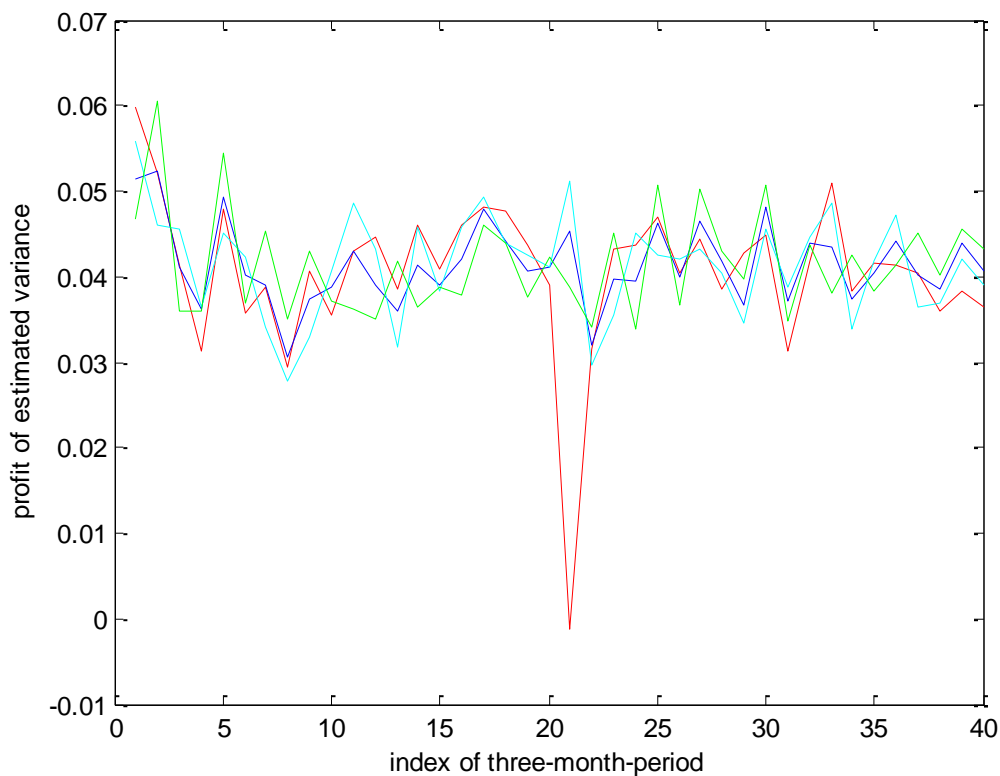


Figure 11 Otherwise same as Figure 9, but added result from using hedging volatility 24% in cyan and 16% in green.

Judging qualitatively from the graph, the wrong volatility used for delta hedging is not a leading source of error. The biggest source of error seems to arise from discrete rather than continuous hedging.

9.8. Quantitative results

To form a more rigorous comparison of the two methods (hedging variance swap replicating portfolio and delta hedging a single call option) 1 000 years of daily data is generated. The tests are performed on this longer set of data, in order to get statistically more significant results. The price of the variance swap replicating portfolio is 0.0415. In the competing strategy, call options worth the same amount 0.0415 are bought. The net profits are calculated (payoff – initial price) and the result is divided by the initial price to get a return. The returns are plotted in Figure 12 and Figure 13.

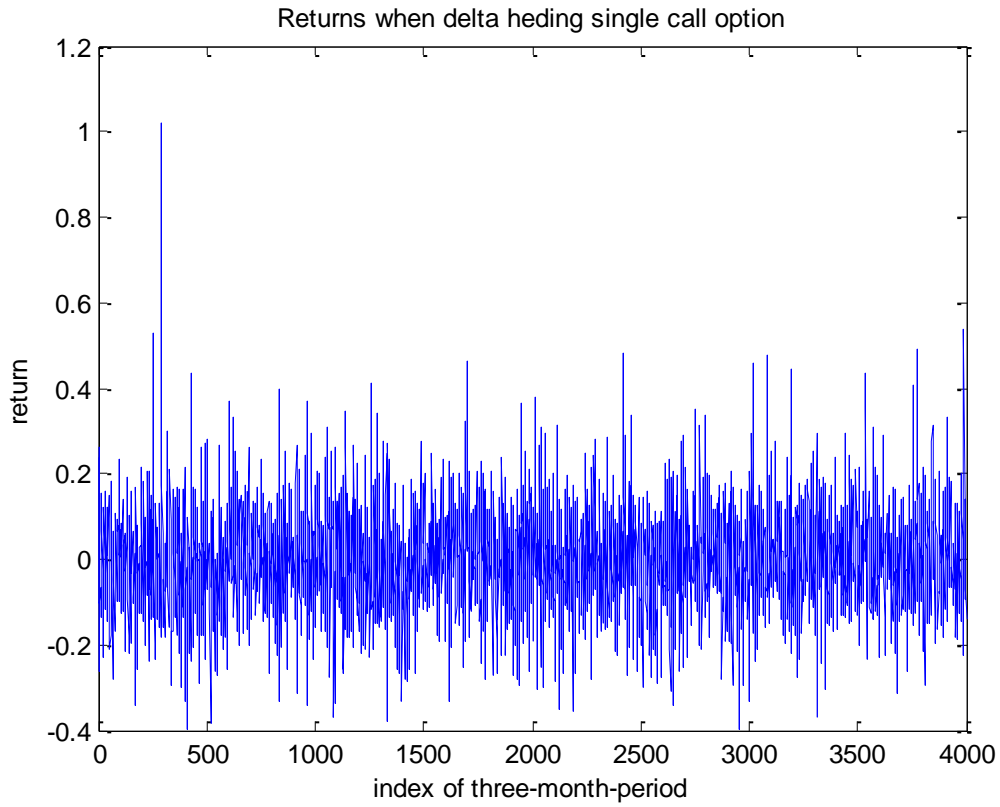


Figure 12 Returns in 4000 periods (10 years of three-month-periods), when buying a single call option and delta hedging.

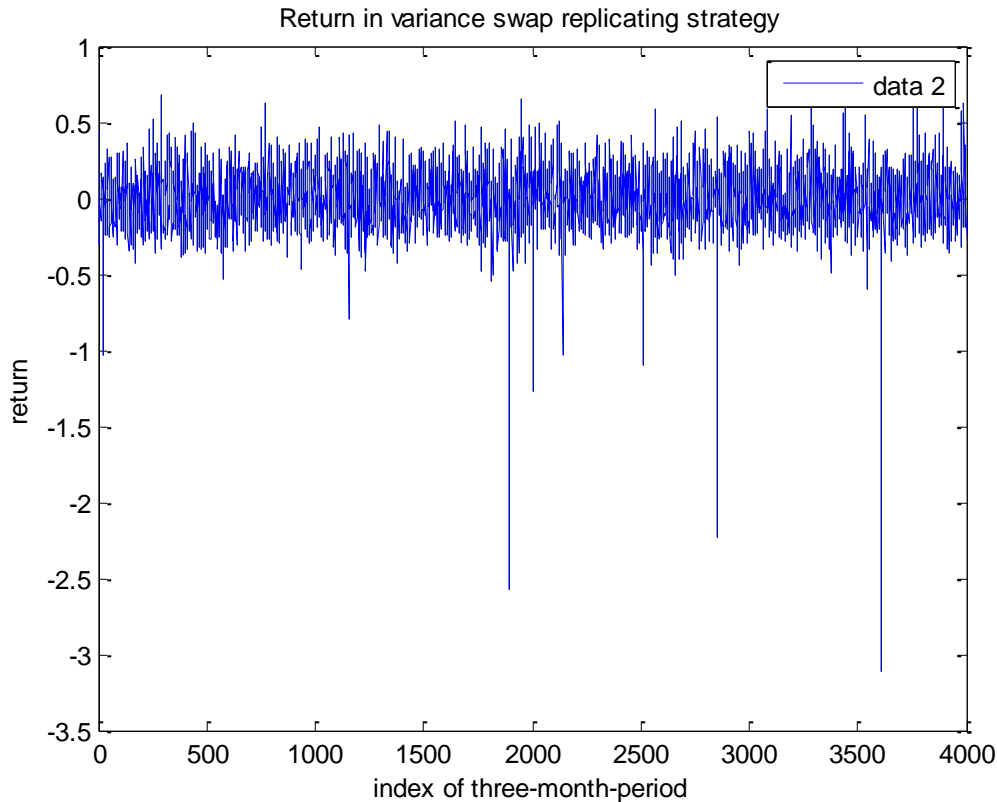


Figure 13 Returns in 4000 periods (10 years of three-month-periods), when buying a variance swap replicating strategy and delta hedging.

The mean return in both the strategies is zero, as expected. The standard deviation is 11% in the simple strategy and 19% in the more advanced strategy. These numbers should be multiplied by 4 to get an annualized standard deviation. The standard deviations are large, which means that the strategies carry a lot of risk.

Here we knew to hedge with the correct volatility of 20%, a luxury we don't have in real life. However, in the previous section our assessment was that the correct hedging volatility was not a leading source of error.

Clearly, hedging at the Black-Scholes delta is a safer method, since there is less risk for large discrepancies between the ideal delta to be used and the one that in fact is in use. This means a more pure exposure to variance.

Intuitively, it is clear that having options of several strikes is safer than just having a single call option. The benefit from this should be even larger in the real world, where returns aren't normally distributed. Perhaps the ideal way to trade is to buy a few options and hedge with the Black-Scholes delta.

10. Conclusions

A variance swap is defined so that the buyer receives the daily estimated realized variance and pays fixed variance. We have gone through the theory of how such a position is hedged. The problem seems to be that options lose value according to a continuous model, while discrete hedging delivers a random payoff. Daily delta hedging carries a significant amount of risk, even if you knew all the necessary variables.

Financial institutions are better at bearing this risk than single investors. Hence there is value in buying a variance swap from an institution rather than hedging yourself.

Banks also do not have to replicate all their positions, since a large part of their positions will be opposite and cancel. It is enough to hedge the smaller net exposure.

In the tests, using 10 options did not provide far better performance than buying a single call option and delta-hedging that. We conclude that it is not at least clear that the more sophisticated way would be worth the extra effort.

During the writing of this thesis I have become less critical towards the Black-Scholes model and its assumptions. It actually does not seem like such a bad model, provided that you put effort in understanding the assumptions. It is not a bad comparison that Derman makes in his paper from 2010, when he compares models in finance to more simple models like estimating flat prices by price per square meter. Simplified models help organize your thought. Then you adjust the result by taking into consideration factors not covered by the model.

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Appendix

Here is the code for the Matlab script used in the empirical part.

```
function [S]=logstock(s0, mu, sigma, years)
    %Function for creating daily lognormal price data for testing. s0 is the
    %starting price, mu is drift, sigma is standard deviation and years the
    %length of the desired time period in years.

    %number of trading days in a year
    h=252;

    S=zeros(1,h*years);

    %time step 1 day expressed in years
    dt=1/h;

    %starting price
    S(1)=s0;

    for i=1:1:years*h-1

        %Normal random variable, mean 0 and variance 1
        norm=normrnd(0,1);

        %The following value of the stock is evaluated
        S(i+1)=S(i)*exp( (mu-sigma^2/2)*dt+sigma*sqrt(dt)*norm );

    end;

    S=S';

end

function [ profit_balancing, profit_options, profit_futures, captured, estimated, calls, puts, futures_strikes,
split_data, daily_balancing_profits, total_captured_stdev, total_estimated_stdev, mean_e, median_e,
balancing_costs, special_cost, captured_with_costs, error, captured_stdev, estimated_stdev,
captured_with_costs_stdev, prices, vegas] = payoff4( data, T, step, amount, cost )
%function that gathers all the other functions. Goal is to build a
%variance swap replicating portfolio, delta hedge it, and realize a profit
%directly proportional to the period's realized variance. The test does not
%take into consideration the cost of buying the options portfolio, only
%whether the portfolio can deliver a payoff directly proportional to the
%periods realized variance given different assumptions about options used
%and cost from delta-hedging. In other words, it does not take a stance on
%whether the options are fairly priced.
```

```

%First step is to divide the given data into test periods of desired
%length. The one column of daily closing prices (data) is split into
%periods of T years. Every period is in a separate column after split. The
%last price of the columns is the same as the first price in the next.
split_data=dele2(data, T);

%The futures strikes are determined as the first price in every period. The
%futures are needed for the replicating strategy.
futures_strikes=future2(split_data);

%The payoffs from daily balancing of stock position, payoff from options
%and payoffs from futures contract are gathered separately for every
%period. Here is the payoff from balancing the stock position.
[daily_balancing_profits]=balancing2(split_data);
profit_balancing=sum(daily_balancing_profits);

%option strikes, weights, payoffs, prices and vegas calculated by function
%options4
[calls, puts, prices, vegas]=options4(split_data, step, amount, futures_strikes(1,:));
profit_options=sum(calls(:,3))+sum(puts(:,3));

%profit from futures
profit_futures=futures_strikes(2,:);

%The payoff from the investment strategy in each period is the sum of the
%three parts. In theory, the payoff should equal the periods' realized
%variance. The payoff is here called captured as in "captured variance".
%The variance swap replicating strategy is weighted in such a way that the
%payoff is the annualized relized variance (periods variance times 12/3 for
%a three-month period).
captured=profit_futures+profit_options+profit_balancing;
captured_stdev=sqrt(profit_futures+profit_options+profit_balancing);

%estimated variance for each period
estimated=var(returns2(split_data))*(252);
estimated_stdev=sqrt(var(returns2(split_data))*(252));

%total captured variance expressed as annualized standard deviation.
%sum(captured) is the sum of the annualized variances from all the periods.
%We multiply this by T to get rid of the annualization made in strategy.
%The result is divided by the total length of the time horizon. The result
%is hence the annualized variance of the entire time horizon. Taking the
%square root of this gives the equivalent standard deviation (which is not
%yet adjusted for Jensen's equality to be exact)
total_captured_stdev=sqrt(sum(captured)*T/(length(data)/252));

%total estimated variance expressed as annualized standard deviation
total_estimated_stdev=sqrt(sum(estimated)*T/(length(data)/252));

```

```

%difference between captured and estimated variance in each period
error=captured-estimated;

%mean error
mean_e=mean(error);

%median error
median_e=median(error);

%trading costs. daily balancing costs are the costs from adjusting a
%position. Special costs are costs from entering the initial stock position
%or liquidating the final stock position.
[daily_balancing_costs, special_cost]=trading_costs(split_data, cost);

%sum of balancing costs
balancing_costs=sum(daily_balancing_costs);

%captured variance when balancing costs are accounted for. This does not
%yet take into consideration cost from buying the initial position and
%liquidating the position in the end.
captured_with_costs=captured+balancing_costs;
captured_with_costs_stdev=sqrt(captured+balancing_costs);

end

```

```

function [split] = dela2(data,T)
%This function splits the daily prices into shorter periods, specified
%by T, which is the period's length in years. The variance swap
%replicating strategy is performed on these periods separately. The
%vector "data" contains a list of an arbitrary amount of daily prices.
%A time period of three months resembles a T of 3/12.

%How many daily returns per period.
returns_per_period=T*252;

%Amount of daily prices available
total_daily_prices = length(data);

%How many whole periods of specified length can be created from the
%given data. (total_daily_prices-1) because n prices give n-1 returns.
total_periods_possible=floor((total_daily_prices-1)/returns_per_period);

%check if input data has enough days to compute at least one period.
if total_periods_possible==0
    split='two short period of data for that'
    return
end

```

```

% Amount of price observations per period needed.
days_per_period=returns_per_period+1;

% Indexing variable used for printing the last price of every period
% also as the first price of the next period. It is assumed that the
% investor puts on a new position immediately when the old position
% expires, using the same underlying price. Else the investor would have
% a day with no position between every period.
day=1;

% for loops that split the data in periods containing T*252+1 price
% observations, which are converted to T*252 returns by another function
for k=1:total_periods_possible
    day=day-1;

    for i = 1:days_per_period;
        day=day+1;
        split(i,k)=data(day,1);
    end

end

end

function [ futures, bond] = future2( split_data)

% Function that gives the futures strikes for each period as the stock price
% in the beginning of each period. The futures prices are saved on the first
% row of the array "futures". On the second row are saved the payoffs from
% owning 2/(TK) futures contracts at the end of each period. T is the length
% of the periods in years and K is the futures level.

% In "bond" the payoffs from the bonds that are used for the replication are
% saved. Since the futures prices and option strikes that separate liquid
% calls and liquid calls are assumed to be the same in this testing
% environment, the bond payoffs will be zero.

% amount of rows and columns in input data
[down, sideways]=size(split_data);

T=(down-1)/252;

% the strikes
for i=1:sideways

    futures(1, i)=split_data(1,i);

```



```

end

s0=futures;

%the payoffs from a short futures contract, weighted by 2/T and
%bondpayoff(which will be zero in this example)
for i=1:sideways

    futures(2,i)=-(split_data(down,i)-futures(1,i))/futures(1,i)*(2/T);
    bond(1,i)=log( s0(1,i)/futures(1,i))*2/T;
end

end

function [ profit ] = balancing2( data)
    %Function that gives the payoff from daily balancing of stock position
    %to contain 1/stock_price of stock. In other words, the stock position
    %is daily balanced to be worth 1 dollar. Takes as input the stock
    %prices split in periods by the function dela2. The sum of the
    %balancing profits is how much more or less money you have in the end
    %than in the beginning in dollar terms.

    %size of input data
    [down, sideways]=size(data);

    %length of period in years. Note that there is (x-1)/252 years between
    %the first closing price and the last closing price, assuming 252
    %trading days in a year.
    T=(down-1)/252;

    %for each period
    for n=1:sideways;

        %daily profits
        for i=1:down-1
            profit(i,n)=2/(data(i,n)*T)*( data(i+1,n)-data(i,n));

        end

    end

end

function [calls, puts, prices, vegas] = options4(data, step, amount, futures_strikes )
    %"calls" and "puts" are three dimensional arrays. The first layer contains

```

```

%the strikes of the options. The second layer contains their weight in
%option portfolio. The third layer contains their payoffs at maturity. The
%forth layer stores the option price and the fifth their vegas. Step is the
%distance between available option strikes as percentage of stock price
%level. Amount is how many calls and how many puts that are bought for the
%replicating strategy. Futures_strikes are the futures strikes at the
%beginning of each period, calculated by another function.

```

```

%size of input data

```

```

[down, sideways]=size(data);

```

```

%length of time horizon

```

```

T=(down-1)/252;

```

```

%the forward level for each period

```

```

forward_strike=futures_strikes(1,:);

```

```

%for both calls and puts, the strike of the first option is the forward
%level

```

```

calls(1,:,1)=futures_strikes(1,:);

```

```

puts(1,:,1)=futures_strikes(1,:);

```

```

%better to use step relative to index level:

```

```

step1=step/100;

```

```

%gives the strikes

```

```

for n=1:sideways;

```

```

    step(1,n)=step1*futures_strikes(1,n);

```

```

    for i=2:amount

```

```

        calls(i,n,1)=calls(i-1,n,1)+step(1,n);

```

```

        puts(i,n,1)=puts(i-1,n,1)-step(1,n);

```

```

    end

```

```

end

```

```

%gives the weights of the . Help function "value" is used, see further

```

```

%down.

```

```

%for every period

```

```

for n=1:sideways;

```

```

    %weights of the first call and put options (at the money)

```

```

    calls(1,n,2)=(value(calls(2,n,1), forward_strike(1,n), T)-value(calls(1,n,1), forward_strike(1,n),
T))/(calls(2,n,1)-calls(1,n,1));

```

```

    puts(1,n,2)=(value(puts(2,n,1), forward_strike(1,n), T)-value(puts(1,n,1), forward_strike(1,n), T))/-
(puts(2,n,1)-puts(1,n,1));

```

```

% weights of the other options except the last
for i=2:amount-1

    calls(i,n,2)=(value(calls(i+1,n,1), forward_strike(1,n), T)-value(calls(i,n,1), forward_strike(1,n),
T))/(calls(i+1,n,1)-calls(i,n,1))-sum(calls(1:i-1,n,2));
    puts(i,n,2)=(value(puts(i+1,n,1), forward_strike(1,n,1), T)-value(puts(i,n,1), forward_strike(1,n),
T))/-((puts(i+1,n,1)-puts(i,n,1))-sum(puts(1:i-1,n,2)));

end

% weights of the last options in the portfolio
calls(amount,n,2)=(value(calls(amount,n,1)+step(1,n), forward_strike(1,n), T)-value(calls(amount,n,1),
forward_strike(1,n), T))/step(1,n)-sum(calls(1:amount-1,n,2));
puts(amount,n,2)=(value(puts(amount,n,1)-step(1,n), forward_strike(1,n), T)-value(puts(amount,n,1),
forward_strike(1,n), T))/step(1,n)-sum(puts(1:amount-1,n,2));

end

% gives the payoffs from the options (payoff per option times weight)

% for each period
for n=1:sideways;

    % for each option
    for i=1:amount
        calls(i,n,3)=max( data(down,n)-calls(i,n,1),0)*calls(i,n,2);
        puts(i,n,3)=max( puts(i,n,1)- data(down,n),0)*puts(i,n,2);
    end

end

% for each period
for n=1:sideways;

    % for each option. option prices
    for i=1:amount
        [calls(i,n,4),~]=blsprice(calls(1,n,1), calls(i,n,1),0,T,0.2);
        [~,puts(i,n,4)]=blsprice(puts(1,n,1), puts(i,n,1),0,T,0.2);
    end

end

% for each period
for n=1:sideways;

    % for each option. option prices
    for i=1:amount
        [calls(i,n,5)]=blsvega(calls(1,n,1), calls(i,n,1),0,T,0.2);
    end
end

```

```

    [puts(i,n,5)]=blsvega(puts(1,n,1), puts(i,n,1),0,T,0.2);
end

end

prices=zeros(1,sideways);

    %for each period
for n=1:sideways;

prices(n)=sum(calls(:,n,2).*calls(:,n,4)+puts(:,n,2).*puts(:,n,4));

end

vegas=zeros(1,sideways);

    %for each period
for n=1:sideways;

vegas(n)=sum(calls(:,n,2).*calls(:,n,5)+puts(:,n,2).*puts(:,n,5));

end

%helpfunctions:

    %Gives the value at maturity of the function that is replicated. See
    %section "Replicating a log contract" in the thesis for explanations
function [value] = value(strike, strike_forward, T)

value=2/T*((strike-strike_forward)/strike_forward-log(strike/strike_forward) );

end

end

function [ cdelta, totprofit, bprofit, oprofit, cprice, wc, cvega, impliedprofit ] = balancing4( data, sigma)

    %Function for testing the performance of buying one at the money call
    %option and delta hedging it. Input "data" is any amount of daily
    %closing prices. Data should be split into testing periods of desired
    %length, with periods in separate columns.Sigma is the Black-Scholes
    %volatility used for delta hedging.

    %size of input data
    [down, sideways]=size(data);

    %allocation of memory for speed

```

```

%call option delta
cdelta=zeros(down-1, sideways);

%call option vega
cvega=zeros(down-1, sideways);

%option payoff at maturity
oprofit=zeros(1,sideways);

%balancing profit from delta hedging
bprofit=zeros(1,sideways);

%call option price
cprice=zeros(down-1,sideways);

%change in option value over one time step.
impliedprofit=zeros(down-2,sideways);

%weight for call option in order to have price 0.0415. This is since the
%option portfolios in the variance swap replicating strategy have this
%price. We want to compare the two strategies.
wc=zeros(1,sideways);

%Let us first solve the deltas
for n=1:sideways;

    %suitable call option weight, so that price is 0.0415
    [wc(n),~]=blsprice(data(1,n), data(1,n),0,3/12,0.2);
    wc(n)=0.0415/wc(n);

    %call option profit at maturity
    oprofit(n)=max( data(down,n)-data(1,n),0)*wc(n);

    %call option prices, priced att assumed implied volatility of 20%
    for i=1:down-1
        [cprice(i,n),~]=blsprice(data(i,n), data(1,n),0,3/12,0.2);
        cprice(i,n)=cprice(i,n)*wc(n);
    end

    %change in option value over one time step
    for i=2:down-1
        impliedprofit(i,n)=cprice(i,n)-cprice(i-1,n);
    end

    %deltas calculated at Black-Scholes volatility "sigma", specified
    %in input

```

```

for i=1:down-1
    [cdelta(i,n),~]=blsdelta(data(i,n),data(1,n),0.0001,(down-i)/252,sigma);
    cdelta(i,n)=cdelta(i,n)*wc(n);

```

```
end
```

```
% vegas just for curiosity
```

```

for i=1:down-1
    [cvega(i,n)]=blsvega(data(i,n),data(1,n),0.0001,(down-i)/252,sigma);
    cvega(i,n)=cvega(i,n)*wc(n);

```

```
end
```

```
%Profit from trading the stock
```

```

for i=1:down-1
    bprofit(i,n)=-cdelta(i,n)*( data(i+1,n)-data(i,n));

```

```
end
```

```
end
```

```
%Total profit consists of balancing profit (profit from trading the
%stock) and payoff at maturity
```

```
totprofit=sum(bprofit)+oprofit;
```

```
end
```

```
function [ returns ] = returns2 ( data )
```

```
%funtion than calculates returns. Returns are used for calculating
%volatilities
```

```
[down, sideways]=size(data);
```

```
for n=1:sideways
```

```

    for i=1:down-1
        returns(i,n)=data(i+1,n)/data(i,n)-1;

```

```
    end
```

```
end
```

```
end
```

```
function [ cdelta, totprofit, bprofit, oprofit, cprice, wc, cvega, impliedprofit ] = balancing4( data, sigma)
```

```

%Function for testing the performance of buying one at the money call
%option and delta hedging it. Input "data" is any amount of daily
%closing prices. Data should be split into testing periods of desired
%length, with periods in separate columns.Sigma is the Black-Scholes
%volatility used for delta hedging.

```

```

%size of input data

```

```

[down, sideways]=size(data);

```

```

%allocation of memory for speed

```

```

%call option delta

```

```

cdelta=zeros(down-1, sideways);

```

```

%call option vega

```

```

cvega=zeros(down-1, sideways);

```

```

%option payoff at maturity

```

```

oprofit=zeros(1,sideways);

```

```

%balancing profit from delta hedging

```

```

bprofit=zeros(1,sideways);

```

```

%call option price

```

```

cprice=zeros(down-1,sideways);

```

```

%change in option value over one time step.

```

```

impliedprofit=zeros(down-2,sideways);

```

```

%weight for call option in order to have price 0.0415. This is since the

```

```

%option portfolios in the variance swap replicating strategy have this

```

```

%price. We want to compare the two strategies.

```

```

wc=zeros(1,sideways);

```

```

%Let us first solve the deltas

```

```

for n=1:sideways;

```

```

    %suitable call option weight, so that price is 0.0415

```

```

    [wc(n),~]=blsprice(data(1,n), data(1,n),0,3/12,0.2);

```

```

    wc(n)=0.0415/wc(n);

```

```

    %call option profit at maturity

```

```

    oprofit(n)=max( data(down,n)-data(1,n),0)*wc(n);

```

```

    %call option prices, priced att assumed implied volatility of 20%

```

```

    for i=1:down-1

```

```

        [cprice(i,n),~]=blsprice(data(i,n), data(1,n),0,3/12,0.2);

```

```

        cprice(i,n)=cprice(i,n)*wc(n);
    end
end

```

end

%change in option value over one time step

for i=2:down-1

impliedprofit(i,n)=cprice(i,n)-cprice(i-1,n);

end

%deltas calculated at Black-Scholes volatility "sigma", specified

%in input

for i=1:down-1

[cdelta(i,n),~]=blsdelta(data(i,n),data(1,n),0.0001,(down-i)/252,sigma);

cdelta(i,n)=cdelta(i,n)*wc(n);

end

%vegas just for curiosity

for i=1:down-1

[cvega(i,n)]=blsvega(data(i,n),data(1,n),0.0001,(down-i)/252,sigma);

cvega(i,n)=cvega(i,n)*wc(n);

end

%Profit from trading the stock

for i=1:down-1

bprofit(i,n)=-cdelta(i,n)*(data(i+1,n)-data(i,n));

end

end

%Total profit consists of balancing profit (profit from trading the

%stock) and payoff at maturity

totprofit=sum(bprofit)+oprofit;

end