Department of Mathematics and Systems Analysis

On Hypercomplex and Time-Frequency Analysis

Vesa Vuojamo

Aalto University

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On Hypercomplex and Time-Frequency Analysis

Vesa Vuojamo

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Abstract

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Abstract

In this thesis we study two topics in harmonic analysis. In the first half we concentrate on Clifford analysis and, in particular, derive Cauchy-type formulas for certain regular functions. In the second half of this thesis we focus on time-frequency analysis and prove characterizations of properties quadratic time-frequency transforms.

Clifford analysis is a branch of mathematical analysis applying Clifford algebras to study generalizations of complex analysis. These algebras are used to construct higher dimensional analogues to complex numbers. In this context the complex analytic functions are generalized to monogenic functions which are null-solutions of certain Cauchy-Riemann or Dirac operators. Many of the results in complex analysis may be translated into higher dimensions. However, these results depend on the choice of the operator defining the family of monogenic functions.

We study the theory known as modified Clifford analysis. This theory is based on the modified Cauchy-Riemann operator which is closely connected to the hyperbolic space. Working in the Poincaré upper half-space model of hyperbolic geometry, we find the k-hyperbolic harmonic fundamental solutions. Using these solutions we also prove a Cauchy-type integral formula for k-hypermonogenic functions.

In the second part of this thesis we focus on time-frequency analysis. The goal of this field of study is to find representations which combine the features of both the signal and its Fourier transform. Using time-frequency representations such as time-frequency transforms signals can be described and manipulated jointly in time and in frequency. If the signal is music, a time-frequency transforms acts as its mathematical musical score.

We study quadratic time-frequency transforms which may be interpreted as time-frequency energy densities of a given signal. By the Heisenberg uncertainty relation, a signal cannot be perfectly localized in time and simultaneously have a definite frequency. This precludes the existence of a perfect time-frequency energy density. Nevertheless, such an energy density may be approximated in some sense using quadratic time-frequency transforms. We consider the Cohen class of covariant time-frequency transforms and prove characterizations of several properties linked to energy densities and transformations of signals. Most of these properties are characterized in terms of the quantization, the integral kernel and the evaluation at the time-frequency origin of the given transform.

Keywords Clifford analysis, hypermonogenic functions, Cauchy formula, hyperbolic geometry, time-frequency analysis, Cohen class, quadratic time-frequency transform

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Tiivistelmä

Väitöskirjassa tutkitaan kahta harmonisen analyysin osa-aluetta. Ensimmäisessä osiossa keskitytään Cliffordin analyysiin ja johdetaan Cauchy-tyyppisiä kaavoja tietyille säännöllisille funktioille. Toisessa osiossa tutkitaan aikataajuusanalyysiä ja todistetaan neliöllisten aikataajuusmuunnosten ominaisuuksien karakterisointeja.

Cliffordin analyysi on matemaattisen analyysin osa-alue, jossa tutkitaan kompleksianalyysin yleistyksiä Cliffordin algebroiden avulla. Näitä algebroita voidaan käyttää kompleksilukujen korkeampiulotteisina vastineina. Cliffordin analyysissä analyyttiset funktiot yleistetään monogeenisiksi funktioiksi, jotka ovat tiettyjen Cauchy-Riemannin tai Diracin operaattoreiden nollaratkaisuja. Useat kompleksianalyysin tulokset voidaan yleistää korkeampiin ulottuvuuksiin. Nämä yleistykset riippuvat kuitenkin valitusta operaattorista, joka määrittelee monogeeniset funktiot.

Tutkimuksessa keskitytään teoriaan, joka tunnetaan yleisesti modifioituna Cliffordin analyysinä. Tämä teoria pohjautuu modifioituun Cauchy-Riemannin operaattoriin, joka on myös läheisesti kytköksissä hyperboliseen geometriaan. Tutkimuksessa johdetaan k-hyperbolisesti harmoniset perusratkaisut Poincarén ylemmän puoliavaruuden mallissa. Näitä ratkaisuja soveltaen todistetaan Cauchy-tyyppinen integraalikaava k-hypermonogeenisille funktioille.

Väitöskirjan jälkimmäisessä osiossa tutkitaan aikataajuusanalyysiä. Aikataajuusanalyysin tavoitteena on kehittää signaaleille esityksiä, jotka yhdistävät sekä signaalin että sen Fourier-muunnoksen ominaisuuksia. Aikataajuusesitysten, kuten aikataajuusmuunnosten, avulla signaaleja voidaan kuvailla ja käsitellä yhteisesti sekä ajassa että taajuudessa. Jos signaali on musiikkia, aikataajuusmuunnos on sille eräänlainen matemaattinen nuottikirjoitus.

Tutkimuksen pääpainona ovat neliölliset aikataajuusmuunnokset, jotka voidaan tulkita signaalien energiatiheyksiksi ajassa ja taajuudessa. Heisenbergin epätarkkuusperiaatteen mukaan signaali ei voi olla samalla täysin keskittynyt sekä ajassa että taajuudessa. Tästä johtuen täydellistä aikataajuusenergiatiheyttä ei ole olemassa. Energiatiheyttä voi kuitenkin approksimoida neliöllisillä aikataajuusmuunnoksilla. Tässä tutkimuksessa todistetaan sekä signaalien muunnoksiin että energiatiheysominaisuuksiin liittyviä karakterisointeja Cohenin luokkaan kuuluville kovarianteille aikataajuusmuunnoksille. Muunnosten ominaisuuksille johdetaan yhtäpitäviä ehtoja kvantisoinnin, integraaliytimen sekä aikataajuusorigoevaluaation suhteen.

Avainsanat Cliffordin analyysi, hypermonogeeniset funktiot, Cauchyn kaava, hyperbolinen geometria, aikataajuusanalyysi, Cohenin luokka, neliöllinen aikataajuusmuunnos

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Preface

The research presented in this thesis was carried out at the Department of Mathematics and Systems Analysis at Aalto University and at the Department of Mathematics at Tampere University of Technology.

First, I would like to thank my thesis advisors. I started to work on this thesis at Tampere University of Technology with Dr. Sirkka-Liisa Eriksson as my supervisor. I thank her for being my mentor in Clifford analysis. Thanks to her active role as the leader of the Clifford analysis group, I had many possibilities to participate in international conferences, to visit collaborators and to teach in graduate courses. I thank my advisor at Aalto University, Dr. Ville Turunen, for insightful discussions and a wealth of research ideas in time-frequency analysis. In addition, I wish to thank Dr. Heikki Orelma for collaboration both in Clifford analysis and time-frequency analysis.

I would like to thank Prof. Juha Kinnunen for supervision of this thesis and for the opportunity to work on time-frequency analysis at the Department of Mathematics and Systems Analysis at Aalto University.

I wish to express my gratitude to Prof. Swanhild Bernstein and Sen. Ass. Prof. Eckhard Hitzer for acting as preliminary examiners of this thesis and to Ass. Prof. Paula Cerejeiras for agreeing to act as my opponent in the public examination.

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Finally, I would like to express my deepest gratitude to my family and friends. Lämpimin kiitos jatkuvasta rohkaisusta ja tuesta kuuluu rakkaalleni Saaralle.

Helsinki, November 18, 2020,

Vesa Vuojamo

Contents

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I Sirkka-Liisa Eriksson, Heikki Orelma and Vesa Vuojamo. Generalized Hyperbolic Harmonic Functions in the Plane. In *AIP Conference Proceedings*, Volume 1648, 440007, 3 pages, DOI: 10.1063/1.4912658, April 2015.

II Sirkka-Liisa Eriksson and Vesa Vuojamo. Integral Kernels for k-hypermonogenic Functions. *Complex Variables and Elliptic Equations*, Volume 62, Issue 9, Pages 1254–1265, DOI: 10.1080/17476933.2016.1250402, February 2017.

III Heikki Orelma, Ville Turunen and Vesa Vuojamo. Time-Frequency Analysis in $\mathbb{R}^n$. *Submitted to a journal*, 37 pages, September 2020.

Author's Contribution

Publication I: "Generalized Hyperbolic Harmonic Functions in the Plane"

The author contributed to the ideas and presented the work in the ICNAAM2014 conference.

Publication II: "Integral Kernels for k-hypermonogenic Functions"

The research problem was posed by Sirkka-Liisa Eriksson and Heikki Orelma based on previous work in $\mathbb{R}^3$. The author wrote the article, found the solutions in Section 4 and proved Theorems 6.2, 6.3 and 6.4.

Publication III: "Time-Frequency Analysis in $\mathbb{R}^n$"

The author wrote the article and proved the Lemmas 4.5 and 5.21, Propositions 4.6, 5.13 and 6.2 and Theorems 5.3 and 5.6. All the authors contributed equally to the other parts of the article.

1. Introduction

Rapid development of quantum mechanics in the early 20th century brought on a wealth of new mathematical ideas. Novel mathematical methods were needed to transform concepts from classical mechanics into their quantized counterparts. Many of the physically motivated problems had mathematical interest in their own right and have since developed into independent disciplines.

Both Clifford analysis and time-frequency analysis have strong connections to quantum mechanics. The algebras defined by W. Clifford re-emerged in P. Dirac's relativistic theory of the electron. Defining first-order operators to factor the Klein–Gordon equation he found the relativistic equation describing massive spin-$\frac{1}{2}$ particles. In modern Clifford analysis or hypercomplex analysis the first-order vector differential operators

$$\partial_{\mathbf{x}} f(\mathbf{x}) = \sum_{k=1}^{n} e_k \frac{\partial}{\partial x_k} f(\mathbf{x}) \tag{1.0.1}$$

are referred to as Dirac operators. The solutions to the Dirac equation

$$\partial_{\mathbf{x}} f(\mathbf{x}) = 0 \tag{1.0.2}$$

and its various modifications are the central objects of study in Clifford analysis. For a general overview with historical comments we refer to [59].

Time-frequency analysis has its roots in the interpretation of classical physical observables in terms of quantum mechanics. In quantum mechanics physical observables $(q,p) \mapsto a(q,p)$ are mapped to Hermitian operators $\psi \mapsto \mathrm{Op}_Q(a)\psi$, where ψ is a wave function. Eigenvalues of the operator $\mathrm{Op}_Q(a)$ are then interpreted as the possible values of the observable. Rules $a \mapsto \mathrm{Op}_Q(a)$ assigning an operator $\mathrm{Op}_Q(a)$ to a given function a are called quantizations. It turns out that defining a quantization is equivalent to defining a suitable position-momentum pseudo-probability distribution $Q(\psi,\psi)$ for the wave function ψ.

The pseudo-probability distribution $W(\psi,\psi)$ proposed by E. Wigner in the context of quantum thermodynamics yields the Weyl–Wigner quantization rule Op_W defined earlier by H. Weyl. The same distribution $W(u,v)$ was later discovered in signal analysis by J. Ville to describe the distribution of energy in a signal in terms of time and frequency. Weyl's quantization rule has since developed into the mathematical theory of Weyl–Wigner pseudo-differential operators. The physical interpretations of quantum mechanics and time-frequency analysis are totally different. However, they share the same mathematical framework developed for understanding quantum mechanics, see [39, Section 2.4].

Although Clifford analysis and time-frequency analysis have grown into independent fields of mathematics, the initial contributions and motivation live on in their terminology and fundamental theorems. The present thesis looks into problems in these fields, studying both of them as topics of their own.

This thesis is organized as follows. In Chapter 2 we present an overview of the theory related to Publication I and Publication II. We introduce the modified Clifford analysis initiated by H. Leutwiler in [51]. Following S.-L. Eriksson's work we study k-hypermonogenic functions and prove formulas for k-hyperbolic harmonic fundamental solutions. With these solutions we obtain a Cauchy-type integral formula proven in Publication II. We will also compare the methods for finding k-hyperbolic harmonic fundamental solutions.

In Chapter 3 we present the concepts and methods of time-frequency analysis employed in Publication III. We give a general introduction to time-frequency analysis and discuss the fundamental ideas related to quadratic time-frequency transforms. We introduce the Cohen class of covariant time-frequency transforms and some of their properties that have been characterized in Publication III.

2. Clifford analysis

In this chapter we present an overview of the topics studied in Publication I and Publication II. We introduce the generalization of complex analysis into higher dimensions using Clifford algebras. After presenting the basic ideas in Clifford analysis we study the theory called modified Clifford analysis. Lastly, we discuss the results obtained in Publication I and Publication II.

2.1 Clifford algebras

Clifford algebras are a family of associative algebras on bilinear spaces which may be considered as generalizations to complex numbers in higher dimensions. As an introductory example to Clifford algebras we consider the algebra of quaternions using the monograph [62] as our main reference. As an accessible and thorough introduction to various aspects of Clifford algebras and analysis we refer also to [57].

2.1.1 Quaternions and geometric multiplication

Multiplications in the complex plane give rise to scalings and rotations in the two-dimensional space. A lot of effort was put to finding ways to multiply objects in higher dimensions to yield similar results. Hamilton's quaternions make it possible to represent geometric mappings such as rotations in three-dimensional space using the quaternionic multiplication. The space of quaternions is the four-dimensional associative algebra over the real numbers with basis $1, i, j, k$ and the multiplication defined by

$$\begin{aligned} i^2 =& j^2 = k^2 = -1, \\ ijk =& -1, \\ 1x =& x1 = x, \end{aligned}$$

see [62, p. 58].

In calculations it is useful to represent quaternions q as

$$q = q_0 + \mathbf{q},$$

where $\mathbf{q} = q_1 i + q_2 j + q_3 k$ is the imaginary part of the quaternion q. If $q_0 = 0$ we call the quaternion q pure. From the multiplication of two pure quaternions (or vectors in $\mathbb{R}^3$) we obtain

$$\mathbf{pq} = -\langle \mathbf{p}, \mathbf{q} \rangle + \mathbf{p} \times \mathbf{q},$$

where $\langle \cdot, \cdot \rangle$ is the inner product and $\times$ the cross product in $\mathbb{R}^3$. These two operations are the ones seen in contemporary engineering mathematics curriculum in contrast to the full algebra of quaternions.

Similarly to the complex numbers we define the conjugation of a quaternion q as

$$\overline{q} = q_0 - \mathbf{q}.$$

The modulus $|q|$ becomes

$$|q|^2 := q_0^2 + \|\mathbf{q}\|^2 = q\overline{q}.$$

With these definitions we define the inverse of a quaternion q by

$$q^{-1} = \frac{\overline{q}}{|q|^2}.$$

Since each non-zero quaternion q is invertible, the quaternions form a division algebra.

Basic geometric mappings in $\mathbb{R}^3$ can be expressed concisely in terms of quaternions. The reflection mapping $R_a : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by the formula

$$R_a(x) := x - 2\frac{\langle x, a \rangle}{\|a\|^2} a. \tag{2.1.1}$$

This formula may be written using the quaternionic multiplication as

$$R_a(q) = -aqa^{-1}, \tag{2.1.2}$$

where a and q are pure quaternions and a is a unit quaternion representing the axis of reflection, see [62, Proposition 8.19]. By the theorem of Cayley and Hamilton any orthogonal map in $\mathbb{R}^n$ can be represented as a composition of at most n reflections. Any rotation is a composition of an even number of reflections. Applying (2.1.2), a single rotation in $\mathbb{R}^3$ is given by

$$R_b \circ R_a(q) = rqr^{-1},$$

where $r = ba$. This formula makes it possible to give any rotation in $\mathbb{R}^3$ in terms of the quaternionic multiplication. This idea carries over to higher dimensions using general Clifford algebras.

2.1.2 General definition of Clifford algebras

The universal Clifford algebra associated to the vector space V with the quadratic form Q is defined to be the unital associative algebra $C\ell(V, Q)$ such that for any unital associative algebra A and a map $j : V \to A$ satisfying $j(v)^2 = Q(v)1_A$ the diagram

$$\begin{array}{ccc} V & \xrightarrow{\ i\ } & C\ell(V,Q) \\ & {\scriptstyle j}\searrow & \downarrow {\scriptstyle f} \\ & & A \end{array}$$

commutes. Given explicitly in terms of a basis this definition may be given for the real quadratic space $\mathbb{R}_{p,q}$ as follows.

Definition 2.1.1. Let $p, q \in \mathbb{N}_0$ and let V be a vector space with the basis $\{e_1, \dots, e_{p+q}\}$. The universal real Clifford algebra $C\ell_{p,q}$ is the real associative algebra with a unit $1 := e_0$ generated by the vectors $e_1, \dots, e_{p+q}$ satisfying the properties

$$e_i^2 = 1, \quad i = 1, \dots, p,$$

$$e_i^2 = -1, \quad i = p+1, \dots, p+q,$$

$$e_i e_j = -e_j e_i, \quad i \neq j, \quad i, j \neq 0.$$

We will mostly deal with the Clifford algebra $C\ell_{0,n}$. However, the general case $C\ell_{p,q}$ appears often in related research.

The product in a Clifford algebra is commonly referred to as the *geometric product* or *Clifford product*. For products of vectors we will use the multi-index notation

$$e_\alpha = e_{\alpha_1} \dots e_{\alpha_k}, \tag{2.1.3}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ and $|\alpha| = k$. As a vector space the real Clifford algebra has a basis

$$\{e_\alpha = e_{\alpha_1} \dots e_{\alpha_k} : 1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_k \leq n\} \cup \{1\}$$

and dimension 2^n. Using the convention $e_\emptyset = 1$, any element $a \in C\ell_{p,q}$ is a real linear combination of these basis elements written as

$$a = \sum_\alpha a_\alpha e_\alpha.$$

The subspace of k-vectors is the space

$$C\ell_{p,q}^{(k)} = \left\{ a \in C\ell_{p,q} : a = \sum_{|\alpha|=k} a_\alpha e_\alpha \right\},$$

and we denote the subspace of all elements with $|\alpha|$ even as $C\ell_{p,q}^+$. This subspace is also a subalgebra of $C\ell_{p,q}$. The space $\mathbb{R}^{n+1}$ is identified inside the Clifford algebra as $C\ell_{p,q}^{(0)} \oplus C\ell_{p,q}^{(1)}$. These elements of the Clifford algebra are referred to as *paravectors*.

As an example, complex numbers and quaternions can be regarded as Clifford algebras by

$$\mathbb{C} \simeq C\ell_{0,2}^+ \simeq C\ell_{0,1},$$

$$\mathbb{H} \simeq C\ell_{0,3}^+ \simeq C\ell_{0,2}.$$

In analogy to the complex and quaternionic conjugation, there are natural involutions defined in Clifford algebras.

Definition 2.1.2. The *grade involution* or *main involution* is the map $(\cdot)' : C\ell_{p,q} \to C\ell_{p,q}$ defined in terms of the basis elements by

$$(e_{\alpha_1\alpha_2\ldots\alpha_k})' = (-1)^k e_{\alpha_1\alpha_2\ldots\alpha_k}.$$

Reversion is an anti-involution $(\cdot)^* : C\ell_{p,q} \to C\ell_{p,q}$ is defined by

$$(e_{\alpha_1\alpha_2\ldots\alpha_k})^* = e_{\alpha_k\ldots\alpha_2\alpha_1}.$$

We will call the composition of these the *conjugation*

$$\overline{a} := (a')^* = (a^*)'.$$

These involutions are extended linearly to the whole algebra.

For paravectors in $C\ell_{0,n}$ the norm may be defined formally exactly as in the case of quaternions using the conjugation

$$x\overline{x} = \overline{x}x = |x|^2 = \sum_{i=0}^{n} x_i^2.$$

We define the *Clifford group* Γ_n as the group of the elements in $C\ell_{0,n}$ which may be written as a finite product of invertible paravectors. Using this group we introduce Möbius transformations in $\mathbb{R}^{n+1}$.

They may be represented by Clifford matrices as was first proven by Vahlen [68] and later rediscovered by Ahlfors [1]. This relatively simple description is one of the motivations for using Clifford algebras in higher dimensional analysis. We state the basic results following Waterman [69].

Definition 2.1.3 ([69], p. 91)**.** Denote by $\mathrm{GL}(2, C\ell_{0,n})$ the group of matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in C\ell_{0,n}, \tag{2.1.4}$$

such that the map $x \mapsto gx$ defined by

$$gx = (ax + b)(cx + d)^{-1}, \tag{2.1.5}$$

is a bijection from $\overline{\mathbb{R}^{n+2}} := \mathbb{R}^{n+2} \cup \{\infty\}$ to $\overline{\mathbb{R}^{n+2}}$. These matrices are called *Clifford matrices*.

The Clifford matrices may also be characterized by their elements, see [69, Theorem 6]. Clifford matrices act on paravectors $x \in \mathbb{R}^{n+1}$ by the formula (2.1.5) which is interpreted in the extended space $\overline{\mathbb{R}^{n+1}} = \mathbb{R}^{n+1} \cup \{\infty\}$ in the same fashion as in the complex case. The matrix (2.1.4) inducing (2.1.5) is referred to as the *Vahlen matrix* associated to (2.1.5).

By a theorem of originally due to Vahlen [68], the group of Clifford matrices (2.1.4) is a double cover of the group of Möbius transformations in $\mathbb{R}^{n+1}$:

Theorem 2.1.4 ([69], Theorem 5.). *The Clifford matrices form a group whose quotient modulo $\pm I$ is isomorphic to the group of sense-preserving Möbius transformations in $\mathbb{R}^{n+1}$.*

Particular examples of the map induced by Vahlen matrices are rotations

$$x \mapsto axa^*, \quad g = \begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix}, \tag{2.1.6}$$

where $a \in \Gamma_n$, $|a| = 1$, and translations

$$x \mapsto x + \mu, \quad g = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \tag{2.1.7}$$

where $\mu \in \mathbb{R}^{n+1}$, see [69, Lemma 10].

2.2 Clifford analysis

As the quaternions seem to be a reasonable generalization of the complex numbers, it is natural to consider the generalization of the theory of analytic functions to quaternions. The first attempts at a definition of quaternionic analytic function by the limit of a difference quotient or a power series expansion fail. Only linear quaternionic functions are differentiable in the sense of a difference quotient. On the other hand, power series with quaternion coefficients yield too large a family of functions, see [57, p. 74].

A more promising candidate for a definition of regular quaternionic functions arises from the Cauchy-Riemann equation for a complex analytic function $f(z) = u(z) + \mathrm{i}v(z)$ with $z = x + \mathrm{i}y$

$$2\partial_{\overline{z}} f(z) := (\partial_x + \mathrm{i}\partial_y)\, f(z) = 0.$$

This equation can be readily generalized to quaternionic functions by setting

$$D_x f(x) := (\partial_{x_0} + \mathrm{i}\partial_{x_1} + \mathrm{j}\partial_{x_2}) f(x),$$

where $x = x_0 + \mathrm{i}x_1 + \mathrm{j}x_2$ and considering the null-solutions of this operator.

The Dirac operator was initially used by Dirac to account for the electron spin in the relativistic wave equation [18]. The idea is to factor the wave operator into two linear first-order differential operators. This results in the Dirac operator in $C\ell_{1,3}$. In his work Dirac did not explicitly mention general Clifford algebras but used a matrix representation instead.

Analysis of quaternion-valued functions was initiated by Gr. C. Moisil, N. Teodorescu and R. Fueter in the 1930's. In the 1970's R. Delanghe's work on analysis using Clifford algebras and consequently the book [6] together with F. Brackx and F. Sommen gave rise to the field now called Clifford analysis. The basic theory is developed further in [16]. For a general overview on Clifford algebras and analysis we refer to R. Delanghe's article [15].

2.2.1 Dirac- and Cauchy–Riemann operators

In analogy to the holomorphic functions in complex analysis we define the corresponding hyperholomorphic functions starting with the C-R equations as has been done in the monograph [6]. Clifford algebra-valued functions are of the form

$$f(x) = \sum_{\alpha} e_\alpha f_\alpha(x),$$

where α is the multi-index with indices in $0, 1, \ldots, n$ and the component functions f_α are real-valued. Let $\Omega \subset \mathbb{R}^{n+1}$ be open. The left and the right Cauchy–Riemann operators are defined by

$$D_x^l f(x) = \sum_{i=0}^{n} e_i \partial_{x_i} f(x),$$

$$D_x^r f(x) = \sum_{i=0}^{n} \partial_{x_i} f(x) e_i,$$

respectively with the convention $D_x := D_x^l$. The Cauchy–Riemann operator has a scalar part and a vector part

$$D_x = e_0 \partial_{x_0} + \sum_{i=1}^{n} e_i \partial_{x_i} = \partial_{x_0} + \partial_{\mathbf{x}},$$

where $\partial_{\mathbf{x}}$ is usually referred to as the Dirac operator.

The generalized Cauchy–Riemann equations for differentiable functions $f : \Omega \to C\ell_{p,q}$ are

$$D_x f = 0. \tag{2.2.1}$$

The functions which satisfy (2.2.1) are called hyperholomorphic or monogenic [1]. We will mainly concentrate on the analysis of the Cauchy–Riemann operator and its modification in $C\ell_{0,n}$, where paravectors act as higher dimensional complex numbers.

2.2.2 Operator identities

Referring to the monographs [6, 16], we present some basic identities for the Cauchy–Riemann and Dirac operators. The Cauchy–Riemann operator in $C\ell_{0,n}$ factorizes the Laplace operator in $\mathbb{R}^{n+1}$ as can be seen by

$$\begin{aligned} D_x \overline{D}_x &= (\partial_{x_0} + \partial_{\mathbf{x}})\overline{(\partial_{x_0} + \partial_{\mathbf{x}})} \\ &= (\partial_{x_0} + \partial_{\mathbf{x}})(\partial_{x_0} - \partial_{\mathbf{x}}) \\ &= \partial_{x_0}^2 + \partial_{x_0}\partial_{\mathbf{x}} - \partial_{x_0}\partial_{\mathbf{x}} - \partial_{\mathbf{x}}^2 \\ &= \Delta_x, \end{aligned}$$

[1]There is some flexibility in terminology in the literature since either of the operators $\partial_{\mathbf{x}}$ and D_x may be referred to as Dirac operators and their null-solutions as monogenic functions.

where the Dirac operator satisfies

$$\partial_{\mathbf{x}}^2 = \sum_{j=1}^{n} \sum_{i=1}^{n} e_i e_j \partial_{x_i} \partial_{x_j} = -\sum_{i=1}^{n} \partial_{x_i}^2 = -\Delta_{\mathbf{x}}.$$

We derive some formulas for the variable x and the operator D_x. The anticommutator of x and D_x is given by

$$D_x[xF(x)] = -2E_{\mathbf{x}}F(x) - xD_xF(x),$$

where $E_{\mathbf{x}}$ is a part of the Euler operator

$$E_x f(x) = (E_{x_0} + E_{\mathbf{x}})f(x) = x_0 \frac{\partial f}{\partial x_0} + \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}.$$

The Dirac operator satisfies further

$$\mathbf{x}\partial_{\mathbf{x}} = -E_{\mathbf{x}} - \Gamma_{\mathbf{x}}$$

where the gamma operator $\Gamma_{\mathbf{x}}$ is the bivector part given by

$$\Gamma_{\mathbf{x}} = -\mathbf{x} \wedge \partial_{\mathbf{x}} = -\sum_{i<j} e_{ij}(x_i \partial_{x_j} - x_j \partial_{x_i}).$$

2.2.3 Examples of monogenic functions

The usual polynomials $p(x)$ are not, in general, monogenic functions. For instance, for the variable x we find

$$D_x x = \sum_{i=0}^{n} e_i^2 = 1 - n.$$

The standard building blocks for the theory of monogenic functions must be chosen in another manner. Fueter considered the polynomials built from the variables

$$z_i = x_i - x_0 e_i.$$

From these so-called hypercomplex variables one may construct the Fueter polynomials which are monogenic, see [41, Satz 6.2].

A general method for constructing monogenic functions is the *Cauchy–Kovalevska extension*.

Theorem 2.2.1 ([41], Satz 11.38)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f(\mathbf{x})$ be a Clifford algebra-valued real analytic function in Ω. Then f admits a Cauchy–Kovalevska extension $\widetilde{f}(x_0, \mathbf{x})$ given formally by*

$$\widetilde{f}(x_0, \mathbf{x}) = \exp(-x_0 \partial_{\mathbf{x}}) f(\mathbf{x}), \tag{2.2.2}$$

where $\widetilde{f}$ is monogenic on some open neighborhood Ω' of Ω in $\mathbb{R}^{n+1}$.

The hypercomplex variables z_i are such extensions, for if we define $f(\mathbf{x}) = x_i$ such that $i \in \{1, \ldots, n\}$, then the Cauchy–Kovalevska extension of f satisfies

$$\widetilde{f}(x_0, \mathbf{x}) = x_i - x_0 e_i = z_i. \tag{2.2.3}$$

An important particular case of monogenic functions is the set of paravector-valued monogenic functions.

Example 2.2.2 (Paravector-valued functions)**.** In analogy to complex analysis we consider functions of the form

$$f(x) = \sum_{i=0}^{n} u_i(x) e_i = u_0(x) + \mathbf{u}(x).$$

The Cauchy-Riemann system $D_x f = 0$ is now given as

$$\begin{aligned} \partial_{x_0} u_0 - \partial_{\mathbf{x}} \cdot \mathbf{u} &= 0, \\ \partial_{\mathbf{x}} \wedge \mathbf{u} &= 0, \\ \partial_{\mathbf{x}} u_0 + \partial_{x_0} \mathbf{u} &= 0, \end{aligned}$$

or in components

$$\begin{cases} \partial_{x_0} u_0 - \displaystyle\sum_{i=1}^{n} \partial_{x_i} u_i = 0, \\ \quad \partial_{x_k} u_i - \partial_{x_i} u_k = 0, \quad (i, k = 1, \ldots, n), \\ \quad \partial_{x_k} u_0 + \partial_{x_0} u_k = 0, \quad (k = 1, \ldots, n). \end{cases} \tag{2.2.4}$$

This system is often referred to as the Moisil–Teodorescu system or the Stein–Weiss system, see [64].

2.2.4 Integral formulas

Our main result in Publication II is the Cauchy-type integral formula for k-hypermonogenic functions. In this section we introduce the ideas leading to the Cauchy formula for monogenic functions. Similar methods are used also in the k-hypermonogenic case.

We define first the Cauchy kernel as

$$E(x) = \frac{1}{\omega_{n+1}} \frac{\overline{x}}{|x|^{n+1}}, \quad x \neq 0,$$

where ω is the area of the unit sphere S^n given by

$$\omega_{n+1} = 2\pi^{(n+1)/2} \frac{1}{\Gamma\left(\frac{n+1}{2}\right)}.$$

The Cauchy kernel $E(x)$ is a fundamental solution for the left and the right Cauchy–Riemann operator in $\mathbb{R}^{n+1}$. The Clifford-algebra-valued surface form $\mathrm{d}\sigma_x$ is defined by

$$\mathrm{d}\sigma_x = \sum_{i=1}^{n} (-1)^i e_i \, \mathrm{d}\widehat{x}_i$$

where $\mathrm{d}\widehat{x}_i = \mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_n$. This form is given by the usual surface area form of a sufficiently smooth surface with the outward normal ν as

$$\mathrm{d}\sigma_x = \nu \, \mathrm{d}S.$$

Let $U \subset \mathbb{R}^{n+1}$ be an open set, $f, g : U \to C\ell_{0,n}$ smooth functions and $\Omega \subset U$ a smooth domain. Taking the exterior derivative of the form $f \, \mathrm{d}\sigma_x g$ we find

$$\mathrm{d}(f \, \mathrm{d}\sigma_x g) = [(fD)g + f(Dg)] \; \mathrm{d}x \tag{2.2.5}$$

and as an application of the Stokes theorem we find the integral formula

$$\int_{\partial\Omega} f \, \mathrm{d}\sigma_x g = \int_{\Omega} [(fD)g + f(Dg)] \; \mathrm{d}x, \tag{2.2.6}$$

as has been shown in [6, Proposition 9.2].

Setting g to be the fundamental solution $E(x-y)$ with y in the interior of Ω we may state the Cauchy integral formula:

Proposition 2.2.3 ([6], Corollary 9.6)**.** *Let $\Omega \subset U$ be an $(n+1)$-dimensional compact differentiable oriented manifold-with-boundary. If f is left-monogenic in U and the point y is in the interior of Ω, then*

$$f(x) = \int_{\partial\Omega} E(y-x) \, \mathrm{d}\sigma_x f(y). \tag{2.2.7}$$

The detailed proof of this formula with its corollaries can be found in [6]. Many of the important theorems in complex analysis are corollaries of the Cauchy integral formula and such theorems can be transferred in several cases to the Clifford analysis setting.

2.3 Modified Clifford analysis

In his articles starting with [51] Leutwiler proposed a modification to the usual Cauchy–Riemann system which also the integer powers x^n would satisfy. This led to the theory known as modified Clifford analysis. Initially, the objects of study were paravector-valued functions defined on $\mathbb{R}^{n+1}_{+} := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_n > 0\}$ and also the corresponding theory in the context of quaternionic analysis. Modified quaternionic analysis was studied by Leutwiler in [52, 53, 54, 55] and together with Eriksson-Bique in [36]. Modified quaternionic analysis on $\mathbb{R}^4$ was studied by Leutwiler and T.

Hempfling in [42]. Paravector-valued hypermonogenic functions were also studied by Cnops in [11].

In the article [35] a modified Cauchy–Riemann operator was introduced to study general $C\ell_{0,n}$-valued functions in modified Clifford analysis. This generalization was developed further by Eriksson and Leutwiler [25, 26, 27, 28], Eriksson [21, 22, 23, 32] and Eriksson and Orelma [29, 30, 31].

2.3.1 Modified Cauchy–Riemann equations

We consider the paravector-valued function

$$f(x) = \sum_{i=0}^{n} u_i(x) e_i$$

which is also monogenic. The Cauchy–Riemann system can be seen equivalently as the condition for harmonicity of the differential form

$$\sigma = u_0(x)\,\mathrm{d}x^0 - u_1(x)\,\mathrm{d}x^1 - \cdots - u_n(x)\,\mathrm{d}x^n.$$

The differential form σ is *harmonic* if it satisfies the system

$$\begin{cases} \mathrm{d}\sigma = 0, \\ \mathrm{d}^*\sigma = 0, \end{cases} \tag{2.3.1}$$

where d is the differential map and d^* is the codifferential map defined in 2.4.4. Substituting the expression for σ and using the Euclidean metric $g_{ij}(x) = \delta_{ij}$ we find the usual Cauchy–Riemann system (2.2.4).

H. Leutwiler noticed in [51] that by considering the harmonicity in upper half-space model of the hyperbolic space with the metric $g_{ij}(x) = x_n^{-2}\delta_{ij}$ one obtains the system

$$\begin{cases} x_n \left(\partial_{x_0} u_0 - \sum_{i=1}^{n} \partial_{x_i} u_i \right) + (n-1) u_n = 0, \\ \partial_{x_k} u_i - \partial_{x_i} u_k = 0, \quad (i, k = 1, \ldots, n), \\ \partial_{x_k} u_0 + \partial_{x_0} u_k = 0, \quad (k = 1, \ldots, n). \end{cases} \tag{2.3.2}$$

The functions which satisfy this system are called H_n*-solutions*. These are the paravector-valued functions in the family of hypermonogenic functions defined below using (2.3.5) and (2.3.6). As a particular case of both (2.2.4) and (2.3.2) setting $n = 1$ we recover the usual Cauchy–Riemann equations in $\mathbb{C}$.

The components of H_n-solutions are related to the eigenfunctions of the Laplace-Beltrami operator in the hyperbolic upper half-space. The components satisfy

$$\Delta^H u_i = 0, \quad (i = 0, \ldots, n-1),$$

$$\Delta^H u_n + (n-1) u_n = 0,$$

where Δ^H is the Laplace–Beltrami operator in the hyperbolic upper half-space defined in (2.4.5).

2.3.2 H_n-solutions and hypermonogenic functions

We proceed by presenting some examples given in [51]. There is a family of examples of functions related to the construction by Fueter in quaternionic analysis.

Lemma 2.3.1 ([51], Lemma 2.2)**.** *Let $f = u_0 + u_1 e_1 + \cdots + u_k e_k$, $k \in \mathbb{N}$ be an H_k-solution defined on an open set $\Omega \subset \mathbb{R}^{k+1}_+$. Then for any natural number p the function*

$$g(x_0, \ldots, x_{k+p}) = \sum_{i=0}^{k-1} u_i(x_0, \ldots, x_{k-1}, \rho) e_i + \frac{\widetilde{x}}{\rho} u_k(x_0, \ldots, x_{k-1}, \rho), \qquad (2.3.3)$$

where $\widetilde{x} = x_k e_k + \ldots x_{k+p} e_{k+p}$ and $\rho = \|\widetilde{x}\|$, is an H_{k+p}-solution defined on the set

$$\left\{(x_0, \ldots, x_{k+p}) \in \mathbb{R}^{k+p+1} : (x_0, \ldots, x_{k-1}, \rho) \in \Omega\right\}$$

This lemma yields as the particular case $k = 1, p = n - 1$ the construction which gives an H_n-solution for any holomorphic complex function.

Example 2.3.2 (Fueter construction)**.** Let $f = u + \mathrm{i}v$ be holomorphic in $\mathbb{C}$, $r = \|\mathbf{x}\|$ and define

$$g(x) = u(x_0, r) + \mathbf{x}\frac{v(x_0, r)}{r}.$$

With this notation the Cauchy–Riemann equations for f are

$$\begin{cases} \partial_{x_0} u - \partial_r v = 0, \\ \partial_r u + \partial_{x_0} v = 0. \end{cases}$$

The function g is an H_n-solution in the sense of (2.3.2) by the calculation

$$\begin{aligned} D_x g &= (\partial_{x_0} + \partial_{\mathbf{x}}) \left(u + \mathbf{x}\frac{v}{r}\right) \\ &= \partial_{x_0} u + \frac{\mathbf{x}}{r}\partial_r u + \frac{\mathbf{x}}{r}\partial_{x_0} v + \partial_{\mathbf{x}}\mathbf{x}\frac{v}{r} \\ &= (\partial_r u + \partial_{x_0} v)\frac{\mathbf{x}}{r} + \partial_{x_0} u - n\frac{v}{r} - 2E_{\mathbf{x}}\frac{v}{r} - \mathbf{x}\partial_{\mathbf{x}}\frac{v}{r} \\ &= (\partial_r u + \partial_{x_0} v)\frac{\mathbf{x}}{r} + \partial_{x_0} u - n\frac{v}{r} - 2\partial_r v + 2\frac{v}{r} + \partial_r v - \frac{v}{r} \\ &= (1 - n)\frac{v}{r} = -\frac{n-1}{x_n} g_n, \end{aligned}$$

where g_n is the e_n-component of the paravector-valued function g.

Applying this construction to the complex power function $z \mapsto z^k$ we find that natural powers x^k are H_n-solutions. By the Möbius invariance also the negative powers are H_n-solutions. This enables us to introduce hypermonogenic functions by defining them in terms of power series.

Example 2.3.3 ([51], p. 159)**.** The function $f(x) = (1-x)^{-1}$ is an H_n-solution as the sum of the geometric series

$$\sum_{k=0}^{\infty} x^k = (1-x)^{-1},$$

convergent for $\|x\| < 1$. A hypermonogenic version of the exponential function is defined by the usual series expansion

$$\mathrm{e}^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Further examples of H_n-solutions are furnished by the homogeneous hypermonogenic polynomials. These were studied, for instance, in [33].

The system (2.3.2) can be generalized to functions taking values in the full Clifford algebra. To achieve this we define first the projection-type operators P and Q. Any element a in the Clifford algebra $C\ell_{0,n}$ may be decomposed uniquely as

$$a = p + qe_n, \quad p, q \in C\ell_{0,n-1},$$

and we set

$$Pa = p, \quad Qa = q. \tag{2.3.4}$$

Lemma 2.3.4 ([34], p. 290)**.** *Let a and b be elements of the Clifford algebra $C\ell_{0,n}$. The operators P and Q satisfy the following properties.*

$$P(ab) = PaPb - Qa(Qb)',$$

$$Q(ab) = aQb + (Qa)b',$$

$$P(a') = (Pa)',$$

$$Q(a') = -(Qa)'.$$

Using the operator Q we define the left and right *modified Cauchy-Riemann operator* introduced in [35] by

$$M^l_{n-1}f = D^l_x f + \frac{n-1}{x_n}(Qf)', \tag{2.3.5}$$

$$M^r_{n-1}f = D^r_x f + \frac{n-1}{x_n}Qf. \tag{2.3.6}$$

The functions f satisfying $M_{n-1}f = 0$ in $\mathbb{R}^{n+1}_+$ are referred to as *hypermonogenic* functions. This operator can be characterized by simple requirements on its null-solutions.

Theorem 2.3.5 ([27], Theorem 3.6)**.** *Let $L : C\ell_{0,n} \to C\ell_{0,n}$ be a real-linear mapping. Suppose that the system*

$$x_n Du + Lu = 0 \tag{2.3.7}$$

has the following properties:

1. *The constant function $x \mapsto 1$ is a solution.*

2. *The identity function $x \mapsto x$ is a solution.*

3. *If u is a solution, so is ue_i for $i = 1, \dots, n-1$.*

Then the operator L satisfies $Lu = (n-1)(Qu)'$.

Any hypermonogenic function may be presented in components all of which are H_n-solutions.

Theorem 2.3.6 ([35], Theorem 4)**.** *A mapping f is hypermonogenic if and only if it can be locally presented as*

$$f = \sum_{\alpha} g_\alpha e_\alpha,$$

where g_α is an H_n-solution for each multi-index $\alpha = (\alpha_1, \dots, \alpha_k)$ such that any α_i in α satisfies $\alpha_i \in \{1, \dots, n-1\}$.

More generally, we will replace the constant $n-1$ in the modified operator by a real parameter k and call the functions satisfying the equation

$$M_k f := D_x f + \frac{k}{x_n}(Qf)' = 0, \tag{2.3.8}$$

k-hypermonogenic. These functions were introduced in [32].

2.3.3 Properties of k-hypermonogenic functions

The modified Cauchy–Riemann operator satisfies the following lemma.

Lemma 2.3.7 ([32], Lemma 4)**.** *Let $f : \Omega \to C\ell_{0,n}$ be a twice continuous differentiable function on the open set $\Omega \in \mathbb{R}^{n+1}_+$. Then*

$$\overline{M}_k M_k f(x) = M_k \overline{M}_k f(x) = \left(\Delta^E - \frac{k}{x_n}\frac{\partial}{\partial x_n}\right) Pf(x) \tag{2.3.9}$$

$$+ \left(\Delta^E - \frac{k}{x_n}\frac{\partial}{\partial x_n} + \frac{k}{x_n^2}\right) Qf(x), \tag{2.3.10}$$

where Δ^E is the Euclidean Laplacian given by

$$\Delta^E = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}. \tag{2.3.11}$$

The solutions of the equation $\overline{M}_k M_k f(x) = 0$ given in components as

$$\begin{cases} \Delta^E Pf - \dfrac{k}{x_n}\dfrac{\partial Pf}{\partial x_n} = 0, \\ \Delta^E Qf - \dfrac{k}{x_n}\dfrac{\partial Qf}{\partial x_n} + \dfrac{k}{x_n^2} Qf = 0, \end{cases} \tag{2.3.12}$$

are called k-hyperbolic harmonic functions. We collect some useful properties of these functions in the following proposition.

Proposition 2.3.8. *Let f be a $C\ell_{0,n}$-valued function defined in an open subset of the upper half-space $\mathbb{R}^{n+1}_+$.*

1. *If f is k-hypermonogenic, then $\overline{M}_k f$ is k-hypermonogenic.*
2. *f is k-hypermonogenic if and only if $x_n^{-k} f e_n$ is $-k$-hypermonogenic.*
3. *f is k-hypermonogenic if and only if there exists a locally k-hyperbolic harmonic function h such that $Qh = 0$ and $\overline{D}h = f$.*

Proposition 2.3.9. *Let Ω be an open subset of $\mathbb{R}^{n+1} \setminus \{x_n = 0\}$ and f a $C\ell_{0,n}$-valued function on Ω. Then*

$$M^l_{-k}\left(f\frac{e_n}{x_n^k}\right) = (M^l_k f)\frac{e_n}{x_n^k}, \tag{2.3.13}$$

$$M^r_{-k}\left(\frac{e_n}{x_n^k} f\right) = \frac{e_n}{x_n^k}(M^r_k f). \tag{2.3.14}$$

Möbius transforms introduce an additional factor when mapping k-hypermonogenic functions:

Theorem 2.3.10 ([23], Theorem 22)**.** *Let Ω be an open set in $\mathbb{R}^{n+1}_+$, and $T : \overline{\mathbb{R}^{n+1}} \to \overline{\mathbb{R}^{n+1}}$ be a Möbius transformation induced by an element in $\mathrm{GL}(2, \Gamma_{n-1})$ mapping the upper half space $\mathbb{R}^{n+1}_+$ bijectively onto itself. If f is k-hypermonogenic on $T(\Omega)$, then the function F defined by*

$$F(x) = \frac{(cx+d)^{-1}}{|cx+d|^{n-k-1}} f(T(x)) \tag{2.3.15}$$

is k-hypermonogenic on Ω.

For k-hyperbolic harmonic functions the same result holds with the modification

$$F(x) = \frac{1}{|cx+d|^{n-k-1}} f(T(x)),$$

as was proven in [56].

2.4 Hyperbolic function theory

Hyperbolic function theory uses Clifford analysis methods to study functions defined in the hyperbolic space. The central objects of study are the Laplace–Beltrami operator and the corresponding Cauchy–Riemann or Dirac operator. The modified Cauchy–Riemann system (2.3.2) arises naturally in the upper half-space model of hyperbolic geometry and the system (2.3.12) has a simple interpretation in terms of eigenfunction of the Laplace–Beltrami operator in the hyperbolic upper half-space.

2.4.1 Models of hyperbolic geometry

Of the different models of hyperbolic geometry we will mainly work in the upper half-space model. The ball-model is also used in conjunction with the Möbius transform mapping it to the upper half-space. We introduce these models briefly and refer to the book by B. Iversen [46] for more details.

Plane model
The Poincaré upper half space model is the set $\mathbb{R}^n_+$ with the Riemannian metric

$$g_{ij}(x) = \frac{1}{x_n^2}\delta_{ij}.$$

The distance between points x and y is defined by

$$d_h(x, y) := \inf_\gamma \int_\gamma \mathrm{d}s,$$

where the infimum is taken over all the paths connecting the points x and y. It can be expressed in a simple form as

$$d_h(x, y) = \cosh^{-1}(\lambda(x, y)),$$

where the *hyperbolic distance parameter* λ is defined by

$$\lambda(x, y) := \frac{\|x - y\|^2 + 2x_n y_n}{2x_n y_n} = \frac{\|x - y\|^2}{2x_n y_n} + 1.$$

Using this parameter as the variable in functions depending only on the hyperbolic distance yields simpler expressions in the plane. We will need the following lemma stated in [29, Lemma 3.3]:

Lemma 2.4.1. *If x and y are points in the upper half-space, then*

$$\|x - y\|^2 = 2x_n y_n(\lambda(x, y) - 1),$$
$$\|x - \widehat{y}\|^2 = 2x_n y_n(\lambda(x, y) + 1),$$
$$\frac{\|x - y\|^2}{\|x - \widehat{y}\|^2} = \frac{\lambda(x, y) + 1}{\lambda(x, y) - 1},$$

where $\widehat{y} = (y_0, y_1, \ldots, y_{n-1}, -y_n)$.

The function $\lambda(x, y)$ is invariant under Möbius transformations which leads to the following theorem, see [46, Theorem 7.3]:

Theorem 2.4.2 (Möbius transforms)**.** *The group $M(\mathbb{R}^n_+)$ of sense preserving Möbius transformations mapping $\mathbb{R}^n_+$ bijectively to itself is the group of isometries of the hyperbolic upper half space, that is, the group of mappings f satisfying*

$$d_h(f(x), f(y)) = d_h(x, y)$$

for any $x, y \in \mathbb{R}^n_+$.

Ball model

The Poincaré unit ball model of hyperbolic geometry is given by the unit ball in $\mathbb{R}^n$ equipped with the metric

$$g_{ij}(x) = \frac{4}{(1-\|x\|^2)^2}\delta_{ij}.$$

The upper half-space and the ball are connected by a Möbius transform.

Lemma 2.4.3 ([23], Lemma 23)**.** *The Möbius transform φ mapping the unit ball B_n to the upper half-space $\mathbb{R}^n_+$ is defined by*

$$\varphi(x) = (x + e_n)(e_n x + 1)^{-1}. \tag{2.4.1}$$

The inverse transform has the formula

$$\varphi^{-1}(x) = (x - e_n)(-e_n x + 1)^{-1}. \tag{2.4.2}$$

Some calculations are simpler in one of the models and the Möbius transform (2.4.1) can be used to move hyperbolic harmonic functions between the models. This method has been used, for instance, in [56] and [20].

The unit ball model is closely related by projections in the future cone of the Minkowski space to the Klein model and the hyperboloid model. Hyperbolic function theory related to these models different to modified Clifford analysis has been pursued in the dissertation of D. Eelbode [19].

2.4.2 Laplace–Beltrami and related operators

The modified Clifford analysis arises naturally from hyperbolic geometry. The modified Cauchy–Riemann operators factor the Laplace–Beltrami operator corresponding to the metric of the space.

For the definition we need the differential and codifferential maps on a Riemannian manifold M. For the basic definitions we use J. Jost's book [48, Section 3.3]. The L^2-inner product on the Riemannian manifold M for the differential forms $\rho, \sigma \in \Omega^p(M)$ is defined by

$$(\rho, \sigma) := \int_M \langle \rho, \sigma\rangle * (1) = \int_M \rho \wedge *\sigma.$$

Definition 2.4.4. The *codifferential* d^* is the formal adjoint to the differential map d on $\Omega(M)$ given by

$$(\mathrm{d}\alpha, \beta) = (\alpha, \mathrm{d}^*\beta).$$

Definition 2.4.5. The *Laplace–Beltrami operator* is defined for the differential forms $\omega \in \Omega(M)$ by

$$\Delta\omega = (\mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d})\omega. \tag{2.4.3}$$

The form ω is called *harmonic* if $\Delta\omega = 0$. For clarity, the manifold may be explicitly given by the notation Δ^M.

We consider Riemannian manifolds which have metrics of the form

$$g_{ij}(x) = \rho(x)^2 \delta_{ij} \tag{2.4.4}$$

for some non-vanishing ρ, that is, manifolds which are conformally equivalent to the Euclidean space. Particular examples of such manifolds are the previously mentioned Poincaré ball and upper half-space models of hyperbolic geometry.

The expression for the Laplace-Beltrami operator for smooth functions in local coordinates is given by

$$\Delta = \mathrm{d}^* \mathrm{d} = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j \right),$$

and specializing to the metric (2.4.4) we obtain

$$\begin{aligned} \Delta &= \rho^{-n} \partial_i \left(\rho^{n-2} \delta^{ij} \partial_j \right) \\ &= \frac{1}{\rho^2} \left(\Delta^E + \frac{n-2}{\rho} \sum (\partial_i \rho) \partial_i \right), \end{aligned}$$

where $\Delta^E = \sum_{i=1}^n \partial_i^2$ is the Euclidean Laplacian.

In the sequel we are going to need the following lemma which appears usually by the name "Correspondence of solutions" in the particular cases of metrics, for instance, in [50, Lemma 2.1, Lemma 3.4].

Lemma 2.4.6. *The Laplacian of the function $\rho^{-\alpha} u$ is given by*

$$\begin{aligned} \Delta(\rho^{-\alpha} u) =& \frac{1}{\rho^{2+\alpha}} \left(\Delta^E u + \frac{n-2-2\alpha}{\rho} \sum (\partial_i \rho) \partial_i u \right. \\ & \left. - \left[\alpha(n-3-\alpha) \frac{\|\partial \rho\|^2}{\rho^2} + \alpha \frac{\Delta^E \rho}{\rho} \right] u \right). \end{aligned}$$

Proof. By a direct calculation we find

$$\begin{aligned} \Delta(\rho^{-\alpha} u) &= \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j \right) \rho^{-\alpha} u \\ &= \rho^{-n} \partial_i \left(\rho^{n-2} \delta^{ij} (-\alpha \rho^{-\alpha-1} (\partial_j \rho) u + \rho^{-\alpha} \partial_j u) \right) \\ &= \frac{1}{\rho^2} \left(\rho^{-\alpha} \Delta^E u + \rho^{-\alpha} \frac{n-2-\alpha}{\rho} \sum (\partial_i \rho) \partial_i u \right. \\ &+ \rho^{-\alpha} \frac{-\alpha(n-2-\alpha-1)}{\rho^2} \|\partial \rho\|^2 u \\ &\left. - \rho^{-\alpha} \frac{\alpha}{\rho} \Delta^E \rho u - \rho^{-\alpha} \frac{\alpha}{\rho} \sum (\partial_i \rho) \partial_i u \right), \end{aligned}$$

which simplifies to the result. □

In the upper half-space model of hyperbolic geometry with and $\rho = x_n^{-1}$ the Laplace–Beltrami operator is

$$\Delta^H = x_n^2 \left(\Delta^E - \frac{(n-2)}{x_n} \partial_n \right). \tag{2.4.5}$$

We define the k-Laplacian[2] as

$$\Delta_k^H u = x_n^2 \left(\Delta^E u - \frac{k}{x_n} \partial_n u \right). \tag{2.4.6}$$

Using this notation we have the following particular form of Lemma 2.4.6,

$$\Delta_{n-2}^H (x_n^{-\alpha} u) = x_n^{2-\alpha} \left(\Delta^E u - \frac{n-2+2\alpha}{x_n} \partial_n u \right) + \alpha(n-1+\alpha)u,$$

which can be rearranged to

$$(\Delta_{n-2}^H - \alpha(n-1+\alpha))(x_n^{-\alpha} u) = x_n^{-\alpha} \Delta_{n-2+2\alpha}^H u. \tag{2.4.7}$$

Hence, if a function u is $n-2+2\alpha$-hyperbolic harmonic, then the function $x_n^{-\alpha} u$ is an eigenfunction of the Laplace–Beltrami operator with the eigenvalue $\alpha(n-1-\alpha)$.

In the unit ball model of hyperbolic geometry the corresponding k-Laplace operator is

$$\Delta_k^B u = \frac{(1-\|x\|^2)^2}{4} \left(\Delta^E u + \frac{2k}{(1-\|x\|^2)} \left[E + \frac{n-2-k}{2} \right] u \right), \tag{2.4.8}$$

which yields the Laplace–Beltrami operator as the special case Δ_{n-2}^B. Lemma 2.4.6 formulated in the ball is formally similar to the equation (2.4.7) in the plane. In the unit ball we have

$$(\Delta_{n-2}^B - \alpha(n-1+\alpha))(\rho^{-\alpha} u) = \rho^{-\alpha} \Delta_{n-2+2\alpha}^B u, \tag{2.4.9}$$

where $\rho = (1-\|x\|^2)/2$. The operators Δ_k^B and Δ_k^H are connected by the Möbius transform φ in (2.4.1) as is stated in the following theorem[3]:

Theorem 2.4.7 ([50] Theorem 3.1)**.** *The function u is a solution of $\Delta_k^H u - lu = 0$ if and only if $v(x) = \|x - e_n\|^{2-n+k} (u \circ \varphi)(x)$ is a solution of $\Delta_k^B v - lv = 0$, where $k, l \in \mathbb{R}$ such that $(k+1)^2 + 4l \geq 0$.*

Remark 2.4.8 (Hyperbolic function theory on the Poincaré ball). The operator

$$\frac{4}{(1-\|x\|^2)^2} \Delta_{n-2+2\alpha}^B = \Delta^E + \frac{2(n-2+2\alpha)}{(1-\|x\|^2)} (E - \alpha)$$

with $k = n-2+2\alpha$ can be factored in the Clifford algebra $C\ell_{1,n}$ using the Dirac operators

$$D_\alpha = -\partial + 2 \frac{x+\epsilon}{1-\|x\|^2} (\alpha - E)$$

where $x = \sum_{i=1}^n x_i e_i$, $e_i^2 = -1$ for $i = 1, \ldots, n$ and $\epsilon^2 = 1$. These operators satisfy

$$D_{\alpha-1} D_\alpha = -\frac{4}{(1-\|x\|^2)^2} \Delta_{n-2+2\alpha}^B.$$

The function theory based on the Dirac operator D_α was developed in the articles of P. Cerejeiras, U. Kähler and F. Sommen [7, 8] and in the doctoral thesis of D. Eelbode [19].

[2] This operator is not a Laplace–Beltrami operator except for the case $k = n-2$.

[3] We use slightly different sign conventions than in the reference.

2.4.3 Radial hyperbolic harmonic functions

We set out to find the radial version of the equation

$$\Delta^E u - \frac{k}{x_n}\partial_n u = 0. \tag{2.4.10}$$

Radial functions with respect to a point $a \in \mathbb{R}^{n+1}_+$ are those functions u which can be represented by

$$u(x) = (\widetilde{u} \circ \lambda)(x, a),$$

where λ is the hyperbolic distance parameter and $\widetilde{u} : \mathbb{R} \to C\ell_{0,n}$ is a path in $C\ell_{0,n}$. We consider here the real valued case $\widetilde{u} : \mathbb{R} \to \mathbb{R}$. By a straightforward application of the chain rule we find the necessary differential operators in terms of the hyperbolic distance parameter $x \mapsto \lambda(x, a)$. The derivatives are

$$\partial_{x_i}\lambda(x,a) = \frac{x_i - a_i}{x_n a_n}, \quad \partial_{x_n}\lambda(x,a) = \frac{1}{a_n} - \frac{\lambda}{x_n},$$
$$\partial^2_{x_i}\lambda(x,a) = \frac{1}{a_n x_n}, \quad \partial^2_{x_n}\lambda(x,a) = -\frac{1}{a_n x_n} + \frac{2\lambda}{x_n^2},$$

and for the Euclidean Laplacian we obtain the formula

$$\begin{aligned}
\sum_{i=1}^{n}\partial^2_{x_i} &= \sum_{i=1}^{n}\frac{\partial^2\lambda}{\partial x_i^2}\partial_\lambda + \left(\frac{\partial\lambda}{\partial x_i}\right)^2\partial^2_\lambda \\
&= \left(\frac{n-2}{a_n x_n} + \frac{2\lambda}{x_n^2}\right)\partial_\lambda + \left[\frac{\|x-a\|^2 - (x_n - a_n)^2}{(a_n x_n)^2} + \left(\frac{1}{a_n} - \frac{\lambda}{x_n}\right)^2\right]\partial^2_\lambda \\
&= \left(\frac{n-2}{a_n x_n} + \frac{2\lambda}{x_n^2}\right)\partial_\lambda + \left(\frac{\lambda^2 - 1}{x_n^2}\right)\partial^2_\lambda.
\end{aligned}$$

For k-hyperbolic harmonic functions we find now the equation

$$(\lambda^2 - 1)\partial^2_\lambda\widetilde{u} + \left((n-2-k)\frac{x_n}{a_n} + (2+k)\lambda\right)\partial_\lambda\widetilde{u} = 0,$$

which depends on λ only if $k = n-2$. In that case, the previous equation becomes

$$\Delta^H_{n-2}\widetilde{u} = (\lambda^2 - 1)\partial^2_\lambda\widetilde{u} + n\lambda\partial_\lambda\widetilde{u} = 0.$$

By the correspondence of k-hyperbolic harmonic functions and eigenfunctions of the hyperbolic Laplacian we may compute the k-hyperbolic harmonic functions which correspond to the radial eigenfunctions of the hyperbolic Laplacian.

By (2.4.7) we find k-hyperbolic harmonic functions as

$$v(x) = x_n^{(k-n+2)/2}u(\lambda(x,a)),$$

where the radial function u satisfies

$$(\lambda^2 - 1)\partial^2_\lambda u + n\lambda\partial_\lambda u - \frac{(k+1)^2 - (n-1)^2}{4}u = 0,$$

which can be transformed to an associated Legendre equation. The solutions of the eigenvalue problem are presented in Section 2.5.1.

2.5 Integral formulas in the upper half-space

To obtain a Cauchy-type formula which is similar to the classical case (2.2.7) we need to find fundamental solutions to the modified Cauchy–Riemann operator. From the form of the Stokes formula for hypermonogenic functions it becomes evident that fundamental solutions are needed given $k \in \mathbb{R}$ for both k and $(-k)$. In this section we will work with paravectors $x \in \mathbb{R}^{n+1}$.

2.5.1 Methods for finding k-hypermonogenic fundamental solutions

Transplanting the solution from the unit ball
To illustrate transplanting of solutions we present only the derivation in the hypermonogenic case. It admits simple particular solutions yielding fairly explicit formulas. Using the particular Möbius transformation (2.4.1) and the observation in Theorem 2.4.7 the k-hyperbolic harmonic functions in the ball are transplanted to the upper half-space. We formulate this along Theorem 11 in [20].

Theorem 2.5.1. *Let f be a k-hyperbolic harmonic function in the unit ball. Then the function $x \mapsto h(x, e_n)$, where*

$$h(x, e_n) = \|x + e_n\|^{k-n+1} f(\varphi^{-1}(x)) \tag{2.5.1}$$

$$= \|x + e_n\|^{k-n+1} f\left(\frac{\|x - e_n\|}{\|x + e_n\|}\right) \tag{2.5.2}$$

is k-hyperbolic harmonic on $\mathbb{R}^{n+1}_+ \setminus \{e_n\}$. Furthermore, the function $x \mapsto h(x, y)$, where

$$h(x, y) = \|x - \widehat{y}\|^{k-n+1} f\left(\frac{\|x - y\|}{\|x - \widehat{y}\|}\right) \tag{2.5.3}$$

is k-hyperbolic harmonic on $\mathbb{R}^{n+1}_+ \setminus \{y\}$.

We may now use this result to obtain k-hyperbolic harmonic functions in the upper half-space. We illustrate this by transplanting the hyperbolic harmonic and $(1-n)$ hyperbolic harmonic fundamental solutions. The following lemma was presented in [51] using the results proven in [1]:

Lemma 2.5.2. *The function $x \mapsto p(x, y)$, where*

$$\begin{aligned} p(x, y) &= \frac{1}{2^{n-1}} \overline{D}_x \left(\int_{\frac{\|x-y\|}{\|x-\widehat{y}\|}}^{1} \frac{(1-s^2)^{n-1}}{s^n} \,\mathrm{d}s \right) \\ &= 2^n x_n^{n-1} y_n^{n-1} \frac{(x-y)^{-1}}{\|x-y\|^{n-1}} y_n e_n \frac{(x-\widehat{y})^{-1}}{\|x-\widehat{y}\|^{n-1}} \\ &= 2^{n-1} x_n^{n-1} y_n^{n-1} \frac{(x-y)^{-1} - (x-\widehat{y})^{-1}}{\|x-y\|^{n-1}\|x-\widehat{y}\|^{n-1}} \end{aligned}$$

is hypermonogenic on $\mathbb{R}^{n+1} \setminus \{y, \widehat{y}\}$ for each y such that $y_n \neq 0$.

The corresponding $(1-n)$-hypermonogenic fundamental solution is found by observing that $\|x\|^{1-n}$ is a null-solution to the operator (2.4.8) in the unit ball of $\mathbb{R}^{n+1}$ when $k = 1-n$. Transplanting this to the upper half-space and applying the conjugate Cauchy–Riemann operator we obtain the following result.

Lemma 2.5.3 ([22], Lemma 14)**.** *The function $x \mapsto q(x,y)$, where*

$$\begin{aligned} q(x,y) &= \frac{2^{n-1}}{1-n}\overline{D}_x\left(\frac{1}{\|x-y\|^{n-1}\|x-\widehat{y}\|^{n-1}}\right) \\ &= 2^n\frac{(x-y)^{-1}}{\|x-y\|^{n-1}}(x-Py)\frac{(x-\widehat{y})^{-1}}{\|x-\widehat{y}\|^{n-1}} \\ &= 2^{n-1}\frac{(x-y)^{-1}+(x-\widehat{y})^{-1}}{\|x-y\|^{n-1}\|x-\widehat{y}\|^{n-1}} \end{aligned}$$

is $(1-n)$-hypermonogenic on $\mathbb{R}^{n+1}\setminus\{y,\widehat{y}\}$ for any y such that $y_n \neq 0$.

For the general case of k-hyperbolic harmonic functions this procedure has been studied in [20].

Eigenfunctions of the Laplace–Beltrami operator in the half-space
For a twice differentiable function $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ depending only on the hyperbolic distance parameter $x \mapsto \lambda(x,a)$ we have the Laplace–Beltrami operator

$$\Delta^H u = (\lambda^2-1)\partial_\lambda^2 u + (n+1)\lambda\partial_\lambda u.$$

By (2.3.12), both P- and Q-parts of a k-hyperbolic harmonic function satisfy the Leutwiler–Weinstein equation

$$\Delta^E g - \frac{k}{x_n}\frac{\partial}{\partial x_n}g + \frac{\gamma}{x_n^2}g = 0. \tag{2.5.4}$$

They are thus eigenfunctions of the hyperbolic k-Laplacian and may be transformed to eigenfunctions of the ordinary hyperbolic Laplace–Beltrami operator by the equation (2.4.7). We need to solve the equation

$$(\lambda^2-1)\partial_\lambda^2 u + (n+1)\lambda\partial_\lambda u + \gamma' u = 0, \tag{2.5.5}$$

which becomes a Legendre differential equation by the substitution

$$u(\lambda) = (\lambda^2-1)^{(1-n)/4}v(\lambda).$$

Finally, we find that the function v satisfies the equation

$$(1-\lambda^2)\partial_\lambda^2 v - 2\lambda\partial_\lambda v + \left(\gamma' + \frac{n^2-1}{4} - \frac{(n-1)^2}{4(1-\lambda^2)}\right)v = 0, \tag{2.5.6}$$

which is an associated Legendre equation in standard form

$$(1-\lambda^2)\partial_\lambda^2 v - 2\lambda\partial_\lambda v + \left(\nu(\nu+1) - \frac{\mu^2}{1-\lambda^2}\right)v = 0, \tag{2.5.7}$$

with the coefficients satisfying

$$\nu(\nu+1) = \gamma' + \frac{n^2-1}{4}, \quad \mu^2 = \frac{(n-1)^2}{4}.$$

This equation yields k-hyperbolic harmonic fundamental solutions H_k and H_{-k} as associated Legendre functions $\mathbf{Q}_\nu^\mu(z)$ with the formulas

$$H_k = 2^{\frac{3-n}{2}} \frac{\Gamma\left(\frac{k+n+1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} a_n^{\frac{k+1-n}{2}} x_n^{\frac{k+1-n}{2}} (\lambda^2-1)^{\frac{1-n}{4}} \mathbf{Q}_{\frac{k}{2}}^{\frac{n-1}{2}}(\lambda), \tag{2.5.8}$$

$$H_{-k} = 2^{\frac{3-n}{2}} \frac{\Gamma\left(\frac{k+n-1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} a_n^{\frac{-k+1-n}{2}} x_n^{\frac{-k+1-n}{2}} (\lambda^2-1)^{\frac{1-n}{4}} \mathbf{Q}_{\frac{k}{2}-1}^{\frac{n-1}{2}}(\lambda), \tag{2.5.9}$$

as we have shown in Publication II.

Fundamental solutions in the complex plane $C\ell_{0,1}$ were found previously in Publication I. The solutions are given in an integral form which are, in fact, also expressible as associated Legendre functions. Using the integral representation for $Q_\nu^\mu(z)$ in [4, 3.7(5)] with $\mu = 0$ and $\nu = k/2 - 1$ we find

$$Q^0_{\frac{k}{2}-1}(\lambda) = 2^{-k/2} \int_0^\pi (\lambda + \cos\alpha)^{-k/2} (\sin\alpha)^{k-1} \, \mathrm{d}\alpha. \tag{2.5.10}$$

Comparing this formula to the solutions obtained in Publication I, we notice that the solutions in the complex plane may be given using (2.5.8) and (2.5.9) as well.

2.5.2 Cauchy formulas

Deriving Cauchy formulas in different settings begins with an application of the usual Stokes' theorem to a variant of the identity (2.2.5). By using the surface area form $x_n^{-k} \, \mathrm{d}\sigma$ and applying the Stokes formula for the Cauchy–Riemann operator (2.2.6) we find two formulas following the calculations in Theorems 6 and 9 of [22]:

Theorem 2.5.4. *Let f and g be differentiable Clifford algebra-valued functions in an open set $\Omega \subset \mathbb{R}^{n+1} \setminus \{x_n = 0\}$. Then the following generalized Stokes formulas hold:*

$$\int_{\partial\Omega} f \, \mathrm{d}\sigma_k g = \int_\Omega (f M_k^r) g + f(M_k^l g) - \frac{k}{x_n} P(fg') e_n \, \mathrm{d}m_k,$$

$$\int_{\partial\Omega} f \, \mathrm{d}\sigma g = \int_\Omega (f M_{-k}^r) g + f(M_k^l g) + \frac{k}{x_n} Q(fg') \, \mathrm{d}m,$$

where the differentials in the first equation are $\mathrm{d}\sigma_k := x_n^{-k} \, \mathrm{d}\sigma$ and $\mathrm{d}m_k := x_n^{-k} \, \mathrm{d}m$.

Separate formulas for the P- and Q-parts of the left-hand-sides are a direct corollary of these. According to the following corollary presented in Theorems 7 and 10 of [22]:

Corollary 2.5.5. *Let f and g be differentiable Clifford algebra-valued functions in an open set $\Omega \subset \mathbb{R}^{n+1} \setminus \{x_n = 0\}$. Then the following formulas hold*

$$\int_{\partial\Omega} P(f \, \mathrm{d}\sigma_k g) = \int_\Omega P\left((f M_k^r) g + f(M_k^l g)\right) \mathrm{d}m_k, \tag{2.5.11}$$

$$\int_{\partial\Omega} Q(f \, \mathrm{d}\sigma g) = \int_\Omega Q\left((f M_{-k}^r) g + f(M_k^l g)\right) \mathrm{d}m. \tag{2.5.12}$$

For hypermonogenic functions we find Cauchy formulas separately for P- and Q-parts by using the k-hyperbolic harmonic fundamental solutions. Denote by $H_k(x,a)$ and $H_{-k}(x,a)$ the fundamental solutions to the k-hyperbolic and $-k$-hyperbolic Laplace equation given by (2.5.8) and (2.5.9), respectively. These considerations result in the following theorem:

Theorem 2.5.6 (II, Theorem 6.2 and Theorem 6.3)**.** *Let Ω be an open subset of $\mathbb{R}^{n+1}_+$ and $K \subset \Omega$ be a smoothly bounded compact set. If f is k-hypermonogenic in Ω and $a \in K$, then*

$$Pf(a) = -\frac{1}{\omega_n}\int_{\partial K} P\left(\overline{M}^x_k H_k(x,a)\,\mathrm{d}\sigma_k f\right), \tag{2.5.13}$$

$$Qf(a) = -\frac{a_n^k}{\omega_n}\int_{\partial K} Q\left(\overline{M}^x_{-k} H_{-k}(x,a)\,\mathrm{d}\sigma f\right). \tag{2.5.14}$$

Combining these formulas we find by Lemma 2.3.4 that

$$\begin{aligned} f(a) = &-\frac{1}{\omega_n}\int_{\partial K}\left[x_n^{-k}P(\overline{M}_k H_k) + a_n^k Q(\overline{M}_{-k}H_{-k})e_n\right] P(\,\mathrm{d}\sigma f) \\ &-\frac{1}{\omega_n}\int_{\partial K}\left[a_n^k P(\overline{M}_{-k}H_{-k}) + x_n^{-k} Q(\overline{M}_k H_k)e_n\right] Q'(\,\mathrm{d}\sigma f), \end{aligned} \tag{2.5.15}$$

which simplifies further to a formula containing two hypermonogenic kernels. Hence, as our main theorem in Publication II we have the Cauchy-type formula for k-hypermonogenic functions.

Theorem 2.5.7 (II, Theorem 6.4)**.** *Let Ω be an open subset of $\mathbb{R}^{n+1}_+$ and $K \subset \Omega$ be a smoothly bounded compact set. If f is k-hypermonogenic in Ω and $a \in K$, then*

$$f(a) = \frac{1}{\omega_n}\int_{\partial\Omega} h_k^a(x,a)P(\,\mathrm{d}\sigma f) + h_{-k}^a(x,a)Q'(\,\mathrm{d}\sigma f), \tag{2.5.16}$$

where $h_k^a(x,a) = x_n^{-k}\overline{D}^a H_k(x,a)$ and $h_{-k}^a(x,a) = a_n^k\overline{D}^a H_{-k}(x,a)e_n$ are k-hypermonogenic with respect to a.

An important observation in the proof of this theorem is that in the fundamental solutions the variables a and x are interchangeable, that is, $H_k(x,a) = H_k(a,x)$. Hence, also the equation $M_k^a\overline{M_k^a}H_k(x,a) = 0$ holds.

Let us present as an example the kernels for the hypermonogenic case using the fundamental solutions $H_k(x,a)$ found by the eigenfunction-approach. These kernels agree with the ones transplanted from the ball.

Example 2.5.8. The hypermonogenic kernels are

$$h_{n-1}^y(x,y) = -2^{n-1}y_n^{n-1}\frac{(x-y)^{-1} - (\widehat{x}-y)^{-1}}{\|x-y\|^{n-1}\|x-\widehat{y}\|^{n-1}}, \tag{2.5.17}$$

$$h_{1-n}^y(x,y) = -2^{n-1}y_n^{n-1}\frac{(x-y)^{-1} + (\widehat{x}-y)^{-1}}{\|x-y\|^{n-1}\|x-\widehat{y}\|^{n-1}}e_n, \tag{2.5.18}$$

which agree with the kernels found in [25].

2.6 Further research

The theory of hyperbolic harmonic functions has been recently applied to stochastics by S.-L. Eriksson and T. Kaarakka in [24]. In the article the authors study connections between generalized hyperbolic Brownian motion and α-hyperbolic harmonic functions. The Leutwiler–Weinstein equation (2.5.4) and the Lie symmetries of its fundamental solutions are studied by A. Aksenov and H. Orelma in the article [2]. A generalization of the k-hypermonogenic functions is studied by D. C. Dinh in [17]. He studies k_i.hypermonogenic functions, which are solutions to the system

$$Df + \sum_{i=1}^{n} \frac{k_i e_i}{x_i} T_i(f) = 0, \tag{2.6.1}$$

where the operators T_i are similar to the operator Q defined in (2.3.4).

3. Time-Frequency Analysis

In this chapter we present an overview of time-frequency analysis focusing mainly on quadratic time-frequency transforms. We also summarize the characterization results that are the main content of Publication III.

There are many introductory texts available in time-frequency analysis both from a mathematical and an engineering perspective. The book by L. Cohen [13] is an accessible introduction to the subject. The book by P. Flandrin [37] has a broader scope discussing also wavelets. As a comprehensive reference in signal analysis we refer to the book [3]. For a more mathematical view on the subject we recommend the books by K. Gröchenig [39] and G. Folland [38].

3.1 Representations of signals

Time-frequency analysis studies methods of representing and manipulating signals using tools from Fourier analysis. Time-frequency representations include, for instance, linear decompositions of signals in terms of Gabor frames and quadratic time-frequency transforms yielding approximate time-frequency energy densities of signals.

By a signal we mean in this context either a function $u : \mathbb{R}^n \to \mathbb{C}$ or a distribution $v(\varphi)$ in terms of some suitable space of test functions φ. The Fourier transform $\widehat{u} : \mathbb{R}^n \to \mathbb{C}$ is defined for absolutely integrable signals $u \in L^1(\mathbb{R}^n)$ by

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}2\pi x\cdot\xi} u(x)\,\mathrm{d}x. \tag{3.1.1}$$

It can be used to describe the distribution of frequencies in the given signal u. In signal processing, the expression $x \mapsto u(x)$ is often referred to as the *time description* and $\xi \mapsto \widehat{u}(\xi)$ as the *frequency description* of the signal u.

3.1.1 Energy

For a square-integrable signal $u \in L^2(\mathbb{R}^n)$, the *energy* E_u is defined as

$$E_u := \|u\|^2 = \int_{\mathbb{R}^n} |u(x)|^2\,\mathrm{d}x. \tag{3.1.2}$$

Hence, the function $|u(x)|^2$ may be interpreted as the energy density of the signal in time. By Plancherel's theorem in Fourier analysis, the energy density with respect to frequency is given by $|\widehat{u}(\xi)|^2$. These energy densities describe the signal either in time or frequency. Computing, for example, the spectral energy density $|\widehat{u}(\xi)|^2$ of a piece of music u reveals the frequency distribution of all the notes but conveys no information on when they were playing. This description is thus insufficient for describing how the spectrum of a given signal changes in time. The problem of finding a joint representation describing the signal in both time and frequency is the starting point of time-frequency analysis.

3.1.2 Short-time Fourier transform

Put in simple terms, time-frequency analysis deals simultaneously with the questions "Which frequencies are present in a signal?" and "When do these frequencies appear?". The basic approach to answer these questions is to decompose the signal u into simpler signals with short duration and narrow bandwidth.

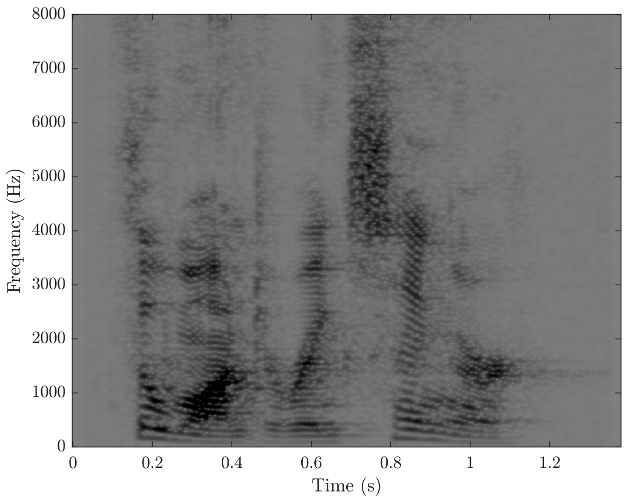

Figure 3.1. Spectrogram with a Gaussian window of the example signal.

We introduce time-frequency analysis by defining the *short-time Fourier transform* (STFT). To concentrate only on a short time interval we multiply our original signal u, for instance, by a fast-decaying smooth function such as the Gaussian $\phi_a(x) = \mathrm{e}^{-\pi\|x\|^2/a}$ with a real parameter $a > 0$. Applying then the Fourier transform, we find the STFT $u \mapsto V_\phi u$ which is given by the formula

$$V_\phi u(y, \xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}2\pi x\cdot\xi} u(x)\phi^*(x - y)\,\mathrm{d}x. \tag{3.1.3}$$

The function ϕ in the transform is referred to as the *window function* and the STFT is often also called the windowed Fourier transform. The squared absolute value of the

STFT given by

$$u \mapsto |V_\phi u|^2, \tag{3.1.4}$$

is called the *spectrogram* of u with respect to the window ϕ. Owing to its simplicity, the spectrogram is widely used as a means to discern different components in a given signal. The spectrogram of a male voice speaking “The wandering singer” is shown in Figure 3.1. The STFT requires one to choose a window function and this choice affects the form of the spectrogram. The choice is dictated by the balance between the decay of the window and its Fourier transform described by the Heisenberg uncertainty principle. It states that the duration and the bandwidth of a signal cannot simultaneously be arbitrarily small. We illustrate this using the Gaussian spectrogram defined above.

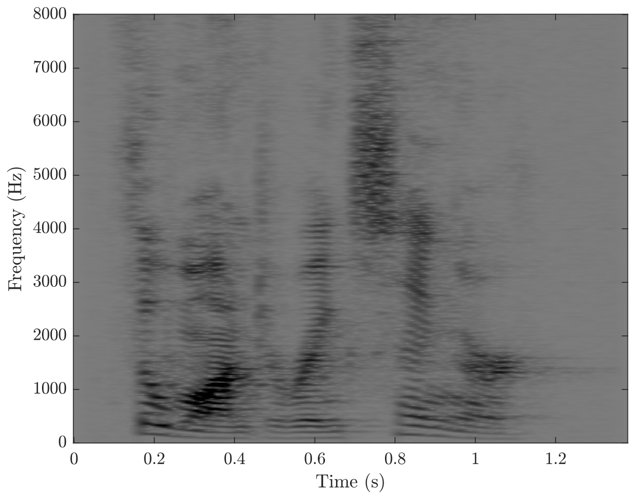

Figure 3.2. Spectrogram with a long Gaussian window of the example signal.

In Figure 3.2 we observe that a long window makes the frequencies clearer but smears the details in time.

In Figure 3.3 the short window sharpens the short details in time but makes frequencies harder to discern.

It is also a consequence of the uncertainty principle that a perfect point-wise time-frequency energy density $(x, \eta) \mapsto E_u(x, \eta)$ cannot exist. The spectrogram is one example of the many possible approximate energy densities satisfying a subset of the properties of an ideal energy density. In Publication III we study characterizations of these properties for quadratic time-frequency transforms.

3.1.3 Time-scale analysis

Time-scale analysis or wavelet analysis complements time-frequency analysis. The STFT is replaced by the continuous wavelet transform CWT. The CWT of a signal

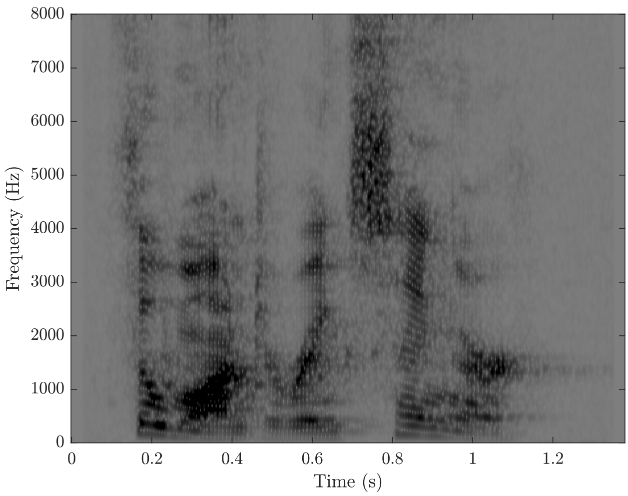

Figure 3.3. Spectrogram with a short Gaussian window of the example signal.

$u : \mathbb{R} \to \mathbb{C}$ with respect to the wavelet ψ is defined by

$$CWT_\psi u(y, s) = |s|^{-1/2} \int_{\mathbb{R}} u(x)\psi^* \left(\frac{x-y}{s}\right) \mathrm{d}x, \tag{3.1.5}$$

where ψ satisfies the admissibility condition

$$\int_0^\infty \frac{|\widehat{\psi}(\xi)|^2}{\xi} \mathrm{d}\xi < \infty, \tag{3.1.6}$$

see [58, Theorem 4.3].

Formula (3.1.5) can also be interpreted as the expression for the matrix coefficients $\langle u, \pi(y, s)\psi\rangle$ of the unitary representation $\pi(y, s) = T_y D_s$ of the affine group. In the short-time Fourier transform $\langle u, \sigma(y, \xi)\phi\rangle$ the underlying representation is $\sigma(y, \xi) = T_y M_\xi$ of the (polarized) Heisenberg group, see the definitions in Sections 3.2 and 3.3. Although formally quite similar, these theories find uses in different contexts and yield fairly different results.

For a tour of using wavelets in signal processing we refer to the book by S. Mallat [58]. Wavelet-type techniques such as the Littlewood–Paley decomposition were used in harmonic analysis long before wavelet analysis existed as an independent field. For an overview of wavelets in harmonic analysis we refer to the excellent treatises by Y. Meyer [60] and Y. Meyer and R. Coifman [61].

3.2 Fourier analysis

Fourier analysis is the main tool in time-frequency analysis. In this section we recall the basics of the Fourier transform and its extensions. For a more detailed treatise we refer to the book by E. Stein [65].

3.2.1 Fourier transform

An analog signal is modeled by a complex-valued function or distribution. Provided that $u \in L^1(\mathbb{R}^n)$ we may define the Fourier transform $\widehat{u} : \mathbb{R}^n \to \mathbb{C}$ directly as the (Lebesgue) integral

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}2\pi x\cdot\xi} u(x)\,\mathrm{d}x. \tag{3.2.1}$$

We will also use the operator notation $\mathcal{F}u := \widehat{u}$. In the context of engineering this is the Fourier transform for non-periodic analog signals.

Remark 3.2.1. It is worth pointing out that there is no agreement on the exact form of Fourier transform of functions $u \in L^1(\mathbb{R}^n)$ in the literature. However, these definitions differ only by having constants in different places. The mapping properties of these transforms are the same. For instance, L. Hörmander [43] defines the Fourier transform $\widetilde{\mathcal{F}}u$ as

$$\widetilde{\mathcal{F}}u(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi} u(x)\,\mathrm{d}x.$$

Perhaps the most important example of a Fourier transform is that of the Gauss function:

Example 3.2.2. Let A be a symmetric matrix in $\mathrm{GL}(\mathbb{R}, n)$. We define the Gaussian functions by

$$\varphi_A(x) = \mathrm{e}^{-\pi\langle Ax,x\rangle}. \tag{3.2.2}$$

The Fourier transform of φ_A is given by

$$\widehat{\varphi_A}(\xi) = |\det A|^{-1/2}\mathrm{e}^{-\pi\langle A^{-1}\xi,\xi\rangle}. \tag{3.2.3}$$

In particular, $\widehat{\varphi_I}(\xi) = \varphi_I(\xi)$. The proof follows simply from the one dimensional case presented in [39, p. 17].

Let u and v be integrable functions. We define their *convolution* by the formula

$$u * v(x) = \int_{\mathbb{R}^n} u(y)v(x-y)\,\mathrm{d}y. \tag{3.2.4}$$

Using Fubini's theorem we find that the Fourier transform of the convolution $u * v$ satisfies

$$\widehat{u * v}(\xi) = \widehat{u}(\xi)\widehat{v}(\xi). \tag{3.2.5}$$

Another consequence of Fubini's theorem is the fundamental identity

$$\int_{\mathbb{R}^n} \widehat{u}(x)v(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} u(x)\widehat{v}(x)\,\mathrm{d}x \tag{3.2.6}$$

known as duality or multiplication formula, see [65, p. 8].

A crucial result in the mathematical theory of Fourier analysis related also to the physical interpretation of the transform is the Plancherel theorem.

Theorem 3.2.3 (Plancherel Theorem)**.** *If $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then the equality*

$$\|u\|_{L^2(\mathbb{R}^n)} = \|\widehat{u}\|_{L^2(\mathbb{R}^n)}$$

holds. This allows the definition of the Fourier transform on $L^2(\mathbb{R}^n)$ as the bounded extension of the transform on $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. The transform thus defined is a unitary operator with the inverse

$$v(x) \mapsto \widehat{v}(-x).$$

A concise proof based on the Gauss function, originally due to Igusa, can be found in Gröchenig's book [39, Section 1.5]. In physical terms, the Plancherel theorem is the statement of conservation of energy under the Fourier transform. Hence, the squared magnitude of the Fourier transform may be interpreted as the spectral energy density of the signal.

If both u and $\widehat{u}$ are absolutely integrable, the *inverse Fourier transform* $\widehat{u} \mapsto u$ may be written as the integral

$$u(x) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}2\pi x\cdot\xi}\widehat{u}(\xi)\,\mathrm{d}\xi. \tag{3.2.7}$$

Lastly, we present some elementary operations on signals and how they act on the Fourier transform. Let $y, \xi \in \mathbb{R}^n$. We define the *modulation*, *translation* and *dilation* of a signal $u : \mathbb{R}^n \to \mathbb{C}$ by

$$M_\xi u(x) = \mathrm{e}^{\mathrm{i}2\pi x\cdot\xi}u(x), \tag{3.2.8}$$

$$T_y u(x) = u(x-y), \tag{3.2.9}$$

$$D_s u(x) = |s|^{-n/2}u(x/s), \quad s \in \mathbb{R}\setminus\{0\}, \tag{3.2.10}$$

respectively. It follows directly the definition (3.2.1) that these operations satisfy

$$\mathcal{F}\circ M_\xi = T_\xi \circ \mathcal{F},$$

$$\mathcal{F}\circ T_y = M_{-y} \circ \mathcal{F},$$

$$\mathcal{F}\circ D_s = D_{1/s} \circ \mathcal{F}$$

as operators in $L^1(\mathbb{R}^n)$.

The notation ι is used throughout for the reflection operator $\iota u(x) = u(-x)$. For functions of two variables $x, y \in \mathbb{R}^n$ the separate reflections in each variable are denoted by

$$\iota_1 u(x,y) = u(-x,y),$$

$$\iota_2 u(x,y) = u(x,-y).$$

3.2.2 Fourier transform of distributions

The definition of the Fourier transform for integrable or square integrable functions leaves still something to be desired. Such basic signals as the constant function or a sudden impulse are left without a proper Fourier transform even though these signal are often encountered in practice. For these signals we must extend the notion of the Fourier transform to distributions. We present here the basic definitions and refer to [40, 43, 66] for treatises on general theory.

Definition 3.2.4. Consider the vector space of functions $\varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$ satisfying

$$\sup_{x\in\mathbb{R}^n} |x^\beta \partial_x^\alpha \varphi(x)| < \infty \tag{3.2.11}$$

for all multi-indices α and β. This space equipped with the topology given by the family of seminorms defined by the left-hand side of (3.2.11) is called the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ of rapidly decreasing smooth functions, also known as the space of Schwartz test functions.

Definition 3.2.5. The space $\mathscr{S}'(\mathbb{R}^n)$ of temperate[1] distributions is the space of continuous linear functionals on $\mathscr{S}(\mathbb{R}^n)$. The notation

$$u(\varphi) =: \langle u, \varphi \rangle_{\mathscr{S}',\mathscr{S}}$$

means the evaluation of the temperate distribution u at φ.

The Fourier transform may now be defined motivated by equation (3.2.6).

Definition 3.2.6 (Fourier transform for temperate distributions)**.** For any $\varphi \in \mathscr{S}$ we define the Fourier transform of $u \in \mathscr{S}'$ as a temperate distribution by

$$\langle \widehat{u}, \varphi \rangle_{\mathscr{S}',\mathscr{S}} := \langle u, \widehat{\varphi} \rangle_{\mathscr{S}',\mathscr{S}}.$$

This definition encompasses virtually all interesting signals and for instance, the constant signal $\mathbf{1} : \mathbb{R}^n \to \mathbb{C}$, $\mathbf{1}(x) \equiv 1$, is a temperate distribution and it has the Fourier transform

$$\widehat{\mathbf{1}}(\xi) = \delta_0(\xi).$$

In defining time-frequency transforms, we require that the transform $Q(u, v)$ of signals $u, v \in \mathscr{S}(\mathbb{R}^n)$ is also a Schwartz test function. To this end we need to define the multipliers and convolutors of the space $\mathscr{S}(\mathbb{R}^n)$.

Definition 3.2.7. The space of multipliers $\mathscr{O}_M(\mathbb{R}^n)$ is the space of functions $\varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$ with polynomially bounded derivatives of all orders. The space of very slowly increasing smooth functions $\mathscr{O}_C(\mathbb{R}^n)$ is the space of smooth functions φ for which there is a number $N \in \mathbb{N}$ such that

$$\sup_{x\in\mathbb{R}^n} (1 + |x|)^N |\partial^\alpha \varphi(x)| < \infty$$

for any multi-index $\alpha \in \mathbb{N}_0^{|\alpha|}$. The dual space $\mathscr{O}'_C(\mathbb{R}^n)$ is called the space of convolutors of $\mathscr{S}(\mathbb{R}^n)$.

[1] Also known as tempered distributions, see the note in [40, p. 103].

The space of convolutors is precisely the subspace of temperate distributions such that Schwartz functions are mapped to Schwartz functions in convolution, that is, $f \in \mathscr{O}'_C(\mathbb{R}^n)$ if and only if

$$f * \varphi \in \mathscr{S}(\mathbb{R}^n)$$

for all $\varphi \in (\mathbb{R}^n)$. The Fourier transform is a bijective linear map from $\mathscr{O}_M(\mathbb{R}^n)$ to $\mathscr{O}'_C(\mathbb{R}^n)$ and from $\mathscr{O}'_C(\mathbb{R}^n)$ to $\mathscr{O}_M(\mathbb{R}^n)$.

3.2.3 Uncertainty principles

Results describing the relationship between the concentration of the signal and its Fourier transform are called uncertainty principles. Heisenberg uncertainty principle is the most well-known one:

Theorem 3.2.8 ([39], Theorem 2.2.1)**.** *If $u \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$ are arbitrary, then*

$$\left(\int_{\mathbb{R}} (x-a)^2 |u(x)|^2 \,\mathrm{d}x\right)^{1/2} \left(\int_{\mathbb{R}} (\xi-b)^2 |\widehat{u}(\xi)|^2 \,\mathrm{d}\xi\right)^{1/2} \geq \frac{1}{4\pi}\|u\|^2. \quad (3.2.12)$$

Equality holds if and only if u is a multiple of the translated and modulated Gaussian $T_a M_b \varphi_c(x) = \exp(\mathrm{i}2\pi b(x-a) - \pi(x-a)^2/c)$ for some $a, b \in \mathbb{R}$ and $c > 0$.

This theorem underlines the special role of the Gauss function in Fourier analysis as the function with minimal spread in time and frequency. The Heisenberg uncertainty principle implies further that a time-frequency point (x, η) cannot be interpreted as the frequency η present at time x. The same principle in the context of quantum mechanics states that the momentum and the position of a particle cannot be determined to arbitrary precision.

3.3 Time-frequency plane and the Heisenberg group

In time-frequency analysis the Heisenberg group is represented as the group of unitary operators on $L^2(\mathbb{R}^n)$ generated by modulations and translations. These operators are only commutative up to phase so there are different ways of defining the time-frequency shift corresponding to $(y, \xi) \in \mathbb{R}^{2n}$. We use the symmetric[2] definition

$$\rho(y, \xi) = M_{\xi/2} T_y M_{\xi/2}, \quad (3.3.1)$$

which yields the multiplication formula

$$\rho(y, \xi)\rho(y', \xi') = \mathrm{e}^{\mathrm{i}\pi(y'\cdot\xi - \xi'\cdot y)}\rho(y + y', \xi + \xi'). \quad (3.3.2)$$

Due to the phase factor appearing in the formula (3.3.1) also the phase shifts are needed for the definition of the Heisenberg group. Using the representation ρ with the

[2]Different ways of ordering the operations may be obtained using the operator $M_{\tau\xi} T_y M_{(1-\tau)\xi}$ with a parameter $\tau \in \mathbb{R}$. Besides the symmetric version satisfying $\tau = 1/2$, the value $\tau = 0$ is of special significance and it yields the polarized Heisenberg group.

phase shifts,

$$(\theta, y, \xi) \mapsto \mathrm{e}^{\mathrm{i}2\pi\theta}\rho(y, \xi) \tag{3.3.3}$$

we define the (full) Euclidean Heisenberg group $\mathbb{H}(n)$ in $\mathbb{R}^{2n+1}$.

Definition 3.3.1 (Heisenberg Group $\mathbb{H}(n)$)**.** [39, Definition 9.1.2] The Euclidean space $\mathbb{R}^{2n} \times \mathbb{R}$ equipped with the multiplication

$$(\theta, y, \xi) \cdot (\theta', y', \xi') = \left(\theta + \theta' + \frac{y'\xi - y\xi'}{2}, y + y', \xi + \xi'\right) \tag{3.3.4}$$

is called the full Heisenberg group $\mathbb{H}$. The quotient with respect to the subgroup $\{0\} \times \{0\} \times \mathbb{Z}$ is the reduced Heisenberg group $\mathbb{H}_r$.

It is worth pointing out that the formula (3.3.4) contains the standard symplectic form

$$B((y, \xi), (y', \xi')) = y \cdot \xi' - \xi' \cdot y. \tag{3.3.5}$$

Schrödinger representations are the usual representations of the Heisenberg group on the Hilbert space $L^2(\mathbb{R}^n)$. There is a scale of irreducible unitary representations depending on a parameter $\hbar$.

Definition 3.3.2 (Schrödinger Representation)**.** Let $\hbar \in \mathbb{R} \setminus \{0\}$. Schrödinger representations are defined by the formula

$$\rho_\hbar(\theta, y, \xi)u(x) = \mathrm{e}^{\mathrm{i}2\pi\hbar\theta}\mathrm{e}^{\mathrm{i}\pi\hbar y\cdot\xi}T_{\hbar y}M_\xi u(x). \tag{3.3.6}$$

This scale of representations gives all the irreducible unitary representations which are nontrivial on the center. This result is known as the Stone–von Neumann theorem [39, Theorem 9.3.1]. We will use the short notation $\rho := \rho_1$ for the representation encountered in (3.3.3). The matrix coefficients of the representation ρ defined by

$$\chi(u, v)(y, \xi) := \langle u, \rho(y, \xi)v\rangle \tag{3.3.7}$$

have a special role in defining time-frequency transforms as will be seen in Section 3.4.

Remark 3.3.3. In addition to our notation of ρ agreeing with Gröchenig's book [39], the version $\widetilde{\rho}(\theta, x, \xi) = \rho(\theta, -x, \xi)$ is used in several sources, for instance, in [38] and [70].

3.3.1 Time-frequency plane

Time-frequency transforms $Q(u, v)$ are defined as functions $(x, \eta) \mapsto Q(u, v)(x, \eta)$ on the time-frequency plane. In applications, time-frequency plane is often called the phase space or the position-momentum space. We introduce the structure of time-frequency plane and present some basic properties of functions on it and the symplectic Fourier transform. In our discussion, we use [38, 49] as basic references. For a comprehensive overview of symplectic methods in harmonic analysis, we refer to the monograph by M. de Gosson [14].

Definition 3.3.4. Time-frequency plane is defined to be the direct sum of vector spaces

$$\mathbb{R}^n \oplus \widehat{\mathbb{R}^n},$$

where $\widehat{\mathbb{R}^n}$ is the unitary dual of $\mathbb{R}^n$ consisting of the functions $e_\xi(x) = \mathrm{e}^{\mathrm{i}2\pi x\cdot\xi}$ with $\xi \in \mathbb{R}^n$. As the mapping $\xi \mapsto e_\xi$ is an isomorphism $\mathbb{R}^n \simeq \widehat{\mathbb{R}^n}$ of groups we will regard the time-frequency plane as the space $\mathbb{R}^n \oplus \mathbb{R}^n \simeq \mathbb{R}^{2n}$. This version of the time-frequency plane is endowed with the symplectic form (3.3.5).

Of particular importance in the time-frequency plane are those coordinate transforms which preserve the symplectic form. The set of these transforms forms the symplectic group.

Definition 3.3.5. Symplectic group $\mathrm{Sp}(2n)$ is the group of linear transformations $\mathcal{A}$ on $\mathbb{R}^{2n}$ satisfying

$$\mathcal{A}^*\sigma\mathcal{A} = \sigma,$$

where

$$\sigma = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is the matrix representing the symplectic form.

The symplectic group is generated by the mappings

$$(x,\eta) \mapsto (Tx, (T^{-1})^*\eta), \tag{3.3.8}$$

$$(x_k,\eta_k) \mapsto (\eta_k, -x_k), \quad k = 1,\dots,n, \text{ other coordinates remain fixed,} \tag{3.3.9}$$

$$(x,\eta) \mapsto (x, \eta + Sx), \tag{3.3.10}$$

where T is an automorphism of $\mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^n$ is a symmetric linear mapping.

The standard symplectic form gives rise to a version of the Fourier transform which we call the *symplectic Fourier transform*.

Definition 3.3.6. We define the symplectic Fourier transform $Fa \in \mathscr{S}(\mathbb{R}^{2n})$ for functions $a \in \mathscr{S}(\mathbb{R}^{2n})$ in the time-frequency plane as

$$\begin{aligned}(Fa)(y,\xi) &:= \iint_{\mathbb{R}^{2n}} \mathrm{e}^{\mathrm{i}2\pi B((x,\eta),(y,\xi))} a(x,\eta)\,\mathrm{d}x\,\mathrm{d}\eta \\ &= \iint_{\mathbb{R}^{2n}} \mathrm{e}^{-\mathrm{i}2\pi(x\cdot\xi - y\cdot\eta)} a(x,\eta)\,\mathrm{d}x\,\mathrm{d}\eta.\end{aligned}$$

This transform is also given by

$$F = J \circ \mathcal{F}_1 \circ \mathcal{F}_2^{-1}, \tag{3.3.11}$$

where the partial Fourier transforms $\mathcal{F}_i$ are defined by

$$\mathcal{F}_1 a(\xi, y) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}2\pi x\cdot\xi} a(x,y)\,\mathrm{d}x, \tag{3.3.12}$$

$$\mathcal{F}_2 a(x, \eta) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}2\pi y\cdot\eta} a(x,y)\,\mathrm{d}y, \tag{3.3.13}$$

and the flip of variables J by $J : a(\xi, y) \mapsto a(y,\xi)$.

We denote the points in the time-frequency plane by (x, η). The Fourier dual of this space is called *ambiguity plane* or *delay-Doppler plane* $\mathbb{R}^{2n} \ni (y, \xi)$.

A direct consequence of the definition is that the symplectic Fourier transform is its own inverse, $F^{-1} = F$, and that it commutes with symplectic coordinate transforms, that is,

$$F \circ T_{\mathcal{A}} = T_{\mathcal{A}} \circ F, \tag{3.3.14}$$

where $T_{\mathcal{A}} a(x, \eta) = a(\mathcal{A}(x, \eta))$ and $\mathcal{A} \in \mathrm{Sp}(2n)$.

Lemma 3.3.7. *The symplectic Fourier transform is an automorphism of $L^2(\mathbb{R}^{2n})$, $\mathscr{S}(\mathbb{R}^{2n})$ and $\mathscr{S}'(\mathbb{R}^{2n})$. It is also a bijective map $\mathscr{O}_M(\mathbb{R}^{2n}) \to \mathscr{O}'_C(\mathbb{R}^{2n})$ and satisfies the Plancherel formula*

$$\langle Fa, Fb \rangle_{L^2(\mathbb{R}^{2n})} = \langle a, b \rangle_{L^2(\mathbb{R}^{2n})}$$

and the identities

$$F \circ T_{(z,\zeta)} = M_{(z,\zeta)} \circ F,$$
$$(Fa)^* = \iota F(a^*),$$
$$F(a * b) = Fa \cdot Fb,$$

where $a, b \in \mathscr{S}(\mathbb{R}^{2n})$, translations and modulations in time and frequency are

$$T_{(z,\zeta)} a(x, \eta) := a(x - z, \eta - \zeta),$$
$$M_{(z,\zeta)} a(x, \eta) := \mathrm{e}^{\mathrm{i}2\pi(x\cdot\zeta - z\cdot\eta)} a(x, \eta) = \mathrm{e}^{\mathrm{i}2\pi B((z,\zeta),(x,\eta))} a(x, \eta),$$

and the time-frequency convolution is an ordinary convolution with respect to both time and frequency.

3.4 Quadratic time-frequency transforms

A time-frequency transform Q is a function in time and frequency $(x, \eta) \mapsto Q(u)(x, \eta)$ depending on the analyzed signal u. There are several families of such transforms according to the dependence on the signal. The STFT $u \mapsto V_\phi u$ defined in (3.1.3) is a linear time-frequency transform. We will consider *quadratic transforms* $Q(u, v)$ depending on the signals u and v so that the map

$$(u, v) \mapsto Q(u, v) \tag{3.4.1}$$

is sesquilinear. These transforms constitute a suitable family for approximate energy distributions of the signal. We call $Q(u, u)$ the *time-frequency distribution* of the signal u and use the short notation

$$Q[u](x, \eta) := Q(u, u)(x, \eta)$$

The family of suitable sesquilinear maps Q may be restricted by the requirement that the maps

$$(x,\eta) \mapsto Q(u,v)(x,\eta) \tag{3.4.2}$$

$$(u,v) \mapsto Q(u,v) \tag{3.4.3}$$

satisfy additional properties characteristic to an energy distribution. For example, a natural requirement is that the integral of the distribution $Q[u]$ over the time-frequency plane gives the energy of the signal, that is,

$$\iint_{\mathbb{R}^{2n}} Q[u](x,\eta)\,\mathrm{d}x\,\mathrm{d}\eta = \|u\|^2. \tag{3.4.4}$$

Compared to linear representations, quadratic transforms are more complicated to interpret partially due to the presence of interference terms. Let us consider a quadratic time-frequency distribution Q of a sum of the signals u and v. The distribution $Q[u+v]$ of the superposed signal $\alpha u + \beta v$, where $\alpha, \beta \in \mathbb{C}$, satisfies

$$\begin{aligned} Q[\alpha u + \beta v] &= Q(\alpha u + \beta v) \\ &= |\alpha|^2 Q[u] + |\beta|^2 Q[v] + \alpha\beta^* Q(u,v) + \beta\alpha^* Q(v,u). \end{aligned}$$

The terms $|\alpha|^2 Q[u]$ and $|\beta|^2 Q[v]$ on the right-hand side are called *autoterms* and the rest of the terms $\alpha\beta^* Q(u,v) + \beta\alpha^* Q(v,u)$ are *interference terms*. A lot of effort has been put to understanding the interferences and defining distributions which reduce these terms, see [9]. A quantitative analysis of the interference terms has been done in [5].

We introduce next the Wigner transform, which is often taken as the starting point for defining quadratic time-frequency transforms.

3.4.1 Wigner transform

In 1932 E. Wigner defined an energy distribution in the context of quantum thermodynamics which satisfies a number of desirable properties for an energy density. In the context of signal analysis the same distribution was derived independently by J. Ville in 1948. It has become the fundamental time-frequency transform in signal analysis.

Definition 3.4.1 ([39], Definition 4.3.1)**.** The Wigner transform $W(u,v)$ is defined for $u, v \in \mathscr{S}(\mathbb{R}^n)$ by the formula

$$W(u,v)(x,\eta) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}2\pi y\cdot\eta} u(x+y/2) v^*(x-y/2)\,\mathrm{d}y. \tag{3.4.5}$$

The properties of the Wigner transform have been well studied and can be found in many sources, for instance, in [13, 38, 39]. As an illustration we present some of the fundamental properties of the transform.

The first basic property the Wigner transform satisfies is the way it transforms in terms of modulations and translations of a signal.

$$W(T_{x_0}u, T_{x_0}v)(x,\eta) = W(u,v)(x-x_0,\eta) \tag{3.4.6}$$

$$W(M_{\eta_0}u, M_{\eta_0}v)(x,\eta) = W(u,v)(x,\eta-\eta_0) \tag{3.4.7}$$

These properties together are called time-frequency covariance, see Definition 3.4.15.

The Wigner transform satisfies also the following properties, see [38, Section 1.8]:

- Normalization.
- Correct marginal densities.
- Real-valued distribution.
- Full symplectic covariance.
- Unitarity.

There are many articles and monographs where these properties have been proven for the Wigner transform and other time-frequency covariant transforms. For reference, we mention [13, 37, 39].

An ideal energy distribution $Q[u](x, \eta)$ would be positive for all signals u and for all points (x, η) in the time-frequency plane. The Wigner distribution $W[u]$ is real-valued but it is not positive, in general, as it satisfies

$$W[u](0, 0) = -2^n \|u\|^2,$$

where u is an odd real-valued function. By a theorem due to Hudson [45], the Wigner distribution $W[u]$ is positive if and only if u is a generalized Gaussian function.

We present next some examples of Wigner distributions.

Example 3.4.2 ([39], Chapter 4)**.** The Wigner distribution of the Gaussian $\phi_a(x) = \mathrm{e}^{-\pi x^2/a}$ is

$$W[\phi_a](x, \eta) = (2a)^{n/2} \phi_{\frac{a}{2}}(x) \phi_{\frac{1}{2a}}(\eta). \tag{3.4.8}$$

For the delta distribution and pure frequency we have

$$W[\delta_{x_0}] = \delta_{x_0} \otimes \mathbf{1},$$

$$W[e_{\eta_0}] = \mathbf{1} \otimes \delta_{\eta_0},$$

where the equality is in the sense of temperate distributions.

Lastly, we discuss the full symplectic covariance of the Wigner transform. Let $\mathcal{A}$ be a symplectic map defined in 3.3.5. The *symplectic operator* $\mu(\mathcal{A})$ is defined up to a complex phase factor by applying Schur's lemma to the Schrödinger representation ρ of the Heisenberg group. It is the unitary operator satisfying

$$\rho(\mathcal{A}(x, \eta)) = \mu(\mathcal{A}) \rho(x, \eta) \mu(\mathcal{A})^{-1}, \tag{3.4.9}$$

see [39, Section 9.4]. The map $\mathcal{A} \mapsto \mu(\mathcal{A})$ from symplectic maps to unitary operators on $L^2(\mathbb{R}^n)$ is called the *metaplectic representation*. The group of all symplectic operators $\mu(\mathcal{A})$ is the *metaplectic group*, see [49, Section 4.4.2].

Definition 3.4.3. The metaplectic group $\mathrm{Mp}(n)$ is the subgroup of unitary operators on $L^2(\mathbb{R}^n)$ generated by

$$Pu(x) = \mathrm{e}^{\mathrm{i}\theta} u(x), \tag{3.4.10}$$

$$Au(x) = |\det T|^{-1/2} u(T^{-1}x), \tag{3.4.11}$$

$$B_k u(\check{x}_k) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi x_k \xi_k} u(x)\,\mathrm{d}x_k, \quad k = 1, \ldots, n, \tag{3.4.12}$$

$$Cu(x) = \mathrm{e}^{\mathrm{i}\pi\langle Sx, x\rangle} u(x), \tag{3.4.13}$$

where $\theta \in \mathbb{R}$, T is an automorphism of $\mathbb{R}^n$, $\check{x}_k = (x_1, \ldots, x_{k-1}, \xi_k, x_{k+1}, \ldots, x_n)$ and S is a symmetric matrix.

Up to a complex phase factor, the symplectic mappings (3.3.8), (3.3.9) and (3.3.10) correspond to the operators A, B_k and C, respectively, see [38, Section 4.2]. The *full symplectic covariance* is the property given by the following Lemma:

Lemma 3.4.4 ([38], p. 180)**.** *The Wigner transform satisfies*

$$W(u, v)(\mathcal{A}^{-1}(x, \eta)) = W(\mu(\mathcal{A})u, \mu(\mathcal{A})v)(x, \eta) \tag{3.4.14}$$

for any $u, v \in \mathscr{S}'(\mathbb{R}^n)$ *and for any* $\mathcal{A} \in \mathrm{Sp}(2n)$.

The Wigner transform is also characterized by this property as has been proven, for instance, in [70, Chapter 30]. We use the formula (3.4.14) as a source of transformation properties that other time-frequency transform should ideally have.

In spite of the many good properties of the Wigner transform, it has also strong interference terms. This makes it difficult to apply in practice. The interference terms may be decreased by smoothing the distribution in time and frequency. We define next a family of such transforms.

3.4.2 Definition of the Cohen class

We consider transforms which can be defined as convolution smoothings of the Wigner transform. In the usual terminology these transforms are referred to as the *Cohen class* transforms named after Leon Cohen who first studied such transforms systematically in [12]. The main motivation of the definition is the requirement of time-frequency covariance introduced previously for the Wigner transform.

Definition 3.4.5. A sesquilinear map $(u, v) \mapsto Q(u, v)(x, \eta)$ is *time-frequency covariant* if

$$Q(T_{x_0} M_{\eta_0} u, T_{x_0} M_{\eta_0} v)(x, \eta) = Q(u, v)(x - x_0, \eta - \eta_0) \tag{3.4.15}$$

for all $(x, \eta), (x_0, \eta_0) \in \mathbb{R}^{2n}$ and $u, v \in \mathscr{S}(\mathbb{R}^n)$.

Time-frequency convolution becomes a multiplication operator in the ambiguity plane after a symplectic Fourier transform. It is easiest to define the transforms in the Cohen class by these multiplication kernels $(y,\xi)\mapsto\phi_Q(y,\xi)$ and the symplectic Fourier transform of the Wigner transform $FW(u,v)$. The symplectic Fourier transform $FW(u,v)$ of the Wigner transform is the (narrow-band) *ambiguity transform*

$$FW(u,v)(y,\xi)=\chi(u,v)(y,\xi):=\int_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}2\pi x\cdot\xi}u(x+y/2)v^*(x-y/2)\,\mathrm{d}x. \tag{3.4.16}$$

We define the Cohen class time-frequency transforms as follows:

Definition 3.4.6. A sesquilinear map $Q:\mathscr{S}(\mathbb{R}^n)\times\mathscr{S}(\mathbb{R}^n)\to\mathscr{S}(\mathbb{R}^{2n})$ is a *time-frequency transform* if it satisfies

$$FQ(u,v)(y,\xi)=\phi_Q(y,\xi)\chi(u,v)(y,\xi)$$

for some kernel function $\phi_Q\in\mathscr{O}_M(\mathbb{R}^{2n})\cap L^\infty(\mathbb{R}^{2n})$. In other words, the transform Q is a convolution of the Wigner transform,

$$Q(u,v)=\psi_Q*W(u,v),$$

such that $\psi_Q\in\mathscr{O}'_C(\mathbb{R}^{2n})$ and $Q:L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^{2n})$.

As convolutions of the Wigner transform, all transforms in the Cohen class are time-frequency covariant.

The Wigner transform is a particular example $W_{1/2}$ of the τ-Wigner transforms defined by

$$W_\tau(u,v)(x,\eta)=\int_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}2\pi y\cdot\eta}u(x+\tau y)v^*(x-(1-\tau)y)\,\mathrm{d}y, \tag{3.4.17}$$

for $\tau\in\mathbb{R}$. Any τ-Wigner transform could be taken as the fundamental time-frequency transform as well. This is shown by our version of [39, Theorem 4.5.1].

Lemma 3.4.7. *Let Q be a time-frequency transform according to Definition 3.4.6. The evaluation at the time-frequency origin Q_0 given by*

$$Q(u,v)(0,0)=\langle u,Q_0v\rangle \tag{3.4.18}$$

defines the distribution kernel $\delta^Q\in\mathscr{S}'(\mathbb{R}^{2n})$ and

$$Q(u,v)(x,\eta)=\langle\delta^Q,\rho(x,\eta)^{-1}u\otimes(\rho(x,\eta)^{-1}v)^*\rangle_{\mathscr{S}',\mathscr{S}}, \tag{3.4.19}$$

where $\rho(x,\eta)=M_{\eta/2}T_xM_{\eta/2}$ is the symmetric time-frequency shift defining the Schrödinger representation. The corresponding time-frequency distribution can be given in terms of the τ-Wigner transform as

$$Q(u,v)(x,\eta)=(W_\tau(u,v)*\iota_1\mathcal{F}_2P_\tau\delta^Q)(x,\eta), \tag{3.4.20}$$

where $P_\tau u(x,y)=u(x+\tau y,x+(\tau-1)y)$ with $\tau\in\mathbb{R}$ and $u\in\mathscr{S}(\mathbb{R}^n)$.

In Publication III we have used the Wigner transform as the basic time-frequency transform to study the properties of other transforms.

3.4.3 Quantization

Any Cohen class time-frequency transform defines a *quantization*, which is a map from functions or distributions in the time-frequency plane to linear operators on a suitable space of test functions, say, $\mathscr{S}(\mathbb{R}^n)$.

Definition 3.4.8. Let $(u, v) \mapsto Q(u, v)$ be a Cohen class time-frequency transform. The Q-quantization is the map $a \mapsto \operatorname{Op}_Q a$ taking temperate distributions $a \in \mathscr{S}'(\mathbb{R}^{2n})$ to linear operators from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$ defined by

$$\langle Q(u, v), a^* \rangle_{\mathscr{S}(\mathbb{R}^{2n}),\mathscr{S}'(\mathbb{R}^{2n})} = \langle u, \left[(\operatorname{Op}_Q a)v\right]^* \rangle_{\mathscr{S}(\mathbb{R}^n),\mathscr{S}'(\mathbb{R}^n)}. \tag{3.4.21}$$

The distribution a is called a *symbol* of the operator $\operatorname{Op}_Q a$.

As an example we give the formulas for the quantizations related to the Rihaczek and Wigner transforms.

Example 3.4.9. The Rihaczek transform yields the quantization formula

$$\langle R(u, v), a \rangle = \langle u, \operatorname{Op}_R v \rangle, \tag{3.4.22}$$

where the formal expression

$$\operatorname{Op}_R v(x) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}2\pi x\cdot\eta} a(x, \eta)\widehat{v}(\eta)\,\mathrm{d}\eta, \tag{3.4.23}$$

defines the *Kohn–Nirenberg quantization*. The Kohn–Nirenberg operator $\operatorname{Op}_R a$ is traditionally called the pseudo-differential operator given by the symbol a. A comprehensive study of the theory of pseudo-differential operators can be found, for instance, in [44, 63].

Using the Wigner transform, the symbol a defines formally the operator

$$\operatorname{Op}_W v(x) = \int_{\mathbb{R}^{2n}} \mathrm{e}^{\mathrm{i}2\pi(x-y)\cdot\eta} a([x+y]/2, \eta)v(y)\,\mathrm{d}y\,\mathrm{d}\eta. \tag{3.4.24}$$

This formula is the *Weyl quantization* of the symbol a. Weyl operators have been studied in [44, 70]. See also the discussion in [63, Chapter XII].

Properties of time-frequency transforms are reflected in the properties of the corresponding quantization and vice-versa. For instance, the Weyl operators are symplectic invariant, see [70, Theorem 29.13]. This is equivalent with the full symplectic covariance of the Wigner transform. In Publication III we study among other characterizations such equivalent conditions between time-frequency transforms Q and their quantizations Op_Q.

3.4.4 Properties of time-frequency transforms

Different desirable properties for time-frequency transforms have been studied previously on several occasions, starting with Cohen [12] and later continued by Claasen and Mecklenbräuker [10]. An overview in the one-dimensional case is given in the tables in [3, pp. 146–147]. The properties have been studied in terms of the Wigner ambiguity kernel ϕ_Q in [47] and in [37, Section 2.3].

In Publication III we prove characterizations for the properties of Cohen class time-frequency transforms in terms of the following objects:

- The transform Q itself.
- The operator Q_0, see Lemma 3.4.7.
- Some of the kernels e.g. ϕ_Q or φ_Q.
- The quantization Op_Q.

As an example of the energy density properties, let us consider the correct marginal densities in terms of the quantization Op_Q and the ambiguity kernel ϕ_Q.

Definition 3.4.10. A time-frequency transform has the *correct frequency margin* if it satisfies

$$\int_{\mathbb{R}^n} Q[u](x,\eta)\,\mathrm{d}x = |\widehat{u}(\eta)|^2 \tag{3.4.25}$$

for all $u \in \mathscr{S}(\mathbb{R}^n)$, and the *correct time margin* if

$$\int_{\mathbb{R}^n} Q[u](x,\eta)\,\mathrm{d}\eta = |u(x)|^2 \tag{3.4.26}$$

for all $u \in \mathscr{S}(\mathbb{R}^n)$.

The margins are correct in the sense that if $Q[u]$ is seen as an energy distribution of the signal u, then integrating over time should yield the spectral energy density $|\widehat{u}(\eta)|^2$. The frequency integral should result in the energy density in time $|u(x)|^2$.

Considering the Q-quantization, the correct frequency margin is equivalent to the property that the symbols $a(x,\eta) = \widehat{f}(\eta)$ map to multiplier operators

$$\widehat{v}(\eta) \mapsto \widehat{f}(\eta)\widehat{v}(\eta). \tag{3.4.27}$$

Correct time margins correspond to the mapping of the symbols $a(x,\eta) = g(x)$ to multiplication operators

$$v(x) \mapsto g(x)v(x). \tag{3.4.28}$$

For example, both Weyl and Kohn–Nirenberg quantizations satisfy these properties.

The conditions in terms of the ambiguity kernel ϕ_Q for the correct frequency and time margins become

$$\varphi_Q(y,0) = 1, \text{ for all } y \in \mathbb{R}^n, \tag{3.4.29}$$

$$\varphi_Q(0,\xi) = 1, \text{ for all } \xi \in \mathbb{R}^n, \tag{3.4.30}$$

respectively.

3.5 Generalizations

The Wigner transform does not directly generalize to the more general context of locally compact groups. This is due to the dilation $y \mapsto y/2$ in the definition of the transform. As we have discussed in Lemma 3.4.7, Cohen class time-frequency transforms may be defined using any of the τ-Wigner transforms. In particular, the Rihaczek transform $W_0(u, v) = R(u, v)$, where

$$R(u, v) = u(x)\mathrm{e}^{-\mathrm{i}2\pi x\cdot\eta}\widehat{v}^*(\eta), \tag{3.5.1}$$

is a good candidate for the basic time-frequency transform. Many of the properties discussed in Publication III may be considered in the more general setting of locally compact groups. The Rihaczek transform is the starting point for defining Cohen class time-frequency transforms on groups in the article by V. Turunen [67].

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