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Approximate distribution of capacity of Rayleigh fading MIMO channels

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Approximate closed-form density and distribution functions for the capacity of multiple-input multiple-output (MIMO) radio systems in spatially semi-correlated Rayleigh fading channels are derived. The approximations are given in terms of the Meijer G-function, hence allowing easy numerical evaluation of capacity outage probabilities with, for example, Maple or Mathematica.

Introduction: We consider a MIMO system with n_t transmit and n_r receive antennas and denote $K = \min(n_r, n_t)$, $L = \max(n_r, n_t)$. When the MIMO channel matrix is random, the capacity becomes a random variable [1]. To our knowledge, no closed-form expressions for the exact density (pdf) or distribution (cdf) functions of channel capacity have been found even for the simplest case of Rayleigh i.i.d. channel statistics. In [2] an exact expression for the characteristic function of the capacity was derived, and in [3] the exact distribution functions (pdf and cdf) for the special case $K \leq 3$ are given in a form involving numerical integration.

In this Letter we derive closed-form approximations for the pdf and cdf of MIMO capacity by using an inverse Mellin transform technique. Spatial correlation is allowed at one end of the link; this is called semi-correlated fading [2].

Channel model: Assume that the distribution of the transmitted signal is complex Gaussian with correlation matrix $P/n_t \boldsymbol{I}_{n_t}$, where P is the total power of the transmitted signal. Then, for a given $n_r \times n_t$ channel matrix $\boldsymbol{H}$, the mutual information is given by $C_{\boldsymbol{H}} = \log_2|\boldsymbol{I}_K + \rho/n_t R^{1/2}\boldsymbol{W}\boldsymbol{R}^{(1/2)H}$, where

$$\boldsymbol{W} = \begin{cases} \boldsymbol{H}\boldsymbol{H}^H, & \text{if } n_r \leq n_t \\ \boldsymbol{H}^H\boldsymbol{H}, & \text{if } n_r > n_t \end{cases}$$

and ρ is the average SNR at the output of each of the n_r receiver antennas [1]. We assume that the entries of $\boldsymbol{H}$ are independent zero-mean circularly symmetric complex Gaussian random variables with unit variance. Spatial correlation is allowed at the link end with K antennas and it is governed by the $K \times K$ correlation matrix $\boldsymbol{R}$, which is assumed to have full rank. Complex additive white Gaussian noise, i.i.d. in spatial and time domains, is also assumed throughout this Letter.

Assuming that the receiver knows $\boldsymbol{H}$ perfectly while the transmitter has no knowledge of it, $C_{\boldsymbol{H}}$ is also the instantaneous maximum of mutual information, i.e. the channel capacity associated with $\boldsymbol{H}$. The capacity outage is defined as the event $\{\boldsymbol{H} : C_{\boldsymbol{H}} < t\}$ with the corresponding outage probability $\Pr(C_{\boldsymbol{H}} < t)$. To evaluate this probability, one needs to find the distribution of $C_{\boldsymbol{H}}$, which is, in general, a difficult problem. In the sequel, we derive an approximate distribution in terms of the Meijer G-function, defined below.

Meijer G-function: The Meijer G-function is a generalisation of the generalised hypergeometric function and may be defined using the contour integral representation [4]:

$$G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} z_{a_1}, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) = \frac{1}{j2\pi}\int_{\mathcal{L}} \frac{\prod_{i=1}^{m}\Gamma(b_i + s)\prod_{i=1}^{n}\Gamma(1 - a_i - s)}{\prod_{i=n+1}^{p}\Gamma(a_i + s)\prod_{i=m+1}^{q}\Gamma(1 - b_i - s)} z^{-s}\, ds \tag{1}$$

where z, $\{a_i\}_{i=1}^{p}$, and $\{b_i\}_{i=1}^{q}$ are, in general, complex-valued. The Meijer G-function has been implemented in commercial mathematics software packages, such as Mathematica and Maple.

Approximate pdf and cdf: Let us denote $X = |\boldsymbol{W}|$. From the Minkowski determinant inequality [5, Th 7.8.8] we have the bound $C_{\boldsymbol{H}} \geq K \log_2(1 + (\rho_{eff}/n_t)X^{1/K}) = Y$, where $\rho_{eff} = \rho|\boldsymbol{R}|^{1/K}$ is the SNR modified by the geometric mean of the eigenvalues of $\boldsymbol{R}$. The lower bound is tight for large ρ/n_t. We are interested in finding the distribution of Y. Towards this end, we can make use of the well-known fact that the random variable X is distributed as the product of K independent $1/2\chi^2_{\mu_k}$ variates with $\mu_k = 2(L - k + 1)$, $k = 1, \ldots, K$, degrees of freedom [6]. The pdf of X can be found as the inverse Mellin transform of $v_h = E[X^h]$, defined by the contour integral [7]:

$$f_X(x) = \frac{1}{j2\pi}\int_{\mathcal{L}} v_h x^{-(h+1)}\, dh \tag{2}$$

Note that h does not need to be integer.

The moments of a $(1/2)\chi^2_{\mu}$ variate random variable are given by $E[((1/2)\chi^2_{\mu})^h] = [\Gamma((1/2)\,\mu)]^{-1}\Gamma((1/2)\,\mu + h)$ from which it follows that $v_h = [\prod_{k=1}^{K}\Gamma(1/2\,\mu_k)]^{-1}\prod_{k=1}^{K}\Gamma(1/2\,\mu_k + h)$. By setting $s = h + 1$, substituting v_h in (2), and using the definition of the Meijer G-function from (1) we have

$$f_X(x) = \frac{1}{\prod_{k=1}^{K}\Gamma(L - k + 1)} G^{K,0}_{0,K}\left(x \,\middle|\, \begin{matrix} - \\ L-1, \ldots, L-K \end{matrix}\right) \tag{3}$$

It can be shown that the Jacobian of the transformation between X and Y is $[g(Y)]^{K-1}2^{Y/K}\ln 2(\rho_{eff}/n_t)^{-1}$, where we abbreviated $g(y) = (\rho_{eff}/n_t)^{-1}(2^{y/K} - 1)$. Consequently, after a change of variable, the density of Y is given as:

$$f_Y(y) = \frac{[g(y)]K^{-1}2^{y/K}\ln 2}{(\rho_{eff}/n_t)\prod_{k=1}^{K}\Gamma(L - k + 1)} \times G^{K,0}_{0,K}\left([g(y)]^K \,\middle|\, \begin{matrix} - \\ L-1, \ldots, L-K \end{matrix}\right) \tag{4}$$

The cdf $F_Y(t) = \int_0^t f_Y(y)\, dy$ is obtained by integrating $f_Y(y)$ with respect to y inside the contour integral by using:

$$\int (2^{y/K} - 1)^{K(1-s)-1}2^{y/K}\, dy = \frac{1}{\ln 2}\frac{1}{1 - s}(2^{y/K} - 1)^{K(1-s)}$$

setting $1 - s = \Gamma(2 - s)/\Gamma(1 - s)$, and again using the definition of the Meijer G-function (1). The result, an upper bound on outage probability, is:

$$F_Y(t) = \frac{[g(t)]^K}{\prod_{k=1}^{K}\Gamma(L - k + 1)} \times G^{K,1}_{1,K+1}\left([g(t)]^K \,\middle|\, \begin{matrix} 0 \\ L-1, \ldots, L-K, -1 \end{matrix}\right) \tag{5}$$

The closed-form pdf and cdf in (4) and (5) are the main results of this Letter.

Note that, at high SNR, the capacity loss incurred by spatial correlation may be interpreted as SNR loss, which is equal to the geometric mean of the eigenvalues of the spatial correlation matrix. To give a simple example, suppose that $K = 3$ and that the ijth element of $\boldsymbol{R}$ is $r^{|i-j|}$ where r is a spatial correlation parameter. For purposes of this example, we choose $r = 0.5$, which results in $|\boldsymbol{R}|^{1/K} \simeq 0.83$; hence spatial correlation results in $10\log(0.83) \simeq -0.83$ dB effective loss in SNR compared to the Rayleigh i.i.d. case.

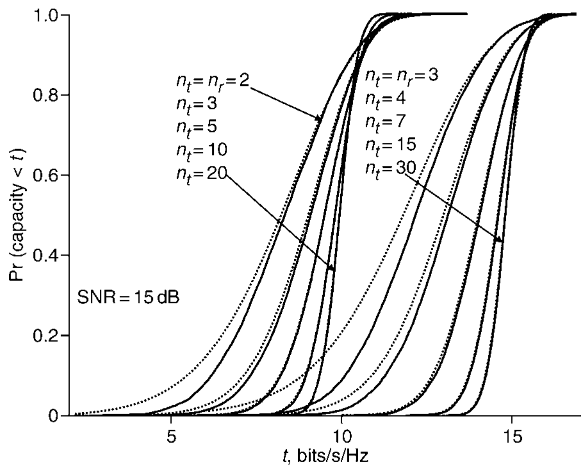

Fig. 1 *Comparison of approximate (dotted) cdf (5) and exact (solid) cdf obtained by Monte Carlo simulation (10^6 realisations)*

For curves on left $n_r = 2$, curves on right $n_r = 3$

Numerical examples: We consider the Rayleigh i.i.d. case, i.e. $\boldsymbol{R} = \boldsymbol{I}_K$. The approximate cdf from (5) is plotted in Fig. 1 for $n_r = 2$ and $n_r = 3$ at $\rho = 15$ dB. We note that the approximation is reasonable when $n_t = n_r$, and quite good when $n_t > n_r$. Hence, (5) provides a tight upper bound on capacity outage probability at the high SNR regime; the threshold of 'high' SNR decreases as K/L becomes smaller.

In Table 1 we compare the approximate cdf in (5) to the exact outage probability. In all cases the exact outage probability is 1% and the corresponding exact outage capacity t is estimated from Monte Carlo simulation with 10^6 channel realisations. We also provide comparison to the best of the five closed-form Gaussian approximations given in [8]. The upper bound is better than the Gaussian approximation when $n_t > n_r$ and about as good at $\rho = 30$ dB when $n_r = n_t$. Note that Fig. 1 and Table 1 show the worst-case results since (5) is considerably tighter for $n_r > n_t$. Furthermore, we note that, unlike the Gaussian approximation, (5) is consistent in the sense that it approaches the true outage probability as $\rho \to \infty$ or $K/L \to 0$, and it also incorporates the effect of spatial correlation. However, at low SNR the best of the Gaussian approximations is tighter. In general, our approximation works best when K/L is small and/or SNR is large. The approximation may be poor when $K = L$ and SNR is low.

Table 1: Comparison of approximate cdf (5), exact outage probability (=1%) and Gaussian approximations

(n_r, n_t)	ρ, dB	Equation (5)	Gaussian approx. [8]
(2, 2)	15	5.2%	2.9% [8, (18)]
(2, 2)	30	1.3%	0.8% [8, (23)]
(2, 10)	15	1.1%	1.4% [8, (23)]
(2, 10)	30	1.0%	0.7% [8, (23)]

Conclusions: We have derived closed-form approximations for the density and distribution functions of MIMO capacity over semi-correlated Rayleigh fading channels. The functions were given in terms of the Meijer G-function and were shown to be accurate at high SNR and/or when the number of transmit and receive antennas are different.

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