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The Distribution of the Product of Independent Rayleigh Random Variables

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***Abstract*—We derive the exact probability density functions (pdf) and distribution functions (cdf) of a product of n independent Rayleigh distributed random variables. The case $n = 1$ is the classical Rayleigh distribution, while $n \geq 2$ is the n-Rayleigh distribution that has recently attracted interest in wireless propagation research. The distribution functions are derived by using an inverse Mellin transform technique from statistics, and are given in terms of a special function of mathematical physics, the Meijer G-function. Series forms of the distribution function are also provided for $n = 3, 4, 5$. We also derive a computationally simple moment-based estimator for the parameter occurring in the distribution, and evaluate its variance.**

***Index Terms*—Fading channels, radio propagation, Rayleigh distributions.**

I. Introduction

Recently, the so called multiple Rayleigh (cascaded Rayleigh, n-Rayleigh) distribution has been found to explain well the amplitude behavior in certain type of measured radio channels [1]–[5]. The physical explanation for the n-Rayleigh model follows by considering a cascade of n statistically independent Rayleigh fading processes connected via narrow pipes; this model has been shown to agree very well with measurements made in a forest environment [2], [3]. The amplitude model arises also in keyhole propagation with $n-1$ keyholes [6], and propagation via diffracting street corners [1]. The probability density and distribution functions for the case of $n = 2$ (double-Rayleigh) have been given in [1]. However, for general n, the distribution functions appear to be absent in the literature. Consequently, empirical distribution functions obtained from Monte Carlo simulations have been used in previous studies. In this paper we derive the exact probability density and distribution functions of a multiple Rayleigh, or n-Rayleigh, random variable. The functions are derived by using an inverse Mellin transform technique from statistics and given in terms of the Meijer G-function. Series forms of the distribution functions are also provided. Based on the method of moments we also derive a computationally simple parameter estimator for the distribution. In addition to [2]–[6], our results may be useful in studies such as [7], where distributions of products of Rayleigh random variables were studied using a Monte Carlo simulation study (due to lack of analytical expressions) in context of investigating fading distributions. They cannot be found even in the advanced handbook [8] hence, completing the available literature in the field. Our results are also a prerequisite for analyzing the more general multiple scattering models proposed in [2], [3]. Further, in order to assess the impact of n-Rayleigh fading on the performance of radio communication systems, analytical expressions of signal amplitude are needed.

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II. Basic Definitions

A. Product of Independent Rayleigh Random Variables

Consider a product of n independent random variables

$$Y = \prod_{i=1}^{n} X_i \tag{1}$$

where X_i is a Rayleigh distributed random variable with probability density function (pdf)

$$f_{X_i}(x) = \frac{x}{\sigma_i^2} \exp\left(-\frac{x^2}{2\sigma_i^2}\right), \; x \geq 0. \tag{2}$$

Because of the way Y is defined we call it an "n-Rayleigh" random variable.

The hth moment of X_i, i.e., $E[X_i^h]$, is

$$\begin{aligned} E\left[X_i^h\right] &= \int_0^\infty \frac{x^{h+1}}{\sigma_i^2} \exp\left(-\frac{x^2}{2\sigma_i^2}\right) dx \\ &= \left(2\sigma_i^2\right)^{\frac{h}{2}} \Gamma\left(\frac{h}{2}+1\right) \end{aligned} \tag{3}$$

where we used the definition of the gamma function $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$.

B. Meijer G-Function

In the sequel, the density and distribution functions of Y will be given in terms of the Meijer G-function, which is a generalization of the generalized hypergeometric function and can be defined using the contour integral representation [9]

$$\begin{aligned} G_{p,q}^{m,n}\left(z\middle|_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}\right) &= \frac{1}{j2\pi} \\ &\times \int_{\mathcal{L}} \frac{\prod_{i=1}^{m}\Gamma(b_i+s)\prod_{i=1}^{n}\Gamma(1-a_i-s)}{\prod_{i=n+1}^{p}\Gamma(a_i+s)\prod_{i=m+1}^{q}\Gamma(1-b_i-s)} z^{-s} ds \end{aligned} \tag{4}$$

where z, $\{a_i\}_{i=1}^p$, and $\{b_i\}_{i=1}^q$ are, in general, complex-valued. The contour $\mathcal{L}$ is chosen so that it separates the poles of the gamma products in the numerator [9]. The Meijer G-function has been implemented in some commercial mathematics software packages.

III. Exact Density and Distribution Functions

In this section we derive the distribution functions of n-Rayleigh random variables. Although the Mellin transform technique is known from advanced texts on statistics, we shall outline the key steps in order to convey to the reader the main idea, which may also be useful in other similar distribution problems.

A. Distribution Functions Using the Meijer G-Function

We can find the pdf of Y as the *inverse Mellin transform*[1] of $\nu_h = E[Y^h]$, defined by the contour integral [10]

$$f_Y(y) = \frac{1}{j2\pi} \int_{\mathcal{L}} \nu_h y^{-(h+1)} dh. \tag{5}$$

Denoting $\sigma^2 = \prod_{i=1}^n \sigma_i^2$ we have from (1) and (3)

$$\nu_h = (2^n \sigma^2)^{\frac{h}{2}} \left[\Gamma\left(\frac{h}{2}+1\right)\right]^n. \tag{6}$$

From (6) it is possible to compute central moments and functions of them, like skewness and kurtosis. For example, the mean is $E[Y] = (2^n\sigma^2)^{1/2}(\sqrt{\pi}/2)^n$, while the variance is given by

$$\mathrm{var}(Y) = 2^n\sigma^2 \left[1 - \left(\frac{\pi}{4}\right)^n\right]. \tag{7}$$

The special case $n = 1$ gives the variance of Rayleigh distribution $2\sigma^2(1-(\pi/4))$.

By substituting ν_h in (5), setting $s = h + 1/2$, and using the definition of the Meijer G-function from (4) we obtain the density of Y as

$$f_Y(y) = 2(2^n\sigma^2)^{-\frac{1}{2}} G_{0,n}^{n,0}\left((2^n\sigma^2)^{-1}y^2 \big|_{\frac{1}{2},\dots,\frac{1}{2}}^{-}\right) \tag{8}$$

which is the density function of an n-Rayleigh random variable. Notice that the density depends only on the parameter $\sigma^2 = \prod_{i=1}^n \sigma_i^2$. This implies that, in theory, the n clusters interacting in cascade affect the n-Rayleigh distribution only through the product of their "size" parameters, the σ_i's.

Different values of σ^2 lead to different normalization. We consider two examples.

- The case $\sigma^2 = 2^{-n}$ is equivalent to the case where the underlying real-valued Gaussian variables have variance 1/2. This is the usual definition of "standard" complex Gaussian variable with zero mean and unit variance. Also with this setting $E[Y^2] = 2^n\sigma^2 = 1$, which is the usual normalization for the radio channel.
- Setting $\sigma^2 = (2/\pi)^n$ results with Y having unit mean, i.e., $\nu_1 = 1$.

[1]The Mellin transform $\int_0^\infty y^{h-1} f_Y(y)dy$, $y > 0$, is just the $(h-1)$th moment of Y. Here we work the other way round: we solve the density $f_Y(y)$ from its moments using the inverse transform.

Note that the pdf (8) gives also the joint pdf of angle and amplitude, $f_{Y\theta}(y,\theta)$, of a product of n circularly symmetric zero-mean complex Gaussian random variables as $f_{Y\theta}(y,\theta) = (1/2\pi)f_Y(y)$.

Cumulative distribution function $F_Y(t) = \int_0^t f_Y(y)dy$ is obtained by integrating (8) with respect to y inside contour integral by using

$$\int_0^t y^{-2s} dy = \frac{t}{2}\left(\frac{1}{2} - s\right)^{-1} t^{-2s}$$

setting $1/2 - s = \Gamma((3/2)-s)/\Gamma((1/2)-s)$, and again using the definition of the Meijer G-function (4). This results in

$$F_Y(t) = (2^n\sigma^2)^{-\frac{1}{2}} t G_{1,n+1}^{n,1}\left((2^n\sigma^2)^{-1}t^2 \big|_{\frac{1}{2},\dots,\frac{1}{2},-\frac{1}{2}}^{\frac{1}{2}}\right). \tag{9}$$

B. Distribution Functions in Series Form

As the Meijer G-function is implemented only in few mathematical software packages, in what follows, we provide series forms of the distribution functions for $n = 3, 4, 5$ for the convenience of the reader. These expressions are easy to program and result also in simple approximations for small argument values. They are also faster to evaluate numerically than the Meijer G-function. The series forms are derived by evaluating the contour integral in the definition of the Meijer G-function using calculus of residues [11]. This results in an infinite series representation for the densities, since the integrand has nth order poles at zero and at negative integer values of its argument. Since the procedure is standard, we omit derivation, and simply state the results. For further details, see [9] or [12].

We denote $x = (2^n\sigma^2)^{-1}y^2$ and $s = (2^n\sigma^2)^{-1}t^2$ for brevity. The distribution functions are as follows:

For $n = 3$

$$f_Y(y) = \sqrt{\frac{x}{8\sigma^2}} \sum_{k=0}^\infty \frac{(-1)^k}{(k!)^3} \times \left[\ln^2(x) - 2\ln(x)A_3(k) + A_3^{(1)}(k) + [A_3(k)]^2\right] x^k \tag{10}$$

$$F_Y(t) = \frac{s}{2}\sum_{k=0}^\infty \frac{(-1)^k}{(k!)^3(k+1)} \times \left[\ln^2(s) - 2\ln(s)C_3(k) + C_3^{(1)}(k) + [C_3(k)]^2\right] s^k. \tag{11}$$

For $n = 4$

$$f_Y(y) = \sqrt{\frac{x}{144\sigma^2}} \sum_{k=0}^\infty \frac{1}{(k!)^4} \times \Big[-\ln^3(x) + 3\ln^2(x)A_4(k) - 3\ln(x)\left[A_4^{(1)}(k) + [A_4(k)]^2\right] + A_4^{(2)}(k) + 3A_4^{(1)}(k)A_4(k) + [A_4(k)]^3\Big] x^k \tag{12}$$

TABLE I
COMPARISON OF ANALYTICAL cdf TO MONTE CARLO SIMULATION, $\sigma^2 = 2^{-n}$

t	$n = 3$ [Eq. (11)/sim.]	$n = 4$ [Eq. (13)/sim.]	$n = 5$ [Eq. (15)/sim.]
10^{-2}	$3.76 \times 10^{-3}/3.89 \times 10^{-3}$	$1.11 \times 10^{-2}/1.11 \times 10^{-2}$	$2.43 \times 10^{-2}/2.44 \times 10^{-2}$
10^{-1}	$1.09 \times 10^{-1}/1.03 \times 10^{-1}$	$1.76 \times 10^{-1}/1.76 \times 10^{-2}$	$2.55 \times 10^{-1}/2.55 \times 10^{-1}$
1	$7.77 \times 10^{-1}/7.76 \times 10^{-1}$	$8.17 \times 10^{-1}/8.17 \times 10^{-1}$	$8.48 \times 10^{-1}/8.48 \times 10^{-1}$
$10^{\dagger}$	$1.60 \times 10^{-5}/1.58 \times 10^{-5}$	$1.42 \times 10^{-4}/1.53 \times 10^{-4}$	$4.17 \times 10^{-4}/4.27 \times 10^{-4}$

† For $t = 10$, the value of $1 - F_Y(t)$ is given.

$$F_Y(t) = \frac{s}{6} \sum_{k=0}^{\infty} \frac{1}{(k!)^4(k+1)} \times \Big[-\ln^3(s) + 3\ln^2(s)C_4(k) - 3\ln(s)\left[C_4^{(1)}(k) + [C_4(k)]^2\right] + C_4^{(2)}(k) + 3C_4^{(1)}(k)C_4(k) + [C_4(k)]^3\Big] s^k. \tag{13}$$

For $n = 5$

$$f_Y(y) = \sqrt{\frac{x}{4608\sigma^2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^5} \times \Big[\ln^4(x) - 4\ln^3(x)A_5(k) + 6\ln^2(x)\left[A_5^{(1)}(k) + [A_5(k)]^2\right] - 4\ln(x)\left[A_5^{(2)}(k) + 3A_5^{(1)}(k)A_5(k) + [A_5(k)]^3\right] + A_5^{(3)}(k) + 4A_5^{(2)}(k)A_5(k) + 3\left[A_5^{(1)}(k)\right]^2 + 6[A_5(k)]^2 A_5^{(1)}(k) + [A_5(k)]^4\Big] x^k \tag{14}$$

$$F_Y(t) = \frac{s}{24} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^5(k+1)} \times \Big[\ln^4(s) - 4\ln^3(s)C_5(k) + 6\ln^2(s)\left[C_5^{(1)}(k) + [C_5(k)]^2\right] - 4\ln(s)\left[C_5^{(2)}(k) + 3C_5^{(1)}(k)C_5(k) + [C_5(k)]^3\right] + C_5^{(3)}(k) + C_5^{(2)}(k)C_5(k) + 3\left[C_5^{(1)}(k)\right]^2 + 6[C_5(k)]^2 C_5^{(1)}(k) + [C_5(k)]^4\Big] s^k. \tag{15}$$

We have denoted $A_n^{(0)}(k) = A_n(k)$ and $[\ln(x)]^n = \ln^n(x)$. The function $A_n^{(l)}(k)$, for $n \geq 2$, is given by

$$A_n^{(l)}(k) = n\left[\psi^{(l)}(1) + (-1)^{l+1}\left[\psi^{(l)}(1) - \psi^{(l)}(k+1)\right]\right]$$

where $\psi^{(m)}(x) = (d^m/x^m)\psi(x) = (d^{m+1}/x^{m+1})\ln\Gamma(x)$ is the mth polygamma function. Similarly, we have denoted

$$C_n^{(l)}(k) = A_n^{(l)}(k) + \frac{l!}{(k+1)^{l+1}}.$$

The series forms of the distribution functions are very fast to compute, since the difference of polygamma functions $\psi^{(l)}(k+1) - \psi^{(l)}(1)$ is given by a finite sum [9].

To verify the derived analytical expressions, we conducted Monte Carlo simulations and estimated few points of the empirical cdfs. The results, given in Table I, were computed with the series forms of the distribution functions. The number of samples for each n was 10^6. The agreement between empirical values is excellent, which is to be expected, since the derived expressions are not approximations, and can, in principle, be used to compute numerical values with arbitrary precision.

C. Special Cases

We now show that, for $n = 1$ and $n = 2$, the G-function form (8) reduces to the well-known cases of Rayleigh and double-Rayleigh distribution.

1) $n = 1$: From the identity $G_{0,1}^{1,0}(z|_{b}^{-}) = z^b e^{-z}$ [13, (Eq. §07.34.03.0228.01)] we obtain the Rayleigh distribution.
2) $n = 2$: From identity $G_{0,2}^{2,0}(z|_{b\ c}^{-}) = 2z^{1/2(b+c)}K_{b-c}(2\sqrt{z})$, where K_μ is a modified Bessel function of the second kind [13, (Eq. §07.34.03.0605.01)], it follows that $f_Y(y) = (y/\sigma^2)K_0(y/\sigma)$. This is the double-Rayleigh distribution [1].

D. Numerical Examples and Discussion

In Fig. 1 densities for varying n and $\sigma^2 = 2^{-n}$ are shown. As n increases the probability mass concentrates close to the origin. On the other hand both tails of density become heavier as n increases. This is better illustrated in Fig. 2 where the cumulative distribution function for different values of n with $\sigma^2 = 2^{-n}$ is shown. The Rayleigh cdf ($n = 1$) appears as a straight line in both subplots, and curvature from straight line indicates deviation from the Rayleigh distribution. It can be seen that the slope of the left tail of the cdf deviates from the Rayleigh cdf only slightly. In fact, from the series forms given in the previous section we note, that the small-t slopes are not equal to that of the Rayleigh distribution, i.e., they are not straight lines in the probability plot of Fig. 2, although for small t and n they may well be approximated as such. The right tail of the n-Rayleigh cdf becomes heavier than that of the Rayleigh cdf for increasing n. For practical measurement data analysis this means that difference between two n-Rayleigh distributions with different n is mostly contained in the tails of the empirical cdf. Hence, a relatively large number of stationary amplitude samples may need to be recorded in order to reliably detect a difference between

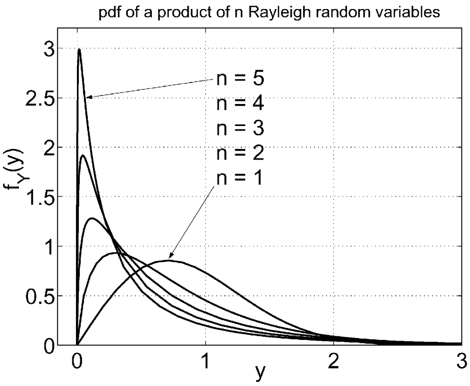

Fig. 1. Probability density functions (8) for different values of n with $\sigma^2 = 2^{-n}$.

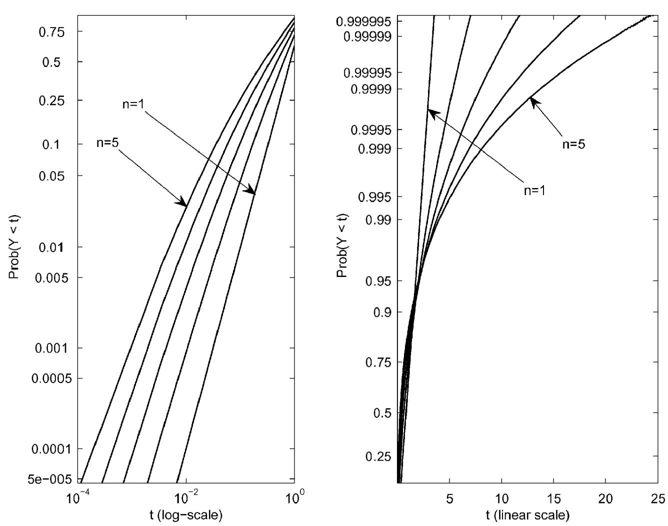

Fig. 2. Rayleigh probability plots of the cdfs (9) for different values of n with $\sigma^2 = 2^{-n}$. The Rayleigh distribution ($n = 1$) appears a straight line in both plots. Left: the left tail of the cdfs. Right: the right tails of the cdfs.

two n-Rayleigh distributions when compared to an empirical data sample.

It is evident that multiple scattering increases the dynamic range of the fading signal. This fact will also have to be taken into account in design of measurements to ensure that the measurement system has sufficient linear dynamic range. For example, assuming that $E[Y^2] = 1$, the required linear dynamic range required for undistorted signal reception 99% of time[2] is 20.1, 41.8, 50.7, 58.3, and 65 dB for $n = 1, 2, 3, 4, 5$, respectively. A receiver with a smaller linear dynamic range will distort the observed amplitude. Obviously, detecting a subtle phenomenon such as multiple scattering in radio channels requires paying some attention to the characteristics of the measurement device.

A simple figure characterizing the severity of fading is the "amount of fading" defined as [14]

$$\mathrm{AF} = \frac{\mathrm{var}[Y^2]}{\left(E[Y^2]\right)^2}.$$

[2]In this case, by dynamic range we mean the ratio t_{max}/t_{min}, given in dB. The thresholds are obtained from $\Pr(Y < t_{min}) = 0.005$ and $\Pr(Y < t_{max}) = 0.995$.

A larger number indicates more severe fading, which translates to greater degradation in communication system performance. For n-Rayleigh distributed amplitude we obtain the simple expression $\mathrm{AF} = 2^n - 1$, i.e., the amplitude fading is more severe than for any other classical small-scale fading distribution, including Nagakami-m and Rice pdfs, whose AF values are $1/m$ and $(1 + 2K)/(1 + K)^2$, respectively.[3] This observation is not only of great philosophical importance, but also lends some motivation for the study of multiple scattering in mobile radio channels. The key question remains: under which environmental conditions does such severe fading occur in nature? Approaching the problem by blindly measuring and analyzing radio channels may lead to false conclusions, since great care should be put to considerations of sufficient receiver dynamics, measurement SNR, effect of signal nonstationarity, and estimation of the amplitude pdf parameters from a measured data record of limited length. The results in this paper present necessary groundwork to facilitate such measurement studies.

IV. A Moment-Based Estimator for σ^2

As already mentioned, the n-Rayleigh distribution is fully characterized by the parameter σ^2. A typical problem in radio propagation research is how to estimate σ^2 from measurement data. For $n = 1$ a maximum-likelihood estimator may be used. However, for general n the optimization of the likelihood function becomes awkward due to the G-function appearing in the density. Therefore, it is desirable to seek alternative estimators. In the following, we present a computationally simple estimator for σ^2 based on the method of moments [15].

A moment-based estimator is obtained by solving σ^2 from (6) and replacing the theoretical hth moment ν_h with its sample estimator $(1/m)\sum_{i=1}^{m} y_i^h$. Here m is the number of independent, identically distributed samples from n-Rayleigh distribution. In this paper we choose h so that the resulting estimator of σ^2 is unbiased. It is easily shown that for an unbiased estimator $h = 2$ is required. Hence, a moment-based based estimator of σ^2 is given by

$$\hat{\sigma}^2 = \frac{1}{2^n m}\sum_{i=1}^{m} y_i^2. \tag{16}$$

We remark that (at least) for $n = 1$ the moment-based estimator coincides with the maximum-likelihood estimator.

The variance of (16) is $\mathrm{var}(\hat{\sigma}^2) = E[(\hat{\sigma}^2)^2] - E[(\hat{\sigma}^2)]^2$, where the second term is just σ^4, and the first term can be evaluated as

$$\begin{aligned} E\left[(\hat{\sigma}^2)^2\right] &= E\left[\left(\frac{1}{2^n m}\sum_{i=1}^{m} y_i^2\right)^2\right] \\ &= \frac{1}{2^{2n}m^2}\left(m\nu_4 + m(m-1)\nu_2^2\right) \end{aligned}$$

where we used the independence of the samples. Substituting from (6) $\nu_4 = 2^{3n}\sigma^4$ and $\nu_2 = 2^n\sigma^2$ it can be shown that

$$E\left[(\hat{\sigma}^2)^2\right] = \frac{(2^n - 1)\sigma^4}{m} + \sigma^4.$$

[3]Here m denotes the parameter of the Nagakami-m pdf and K denotes the Rician K factor.

Hence, the variance of (16) is

$$\mathrm{var}(\hat{\sigma}^2) = \frac{(2^n - 1)\sigma^4}{m}.$$

Therefore, the moment-based estimator (16) also has the desirable property that its variance decreases inversely proportional to m also in the small-sample regime.

Note that there remains the fundamental question of how to select the best n for a given set of measurement data. This is, in general, a difficult statistical modeling problem. Here we merely remark that the derived distribution functions can also be used in construction of statistical hypothesis tests for fading model selection purposes.

V. Conclusion

We have derived the exact probability density and distribution functions for a product of n Rayleigh distributed random variables. The functions were given in terms of the Meijer G-function, for which numerical values can be computed using easily. Series forms of the distribution functions were also supplied for small values of n. A computationally simple unbiased parameter estimator for the distribution was also derived and its variance was evaluated. The results of the paper are expected to be useful for researchers studying fading models for multiple scattering radio propagation scenarios that may occur, for example, in forests and urban microcells via diffracting street corners.

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