Fermionic Fock Spaces in Conformal Field Theory

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Abstract

In this thesis, we investigate a geometric formulation of the free fermion boundary conformal field theory (BCFT). It is believed to represent the scaling limit of the two-dimensional critical planar Ising model, which is the simplest mathematical model of ferromagnetism. In the geometric formulation of Graeme Segal, bulk conformal field theory (CFT) is a functor from the category of bordered Riemann surfaces and their boundary circles to the category of Hilbert spaces and linear operators. Similarly, BCFT is a corresponding functor, except that objects of the source category are collections of unit intervals and the morphisms are planar domains with two types of boundary components.

We propose a definition of the free fermion BCFT inspired by the transfer matrix formalism of the Ising model and compare the definition to a recent construction of the free fermion CFT. One of our main results is that the space of operators associated by the BCFT to a planar domain is isomorphic to the space of vacua of a representation of a Clifford algebra motivated by the Ising model. We also show the existence of the BCFT operators corresponding to rectangular domains of a fixed width and prove that the operators form a contractive semigroup.

To gain insight into the existence problem of the operators in the free fermion BCFT, we present a proof of the Shale-Stinespring equivalence criterion, which is used to relate the existence problem of operators in the free fermion CFT to the geometric properties of the Riemann surfaces. A similar result is also proved for the BCFT functor, but the existence problem in the BCFT case remains elusive.

Most of our results are adapted from the existing literature on the free fermion CFT to the BCFT case. One exception is the Shale-Stinespring equivalence criterion whose proof is a slightly streamlined version of an existing proof.

Keywords Fock Spaces, Fermions, Conformal Field Theory, Ising Model

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Tiivistelmä

Diplomityössä tarkastellaan geometrista muotoilua vapaalle fermioniselle reunakonformikenttäteorialle, jonka uskotaan kuvaavan kaksiulotteisen Isingin mallin skaalausrajaa kriittisessä lämpötilassa. Isingin malli on yksinkertaisin matemaattinen malli ferromagnetismille, ja Graeme Segalin geometrisessa muotoilussa tavallinen konformikenttäteoria on funktori reunallisten Riemannin pintojen ja näiden reunaympyröiden kategoriasta Hilbertin avaruuksien ja lineaarikuvauksien kategoriaan. Reunakonformikenttäteoria on vastaava funktori sillä erotuksella, että lähtökategorian objektit ovat yksikkövälien kokoelmia ja morfismit ovat tasoalueita, joilla on kahdentyyppisiä reunakomponentteja.

Diplomityössä esitetään Isingin mallin siirtomatriisimuotoilun inspiroima määritelmä vapaalle fermioniselle reunakonformikenttäteorialle, ja määritelmää verrataan hiljattain julkaistuun määritelmään tavalliselle vapaalle fermioniselle konformikenttäteorialle. Yksi työn päätuloksista on se, että reunakonformikenttäteorian tasoalueeseen liittämien operaattorien avaruus on isomorfinen Isingin mallin motivoiman Cliffordin algebran esityksen tyhjiövektoreiden avaruuden kanssa. Työssä näytetään myös kiinnitetyn levyisiin suorakaiteisiin liittyvien operaattorien olemassaolo ja osoitetaan että nämä muodostavat kontraktiivisen puoliryhmän.

Reunakonformikenttäteorian operaattorien olemassaolokysymyksen tutkimiseksi työssä esitetään Shalen-Stinespringin ekvivalenssilause, jonka avulla operaattorien olemassaolokysymys voidaan liittää Riemannin pintojen geometriaan tavallisen konformikenttäteorian tapauksessa. Vastaava tulos todistetaan myös reunakonformikenttäteorialle, mutta tästä huolimatta reunakonformikenttäteorian olemassaolokysymys pysyy avoimena.

Työn tulokset ovat pääosin kirjallisuudessa esiintyvien tavallista konformikenttäteoriaa koskevien tulosten sovelluksia reunakonformikenttäteoriaan. Poikkeuksena tähän on Shalen-Stinespringin ekvivalenssilause, jonka todistus on hieman yksinkertaistettu versio kirjallisuudessa esiintyvästä todistuksesta.

Avainsanat Fockin avaruudet, fermionit, konformikenttäteoria , Isingin malli

Contents

1 Introduction

The situation into which we wish to gain insight is the following one. For a bounded domain $\mathscr{D} \subset \mathbb{C}$ and a constant $\delta > 0$, consider the finite lattice $\mathscr{D}^\delta = \mathscr{D} \cap \delta\mathbb{Z}^2$ with the boundary $\partial\mathscr{D}^\delta$ consisting of those $x \in \mathscr{D}^\delta$ with less than four nearest neighbors. Let $S^{\mathrm{Free}}_{\mathscr{D}^\delta} = \{\sigma\colon \mathscr{D}^\delta \to \{\pm 1\}\}$ be the set of $\{\pm1\}$-valued functions on $\mathscr{D}^\delta$ with no prescribed boundary conditions. We define a probability measure $\mathbb{P}^{\mathrm{Free}}_{\mathscr{D}^\delta,\beta}$ on the space $(S^{\mathrm{Free}}_{\mathscr{D}^\delta}, P(S^{\mathrm{Free}}_{\mathscr{D}^\delta}))$ by the formula

$$\mathbb{P}^{\mathrm{Free}}_{\mathscr{D}^\delta,\beta}(\sigma) = \frac{1}{Z^{\mathrm{Free}}_{\mathscr{D}^\delta,\beta}} e^{-\beta\mathscr{H}(\sigma)}, \tag{1}$$

where $\mathscr{H}(\sigma) = -\sum_{x \leftrightsquigarrow y} \sigma(x)\sigma(y)$ is called the Hamiltonian function,

$$Z^{\mathrm{Free}}_{\mathscr{D}^\delta,\beta} = \sum_{\sigma \in S^{\mathrm{Free}}_{\mathscr{D}^\delta}} e^{-\beta\mathscr{H}(\sigma)} \tag{2}$$

is called the partition function, $x \leftrightsquigarrow y$ means that x and y are nearest neighbors in the lattice $\mathscr{D}^\delta$, and $\beta > 0$ is a parameter of the model.

This probability measure is the planar Ising model with free boundary conditions on the lattice $\mathscr{D}^\delta$, and it has the following physical interpretation as a model of ferromagnetism. At each point x in the lattice $\mathscr{D}^\delta$, there is an atom with spin up if $\sigma(x) = 1$ and with spin down if $\sigma(x) = -1$. The value $\mathscr{H}(\sigma)$ of the Hamiltonian is the energy of the spin configuration σ, and β is proportional to the inverse of the temperature.

For unbounded domains $\mathscr{D}$, one considers an exhaustion $\mathscr{D}_1 \subset \mathscr{D}_2 \subset \mathscr{D}_3 \subset \ldots \subset \mathscr{D}$ of $\mathscr{D}$ by an increasing sequence of bounded domains $\mathscr{D}_n$, and it can be shown with the Griffiths correlation inequality that the measures $\mathbb{P}^{\mathrm{Free}}_{\mathscr{D}^\delta_n,\beta}$ have a weak limit $\mathbb{P}^{\mathrm{Free}}_{\mathscr{D}^\delta,\beta}$, which is independent of the approximating sequence $(\mathscr{D}_n)$ [16, Chap. 6]. This limit $\mathbb{P}^{\mathrm{Free}}_{\mathscr{D}^\delta,\beta}$ is the Ising model with free boundary conditions on the infinite lattice $\mathscr{D}^\delta$. To set boundary conditions for the spin configurations σ, let

$$S^{\mathrm{MC}}_{\mathscr{D}^\delta} = \{\sigma\colon \mathscr{D}^\delta \to \{\pm1\} \mid \sigma \text{ constant on each } \gamma \in \pi_0(\partial\mathscr{D}^\delta)\}$$

be the set of those $\{\pm1\}$-valued functions on $\mathscr{D}^\delta$ that are constant on each connected component γ of the boundary $\partial\mathscr{D}^\delta$. Substituting $S^{\mathrm{MC}}_{\mathscr{D}^\delta}$ for $S^{\mathrm{Free}}_{\mathscr{D}^\delta}$ in (1) and (2) defines a probability measure $\mathbb{P}^{\mathrm{MC}}_{\mathscr{D}^\delta,\beta}$, which we call the Ising model with locally monochromatic boundary conditions, see Figure 1. Defining $S^{+}_{\mathscr{D}^\delta}$ as the set of those $\sigma \in S^{\mathrm{Free}}_{\mathscr{D}^\delta}$ that satisfy $\sigma(x) = 1$ for all $x \in \partial\mathscr{D}^\delta$ gives yet another model $\mathbb{P}^{+}_{\mathscr{D}^\delta,\beta}$.

Objects of foremost interest in the Ising model are the correlation functions

$$\langle \sigma(x_1)\sigma(x_2)\ldots\sigma(x_n)\rangle^{\mathrm{BC}}_{\mathscr{D}^\delta,\beta} \coloneqq \mathbb{E}^{\mathrm{BC}}_{\mathscr{D}^\delta,\beta}(\sigma(x_1)\sigma(x_2)\ldots\sigma(x_n)), \tag{3}$$

where BC stands for boundary conditions, with either BC = Free, BC = MC or BC = +. The reason for their importance is that all the macroscopic properties of the model, such as magnetization, can be obtained from them [19]. A remarkable property of the correlation functions is that there exists a value $\beta_C = \frac{1}{2}\log(1+\sqrt{2})$ of the parameter β, called the critical inverse temperature, such that for δ below a threshold value depending on $\mathscr{D}$, we have

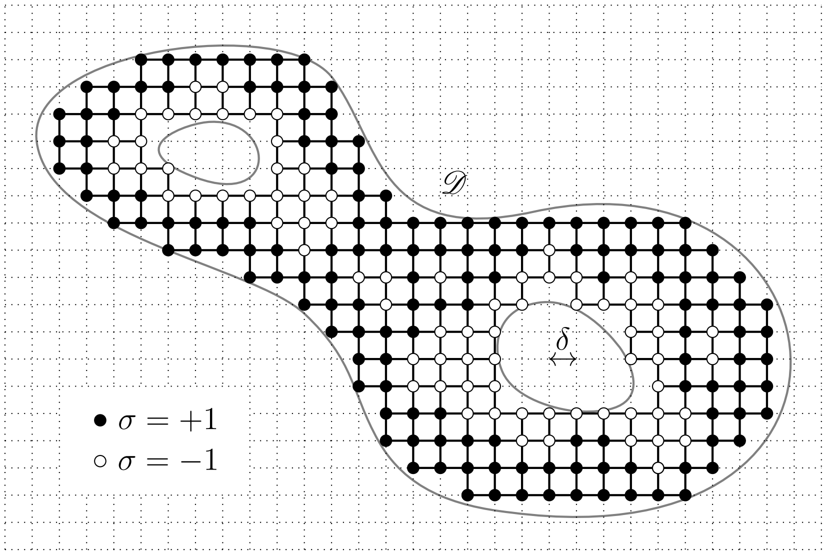

Figure 1: A spin configuration $\sigma \in S^{\mathrm{MC}}_{\mathscr{D}^\delta}$ for a doubly connected domain $\mathscr{D}$.

1. For $\beta > \beta_C$, $\langle \sigma(x)\sigma(y)\rangle^{\mathrm{MC}}_{\mathscr{D}^\delta,\beta} \geq C_1$ for all $x, y \in \mathscr{D}^\delta$

2. For $\beta < \beta_C$, $\langle \sigma(x)\sigma(y)\rangle^{\mathrm{Free}}_{\mathscr{D}^\delta,\beta} \leq C_2 e^{-\eta\frac{|x-y|}{\delta}}$ for all $x, y \in \mathscr{D}^\delta$

for constants $C_1, C_2, \eta > 0$ independent of δ , see e.g. [16, Chap. 3.7] or [18, Chap. 5.4]. One says that the model exhibits a transition from the ordered low-temperature phase at $\beta > \beta_C$ to the disordered high-temperature phase at $\beta < \beta_C$. More precisely, at $\beta > \beta_C$ the model favors configurations with all the spins aligned in the same direction, and when the inverse temperature is lowered past β_C, this property is lost. This is precisely what happens when a real magnet is heated past its Curie temperature [16, Chap. 1.4], and it is this phenomenon that justifies the use of the Ising model as a model of ferromagnetism.

Since the order of magnitude of the number of atoms in a small magnet is 10^{23}, it is reasonable to consider the continuum limit $\delta \to 0$ of the Ising model, where the lattice spacing tends to zero and the number density of the lattice sites tends to infinity. It is a basic underlying assumption in the physics literature that in the limit $\delta \to 0$ at $\beta = \beta_C$, suitably renormalized correlation functions $\frac{1}{R_\delta}\langle \sigma(x_1)\sigma(x_2)\ldots\sigma(x_n)\rangle^{\mathrm{BC}}_{\mathscr{D}^\delta,\beta_C}$ should converge to functions $\langle \sigma(x_1)\sigma(x_2)\ldots\sigma(x_n)\rangle^{\mathrm{BC}}_{\mathscr{D},\beta_C} \colon \times_{i=1}^n \mathscr{D} \to \mathbb{R}$ that are smooth away from coinciding points, see e.g. [15].

Let us briefly sketch an analogue of the type of formal arguments used by physicists to study the continuum limit. The state space $S^{\mathrm{BC}}_{\mathscr{D}^\delta}$ is replaced by a space $\mathrm{Map}(\mathscr{D}, \mathbb{R})$ of fields $\sigma \colon \mathscr{D} \to \mathbb{R}$, that are not defined precisely. The counting measure on $S^{\mathrm{BC}}_{\mathscr{D}^\delta}$ in the definition of the partition function $Z^{\mathrm{MC}}_{\mathscr{D}^\delta,\beta_C}$ is replaced by a formal measure $\mathcal{D}\sigma$ on $\mathrm{Map}(\mathscr{D}, \mathbb{R})$, and the formal definition of the continuum partition function reads

$$Z^{\mathrm{BC}}_{\mathscr{D},\beta_C} = \int_{\mathrm{Map}(\mathscr{D},\mathbb{R})} e^{-\beta_C \mathscr{H}(\sigma)} \mathcal{D}\sigma, \tag{4}$$

where $\mathscr{H}$ is a continuum analogue of the Hamiltonian. Adding a field B to the partition function, one defines the generating functional

$$\Gamma^{\mathrm{BC}}_{\mathscr{D},\beta_C}(B) = \int_{\mathrm{Map}(\mathscr{D},\mathbb{R})} e^{-\beta_C \mathscr{H}(\sigma) - B\sigma} \mathcal{D}\sigma. \tag{5}$$

The continuum limits of the correlation functions should then be given by the formal expression

$$\langle \sigma(x_1)\sigma(x_2)\ldots\sigma(x_n)\rangle^{\mathrm{BC}}_{\mathscr{D},\beta_C} = \frac{1}{Z^{\mathrm{BC}}_{\mathscr{D},\beta_C}} \frac{(-1)^n \delta^n}{\delta B(x_1)\ldots\delta B(x_n)} \Gamma^{\mathrm{BC}}_{\mathscr{D},\beta_C}(B), \tag{6}$$

where $\frac{\delta}{\delta B(x_i)}$ stands for a formal functional derivative. Although none of the objects in (4)-(6) have precise definitions, physicists have been very successful in extracting explicit formulas from them, see e.g. [22, Chap. 9-13]. This is precisely the setting of the path integral formulation of Euclidean quantum field theory. Moreover, one can consider an additional field ψ built from the field σ in a non-local manner, and this results in a quantum field theory known as the massless free fermion, see e.g. [15, Chap. 12].

An important insight came in 1984, when Belavin, Polyakov and Zamolodchikov argued that the fermionic quantum field theory describing the continuum limit of the Ising model at $\beta = \beta_C$ should remain invariant under conformal transformations at the level of the correlation functions (6) and enjoy an additional symmetry known as local conformal invariance [3,4]. Subsequent work by several authors goes by the name of conformal field theory (CFT). The requirement of local conformal invariance restricts the form of the correlation functions (6) very heavily, and it turns out that it allows one to determine them completely. For example, it was predicted in [7] that the continuum limits of the correlation functions in the upper half plane $\mathbb{H}$ should read

$$\langle \sigma(x_1)\sigma(x_2)\ldots\sigma(x_n)\rangle^{+}_{\mathbb{H},\beta_C} = \prod_{i=1}^{n} (2\Im(x_i))^{-\frac{1}{8}} \cdot \left(\frac{1}{2^{\frac{n}{2}}} \sum_{(\mu_1,\ldots,\mu_n)\in\{\pm1\}^n} \prod_{i<j} \left| \frac{x_i - x_j}{x_i - \overline{x}_j} \right|^{\frac{\mu_i\mu_j}{2}} \right)^{\frac{1}{2}}, \tag{7}$$

and in any other simply connected domain $\mathscr{D}$ they should be given by

$$\langle \sigma(x_1)\sigma(x_2)\ldots\sigma(x_n)\rangle^{+}_{\mathscr{D},\beta_C} = \left(\prod_{i=1}^{n} |\phi'(x_i)|^{\frac{1}{8}} \right) \langle \sigma(\phi(x_1))\sigma(\phi(x_2))\ldots\sigma(\phi(x_n))\rangle^{+}_{\mathbb{H},\beta_C}, \tag{8}$$

where $\phi\colon \mathscr{D} \to \mathbb{H}$ is a conformal map guaranteed by the Riemann mapping theorem. However, it was only two decades later that the convergence of the correlation functions $\frac{1}{R_\delta}\langle \sigma(x_1)\sigma(x_2)\ldots\sigma(x_n)\rangle^{+}_{\mathscr{D}^\delta,\beta_C}$ and the validity of (7) and (8) was proven in [10].

Although the path integral formulas such as (4) and (5) have no precise mathematical meaning, there exists a rigorous mathematical framework of Euclidean quantum field theory under the name of Osterwalder-Schrader axioms, where the fields σ in (4) are random distributions and the informal measure $\mathcal{D}\sigma$ is the law of the random distribution [18]. It has indeed been recently shown in [8,9] that a suitably normalized Ising model on the lattice $\delta\mathbb{Z}^2$ converges to a random distribution that is believed to be the simplest non-Gaussian quantum field theory. Although the path integral formalism has a clear interpretation as a scaling limit of the Ising model, this is not the direction we will pursue. Rather, we will work in a geometric formulation of conformal field theory given by Segal in [26] since it best exhibits the connection of conformal field theory to the transfer matrix formalism of the Ising model that we review in Chapter 3.2.

In Segal's formulation, conformal field theory is essentially a functor between two categories. To describe the source category, let us consider a compact bordered Riemann surface Σ with boundary circles $\mathcal{C}_i$ that we parametrize by the unit circle S^1. Both the unit circle S^1 and the complex 1-manifold Σ carry a natural orientation, and the orientation on Σ induces an orientation on each of the $\mathcal{C}_i$. This defines a partition of the set of boundary circles $\mathcal{C}_i$ into incoming boundary circles $\mathcal{C}_i$, where the parametrization is orientation-reversing, and outgoing boundary circles $\mathcal{C}_i$, where the parametrization is orientation-preserving. The objects of the source category are collections $\coprod_i S^1$ of circles, and a morphism from a collection $\coprod_{\text{in}} S^1$ to $\coprod_{\text{out}} S^1$ is a Riemann surface with boundary parametrizations such that the set of incoming boundary components is $\coprod_{\text{in}} S^1$ and the set of outgoing boundary components is $\coprod_{\text{out}} S^1$. Such Riemann surfaces can be sewn together along the parametrizations under certain regularity assumptions [24], and this defines the composition of morphisms in the source category, see Figure 2. However, it should be noted that this construction does not give a category in the precise meaning of the word since there are no identity morphisms. The target category is easy to define. Its objects are tensor powers of a fixed complex Hilbert space H and the morphisms are trace class maps.

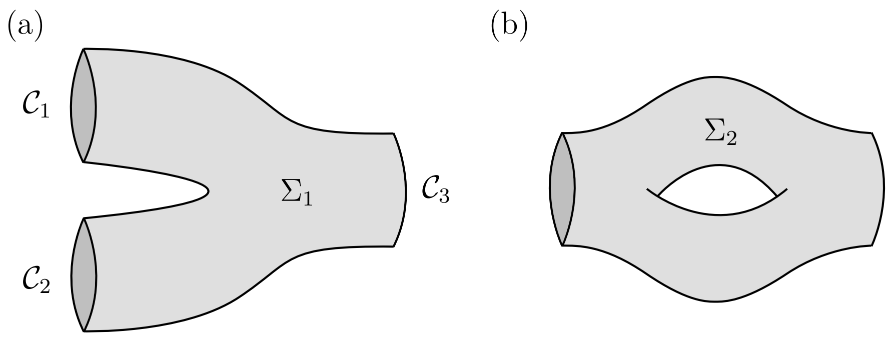

Figure 2: (a) A Riemann surface Σ_1 with boundary components $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$. (b) The Riemann surface Σ_2 is obtained by gluing together a copy of Σ_1 where $\mathcal{C}_1$ and $\mathcal{C}_2$ are outgoing and another one where they are incoming.

The conformal field theory functor associates to each collection $\coprod_i S^1$ a tensor product $\otimes_i H$ and to each Riemann surface a one-dimensional space E_Σ of trace class maps $T_\Sigma \colon \otimes_{\text{in}} H \to \otimes_{\text{out}} H$ subject to some axioms. Segal calls such a map a projective functor in analogy with projective representations. The connection of these projective functors and the path integral formulas, such as (4), is explained, for example, in [17, §. 2.6.].

The previous setting with Riemann surfaces and boundary circles could be called bulk conformal theory to distinguish it from the boundary conformal field theory (BCFT) that we wish to define in order to study the scaling limit of the planar Ising model. The formulation remains essentially the same, except that instead of a Riemann surface Σ with boundary circles $\mathcal{C}_i$, we consider a bounded planar domain $\mathscr{D} \subset \mathbb{C}$ and a subset $\Gamma_{\mathscr{D}} \cong \coprod_i I$ of the boundary $\partial\mathscr{D}$ that is homeomorphic to a collection of unit intervals $I = [-1/2, 1/2] \subset \mathbb{R}$, see Figure 3.

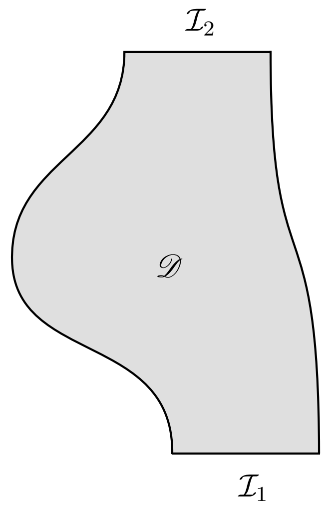

Figure 3: A planar domain $\mathscr{D}$ with $\Gamma_{\mathscr{D}} = \mathcal{I}_1 \cup \mathcal{I}_2$.

Indeed, after reviewing some preliminaries on functional analysis in Chapter 2, in Chapter 3 we use the transfer matrix formalism of the Ising model to motivate the definition of a Segal-type BCFT that serves as a candidate for the scaling limit. More precisely, the operators $T_{\mathscr{D}}$ serve as candidates for the scaling limits of powers of transfer matrices. The first of the two main results of the thesis is Theorem 3.22, which states that the space of the operators $T_{\mathscr{D}}$ satisfying certain properties is isomorphic to the space of vacua of a representation of a Clifford algebra motivated by the Ising model, which is at most 1-dimensional. However, it remains as an open problem whether this space is actually 1-dimensional, i.e., whether our definition gives a well defined projective functor. Nevertheless, we obtain explicit formulae for the 1-dimensional spaces of operators associated to the rectangles $[-1/2, 1/2] \times [0, h]$ for $h > 0$ and show that they form a contractive semigroup. Moreover, our definition of the BCFT is seen to be closely related to the definition of the corresponding bulk CFT that was recently constructed in [27, 28].

In order to better understand the dimension of the space of operators $T_{\mathscr{D}}$, in Chapter 4 we present a proof of the Shale-Stinespring equivalence theorem, which is a classical result that relates the dimension of the space of operators T_Σ to the geometry of the Riemann surfaces Σ in the bulk CFT case. The second main result of the thesis is Theorem 4.4, where we use ideas from the proof of the Shale-Stinespring theorem to reduce the well-definedness of the BCFT functor to spectral properties of operators related to the geometry of the domains $\mathscr{D}$. However, the BCFT case remains open.

Besides the definition of the BCFT, there is no essential novelty in the thesis. Most of the results concerning the BCFT in Chapter 3 are adaptations of the results in [28] concerning the bulk CFT counterpart. The proof of the Shale-Stinespring equivalence criterion is a slightly streamlined version of the proof in [30], but Theorem 4.4 appears in print for the first time in this precise form to the best of the author's knowledge.

2 Background

We begin this section by reviewing some results on Hilbert spaces that will find frequent use in what follows. In Chapter 2.2, we then introduce the concept of a super Hilbert space, which is essential in the formulation of fermionic theories. In Chapter 2.3, we define the Hilbert spaces that the CFT and BCFT functors associate to the boundary components $\coprod_i S^1$ and $\coprod_i I$, respectively. Finally, we take our first steps towards defining the operators T_Σ and $T_{\mathscr{D}}$ associated by the CFT and BCFT functors to the Riemann surfaces Σ and planar domains $\mathscr{D}$, respectively.

2.1 Hilbert spaces

Let $\mathcal{H}$ and $\mathcal{K}$ be real or complex Hilbert spaces, which we always assume to be separable and infinite-dimensional unless otherwise noted. Throughout this thesis, we take the inner products $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{K}}$ to be linear in the first argument and conjugate-linear in the second argument. We denote the space of bounded linear maps from $\mathcal{H}$ to $\mathcal{K}$ by $\mathscr{B}(\mathcal{H},\mathcal{K})$, and we write $\mathscr{B}(\mathcal{H},\mathcal{H}) = \mathscr{B}(\mathcal{H})$. A map $T \in \mathscr{B}(\mathcal{H},\mathcal{K})$ is said to be **compact** if it takes bounded sets in $\mathcal{H}$ to precompact sets in $\mathcal{K}$. The main significance of compact operators to us is that they can be diagonalized much in the same way as square matrices.

Proposition 2.1. *If $T \in \mathscr{B}(\mathcal{H})$ is a self-adjoint compact operator on $\mathcal{H}$, then there exists an orthonormal basis $\{e_k\}_{k=1}^\infty$ of $\mathcal{H}$ and a sequence $(\lambda_k)_{k=1}^\infty$ of real numbers such that $Te_k = \lambda_k e_k$ and $\lambda_k \to 0$ as $k \to \infty$. In particular, the multiplicity of each eigenvalue $\lambda_k \neq 0$ is finite. Conversely, given an orthonomal basis $\{e_k\}_{k=1}^\infty$ of $\mathcal{H}$ and a sequence $(\lambda_k)_{k=1}^\infty$ of real numbers such that $\lambda_k \to 0$ as $k \to \infty$, the formula $Te_k = \lambda_k e_k$ defines a self-adjoint compact operator.*

Proof. See [25, Theorem VI.16]. ■

Proposition 2.1 is the main ingredient in our proof of the Shale-Stinespring theorem (Theorem 4.1), but it also allows us to define subspaces of $\mathscr{B}(\mathcal{H})$ that consist of particularly well-behaved operators. For each $p \geq 1$, let $\mathscr{I}_p(\mathcal{H}) \subset \mathscr{B}(\mathcal{H})$ be the set of compact operators T such that eigenvalues λ_k of the positive self-adjoint compact operator T^*T satisfy $\sum_k (\lambda_k)^{p/2} < \infty$, where T^* is the Hilbert space adjoint of T. The spaces $\mathscr{I}_1(\mathcal{H})$ and $\mathscr{I}_2(\mathcal{H})$ will suffice for our needs, and they are called the space of **trace class** operators and the space of **Hilbert-Schmidt** operators, respectively. The **trace** of an operator $T \in \mathscr{I}_1(\mathcal{H})$ is defined in terms of the eigenvalues λ_k of the operator T^*T as $\operatorname{tr}(T) = \sum_k (\lambda_k)^{1/2}$. More generally, we define the class of Hilbert-Schmidt operators $\mathscr{I}_2(\mathcal{H}, \mathcal{K})$ as the set of compact operators $T \in \mathscr{B}(\mathcal{H}, \mathcal{K})$ that satisfy $T^*T \in \mathscr{I}_1(\mathcal{H})$.

To define the **Hilbert space tensor product** of the spaces $\mathcal{H}$ and $\mathcal{K}$, we start with the algebraic tensor product $\mathcal{H} \otimes_{\mathrm{Alg}} \mathcal{K}$, which can be defined in terms of algebraic bases as follows. If $\{x_k\}$ is an algebraic basis of $\mathcal{H}$ and $\{y_l\}$ is an algebraic basis of $\mathcal{K}$, the algebraic tensor product $\mathcal{H} \otimes_{\mathrm{Alg}} \mathcal{K}$ is the vector space with an algebraic basis consisting of the all the symbols $x_k \otimes y_l$ together with the bilinear map $\otimes \colon \mathcal{H} \times \mathcal{K} \to \mathcal{H} \otimes_{\mathrm{Alg}} \mathcal{K}$ given by

$$(\sum_k a_k x_k) \otimes (\sum_l b_l y_l) = \sum_k \sum_l a_k b_l x_k \otimes y_l$$

on finite linear combinations of the basis elements. On the space $\mathcal{H} \otimes_{\mathrm{Alg}} \mathcal{K}$ we define an inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}\otimes\mathcal{K}}$ by setting

$$\langle x_i \otimes y_j, x_k \otimes y_l \rangle_{\mathcal{H}\otimes\mathcal{K}} = \langle x_i, x_k \rangle_{\mathcal{H}} \langle y_j, y_l \rangle_{\mathcal{K}}$$

and extending linearly in the first argument and conjugate-linearly in the second argument if $\mathcal{H}$ and $\mathcal{K}$ are complex. Completing the space $\mathcal{H} \otimes_{\mathrm{Alg}} \mathcal{K}$ in the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}\otimes\mathcal{K}}$ yields a Hilbert space, which we simply denote by $\mathcal{H} \otimes \mathcal{K}$.

Moreover, the tensor product map $\otimes \colon \mathcal{H} \times \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$ is continuous in the newly defined topology on $\mathcal{H} \otimes \mathcal{K}$, and if $\{e_k\}$ and $\{f_l\}$ are orthonormal bases of $\mathcal{H}$ and $\mathcal{K}$, respectively, then $\{e_k \otimes f_l\}$ is an orthonormal basis of $\mathcal{H} \otimes \mathcal{K}$. The tensor product construction is also associative, so we can define repeated tensor products such as $\mathcal{H} \otimes \mathcal{K} \otimes \mathcal{H}$ unambiguously. Finally, if $\mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_1$ and $\mathcal{K}_2$ are Hilbert spaces, $A \in \mathscr{B}(\mathcal{H}_1, \mathcal{K}_1)$ and $B \in \mathscr{B}(\mathcal{H}_2, \mathcal{K}_2)$, we define $A \otimes B \in \mathscr{B}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{K}_1 \otimes \mathcal{K}_2)$ to be the unique bounded operator satisfying $A \otimes B(e \otimes f) = (Ae) \otimes (Bf)$ for all $e \in \mathcal{H}_1$ and $f \in \mathcal{H}_2$.

If V is a complex vector space, V^* is the algebraic dual of V, and $\langle \cdot, \cdot \rangle_{V,V^*}$ denotes the dual pairing, then the map $v \otimes u \mapsto \langle \cdot, u \rangle v \colon V \otimes_{Alg} V^* \to \operatorname{Hom}(V, V)$ is an isomorphism if and only if $\dim(V) < \infty$. However, the situation is more fortunate in Hilbert spaces.

Proposition 2.2. *The space $\mathscr{I}_2(\mathcal{H}, \mathcal{K})$ is a Hilbert space under the inner product $\langle A, B \rangle = tr(B^*A)$, and the map $\mu_{\mathcal{H},\mathcal{K}} = f \otimes e \mapsto \langle \cdot, e \rangle_{\mathcal{H}} f \colon \mathcal{K} \otimes \mathcal{H} \to \mathscr{I}_2(\mathcal{H}, \mathcal{K})$ defines a Hilbert space isomorphism $\mu_{\mathcal{H},\mathcal{K}} \colon \mathcal{K} \otimes \mathcal{H} \cong \mathscr{I}_2(\mathcal{H}, \mathcal{K})$.*

Proof. See [25, Exercise VI.48]. ■

2.2 Super Hilbert spaces

The basic idea of super Hilbert spaces is that each Hilbert space is divided into two components, and whenever the order of two spaces is exchanged in a formula for ordinary Hilbert spaces, there is an additional sign. Following the first part of this principle, we say that a **super Hilbert space** is a Hilbert space $\mathcal{H}$ together with a direct sum decomposition $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$. One says that the Hilbert space $\mathcal{H}$ is $\mathbb{Z}_2$-graded, and the map $d_{\mathcal{H}} \in \mathscr{B}(\mathcal{H})$ defined by $d_{\mathcal{H}} e = e$ for $e \in \mathcal{H}^0$ and $d_{\mathcal{H}} f = -f$ for $f \in \mathcal{H}^1$ is called the **grading involution**. The elements of $\mathcal{H}^0$ are called **even homogeneous elements** and the elements of $\mathcal{H}^1$ are called **odd homogeneous elements**. Denoting the parity of a homogeneous element f by $P_{\mathcal{H}}(f)$, we set $P_{\mathcal{H}}(f) = 0$ if f is even and $P_{\mathcal{H}}(f) = 1$ if f is odd.

If $\mathcal{H}$ and $\mathcal{K}$ are super Hilbert spaces, we define a $\mathbb{Z}_2$-grading involution $d_{\mathcal{H},\mathcal{K}}$ on $\mathscr{B}(\mathcal{H},\mathcal{K})$ by $d_{\mathcal{H},\mathcal{K}} T = d_{\mathcal{K}} T d_{\mathcal{H}}$. This induces a decomposition $\mathscr{B}(\mathcal{H},\mathcal{K}) = \mathscr{B}^0(\mathcal{H},\mathcal{K}) \oplus \mathscr{B}^1(\mathcal{H},\mathcal{K})$, where $d_{\mathcal{H},\mathcal{K}} T = T$ for $T \in \mathscr{B}(\mathcal{H},\mathcal{K})$ and $d_{\mathcal{H},\mathcal{K}} T = -T$ for $T \in \mathscr{B}^1(\mathcal{H},\mathcal{K})$. The elements of $\mathscr{B}^0(\mathcal{H},\mathcal{K})$ are called **even homogeneous maps** and the elements of $\mathscr{B}^1(\mathcal{K})$ are called **odd homogeneous maps**. We denote the parities by $P_{\mathcal{H},\mathcal{K}}(T)$ similarly as for vectors. An **isomorphism of super Hilbert spaces** is an even unitary map, i.e., a grading-preserving unitary map.

If $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ and $\mathcal{K} = \mathcal{K}^0 \oplus \mathcal{K}^1$ are super Hilbert spaces, we define a super Hilbert space structure on the tensor product $\mathcal{H} \otimes \mathcal{K}$ by setting

$$(\mathcal{H} \otimes \mathcal{K})^k := \bigoplus_{i+j=k} \mathcal{H}^i \otimes \mathcal{K}^j,$$

where the indices are understood as elements of $\mathbb{Z}_2$. Following the second part of the basic principle, we define an isomorphism $\beta_{\mathcal{H},\mathcal{K}} \colon \mathcal{H} \otimes \mathcal{K} \to \mathcal{K} \otimes \mathcal{H}$ by setting

$$\beta_{\mathcal{H},\mathcal{K}}(e \otimes f) = (-1)^{P_{\mathcal{H}}(e) P_{\mathcal{K}}(f)} f \otimes e$$

for homogeneous elements and extending linearly. The map $\beta_{\mathcal{H},\mathcal{K}}$ is called the **braiding** of super Hilbert spaces.

By the associativity of the tensor product, we can suppress parenthesis in multiple tensor products, and it is not difficult to see that

$$(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n)^k = \bigoplus_{i_1+i_2+\ldots+i_n=k} \mathcal{H}_1^{i_1} \otimes \mathcal{H}_2^{i_2} \otimes \ldots \otimes \mathcal{H}_n^{i_n}$$

for super Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$. The individual braidings $\beta_{\mathcal{H}_i,\mathcal{H}_j}$ allow us to define a braiding

$$\beta_\sigma \colon \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n \to \mathcal{H}_{\sigma(1)} \otimes \mathcal{H}_{\sigma(2)} \otimes \ldots \otimes \mathcal{H}_{\sigma(n)}$$

for every permutation $\sigma \in S_n$ in the natural way as follows. Let $\sigma = (i_m j_m) \ldots (i_1 j_1)$ be a decomposition of σ into transpositions with $j_k = i_k + 1$. For each transposition $(i_k j_k)$ let $\sigma_k = (i_k j_k) \ldots (i_1 j_1)$ and define

$$\beta_{i_k,j_k} \colon \mathcal{H}_{\sigma_{(k-1)}(1)} \otimes \ldots \otimes \mathcal{H}_{\sigma_{(k-1)}(n)} \to \mathcal{H}_{\sigma_k(1)} \otimes \ldots \otimes \mathcal{H}_{\sigma_k(n)}$$

by

$$\beta_{i_k,j_k} = \bigotimes_{l=1}^{i_k-1} \mathrm{id}_{\mathcal{H}_{\sigma_{(k-1)}(l)}} \otimes \beta_{\mathcal{H}_{\sigma_{(k-1)}(i_k)},\mathcal{H}_{\sigma_{(k-1)}(j_k)}} \otimes \bigotimes_{l=j_k+1}^{n} \mathrm{id}_{\mathcal{H}_{\sigma_{(k-1)}(l)}}.$$

The braiding β_σ is then defined as $\beta_\sigma = \beta_{i_m,j_m}\beta_{i_{(m-1)},j_{(m-1)}}\dots\beta_{i_1,j_1}$.

Proposition 2.3. *The braiding β_σ is well defined.*

Proof. It suffices to show that the definition of β_σ is independent of the decomposition $\sigma = (i_m j_m)\dots(i_1 j_1)$, and to show this, it suffices to consider homogeneous elements $f_1 \otimes f_2 \otimes \dots \otimes f_n \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_n$. It is clear by the definition of β_σ that

$$\beta_\sigma(f_1 \otimes f_2 \otimes \dots \otimes f_n) = (-1)^N f_{\sigma(1)} \otimes f_{\sigma(2)} \otimes \dots \otimes f_{\sigma(n)}$$

for some integer N, and it is not difficult to see that N is the number of the pairs of indices $i < j$ for which f_i and f_j are both odd and $\sigma^{-1}(i) > \sigma^{-1}(j)$. ■

The braidings β_σ allow us to define tensor products of unordered collections of super Hilbert spaces, which is important since there is no canonical way to order the boundary components of a Riemann surface Σ or a planar domain $\mathscr{D}$. If $\{\mathcal{H}_j\}_{j\in J}$ is a finite collection of super Hilbert spaces, we define their **unordered tensor product** $\bigotimes_{j\in J}\mathcal{H}_j$ as the subspace

$$\bigotimes_{j\in J}\mathcal{H}_j = \left\{ (f_\sigma) \in \bigoplus_{\sigma}\bigotimes_{j=1}^{|J|}\mathcal{H}_{\sigma(j)} \;\middle|\; f_\tau = \beta_{\sigma^{-1}\circ\tau} f_\sigma \;\; \forall\, \tau,\sigma \right\},$$

where σ and τ run through all the bijections $\sigma,\tau\colon \{1,\dots,|J|\} \to J$. Since the maps β_σ are vector space isomorphisms, the projections $\mathrm{pr}_\tau\colon \bigotimes_{j\in J}\mathcal{H}_j \to \bigotimes_{j=1}^{|J|}\mathcal{H}_{\tau(j)}$ are vector space isomorphisms for every τ. Moreover, if we define an inner product on $\bigotimes_{j\in J}\mathcal{H}_j$ by

$$\langle (f_\sigma),(g_\sigma)\rangle = \frac{1}{|J|!}\sum_\sigma \langle f_\sigma, g_\sigma\rangle,$$

it is easy to see that the projections pr_τ become Hilbert space isomorphisms.

If $\{\mathcal{K}_k\}_{k\in K}$ is another finite collection of super Hilbert spaces, we define a **map of unordered tensor products** $T\colon \bigotimes_{j\in J}\mathcal{H}_j \to \bigotimes_{k\in K}\mathcal{K}_k$ to be a collection of maps

$$T_{\sigma,\tau}\colon \mathcal{H}_{\sigma(1)} \otimes \dots \otimes \mathcal{H}_{\sigma(|J|)} \to \mathcal{K}_{\tau(1)} \otimes \dots \otimes \mathcal{K}_{\tau(|K|)} \tag{9}$$

indexed by the bijections $\sigma\colon \{1,\dots,|J|\} \to J$ and $\tau\colon \{1,\dots,|K|\} \to K$ that are compatible with the braiding in the sense that the diagrams

$$\begin{array}{ccc} \bigotimes_{j=1}^{|J|}\mathcal{H}_{\sigma(j)} & \xrightarrow{T_{\sigma,\tau}} & \bigotimes_{k=1}^{|K|}\mathcal{K}_{\tau(k)} \\ \beta_{\widetilde{\sigma}^{-1}\circ\sigma}\uparrow & & \downarrow\beta_{\tau^{-1}\circ\widetilde{\tau}} \\ \bigotimes_{j=1}^{|J|}\mathcal{H}_{\widetilde{\sigma}(j)} & \xrightarrow[T_{\widetilde{\sigma},\widetilde{\tau}}]{} & \bigotimes_{k=1}^{|K|}\mathcal{K}_{\widetilde{\tau}(k)} \end{array} \tag{10}$$

commute for all bijections $\widetilde{\sigma}\colon \{1,\dots,|J|\} \to J$ and $\widetilde{\tau}\colon \{1,\dots,|K|\} \to K$. More precisely, given the collection $\{T_{\sigma,\tau}\}$, we consider T as the linear map $T\colon \bigotimes_{j\in J}\mathcal{H}_j \to \bigotimes_{k\in K}\mathcal{K}_k$ defined by

$$T = \bigoplus_{\sigma} \iota_\tau \circ T_{\sigma,\tau},$$

where ι_τ is the canonical injection $\iota_\tau\colon \bigotimes_{k=1}^{|K|}\mathcal{K}_{\tau(k)} \hookrightarrow \bigotimes_{k\in K}\mathcal{K}_k$.

On the other hand, if we are given super Hilbert spaces $\{\mathcal{H}_j\}_{j=1}^n$ and $\{\mathcal{K}_k\}_{k=1}^m$ and a map $T\colon \bigotimes_{j=1}^n \mathcal{H}_j \to \bigotimes_{k=1}^m \mathcal{K}_k$, we can define maps

$$T_{\sigma,\tau}\colon \mathcal{H}_{\sigma(1)}\otimes\ldots\otimes\mathcal{H}_{\sigma(n)} \to \mathcal{K}_{\tau(1)}\otimes\ldots\otimes\mathcal{K}_{\tau(m)}$$

for all bijections $\sigma \in S_n$ and $\tau \in S_m$ by $T_{\sigma,\tau} = \beta_\tau T \beta_{\sigma^{-1}}$. Since $\beta_{\widetilde{\sigma}}\beta_\sigma = \beta_{\sigma\circ\widetilde{\sigma}}$, it is clear that the collection $\{T_{\tau,\sigma}\}$ satisfies the compatibility condition (10) and thus defines a map of unordered tensor products.

The sum, composition and tensor product of maps of unordered tensor products can be defined in a natural way in terms of suitable representatives as follows.

Definition 2.4. In what follows, $\mathcal{H}$ and $\mathcal{K}$ with any decorations are super Hilbert spaces, and σ,τ and α are understood as bijections as in (9).

1. If $A, B\colon \bigotimes_{j\in J}\mathcal{H}_j \to \bigotimes_{k\in K}\mathcal{K}_k$ are maps of unordered tensor products, $A+B\colon \bigotimes_{j\in J}\mathcal{H}_j \to \bigotimes_{k\in K}\mathcal{K}_k$ is the map of unordered tensor products with representatives $(A+B)_{\sigma,\tau} = A_{\sigma,\tau}+B_{\sigma,\tau}$. Moreover, if $\lambda\in\mathbb{K}$, then $\lambda A\colon \bigotimes_{j\in J}\mathcal{H}_j \to \bigotimes_{k\in K}\mathcal{K}_k$ is the map of unordered tensor products with representatives $(\lambda A)_{\sigma,\tau} = \lambda A_{\sigma,\tau}$, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$ is the common ground field of each $\mathcal{H}_j$ and $\mathcal{K}_k$.

2. If $A\colon \bigotimes_{j\in J}\mathcal{H}^{(1)}_j \to \bigotimes_{k\in K}\mathcal{H}^{(2)}_k$ and $B\colon \bigotimes_{k\in K}\mathcal{H}^{(2)}_k \to \bigotimes_{l\in L}\mathcal{H}^{(3)}_l$ are maps of unordered tensor products, $BA\colon \bigotimes_{j\in J}\mathcal{H}^{(1)}_j \to \bigotimes_{l\in L}\mathcal{H}^{(3)}_l$ is the map of unordered tensor products with representatives $(BA)_{\sigma,\tau} = B_{\alpha,\tau}A_{\sigma,\alpha}$.

3. If $A\colon \bigotimes_{j\in J}\mathcal{H}^{(1)}_j \to \bigotimes_{k\in K}\mathcal{K}^{(1)}_k$ and $B\colon \bigotimes_{l\in L}\mathcal{H}^{(2)}_l \to \bigotimes_{m\in M}\mathcal{K}^{(2)}_m$ are homogeneous maps of unordered tensor products,

$$A\widehat{\otimes}B\colon \bigotimes_{(i,j)\in(\{1\}\times J)\cup(\{2\}\times L)}\mathcal{H}^{(i)}_j \to \bigotimes_{(i,j)\in(\{1\}\times K)\cup(\{2\}\times M)}\mathcal{K}^{(i)}_j$$

is the unique map of unordered tensor products whose representatives contain the collection

$$(A\widehat{\otimes}B)_{(\sigma,\widetilde{\sigma}),(\tau,\widetilde{\tau})} = A_{\sigma,\tau}d^{P(B_{\widetilde{\sigma},\widetilde{\tau}})}_{\otimes_j\mathcal{H}^{(1)}_{\sigma(j)}} \otimes B_{\widetilde{\sigma},\widetilde{\tau}},$$

where we have suppressed the source and target in $P(B_{\widetilde{\sigma},\widetilde{\tau}})$, and

$$(\sigma,\widetilde{\sigma})\colon \{1,\dots,|J|+|L|\} \to (\{1\}\times J)\cup(\{2\}\times L)$$

is defined for $\sigma\colon\{1,\dots,|J|\}\to J$ and $\widetilde{\sigma}\colon\{1,\dots,|L|\}\to L$ by

$$(\sigma,\widetilde{\sigma})(j) = \begin{cases}(1,\sigma(j)) & \text{if } j\le|J|\\ (2,\widetilde{\sigma}(j-|J|)) & \text{if } j>|J|.\end{cases}$$

It is not difficult to show that the maps in Definition 2.4 are well defined, i.e., they satisfy the compatibility condition (10). In particular, the set of maps $A\colon \bigotimes_{j\in J}\mathcal{H}_j \to \bigotimes_{k\in K}\mathcal{K}_k$ of unordered tensor products becomes a vector space in a natural way. The definition of $A\widehat{\otimes}B$ above might seem arbitrary at first, but a straight forward calculation gives that $B\widehat{\otimes}A = (-1)^{P(A)P(B)}A\widehat{\otimes}B$ as maps of unordered tensor products, which is precisely what the basic principle from the beginning of this chapter suggests. Since we will have no use for this fact, we omit the proof.

If $\mathcal{H}$ and $\mathcal{K}$ are super Hilbert spaces, $A \in \mathscr{B}(\mathcal{H})$ and $B \in \mathscr{B}(\mathcal{K})$, it is natural to regard the map $B\widehat{\otimes}A$ of unordered tensor products as a map $B\widehat{\otimes}A \in \mathscr{B}(\mathcal{K}\otimes\mathcal{H})$ of ordered tensor products by picking the corresponding representative. As our final result concerning super Hilbert spaces, we observe how the maps $B\widehat{\otimes}A \in \mathscr{B}(\mathcal{K}\otimes\mathcal{H})$ interact with the map $\mu_{\mathcal{H},\mathcal{K}}$ from Proposition 2.2.

Theorem 2.5. *Let $\mathcal{H}$ and $\mathcal{K}$ be super Hilbert spaces, let $A \in \mathscr{B}(\mathcal{H})$ be homogeneous, let $B \in \mathscr{B}(\mathcal{K})$, and let $\mu_{\mathcal{H},\mathcal{K}}\colon \mathcal{K}\otimes\mathcal{H} \cong \mathscr{I}_2(\mathcal{H},\mathcal{K})$ be the isomorphism from Proposition 2.2. For every $g \in \mathcal{K}\otimes\mathcal{H}$ we have*

$$\mu_{\mathcal{H},\mathcal{K}}((id_{\mathcal{K}}\widehat{\otimes}A)g) = d_{\mathcal{K}}^{P(A)} \circ \mu_{\mathcal{H},\mathcal{K}}(g) \circ A^*$$

and

$$\mu_{\mathcal{H},\mathcal{K}}((B\widehat{\otimes}id_{\mathcal{H}})g) = B \circ \mu_{\mathcal{H},\mathcal{K}}(g),$$

where we have suppressed the source and the target in $P(A)$.

Proof. It suffices to prove the relations for $g = f\otimes e$ with homogeneous $e \in \mathcal{H}$ and $f \in \mathcal{K}$. For every $h \in \mathcal{H}$ we get

$$\begin{aligned}
\mu_{\mathcal{H},\mathcal{K}}((\mathrm{id}_{\mathcal{K}}\widehat{\otimes}A)f\otimes e)h &= \mu_{\mathcal{H},\mathcal{K}}((\mathrm{id}_{\mathcal{K}}\otimes A)(d_{\mathcal{K}}^{P(A)}f\otimes e))h \\
&= (-1)^{P(f)P(A)}\mu_{\mathcal{H},\mathcal{K}}(f\otimes Ae)h \\
&= (-1)^{P(f)P(A)}\langle h, Ae\rangle_{\mathcal{H}} f \\
&= d_{\mathcal{K}}^{P(A)}\langle A^*h, e\rangle_{\mathcal{H}} f \\
&= (d_{\mathcal{K}}^{P(A)} \circ \mu_{\mathcal{H},\mathcal{K}}(f\otimes e)\circ A^*)h
\end{aligned}$$

and

$$\begin{aligned}
\mu_{\mathcal{H},\mathcal{K}}((B\widehat{\otimes}\mathrm{id}_{\mathcal{H}})f\otimes e)h &= \mu_{\mathcal{H},\mathcal{K}}((B\otimes\mathrm{id}_{\mathcal{H}})(d_{\mathcal{K}}^{P(\mathrm{id}_{\mathcal{H}})}f\otimes e))h \\
&= \mu_{\mathcal{H},\mathcal{K}}(Bf\otimes e)h \\
&= \langle h, e\rangle_{\mathcal{H}} Bf \\
&= B\langle h, e\rangle_{\mathcal{H}} f \\
&= (B\circ\mu_{\mathcal{H},\mathcal{K}}(f\otimes e))h,
\end{aligned}$$

so the claim follows. ∎

2.3 Fock spaces

We are now in a position to define the super Hilbert spaces that the CFT and BCFT functors associate to the boundary components of Riemann surfaces and planar domains, respectively. Let us fix a complex Hilbert space $\mathcal{H}$ for the remainder of this section, and for each permutation $\sigma \in S_n$, let $P_\sigma \colon \bigotimes_{j=1}^n \mathcal{H} \to \bigotimes_{j=1}^n \mathcal{H}$ be the map defined by

$$P_\sigma(e_1 \otimes e_2 \otimes \ldots \otimes e_n) = e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \ldots \otimes e_{\sigma(n)}$$

on simple tensors and extended linearly to the whole space. We denote by $\bigwedge^n \mathcal{H}$ the subspace of vectors $f \in \bigotimes_{i=1}^n \mathcal{H}$ satisfying $P_\sigma f = \operatorname{sgn}(\sigma) f$ for every $\sigma \in S_n$. Since the maps P_σ are bounded, it follows that $\bigwedge^n \mathcal{H}$ is a closed subspace and thus a Hilbert space with the inner product inherited from $\bigotimes_{j=1}^n \mathcal{H}$. The space $\bigwedge^0 \mathcal{H}$ is defined as the one-dimensional space spanned by a distinguished vector Ω.

To describe useful orthonormal bases for the spaces $\bigwedge^n \mathcal{H}$, we define the **wedge product** of vectors $e_1, e_2, \ldots, e_n \in \mathcal{H}$ as the vector

$$e_1 \wedge e_2 \wedge \ldots \wedge e_n := \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \ldots \otimes e_{\sigma(n)} \in \bigwedge^n \mathcal{H}.$$

The wedge product is linear in each of its arguments, and the inner products of wedge products are given by

$$\begin{aligned}
&\langle e_1 \wedge \ldots \wedge e_n, f_1 \wedge \ldots \wedge f_n \rangle \\
=&\frac{1}{n!} \sum_{\sigma \in S_n} \sum_{\tau \in S_n} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \langle e_{\sigma(1)}, f_{\tau(1)} \rangle \ldots \langle e_{\sigma(n)}, f_{\tau(n)} \rangle \\
=&\frac{1}{n!} \sum_{\sigma \in S_n} \sum_{\tau \in S_n} \operatorname{sgn}(\tau \circ \sigma^{-1}) \langle e_1, f_{\tau\circ\sigma^{-1}(1)} \rangle \ldots \langle e_n, f_{\tau\circ\sigma^{-1}(n)} \rangle \\
=&\frac{1}{n!} \sum_{\sigma \in S_n} \sum_{\tau \in S_n} \operatorname{sgn}(\tau) \langle e_1, f_{\tau(1)} \rangle \ldots \langle e_n, f_{\tau(n)} \rangle \\
=&\sum_{\tau \in S_n} \operatorname{sgn}(\tau) \langle e_1, f_{\tau(1)} \rangle \ldots \langle e_n, f_{\tau(n)} \rangle \\
=&\det([\langle e_i, f_j \rangle]_{i,j=1}^n).
\end{aligned} \tag{11}$$

Hence, if $\{h_i\}$ is an orthonormal basis of $\mathcal{H}$ indexed by a linearly ordered set, distinct vectors f, g in the collection

$$\left\{ h_{i_1} \wedge h_{i_2} \wedge \ldots \wedge h_{i_n} \;\middle|\; i_1 < i_2 < \ldots < i_n \right\} \tag{12}$$

satisfy $\langle f, f \rangle = \det(I_{n\times n}) = 1$ and $\langle f, g \rangle = 0$ since the corresponding matrix of inner products above has a column of zeros. Hence, the set (12) is orthonormal, and it is actually an orthonormal basis of $\bigwedge^n \mathcal{H}$ if we consider Ω as the empty wedge product, although we do not prove this fact. For any $e \in \mathcal{H}$, the wedge product also defines a linear map $e \wedge \cdot \colon \bigwedge^n \mathcal{H} \to \bigwedge^{n+1} \mathcal{H}$ by setting

$$e \wedge (f_1 \wedge \ldots \wedge f_n) = e \wedge f_1 \wedge \ldots \wedge f_n$$

and extending linearly.

The **Fock space** $\mathscr{F}_{\mathcal{H}}$ is defined as a direct sum of Hilbert spaces by

$$\mathscr{F}_{\mathcal{H}} = \bigoplus_{n=0}^{\infty} \bigwedge^{n} \mathcal{H},$$

and it has a dense subspace $\mathscr{F}_{\mathcal{H}}^0$ consisting of vectors f such that $f \in \bigoplus_{n=0}^{N} \bigwedge^n \mathcal{H}$ for some $N < \infty$. It follows from the previous paragraph that if $\{h_i\}$ is an orthonormal basis of $\mathcal{H}$ indexed by a linearly ordered set, then the collection

$$\left\{h_{i_1} \wedge h_{i_2} \wedge \ldots \wedge h_{i_n} \;\middle|\; n \geq 0 \text{ and } i_1 < i_2 < \ldots < i_n\right\} \tag{13}$$

is an orthonormal basis of $\mathscr{F}_{\mathcal{H}}$. Moreover, $\mathscr{F}_{\mathcal{H}}$ is naturally a super Hilbert space with the $\mathbb{Z}_2$ grading given by

$$(\mathscr{F}_{\mathcal{H}})^i = \bigoplus_{k=0}^{\infty} \bigwedge^{2k+i} \mathcal{H}.$$

For each $f \in \mathcal{H}$, we define a map $a(f) \colon \mathscr{F}_{\mathcal{H}}^0 \to \mathscr{F}_{\mathcal{H}}^0$ by

$$a(f)g = f \wedge g,$$

and these maps have the following properties.

Theorem 2.6. *The maps $a(f)$ extend to bounded operators on $\mathscr{F}_{\mathcal{H}}$ with norm of at most $\|f\|$, and we have*

$$\{a(f), a(g)\} = 0, \quad \{a(f), a(g)^*\} = \langle f, g\rangle id_{\mathcal{H}}, \tag{14}$$

where $\{A, B\} = AB + BA$ is the anticommutator.

Proof. Using the Laplace expansion along the first column in (11) gives

$$\langle a(f)f_1 \wedge \ldots \wedge f_n, e_1 \wedge \ldots \wedge e_{n+1}\rangle = \sum_{k=1}^{n} (-1)^{k+1} \langle f, e_k\rangle \det([\langle f_i, e_j\rangle]_{j\neq k}),$$

so the adjoint is given on $\mathscr{F}_{\mathcal{H}}^0$ by

$$a(f)^*(e_1 \wedge e_2 \wedge \ldots \wedge e_n) = \sum_{k=1}^{n} (-1)^{k+1} \langle f, e_k\rangle^* e_1 \wedge \ldots \wedge \widehat{e}_k \wedge \ldots \wedge e_n,$$

where $\widehat{e}_k$ means that e_k is omitted from the wedge product. Moreover, the first relation in (14) follows from $f \wedge g = -g \wedge f$, and for the second one we observe that the formula for $a(f)^*$ gives

$$\begin{aligned}
&\{a(f), a(g)^*\} e_1 \wedge \ldots \wedge e_n \\
=& \sum_{k=1}^{n} (-1)^{k+1} \langle g, e_k\rangle^* f \wedge e_1 \wedge \ldots \wedge \widehat{e_k} \wedge \ldots \wedge e_n \\
&+ \langle g, f\rangle^* e_1 \wedge \ldots \wedge e_n + \sum_{k=1}^{n} (-1)^{k+2} \langle g, e_k\rangle^* f \wedge e_1 \wedge \ldots \wedge \widehat{e_k} \wedge \ldots \wedge e_n \\
=& \langle f, g\rangle e_1 \wedge \ldots \wedge e_n.
\end{aligned}$$

For $g \in \mathscr{F}^0_{\mathcal{H}}$ the second relation in (14) gives

$$\begin{aligned}\|a(f)g\|^2 + \|a(f)^*g\|^2 &= \langle a(f)^*a(f)g, g\rangle + \langle a(f)a(f)^*g, g\rangle \\ &= \langle \{a(f), a(f)^*\}g, g\rangle = \|f\|^2\|g\|^2,\end{aligned}$$

so $a(f)$ is bounded on $\mathscr{F}^0_{\mathcal{H}}$ with norm of at most $\|f\|$. Hence, it extends to a bounded operator on $\mathscr{F}_{\mathcal{H}}$ with the same norm, and the relations in (14) hold on $\mathscr{F}_{\mathcal{H}}$ by continuity. ∎

The physical interpretation is that the vectors in $\mathcal{H}$ describe the state of a quantum system consisting of a single particle, and similarly the state of a system consisting of n copies of the same particle is described by a subspace of $\bigotimes_{j=1}^n \mathcal{H}$. In physically meaningful cases, the vectors $g \in \bigotimes_{j=1}^n \mathcal{H}$ describing states of the n-particle system all satisfy either $P_\sigma g = g$ or $P_\sigma g = \operatorname{sgn}(\sigma)g$, and we call the corresponding particles **bosons** and **fermions**, respectively. The space $\bigwedge^n \mathcal{H}$ thus describes a system consisting of n identical fermions, and the Fock space $\mathscr{F}_{\mathcal{H}}$ describes a system consisting of an arbitrary number of fermions. Moreover, the subspace $\mathscr{F}^0_{\mathcal{H}}$ contains the vectors describing only finitely many fermions, and the vector Ω describes the system with no fermions. For this reason we call the distinguished vector Ω the **vacuum**. If $f_1, \dots, f_n \in \mathcal{H}$ are the vectors describing the states of n identical fermions, the vector $f_1 \wedge \dots \wedge f_n$ describes the state of the collection of the fermions. The operators $a(f_i)$ and $a(f_i)^*$ thus correspond to creating and annihilating a fermion in state f_i, respectively, and they are usually called the **creation** and **annihilation operators**. For more information, see e.g. [14].

2.4 Clifford algebras

We proceed to discuss a slight variation of the theme encountered in the previous chapter. The discussion is easiest to formulate in terms of the representation theory of universal C^*-algebras, although the concrete formulas for the representations are all we need. A good reference in this regard is [21, Chap. 2.9].

Definition 2.7. Given a real Hilbert space $\mathcal{H}$ and a unitary involution $R \in \mathscr{B}(\mathcal{H})$, i.e. a unitary map R satisfying $R^2 = \mathrm{id}_{\mathcal{H}}$, the Clifford algebra $\mathrm{Cl}(\mathcal{H}, R)$ is the universal C^*-algebra with generators $a(f)$, that are linear in $f \in \mathcal{H}$, and relations

$$a(f)^2 = \frac{1}{2}\langle f, Rf\rangle_{\mathcal{H}} I, \quad a(Rf) = a(f)^* \tag{15}$$

where I is the unit.

Definition 2.8. For a complex Hilbert space $\mathcal{K}$, the algebra $\mathrm{CAR}(\mathcal{K})$ is the universal C^*-algebra with generators $a(f)$, that are linear in $f \in \mathcal{K}$, and relations

$$\{a(f), a(g)\} = 0, \quad \{a(f), a(g)^*\} = \langle f, g\rangle_{\mathcal{K}} I, \tag{16}$$

where I is the unit.

For the remainder of the thesis, we fix the convention that if p is a projection on some Hilbert space, then $p^\perp = (\mathrm{id} - p)$. Moreover, if $\mathcal{H}$ is a real Hilbert space and R is a unitary involution, we say that a projection $p \in \mathscr{B}(\mathcal{H})$ is **R-compatible** if $R\colon p\mathcal{H} \cong p^\perp\mathcal{H}$. For each R-compatible projection p, we define a representation ϕ_p of $\mathrm{Cl}(\mathcal{H}, R)$ on the Fock space $\mathscr{F}_{p\mathcal{H}} := \bigoplus_{n=0}^\infty \bigwedge^n(p\mathcal{H} \otimes_\mathbb{R} \mathbb{C})$ of the complexification of $p\mathcal{H}$ by

$$\phi_p(a(f)) = a(pf) + a(Rp^\perp f)^*, \tag{17}$$

and for each projection q on $\mathcal{K}$, we define a representation π_q of $\mathrm{CAR}(\mathcal{K})$ on $\mathscr{F}_{\mathcal{K},q} := \mathscr{F}_{(q\mathcal{K})^* \oplus (q\mathcal{K})^\perp}$ by

$$\pi_q(a(f)) = a((qf)^*)^* + a(q^\perp f), \tag{18}$$

where $(qf)^* \in (q\mathcal{K})^*$ is the functional $(qf)^*(g) = \langle g, qf\rangle_\mathcal{K}$. To be more precise, the symbol $a(f)$ on the left-hand side of (17) and (18) refers to the element of the corresponding algebra, and on the right-hand side it denotes the creation operator on the corresponding Fock space.

The relations (15) and (16) are verified by a direct computation using (14), and it also follows from Theorem 2.6 that the operators $\phi_p(a(f))$ and $\pi_q(a(f))$ are bounded. The representations ϕ_p and π_q are also irreducible, and the corresponding vacua are cyclic. Indeed, if we denote both vacua by Ω, a reformulation of (13) shows that given an orthonormal basis $\{e_i\}$ of $p\mathcal{H}$ with a linearly ordered index set, the set

$$\left\{ \prod_{k=1}^n \phi_p(a(e_{i_k}))\Omega \;\middle|\; n \geq 0 \text{ and } i_1 < i_2 < \ldots < i_n \right\} \tag{19}$$

is an orthonormal basis of $\mathscr{F}_{p\mathcal{H}}$. To be precise, we take the product above in the order of increasing k, i.e., the rightmost operator is $\phi_p(a(e_{i_1}))$. Similarly given orthonormal bases $\{f_i\}$ and $\{g_i\}$ of $q^\perp\mathcal{K}$ and $q\mathcal{K}$, respectively, the set

$$\left\{ \prod_{k=1}^m \pi_q(a(g_{i_k}))^* \prod_{k=1}^n \pi_q(a(f_{j_k}))\Omega \;\middle|\; m, n \geq 0 \text{ and } i_1 < \ldots < i_m, j_1 < \ldots < j_n \right\} \tag{20}$$

is an orthonormal basis of $\mathscr{F}_{\mathcal{K},q}$, where we order both products as above. More generally, we continue to order noncommutative products by a linear order on the index set without further comments.

Up to a scalar multiple, the vacuum $\Omega \in \mathscr{F}_{p\mathcal{H}}$ is the unique vector satisfying

$$\phi_p(a(f))^*\Omega = 0 \quad \text{for all } f \in p\mathcal{H},$$

and similarly the vacuum $\Omega \in \mathscr{F}_{\mathcal{K},q}$ is the unique vector satisfying

$$\begin{aligned} \pi_q(a(f))\Omega &= 0 \quad \text{for all } f \in q\mathcal{K}, \\ \pi_q(a(f))^*\Omega &= 0 \quad \text{for all } f \in (q\mathcal{K})^\perp. \end{aligned}$$

More generally, we are interested in the following variations.

Definition 2.9. Let ϕ_p be a representation of $\mathrm{Cl}(\mathcal{H}, R)$ given above. Given a projection $\widetilde{p}$ on $\mathcal{H}$, a vector $\Omega_{\widetilde{p}} \in \mathscr{F}_{p\mathcal{H}}$ is said to satisfy the $\widetilde{p}$-vacuum equation if

$$\phi_p(a(f))^*\Omega_{\widetilde{p}} = 0 \quad \text{for all } f \in \widetilde{p}\mathcal{H},$$

and we call such a $\Omega_{\widetilde{p}}$ a $\widetilde{p}$-vacuum.

Definition 2.10. Let π_q be the representation of $\mathrm{CAR}(\mathcal{K})$ given above. Given a projection $\widetilde{q}$ on $\mathcal{K}$, a vector $\Omega_{\widetilde{q}} \in \mathscr{F}_{\mathcal{K},q}$ is said to satisfy the $\widetilde{q}$-vacuum equations if

$$\begin{aligned} \pi_q(a(f))\Omega &= 0 \quad \text{for all } f \in \widetilde{q}\mathcal{K}, \\ \pi_q(a(f))^*\Omega &= 0 \quad \text{for all } f \in (\widetilde{q}\mathcal{K})^\perp, \end{aligned}$$

and we call such a $\Omega_{\widetilde{q}}$ a $\widetilde{q}$-vacuum.

The significance of Definitions 2.9 and 2.10 is that the existence of a $\widetilde{p}$ vacuum in $\mathscr{F}_{p\mathcal{H}}$ for suitable choices of $\mathcal{H}, R, p$ and $\widetilde{p}$ is equivalent to the existence of the BCFT operators, and similarly for a $\widetilde{q}$-vacuum in $\mathscr{F}_{\mathcal{K},q}$ and the CFT operators, as we will see in Chapter 3.4. Moreover, we will see in Chapter 4 that the representations π_q and $\pi_{\widetilde{q}}$ of $\mathrm{CAR}(\mathcal{K})$ are unitarily equivalent if and only if there exists a $\widetilde{q}$-vacuum in $\mathscr{F}_{\mathcal{K},q}$.

Given a Clifford algebra $\mathrm{Cl}(\mathcal{H}, R)$ and an R-compatible projection $p \in \mathscr{B}(\mathcal{H})$, the map R restricts to an isomorphism $R\colon p\mathcal{H} \to p^\perp\mathcal{H}$, which induces an isomorphism $U_R\colon \mathscr{F}_{p\mathcal{H}} \to \mathscr{F}_{p^\perp\mathcal{H}}$ by

$$U_R(e_1 \wedge e_2 \wedge \ldots \wedge e_n) = Re_n \wedge Re_{n-1} \wedge \ldots \wedge Re_1.$$

The map U_R gives rise to the following relation between the representations ϕ_p and $\phi_{p^\perp}$ of $\mathrm{Cl}(\mathcal{H}, R)$.

Theorem 2.11. *For every $f \in \mathcal{H}$, we have*

$$\begin{aligned} U_R^*\phi_{p^\perp}(a(f))U_R &= \phi_p(a((id_\mathcal{H} - 2p)f))^*d, \\ U_R^*\phi_{p^\perp}(a(f))^*U_R &= -\phi_p(a((id_\mathcal{H} - 2p)f))d, \end{aligned}$$

where $d = d_{\mathscr{F}_{p\mathcal{H}}}$ is the $\mathbb{Z}_2$-grading involution.

Proof. For a fixed $f \in \mathcal{H}$, the second relation is obtained from the first one by taking adjoints and vise versa, so it suffices to prove the first one for $f \in p^\perp\mathcal{H}$ and the second one for $f \in p\mathcal{H}$. For $f \in p^\perp\mathcal{H}$ we have $(\mathrm{id}_\mathcal{H} - 2p)f = f$ and

$$\begin{aligned} U_R^*\phi_{p^\perp}(a(f))U_R(e_1 \wedge e_2 \wedge \ldots \wedge e_n) &= U_R^*\phi_{p^\perp}(a(f))(Re_n \wedge Re_{n-1} \wedge \ldots \wedge Re_1) \\ &= U_R^*(f \wedge Re_n \wedge Re_{n-1} \wedge \ldots \wedge Re_1) \\ &= e_1 \wedge e_2 \wedge \ldots \wedge e_n \wedge Rf \\ &= (-1)^n Rf \wedge e_1 \wedge e_2 \wedge \ldots \wedge e_n \\ &= \phi_p(a(f))^*d(e_1 \wedge e_2 \wedge \ldots \wedge e_n). \end{aligned}$$

Moreover, for $f \in p\mathcal{H}$ we have $(\mathrm{id}_{\mathcal{H}} - 2p)f = -f$ and

$$\begin{aligned} U_R^* \phi_{p^\perp}(a(f))^* U_R(e_1 \wedge e_2 \wedge \ldots \wedge e_n) &= U_R^* \phi_{p^\perp}(a(f))^*(Re_n \wedge Re_{n-1} \wedge \ldots \wedge Re_1) \\ &= U_R^*(Rf \wedge Re_n \wedge Re_{n-1} \wedge \ldots \wedge Re_1) \\ &= e_1 \wedge e_2 \wedge \ldots \wedge e_n \wedge f \\ &= (-1)^n f \wedge e_1 \wedge e_2 \wedge \ldots \wedge e_n \\ &= -\phi_p(a(-f))d(e_1 \wedge e_2 \wedge \ldots \wedge e_n), \end{aligned}$$

so the claim follows. ■

Remark. A similar result holds also for $\mathrm{CAR}(\mathcal{K})$ and the representations π_q with essentially the same proof, see [28, Proposition 2.9]. The same remark applies also to Theorem 2.12 below and the subsequent discussion, see [28, Proposition 2.7].

Finally, we consider an interesting relationship between direct sums and tensor products in the Fock space construction. For the remainder of this chapter, let us fix Clifford algebras $\mathrm{Cl}(\mathcal{H}_i, R_i)$ for $i = 1, 2$ and consider their representations ϕ_{p_i} on the spaces $\mathscr{F}_{p_i\mathcal{H}_i}$, where it is understood that each p_i is R_i-compatible. The map $\mathcal{S}_{i,j} \colon \mathscr{F}_{p_i\mathcal{H}_i} \otimes \mathscr{F}_{p_j\mathcal{H}_j} \to \mathscr{F}_{p_i\mathcal{H}_i \oplus p_j\mathcal{H}_j}$ defined by

$$(e_1 \wedge \ldots \wedge e_m) \otimes (f_1 \wedge \ldots \wedge f_n) \mapsto e_1 \wedge \ldots \wedge e_m \wedge f_1 \wedge \ldots \wedge f_n$$

is clearly grading-preserving, and it follows from (11) that $\mathcal{S}_{i,j}$ is also unitary. Hence, it defines an isomorphism of super Hilbert spaces, and this isomorphism turns out to be compatible with the braiding.

Theorem 2.12. *Let $s \colon \mathscr{F}_{p_1\mathcal{H}_1 \oplus p_2\mathcal{H}_2} \to \mathscr{F}_{p_2\mathcal{H}_2 \oplus p_1\mathcal{H}_1}$ be the isomorphism defined by*

$$s((e_1 \oplus f_1) \wedge \ldots \wedge (e_n \oplus f_n)) = (f_1 \oplus e_1) \wedge \ldots \wedge (f_n \oplus e_n)$$

and let $\beta \colon \mathscr{F}_{p_1\mathcal{H}_1} \otimes \mathscr{F}_{p_2\mathcal{H}_2} \to \mathscr{F}_{p_2\mathcal{H}_2} \otimes \mathscr{F}_{p_1\mathcal{H}_1}$ be the braiding of super Hilbert spaces. The isomorphism induced by s under $\mathcal{S}_{i,j}$ is β, i.e., the following diagram commutes.

$$\begin{array}{ccc} \mathscr{F}_{p_1\mathcal{H}_1 \oplus p_2\mathcal{H}_2} & \xrightarrow{\quad s \quad} & \mathscr{F}_{p_2\mathcal{H}_2 \oplus p_1\mathcal{H}_1} \\ \mathcal{S}_{1,2}\Big\uparrow & & \Big\downarrow \mathcal{S}_{2,1}^{-1} \\ \mathscr{F}_{p_1\mathcal{H}_1} \otimes \mathscr{F}_{p_2\mathcal{H}_2} & \xrightarrow[\quad \beta \quad]{} & \mathscr{F}_{p_2\mathcal{H}_2} \otimes \mathscr{F}_{p_1\mathcal{H}_1} \end{array} \tag{21}$$

Proof. Choose $e_1, \ldots, e_m \in p_1\mathcal{H}_1$ and $f_1, \ldots, f_n \in p_2\mathcal{H}_2$. The bottom row gives

$$\beta((e_1 \wedge \ldots \wedge e_m) \otimes (f_1 \wedge \ldots \wedge f_n)) = (-1)^{nm}(f_1 \wedge \ldots \wedge f_n) \otimes (e_1 \wedge \ldots \wedge e_m),$$

and traversing the diagram the other way around we get

$$\begin{aligned} &\mathcal{S}_{2,1}^{-1} s \mathcal{S}_{1,2}((e_1 \wedge \ldots \wedge e_m) \otimes (f_1 \wedge \ldots \wedge f_n)) \\ =&\mathcal{S}_{2,1}^{-1} s((e_1 \oplus 0) \wedge \ldots \wedge (e_m \oplus 0) \wedge (0 \oplus f_1) \wedge \ldots \wedge (0 \oplus f_n)) \\ =&\mathcal{S}_{2,1}^{-1}((0 \oplus e_1) \wedge \ldots \wedge (0 \oplus e_m) \wedge (f_1 \oplus 0) \wedge \ldots \wedge (f_n \oplus 0)) \\ =&(-1)^{nm} \mathcal{S}_{2,1}^{-1}((f_1 \oplus 0) \wedge \ldots \wedge (f_n \oplus 0) \wedge (0 \oplus e_1) \wedge \ldots \wedge (0 \oplus e_m)) \\ =&(-1)^{nm}(f_1 \wedge \ldots \wedge f_n) \otimes (e_1 \wedge \ldots \wedge e_m), \end{aligned}$$

so the diagram commutes. ■

Theorem 2.12 has an immediate generalization to the case of Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ and projections $p_1, \ldots, p_n$. If $\sigma \in S_n$ is any permutation, iterating Diagram (21) gives the diagram

$$\begin{array}{ccc} \mathscr{F}_{p_1\mathcal{H}_1 \oplus \ldots \oplus p_n \mathcal{H}_n} & \xrightarrow{s_\sigma} & \mathscr{F}_{p_{\sigma(1)}\mathcal{H}_{\sigma(1)} \oplus \ldots \oplus p_{\sigma(n)}\mathcal{H}_{\sigma(n)}} \\ \mathcal{S}\uparrow & & \downarrow \mathcal{S}^{-1} \\ \mathscr{F}_{p_1\mathcal{H}_1} \otimes \ldots \otimes \mathscr{F}_{p_n\mathcal{H}_n} & \xrightarrow[\beta_\sigma]{} & \mathscr{F}_{p_{\sigma(1)}\mathcal{H}_{\sigma(1)}} \otimes \ldots \otimes \mathscr{F}_{p_{\sigma(n)}\mathcal{H}_{\sigma(n)}} \end{array} \tag{22}$$

where s_σ and $\mathcal{S}$ have the obvious meanings. Diagram (22) gives a vector space isomorphism between the space of maps $\bigotimes_{j \in J} \mathscr{F}_{p_j \mathcal{H}_j} \to \bigotimes_{k \in K} \mathscr{F}_{p_k \mathcal{H}_k}$ of unordered tensor products and the space of maps $\mathscr{F}_{\bigoplus_{j \in J} p_j \mathcal{H}_j} \to \mathscr{F}_{\bigoplus_{k \in K} p_k \mathcal{H}_k}$.

The representations ϕ_{p_i} behave under the isomorphism $\mathcal{S}_{i,j}$ as expected.

Theorem 2.13. *The representation of $Cl(\mathcal{H}_1 \oplus \mathcal{H}_2, R_1 \oplus R_2)$ induced by $\phi_{p_1 \oplus p_2}$ under $\mathcal{S}_{i,j}$ is given by $a(f \oplus g) \mapsto \phi_{p_1}(a(f)) \widehat{\otimes} id + id \widehat{\otimes} \phi_{p_2}(a(g))$, i.e., the following diagram commutes.*

$$\begin{array}{ccc} \mathscr{F}_{p_1\mathcal{H}_1 \oplus p_2\mathcal{H}_2} & \xrightarrow{\phi_{p_1 \oplus p_2}(a(f \oplus g))} & \mathscr{F}_{p_1\mathcal{H}_1 \oplus p_2\mathcal{H}_2} \\ \mathcal{S}_{1,2}\uparrow & & \downarrow \mathcal{S}_{2,1}^{-1} \\ \mathscr{F}_{p_1\mathcal{H}_1} \otimes \mathscr{F}_{p_2\mathcal{H}_2} & \xrightarrow[\phi_{p_1}(a(f)) \widehat{\otimes} id + id \widehat{\otimes} \phi_{p_2}(a(g))]{} & \mathscr{F}_{p_1\mathcal{H}_1} \otimes \mathscr{F}_{p_2\mathcal{H}_2} \end{array} \tag{23}$$

Proof. A routine verification similar to the proof of Theorem 2.12. ■

3 The free fermion CFT and BCFT

Having defined all the necessary objects, we proceed to discuss the CFT and BCFT functors that we anticipated in Chapter 1. First, we introduce the free fermion CFT constructed in [28] that serves as a model for the BCFT case. In Chapter 3.2, we then introduce the transfer matrix formalism of the Ising model in a rectangular domain and use it to motivate the definition of the BCFT in a rectangle. The BCFT operators corresponding to rectangles are shown to form a contractive semigroup, and we also consider slight deformations to the rectangular domains. Finally, we propose a definition of the free fermion BCFT in Chapter 3.4 and reduce the existence of the BCFT operators to the existence of a suitable $\widetilde{p}$-vacuum in a representation ϕ_p of a Clifford algebra $\mathrm{Cl}(\mathcal{H}, R)$.

3.1 The free fermion CFT

We begin by presenting a slight simplification of the free fermion CFT constructed in [28], highlighting the properties that will be of interest to us in the BCFT case. The setup is a refinement of the one in Chapter 1: one considers a bordered Riemann surface Σ equipped with a complex line bundle L, which induces complex line bundles $L|_{\mathcal{C}_j}$ on the boundary circles $\mathcal{C}_j$ by restriction. The bundles $L|_{\mathcal{C}_j}$ are assumed to come with smooth trivializations $\beta_j \colon S^1 \times \mathbb{C} \to L|_{\mathcal{C}_j}$, and the restrictions

$\beta_j|_{S^1}\colon S^1 \to \mathcal{C}_j$ induce a partition of the the boundary circles $\mathcal{C}_j$ into incoming and outgoing components as in Chapter 1. That is, a boundary circle $\mathcal{C}_j$ is outgoing if and only if the orientation on $\mathcal{C}_j$ induced by $\beta_j|_{S^1}$ from the standard counterclockwise orientation on S^1 agrees with the orientation on $\mathcal{C}_j$ induced by the orientation on the complex base manifold Σ. In this chapter, we index the set of boundary components by a set J_Σ, and the incoming and outgoing boundary components are indexed by J_Σ^{in} and J_Σ^{out}, respectively.

Given triples $X = (\Sigma^{(1)}, L^{(1)}, \{\beta_j^{(1)}\})$ and $Y = (\Sigma^{(2)}, L^{(2)}, \{\beta_j^{(2)}\})$ as above and indices $j^{\mathrm{in}} \in J^{\mathrm{in}}_{\Sigma^{(2)}}$ and $j^{\mathrm{out}} \in J^{\mathrm{out}}_{\Sigma^{(1)}}$, the manifolds $\Sigma^{(1)}$ and $\Sigma^{(2)}$ can be sewn together along the map $\beta^{(2)}_{j^{\mathrm{out}}}|_{S^1} \circ (\beta^{(1)}_{j^{\mathrm{in}}}|_{S^1})^{-1}\colon \mathcal{C}_{j^{\mathrm{in}}} \to \mathcal{C}_{j^{\mathrm{out}}}$, yielding a Riemann surface $\Sigma^{(3)}$ [24]. Much in the same way, the bundles $L^{(1)}$ and $L^{(2)}$ can be sewn together along the map $\beta^{(2)}_{j^{\mathrm{out}}} \circ (\beta^{(1)}_{j^{\mathrm{in}}})^{-1}$, resulting in a line bundle $L^{(3)}$ over $\Sigma^{(3)}$. Hence, we get another triple $Z = (\Sigma^{(3)}, L^{(3)}, \{\beta_j^{(1)}\}_{j \neq j^{\mathrm{out}}} \cup \{\beta_j^{(2)}\}_{j \neq j^{\mathrm{in}}})$, which is said to be obtained by sewing X and Y along $\mathcal{C}_{j^{\mathrm{in}}}$ and $\mathcal{C}_{j^{\mathrm{out}}}$. More generally, given subsets $A \subset J^{\mathrm{out}}_{\Sigma^{(1)}}$ and $B \subset J^{\mathrm{in}}_{\Sigma^{(2)}}$ and a bijection $s\colon A \to B$, applying the previous sewing construction to each pair $(\mathcal{C}_j, \mathcal{C}_{s(j)})$ gives a triple Z, which is said to be obtained by sewing X and Y along s, see Figure 4.

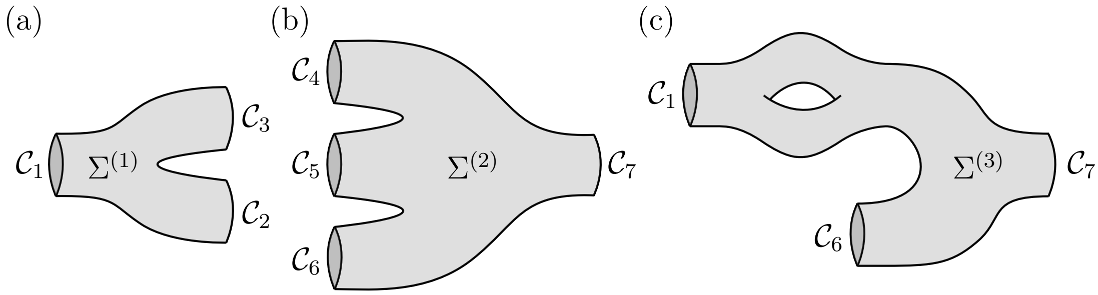

Figure 4: (a) The Riemann surface $\Sigma^{(1)}$ has the incoming boundary component $\mathcal{C}_1$ and outgoing components $\mathcal{C}_2$ and $\mathcal{C}_3$. (b) The surface $\Sigma^{(2)}$ has incoming boundary components $\mathcal{C}_4$-$\mathcal{C}_6$ and the outgoing boundary component $\mathcal{C}_7$. (c) The surface $\Sigma^{(3)}$ is obtained by sewing $\Sigma^{(1)}$ and $\Sigma^{(2)}$ together along the bijection taking $\mathcal{C}_3$ to $\mathcal{C}_4$ and $\mathcal{C}_2$ to $\mathcal{C}_5$.

Given a triple $X = (\Sigma, L, \{\beta_j\})$, let

$$\mathcal{K}_X = \bigoplus_{j \in J_\Sigma^{\mathrm{out}}} \mathcal{K} \oplus \bigoplus_{j \in J_\Sigma^{\mathrm{in}}} \mathcal{K},$$

where $\mathcal{K} = L^2(S^1)$ is the space of square integrable complex-valued functions on the circle S^1. Denoting by $A^\infty(\Sigma, L)$ the space of sections of L that are holomorphic in the interior of Σ and have restrictions to smooth sections of each $L|_{\mathcal{C}_j}$, define the Hardy space $H^2(X) \subset \mathcal{K}_X$ by

$$H^2(X) = \mathrm{cl}(\{\beta^* f \mid f \in A^\infty(\Sigma, L)\})$$

in analogy with the planar case, see e.g. [6]. Moreover, let $H^2(S^1) \subset L^2(S^1)$ be the standard Hardy space of the L^2 boundary values of holomorphic functions in the unit disk, given by

$$H^2(S^1) = \mathrm{cl}(\mathrm{span}\{\theta \mapsto e^{\mathbb{i}\theta k} \mid k \in \mathbb{Z}_{\geq 0}\}),$$

and let $q \in \mathscr{B}(L^2(S^1))$ be the projection onto $H^2(S^1)$.

The free fermion CFT is then defined as follows. Given a triple $X = (\Sigma, L, \{\beta_j\})$ as above, we associate to the boundary components $\coprod_{j\in J_\Sigma^{\mathrm{in}}} \mathcal{C}_j$ and $\coprod_{j\in J_\Sigma^{\mathrm{out}}} \mathcal{C}_j$ the unordered tensor products

$$\bigotimes_{j\in J_\Sigma^{\mathrm{in}}} \mathscr{F}_{\mathcal{K},q}, \bigotimes_{j\in J_\Sigma^{\mathrm{out}}} \mathscr{F}_{\mathcal{K},q},$$

respectively, where $\mathscr{F}_{\mathcal{K},q} = \mathscr{F}_{(q\mathcal{K})^*\oplus(q\mathcal{K})^\perp}$ is the Fock space from Chapter 2.4, and $\mathcal{K} = L^2(S^1)$ as above. To the bundle (Σ, L) we associate the vector space E_X of trace class maps

$$T_X \colon \bigotimes_{j\in J_\Sigma^{\mathrm{in}}} \mathscr{F}_{\mathcal{K},q} \to \bigotimes_{j\in J_\Sigma^{\mathrm{out}}} \mathscr{F}_{\mathcal{K},q} \tag{24}$$

of unordered tensor products, whose homogeneous parts satisfy the commutation relations

$$\pi_q(a(f^{\mathrm{out}}))T_X = (-1)^{P(T_X)}T_X\pi_q(a(f^{\mathrm{in}})) \quad \text{for all } (f^{\mathrm{out}}, f^{\mathrm{in}}) \in H^2(X), \tag{25}$$

$$\pi_q(a(g^{\mathrm{out}}))^*T_X = -(-1)^{P(T_X)}T_X\pi_q(a(g^{\mathrm{in}}))^* \quad \text{for all } (g^{\mathrm{out}}, g^{\mathrm{in}}) \in H^2(X)^\perp, \tag{26}$$

where π_q is the representation of the C^*-algebra $\mathrm{CAR}(\mathcal{K})$ from Chapter 2.4 and $P(T_X)$ is the parity of the operator T_X.

In particular, we stress that the Fock spaces $\mathscr{F}_{\mathcal{K},q}$ come with the super Hilbert space structure defined in Chapter 2.3. Moreover, by the discussion after Theorem 2.12, the maps in (24) are in a natural bijective correspondence with the maps

$$T_X \colon \mathscr{F}_{\bigoplus_{j\in J_\Sigma^{\mathrm{in}}}\mathcal{K}, \bigoplus_{j\in J_\Sigma^{\mathrm{in}}} q} \to \mathscr{F}_{\bigoplus_{j\in J_\Sigma^{\mathrm{out}}}\mathcal{K}, \bigoplus_{j\in J_\Sigma^{\mathrm{out}}} q}, \tag{27}$$

and the commutation relations in (25) and (26) are to be understood in this sense. To ease up the notation, we denote the spaces in (27) by $\mathscr{F}_\Sigma^{\mathrm{in}}$ and $\mathscr{F}_\Sigma^{\mathrm{out}}$, respectively.

The free fermion CFT can be shown to satisfy, among other things, the following properties.

Theorem 3.1 (Tener [28, Theorem 4.5]). *Let $X = (\Sigma^{(1)}, L^{(1)}, \{\beta_j^{(1)}\})$ and $Y = (\Sigma^{(2)}, L^{(2)}, \{\beta_j^{(2)}\})$ be as above. We have*

1. *The spaces E_X and E_Y are 1-dimensional.*
2. *If $A \subset J_{\Sigma^{(1)}}^{out}$, $B \subset J_{\Sigma^{(2)}}^{in}$ and $s \colon A \to B$ is a bijection, the triple Z obtained by sewing X and Y along s satisfies*

$$E_Z = \{(T_Y \widehat{\otimes} id_{A^C})(T_X \widehat{\otimes} id_{B^C}) \mid T_X \in E_X, T_Y \in E_Y\},$$

where

$$id_{A^C} = \bigotimes_{j \in J^{out}_{\Sigma^{(1)}} \setminus A} id_{\mathscr{F}_{\mathcal{K},q}}$$

and similarly for id_{B^C}. *In particular, if* $A = J^{out}_{\Sigma^{(1)}}$ *and* $B = J^{in}_{\Sigma^{(2)}}$, *we have*

$$E_Z = \{T_Y T_X \,\big|\, T_X \in E_X, T_Y \in E_Y\}.$$

3. *Suppose that there exists a bijection* $s\colon J_{\Sigma^{(1)}} \to J_{\Sigma^{(2)}}$ *and a holomorphic isomorphism* $\varphi\colon L^{(1)} \to L^{(2)}$ *such that* $\beta^{(2)}_{s(j)} = \varphi \circ \beta^{(1)}_j$ *for every* $j \in J_{\Sigma^{(1)}}$. *If* $U_{in}\colon \mathscr{F}^{in}_{\Sigma^{(1)}} \to \mathscr{F}^{in}_{\Sigma^{(2)}}$ *is the unitary map obtained by identifying* $J^{in}_{\Sigma^{(1)}}$ *and* $J^{in}_{\Sigma^{(2)}}$ *by* s *and similarly for* U_{out}, *then* $E_Y = U_{out} E_X U^*_{in}$.

4. *Suppose that for each* $j \in J_{\Sigma^{(1)}}$ *we are given a smooth isomorphism* $\varphi_j\colon S^1 \times \mathbb{C} \to S^1 \times \mathbb{C}$ *of line bundles such that* $\varphi_j|_{S^1}$ *is orientation-preserving. If* $Z = (\Sigma^{(1)}, L^{(1)}, \{\beta^{(1)}_j \circ \varphi_j^{-1}\})$, *then there exist unitary maps* $U_{in}\colon \mathscr{F}^{in}_{\Sigma^{(1)}} \to \mathscr{F}^{in}_{\Sigma^{(1)}}$ *and* $U_{out}\colon \mathscr{F}^{out}_{\Sigma^{(1)}} \to \mathscr{F}^{out}_{\Sigma^{(1)}}$ *such that* $E_Z = U_{out} E_X U^*_{in}$.

Properties 1. and 2. simply say that the construction is indeed functorial, and properties 3. and 4. are related to conformal invariance. Although we do not prove Theorem 3.1, we will show in Chapter 4 how property 1. can be reduced to the properties of the Hardy spaces $H^2(X)$. Moreover, we will investigate the correct BCFT analogies of properties 3. and 4. in Chapter 3.4.

Although it is not directly of interest to us, we briefly describe the physical motivation of the CFT construction. The idea is that we have a collection of strings in the form of circles $\mathcal{C}_j$ index by J^{in}_Σ, propagating through some manifold. The strings are allowed to merge into each other, and a single string can split into two, see Figure 4 (c). Eventually, we are left with a collection of strings indexed by J^{out}_Σ, and the strings have traced the surface Σ. Each string can vibrate in a number of different elementary modes, each describing an elementary particle. The vectors in the Fock spaces $\mathscr{F}_{\mathcal{H},q}$ have the usual interpretation as the vibrational states of the string, and the operators $\pi_q(a(f))$ and $\pi_q(a(f))^*$ create and annihilate vibrations in the elementary mode f, respectively. Moreover, the spaces $\mathscr{F}^{\text{in}}_\Sigma$ and $\mathscr{F}^{\text{out}}_\Sigma$ describe the states of the entire collections of incoming and outgoing strings, respectively. Finally, the operators $T\colon \mathscr{F}^{\text{in}}_\Sigma \to \mathscr{F}^{\text{out}}_\Sigma$ describe the time evolution of the state of the strings given the surface Σ they have traced. For more information, see e.g. [12] and references therein.

3.2 The transfer matrix on a Fock space

It turns out that if one writes the Ising model in a clever way as in [1,2], one can find natural counterparts for the state spaces $\mathscr{F}^{\text{in}}_\Sigma$ and $\mathscr{F}^{\text{out}}_\Sigma$, the time evolution operator T_X and the creation and annihilation operators $\pi_q(a(f))$ and $\pi_q(a(f))^*$ from the previous chapter. The goal of this section is to first show that the transfer matrix of the Ising model, that we will soon define, satisfies a commutation relation similar to (25), and that this relation uniquely determines the transfer matrix up to a scalar multiple. This is the content of Theorems 3.6 and 3.9, respectively. For simplicity, we work in a rectangular domain where the transfer matrix formalism is especially clear, but the results should generalize to more complicated domains as well. Moreover, we will impose locally monochromatic boundary conditions, as explained in Chapter 1, and set $\beta = \beta_C$ without further comments.

Apart from Theorems 3.6 and 3.9, this chapter can be regarded as a brief overview of the results in [1,2] that are of interest to us. In particular, we use the same notation for the ease of reference whenever it is not in conflict with our earlier conventions. Moreover, the present chapter is mostly self-contained apart from Lemmas 3.5 and 3.7 for the convenience of the reader.

Definitions

For the remainder of the chapter, let us fix an even $\ell \in \mathbb{Z}_{>0}$ and an arbitrary $h \in \mathbb{Z}_{>0}$. We will work with the Ising model on the subgraph of $\mathbb{Z}^2$ with nearest neighbor edges and the vertex set

$$\mathbb{S}^{(\ell,h)} = \mathcal{I} \times \{0, 1, \ldots, h\},$$

where $\mathcal{I} = \{-\ell/2, -\ell/2+1, \ldots, \ell/2-1, \ell/2\}$ is interpreted as the bottom row. The set of edges is denoted by $E(\mathbb{S}^{(\ell,h)})$, and we freely identify the vertices $w \in \mathbb{S}^{(\ell,h)}$ with the corresponding complex numbers and the edges $\{w_i, w_j\} \in E(\mathbb{S}^{(\ell,h)})$ with their midpoints. In particular, we write $\mathcal{I}^* = \{-\ell/2+1/2, -\ell/2+3/2, \ldots, \ell/2-1/2\}$ for the edges on the bottom row, and the left boundary of the edge set is

$$\partial_L E(\mathbb{S}^{(\ell,h)}) = \{-\ell/2 + \mathbb{i}y' \mid y' \in (\mathbb{Z}+1/2) \cap [0,h]\}$$

and similarly for the right boundary $\partial_R E(\mathbb{S}^{(\ell,h)})$ and the boundaries $\partial_L \mathbb{S}^{(\ell,h)}$ and $\partial_R \mathbb{S}^{(\ell,h)}$ of the vertex set. We write $\partial E(\mathbb{S}^{(\ell,h)})$ and $\partial \mathbb{S}^{(\ell,h)}$ for the union of the left and right boundaries of $E(\mathbb{S}^{(\ell,h)})$ and $\mathbb{S}^{(\ell,h)}$, respectively. We will also consider the graph with nearest neighbor edges and the vertex set $\mathbb{S}^{(\ell)} = \mathcal{I} \times \mathbb{Z}$, and we extend all the previous definitions to $\mathbb{S}^{(\ell)}$ in the obvious way.

In what follows, we will consider various maps $E(\mathbb{S}^{(\ell,h)}) \to \mathbb{C}$ and $E(\mathbb{S}^{(\ell)}) \to \mathbb{C}$. In particular, we are interested in the space $\mathbb{C}^{\mathcal{I}^*}$, which becomes a real Hilbert space under the inner product

$$\langle f, g \rangle = \Re \left(\sum_{x' \in \mathcal{I}^*} f(x') \overline{g(x')} \right). \tag{28}$$

The sum in (28) can be understood as a discrete integral

$$\sum_{x' \in \mathcal{I}^*} f(x') \overline{g(x')} =: \int_{\gamma_0}^{\#} f(z) \overline{g(z)} d^{\#} z,$$

where γ_0 is a straight path on $\mathcal{I}$ to be defined more precisely below, and the superscripts # remind us that we are working on a square lattice.

More generally, if $\mathbb{V}$ is a vector space, a **$\mathbb{V}$-valued discrete 1-form** ω on $E(\mathbb{S}^{(\ell,h)})$ is a formal expression

$$\omega = f_1 d^{\#} z + f_2 d^{\#} \overline{z},$$

where f_1 and f_2 are $\mathbb{V}$-valued functions on $E(\mathbb{S}^{(\ell,h)})$. Given a sequence $\gamma = (w_0, w_1, \dots, w_m) \subset (\mathbb{S}^{(\ell,h)})^{m+1}$ such that $z_j = \{w_j, w_{j-1}\} \in E(\mathbb{S}^{(\ell,h)})$ for every $j = 1, 2, \dots, m$, we define the integral of ω along γ as

$$\int_\gamma^{\#} \omega = \sum_{j=1}^{m} \Big(f_1(z_j)(w_j - w_{j-1}) + f_2(z_j)(\overline{w_j} - \overline{w_{j-1}}) \Big).$$

Moreover, the previous definition makes sense also when f_1 and f_2 are only defined on the edges z_j along the path γ, and we extend the definition to such cases.

A 1-form ω is said to be **closed** if $\int_\gamma^{\#} \omega = 0$ for all paths of the form

$$\gamma = (w, w+1, w+1+\mathbb{i}, w+\mathbb{i}, w),$$

and it is said to be **vertically slidable** if $\gamma \subset \partial_L \mathbb{S}^{(\ell,h)}$ and $\gamma \subset \partial_R \mathbb{S}^{(\ell,h)}$ both imply $\int_\gamma^{\#} \omega = 0$. It is easy to see that if $\gamma_0 = (-\ell/2, -\ell/2+1, \dots, \ell/2)$ is the path tracing $\mathcal{I}$ from left to right and γ_h is the corresponding path on $\mathcal{I} + \mathbb{i}h$, any closed and vertically slidable 1-form ω satisfies

$$\int_{\gamma_0}^{\#} \omega = \int_{\gamma_h}^{\#} \omega. \tag{29}$$

Moreover, the preceding discussion on 1-forms extends directly to the graph $\mathbb{S}^{(\ell)}$.

We will soon see that for a family of discrete 1-forms ω related to the Ising model, the closedness of ω is related to a discrete counterpart of holomorphicity and the vertical slidability of ω is related to boundary conditions. The suitable notion of holomorphicity in the context of the Ising model is the concept of s-holomorphicity.

Definition 3.2. A function $F\colon E(\mathbb{S}^{(\ell,h)}) \to \mathbb{C}$ is s-holomorphic if for every $z_1, z_2 \in E(\mathbb{S}^{(\ell,h)})$ that are adjacent to the same vertex $v \in \mathbb{S}^{(\ell,h)}$ and the same midpoint p of a face, we have

$$F(z_1) + \frac{\mathbb{i}|v-p|}{v-p}\overline{F(z_1)} = F(z_2) + \frac{\mathbb{i}|v-p|}{v-p}\overline{F(z_2)}. \tag{30}$$

On the other hand, the suitable boundary conditions are Riemann boundary values (RBV) on the boundary $\partial E(\mathbb{S}^{(\ell,h)})$.

Definition 3.3. A function $F\colon E(\mathbb{S}^{(\ell,h)}) \to \mathbb{C}$ has a Riemann boundary value at a point $x' \in \partial E(\mathbb{S}^{(\ell,h)})$ if

$$F(x') \in \mathbb{i}\tau(x')^{-1/2}\mathbb{R},$$

where $\tau(x') \in S^1$ is the complex number representing the unit tangent vector of the positively oriented boundary $\partial E(\mathbb{S}^{(\ell,h)})$.

In the present case, the previous definition clearly reduces to the requirement that $F(x') \in e^{-\mathbb{i}\pi/4}\mathbb{R}$ on $\partial_L E(\mathbb{S}^{(\ell,h)})$ and $F(x') \in e^{\mathbb{i}\pi/4}\mathbb{R}$ on $\partial_R E(\mathbb{S}^{(\ell,h)})$, but we will also use the same definition for more general domains in Chapters 3.3 and 3.4. Moreover, Definitions 3.2 and 3.3 have natural counterparts for $E(\mathbb{S}^{(\ell)})$.

The transfer matrix

We proceed to discuss the transfer matrix formalism of the Ising model. The basic rationale is that since the number of summations in Equations (2) and (3) for the partition function and the correlation functions increases exponentially with the size of the domain, one seeks more systematic ways of computation. Specializing to the case of the Ising model on $\mathbb{S}^{(\ell,h)}$, we can summarize the procedure as follows. One first writes the summation over $\sigma \in S^{\mathrm{BC}}_{\mathbb{S}^{(\ell,h)}}$ in (2) and (3) as repeated summation over spin configurations on $\mathcal{I}+y$ for $y=0,\ldots,h$, and then interprets resulting repeated sums as products of a matrix T, known as the transfer matrix. For more information, see e.g. [20, Chapter 1.1].

More precisely, we have a finite-dimensional complex Hilbert space $\widetilde{V}$ with a basis consisting of vectors u_ρ indexed by the spin configurations ρ on $\mathcal{I}$, that is

$$\widetilde{V} = \operatorname{span}_{\mathbb{C}} \left\{ u_\rho \;\middle|\; \rho \in \{\pm 1\}^{\mathcal{I}} \right\}.$$

Following the notation in [1], we denote the Hermitian conjugate of a matrix A by $A^\dagger$, so that the inner product of u_ρ and u_τ is $u_\rho^\dagger u_\tau$. Specializing further to the case of locally monochromatic boundary conditions on $\partial\mathbb{S}^{(\ell,h)}$, the transfer matrix $T\colon \widetilde{V} \to \widetilde{V}$ is given by $T = T_{\mathrm{hor}}^{1/2} T_{\mathrm{ver}} T_{\mathrm{hor}}^{1/2}$, where T_{hor} is the matrix with elements

$$u_\tau^\dagger T_{\mathrm{hor}} u_\rho = \exp\Big(\beta \sum_{x=-\ell/2}^{\ell/2-1} \rho(x)\rho(x+1)\Big)\delta_{\tau,\rho} \tag{31}$$

describing the energy contribution of the configuration ρ on the horizontal row $\mathcal{I}$, and T_{ver} is the matrix with elements

$$u_\tau^\dagger T_{\mathrm{ver}} u_\rho = \exp\Big(\beta \sum_{x=-\ell/2}^{\ell/2} \tau(x)\rho(x)\Big)\delta_{\tau(-\ell/2),\rho(-\ell/2)}\delta_{\tau(\ell/2),\rho(\ell/2)} \tag{32}$$

describing the interaction of adjacent horizontal rows with spin configurations τ and ρ, respectively. For concrete expressions of the partition function and the correlation functions in terms of T, see [1, Theorem 5.1].

It is clear from (31) and (32) that T is invertible since T_{hor} and T_{ver} are positive-definite. Since the Ising model is symmetric under the global spin flip $\sigma \mapsto -\sigma$, we restrict our attention to the subspace $V \subset \widetilde{V}$ spanned by those u_ρ with $\rho(\ell/2) = 1$. It is clear from (31) and (32) that T restricts to a map $T\colon V \to V$, and the restrictions of T to V and $V^\perp$ are unitarily equivalent. Since $T\colon V \to V$ is symmetric and has positive entries, it follows from the Perron-Frobenius theorem that, up to a positive scalar multiple, T has a unique eigenvector $v_\emptyset \in V$ with positive entries, and the corresponding eigenvalue $\mu_\emptyset$ is maximal.

A Clifford algebra representation

The analogue of the state spaces $\mathscr{F}_\Sigma^{\mathrm{in}}$ and $\mathscr{F}_\Sigma^{\mathrm{out}}$ in the present setting is the space V, and the analogue of the time evolution operator is a suitable power of transfer matrix $T\colon V \to V$. However, it takes some work to define a Fock space structure on V and find the corresponding creation and annihilation operators. We start by defining a representation of a suitable finite-dimensional Clifford algebra on V. For each $x' \in \mathcal{I}^*$ we define a map $\mathsf{fold}_{<x'}\colon \{\pm 1\}^{\mathcal{I}} \to \{\pm 1\}^{\mathcal{I}}$ by $\mathsf{fold}_{<x'}(\rho) = \rho'$, where $\rho'(x) = -\rho(x)$ if $x < x'$ and $\rho'(x) = \rho(x)$ if $x > x'$. This allows us to define maps $\psi_{x'}, \psi^*_{x'}\colon \widetilde{V} \to \widetilde{V}$ for each $x' \in \mathcal{I}$ by setting

$$\begin{aligned} \psi_{x'} u_\rho &= \frac{-\rho(x'-\frac{1}{2}) + \mathbb{i}\rho(x'+\frac{1}{2})}{\sqrt{2}} u_{\mathsf{fold}_{<x'}(\rho)}, \\ \psi^*_{x'} u_\rho &= \frac{-\mathbb{i}\rho(x'-\frac{1}{2}) + \rho(x'+\frac{1}{2})}{\sqrt{2}} u_{\mathsf{fold}_{<x'}(\rho)} \end{aligned}$$

and extending linearly. A quick computation gives the anticommutators

$$\{\psi_{x'_1}, \psi_{x'_2}\} = -2\delta_{x'_1,x'_2}\mathrm{id}_{\widetilde{V}}, \quad \{\psi^*_{x'_1}, \psi^*_{x'_2}\} = 2\delta_{x'_1,x'_2}\mathrm{id}_{\widetilde{V}}, \quad \{\psi_{x'_1}, \psi^*_{x'_2}\} = 0 \tag{33}$$

and the Hilbert space adjoints

$$(\psi_{x'})^\dagger = -\psi_{x'}, \quad (\psi^*_{x'})^\dagger = \psi^*_{x'}. \tag{34}$$

Following [1], we define

$$\mathrm{CliffGen} = \mathrm{span}_{\mathbb{C}}\left(\{\psi_{x'} \;\middle|\; x' \in I^*\} \cup \{\psi^*_{x'} \;\middle|\; x' \in I^*\}\right),$$

where the rationale is that CliffGen is an analogue of the Clifford algebra in Definition 2.7 with the exception that we allow the space $\mathcal{H}$ to be complex, and the bilinear form $\langle \cdot, R\cdot\rangle$ is replaced with the anticommutator $\{\cdot,\cdot\}$. However, we will not use this fact.

The operators $\psi_{x'}$ and $\psi^*_{x'}$ allow us to turn the space V into a Fock space representation of the Clifford algebra $\mathrm{Cl}(\mathbb{C}^{\mathcal{I}^*}, \mathsf{R})$ as in Definition 2.7 and (17), where $\mathsf{R}\colon \mathbb{C}^{\mathcal{I}^*} \to \mathbb{C}^{\mathcal{I}^*}$ is the map $f \mapsto -\mathbb{i}\overline{f}$. For every $f \in C^{\mathcal{I}^*}$, we define $\phi(f)\colon V \to V$ by

$$\phi(f) = \frac{e^{\mathbb{i}\frac{\pi}{4}}}{2} \int_{\gamma_0}^{\#} \left(\mathbb{i}f(z)\psi(z)d^{\#}z - \mathbb{i}\overline{f(z)}\psi^*(z)d^{\#}\overline{z}\right), \tag{35}$$

where $\gamma_0 = (-\ell/2, -\ell/2+1, \ldots, \ell/2)$. Indeed, the operators $\phi(f)$ define a representation of the Clifford algebra $\mathrm{Cl}(\mathbb{C}^{\mathcal{I}^*}, \mathsf{R})$ and moreover, they satisfy a further important property of the representations ϕ_p from Chapter 2.4.

Theorem 3.4. *For every $f, g \in \mathbb{C}^{\mathcal{I}^*}$, we have $\{\phi(f), \phi(g)\} = \langle \mathsf{R}f, g\rangle$ and $(\phi(f))^\dagger = \phi(\mathsf{R}f)$. In particular, we have $\phi(f)^2 = \frac{1}{2}\langle \mathsf{R}f, f\rangle$.*

Proof. The property $(\phi(f))^\dagger = \phi(\mathsf{R}f)$ follows directly from the formulas for the adjoints $(\psi_{x'})^\dagger$ and $(\psi^*_{x'})^\dagger$ in (34). Moreover, using the anticommutators of the operators $(\psi_{x'})^\dagger$ and $(\psi^*_{x'})^\dagger$ given in (33), we compute

$$\begin{aligned}\{\phi(f),\phi(g)\} &= \frac{\mathbb{i}}{4}\int_{\gamma_0}^{\#}\left(2f(z)g(z)d^{\#}z - 2\overline{f(z)g(z)}d^{\#}\overline{z}\right)\\ &= \frac{1}{2}\int_{\gamma_0}^{\#}\left(\overline{\mathsf{R}f(z)}g(z) + \mathsf{R}f(z)\overline{g(z)}\right)d^{\#}z\\ &= \Re\Big(\int_{\gamma_0}^{\#}\mathsf{R}f(z)\overline{g(z)}d^{\#}z\Big)\\ &= \langle\mathsf{R}f, g\rangle.\end{aligned}$$

■

A necessary condition for the transfer matrix

As the notation in (35) might suggest, the operators $\phi(f)$ can indeed be defined as integrals of globally defined CliffGen-valued discrete 1-forms, and this fact lies at the heart of the present chapter. Hence, we would like to extend $\psi_{x'}$ and $\psi^*_{x'}$ to maps $\psi, \psi^*\colon E(\mathbb{S}^{(\ell)}) \to$ CliffGen. The first step is to note that $T^{-1}\psi_{x'}T, T^{-1}\psi^*_{x'}T \in$ CliffGen for every $x' \in \mathcal{I}^*$, which can be shown by a direct but rather involved computation. Alternatively, see [1, Proposition 3.4]. This allows us to define

$$\psi(x' + \mathbb{i}y) = T^{-y}\psi_{x'}T^{y} \quad \text{and} \quad \psi^*(x' + \mathbb{i}y) = T^{-y}\psi^*_{x'}T^{y} \tag{36}$$

on the horizontal lines $\mathcal{I}^* + \mathbb{i}y$ for $y \in \mathbb{Z}$. It turns out that the maps in (36) have very convenient extensions.

Lemma 3.5. *At $\beta = \beta_C$, the maps in (36) have extensions $\psi, \psi^*\colon E(\mathbb{S}^{(\ell)}) \to$ CliffGen such that for every $F\colon E(\mathbb{S}^{(\ell)}) \to \mathbb{C}$, the CliffGen-valued discrete 1-form*

$$\mathbb{i}F(z)\psi(z)d^{\#}z - \mathbb{i}\overline{F(z)}\psi^*(z)d^{\#}\overline{z}$$

is closed if F is s-holomorphic and vertically slidable if F has RBV on $\partial E(\mathbb{S}^{(\ell)})$.

Proof. See [1, Chapter 3.2-3.3]. ■

Given a function $F\colon E(\mathbb{S}^{(\ell,h)}) \to \mathbb{C}$, the restrictions $f^{\mathrm{in}} = F|_{\mathcal{I}^*}$ and $f^{\mathrm{out}} = F|_{\mathcal{I}^*+\mathbb{i}h}$ have a clear interpretation as maps $f^{\mathrm{in}}, f^{\mathrm{out}}\colon \mathcal{I}^* \to \mathbb{C}$. Moreover, we can associate the spaces $V^{\mathrm{in}} = V$ and $V^{\mathrm{out}} = V$ to the boundaries $\mathcal{I}^*$ and $\mathcal{I}^* + \mathbb{i}h$ at the bottom and top of $E(\mathbb{S}^{(\ell,h)})$, respectively. The matrix $T^h\colon V \to V$ can then be interpreted as a map $T^h\colon V^{\mathrm{in}} \to V^{\mathrm{out}}$, and it follows from Lemma 3.5 that the operator T^h satisfies a property similar to (25).

Theorem 3.6. *Suppose that $F\colon E(\mathbb{S}^{(\ell,h)}) \to \mathbb{C}$ is an s-holomorphic map with RBV on $\partial E(\mathbb{S}^{(\ell,h)})$ and let $f^{in} = F|_{\mathcal{I}^*}$ and $f^{out} = F|_{\mathcal{I}^*+\mathbb{i}h}$. The operator T^h satisfies the commutation relation*

$$\phi(f^{out})T^h = T^h\phi(f^{in}). \tag{37}$$

Proof. It follows from Lemma 3.5 that the 1-form

$$\mathbb{i}F(z)\psi(z)d^{\#}z - \mathbb{i}\overline{F(z)}\psi^*(z)d^{\#}\overline{z}$$

is closed and vertically slidable, so we can use the observation in (29) and the property $T^h\psi_{x'}T^{-h} = \psi(x' - \mathbb{i}h)$ to compute

$$\begin{aligned}
T^h\phi(f^{\text{in}}) &= T^h\frac{e^{\mathbb{i}\frac{\pi}{4}}}{2}\int_{\gamma_0}^{\#}\left(\mathbb{i}f^{\text{in}}(z)\psi(z)d^{\#}z - \mathbb{i}\overline{f^{\text{in}}(z)}\psi^*(z)d^{\#}\overline{z}\right) \\
&= T^h\frac{e^{\mathbb{i}\frac{\pi}{4}}}{2}\int_{\gamma_0}^{\#}\left(\mathbb{i}F(z)\psi(z)d^{\#}z - \mathbb{i}\overline{F(z)}\psi^*(z)d^{\#}\overline{z}\right) \\
&= T^h\frac{e^{\mathbb{i}\frac{\pi}{4}}}{2}\int_{\gamma_h}^{\#}\left(\mathbb{i}F(z)\psi(z)d^{\#}z - \mathbb{i}\overline{F(z)}\psi^*(z)d^{\#}\overline{z}\right) \\
&= \frac{e^{\mathbb{i}\frac{\pi}{4}}}{2}\int_{\gamma_h}^{\#}\left(\mathbb{i}F(z)\psi(z-\mathbb{i}h)d^{\#}z - \mathbb{i}\overline{F(z)}\psi^*(z-\mathbb{i}h)d^{\#}\overline{z}\right)T^h \\
&= \frac{e^{\mathbb{i}\frac{\pi}{4}}}{2}\int_{\gamma_0}^{\#}\left(\mathbb{i}F(z+\mathbb{i}h)\psi(z)d^{\#}z - \mathbb{i}\overline{F(z+\mathbb{i}h)}\psi^*(z)d^{\#}\overline{z}\right)T^h \\
&= \frac{e^{\mathbb{i}\frac{\pi}{4}}}{2}\int_{\gamma_0}^{\#}\left(\mathbb{i}f^{\text{out}}(z)\psi(z)d^{\#}z - \mathbb{i}\overline{f^{\text{out}}(z)}\psi^*(z)d^{\#}\overline{z}\right)T^h \\
&= \phi(f^{\text{out}})T^h.
\end{aligned}$$

■

A sufficient condition for the transfer matrix

To see that (37) defines the operator T^h up to a scalar multiple, we build an orthonormal basis of V in terms of the operators $\phi(f)$ as in (19). The first step is to find a convenient basis of the real Hilbert space $\mathbb{C}^{\mathcal{I}^*}$. In what follows, let

$$\mathscr{K}^{(\ell)} = \left\{\frac{1}{2}, \frac{3}{2}, \ldots, \left(\ell - \frac{1}{2}\right)\right\} \tag{38}$$

and let

$$\pm\mathscr{K}^{(\ell)} = (\mathscr{K}^{(\ell)}) \cup (-\mathscr{K}^{(\ell)}) = \left\{\pm\frac{1}{2}, \pm\frac{3}{2}, \ldots, \pm\left(\ell - \frac{1}{2}\right)\right\}.$$

Lemma 3.7. *For every $k \in \pm\mathscr{K}^{(\ell)}$, there is an s-holomorphic function $\mathfrak{F}_k\colon E(\mathbb{S}^{(\ell)}) \to \mathbb{C}$ with RBV on $\partial E(\mathbb{S}^{(\ell)})$ and a constant $\Lambda_k^{(\ell)} \in \mathbb{R}$ depending on ℓ such that the following property holds. For every $z \in E(\mathbb{S}^{(\ell)})$ and $h \in \mathbb{Z}$ we have*

$$\mathfrak{F}_k(z + \mathbb{i}h) = (\Lambda_k^{(\ell)})^h\mathfrak{F}_k(z), \tag{39}$$

and the constant $\Lambda_k^{(\ell)}$ satisfies $\Lambda_k^{(\ell)} < 1$ for $k \in -\mathscr{K}^{(\ell)}$ and $\Lambda_k^{(\ell)} > 1$ for $k \in \mathscr{K}^{(\ell)}$. Moreover, the restrictions $\mathfrak{f}_k = \mathfrak{F}_k|_{\mathcal{I}^*}$ satisfy

$$\mathsf{R}\mathfrak{f}_k = \mathfrak{f}_{-k},$$

and the set $\{\mathfrak{f}_k \mid k \in \pm\mathscr{K}^{(\ell)}\}$ is an orthonormal basis of $\mathbb{C}^{\mathcal{I}^*}$.

Proof. See [2, Proposition 3.7]. ■

Property (39) together with the fact that the eigenvalue $\mu_\emptyset$ associated to the Perron-Frobenius eigenvector $v_\emptyset \in V$ of T is maximal allows us to find a very convenient orthonormal basis of V.

Theorem 3.8. *Let $v_\emptyset \in V$ be the Perron-Frobenius eigenvector of T normalized so that $\|v_\emptyset\| = 1$ and let $\mu_\emptyset$ be the associated eigenvalue. For every $k \in \mathscr{K}^{(\ell)}$ we have $\phi(\mathfrak{f}_k)v_\emptyset = 0$, and up to scalar multiple, $v_\emptyset$ is the unique vector with this property. Moreover, the set*

$$\left\{ \prod_{j=1}^{m} \phi(\mathfrak{f}_{k_j})v_\emptyset \;\middle|\; \{k_1, \ldots, k_m\} \subseteq -\mathscr{K}^{(\ell)}, k_1 > \ldots > k_m \right\} \tag{40}$$

is an orthonormal basis of V, and $\prod_{j=1}^{m} \phi(\mathfrak{f}_{k_j})v_\emptyset$ is an eigenvector of T with eigenvalue $\mu_\emptyset \prod_{j=1}^{m} \Lambda_{k_j}^{(\ell)}$.

Proof. We begin by showing that $\phi(\mathfrak{f}_k)v_\emptyset = 0$ for every $k \in \mathscr{K}^{(\ell)}$. Given the function $\mathfrak{F}_k$ from Lemma 3.7, we have $\mathfrak{F}_k|_{\mathcal{I}^*} = \mathfrak{f}_k$ and $\mathfrak{F}_k|_{\mathcal{I}^*+\mathbb{i}} = \Lambda_k^{(\ell)}\mathfrak{f}_k$, so it follows from Theorem 3.6 that $T\phi(\mathfrak{f}_k)T^{-1} = \Lambda_k^{(\ell)}\phi(\mathfrak{f}_k)$. Hence, we obtain

$$T\phi(\mathfrak{f}_k)v_\emptyset = (T\phi(\mathfrak{f}_k)T^{-1})Tv_\emptyset = \Lambda_k^{(\ell)}\mu_\emptyset\phi(\mathfrak{f}_k)v_\emptyset,$$

so we must have $\phi(\mathfrak{f}_k)v_\emptyset = 0$ since the eigenvalue $\mu_\emptyset$ is maximal and $\Lambda_k^{(\ell)} > 1$ for $k \in \mathscr{K}^{(\ell)}$. Iterating the same argument, we see that $\prod_{j=1}^{m} \phi(\mathfrak{f}_{k_j})v_\emptyset$ is an eigenvector of T with eigenvalue $\mu_\emptyset \prod_{j=1}^{m} \Lambda_{k_j}^{(\ell)}$ unless $\prod_{j=1}^{m} \phi(\mathfrak{f}_{k_j})v_\emptyset = 0$.

To show that (40) is an orthonormal basis of V, we first observe that $\phi(\mathfrak{f}_k)^\dagger = \phi(\mathsf{R}\mathfrak{f}_k) = \phi(\mathfrak{f}_{-k})$ and

$$\{\phi(\mathfrak{f}_{k_1}), \phi(\mathfrak{f}_{k_2})\} = \langle \mathsf{R}\mathfrak{f}_{k_1}, \mathfrak{f}_{k_2}\rangle \mathrm{id}_{\mathbb{C}^{\mathcal{I}^*}} = \langle \mathfrak{f}_{-k_1}, \mathfrak{f}_{k_2}\rangle \mathrm{id}_{\mathbb{C}^{\mathcal{I}^*}} = \delta_{-k_1,k_2}\mathrm{id}_{\mathbb{C}^{\mathcal{I}^*}} \tag{41}$$

by Theorem 3.4. Hence, given subsets $\mathscr{K}_1 = \{k_1^{(1)}, \ldots, k_m^{(1)}\}, \mathscr{K}_2 = \{k_1^{(2)}, \ldots, k_n^{(2)}\} \subseteq -\mathscr{K}^{(\ell)}$, the inner products

$$\left\langle \prod_{i=1}^{m} \phi(\mathfrak{f}_{k_i^{(1)}})v_\emptyset, \prod_{j=1}^{n} \phi(\mathfrak{f}_{k_j^{(2)}})v_\emptyset \right\rangle = \left\langle \prod_{j=n}^{1} \phi(\mathfrak{f}_{-k_j^{(2)}}) \prod_{i=1}^{m} \phi(\mathfrak{f}_{k_i^{(1)}})v_\emptyset, v_\emptyset \right\rangle \tag{42}$$

can be computed by moving the operators $\phi(\mathfrak{f}_{-k_j^{(2)}})$ all the way to the right using (41), where they annihilate the vector $v_\emptyset$, i.e. $\phi(\mathfrak{f}_{-k_j^{(2)}})v_\emptyset = 0$. If there exists $k_j^{(2)} \notin \mathscr{K}_1$, moving $\phi(\mathfrak{f}_{-k_j^{(2)}})$ to the right gives no non-vanishing anticommutators, and the inner product (42) is seen to be 0. On the other hand, if $\mathscr{K}_1 = \mathscr{K}_2$, the inner product (42) is easily seen to be $\langle v_\emptyset, v_\emptyset\rangle = 1$. Hence, (40) is at least an orthonormal set, and since the dimension of V and the cardinality of (40) are both 2^ℓ, we see that (40) is actually an orthonormal basis of V.

Finally, to show that the relation $\phi(\mathfrak{f}_k)v_\emptyset = 0$ for every $k \in \mathscr{K}^{(\ell)}$ defines $v_\emptyset$ up to a scalar multiple, we observe that for any $k \in \mathscr{K}^{(\ell)}$ and $k_1, \ldots, k_n \in -\mathscr{K}^{(\ell)}$ with $k_1 > \ldots > k_n$, the anticommutation relation (41) gives

$$\phi(\mathfrak{f}_k) \prod_{j=1}^{n} \phi(\mathfrak{f}_{k_j}) v_\emptyset = \sum_{j=1}^{n} \delta_{-k,k_j} (-1)^{j-1} \prod_{i \in \{1,\ldots,n\} \setminus \{j\}} \phi(\mathfrak{f}_{k_i}) v_\emptyset.$$

Hence, each $\phi(\mathfrak{f}_k)$ maps each basis vector in (40) to zero or another vector in (40) up to a sign, and two distinct basis vectors never get mapped to the same basis vector. Since $v_\emptyset$ is the only vector in (40) annihilated by every $\phi(\mathfrak{f}_k)$ with $k \in \mathscr{K}^{(\ell)}$, the claim follows. ∎

If $p^{\#} \colon \mathbb{C}^{\mathcal{I}^*} \to \mathbb{C}^{\mathcal{I}^*}$ is the projection onto $\mathrm{span}_{\mathbb{R}}\{\mathfrak{f}_k \mid k \in -\mathscr{K}^{(\ell)}\}$, Theorem 3.8 gives an isomorphism $V \cong \mathscr{F}_{p^{\#}\mathbb{C}^{\mathcal{I}^*}}$ defined by $\prod_{j=1}^{n} \phi(\mathfrak{f}_{k_j}) v_\emptyset \mapsto \mathfrak{f}_{k_1} \wedge \ldots \wedge \mathfrak{f}_{k_n}$. Hence, the operators $\phi(f)$ give the representation $\phi_{p^{\#}}$ of $\mathrm{Cl}(\mathbb{C}^{\mathcal{I}^*}, \mathsf{R})$ from Chapter 2.4, and $v_\emptyset$ is the vacuum vector. It is now easy to see that (37) defines the operator T^h up to a scalar multiple.

Theorem 3.9. *Suppose that $A \colon V \to V$ is a linear map satisfying*

$$\phi(f^{out})A = A\phi(f^{in}) \tag{43}$$

for every s-holomorphic $F \colon E(\mathbb{S}^{(\ell,h)}) \to \mathbb{C}$ with RBV on $\partial E(\mathbb{S}^{(\ell,h)})$, where $f^{in} = F|_{\mathcal{I}^*}$ and $f^{out} = F|_{\mathcal{I}^* + \mathbb{i}h}$. Then we have $A = CT^h$ for some $C \in \mathbb{C}$.

Proof. Choosing $F = \mathfrak{F}_k$ for $k \in \mathscr{K}^{(\ell)}$, we get

$$0 = A\phi(\mathfrak{f}_k)v_\emptyset = \Lambda_k^{(\ell)} \phi(\mathfrak{f}_k) A v_\emptyset$$

by Lemma 3.7, which implies that $\phi(\mathfrak{f}_k)Av_\emptyset = 0$ for every $k \in \mathscr{K}^{(\ell)}$. Hence, it follows from Theorem 3.8 that $Av_\emptyset = Cv_\emptyset$ for some $C \in \mathbb{C}$. Similarly, we have for the elements in the basis (40) that

$$A \prod_{j=1}^{n} \phi(\mathfrak{f}_{k_j}) v_\emptyset = \prod_{j=1}^{n} \Lambda_{k_j}^{(\ell)} \phi(\mathfrak{f}_{k_j}) A v_\emptyset = C \left(\prod_{j=1}^{n} \Lambda_{k_j}^{(\ell)} \right) \prod_{j=1}^{n} \phi(\mathfrak{f}_{k_j}) v_\emptyset = \frac{C}{\mu_\emptyset} T^h \prod_{j=1}^{n} \phi(\mathfrak{f}_{k_j}) v_\emptyset,$$

so we get $A = C/\mu_\emptyset T^h$. ∎

3.3 The BCFT in a rectangle

In Chapter 3.1 we have seen the free fermion CFT, and in Chapter 3.2 we have seen that all the objects in the free fermion CFT have natural counterparts in the Ising model, at least on the graph $\mathbb{S}^{(\ell,h)}$. In this section, we will then consider the natural continuum counterpart of the picture that we obtained for the Ising model in the previous chapter. Until Chapter 3.4, we restrict our attention to simply connected domains with only one incoming and one outgoing boundary component for simplicity. This allows us to forget the super Hilbert space structure on the state spaces associated with the boundary components.

Furthermore, we restrict our attention to domains $\mathscr{D} \subset \mathbb{C}$ that are bounded by line segments $\mathcal{I}^{\text{in}}$ and $\mathcal{I}^{\text{out}}$ parallel to the real axis $\mathbb{R} \subset \mathbb{C}$ and analytic curves $\partial_L \mathscr{D}$ and $\partial_R \mathscr{D}$ as in Figure 5. Moreover, we require that the line segments $\mathcal{I}^{\text{in}}$ and $\mathcal{I}^{\text{out}}$ are isometric copies of the unit interval $I := [-1/2, 1/2] \subset \mathbb{R}$, and we equip them with parametrizations $\beta^{\text{in}} \colon I \to \mathcal{I}^{\text{in}}$ and $\beta^{\text{out}} \colon I \to \mathcal{I}^{\text{out}}$. In principle, we could allow arbitrary smooth parametrizations, and this possibility is discussed in Chapter 3.4. For now, however, we parametrize $\mathcal{I}^{\text{in}}$ and $\mathcal{I}^{\text{out}}$ by arc length so that β^{in} is orientation-preserving and β^{out} is orientation-reversing, where $\partial \mathscr{D}$ is given the standard counterclockwise orientation. We consider $\mathcal{I}^{\text{in}}$ as an incoming boundary component and $\mathcal{I}^{\text{out}}$ as an outgoing boundary component, so the boundary components are divided into incoming and outgoing ones according to the orientation induced by β in a way precisely opposite to the one in Chapter 3.1. For the remainder of this chapter, we call a pair $X = (\mathscr{D}, \{\beta^{\text{in}}, \beta^{\text{out}}\})$ as above a **domain with two parametrized boundary components**.

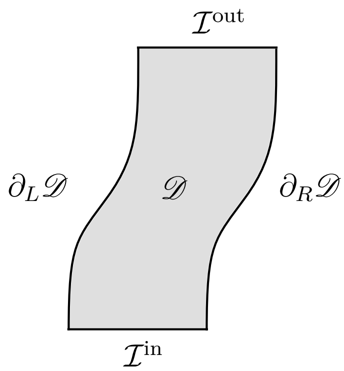

Figure 5: A simply connected domain $\mathscr{D}$ with incoming boundary component $\mathcal{I}^{\text{in}}$ and outgoing boundary component $\mathcal{I}^{\text{out}}$.

At the end of Chapter 3.2, we identified the transfer matrix state space with the Fock space $\mathscr{F}_{p^\# \mathbb{C}^{\mathcal{I}^*}}$, where $p^\# \colon \mathbb{C}^{\mathcal{I}^*} \to \mathbb{C}^{\mathcal{I}^*}$ is the projection to the subspace $\operatorname{span}\{\mathfrak{f}_k \mid k \in -\mathscr{K}^{(\ell)}\}$. The natural continuum analogue of the set $\mathcal{I}^*$ of edges in a horizontal row is the unit interval I, and we take the strip $\mathbb{S} = I \times \mathbb{R} \subset \mathbb{C}$ as the continuum counterpart of the graph $\mathbb{S}^{(\ell)}$, so that the continuum counterpart of the graph $\mathbb{S}^{(\ell,h)}$ is the rectangular domain $\mathbb{S}^{(h)} = I \times [0, h] \subset \mathbb{C}$. As the continuum counterpart of the real Hilbert space $\mathbb{C}^{\mathcal{I}^*}$, we consider the real Hilbert space $\mathcal{H} := L^2(I, \mathbb{C})$ with the inner product

$$\langle f, g \rangle = \Re \left(\int_I f(x) \overline{g(x)} dx \right),$$

where dx stands for the Lebesgue measure.

To find the continuum analogue of the projection $p^\# \colon \mathbb{C}^{\mathcal{I}^*} \to \mathbb{C}^{\mathcal{I}^*}$, we recall that for each $k \in \pm\mathscr{K}^{(\ell)}$, the function $\mathfrak{f}_k \in \mathbb{C}^{\mathcal{I}^*}$ has the s-holomorphic extension $\mathfrak{F}_k \colon E(\mathbb{S}^{(\ell)}) \to \mathbb{C}$ with RBV on $\partial E(\mathbb{S}^{(\ell)})$, and it follows from (30) that $\mathfrak{F}_k$ is the unique s-holomorphic extension of $\mathfrak{f}_k$ with RBV on $\partial\mathbb{S}^{(\ell)}$. In addition, for every $z \in E(\mathbb{S}^{(\ell)})$ and $h \in \mathbb{Z}$, we have

$$\mathfrak{F}_k(z + \mathbb{i}h) = (\Lambda_k^{(\ell)})^h \mathfrak{F}_k(z),$$

where $\Lambda_k^{(\ell)} < 1$ for $k \in -\mathscr{K}^{(\ell)}$ and $\Lambda_k^{(\ell)} > 1$ for $k \in \mathscr{K}^{(\ell)}$. It follows that $\mathfrak{F}_k(x' + \mathbb{i}h) \to 0$ as $h \to +\infty$ for every $x' \in \mathcal{I}^*$ if $k \in -\mathscr{K}^{(\ell)}$ and $|\mathfrak{F}_k(x + \mathbb{i}h)| \to +\infty$ if $k \in \mathscr{K}^{(\ell)}$. Given that the set $\{\mathfrak{f}_k \mid k \in \pm\mathscr{K}^{(\ell)}\}$ is an orthonormal basis of $\mathbb{C}^{\mathcal{I}^*}$, it is not difficult to see that $p^\#$ is the projection onto the subspace of functions $F|_{\mathcal{I}^*}$, where $F\colon E(\mathbb{S}^{(\ell)}) \to \mathbb{C}$ is s-holomorphic with RBV on $\partial E(\mathbb{S}^{(\ell)})$ and $F(x' + \mathbb{i}h) \to 0$ as $h \to +\infty$ for all $x' \in \mathcal{I}^*$. Indeed, the proof of this fact is essentially the same as that of Lemma 3.11 below.

The previous characterization of the range of $p^\#$ has an immediate continuum version, but we first have to define Riemann boundary values for maps on planar domains.

Definition 3.10. Let $\mathscr{D} \subset \mathbb{C}$ be a planar domain with piecewise smooth boundary $\partial\mathscr{D}$ and let $x \in \partial\mathscr{D}$ be a point such that $\partial\mathscr{D}$ is smooth in a neighborhood of x. A function $F\colon \overline{\mathscr{D}} \to \mathbb{C}$ has a Riemann boundary value at the point x if

$$F(x) \in \mathbb{i}\tau(x)^{-1/2}\mathbb{R},$$

where $\tau(x) \in S^1$ is the complex number representing the unit tangent vector of the positively oriented boundary $\partial\mathscr{D}$.

If $A^\infty(\mathbb{S})$ is the real vector space of holomorphic functions $F\colon \mathbb{S} \to \mathbb{C}$ with RBV on $\partial\mathbb{S}$ that satisfy $F(x + \mathbb{i}h) \to 0$ as $h \to +\infty$ for every $x \in I$, we define

$$H^2(I) := \mathrm{cl}(\{F|_I \mid F \in A^\infty(\mathbb{S})\}) \subset \mathcal{H},$$

where we recall that $\mathcal{H} = L^2(I, \mathbb{C})$ as above. For the remainder of this chapter, we denote by p the projection onto the subspace $H^2(I) \subset \mathcal{H}$. Moreover, we define a unitary involution $\mathsf{R}\colon \mathcal{H} \to \mathcal{H}$ by $Rf = -\mathbb{i}\overline{f}$.

To make concrete computations, we need a practical orthonormal basis of $H^2(I)$. Fortunately, an orthonormal basis similar to that in Lemma 3.7 is given in [2]. Let

$$\mathscr{K} := (\mathbb{Z} + \frac{1}{2}) \cap [0, +\infty) = \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\right\}$$

and let

$$\pm\mathscr{K} := \mathscr{K} \cup (-\mathscr{K}) = \left\{\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \dots\right\}.$$

For each $k \in \pm\mathscr{K}$, define a function $E_k\colon \mathbb{S} \to \mathbb{C}$ by

$$E(x + \mathbb{i}y) := C_k e^{-\mathbb{i}\pi k x + \pi k y}, \tag{44}$$

where the constant $C_k := e^{\mathbb{i}\pi(-k/2-1/4)}$ is chosen so that E_k has RBV on $\partial\mathbb{S}$ and the restrictions $e_k := E_k|_I$ satisfy $\|e_k\|_{\mathcal{H}} = 1$.

Lemma 3.11. *The collection* $\{e_k \mid k \in \pm\mathscr{K}\}$ *is an orthonormal basis of* $\mathcal{H}$*, and we have*

$$H^2(I) = cl(\mathrm{span}_{\mathbb{R}}\{e_k \bigm| k \in -\mathscr{K}\}).$$

Moreover, the map $\mathsf{R}\colon \mathcal{H} \to \mathcal{H}$ *satisfies* $\mathsf{R}e_k = e_{-k}$ *for every* $k \in \pm\mathscr{K}$*.*

Proof. For the proof of the fact that $\{e_k \mid k \in \pm\mathscr{K}\}$ is an orthonormal basis of $L^2(I, \mathbb{C})$, see [2, Proposition 2.4]. To see that $H^2(I) = \mathrm{cl}(\mathrm{span}_{\mathbb{R}}\{e_k \mid k \in -\mathscr{K}\})$, we first note that for every $k \in -\mathscr{K}$, the function e_k has the holomorphic extension E_k with RBV on $\partial\mathbb{S}$, and we have $E_k(x + \mathbb{i}h) \to 0$ as $h \to +\infty$ for every $x \in I$ by the explicit formula in (44). This implies $\mathrm{cl}(\mathrm{span}_{\mathbb{R}}\{e_k \mid k \in -\mathscr{K}\}) \subseteq H^2(I)$ since RBV is a real-linear condition.

Moreover, if $k_1, \ldots, k_n \in \mathscr{K}$ with $k_1 < \ldots < k_n$ and $c_1, \ldots, c_n \in \mathbb{R}$, the function $\sum_{j=1}^n c_j E_{k_j}$ has RBV on $\partial\mathbb{S}$ and it is the unique holomorphic extension of $\sum_{j=1}^n c_j e_{k_j}$ to $\mathbb{S}$. For every $x \in I$, we have

$$\left|\sum_{j=1}^{n} c_j E_{k_j}(x + \mathbb{i}h)\right| \geq |c_n| e^{\pi k_n h} - \sum_{j=1}^{n-1} |c_j| e^{\pi k_j h}$$

by (44), so $\left|\sum_{j=1}^n c_j E_{k_j}(x + \mathbb{i}h)\right| \to \infty$ as $h \to +\infty$ for every $x \in I$. This implies that $\mathrm{cl}(\mathrm{span}_{\mathbb{R}}\{e_k \mid k \in \mathscr{K}\}) \subseteq H^2(I)^\perp$. It thus follows that $H^2(I) \subseteq \mathrm{cl}(\mathrm{span}_{\mathbb{R}}\{e_k \mid k \in -\mathscr{K}\})$ since $\{e_k \mid k \in \pm\mathscr{K}\}$ is an orthonormal basis of $\mathcal{H}$.

Finally, the property $\mathsf{R}e_k = e_{-k}$ follows by a simple computation. ∎

If $X = (\mathscr{D}, \{\beta^{\mathrm{in}}, \beta^{\mathrm{out}}\})$ is a planar domain with two parametrized boundary components, we define the Hardy space of X in analogy with Theorems 3.6 and 3.9 as follows. If $A^\infty(X)$ is the set of holomorphic functions $F\colon \mathscr{D} \to \mathbb{C}$ that extend to continuous functions on $\partial\mathscr{D}$ that are C^∞ on smooth components of $\partial\mathscr{D}$ and have RBV on $\partial\mathscr{D} \setminus (\mathcal{I}^{\mathrm{in}} \cup \mathcal{I}^{\mathrm{out}})$, we define

$$H^2(X) \coloneqq \mathrm{cl}\left(\mathrm{span}\left\{\left((\beta^{\mathrm{out}})^* F, (\beta^{\mathrm{in}})^* F\right) \bigm| F \in A^\infty(X)\right\}\right) \subset \mathcal{H} \oplus \mathcal{H},$$

where $((\beta^{\mathrm{in}})^* F)(x) = F(\beta^{\mathrm{in}}(x))$ and similarly for $(\beta^{\mathrm{out}})^* F$. This allows us to give a provisional definition of the BCFT functor.

Definition 3.12. Let $X = (\mathscr{D}, \{\beta^{\mathrm{in}}, \beta^{\mathrm{out}}\})$ be a planar domain with two parametrized boundary components and let $p\colon \mathcal{H} \to \mathcal{H}$ be the projection to the subspace $H^2(I) \subset \mathcal{H}$. The BCFT functor associates to X the space E_X of bounded linear maps $T_X\colon \mathscr{F}_{p\mathcal{H}} \to \mathscr{F}_{p\mathcal{H}}$ satisfying

$$\phi_p(a(f^{\mathrm{out}}))T_X = T_X \phi_p(a(f^{\mathrm{in}})) \tag{45}$$

for every $(f^{\mathrm{out}}, f^{\mathrm{in}}) \in H^2(X)$, where ϕ_p is the representation of the Clifford algebra $\mathrm{Cl}(\mathcal{H}, \mathsf{R})$ on $\mathscr{F}_{p\mathcal{H}}$ as in Chapter 2.4.

To investigate the continuum analogues of the operators T^h from Chapter 3.2, we define a one parameter family of planar domains with two parametrized boundary components.

Definition 3.13. For every $h > 0$, X_h is the pair $X_h = (\mathscr{D}, \{\beta^{\text{in}}, \beta^{\text{out}}\})$, where $\mathscr{D} = I \times [0, h]$ and

$$\beta^{\text{in}}(x) = \mathrm{id}_I(x) \quad \text{and} \quad \beta^{\text{out}}(x) = \mathring{\imath} h + \mathrm{id}_I(x),$$

where id_I is interpreted as a map $\mathrm{id}_I : I \to \mathbb{C}$.

The operators T_{X_h} can be written explicitly in the basis given in Lemma 3.11 much in the same way that we diagonalized the operators T^h in Chapter 3.2.

Theorem 3.14. *For every $h > 0$, the space E_{X_h} in Definition 3.12 is one-dimensional, and every $T_{X_h} \in E_{X_h}$ is trace class. Moreover, the operators T_{X_h} form a semigroup in the sense that for any $h_1, h_2 > 0$, we have*

$$E_{X_{h_1+h_2}} = \{T_{X_{h_2}} T_{X_{h_1}} \mid T_{X_{h_1}} \in E_{X_{h_1}}, T_{X_{h_2}} \in E_{X_{h_2}}\}.$$

Proof. Let us begin by finding an orthonormal basis for $H^2(X_h)$. For every $k \in \pm\mathscr{K}$, the function E_k in (44) is the unique holomorphic extension of the function e_k to $I \times [0, h]$, and we have $\big((\beta^{\text{out}})^* E_k, (\beta^{\text{in}})^* E_k\big) = (e^{\pi k h} e_k, e_k)$. Given the function e_k, the only vector $f \in \mathcal{H}$ satisfying $(f, e_k) \in H^2(X_h)$ is thus $f = e^{\pi k h} e_k$. Since the set $\{e_k \mid k \in \pm\mathscr{K}\}$ is an orthonormal basis of $\mathcal{H}$ by Lemma 3.11 and $(f, 0) \notin H^2(X_h)$ for every $f \neq 0$, it follows that the set

$$\{(1 + e^{2\pi k h})^{-1/2} (e^{\pi k h} e_k, e_k) \mid k \in \pm\mathscr{K}\} \tag{46}$$

is an orthonormal basis of $H^2(X_h)$.

To show that the space E_{X_h} is one-dimensional, let us assume that there exists a bounded linear map T_{X_h} satisfying (45). If $\Omega \in \mathscr{F}_{p\mathcal{H}}$ is the vacuum vector, it follows from (45) and the previous observations that for every $k \in \mathscr{K}$ we have

$$0 = T_{X_h} \phi_p(a(e_k))\Omega = e^{\pi k h} \phi_p(a(e_k)) T_{X_h} \Omega.$$

Hence, it follows that $T_{X_h}\Omega = C\Omega$ for some $C \in \mathbb{C}$, since up to a multiplicative constant, Ω is the only vector annihilated by $\phi_p(a(e_k))$ for every $k \in \mathscr{K}$. Similarly, if $\{k_1, \ldots, k_n\} \subset -\mathscr{K}$ and $k_1 > \ldots > k_n$, we get

$$T_{X_h} \prod_{j=1}^{n} \phi_p(a(e_{k_j}))\Omega = \prod_{j=1}^{n} e^{\pi k_j h} \phi_p(a(e_{k_j})) T_{X_h} \Omega = C \left(\prod_{j=1}^{n} e^{\pi k_j h} \right) \prod_{j=1}^{n} \phi_p(a(e_{k_j}))\Omega. \tag{47}$$

Recalling from (19) that the set

$$\left\{ \prod_{j=1}^{n} \phi_p(a(e_{k_j}))\Omega \;\middle|\; n \geq 0, \{k_1, \ldots, k_n\} \subset -\mathscr{K}, k_1 > \ldots > k_n \right\} \tag{48}$$

is an orthonormal basis of $\mathscr{F}_{p\mathcal{H}}$, we see that (45) defines the bounded linear map T_{X_h} up to the multiplicative constant C, so $\dim(E_{X_h}) \leq 1$. Conversely, the bounded linear maps T_{X_h} defined by (47) satisfy (45) for $(f^{\text{out}}, f^{\text{in}})$ taken from the orthonormal basis of $H^2(X_h)$ given in (46). Since the map $f \mapsto \phi_p(a(f))$ is linear and continuous in the operator norm by Theorem 2.6, we see that the maps T_{X_h} defined by (47) actually satisfy (45) for any $(f^{\text{out}}, f^{\text{in}}) \in H^2(X_h)$, so $\dim(E_{X_h}) \geq 1$. Hence, we see that E_{X_h} is one-dimensional.

Moreover, the property $E_{X_{h_1+h_2}} = \{T_{X_{h_2}} T_{X_{h_1}} \mid T_{X_{h_1}} \in E_{X_{h_1}}, T_{X_{h_2}} \in E_{X_{h_2}}\}$ follows directly from the explicit expression for T_{X_h} in (47).

To see that the maps T_{X_h} are trace class, we introduce the following notation. For each $m \in \mathbb{Z}_{\geq 0}$, we denote by L_m the set of increasing sequences of m positive odd integers. Moreover, for every $k \in \mathbb{Z}_{\geq 0}$ we denote by $S_{m,k} \subset L_m$ the subset of sequences that sum up to k. Finally, for every $k \in \mathbb{Z}_{\geq 0}$ we let S_k be the set of increasing sequences of positive integers that sum up to k. It is clear that $|S_{m,k}| \leq |S_k|$, and since the sum of the first k positive integers is $k(k+1)/2$, a sequence in S_k cannot have more than $3\sqrt{k}$ elements. Since none of the integers in a sequence in S_k is larger than k, it follows that $|S_k| \leq k^{3\sqrt{k}} = e^{3\log(k)\sqrt{k}}$. Hence, up to the constant C, the eigenvalues of T_{X_h} given in (47) satisfy

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{\substack{\{k_1,\ldots,k_n\} \subset -\mathscr{K} \\ k_1 > \ldots > k_n}} \prod_{j=1}^{n} e^{\pi k_j h} &= \sum_{n=0}^{\infty} \sum_{s \in L_n} e^{-\frac{1}{2}\pi h \sum s} \\
&= \sum_{n=0}^{\infty} \sum_{k=n(n+1)/2}^{\infty} |S_{n,k}| e^{-\frac{1}{2}\pi h k} \\
&\leq \sum_{n=0}^{\infty} \sum_{k=n(n+1)/2}^{\infty} e^{-\frac{1}{2}\pi h k + 3\log(k)\sqrt{k}} \\
&\leq 1 + \sum_{k=1}^{\infty} k e^{-\frac{1}{2}\pi h k + 3\log(k)\sqrt{k}} < \infty,
\end{aligned}$$

where it is understood that $0\log(0) = 0$. Hence, T_{X_h} is trace class. ■

If we choose $C = 1$ in (47) for every $h > 0$, the the largest eigenvalue of T_{X_h} is 1, corresponding to the vacuum Ω. Moreover, we have $T_{X_{h_2}} T_{X_{h_1}} = T_{X_{h_1+h_2}}$, so the operators T_h form a contractive semigroup $\{e^{-hL_0}\}_{h \geq 0}$, where the generator $L_0 \geq 0$ is given in the basis (48) by

$$L_0 \prod_{j=1}^{n} \phi_p(a(e_{k_j}))\Omega = -\pi h \left(\sum_{j=1}^{n} k_j \right) \prod_{j=1}^{n} \phi_p(a(e_{k_j}))\Omega$$

and $L_0 \Omega = 0$. The operator L_0 has a natural interpretation as the 0th generator of a Virasoro algebra associated to the BCFT, although the geometric interpretation of the rest of the generators is unclear at this stage. For more information, see [17, Chapter 2.6].

Our success in Theorems 3.9 and 3.14 was based on the following phenomena. First, up to a scalar multiple, the operators T^h and T_{X_h} were easily seen to map the vacua $v_\emptyset \in \mathscr{F}_{p\#\mathbb{C}^{\mathcal{I}^*}}$ and $\Omega \in \mathscr{F}_{p\mathcal{H}}$ to $v_\emptyset$ and Ω, respectively. Secondly, the commutation relations (43) and (45) then allowed us to define the operators T^h and T_{X_h} on the bases (40) and (48). However, the first of these observations might not necessarily hold if instead of X_h we consider a general X, which is planar domain with two parametrized boundary components. Hence, the dimension of the space E_X for a general X is an open question at this stage. The dimension of E_X turns out to be equal to the dimension of the space of $\widetilde{q}$-vacua $\Omega_{\widetilde{q}} \in \mathscr{F}_{p^\perp\mathcal{H}\oplus p\mathcal{H}}$ of the representation $\phi_{p^\perp\oplus p}$ of the Clifford algebra $\mathrm{Cl}(\mathcal{H}\oplus\mathcal{H}, \mathsf{R}\oplus\mathsf{R})$ from Definition 2.9, where $\widetilde{q} = r\widehat{q}r$, $\widehat{q}\colon \mathcal{H}\oplus\mathcal{H} \to \mathcal{H}\oplus\mathcal{H}$ is the projection onto $H^2(X)$ and $r = (2p - \mathrm{id}_{\mathcal{H}}) \oplus \mathrm{id}_{\mathcal{H}}$. However, we will prove this fact only after we have defined the BCFT for general domains in the next chapter.

To see what kind of complications arise when we deform the domains X_h, we give a more explicit formula for the commutation relation (45) for a general domain with two parametrized boundary components. First, we note the following conformal mapping property.

Lemma 3.15. *Let $X = (\mathscr{D}, \{\beta^{in}, \beta^{out}\})$ be a planar domain with two parametrized boundary components. For some $h > 0$, there exists a conformal map $\varphi\colon cl(\mathscr{D}) \to I \times [0, h]$ taking $\mathcal{I}^{in} = \beta^{in}(I)$ to I and $\mathcal{I}^{out} = \beta^{out}(I)$ to $I + \mathbb{i}h$.*

Proof. If $\partial_L\mathscr{D}$ and $\partial_R\mathscr{D}$ are as in Figure 5, let $u\colon \mathscr{D} \to \mathbb{R}$ be a solution of the mixed boundary value problem

$$\begin{cases} \Delta u(z) = 0, & z \in \mathscr{D} \\ u(z) = -1/2, & z \in \partial_L\mathscr{D} \\ u(z) = +1/2, & z \in \partial_R\mathscr{D} \\ \nabla_\nu u(z) = 0, & z \in \mathcal{I}^{\mathrm{in}} \cup \mathcal{I}^{\mathrm{out}}, \end{cases}$$

where $\nabla_\nu u = (\nabla u)\cdot\nu$ and ν is the outward pointing unit normal to $\partial\mathscr{D}$. If v is a harmonic conjugate of u, i.e. a harmonic function such that $u + \mathbb{i}v$ is holomorphic in $\mathscr{D}$, v satisfies a similar mixed boundary value problem with a Dirichlet boundary condition on $\mathcal{I}^{\mathrm{in}} \cup \mathcal{I}^{\mathrm{out}}$ and a Neumann boundary condition on $\partial_L\mathscr{D} \cup \partial_R\mathscr{D}$. In particular, we have

$$\begin{cases} v(z) = a, & z \in \mathcal{I}^{\mathrm{in}} \\ v(z) = a + h, & z \in \mathcal{I}^{\mathrm{out}} \end{cases}$$

for some $a \in \mathbb{R}$ and $h > 0$. Hence, $\phi = u + \mathbb{i}v$ defines a holomorphic map $\phi\colon \mathrm{cl}(\mathscr{D}) \to I \times [a, a+h]$, and post-composing ϕ with a suitable translation gives a holomorphic map $\varphi\colon \mathrm{cl}(\mathscr{D}) \to I \times [0, h]$.

To see that φ is conformal, it suffices to show that it is bijective. To see this, pick $w \in \mathrm{Int}(I \times [0, h])$ and denote by N_w the number of roots of $\varphi(z) - w$ in $\mathscr{D}$. Since φ is continuous up to the boundary $\partial\mathscr{D}$, the argument principle implies

$$N_w = \frac{1}{2\pi\mathbb{i}} \oint_{\partial\mathscr{D}} \frac{\varphi'(z)}{\varphi(z) - w} dz = \frac{1}{2\pi\mathbb{i}} \oint_{\varphi(\partial\mathscr{D}) - w} \frac{1}{z} dz = 1,$$

which shows that φ is bijective on $\mathscr{D}$. ■

Since the Hardy spaces in Definition 3.12 are defined in terms of boundary values of holomorphic functions, we observe the following technical result concerning the boundary behavior of conformal maps.

Proposition 3.16. *Let $\mathscr{D}_1, \mathscr{D}_2 \subset \mathbb{C}$ be simply connected domains with piecewise C^∞ boundaries that are homeomorphic to S^1. Any conformal map $\varphi\colon \mathscr{D}_1 \to \mathscr{D}_2$ extends to a homeomorphism $\widetilde{\varphi}\colon cl(\mathscr{D}_1) \to cl(\mathscr{D}_2)$, and if $z \in \partial\mathscr{D}_1$ is a point such that $\partial\mathscr{D}_1$ is C^∞ in a neighborhood of z and $\partial\mathscr{D}_2$ is C^∞ in a neighborhood of $\widetilde{\varphi}(z)$, there exists a neighborhood U of z in $\mathbb{C}$ such that φ extends as a C^∞ function to U.*

Proof. For a simple proof of the fact that φ extends to a homeomorphism $\widetilde{\varphi}\colon \mathrm{cl}(\mathscr{D}_1) \to \mathrm{cl}(\mathscr{D}_2)$, see [23, Chap. 2]. The fact that φ has a C^∞ extension to the boundary is proven in [5, Theorem 3.1] for a domain with C^∞ boundary, and the same proof applies to the case of a piecewise C^∞ boundary. ∎

Given the conformal map $\varphi\colon \mathrm{cl}(\mathscr{D}) \to I \times [0, h]$ from Lemma 3.15, the following result lets us characterize the Hardy space $H^2(X)$.

Theorem 3.17. *Suppose that $\mathscr{D}_1, \mathscr{D}_2 \subset \mathbb{C}$ are simply connected domains with $\partial\mathscr{D}_1, \partial\mathscr{D}_2 \approx S^1$ and $\varphi\colon \mathscr{D}_1 \to \mathscr{D}_2$ is a conformal map, and let $\widetilde{\varphi}\colon cl(\mathscr{D}_1) \to cl(\mathscr{D}_2)$ be the extension given by Proposition 3.16. If $E\colon \mathscr{D}_2 \to \mathbb{C}$ is a holomorphic map with RBV at $z_0 \in \partial\mathscr{D}_2$ and $\partial\mathscr{D}_1$ is C^∞ in a neighborhood of $\widetilde{\varphi}^{-1}(z_0)$, then*

$$F(z) = \sqrt{\varphi'(z)}E(\varphi(z))$$

gives a well defined holomorphic map $F\colon \mathscr{D}_1 \to \mathbb{C}$ with RBV at $\widetilde{\varphi}^{-1}(z_0)$ for a suitable choice of the branch of the square root.

Proof. Since φ is injective and holomorphic, $\varphi'(z) \neq 0$ for all $z \in \mathscr{D}_1$. Hence, the square root has a branch such that $\sqrt{\varphi'}$ is holomorphic in the simply connected domain $\mathscr{D}_1$. This shows that F is well defined and holomorphic in $\mathscr{D}_1$. Moreover, Proposition 3.16 gives a neighborhood U of $w_0 = \widetilde{\varphi}^{-1}(z_0)$ such that φ extends to a map $\varphi \in C^\infty(U)$. Hence, φ and φ' have continuous extensions to w_0, and since E has a continuous extension to z_0, it follows that F has a continuous extension to w_0.

Let $\tau_{\mathscr{D}_1}(w_0), \tau_{\mathscr{D}_2}(z_0) \in S^1$ be the complex numbers representing the unit tangent vectors to $\partial\mathscr{D}_1$ at w_0 and to $\partial\mathscr{D}_2$ at z_0, respectively. Moreover, let $\gamma\colon I \to \partial\mathscr{D}_1 \cap U$ be a C^∞ parametrization such that

$$\tau_{\mathscr{D}_1}(w_0) = \frac{d}{dt}\bigg|_{t=t_0} \gamma(t)$$

for some $t_0 \in I$. Denoting by φ' the extension of φ' to U, we get

$$\tau_{\mathscr{D}_2}(z_0) \in \frac{d}{dt}\bigg|_{t=t_0} \varphi(\gamma(t))\mathbb{R} = \varphi'(w_0)\tau_{\mathscr{D}_1}(w_0)\mathbb{R}.$$

Since $E(\varphi(w_0)) \in \mathbb{i}\tau_{\mathscr{D}_2}(z_0)^{-1/2}$ by assumption, it follows that

$$\sqrt{\varphi'(w_0)}E(\varphi(w_0)) \in \mathbb{i}\tau_{\mathscr{D}_1}^{-1/2}(w_0)\mathbb{R},$$

which finishes the proof. ∎

For the remainder of this chapter, let $X = (\mathscr{D}, \{\beta^{\text{in}}, \beta^{\text{out}}\})$ be a planar domain with two parametrized boundary components and let $\varphi\colon \text{cl}(\mathscr{D}) \to I \times [0,h]$ be the conformal map from Lemma 3.15. Moreover, let $A\colon H^2(X_h) \to H^2(X)$ be the linear map defined by

$$A\big((\beta^{\text{out}}_{X_h})^* F, (\beta^{\text{in}}_{X_h})^* F\big) = \big((\beta^{\text{out}})^*(\sqrt{\varphi'} \cdot F \circ \varphi), (\beta^{\text{in}})^*(\sqrt{\varphi'} \cdot F \circ \varphi)\big) \qquad (49)$$

on the elements $\big((\beta^{\text{out}}_{X_h})^* F, (\beta^{\text{in}}_{X_h})^* F\big)$ that are boundary values of holomorphic functions $F\colon I \times [0,h] \to \mathbb{C}$, where $\beta^{\text{in}}_{X_h}$ and $\beta^{\text{out}}_{X_h}$ are the parametrizations of X_h from Definition 3.13. Theorem 3.17 implies that A is well defined, bounded and invertible with a bounded inverse unless φ' or $(\varphi^{-1})'$ fails to have a bounded extension to a corner point of $\mathscr{D}$ or $I \times [0,h]$, respectively.

If ζ is a corner point of $\mathscr{D}$, where the boundary curves meet at an angle θ, and $\xi = \varphi(\zeta)$, there exists a neighborhood of ζ in $\mathscr{D}$ and a neighborhood of ξ in $I \times [0,h]$ such that φ' and $(\varphi^{-1})'$ can be written as

$$\varphi'(z) = \frac{\psi_1(z)}{(z-\zeta)^{\theta-\pi/2}}, \quad (\varphi^{-1})'(z) = \frac{\psi_2(z)}{(z-\xi)^{\pi/2-\theta}},$$

where ψ_1 and ψ_2 are continuous and nonvanishing up to the boundary, see [23, Theorem 3.9].

Hence, if the boundary curves of $\mathscr{D}$ meet at angles of $\pi/2$, we have $H^2(X) = A(H^2(X_h))$. In this case, the commutation relation (45) can be written as

$$e^{\pi k h}\phi_p(a(\sqrt{\varphi' \circ \beta^{\text{out}}} \cdot e_k \circ \varphi \circ \beta^{\text{out}}))T_X = T_X \phi_p(a(\sqrt{\varphi' \circ \beta^{\text{in}}} \cdot e_k \circ \varphi \circ \beta^{\text{in}})) \qquad (50)$$

for all $k \in \pm\mathscr{K}$, where e_k are as in Lemma 3.11.

As an immediate corollary of (50), we note the following conformal invariance property of the operators T_X.

Proposition 3.18. *Let $X = (\mathscr{D}, \{\beta^{in}, \beta^{out}\})$ be a planar domain with two parametrized boundary components and let $\mathcal{T}\colon \mathbb{C} \to \mathbb{C}$ be a translation, i.e. $\mathcal{T}(z) = z + a$ for some $a \in \mathbb{C}$. If $Y = (\mathcal{T}(\mathscr{D}), \{\mathcal{T} \circ \beta^{in}, \mathcal{T} \circ \beta^{out}\})$ is the planar domain with two parametrized boundary components obtained by applying $\mathcal{T}$ to X, then we have $E_Y = E_X$.*

Proof. By (50), the operators T_X and T_Y are both subject to the same system of equations. ∎

3.4 The BCFT in general domains

To extend Definition 3.12 to more general domains, we begin by determining the correct class of domains and the appropriate notion of sewing. The basic setup is as in Chapter 3.3: we consider connected bounded domains $\mathscr{D} \subset \mathbb{C}$ with boundaries consisting of isometric copies $\mathcal{I}_j$ of the unit interval $I = [-1/2, 1/2]$ parallel to the real line $\mathbb{R} \subset \mathbb{C}$ and analytic curves γ_j such that $\mathcal{I}_i \cap \mathcal{I}_j = \emptyset$ and $\gamma_i \cap \gamma_j = \emptyset$ if $i \neq j$. Moreover, we attach a smooth parametrization $\beta_j\colon I \to \mathcal{I}_j$ to each boundary component $\mathcal{I}_j$. Given a pair $X = (\mathscr{D}, \{\beta_j\})$ as above, we index the boundary components $\mathcal{I}_j = \beta_j(I)$ by a set J_X. As in Chapter 3.1, the parametrizations β_j induce a partition $J_X = J_X^{\text{in}} \coprod J_X^{\text{out}}$ where $j \in J_X^{\text{in}}$ if and only if β_j is orientation-preserving, and we call a boundary component $\mathcal{I}_j$ incoming if $j \in J_X^{\text{in}}$ and outgoing if $j \in J_X^{\text{out}}$.

Based on our experience at the end of Chapter 3.3, we restrict our attention to the case where the curves γ_i intersect the lines $\mathcal{I}_j$ at angles of $\pi/2$. Moreover, we take the parametrizations β_j to be the natural linear isometries obtained by parametrizing the boundary components $\mathcal{I}_j$ by arc length. The set of pairs $X = (\mathscr{D}, \{\beta_j\})$ satisfying these two properties is denoted by $\mathcal{G}$, and it is the class of domains that we will work with.

Given $X = (\mathscr{D}_1, \{\beta_j^{(1)}\}), Y = (\mathscr{D}_2, \{\beta_j^{(2)}\}) \in \mathcal{G}$, we can sew any number of the outgoing boundary components of $\mathscr{D}_1$ to incoming boundary components of $\mathscr{D}_2$ along the maps $\beta_i^{(2)} \circ (\beta_j^{(1)})^{-1}$ as in [24]. However, the resulting Riemann surface may not be conformally equivalent to a bounded domain in $\mathbb{C}$, so we consider the following more cumbersome notion of sewing.

Definition 3.19. Given $X = (\mathscr{D}_1, \{\beta_j^{(1)}\}), Y = (\mathscr{D}_2, \{\beta_j^{(2)}\}) \in \mathcal{G}$, subsets $A \subset J_X^{\text{out}}$, $B \subset J_Y^{\text{in}}$ and a bijection $s\colon A \to B$, we say that X and Y are s-compatible if there exists a translation $\mathcal{T}\colon \mathbb{C} \to \mathbb{C}$ such that $\mathcal{T}(\beta_j^{(1)}(I)) = \beta_{s(j)}^{(2)}(I)$ for every $j \in A$ and $\mathcal{T}(\text{cl}(\mathscr{D}_1)) \cap \text{cl}(\mathscr{D}_2) = \cup_{j \in B} \beta_j^{(2)}(I)$. If X and Y are s-compatible, we define by

$$s(X,Y) = \Big(\text{Int}\left(\mathcal{T}(\text{cl}(\mathscr{D}_1)) \cup \text{cl}(\mathscr{D}_2)\right), \{\mathcal{T} \circ \beta_j^{(1)}\}_{j \in J_X \setminus A} \cup \{\beta_j^{(2)}\}_{j \in J_Y \setminus B}\Big)$$

the element of $\mathcal{G}$ obtained by sewing X to Y along the bijection s.

A few remarks concerning the previous definition are in order. First, this notion of sewing is compatible with that in [24] if and only if we parametrize the boundary components $\mathcal{I}_j$ by arc length, which explains our previous choice of the parametrizations β_j. Secondly, although Definition 3.19 seems to be asymmetric in X and Y, the elements $s(X,Y)$ and $s^{-1}(Y,X)$ are related by a translation as in Proposition 3.18. Finally, it is clear that this notion of sewing does not define an actual category with $\mathcal{G}$ as the set of morphisms (see Figure 6). However, we still retain this terminology as in Chapter 1.

For the remainder of this chapter, we write $\mathcal{H} = L^2(I, \mathbb{C})$ as in Chapter 3.3. In analogy to Chapters 3.1 and 3.3, we define the Hardy space of $X \in \mathcal{G}$ as follows.

Definition 3.20. Given $X = (\mathscr{D}, \{\beta_j\}) \in \mathcal{G}$, let $A^\infty(X)$ be the set of holomorphic functions $F\colon \mathscr{D} \to \mathbb{C}$ extending to continuous functions on $\partial\mathscr{D}$ that are smooth on smooth components of $\partial\mathscr{D}$ and have RBV on $\partial\mathscr{D} \setminus (\cup_{j \in J_X} \mathcal{I}_j)$. The Hardy space $H^2(X) \subset \bigoplus_{j \in J_X^{\text{out}}} \mathcal{H} \oplus \bigoplus_{j \in J_X^{\text{in}}} \mathcal{H}$ of X is

$$H^2(X) = \text{cl}(\{\beta^* F \mid F \in A^\infty(X)\}).$$

In analogy to Definition 3.12, the BCFT functor is then defined as follows.

Definition 3.21. Let $X = (\mathscr{D}, \{\beta_j\}) \in \mathcal{G}$ and let $p\colon \mathcal{H} \to \mathcal{H}$ be the projection to the subspace $H^2(I) \subset \mathcal{H}$ from Chapter 3.3. The BCFT functor associates to the boundary

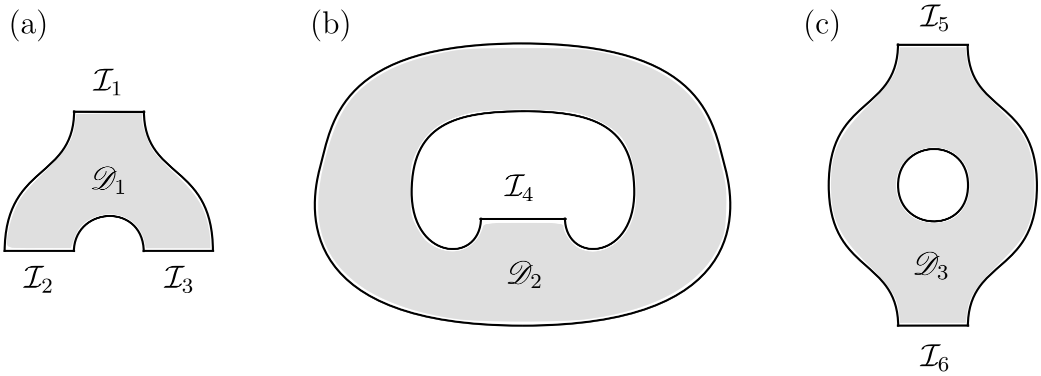

Figure 6: (a) The domain $\mathscr{D}_1$ has parametrized boundary components $\mathcal{I}_1 - \mathcal{I}_3$. (b) The domain $\mathscr{D}_1$ cannot be sewn to the domain $\mathscr{D}_2$ by any translation, so the composition of domains is not always defined. (c) The domain $\mathscr{D}_3$ is obtained by sewing a copy of $\mathscr{D}_1$ with $\mathcal{I}_2$ and $\mathcal{I}_3$ incoming to a reflected copy of $\mathscr{D}_1$ where $\mathcal{I}_2$ and $\mathcal{I}_3$ are outgoing, cf. Figure 2.

components $\cup_{j\in J_X^{\text{in}}}\mathcal{I}_j$ and $\cup_{j\in J_X^{\text{out}}}\mathcal{I}_j$ the unordered tensor products $\bigotimes_{j\in J_X^{\text{in}}}\mathscr{F}_{p\mathcal{H}}$ and $\bigotimes_{j\in J_X^{\text{out}}}\mathscr{F}_{p\mathcal{H}}$, respectively. Moreover, the BCFT functor associates to X the space E_X of homogeneous Hilbert-Schmidt operators $T_X\colon \bigotimes_{j\in J_X^{\text{in}}}\mathscr{F}_{p\mathcal{H}} \to \bigotimes_{j\in J_X^{\text{out}}}\mathscr{F}_{p\mathcal{H}}$ of unordered tensor products satisfying

$$\phi_p(a(f^{\text{out}}))T_X = T_X\phi_p(a(f^{\text{in}})) \tag{51}$$

for every $(f^{\text{out}}, f^{\text{in}}) \in H^2(X)$, where ϕ_p is the representation of the Clifford algebra $\mathrm{Cl}(\mathcal{H}, \mathsf{R})$ on $\mathscr{F}_{p\mathcal{H}}$ from Chapter 2.4 and (51) is understood as in Theorem 2.13.

Before addressing the question of whether Definition 3.21 gives a well defined projective functor, i.e. whether $\dim(E_X) = 1$, we consider the analogues of the invariance properties in Theorem 3.1. As for property 3, we see that given $X = (\mathscr{D}_1, \{\beta_j^{(1)}\}), Y = (\mathscr{D}_2, \{\beta_j^{(2)}\}) \in \mathcal{G}$, a bijection $s\colon J_X \to J_Y$ and a conformal map $\varphi\colon \mathrm{cl}(\mathscr{D}_1) \to \mathrm{cl}(\mathscr{D}_2)$ such that $\beta_{s(j)}^{(2)} = \varphi\circ\beta_j^{(1)}$, we necessarily have that φ is a translation, possibly composed with the map $z \mapsto -z$. As an analogue of property 4, one can ask whether the spaces E_X and E_Y are related by unitary maps if we drop the requirement that $\beta_{s(j)}^{(2)} = \varphi\circ\beta_j^{(1)}$ but instead require that $\beta_{s(j)}^{(2)}(I) = \varphi\circ\beta_j^{(1)}(I)$.
Hence, Definition 3.21 raises the following four questions.

1. Given $X \in \mathcal{G}$, what is $\dim(E_X)$?

2. Given $X = (\mathscr{D}_1, \{\beta_j^{(1)}\}), Y = (\mathscr{D}_2, \{\beta_j^{(2)}\}) \in \mathcal{G}$, subsets $A \subset J_X^{\text{out}}, B \subset J_Y^{\text{in}}$ and a bijection $s\colon A \to B$, does it follow that

   $$E_{s(X,Y)} = \{(T_Y\widehat{\otimes}\mathrm{id}_{A^C})(T_X\widehat{\otimes}\mathrm{id}_{B^C}) \mid T_X \in E_X, T_Y \in E_Y\},$$

   where

   $$\mathrm{id}_{A^C} = \bigotimes_{j\in J_X^{\text{out}}\setminus A}\mathrm{id}_{\mathscr{F}_{p\mathcal{H}}}$$

   and similarly for id_{B^C}?

3. Given $X = (\mathscr{D}_1, \{\beta_j^{(1)}\}), Y = (\mathscr{D}_2, \{\beta_j^{(2)}\}) \in \mathcal{G}$, a bijection $s\colon J_X \to J_Y$ and a translation $\mathcal{T}\colon \mathbb{C} \to \mathbb{C}$ such that $\beta_{s(j)}^{(2)} = \mathcal{T} \circ \beta_j^{(1)}$, does it follow that

$$E_Y = U_{\text{out}} E_X U_{\text{in}}^*,$$

where $U_{\text{in}}\colon \bigotimes_{j\in J_X^{\text{in}}} \mathscr{F}_{p\mathcal{H}} \to \bigotimes_{j\in J_Y^{\text{in}}} \mathscr{F}_{p\mathcal{H}}$ is the unitary map obtained by identifying J_X^{in} and J_Y^{in} by s and similarly for U_{out}?

4. If we modify question 3 by assuming that there exists a conformal map $\varphi\colon \text{cl}(\mathscr{D}_1) \to \text{cl}(\mathscr{D}_2)$ such that $\beta_{s(j)}^{(2)}(I) = \varphi \circ \beta_j^{(1)}(I)$, do there exist unitary maps U_{in} and U_{out} such that $E_Y = U_{\text{out}} E_X U_{\text{in}}^*$?

We have seen in Chapter 3.3 that for the rectangular domains X_h, $\dim(E_{X_h}) = 1$ and the answer to questions 2 and 3 is in the affirmative. Moreover, a quick adaptation of the arguments after Theorem 3.17 shows that the relation in 3 holds generally in $\mathcal{G}$. The most pressing issue is of course question 1, which can be reduced to a question about the space of vacua of the representations ϕ_p.

Theorem 3.22. *Given $X \in \mathcal{G}$, $\dim(E_X)$ is equal to the dimension of the space of $\widetilde{q}$-vacua $\Omega_{\widetilde{q}} \in \mathscr{F}_{\bigoplus_{out} p^\perp \mathcal{H} \oplus \bigoplus_{in} p\mathcal{H}}$ of the representation $\phi_{\bigoplus_{out} p^\perp \oplus \bigoplus_{in} p}$ of the Clifford algebra $Cl(\bigoplus_{j\in J_X} \mathcal{H}, \bigoplus_{j\in J_X} \mathsf{R})$ from Definition 2.9, where $\widetilde{q} = rqr$, $q\colon \bigoplus_{j\in J_X} \mathcal{H} \to \bigoplus_{j\in J_X} \mathcal{H}$ is the projection onto $H^2(X)$, $r = (2\bigoplus_{out} p - id_{\bigoplus_{out} \mathcal{H}}) \oplus id_{\bigoplus_{in} \mathcal{H}}$, and $\bigoplus_{in}$ and $\bigoplus_{out}$ have the obvious meanings.*

Corollary. *For every $X \in \mathcal{G}$, we have* $\dim(E_X) \leq 1$.

Proof. Using the notation of Theorem 3.22, we first note that the irreducibility of the representation $\phi_{\bigoplus_{\text{out}} p^\perp \oplus \bigoplus_{\text{in}} p}$ implies that each nonzero $\widetilde{q}$-vacuum is cyclic. Hence, the argument in the last part of the proof of Theorem 3.8 shows that all the $\widetilde{q}$-vacua are scalar multiples of each other. ■

To prove Theorem 3.22, we record the following lemma.

Lemma 3.23. *Let $\mathcal{H}^{in}$ and $\mathcal{H}^{out}$ be real Hilbert spaces, let $R^{in}\colon \mathcal{H}^{in} \to \mathcal{H}^{in}$ and $R^{out}\colon \mathcal{H}^{out} \to \mathcal{H}^{out}$ be unitary involutions, and let $p^{in}\colon \mathcal{H}^{in} \to \mathcal{H}^{in}$, $p^{out}\colon \mathcal{H}^{out} \to \mathcal{H}^{out}$ and $q\colon \mathcal{H}^{out} \oplus \mathcal{H}^{in} \to \mathcal{H}^{out} \oplus \mathcal{H}^{in}$ be projections such that p^{in} is R^{in}-compatible and p^{out} is R^{out}-compatible. If $r_0 = (2p^{out} - id_{\mathcal{H}^{out}})$, $r = r_0 \oplus id_{\mathcal{H}^{in}}$ and $\widetilde{q} = rqr$, the space of homogeneous $\widetilde{q}$-vacua $\Omega_{\widetilde{q}} \in \mathscr{F}_{(p^{out})^\perp \mathcal{H}^{out} \oplus p^{in} \mathcal{H}^{in}}$ of the representation $\phi_{(p^{out})^\perp \oplus p^{in}}$ of the Clifford algebra $Cl(\mathcal{H}^{out} \oplus \mathcal{H}^{in}, R^{out} \oplus R^{in})$ is isomorphic with the space of homogeneous Hilbert-Schmidt operators $T_q\colon \mathscr{F}_{p^{in}\mathcal{H}^{in}} \to \mathscr{F}_{p^{out}\mathcal{H}^{out}}$ satisfying the commutation relation*

$$\phi_{p^{out}}(a(f^{out}))T_q = T_q \phi_{p^{in}}(a(f^{in}))$$

for every $(f^{out}, f^{in}) \in q(\mathcal{H}^{out} \oplus \mathcal{H}^{in})$.

Proof. To streamline the notation, let us write $\mathscr{F}^{\text{in}} := \mathscr{F}_{p^{\text{in}}\mathcal{H}^{\text{in}}}$, $\mathscr{F}^{\text{out}} := \mathscr{F}_{p^{\text{out}}\mathcal{H}^{\text{out}}}$ and $\mathscr{F}_\perp^{\text{out}} := \mathscr{F}_{(p^{\text{out}})^\perp \mathcal{H}^{\text{out}}}$ for the spaces on which the operators $\phi_{p^{\text{in}}}$, $\phi_{p^{\text{out}}}$ and $\phi_{(p^{\text{out}})^\perp}$ are defined.

To begin, let us assume that there exists a $\widetilde{q}$-vacuum $\Omega_{\widetilde{q}} \in \mathscr{F}_{(p^{\mathrm{out}})^\perp\mathcal{H}^{\mathrm{out}}\oplus p^{\mathrm{in}}\mathcal{H}^{\mathrm{in}}}$. Since r_0 is invertible and $(r_0 f^{\mathrm{out}}, f^{\mathrm{in}}) \in \widetilde{q}(\mathcal{H}^{\mathrm{out}} \oplus \mathcal{H}^{\mathrm{in}})$ if and only if $(f^{\mathrm{out}}, f^{\mathrm{in}}) \in q(\mathcal{H}^{\mathrm{out}} \oplus \mathcal{H}^{\mathrm{in}})$, the $\widetilde{q}$-vacuum equation from Definition 2.9 becomes

$$\phi_{(p^{\mathrm{out}})^\perp\oplus p^{\mathrm{in}}}(a(r_0 f^{\mathrm{out}}, f^{\mathrm{in}}))^*\Omega_{\widetilde{q}} = 0$$

for every $(f^{\mathrm{out}}, f^{\mathrm{in}}) \in q(\mathcal{H}^{\mathrm{out}} \oplus \mathcal{H}^{\mathrm{in}})$. Identifying the space $\mathscr{F}_{(p^{\mathrm{out}})^\perp\mathcal{H}^{\mathrm{out}}\oplus p^{\mathrm{in}}\mathcal{H}^{\mathrm{in}}}$ with the space $\mathscr{F}^{\mathrm{out}}_\perp \otimes \mathscr{F}^{\mathrm{in}}$ as in Theorem 2.13, the $\widetilde{q}$-vacuum equation becomes

$$\Big(\phi_{(p^{\mathrm{out}})^\perp}(a(r_0 f^{\mathrm{out}}))^*\widehat{\otimes}\mathrm{id}_{\mathscr{F}^{\mathrm{in}}} + \mathrm{id}_{\mathscr{F}^{\mathrm{out}}_\perp}\widehat{\otimes}\phi_{p^{\mathrm{in}}}(a(f^{\mathrm{in}}))^*\Big)\Omega_{\widetilde{q}} = 0,$$

and setting $U := U^*_{R^{\mathrm{out}}} \otimes \mathrm{id}_{\mathscr{F}^{\mathrm{in}}}$, we see from Theorem 2.11 that the previous equation is equivalent with

$$U^*\Big(\phi_{p^{\mathrm{out}}}(a(f^{\mathrm{out}}))d_{\mathscr{F}^{\mathrm{out}}}\widehat{\otimes}\mathrm{id}_{\mathscr{F}^{\mathrm{in}}} + \mathrm{id}_{\mathscr{F}^{\mathrm{out}}}\widehat{\otimes}\phi_{p^{\mathrm{in}}}(a(f^{\mathrm{in}}))^*\Big)U\Omega_{\widetilde{q}} = 0.$$

Since the operator U^* is injective, the previous equation is equivalent with

$$-\left(\phi_{p^{\mathrm{out}}}(a(f^{\mathrm{out}}))d_{\mathscr{F}^{\mathrm{out}}}\widehat{\otimes}\mathrm{id}_{\mathscr{F}^{\mathrm{in}}}\right)U\Omega_{\widetilde{q}} = \left(\mathrm{id}_{\mathscr{F}^{\mathrm{out}}}\widehat{\otimes}\phi_{p^{\mathrm{in}}}(a(f^{\mathrm{in}}))^*\right)U\Omega_{\widetilde{q}}.$$

Letting $T_q = \mu(U\Omega_{\widetilde{q}})$, where μ is the Hilbert space isomorphism $\mu\colon \mathscr{F}^{\mathrm{out}} \otimes \mathscr{F}^{\mathrm{in}} \to \mathscr{I}_2(\mathscr{F}^{\mathrm{in}}, \mathscr{F}^{\mathrm{out}})$ from Proposition 2.2 and applying Theorem 2.5, we see that the previous equation is equivalent to

$$-\phi_{p^{\mathrm{out}}}(a(f^{\mathrm{out}}))d_{\mathscr{F}^{\mathrm{out}}}T_q = d_{\mathscr{F}^{\mathrm{out}}}^{P\left(\phi_{p^{\mathrm{in}}}(a(f^{\mathrm{in}}))^*\right)}T_q\phi_{p^{\mathrm{in}}}(a(f^{\mathrm{in}})),$$

where note that the map T_q is homogeneous since the vector $\Omega_{\widetilde{q}}$ was assumed to be homogeneous. Since $P\big(\phi_{p^{\mathrm{in}}}(a(f^{\mathrm{in}}))^*\big) = 1$, we see that the previous equation is equivalent with

$$\phi_{p^{\mathrm{out}}}(a(f^{\mathrm{out}}))T_q = T_q\phi_{p^{\mathrm{in}}}(a(f^{\mathrm{in}})).$$

Since the maps U and μ are injections, it follows that the map $\Omega_{\widetilde{q}} \mapsto T_q = \mu(U\Omega_{\widetilde{q}})$ is injective. Conversely, given a map T_q satisfying the previous equation, we can repeat the preceding computations in reverse order to get a vector $\Omega_{\widetilde{q}}$ satisfying the $\widetilde{q}$-vacuum equation and $T_q = \mu(U\Omega_{\widetilde{q}})$. Hence, the map $\Omega_{\widetilde{q}} \mapsto T_q = \mu(U\Omega_{\widetilde{q}})$ gives the desired isomorphism. ∎

Theorem 3.22 follows now as a simple corollary.

Proof of Theorem 3.22. It suffices to take

$$\mathcal{H}^{\mathrm{in}} = \bigoplus_{\mathrm{in}}\mathcal{H}, \quad \mathcal{H}^{\mathrm{out}} = \bigoplus_{\mathrm{out}}\mathcal{H},$$
$$R^{\mathrm{in}} = \bigoplus_{\mathrm{in}}\mathsf{R}, \quad R^{\mathrm{out}} = \bigoplus_{\mathrm{out}}\mathsf{R},$$
$$p^{\mathrm{in}} = \bigoplus_{\mathrm{in}}p, \quad p^{\mathrm{out}} = \bigoplus_{\mathrm{out}}p$$

and q as the projection onto $H^2(X)$ in the lemma. ∎

Finally, we note that there is an analog of Theorem 3.22 for the CFT functor, since Theorems 2.11 and 2.13 have analogs for the representations π_p of the CAR algebra, as noted in the remark after Theorem 2.11. More precisely, using the language of Chapter 3.1 we have

Theorem 3.24. *For every $X = (\Sigma, L, \{\beta_j\})$, the space of Hilbert-Schmidt maps satisfying (25,26) is isomorphic to the space of $\widetilde{q}$-vacua $\Omega_{\widetilde{q}} \in \mathscr{F}_{\bigoplus \mathcal{K}, \bigoplus_{out} q^{\perp} \oplus \bigoplus_{in} q}$ of the representation $\pi_{\bigoplus_{out} q^{\perp} \oplus \bigoplus_{in} q}$ of the algebra $CAR(\bigoplus_{j \in J_\Sigma} \mathcal{K})$ from Definition 2.10, where $\widetilde{q} = r\widehat{q}r$, $\widehat{q} \colon \bigoplus_{j \in J_\Sigma} \mathcal{K} \to \bigoplus_{j \in J_\Sigma} \mathcal{K}$ is the projection onto $H^2(X)$, $r = (id_{\bigoplus_{out} \mathcal{K}} - 2 \bigoplus_{out} q) \oplus id_{\bigoplus_{in} \mathcal{K}}$, and $\bigoplus_{in}$ and $\bigoplus_{out}$ have the obvious meanings.*

Proof. See [28, Lemma 4.8]. ■

As we will see in Theorem 4.1, the space of $\widetilde{q}$-vacua in Theorem 3.24 can be shown to be one-dimensional provided that the operator $\widetilde{q} - \bigoplus_{\mathrm{out}} q^{\perp} \oplus \bigoplus_{\mathrm{in}} q$ is Hilbert-Schmidt, which is easily seen to be equivalent with $\widehat{q} - \bigoplus_{\mathrm{out}} q^{\perp} \oplus \bigoplus_{\mathrm{in}} q$ being Hilbert-Schmidt. In terms of the triple X, this result says that the projective CFT functor is well defined as long as the geometry of the surface Σ in a neighborhood of each boundary circle does not differ too much from that of the complement of the unit disk $\mathbb{D} \subset \mathbb{C}$, see [28, Chapter 6]. A similar result holds for the BCFT functor, see Theorem 4.4.

4 The Shale-Stinespring theorem

We have seen in Theorem 3.22 that for the BCFT functor, the space E_X is isomorphic to the space of vacua of a Clifford algebra representation, but the dimension of the latter space remains open. To gain insight into this problem and to illustrate the difficulties that arise in the BCFT case, we present a classical result of Shale and Stinespring that sheds light on the corresponding question in the CFT case. At the end of this chapter, we modify the proof of the Shale-Stinespring theorem to obtain a corresponding result in the BCFT case.

Theorem 4.1 (Shale-Stinespring)**.** *Let $\mathcal{K}$ be a complex Hilbert space, let $p, q \colon \mathcal{K} \to \mathcal{K}$ be projections and let π_p, π_q be the representations of $CAR(\mathcal{K})$ on $\mathscr{F}_{\mathcal{K},p}$ and $\mathscr{F}_{\mathcal{K},q}$ from Chapter 2.4, respectively. If the operator $p - q$ is Hilbert-Schmidt, there exists a cyclic q-vacuum $\widetilde{\Omega}_q \in \mathscr{F}_{\mathcal{K},p}$.*

Remark. The following converse of Theorem 4.1 also holds: if there exists a q-vacuum $\widetilde{\Omega}_q \in \mathscr{F}_{\mathcal{K},p}$, then the operator $p - q$ is Hilbert-Schmidt, see [29] or [11].

Theorem 4.1 is usually known as the Shale-Stinespring equivalence criterion or Segal's equivalence criterion for the following reason.

Proposition 4.2. *Let $\mathcal{K}$ be a complex Hilbert space and let $p, q \colon \mathcal{K} \to \mathcal{K}$ be projections. The representations π_p and π_q are unitarily equivalent if and only if there exists a cyclic q-vacuum $\widetilde{\Omega}_q \in \mathscr{F}_{\mathcal{K},p}$.*

Proof. If π_p and π_q are unitarily equivalent, there exists a unitary map $U\colon \mathscr{F}_{\mathcal{K},q} \to \mathscr{F}_{\mathcal{K},p}$ such that $\pi_p(a(f)) = U\pi_q(a(f))U^*$ for every $f \in \mathcal{K}$. If $\Omega_q \in \mathscr{F}_{\mathcal{K},q}$ is the vacuum, it is easy to see that $\widetilde{\Omega}_q = U\Omega_q$ is a cyclic q-vacuum in $\mathscr{F}_{\mathcal{K},p}$ since Ω_q is cyclic.

Conversely, if $\widetilde{\Omega}_q$ is a cyclic q-vacuum in $\mathscr{F}_{\mathcal{K},p}$, let $\{f_j\}_{j=1}^{\infty}$ be an orthonormal basis of $\mathcal{K}$. Our goal is to define a map $U\colon \mathscr{F}_{\mathcal{K},q} \to \mathscr{F}_{\mathcal{K},p}$ by

$$U \prod_{k=1}^{n} \pi_q(a(f_{j_k}))\Omega_q = \prod_{j=1}^{n} \pi_p(a(f_{j_k}))\widetilde{\Omega}_q,$$

where we recall that we continue to arrange noncommutative products in the order of increasing index. The inner products

$$\langle \prod_{k=1}^{n} \pi_q(a(f_{j_k}))\Omega_q, \prod_{l=1}^{m} \pi_q(a(f_{j_l}))\Omega_q \rangle$$

and

$$\langle \prod_{k=1}^{n} \pi_p(a(f_{j_k}))\widetilde{\Omega}_q, \prod_{l=1}^{m} \pi_p(a(f_{j_l}))\widetilde{\Omega}_q \rangle$$

are easily seen to be equal by using the anticommutation relations of $\mathrm{CAR}(\mathcal{K})$ as in the proof of Theorem 3.8. Hence, it follows from the cyclicity of $\widetilde{\Omega}_q$ that U maps an orthonormal basis onto an orthonormal basis, so U is a well defined unitary map. Moreover, we have $\pi_p(a(f)) = U\pi_q(a(f))U^*$ for every $f \in \mathcal{K}$. ∎

Our proof of Theorem 4.1 follows [30] very closely. In particular, we begin with the following observation.

Lemma 4.3. *Let $\mathcal{K}$ be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$ and let $p, q\colon \mathcal{K} \to \mathcal{K}$ be projections. If $p - q$ is Hilbert-Schmidt, there exists a decomposition*

$$\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_p \oplus \mathcal{K}_q \oplus \mathcal{K}_1 \tag{52}$$

and a summable sequence (μ_j) with $0 < \mu_j < 1$ such that following properties hold. The operators p and q restrict to projections on each of the subspaces in (52),

$$\mathcal{K}_0 = \{f \in \mathcal{K}\colon pf = qf\}, \quad \mathcal{K}_p = \mathrm{Ran}(p) \cap \ker(q), \quad \mathcal{K}_q = \ker(p) \cap \mathrm{Ran}(q)$$

and $\dim(\mathcal{K}_p), \dim(\mathcal{K}_q) < \infty$. Moreover, $\mathcal{K}_1$ has a decomposition

$$\mathcal{K}_1 = \bigoplus_j V_j \tag{53}$$

into two-dimensional subspaces V_j such that p and q restrict to projections on each V_j, and for each V_j there exist unit vectors $g_j, h_j \in V_j$ such that

$$\mathrm{Ran}(p|_{V_j}) = \mathrm{span}\{g_j\}, \quad \mathrm{Ran}(q|_{V_j}) = \mathrm{span}\{h_j\}, \quad V_j = \mathrm{span}\{g_j, h_j\}$$

and $\langle g_j, h_j\rangle = \sqrt{1-\mu_j}$.

Proof. By complexfication, it suffices to consider the case where $\mathcal{K}$ is complex. Since a product of Hilbert-Smith operators is trace class, we have $(p-q)^2 \in \mathscr{I}_1$. Moreover, $(p-q)^2$ is clearly self-adjoint, and the spectrum of $(p-q)^2$ is contained in $[0,1] \subset \mathbb{R}$. Hence, Proposition 2.1 implies that there exists an orthonormal basis $\{u_j\}$ of $\mathcal{K}$ and a summable sequence (λ_j) such that $\lambda_j \in [0,1]$ and $(p-q)^2 u_j = \lambda_j u_j$. Hence, we have

$$\mathcal{K} = \bigoplus_{\lambda} \mathcal{K}^{(\lambda)},$$

where the subspaces $\mathcal{K}^{(\lambda)} = \operatorname{span}\{u_j \mid \lambda_j = \lambda\}$ are finite-dimensional for each $\lambda > 0$.

A direct computation shows that

$$[(p-q)^2, p] = 0 = [(p-q)^2, q],$$

so for $f \in \mathcal{K}^{(\lambda)}$ we have

$$(p-q)^2 pf = \lambda pf, \quad (p-q)^2 qf = \lambda qf.$$

In particular, it follows that $p|_{\mathcal{K}^{(\lambda)}}$ and $q|_{\mathcal{K}^{(\lambda)}}$ are projections. Let us consider the cases $\lambda = 0, \lambda = 1$ and $0 < \lambda < 1$ separately.

$\boxed{\lambda = 0}$ If $f \in \mathcal{K}^{(0)}$, we have

$$0 = \langle (p-q)^2 f, f \rangle = \langle (p-q)f, (p-q)f \rangle = \|(p-q)f\|^2,$$

which implies $(p-q)f = 0$. Conversely, it is clear that $pf = qf$ implies $f \in \mathcal{K}^{(0)}$. We can thus take $\mathcal{K}_0 = \mathcal{K}^{(0)}$ in (52).

$\boxed{\lambda = 1}$ Let $f \in \mathcal{K}^{(1)}$ be a unit vector. Since $p|_{\mathcal{K}^{(1)}}$ is a projection, we have $\mathcal{K}^{(1)} = \operatorname{Ran}(p|_{\mathcal{K}^{(1)}}) \oplus \ker(p|_{\mathcal{K}^{(1)}})$. If $f \in \ker(p|_{\mathcal{K}^{(1)}})$, we have

$$1 = \langle f, f \rangle = \langle (p-q)^2 f, f \rangle = \langle (p-q)f, (p-q)f \rangle = \langle qf, qf \rangle = \langle f, qf \rangle,$$

which implies that $f \in \operatorname{Ran}(q|_{\mathcal{K}^{(1)}})$. Similarly, if $f \in \operatorname{Ran}(q|_{\mathcal{K}^{(1)}})$, we have

$$0 = 1 - \langle (p-q)f, (p-q)f \rangle = 1 - \langle pf - f, pf - f \rangle = \langle f, pf \rangle,$$

which implies $f \in \ker(p|_{\mathcal{K}^{(1)}})$, and we thus get $\ker(p|_{\mathcal{K}^{(1)}}) = \operatorname{Ran}(q|_{\mathcal{K}^{(1)}})$. Hence, we have $\mathcal{K}^{(1)} = \operatorname{Ran}(p|_{\mathcal{K}^{(1)}}) \oplus \operatorname{Ran}(q|_{\mathcal{K}^{(1)}})$.

Conversely, it is easy to see that for any $f \in \mathcal{K}$, the conditions $pf = 1, qf = 0$ and $pf = 0, qf = 1$ both imply $f \in \mathcal{K}^{(1)}$. Hence, we can take $\mathcal{K}_p = \operatorname{Ran}(p|_{\mathcal{K}^{(1)}})$ and $\mathcal{K}_q = \operatorname{Ran}(q|_{\mathcal{K}^{(1)}})$ in (52). Moreover, it is clear that $\dim(\mathcal{K}_p), \dim(\mathcal{K}_q) < \infty$ since $p-q$ is Hilbert-Schmidt.

$\boxed{0 < \lambda < 1}$ Let $\mathcal{A}_\lambda \subset \operatorname{End}(K^{(\lambda)})$ be the algebra generated by the self-adjoint operators $p|_{\mathcal{K}^{(\lambda)}}$ and $q|_{\mathcal{K}^{(\lambda)}}$ on the Hilbert space $K^{(\lambda)}$. If $W \subset K^{(\lambda)}$ is an irreducible submodule, it follows from the self-adjointness of the generators that so is $W^\perp$. Hence, by induction on dimension and the finite-dimensionality of $\mathcal{K}^{(\lambda)}$ we obtain a decomposition $\mathcal{K}^{(\lambda)} = \bigoplus_j V_j^{(\lambda)}$ of $\mathcal{K}^{(\lambda)}$ into irreducible submodules $V_j^{(\lambda)}$. Since

$$\mathcal{A}_\lambda = \operatorname{span}\{A_1 \dots A_n \bigm| n \in \mathbb{Z}_{\geq 0}, A_i = p|_{\mathcal{K}^{(\lambda)}} \text{ or } A_i = q|_{\mathcal{K}^{(\lambda)}}\}$$

and we have the relations $p^2 = p$, $q^2 = q$ and

$$\lambda \mathrm{id}_{\mathcal{K}^{(\lambda)}} = (p-q)^2|_{\mathcal{K}^{(\lambda)}} = p|_{\mathcal{K}^{(\lambda)}} - p|_{\mathcal{K}^{(\lambda)}} q|_{\mathcal{K}^{(\lambda)}} - q|_{\mathcal{K}^{(\lambda)}} p|_{\mathcal{K}^{(\lambda)}} + q|_{\mathcal{K}^{(\lambda)}},$$

it follows that $\dim(\mathcal{A}_\lambda) \le 4$. Hence, it follows from basic representation theory of finite-dimensional algebras (see e.g. [13, Corollary 3.5.5]) that $\dim(V_j^{(\lambda)}) \le 2$ for each j.

Since $p|_{V_j^{(\lambda)}}$ and $q|_{V_j^{(\lambda)}}$ are projections, $\dim(V_j^{(\lambda)}) = 1$ implies that we have either $p|_{V_j^{(\lambda)}} = q|_{V_j^{(\lambda)}}$, $\mathrm{Ran}(p|_{V_j^{(\lambda)}}) = \ker(q|_{V_j^{(\lambda)}})$ or $\ker(p|_{V_j^{(\lambda)}}) = \mathrm{Ran}(q|_{V_j^{(\lambda)}})$. However, each of these three conditions were shown to be equivalent with $V_j^{(\lambda)} \subset \mathcal{K}^{(0)}$ or $V_j^{(\lambda)} \subset \mathcal{K}^{(1)}$ above. Hence, we have $\dim(V_j^{(\lambda)}) = 2$.

Moreover, irreducibility of $V_j^{(\lambda)}$ implies that

$$\dim\left(\mathrm{Ran}(p|_{V_j^{(\lambda)}})\right) = 1 = \dim\left(\mathrm{Ran}(q|_{V_j^{(\lambda)}})\right)$$

and

$$\mathrm{Ran}(p|_{V_j^{(\lambda)}}) \not\perp \mathrm{Ran}(q|_{V_j^{(\lambda)}}), \quad \mathrm{Ran}(p|_{V_j^{(\lambda)}}) \nparallel \mathrm{Ran}(q|_{V_j^{(\lambda)}}),$$

so there exist unit vectors $e_j^{(\lambda)}, f_j^{(\lambda)} \in V_j^{(\lambda)}$ such that

$$\mathrm{Ran}(p|_{V_j^{(\lambda)}}) = \mathrm{span}\{e_j^{(\lambda)}\}, \quad \mathrm{Ran}(q|_{V_j^{(\lambda)}}) = \mathrm{span}\{f_j^{(\lambda)}\}, \quad V_j^{(\lambda)} = \mathrm{span}\{e_j^{(\lambda)}, f_j^{(\lambda)}\}$$

and $\langle e_j^{(\lambda)}, f_j^{(\lambda)}\rangle > 0$. For every $g = \alpha e_j^{(\lambda)} + \beta f_j^{(\lambda)} \in V_j^{(\lambda)}$ with $\alpha, \beta \in \mathbb{C}$ we have

$$p|_{V_j^{(\lambda)}} g = \langle g, e_j^{(\lambda)}\rangle e_j^{(\lambda)}, \quad q|_{V_j^{(\lambda)}} g = \langle g, f_j^{(\lambda)}\rangle f_j^{(\lambda)},$$

so a short computation gives

$$\begin{aligned}\lambda(\alpha e_j^{(\lambda)} + \beta f_j^{(\lambda)}) &= (p-q)^2|_{V_j^{(\lambda)}}(\alpha e_j^{(\lambda)} + \beta f_j^{(\lambda)}) \\ &= \alpha(1 - \langle e_j^{(\lambda)}, f_j^{(\lambda)}\rangle^2) e_j^{(\lambda)} + \beta(1 - \langle e_j^{(\lambda)}, f_j^{(\lambda)}\rangle^2) f_j^{(\lambda)}.\end{aligned}$$

By linear independence of the set $\{e_j^{(\lambda)}, f_j^{(\lambda)}\}$ we thus have $\langle e_j^{(\lambda)}, f_j^{(\lambda)}\rangle = \sqrt{1-\lambda}$.

Hence, we can take $\mathcal{K}_1 = \bigoplus_{0<\lambda<1} \mathcal{K}^{(\lambda)}$ in (52). Moreover, we can take each V_j to be some $V_i^{(\lambda)}$ in (53) and let g_j, h_j and μ_j be the corresponding $e_j^{(\lambda)}, f_j^{(\lambda)}$ and λ, respectively. ■

Lemma 4.3 allows us to construct the cyclic vacuum vector in Theorem 4.1 quite explicitly.

Proof of Theorem 4.1. Let $\{g_j\}, \{h_j\}$ and (μ_j) be as in Lemma 4.3 and let

$$\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_p \oplus \mathcal{K}_q \oplus \mathcal{K}_1$$

and

$$\mathcal{K}_1 = \bigoplus_j V_j$$

be the corresponding decompositions of $\mathcal{K}$ and $\mathcal{K}_1$. Moreover, let $\{\widetilde{g}_j\}_{j=1}^M$ and $\{\widetilde{h}_j\}_{j=1}^N$ be orthonormal bases of $\mathcal{K}_p$ and $\mathcal{K}_q$, respectively. Finally, let $e_j = \widetilde{g}_j$ if $1 \le j \le M$ and $e_j = g_{j-M}$ if $j > M$. Similarly, let $f_j = \widetilde{h}_j$ if $1 \le j \le N$ and $f_j = h_{j-N}$ if $j > N$.

For each $n \in \mathbb{Z}_{>0}$, define an element $\eta_n \in \mathscr{F}_{\mathcal{K},p}$ by

$$\eta_n = \alpha_n \prod_{i=1}^{N+n} \pi_p(a(f_i)) \prod_{j=1}^{M+n} \pi_p(a(e_j))^* \Omega_p,$$

where $\alpha_n = (-1)^{nN+n(n-1)/2}$ and $\Omega_p \in \mathscr{F}_{\mathcal{K},p}$ is the vacuum vector. We proceed in two steps. First, we show that η_n converges to a vector $\eta \in \mathscr{F}_{\mathcal{K},p}$ as $n \to \infty$ by showing that (η_j) is a Cauchy sequence. Then we verify that η is a q-vacuum.

Step 1: Convergence of (η_n). It follows from the anticommutation relations in (16) that $\pi_p(a(e_j))\eta_n = 0$ if $j > n + M$. Hence, for $n > m$ we have

$$\begin{aligned}
\langle \eta_n, \eta_m \rangle &= \alpha_n \alpha_m \left\langle \prod_{i=1}^{N+n} \pi_p(a(f_i)) \prod_{j=1}^{M+n} \pi_p(a(e_j))^* \Omega_p, \prod_{i=1}^{N+m} \pi_p(a(f_i)) \prod_{j=1}^{M+m} \pi_p(a(e_j))^* \Omega_p \right\rangle \\
&= (-1)^{(n-m)(m+N)} \alpha_n \alpha_m \left\langle \prod_{i=N+m+1}^{N+n} \pi_p(a(f_i)) \prod_{j=M+m+1}^{M+n} \pi_p(a(e_j))^* \eta_m, \eta_m \right\rangle \\
&= (-1)^{(n-m)(n-m-1)/2+(n-m)(m+N)} \alpha_n \alpha_m \left(\prod_{j=m+1}^{n} \sqrt{1-\mu_j} \right) \langle \eta_m, \eta_m \rangle \\
&= \left(\prod_{j=m+1}^{n} \sqrt{1-\mu_j} \right) \langle \eta_m, \eta_m \rangle,
\end{aligned}$$

where in the third equality we used (16) to move each $\pi_p(a(e_j))^*$ all the way to the left so that its adjoint annihilates the vector η_m in the second slot of the inner product.

For each $j > M$, let $e_j^\perp$ be a unit vector in V_{j-M} such that $\langle e_j, e_j^\perp \rangle = 0$ and $\langle e_j^\perp, f_{N+j-M} \rangle > 0$. It follows from $\langle e_j, f_{N+j-M} \rangle = \sqrt{1-\mu_{j-M}}$ that for $j > N$ we have

$$f_j = \sqrt{1-\mu_{j-N}}\, e_{M+j-N} + \sqrt{\mu_{j-N}}\, e_{M+j-N}^\perp,$$

so we get

$$\begin{aligned}
\eta_m =& \alpha_m \prod_{i=M+1}^{M+m} \left(\sqrt{1-\mu_{i-M}}\, \pi_p(a(e_i)) + \sqrt{\mu_{i-M}}\, \pi_p(a(e_i^\perp)) \right) \\
& \times \prod_{j=1}^{N} \pi_p(a(f_j)) \prod_{k=1}^{M+m} \pi_p(a(e_k))^* \Omega_p \\
=& \sum_{l=0}^{m} \sum_{\substack{S \subset \{M+1,\dots,M+m\} \\ |S|=l}} (-1)^{\#} \xi_S,
\end{aligned}$$

where

$$\xi_S = \prod_{i\in S}\sqrt{1-\mu_{i-M}}\pi_p(a(e_i))\prod_{j\in S^c}\sqrt{\mu_{j-M}}\pi_p(a(e_j^\perp))\prod_{j=1}^{N}\pi_p(a(f_j))\prod_{k=1}^{M+m}\pi_p(a(e_k))^*\Omega_p$$

and # stands for an integer constant that is not important to us. It follows from (16) that the vectors ξ_S satisfy

$$\langle \xi_S, \xi_{\widetilde{S}}\rangle = \delta_{S,\widetilde{S}}C_S$$

for some $C_S \geq 0$, and it follows from (11) that we actually have

$$C_S = \prod_{i\in S}(1-\mu_{i-M})\prod_{j\in S^c}\mu_{j-M}.$$

Hence, we get

$$\langle \eta_m, \eta_m\rangle = \sum_{l=0}^{m}\sum_{\substack{S\subset\{1,\ldots,m\}\\|S|=l}}\prod_{i\in S}(1-\mu_i)\prod_{j\in S^c}\mu_j = 1.$$

For $0<\mu<1$ we have the elementary inequality $-\mu/(1-\mu) < \log(1-\mu) < -\mu$, so it follows from $\sum_j \mu_j < \infty$ that $\sum_{j=1}^{\infty}\log(1-\mu_j) > -\infty$. Hence, we get

$$\prod_{j=m+1}^{n}\sqrt{1-\mu_j} = e^{\frac{1}{2}\sum_{j=m+1}^{n}\log(1-\mu)} \to 1$$

as $n, m\to\infty$. It follows that

$$\|\eta_n - \eta_m\| = \|\eta_n\|^2 - \langle\eta_n,\eta_m\rangle - \langle\eta_m,\eta_n\rangle + \|\eta_m\| = 2\left(1-\prod_{j=m+1}^{n}\sqrt{1-\mu_j}\right)\to 0$$

as $n, m\to\infty$ so (η_n) is a Cauchy sequence converging to a limit $\eta\in\mathscr{F}_{\mathcal{K},p}$.

Step 2: η is a q-vacuum. We first note that to establish the vacuum equations

$$\pi_p(a(f))\eta = 0, \quad f\in q\mathcal{K} \tag{54}$$
$$\pi_p(a(f))^*\eta = 0, \quad f\in q^\perp\mathcal{K}, \tag{55}$$

it suffices to show that they hold for f in a set with dense span in $q\mathcal{K}$ and $q^\perp\mathcal{K}$, respectively, since the map $f\to\pi_p(a(f))$ is linear and continuous in operator topology by Theorem 2.6.

It follows from (16) that $\pi_p(a(f))\eta_n = 0$ for all $f\in\mathcal{K}_0\cap\mathrm{Ran}(q)$ and $f\in\mathcal{K}_q$ and any $n\geq 1$. Moreover, we have $\pi_p(a(f_j))\eta_n = 0$ as soon as $n > j$, so (54) follows from boundedness of $\pi_p(a(f))$.

Moreover, we clearly have $\pi_p(a(f))^*\eta_n = 0$ for all $f\in\mathcal{K}_0\cap\ker(q)$ and $f\in\mathcal{K}_p$ and any $n\geq 1$. For every $j > N$, let $f_j^\perp$ be a unit vector in V_{j-N} such that $\langle f_j, f_j^\perp\rangle = 0$

and $V_{j-N} = \operatorname{span}\{f_j, f_j^\perp\}$. It suffices to show that for every j there exists an index n_j such that $\pi_p(a(f_j^\perp))^*\eta_n = 0$ for $n > n_j$. It follows from (16) that

$$\pi_p(a(f_j^\perp))^*\eta_n = (-1)^{\#}\alpha_n \left(\prod_{i=1}^{N+n} \pi_p(a(f_i))\right) \pi_p(a(f_j^\perp))^* \prod_{j=1}^{M+n} \pi_p(a(e_j))^*\Omega_p,$$

and since $f_j^\perp = \sqrt{\mu_{j-N}} e_{M+j-N} + \sqrt{1-\mu_{j-N}} e_{M+j-N}^\perp$, we actually have that

$$\pi_p(a(f_j^\perp))^*\eta_n = (-1)^{\#}\alpha_n \left(\prod_{i=1}^{N+n} \pi_p(a(f_i))\right) \sqrt{\mu_{j-N}}\pi_p(a(e_{M+j-N}))^* \prod_{j=1}^{M+n} \pi_p(a(e_j))^*\Omega_p.$$

Since $\pi_p(a(e_{M+j-N}))^* \prod_{j=1}^{M+n} \pi_p(a(e_j))^*\Omega_p = 0$ as soon as $n > j - N$, we can take $n_j = j - N$ above. Hence, (55) follows as above.

Finally, cyclicity of η follows from irreducibility of π_p.

■

Remark. It is easy to see that Theorem 4.1 does not hold for the representation ϕ_p of the Clifford algebra $\mathrm{Cl}(\mathcal{H}, R)$ from (17) without further assumptions on q in the sense that there may not be a q-vacuum in $\mathscr{F}_{p\mathcal{H}}$ even if $p - q$ is Hilbert-Schmidt.

As a concrete example, let $\mathcal{H}$ be an infinite-dimensional real Hilbert space and let $\{g_i\}_{i=1}^\infty$ be an orthonormal basis of $\mathcal{H}$. Define vectors

$$\begin{aligned} e_i &= g_{2i-1} \\ f_i &= \sqrt{1-2^{-i}}g_{2i-1} + \sqrt{2^{-i}}g_{2i} \end{aligned}$$

for $i \in \mathbb{Z}_{>0}$ and let p be the projection onto $\mathrm{cl}(\operatorname{span}\{e_i\}_{i=1}^\infty)$ and let q be the projection onto $\mathrm{cl}(\operatorname{span}\{f_i\}_{i=1}^\infty)$. Finally, define a linear map $R\colon \mathcal{H} \to \mathcal{H}$ by $Re_{2i} = e_{2i-1}$ and $Re_{2i-1} = e_{2i}$ for every $i \in \mathbb{Z}_{>0}$.

The map R is clearly self-adjoint and unitary, and we have $R\colon p\mathcal{H} \cong p^\perp\mathcal{H}$. Moreover, $\operatorname{span}\{e_{2i-1}, e_{2i}\}$ is an eigenspace of $(p-q)^2$ with eigenvalue 2^{-i} for every $i \in \mathbb{Z}_{>0}$, so $p - q$ is definitely Hilbert-Schmidt.

If S denotes the collection of all finite sets of strictly positive integers, the prospective q-vacuum $\widetilde{\Omega}_q \in \mathscr{F}_{p\mathcal{H}}$ can be written in terms of the basis (19) as

$$\widetilde{\Omega}_q = \sum_{s\in S} C_s \bigwedge_{i\in s} e_i,$$

where $C_s \in \mathbb{C}$, the wedge products in $\bigwedge_{i\in s} e_i$ are taken in order of increasing i and the series is understood to converge in the Hilbert space topology on $\mathcal{H}$. Moreover, the vacuum equation from Definition 2.9 can be written equivalently as

$$\phi_p(a(f_i))^*\widetilde{\Omega}_q = 0 \tag{56}$$

for every $i \in \mathbb{Z}_{>0}$.

However, we are in trouble already at $i = 1$ in (56). Indeed, for $i = 1$ the vacuum equation reads

$$\begin{aligned} 0 &= \frac{1}{\sqrt{2}}(a(e_1) + a(e_1)^*) \sum_{s \in S} C_s \bigwedge_{i \in s} e_i \\ &= \frac{1}{\sqrt{2}} \sum_{\substack{s \in S \\ 1 \notin s}} C_s \bigwedge_{i \in \{1\} \cup s} e_i + \frac{1}{\sqrt{2}} \sum_{\substack{s \in S \\ 1 \in s}} C_s \bigwedge_{i \in s \setminus \{1\}} e_i, \end{aligned}$$

so we get $C_s = 0$ for all $s \in S$, which implies $\widetilde{\Omega}_q = 0$.

The problem in the previous remark is that the operators $\phi_p(a(f))$ and $\phi_p(a(g))$ do not necessarily anticommute even if $f, g \in q\mathcal{H}$. However, if the projections p and q are both R-compatible, one can prove an analog of the Shale-Stinespring theorem for the representations ϕ_p.

Theorem 4.4. *Let $\mathcal{H}$ be a real Hilbert space, let $p, q \colon \mathcal{H} \to \mathcal{H}$ be projections and let $R \colon \mathcal{H} \to \mathcal{H}$ be a unitary involution. If p and q are R-compatible, $p - q$ is trace class, and the eigenvalues λ of $p - q$ with $0 < \lambda < 1$ have multiplicity 4, then there exists a cyclic q-vacuum in $\mathscr{F}_{p\mathcal{H}}$.*

Proof. We begin as in the proof of Theorem 4.1. If $\{g_j\}, \{h_j\}$ and (μ_j) are as in Lemma 4.3, and

$$\mathcal{K} = \mathcal{H}_0 \oplus \mathcal{H}_p \oplus \mathcal{H}_q \oplus \mathcal{H}_1$$

and

$$\mathcal{H}_1 = \bigoplus_j V_j$$

are the corresponding decompositions of $\mathcal{H}$ and $\mathcal{H}_1$, it follows from the proof of Lemma 4.3 that $\sum_{k=1}^{\infty} \sqrt{\mu_k} < \infty$ since $p - q$ is trace class.

Continuing as in the proof of Theorem 4.1, let $\{\widetilde{g}_j\}_{j=1}^M$ and $\{\widetilde{h}_j\}_{j=1}^N$ be orthonormal bases of $\mathcal{H}_p$ and $\mathcal{H}_q$, respectively. Moreover, let $e_j = \widetilde{g}_j$ if $1 \le j \le M$ and $e_j = g_{j-M}$ if $j > M$. Similarly, let $f_j = \widetilde{h}_j$ if $1 \le j \le N$ and $f_j = h_{j-N}$ if $j > N$. Finally, for each $n \in \mathbb{Z}_{\ge 0}$, define an element $\eta_n \in \mathscr{F}_{p\mathcal{H}}$ by

$$\eta_n = \alpha_n \prod_{i=N+1}^{N+n} \phi_p(a(f_i))^* \prod_{j=1}^{M+n} \phi_p(a(e_j)) \Omega_p,$$

where $\alpha_n = (-1)^{n(n-1)/2}$ and $\Omega_p \in \mathscr{F}_{p\mathcal{H}}$ is the vacuum vector. We begin by showing that if a subsequence of (η_n) converges to a nonzero limit $\eta \in \mathscr{F}_{p\mathcal{H}}$, then η is the cyclic q-vacuum.

It follows easily from (17) that

$$\{\phi_p(a(f)), \phi_p(a(g))\} = \langle f, Rg \rangle \tag{57}$$

$$\{\phi_p(a(f))^*, \phi_p(a(g))\} = \langle f, g \rangle \tag{58}$$

for every $f, g \in \mathcal{H}$. Since q is R-compatible, it follows from (57) that for $n+N > j > N$ we have

$$\begin{aligned}
&\phi_p(a(f_j))^*\eta_n \\
=&\phi_p(a(f_j))^*\alpha_n \prod_{i=N+1}^{N+n} \phi_p(a(f_i))^* \prod_{l=1}^{M+n} \phi_p(a(e_l))\Omega_p \\
=&(-1)^{\#}\alpha_n \left(\prod_{i=j+1}^{N+n} \phi_p(a(f_i))^*\right) (\phi_p(a(f_j))^*)^2 \prod_{k=N+1}^{j-1} \phi_p(a(f_k))^* \prod_{l=1}^{M+n} \phi_p(a(e_l))\Omega_p = 0,
\end{aligned}$$

where we used the fact that $(\phi_p(a(f_j))^*)^2{=}0$.

Moreover, it follows from $Rp = p^{\perp}R$ that

$$(p-q)^2 R = R(p-q)^2, \tag{59}$$

which implies that $R\colon \mathcal{H}_p \cong \mathcal{H}_q$. Hence, we get for $1 \leq j \leq N$ that $Rf_j \in \operatorname{span}\{e_1, \ldots, e_M\}$ and thus

$$\begin{aligned}
&\phi_p(a(f_j))^*\eta_n \\
=&(-1)^{2n}\alpha_n \left(\prod_{i=N+1}^{N+n} \phi_p(a(f_i))^* \prod_{k=M+1}^{M+n} \phi_p(a(e_k))\right) \phi_p(a(f_j))^* \prod_{l=1}^{M} \phi_p(a(e_l))\Omega_p \\
=&(-1)^{2n}\alpha_n \left(\prod_{i=N+1}^{N+n} \phi_p(a(f_i))^* \prod_{k=M+1}^{M+n} \phi_p(a(e_k))\right) \phi_p(a(Rf_j)) \prod_{l=1}^{M} \phi_p(a(e_l))\Omega_p = 0
\end{aligned}$$

for any $n \in \mathbb{Z}_{\geq 0}$.

Finally, it follows from (57) and (58) that for any $f \in \mathcal{H}_0 \cap \operatorname{Ran}(q)$ and any n, we have

$$\phi_p(a(f))^*\eta_n = (-1)^{M+2n}\alpha_n \left(\prod_{i=N+1}^{N+n} \phi_p(a(f_i))^* \prod_{l=1}^{M+n} \phi_p(a(e_l))\right) \phi_p(a(f))^*\Omega_p \ = 0.$$

Hence, if a subsequence of (η_n) converges to a nonzero limit $\eta \in \mathscr{F}_{p\mathcal{H}}$, we see that η is the unique cyclic q-vacuum as in the proof of Theorem 4.1. It thus suffices to prove the convergence of a subsequence of (η_n). We proceed in three steps.

Step 1: recursive formula for $\|\eta_{2n}\|^2$. Anticommuting $\phi_p(a(f_{N+n}))^*$ and $\phi_p(a(e_{M+n}))$ past each other, we get

$$\begin{aligned}
\eta_n &= \alpha_n \prod_{i=N+1}^{N+n} \phi_p(a(f_i))^* \prod_{j=1}^{M+n} \phi_p(a(e_i))\Omega_p \\
&= (-1)^{\#}\sqrt{1-\mu_n}\eta_{n-1} + \xi_n,
\end{aligned}$$

where

$$\xi_n = (-1)^{\#} \left(\prod_{i=N+1}^{N+n-1} \phi_p(a(f_i))^*\right) \phi_p(a(e_{M+n}))\phi_p(a(f_{N+n}))^* \left(\prod_{j=1}^{M+n-1} \phi_p(a(e_j))\right) \Omega_p.$$

If $e_{M+n}^{\perp} \in V_n$ is a unit vector such that $\langle e_{M+n}, e_{M+n}^{\perp}\rangle = 0$ and $\langle f_{N+n}, e_{M+n}^{\perp}\rangle > 0$, we have

$$\phi_p(a(f_{N+n}))^* = \sqrt{1-\mu_n}\phi_p(a(e_{M+n}))^* + \sqrt{\mu_n}\phi_p(a(Re_{M+n}^{\perp})),$$

where the first term on the right-hand side annihilates the vacuum in ξ_n, so we get

$$\xi_n = (-1)^{\#}\sqrt{\mu_n}\phi_p(a(e_{M+n}))\phi_p(a(e^{(M+n)}))\eta_{n-1},$$

where $e^{(M+n)}$ is the projection of $Re_{M+n}^{\perp}$ onto $\operatorname{span}\{e_1,\dots,e_{M+n}\}^{\perp}$.

However, it follows from (59) and the condition on the eigenvalues of $p-q$ that $R\colon V_{2n}\oplus V_{2n-1} \to V_{2n}\oplus V_{2n-1}$, so we have that $\xi_{2n} = 0$ and thus

$$\begin{aligned}\eta_{2n} = {} & (-1)^{\#}\sqrt{1-\mu_{2n}}\sqrt{1-\mu_{2n-1}}\eta_{2(n-1)} \\ & +(-1)^{\#}\sqrt{1-\mu_{2n}}\sqrt{\mu_{2n-1}}\phi_p(a(e_{M+2n-1}))\phi_p(a(e^{(M+2n-1)}))\eta_{2(n-1)}.\end{aligned}$$

The terms above are easily seen to be orthogonal by an application of $R\colon V_{2n}\oplus V_{2n-1} \to V_{2n}\oplus V_{2n-1}$, so we get

$$\|\eta_{2n}\|^2 = (1-\mu_{2n})(1-\mu_{2n-1})\|\eta_{2(n-1)}\|^2 + (1-\mu_{2n})\mu_{2n-1}C_{2n}$$

for a constant $0 \le C_{2n} \le 1$.

Hence, it follows by induction from $\|\eta_0\| = 1$ that

$$\|\eta_{2n}\|^2 = \prod_{j=1}^{2n}(1-\mu_j) + \sum_{k=1}^{n} a_k$$

for some $0 \le a_k \le \mu_{2k-1}$. Since the sequence (μ_j) is summable and $0 < \mu_j < 1$, it follows that $\|\eta_{2n}\|^2$ converges to a nonzero limit as in the proof of Theorem 4.1. In particular, if (η_{2n}) converges to a limit $\eta \in \mathscr{F}_{p\mathcal{H}}$, we have $\eta \neq 0$.

Step 2: Estimate for $\langle \eta_n, \eta_m\rangle$. For $l > n+M$ we have

$$\begin{aligned}\phi_p(a(e_l))^*\eta_n &= \phi_p(a(e_l))^*\alpha_n \prod_{i=N+1}^{N+n}\phi_p(a(f_i))^* \prod_{j=1}^{M+n}\phi_p(a(e_j))\Omega_p \\ &= \alpha_n \sum_{k=N+1}^{N+n}(-1)^{k-N+1}\langle e_l, Rf_k\rangle \prod_{\substack{i=N+1\\ i\neq k}}^{N+n}\phi_p(a(f_i))^* \prod_{j=1}^{M+n}\phi_p(a(e_j))\Omega_p,\end{aligned}$$

so we get for $n > m$ that

$$\begin{aligned}\langle \eta_n, \eta_m\rangle &= \alpha_n\alpha_m \left\langle \prod_{i=N+1}^{N+n}\phi_p(a(f_i))^* \prod_{j=1}^{M+n}\phi_p(a(e_j))\Omega_p, \eta_m\right\rangle \\ &= (-1)^{(n-m)m}\alpha_n\alpha_m \left\langle \prod_{i=N+m+1}^{N+n}\phi_p(a(f_i))^* \prod_{j=M+m+1}^{M+n}\phi_p(a(e_j))\eta_m, \eta_m\right\rangle \\ &= (-1)^{(n-m)m+(n-m)(n-m-1)/2}\alpha_n\alpha_m \prod_{j=m+1}^{n}\sqrt{1-\mu_j}\langle \eta_m, \eta_m\rangle + \mathcal{E}(n,m) \\ &= \|\eta_m\|^2 \prod_{j=m+1}^{n}\sqrt{1-\mu_j} + \mathcal{E}(n,m)\end{aligned}$$

where

$$\begin{aligned}\mathcal{E}(n,m) =&(-1)^{(n-m)(m+1)}\alpha_n\alpha_m \\ &\times\sum_{k=1}^{n-m}\left(\prod_{j=1}^{k-1}(-1)^{n-m-j}\sqrt{1-\mu_{n-j+1}}\right) \\ &\times\left\langle\prod_{i=N+m+1}^{N+n-k+1}\phi_p(a(f_i))^*\prod_{j=M+m+1}^{M+n-k}\phi_p(a(e_j))\eta_m,\right. \\ &\left.\sum_{l=N+1}^{N+m}(-1)^{l-N+1}\langle e_{M+n-k+1}, Rf_l\rangle\prod_{\substack{i=N+1\\ i\neq l}}^{N+m}\phi_p(a(f_i))^*\prod_{j=1}^{M+m}\phi_p(a(e_j))\Omega_p\right\rangle\end{aligned}$$

and the products above are considered empty if the first index is smaller than the last one. Since $R\colon V_{2n}\oplus V_{2n-1}\to V_{2n}\oplus V_{2n-1}$, an application of $\|\phi_p(a(f))\xi\|\leq\|f\|\|\xi\|$ and the Cauchy-Schwarz inequality gives

$$\begin{aligned}|\mathcal{E}(n,m)|&\leq\sum_{k=M+m+1}^{M+n}\sum_{l=N+1}^{N+m}|\langle e_k, Rf_l\rangle| \\ &\leq 2\sum_{j=m+1}^{n}\sqrt{\mu_j},\end{aligned}$$

so we have $|\mathcal{E}(n,m)|\to 0$ as $n,m\to\infty$ since $\sum_{j=1}^{\infty}\sqrt{\mu_j}<\infty$.

Step 3: Conclusion. Similarly as in the proof of Theorem 4.1, we get

$$\begin{aligned}\|\eta_{2n}-\eta_{2m}\|^2&=\|\eta_{2n}\|^2-\langle\eta_{2n},\eta_{2m}\rangle-\langle\eta_{2m},\eta_{2n}\rangle+\|\eta_{2m}\|^2 \\ &=\|\eta_{2n}\|^2+\|\eta_{2m}\|^2-2\|\eta_{2m}\|^2\prod_{j=2m+1}^{2n}\sqrt{1-\mu_j}-2\Re(\mathcal{E}(2n,2m))\to 0\end{aligned}$$

as $n,m\to\infty$. Hence, the sequence (η_{2n}) is Cauchy and thus converges to a nonzero limit $\eta\in\mathscr{F}_{p\mathcal{H}}$. This finishes the proof. ■

Remark. The requirement that the eigenvalues of $p-q$ have multiplicity 4 is somewhat restrictive, but the proof applies almost verbatim also to the case where we only assume that the irreducible submodules of the algebra generated by $\{p,q,R\}$ are 4-dimensional. This latter condition is expected to follow from a more detailed consideration as in Lemma 4.3.

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