Enhanced elementary operations for quantum computing

Joni Ikonen
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Abstract

Superconducting circuits are one of the most promising and rapidly developing platforms for implementing quantum bits and their control operations for the purpose of large-scale quantum computing. A quantum algorithm is run by executing a series of precise operations on the quantum bits, or qubits. The most important of these operations are the quantum gates along with the measurement and initialization of the states of the qubits and auxiliary components. In this dissertation, we develop techniques to enhance each of these three operations. The work employs systems of coupled qubits and resonators in the framework of circuit quantum electrodynamics. Firstly, we derive a theoretical lower bound for the error of a single-qubit gate implemented with a linear oscillator mode and show how to reach this bound. Secondly, we experimentally demonstrate a technique for measuring the state of a qubit by simultaneously driving the qubit and the accompanying readout resonator. Compared to our method, the standard technique involving only the resonator drive yields up to 100% larger error. Moreover, we experimentally and theoretically develop quantum devices and protocols that may be used to initialize or reset auxiliary quantum circuits such as resonators. By coupling a quantum circuit to a strongly dissipative environment through a coupler, the decay rate of the circuit may be tuned on demand by several orders of magnitude. Finally, we propose a general method to analytically solve the dynamics of a complex, open bosonic system that may be used to describe, for example, the experimental systems of this dissertation. The experimental techniques developed in this dissertation may potentially be utilized in future implementations of a quantum computer, and the proposed theoretical protocols pave the way for future experiments aiming to improve qubit control and heat management in large-scale quantum processors.

Keywords superconducting quantum circuits, quantum operations, single-qubit gate, qubit

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Tässä väitöskirjassa kehitetään tapoja näiden kolmen operaation parantamiseksi. Työssä käsitellään kvanttielektrodynamikkaan perustuvia, kubitsien ja resonaattoriin muodostamaa kytkettyjä systeemejä.


Väitöskirjassa kehitettyjä kokeellisia menetelmiä on tulevaisuudessa mahdollista käyttää kvanttitiokoneen prototyppeissä. Esitetty teoreettiset tulokset tukevat tulevia kokeita, jotka täyttävät kvanttipilat ja lämmönhallinnan parantamiseen suuren kokoluokan kvanttiprocesoireissa.
The work for this dissertation was carried out in the Quantum Computing and Devices (QCD) group at the Department of Applied Physics, Aalto University, during the years 2016–2020. Here, I would like to thank all the great people who helped me along the way and made the process possible and enjoyable.

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Helsinki, September 26, 2020,

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This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.


Author’s Contribution

Publication I: “Energy-efficient quantum computing”

The author carried out the published calculations and simulations, and wrote the manuscript with input from the other authors.

Publication II: “Qubit measurement by multichannel driving”

The author proposed the idea for the measurement technique, produced the theory, carried out the simulations, most of the measurements and data analysis, and wrote major contributions to the manuscript.

Publication III: “Exceptional points in tunable superconducting resonators”

The author contributed to the formulation and analysis of the quantum-mechanical model, and commented on the manuscript.

Publication IV: “Accelerated stabilization of coherent photon states”

The author carried out the published calculations and simulations, and wrote the manuscript with input from the other authors.
Author's Contribution

**Publication V: “Reconstruction approach to quantum dynamics of bosonic systems”**

The author provided the initial ideas for the reconstruction approach, actively participated in formulating the method, revised the figures, and commented on the manuscript.
1. Introduction

1.1 Background and motivation

The idea of quantum computing [1–3] has intrigued scientists since its inception in the 1980s, after Richard Feynman [4] and later others [5–7] proposed ideas of carrying out computational tasks by harnessing the laws of quantum mechanics. In theory, a universal quantum computer outperforms classical computers [8–10] in specific tasks such as simulating quantum systems [4, 11, 12], searching large datasets, machine learning, data fitting, solving linear systems of equations, clustering and topological problems, and increasingly many more [13]. For example, quantum computing provides computational shortcuts that allow simulating structures of complex molecules [12, 14, 15] in a time scale that is several orders of magnitude shorter than what is possible with contemporary supercomputers [11]. Although the revolutionary potential of quantum computing was recognized early, it remained a distant vision for many years due to the difficulty of its implementation.

In classical processors, information is processed with binary voltage gates, whereas quantum algorithms require various physical operations, including changing, measuring, and initializing the states of quantum bits, or qubits [1, 3], which are the main storage units for information in quantum processors. One of the outstanding challenges is that all of the operations must be carried out with high precision such that the demanding requirements for quantum computing [16] are met. The main goal of this dissertation is to improve the physical implementation of three of these elementary operations: single-qubit gates, qubit readout, and reset operations.

Immense progress in nanofabrication technology of superconducting devices in the past 20 years has sparked a wide interest around the world to realize a large-scale quantum computer based on superconducting solid-state qubits, or more precisely, circuit quantum electrodynamics.
Introduction

(cQED) [17–20]. In many experimental cQED systems, qubits [21–26] derived from the early Cooper-pair-box and transmon types [27–29] are coupled with electromagnetic modes in coplanar waveguides [17, 30] or three-dimensional (3D) cavities [31] and controlled with signals in the microwave range. To reach superconductivity and combat thermal noise and decoherence, the devices are cooled down to millikelvin temperatures using cryogenic refrigerators.

As a platform, superconducting devices are appealing since they share many characteristics with regular silicon processors, such as scalability, highly developed fabrication methods, and compatibility with established commercial test equipment. Moreover, cQED has the advantage of flexibility in the choice of design parameters and straightforward connectivity between the qubits. At the moment, cQED is without doubt the most advanced platform for quantum computing, as demonstrated in numerous experiments showcasing small-scale quantum algorithms [23, 32–37], recently culminating in the Google AI team’s report on their achievement of quantum supremacy [38], i.e., solving a well-defined computational problem with a quantum computer faster than any classical computer.

Despite the promising advances in cQED, several challenges remain. Due to the analog nature of the quantum operations, errors in quantum computing are inevitable: Small errors accumulate during the execution of an algorithm and eventually lead to erroneous results. In addition to inaccurate control pulses, errors may arise from the coupling between the qubits and their environment, most commonly manifesting as thermal noise and the tendency of the qubits to lose their information over time. Fortunately, it is possible to carry out almost error-free quantum computing on an error-prone hardware by utilizing quantum error correction codes [34, 39–42] which encode the information of a qubit into an ensemble of qubits in a manner resilient to uncorrelated and infrequent errors. The disadvantage of such schemes is that many known error correction codes require a vast number of qubits. For example, to factor a 2000-bit integer using Shor’s algorithm [6] running the surface code, one of the most tolerant correction codes, on a reasonably robust hardware would require $10^5–10^8$ physical qubits [39], depending on the error probabilities and implementation of the hardware. Reduction of the error of the operations also reduces the number of qubits required.

In the near future, it is expected that quantum computers will be upgraded from small-to-intermediate-scale prototypes consisting of tens of qubits [15, 34, 38, 43] to machines capable of tackling real-life problems. To this end, it is not sufficient to only increase the number of qubits, but also to further diminish the existing sources of errors, improve the speed of the operations with respect to the lifetime of the qubits, and prevent issues that emerge due to scaling up.

In this dissertation, we analytically, numerically, and experimentally
investigate ways to improve three elementary operations needed for quantum computing in the framework of cQED: single-qubit gates, readout, and initialization. We mostly focus our research to the well-established platform of superconducting qubits and resonators, although some of the general theory is independent of the platform.

1.2 Single-qubit gates

In cQED, the standard way of changing the state of single qubits is by generating short microwave pulses at room temperature and sending them through dedicated and strongly attenuated lines to a chip containing the qubits. Single-qubit gates are typically the least error-prone operation in quantum algorithms, and fidelities as high as 99.9% are routinely reported [38, 44–46]. Using high input powers, the pulses implementing the gates operate in a semi-classical regime where unwanted quantum mechanical effects are negligible. However, as the number of qubits and physical overhead increases, the current paradigm may become impractical and novel methods for delivering the drive power such as on-chip microwave sources [47], switches [48, 49], phase shifters [50, 51], and circulators [52, 53] may become necessary.

In Publication I, we theoretically investigate the relationship between the error probability and the required power of single-qubit gates, which becomes increasingly important as compromises between power, gate precision, and heat management need to be made for large-scale systems. Specifically, we show that high-precision gates cannot be achieved with arbitrarily low power and derive a lower bound for the gate error within cQED. Such relationships have been investigated by others in preceding work [54–61], but thus far no indisputable expression for the lowest achievable error has been presented for gates implemented with a Jaynes–Cummings-type interaction [17]. In Chapter 3, we outline our derivation, discuss how to reach the lower bound, and comment on the connection to previous work.

Furthermore, we propose in Publication I the possibility of using a single persistent control pulse to drive gates on many qubits consecutively without loss of precision. With such a pulse scheme, we demonstrate that although the error of a single gate is bounded by the energy of the pulse, the bound may be circumvented in a system of many qubits that share the driving source. This challenges the conclusion of Refs. [57, 61] which suggest that reusing a control pulse would not yield any benefit. The control scheme is discussed in Chapter 3.
1.3 Qubit readout

The most common method of measuring the state of a superconducting qubit is by dispersive readout, where the state is mapped to a pointer state of a transversely coupled resonator mode [18, 62]. Although the dispersive readout has been widely successful [63–66], it is one of the slowest and most error-prone quantum operation due to its inherent limitations. The dispersive scheme works ideally only in a perturbative approximation of the physical system; in practice, excessively increasing the number of photons in the resonator introduces complex dynamics and distortions instead of further reducing the error [67, 68].

To improve upon the dispersive scheme, systems exhibiting different types of coupling between the resonator and the qubit have been introduced, including longitudinal [69, 70] and cross-Kerr [67] couplings. A common goal of the novel coupling schemes is to produce a Hamiltonian and dynamics that allow a non-demolition measurement of the qubit state without perturbative approximations. In such a system, the readout power or other control parameters can be freely increased without inducing unwanted transitions.

In Publication II, we theoretically propose and experimentally demonstrate a novel readout method. In our so-called multichannel readout, a second microwave pulse is applied to the qubit simultaneously with the conventional readout pulse to the resonator, which effectively changes the dynamics to that of a longitudinal coupling, leading to faster resolution of the qubit state. Since the multichannel readout is an all-microwave extension for the dispersive readout, it requires only minor changes to conventional experimental setups. The theoretical predictions and the experimental results of Publication II are presented in Chapter 4.

1.4 Initialization and open quantum systems

The third operation needed for quantum algorithms and especially for error correction is the reset and initialization of the state of the qubits and auxiliary resonators to a predetermined state. In simple cases, passively waiting for the system to thermalize to a cold bath is sufficient but may considerably increase the repetition time of algorithms and measurements. Whereas the gate and readout operations are reasonably well established, a vast amount of different protocols and devices for initialization of both qubits and resonators have been developed. The techniques range from pulse sequences [71–74], measurement and active feedback loops [75], and active Purcell filtering [76], to tunable coupling to environments [77–80].

In this dissertation, we investigate two novel methods where a resonator is initialized by coupling it indirectly to a tunable dissipative environment.
through either another resonator in Publication III or through a qubit in Publication IV.

In Publication III, we experimentally realize a system suitable for resetting a resonator, consisting of a frequency-tunable coupler resonator and a normal-metal–insulator–superconductor junction device, referred to as quantum-circuit refrigerator (QCR) [81–85], where the QCR acts as the dissipative environment. By controlling the frequency of the coupler and its decay rate owing to the QCR, we are able to tune the effective energy dissipation rate in the resonator by several orders of magnitude in only tens of nanoseconds. Importantly, by choosing an exceptional point of the parameter space spanned by the control parameters, we obtain critical damping and the most efficient heat transfer from the resonator. A summary of Publication III and a brief description of the QCR are given in Chapter 5.

The protocol proposed in Publication IV allows one to initialize a resonator to a coherent state, or as a special case to the vacuum state, at a faster rate than the resonator decay rate. The protocol utilizes an auxiliary qubit that is connected to a cold dissipative environment such as the QCR. With numerical simulations, we show that repeatedly tuning the qubit and its coupling to the environment, it is possible to cool down the resonator to a temperature lower than the surrounding bath temperature. In addition to cooling, the ability to quickly initialize coherent states may prove useful in applications involving resonators with high quality factor, such as in photonic quantum memories [86]. The results of Publication IV are discussed in Chapter 6.

Most initialization methods, such as those in Publications III and IV and Refs. [71, 72, 76–80], exploit an interaction with a cold dissipative environment. The dynamics of systems coupled to such environments is conventionally obtained by solving the associated Lindblad master equation that governs the state of the total system. However, Lindblad master equations rely on the Born and Markov approximations [87] and are thus not suitable for systems with strongly coupled environments. Moreover, solving master equations for multipartite systems is a demanding task, both numerically and analytically. An alternative approach is presented in Publication V, where a general system of coupled bosonic modes and environments is solved analytically by transforming the time-dependence into the operators of the system and reconstructing the state from the solved operators. To showcase the power of the reconstruction method, we analytically solve, for the first time to our knowledge, the temporal evolution of the density operator of the system studied experimentally in Publication III. An outline of the reconstruction method is presented in Chapter 7.
2. Theoretical foundations and methods

In this chapter, we briefly present the key models and concepts that are essential throughout the dissertation. Specifically, the mathematical treatment of driven and open resonator–qubit systems are outlined from the perspective of cQED in Secs. 2.1–2.5. Detailed derivations of the models, starting from the microscopic theory and utilizing Lagrange–Hamilton formulation and second quantization, are presented in many sources (see e.g. Refs. [17, 18, 29, 88] and references therein) and are therefore not reproduced here. Section 2.6 summarizes standard experimental methods for measuring superconducting devices.

2.1 Resonators

Superconducting resonators, such as coplanar waveguides and 3D cavities, are used for many purposes in cQED such as quantum information storage [86], qubit readout, and amplification [89]. Owing to the high-quality superconducting materials, the resonators are capable of supporting coherent oscillations of electromagnetic modes at gigahertz frequencies persisting for several microseconds or even milliseconds [31], long enough for the purposes of quantum information processing. Moreover, the amplitude of the oscillations is controllable at the single-photon level, making resonators excellent tools for exploring quantum phenomena.

A resonator mode is well approximated by a lumped-element $LC$ circuit [30]. The mathematical treatment of the circuit begins by carrying out the second quantization [90] where the canonical conjugate variables are conventionally taken to be the superconducting flux $\hat{\phi}_r$ and charge $\hat{Q}_r$ which are defined as time integrals of the voltage and current across the circuit, respectively [18]. The Hamiltonian of the mode is that of a quantized harmonic oscillator [91], given by

$$\hat{H}_r = \hbar \omega_0 \hat{a}^\dagger \hat{a},$$  \hspace{1cm} (2.1)
where $\omega_{r0}$ is the bare angular resonance frequency of the mode, $\hbar$ is the reduced Planck constant, and $\hat{a}$ is the annihilation operator acting as a ladder operator in the Fock basis $\{ |n\rangle \}$, the eigenbasis of the photon number operator $\hat{n} = \hat{a}^\dagger \hat{a}$.

States with exactly $n$ photons do not appear naturally without deliberate engineering [92]. A more natural state of the resonator is the eigenstate of the annihilation operator, the coherent state $|\alpha\rangle$ [93], which is produced by driving a resonator with a coherent input signal. It is characterized by the complex eigenvalue $\alpha$, and has an average of $n = |\alpha|^2$ photons with a variance of $|\alpha|^2$. The state is conveniently visualized in the phase space of the conjugate variables $\langle \hat{Q}_r \rangle \propto \langle \hat{a}^\dagger + \hat{a} \rangle$ and $\langle \hat{\phi}_r \rangle \propto i \langle \hat{a}^\dagger - \hat{a} \rangle$, as depicted in Fig. 2.1, or equivalently in the complex plane of $\langle \hat{a} \rangle$. Under the Hamiltonian in Eq. (2.1), the amplitude $\alpha = \langle \hat{a} \rangle$ of a coherent state rotates about the origin at angular frequency $\omega_{r0}$.

2.2 Qubits

In quantum information processing, ideally any controllable two-level quantum system may act as a qubit. However, the mesoscopic devices in cQED exhibit complex level structures, and hence require careful design to effectively restrict the state of the system into a two-level subspace. An effective qubit in such a device may be created by introducing nonlinearity in the energy spectrum of a resonator, conventionally by fabricating Josephson junctions exhibiting nonlinear inductance [21, 94, 95].

For universal quantum computing, the most widely used [21] qubit type is the transmon qubit [26, 29], which consists of one or two parallel Josephson junctions shunted by a large capacitor. An example circuit diagram of a transmon, used in the experiments of Publication II, is shown in Fig. 2.1a. It is described by a Hamiltonian of the form

$$H_q = \hbar \omega_{q0} \hat{q}^\dagger \hat{q} + \frac{1}{2} \hbar \omega^2 \hat{q}^\dagger \hat{q}^2,$$

where $\omega_{q0}$ denotes the bare angular frequency of the qubit, the anharmonicity $\omega^2$ is the frequency difference of the two lowest transition frequencies, and $\hat{q}$ is the annihilation operator acting on the qubit states $\{ |g\rangle, |e\rangle, |f\rangle, \ldots \}$.

The transition energies of the transmon can be further tuned using a SQUID loop [96] formed by the two Josephson junctions. Through the effective inductance $L_q$ of the SQUID, $\omega_{q0}$ is controllable on demand by changing the external magnetic flux through the loop. The tunability allows us to set the qubit frequency to a suitable level with respect to other transition frequencies in the system, and enables more complex applications such as the cooling protocol presented in Publication IV. Similarly, a
Figure 2.1. Models of resonators and qubits, shown in green and red, respectively. The symbols correspond to those defined in the main text. The features are only illustrative and not to scale. (a) A lumped-element circuit diagram of a resonator capacitively coupled to a transmon qubit. The resonator is modeled as an LC oscillator and the qubit consists of a SQUID loop with two Josephson junctions (crossed squares), and a large shunt capacitance $C_q$. Both devices are also capacitively coupled to dedicated drive lines, shown in blue and magenta. (b) Energy diagrams of uncoupled harmonic and anharmonic oscillators. Resonant drive signals are used to excite the systems at frequencies $\omega_{r0}$ and $\omega_{q0}$, and the negative anharmonicity $\mathcal{A}$ prevents the excitation of higher levels of the qubit. (c) A phase space diagram showing the temporal evolution of a coherent state of the resonator. The trajectory of the amplitude $\alpha$ is shown in dark green and the light green area represents the Heisenberg uncertainty in the conjugate variables $\hat{\phi}_r$ and $\hat{Q}_r$. The drive amplitude $\Omega_r$ may excite the state from the vacuum state located at the origin. Without driving, the state decays at rate $\gamma_r$ and rotates at angular frequency $\omega_{r0}$. (d) The Bloch sphere representation of a two-level system. The poles represent the energy eigenstates and points on the shell represent their superpositions. In the laboratory frame, the state vector rotates naturally about the z-axis at $\omega_{q0}$ while the qubit drive $\Omega_q$ enables rotations about the x and y-axes.
waveguide resonator can be made tunable by attaching a SQUID loop to it [97].

The anharmonicity of a transmon qubit is of the order of few percent of the maximum transition frequency, which is sufficient for addressing individual transitions between the energy levels. For the purpose of quantum computing, the dynamics is conventionally restricted to the ground state $|g\rangle$ and to the first excited state $|e\rangle$, while unwanted transitions to higher states are suppressed. Restricting to the ideal two-level subspace $\{ |g\rangle, |e\rangle \}$, the Hamiltonian in Eq. (2.2) is reduced to

$$\hat{H}_{\text{ideal}} = -\frac{1}{2}\hbar\omega_{q0}\hat{\sigma}_z,$$

(2.3)

where $\hat{\sigma}_z = |g\rangle\langle g| - |e\rangle\langle e|$ is the Pauli Z operator and the zero-point energy has been renormalized. The validity of this approximation depends on the ratio of the anharmonicity and the resonance frequency: for example, it is not sufficient to accurately describe the full dynamics of a transmon qubit [29]. To simplify the presentation and to keep the discussion applicable to different types of qubits, we continue with the two-level approximation for the rest of this chapter.

The density operator $\hat{\rho}_q$ of the qubit is conveniently visualized as a three-dimensional Bloch vector [3], see Fig. 2.1d. Quantum gates on a single qubit are equivalent to rotations of the Bloch vector, denoted here as $R_\theta$ with $R \in \{X,Y,Z\}$ and $\theta$ being the axis and the angle of the rotation, respectively. Physically, gates $X_\pi$ and $Y_\pi$ are implemented by driving the $|g\rangle \leftrightarrow |e\rangle$ transition such that the dynamics results in Pauli X and Y operators $\hat{\sigma}_x = |e\rangle\langle g| + |g\rangle\langle e|$ and $\hat{\sigma}_y = i|e\rangle\langle g| - i|g\rangle\langle e|$, respectively. In addition to single-qubit gates described above, gate operations between at least two qubits are also needed for universal quantum computing [3].

### 2.3 Coupled systems

Superconducting components are conveniently coupled to each other by utilizing simple gap and finger capacitors or inductances created by a specific circuit geometry. Carrying out circuit analysis for a joint system where nodes $j$ and $k$ are connected by a capacitor with capacitance $C_{jk}$ reveals that the coupling introduces to the Hamiltonian an interaction term

$$\hat{H}_{jk} = C_{jk}\hat{V}_j\hat{V}_k.$$

(2.4)

Here, the node voltage operators $\hat{V}_j$ and $\hat{V}_k$ are proportional to the charge operators $\hat{Q}_j$ and $\hat{Q}_k$, and hence may be expressed in terms of the ladder operators associated with the corresponding components, following the rules of second quantization. This analysis applies to capacitive qubit–
Theoretical foundations and methods

qubit, qubit–resonator, and resonator–resonator couplings, and may be extended to circuits connected by intermediary coupler circuits [25, 98].

The system of two detuned resonators has a wide range of applications, notably as Purcell filters [66, 99] and tunable couplers [100–102]. For two resonators described by annihilation operators $a_1$ and $a_2$, Eq. (2.4) results in a term of the form $\hat{H}_{12} \propto (a_1 + a_1^\dagger)(a_2 + a_2^\dagger)$. If the resonators are driven with large amplitudes and the modes are well described by coherent states, the analysis is greatly simplified as shown in Publication III. On the other hand, the exact quantum solution of the system, especially in the presence of strongly dissipative environments, requires a more sophisticated treatment as discussed in Ch. 7.

2.3.1 Jaynes–Cummings model

Let us consider a system of a resonator and a capacitively coupled qubit, studied in Publications I, II, and IV. Combining Eqs. (2.1), (2.3), and (2.4), the Hamiltonian $\hat{H}_{JC} = \hat{H}_r + \hat{H}_{q} + \hat{H}_{rq}$ is given by

$$\hat{H}_{JC}/\hbar = \omega_r \hat{a}^\dagger \hat{a} - \frac{\omega_q}{2} \hat{\sigma}_z + g \frac{\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger}{\omega_r - \omega_q} \left| g \right\left| e \right\left| g \right| \left| e \right|,$$

(2.5)

where the aggregate $g$ denotes the strength of the coupling and the frequencies $\omega_{r0}$ and $\omega_{q0}$ have been renormalized to $\omega_r$ and $\omega_q$, respectively. In the last line of Eq. (2.5), we employed the rotating-wave approximation [88], which neglects terms of the form $\hat{a} \left| g \right\left| e \right|$ under the assumption $\omega_r + \omega_q \gg \left| \omega_r - \omega_q \right|$. Equation (2.5) is known as the Jaynes–Cummings Hamiltonian [103, 104], a cornerstone of cQED.

The system has two notable regimes of operation. In the resonant regime, $\omega_r = \omega_q$, the two subsystems swap excitations coherently, i.e., photons in the resonator drive the $|g\rangle \leftrightarrow |e\rangle$ transition, and the dynamics of the resonator are influenced by the state of the qubit. The protocols proposed in Publications I and IV exploit this back-action to control the state of the resonator.

The second important regime is the dispersive regime, where the frequencies are far detuned, $|\omega_r - \omega_q| \gg g$, and where no energy is exchanged between the subsystems. Instead, the coupling slightly modifies the frequencies of the components, as is evident after employing the dispersive approximation [20, 88] that reduces Eq. (2.5) to

$$\hat{H}_{\text{disp}}/\hbar = \left( \omega_r - \chi \hat{\sigma}_z \right) \hat{a}^\dagger \hat{a} - \frac{\omega_q}{2} \hat{\sigma}_z.$$

(2.6)

Here, $\chi = g^2/(\omega_q - \omega_r)$ defines the dispersive shift for a two-level system, an effective change to the resonator angular frequency that depends on the
qubit state. This is the main mechanism used in the dispersive readout scheme, where the state of the qubit is inferred from the frequency shift of the resonator. On the other hand, each photon in the resonator changes the angular frequency of the qubit by $2\chi$. The above also holds for a transmon system with a different expression for $\chi$.

2.4 Driven systems

External control of superconducting circuits is achieved, for example, by capacitively coupling them to superconducting transmission lines [105] through which coherent microwave signals are applied to the system. The amplitude of the external signal is typically large compared to the corresponding quantum fluctuations, and hence may be treated as a classical variable. Following Eq. (2.4), we may write the drive Hamiltonians as

$$H_{rd} = 2\hbar \tilde{\Omega}_r (a^\dagger + a)$$

and

$$H_{qd} = 2\hbar \tilde{\Omega}_q (|e\rangle \langle g| + |g\rangle \langle e|),$$

(2.7)

for a resonator and a qubit, respectively. Here, $\tilde{\Omega}_j \propto \langle \hat{V}_j \rangle$ denotes the driving amplitude of circuit $j$, which is proportional to the voltage of the signal applied to the respective transmission line, typically oscillating at the frequency of some desired transition. However, with low enough amplitudes, such an approximation is not valid and significant deviations from the classical behavior will occur, as shown in Publication I.

For analytical treatment and numerical simulations, it is convenient to separate the oscillation at $\omega_j$ from the pulse envelope. In experimental context, the resonator drive signal is often decomposed to in-phase and quadrature components as $\tilde{\Omega}_r(t) = \text{Re}(\Omega_r) \sin(\omega_r t) - \text{Im}(\Omega_r) \cos(\omega_r t)$, where $\Omega_r$ is a complex-valued amplitude of the drive. If we employ a unitary transformation to a frame rotating at $\omega_r$ along with the rotating-wave approximation, the drive Hamiltonian of the resonator in Eq. (2.7) assumes the form $\dot{H}'_{rd}/\hbar \approx i\Omega_r a^\dagger - i\Omega_r^* a$. From this form, the effect of the drive is conveniently demonstrated by computing the temporal evolution operator under $\dot{H}'_{rd}$ as $\hat{U}(t) = \exp\left(-it\dot{H}'_{rd}/\hbar\right)$. A coherent state evolves as $\hat{U}(t)|a\rangle = |a + \Omega_r t\rangle$, that is, the coherent amplitude is displaced [93] by the drive, with a speed and direction determined by the drive amplitude.

Similarly, we implicitly define $\Omega_q$ for the qubit drive but choose the signs of the quadratures differently from $\Omega_r$ to obtain a conventional form of the Hamiltonian. With $\tilde{\Omega}_q(t) = \text{Re}(\Omega_q) \cos(\omega_q t) + \text{Im}(\Omega_q) \sin(\omega_q t)$, the Hamiltonian in the rotating frame is given by

$$\dot{H}'_{qd}/\hbar \approx \Omega_q |e\rangle \langle g| + \Omega_q^* |g\rangle \langle e| = \text{Re}(\Omega_q) \hat{\sigma}_x + \text{Im}(\Omega_q) \hat{\sigma}_y.$$ 

(2.8)

The temporal evolution operator under $\dot{H}'_{qd}$ is proportional to $\hat{\sigma}_x$ and $\hat{\sigma}_y$, enabling rotations of the Bloch vector about an axis lying on the XY-plane.
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2.5 Open systems

Experimental systems of interest are not completely isolated from their environment and are thus correctly treated as open quantum systems. In an open system, the environment represents unquantifiable, usually macroscopical, degrees of freedom. Coupling to these degrees of freedom causes non-unitary dynamics in the system, leading to loss of quantum information in the form of energy dissipation, thermalization, and decoherence [87]. On the other hand, controllable decay channels such as open transmission lines and the QCR have various uses in qubit readout and initialization.

Although superconducting resonators are non-resistive, the stored energy dissipates to the environment through various mechanisms which are divided into internal and external losses. Internal losses include e.g. radiative losses and couplings to phonons and other quasiparticles [106], whereas external losses account for the energy transfer to coupled components, such as decay to a transmission line.

The Lindblad master equation is a tool for solving the dynamics of open quantum systems in the weak-coupling regime [87]. For a system with Hamiltonian $\hat{H}$ that is coupled to the environment through jump operators $\{\hat{A}_k\}$ with strengths $\{\gamma_k\}$, the master equation for the density operator is given by [87]

$$\frac{d}{dt}\hat{\rho}(t) = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}(t)] + \sum_k \frac{\gamma_k}{2} \hat{L}[\hat{A}_k; \hat{\rho}(t)], \quad (2.9)$$

where $\hat{L}[\hat{A}; \hat{\rho}] = 2\hat{A}\hat{\rho}\hat{A}^\dagger - \hat{A}^\dagger\hat{A}\hat{\rho} - \hat{\rho}\hat{A}^\dagger\hat{A}$ is the Lindblad superoperator. For example, the non-unitary decay and dephasing processes in a resonator and a qubit are described by the jump operators $\{\hat{a}, \hat{a}^\dagger\}$ and $\{|e\rangle\langle g|, \sigma_z\}$, respectively. The coupling strengths corresponding to the annihilation operators $\hat{a}$ and $|e\rangle\langle g|$, denoted here as $\gamma_r$ and $\gamma_q$, describe the exponential decay rates of energy in a resonator and a qubit, respectively. In Publications II–IV, master equations similar to Eq. (2.9) are employed to model the transfer of a readout signal to a transmission line, controlled dissipative element, or both. Due to their complexity, master equations are usually solved numerically.

The derivation of a Lindblad equation from first principles requires multiple assumptions such as Markovianity of the environment, weak coupling to the environment, and no initial correlation between the system and the environment [87]. In Publication V, we present an alternative for the master equation technique that requires fewer approximations and

at an angle $\arg(\Omega_q)$.
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opens the possibility of solving the dynamics analytically.

2.6 Experimental methods

In this section, we briefly present an experimental measurement setup that is commonly used in cQED. Details that are specific to the experiments of Publications II and III are discussed in Chapters 4 and 5, respectively. An essential part of the experimental work, the design and fabrication of the superconducting structures and samples, is beyond the scope of this dissertation.

A schematic of an experimental cQED measurement setup is presented in Fig. 2.2. To reach superconductivity and to protect the quantum devices from thermal excitations and radiation, the samples are shielded and cooled down to temperatures of 10–100 mK using, for instance, a dilution refrigerator. The control pulses typically have a carrier frequency of several gigahertz, matching the fundamental resonance frequencies of the circuits, and a duration that ranges from nanoseconds to microseconds. The envelopes of pulsed signals applied to qubits and resonators, corresponding to $\Omega_q$ and $\Omega_r$ in Sec. 2.4, are generated by arbitrary-waveform generators at room temperature and mixed into continuous-wave signals at respective frequencies $\omega_q$ and $\omega_r$. The input signals are heavily attenuated to suitable levels such that the thermal noise is greatly reduced [107]. After interacting with the resonator, the readout signal is amplified multiple times and digitized. To combat excessive thermal noise in the output signal, the signal requires either averaging or use of a cryogenic amplifier, such as a traveling-wave parametric amplifier [108–110].

The measured signal is divided into in-phase and quadrature components which may be mapped [111, 112] to the phase space of the resonator state (see Fig. 2.1c) to study the temporal evolution of the system.

If steady state measurements are sufficient, the room temperature part of the readout loop in Fig. 2.2 may be replaced with a vector network analyzer. This allows us to determine the scattering matrix and the decay rates of the resonators in Publication III.
Figure 2.2. Simplified diagram of a typical experimental heterodyne measurement setup for a superconducting qubit. The hardware at room temperature is shown at the top of the figure using symbols defined in the legend on the left. The yellow box represents an experimental sample, controlled with microwave and dc lines, inside a dilution refrigerator with stages at various temperatures. Pulses for qubit gates and readout are generated and upconverted to carrier frequencies $\omega_q$ and $\omega_r$ in separate mixing processes, shown as red and green components, respectively. The readout signal emitted from the sample is amplified in several stages, downconverted and recorded by an analog-to-digital converter (ADC), as depicted by the rightmost branch. Optional components, shown in black color, include a pump signal for a parametric amplifier and dc control for sample-specific structures (asterisk), e.g. magnetic flux lines or SQUIDs. For simplicity, auxiliary hardware and components such as filters are not drawn.
3. Achieving the theoretical limit of qubit gate precision

One of the requirements of error-corrected quantum computing is that all gate operations must be carried out with an error probability below a certain threshold. In current state-of-the-art prototypes, the requirement is satisfied with single-qubit gate errors of the order of 0.1%. This is well below the most tolerant threshold of roughly 1% of the surface code which is the most prominent error correction code for superconducting circuits [39]. However, scaling up the number of qubits to thousands or millions required by the surface code raises several issues that necessitate re-examining the achievable error rates of single-qubit gates. On one hand, extending the current method of generating the control pulses at room temperature and delivering them through dedicated, heavily attenuated lines to the cold temperatures would pose challenges in the cryogenic control of heat. On the other hand, reducing the amount of power used in the control pulses directly increases the gate error, as discussed below. Another option for mitigating the scaling issues is to use error correction codes that require fewer physical qubits, at the cost of a much more stringent error tolerance threshold. These concerns motivate the investigation of the relationship between gate error and the energy used for the gate.

Prior research [54–60] has concluded that the error of qubit gate operations, implemented in different physical platforms, tend to be constrained by a lower bound that is inversely proportional to the "size" of the driving system. Refs. [54, 56, 58, 59] present derivations for general Hamiltonians that rely on the conservation of non-commuting quantities, whereas Refs. [55, 57, 60] explore the limits of the Jaynes–Cummings Hamiltonian explicitly. Although the consensus is that gate errors implemented with bosonic drives with $\bar{n}$ average photons in general scale as $\delta \propto 1/\bar{n}$, there has not been a conclusive expression for the lowest achievable error using a Jaynes–Cummings-type interaction. For instance, error bounds derived from conservation laws are not necessarily achievable with physical Hamiltonians of interest. On the other hand, earlier derivations starting from the Jaynes–Cummings model produce more stringent bounds, but are applicable to only specific gates or introduce an unnecessary assumption.
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of a coherent drive state. Exact comparison between different results is further convoluted due to inconsistencies in chosen error measures which include trace distance [54, 58, 59], worst-case infidelity $\delta_{\text{max}}$ [58, 59], and state-dependent infidelity [56, 57, 60].

In this chapter, we discuss the results of Publication I. First, we review the semi-classical and quantum descriptions of the physics of single-qubit gates. Secondly, we show that there is a fundamental lower bound for the single-qubit gate error that arises from the quantum properties of the drive mode. We also show how this bound may be reached by preparing the drive mode in specific quantum states. Finally, we show that a single drive mode may be reused to implement multiple gates without considerable increase in the gate error.

3.1 Semi-classical and quantum treatment of qubit gates

As discussed in Secs. 2.3 and 2.4, single-qubit gates are implemented by driving the $|g\rangle \rightarrow |e\rangle$ transition with a resonant electromagnetic mode $\hat{a}$ through the Jaynes–Cummings interaction in Eq. (2.5). Here, $\hat{a}$ can be associated with a standing wave of a resonator or a propagating wave packet of photons. As the amplitude of the driving oscillation approaches infinity, the dynamics reduce to that of $\hat{H}_{\text{qd}}$ in Eq. (2.7), that is, the operator $\hat{a}$ is replaced by a complex number. We refer to this model as the semi-classical model.

In the semi-classical model, arbitrarily accurate Pauli X and Y gates are achieved by turning the driving interaction on for a suitable amount of time. In cQED, the gates are usually implemented by a high-amplitude microwave pulse of finite duration. Mathematically, this amounts to constructing a temporal evolution operator $\hat{U}(t)$ that satisfies, for example, $\hat{U}(t) \propto \sigma_x$ or $\hat{U}(t) \propto \sigma_y$. Choosing a real constant driving amplitude $\Omega_q$ and an interaction time $t_{\text{int}} = \pi/(2\Omega_q)$ results in the X gate, up to the physically irrelevant global phase, as

$$\hat{U}(t_{\text{int}}) = \exp \left( -i \int_{0}^{t_{\text{int}}} \hat{H}_{\text{qd}}' dt / \hbar \right) = e^{-i\Omega_q t_{\text{int}} \sigma_x} = e^{-i\sigma_x \pi/2} = -i\sigma_x. \quad (3.1)$$

Similarly, rotations of arbitrary angles are produced by varying the interaction time or the drive amplitude.

However, in a case where the drive amplitude corresponds to a coherent state with $\bar{n} \lesssim 100$ photons, quantum fluctuations in the photon number and phase along with the qubit-induced back-action on the driving mode lead to non-negligible increase in the gate error. Including the bosonic drive mode in the quantum description of the system, the reduced density
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operator $\hat{\rho}_q(t)$ of the qubit evolves as

$$\hat{\rho}_q(t) = \text{Tr}_r \left[ e^{-it\hat{H}_JC/\hbar} \hat{\rho}(0) e^{it\hat{H}_JC/\hbar} \right], \quad (3.2)$$

where $\hat{\rho}(0)$ is the initial state of the total system and $\text{Tr}_r$ denotes partial trace over the degrees of freedom of the drive.

Gate error is evaluated by calculating the overlap fidelity [45] with the ideally transformed state. For an ideal gate operator $\hat{K}$, the error is thus given by

$$\mathcal{E} = 1 - \text{Tr} \left[ \hat{\rho}_q(t) \hat{K} \hat{\rho}_q(0) \hat{K}^\dagger \right], \quad (3.3)$$

where the latter term is the overlap fidelity. Note that unlike in the semiclassical model, the initial states of both the qubit and the driving mode affect the resulting dynamics. Other error measures derived from $\mathcal{E}$ include the error averaged over all possible initial states of the qubit, $\bar{\mathcal{E}}$, and the worst-case error, $\mathcal{E}_{\text{max}} = \max_{\hat{\rho}_q(0)} \mathcal{E}$.

### 3.2 Optimal drive states

In Publication I, we employ a novel method to derive the greatest lower bound for arbitrary single-qubit gates within the Jaynes–Cummings model. In contrast to the previous literature, our approach does not assume any particular drive state a priori, which allows determining the optimal quantum state of the drive mode for a gate of given rotation and duration.

The optimization procedure is summarized as follows. The expression for gate error in Eq. (3.3) may be expressed as

$$\mathcal{E} = 1 - \langle \Psi_r | \hat{F} | \Psi_r \rangle, \quad (3.4)$$

where $\langle \Psi_r \rangle$ an arbitrary initial state vector of the drive mode, and $\hat{F}$ is an operator that contains the information about $\hat{\rho}_q(0)$, $\hat{K}$, and the interaction time $t_{\text{int}}$. We derive an expression for $\hat{F}$ for the average error, the worst case error, and error of specific initial states of the qubit, presented in the supplementary information of Publication I. Subsequently, we solve the eigenproblem

$$\hat{F} | \Psi_{r}^{\text{opt}} \rangle = f | \Psi_{r}^{\text{opt}} \rangle, \quad (3.5)$$

and choose the optimal state $| \Psi_{r}^{\text{opt}} \rangle$ corresponding to the largest eigenvalue $f$, or equivalently to the smallest error $\mathcal{E} = 1 - f$.

Examples of the optimal drive states are visualized in Fig. 3.1a using the Wigner quasiprobability distribution [113]. Our method reveals that the states that minimize the average and worst-case errors for qubit rotations $R_\theta$, where $\theta \lesssim \pi/2$, are slightly squeezed coherent states $| \alpha, r_\theta \rangle$ where
Figure 3.1. Optimal drive states and the resulting error. (a) Wigner distributions $W(z)$ of numerically solved initial drive states that minimize the average error of rotations $R_\theta = Y_{\pi/2}$ and $R_\theta = X_\pi$ are shown above and below the dashed line, respectively. The distribution is defined over a complex variable $z$ that corresponds to the expectation value $\langle a^\dagger a \rangle$. The given time constraint for the operations is $t_{\text{opt}} = \theta/(6g)$ which results in a squeezed coherent state for $Y_{\pi/2}$ and a squeezed cat state for $X_\pi$, each with amplitude $|\alpha| = 3$. (b) Gate error for an $X_\pi$ operation as a function of the average photon number $n$ of the drive pulse which is initialized either in the coherent state (red color) or the optimally squeezed cat state (blue color). The highlighted areas indicate the range of error, depending on the initial state of the qubit, and the solid lines show the error averaged over qubit states distributed uniformly on the Bloch sphere, $\mathcal{B}$. Figure adapted from Publication I.
\[ |\alpha| = \sqrt{\pi} = \theta/(2gt_{int}) \] and \( r_\theta \) is a squeezing parameter \([113]\). This is consistent with the semi-classical model, apart from the squeezing which is a pure quantum effect. Interestingly however, the optimal states for rotations of \( \theta \approx \pi \) are squeezed cat states \([113]\), \[ |\Psi_{\text{opt}}^{\theta}\rangle \propto |\alpha, r_\theta\rangle \pm |\alpha, r_\theta\rangle. \]

The optimal states outperform their coherent counterparts because they mitigate two error sources arising from the interaction: quantum fluctuations and back-action. Since the amplitude of the coherent state determines the rotation angle, squeezing the amplitude fluctuations makes the rotation more accurate at the expense of increasing phase fluctuations. Thus, optimal amount of squeezing strikes a balance between uncertainties of the rotation angle and the rotation axis. Furthermore, the amplitude may change during the interaction due to quantum back-action and thus undershoot or overshoot the desired rotation depending on the trajectory of the qubit state on the Bloch sphere. Especially for rotations of \( \pi \), the optimal superposition state essentially causes the qubit to take the average of the undershooting and overshooting paths, hence minimizing the harmful effect from back-action.

In agreement with previous literature, gate errors in the cases discussed above are inversely proportional to the average photon number in the drive mode, as shown in Fig. 3.1b. Comparing \( \pi \) rotations that are implemented using either coherent states or the optimal states, the coherent state yields an approximately 110\% larger worst-case error and 10\% larger average error. For explicit expressions for the lower bounds of error and the optimal states for the various gates, see Publication I.

### 3.3 Qubit gates driven by a persistent bosonic state

According to the results discussed above, implementing single-qubit gates that are accurate enough for error correction algorithms requires the use of drive pulses containing at least hundreds of photons. At first sight, this seems to pose several challenges for the heat control of large-scale systems, as discussed above. We therefore investigate in Publication I the possibility of using a single drive pulse to implement several gates without loss of gate fidelity. Intuitively, one might expect that the back-action arising from the interactions with qubits does not affect a drive mode with hundreds of photons enough to reduce its ability to generate high-fidelity gates. Previous literature \([57, 61]\) suggests that the error scaling of \( 1/\pi \) cannot be improved upon by such reuse of the drive state. However, their definition of gate infidelity includes the irrelevant quantum state of the drive. Here, we theoretically show that it is indeed possible to apply several high-fidelity gates using a persistent drive state, such that the minimum error bound for single interactions is reached with an effectively lower total energy cost.
We investigate the above idea with a toy model presented in Fig. 3.2. The system contains $N$ computational qubits and a persistent bosonic drive mode initially in an error-minimizing state $|\Psi_{opt}^{r1}\rangle$. The drive sequentially interacts with the computational qubits, applying the desired gate $R_\theta$ to each and is hence susceptible to harmful back-action in the process. In addition, the drive state may interact with $M$ ancilla qubits between each computational gate. See Publication I for an illustration of how the circulation might be implemented in practice.

The purpose of the ancilla interactions is to combat the degradation of the drive state by utilizing the back-action from $X_\pi$ operations in a constructive way. In the interaction with an ancilla in state $|+y\rangle = (|g\rangle + i|e\rangle)/\sqrt{2}$, the amplitude of the drive will be steered towards a target value determined by $t_{int}$. For example, if the drive has lost photons such that the amplitude is lower than the target amplitude, the interaction will rotate the Bloch vector from the equator about the x axis to a position above the equator. Namely, the energy of the qubit is decreased and the drive amplitude is increased. To our knowledge, such refreshing interaction has not been reported before.

We simulate the protocol with different values of $M$ and the initial number of photons $\pi$ in the drive state. To imitate the computation of a quantum algorithm, the initial states of the computational qubits are randomized and the results of the simulations are averaged over several initial configu-
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Figure 3.3. Gate errors resulting from the drive-refreshing protocol. (a) Average error $\overline{E}$ of $X_{\pi}$ gates generated by a single drive state which initially had the indicated number of photons and has reached the steady state due to ancilla refreshing. The drive is set to interact with $M$ ideal ancillas $|+\rangle$ for each computational gate as indicated, leading to effective refreshing of the drive state. The dashed line indicates the lower bound of error which is achieved either with a disposable optimal pulse or with a pulse refreshed by infinitely many ideal ancillas. (b) Average gate error as a function of the photon cost per gate, $2n/N$ for $M > 0$ and $\pi/N$ for $M = 0$. Here, $\pi$ is the mean number of photons in the initial pulses and $N$ is the number of $X_{\pi}$ computational gates generated by the protocol. The four families of curves are, from top to bottom, for $\pi = 10$, $\pi = 25$, $\pi = 50$, and $\pi = 100$. The ancilla states are non-ideally prepared by a corrector pulse. During the protocol, the curve advances from right to left as $1/N$ decreases. The dashed line is as in (a). Figure adapted from Publication I.
4. Multichannel method for qubit readout

The most common way of measuring the state of superconducting qubits utilizes a Jaynes–Cummings-type coupling to a resonator with a large decay constant and the dispersive regime. Due to the dispersive frequency shift in Eq. (2.6), the steady state of the driven resonator depends on the state of the qubit. Thus a coherent state of the resonator serves as an indirect indicator of the qubit state and is commonly referred to as the pointer state. The phase and amplitude information of the pointer state is transferred to photons propagating in a transmission line.

The measured signal contains noise arising from the Heisenberg uncertainty of the pointer state and from the amplifier chain of the experimental setup. The quantum uncertainty of the states is inevitable, and therefore the key to fast and precise readout is to separate the pointer states as quickly as possible. The maximal speed of separation is affected by physical parameters such as the coupling strength of the resonator to the transmission line and to the qubit, frequency detuning, and the driving power. Unfortunately, the dispersive scheme is limited by the fact that the number of photons in the resonator must be kept relatively small to suppress unwanted transitions to higher energy levels [68].

In this chapter, we summarize the results of Publication II, in which we demonstrate a modification of the standard dispersive readout scheme where an additional microwave pulse is applied to the qubit. This addition causes the pointer states to separate more quickly, resulting in faster and more accurate resolution of the qubit state. Similar method was also independently discovered and demonstrated in Ref. [114].

4.1 Theoretical principle

In addition to the pulse that populates the resonator, our multichannel readout method features a simultaneous pulse applied to the qubit at the resonance frequency of the resonator. The total Hamiltonian in the
laboratory frame of the system is given by

$$\hat{H}_{mc} = \hat{H}_{JC} + \hat{H}_{rd} + 2\hbar\tilde{\Omega}_{mc}\hat{\sigma}_x,$$  \hspace{1cm} (4.1)

where the Jaynes–Cummings Hamiltonian $\hat{H}_{JC}$ and the resonator drive $\hat{H}_{rd}$ are defined in Eqs. (2.5) and (2.7), respectively, and $\tilde{\Omega}_{mc}$ denotes the amplitude of the additional qubit drive, consisting of an envelope $\Omega_{mc}$ modulated at $\omega_r$. In a frame rotating at $\omega_r$ where the dispersive approximation is employed, the Hamiltonian assumes the form

$$\hat{H}'_{mc}/\hbar = -\frac{\omega_q - \omega_r}{2}\hat{\sigma}_z + \left[\left(\Omega_{mc} + i\Omega_r \frac{\chi}{g}\right)e\langle g | + \text{H.c.}\right]$$

$$-\chi\hat{\sigma}_z\hat{a}^{\dagger}\hat{a} + \left[\left(i\Omega_r - \Omega_{mc} \frac{\chi}{g}\right)\hat{a}^{\dagger} + \text{H.c.}\right].$$ \hspace{1cm} (4.2)

Thus driving the qubit introduces a longitudinal \cite{69, 70} coupling term proportional to $\hat{\sigma}_z(\hat{a}^{\dagger} + \hat{a})$ that is not present in the dispersive Hamiltonian in Eq. (2.6). Whereas the influence of the dispersive term $\chi\hat{\sigma}_z\hat{a}^{\dagger}\hat{a}$ depends on the number of photons present in the resonator and is therefore limited by the decay rate $\gamma_r$, the longitudinal term begins separating the pointer states as soon as the drive is turned on. Although the additional pulse is detuned from the transition frequency of the qubit, it slightly tilts the quantization axis, causing unwanted transitions if the drive is not turned on adiabatically.

Alternatively, the dynamics may be understood such that the drive on the qubit effectively displaces the reference frame of the resonator. We define a displaced annihilation operator as $\hat{b} = \hat{a} - \alpha_{vo}$, where $\alpha_{vo} = -\Omega_{mc}/g$ defines the virtual origin of the displaced frame. With this notation, the part of Eq. (4.2) that affects the resonator is simplified to $\hat{H}'_{mc}/\hbar = -\chi\langle \hat{\sigma}_z \rangle\hat{b}^{\dagger}\hat{b} + \Omega_r\hat{b}^{\dagger} + \Omega_r^{\ast}\hat{b}$. In short, the pointer states rotate about the virtual origin $\alpha_{vo}$ at angular frequency $\chi\langle \hat{\sigma}_z \rangle$ and may be further manipulated with $\Omega_r$. The concept of the virtual origin is further utilized in Publication IV.

Note that the above expressions describe a system with an ideal qubit, i.e., a two-level qubit system. The results may be extended to a transmon qubit, in the case of which the Hamiltonian becomes slightly more complicated. The most notable difference is an additional state-independent drive term proportional to $\Omega_{mc}$ for the resonator. We may cancel its effect by driving the resonator with a suitable amplitude and phase to limit the number of photons in the resonator and obtain symmetric trajectories. In addition, the sign of $\chi$ may be reversed due the more complicated energy level structure.
4.2 Simulations

We simulate the dynamics of the multichannel scheme for a transmon qubit with parameters corresponding to the experimental setup described in Sec. 4.3. We numerically solve the Lindblad master equation corresponding to Eq. (4.2), taking into account 30 and 4 energy levels for the linear resonator mode and the nonlinear transmon, respectively. The simulated equation is explicitly given in the supplemental material of Publication II.

In Fig. 4.1, we compare the dispersive and the multichannel schemes by showing the trajectories of the expectation value $\langle \hat{a}(t) \rangle$. Importantly, the pointer states separate at different speeds in the two readout schemes before saturating towards steady states due to the dissipation. Furthermore, the multichannel method requires less photons in the resonator to reach an approximately equal separation of the steady states, which prevents exceeding the critical photon number of the resonator [20, 68]. In Fig. 4.1c, we demonstrate that if the above-discussed additional drive term is not completely compensated, the resulting trajectories show features of both the dispersive and multichannel readout schemes.

4.3 Experimental realization

We experimentally demonstrate the above idea with a superconducting transmon qubit of the Xmon subtype [26] coupled to a coplanar waveguide resonator, shown in Fig. 4.2. The sample is mounted inside a cryostat with a base temperature of $T = 20$ mK. Readout and qubit control are imple-
Figure 4.2. Simplified measurement setup for the multichannel readout and scanning electron micrographs of the measured sample. The frequency of the Xmon-type qubit is tuned by applying an external magnetic flux to the dc SQUID. The qubit is capacitively coupled to a coplanar waveguide resonator which is, in turn, coupled to a transmission line leading to the amplification and measurement chain. Pulses for the multichannel readout and for the \( \pi \) rotations of the qubit are applied by a single drive line. Figure adapted from Publication II.

mented using a heterodyne detection setup similar to the one introduced in Sec. 2.6. To apply the multichannel pulse \( \Omega_{mc} \) to the qubit, we modify the setup in Fig. 2.2 by splitting the readout pulse and directing half of it to the qubit drive line after adjusting its phase and amplitude. A more detailed description of the setup, sample design, and characterization is given in the supplementary material of Publication II.

To compare the different readout methods qualitatively, we first measure the ensemble-averaged trajectories of the resonator mode \( \hat{a} \), presented in Fig. 4.3. The similarity between the simulated and measured trajectories confirms that our model captures the essential features of the dynamics. We attribute the differences between the simulations and the measurements to slight miscalibrations in the delays, amplitudes, and phases of the drive pulses.

Finally, we show that the readout fidelity is improved by the multichannel pulse. For a single, non-averaged measurement of the qubit state, we compute a weighted time integral of the digitized signal to obtain a point in the phase space of \( \hat{a} \). These single-shot points are shown by the colored dots in Fig. 4.3a–b. In a calibration measurement, we choose a reference line that most accurately differentiates between distributions corresponding to \(|g\rangle \) and \(|e\rangle \), and in further measurements label the measurement outcomes based on which side of the line each single-shot point falls. The total process error \( \epsilon \) is finally calculated as the percentage of measurement outcomes that do not match the intended initial state.

We compare the performance of the dispersive and multichannel readout
Figure 4.3. Evolution of the resonator states during (a) the dispersive readout and (b) the multichannel readout. The blue and red lines show the trajectories $\langle \hat{a}(t) \rangle$ of the amplitudes of the coherent states corresponding to the ground and excited states of the qubit, respectively. Results of the corresponding single-shot measurements are shown by dots. The dotted line indicates the reference for assigning the measurement outcome. (c) Average measurement error as a function of integration time with parameters corresponding to (a) and (b) are shown by blue and yellow markers, respectively. The case where driving is applied only to the qubit is shown in red. The markers show the average and the standard deviation of 10 measurement runs consisting of $10^4$ single-shot measurements each. The stars indicate the shortest time for which the lowest error is obtained for each method. The inset shows the relative increase in the measurement error of the dispersive readout compared with the multichannel scheme. Figure adapted from Publication II.
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by measuring the process error in both schemes and calculate the readout error from $\epsilon$ by accounting for separately measured state preparation errors. We also test a variation of the multichannel method where a readout pulse is only applied to the qubit. For the three readout schemes, the drive power is maximized on the condition that the readout pulses do not significantly excite the third energy level of the transmon qubit. Figure 4.3c presents the readout errors as functions of the integration time. We conclude that compared with the dispersive readout, the multichannel scheme, with or without the compensation pulse, reaches smaller error for equal integration time, or alternatively achieves an equal error probability faster. This is the key result of Publication II.
5. Controllable dissipation for resonator initialization

The remainder of this dissertation focuses on methods related to the initialization of superconducting circuits. In this chapter, we summarize the results of Publication III where a combination of a tunable resonator and a quantum circuit refrigerator (QCR) is utilized to efficiently remove excitations from a primary resonator. The QCR provides a strongly dissipative environment and the tunable resonator further improves the on-off ratio of the on-demand dissipation. Thus the combination of the two devices offers a higher level of control of the dissipative dynamics compared to using either of the devices alone.

5.1 Structure and model of the sample

The experimental sample consists of two coplanar waveguide resonators R1 and R2 and a QCR built from normal-metal–insulator–superconductor (NIS) junctions, as shown in Fig. 5.1. The resonator modes are characterized by the angular resonance frequencies $\omega_1$ and $\omega_2$ and decay rates $\kappa_1$ and $\kappa_2$, respectively, and are mutually coupled with a coupling strength $g$ through a capacitance $C_{12}$. The combination of the frequency-tunable resonator R2 and the QCR acts as a highly-controllable dissipative environment for the primary resonator R1. The decay rate $\kappa_1$ is a constant owing to the internal losses in R1 and to the external coupling $\kappa_{\text{ext}}$ to the transmission lines, whereas $\kappa_2$ is controllable by two orders of magnitude by biasing the QCR.

The operation principle of the QCR is outlined in Fig. 5.1c and presented in detail in Ref. [82]. In short, adjusting the Fermi levels of the NIS structures with bias voltage $V_b$ in relation to the gap of $2\Delta$ in the density of states of the superconductor allows inelastic tunneling of electrons through the junction. The tunneling electrons absorb photons from the capacitively coupled resonator R2 while the reverse photon-emitting process is strongly suppressed, leading to evacuation of the photons in the resonator [81–84].

We model the system as two coupled quantum harmonic oscillators that
are connected to their respective environments. Following the principles of Secs. 2.3 and 2.5, the Markovian master equation describing the system in a frame rotating at \( \omega_1 \) is given by

\[
\frac{d\tilde{\rho}(t)}{dt} = -\frac{i}{\hbar} \left[ \hat{H}'_{12}, \tilde{\rho}(t) \right] + \frac{\kappa_1}{2} \hat{L}(\hat{a}_1; \tilde{\rho}) + \frac{\kappa_2}{2} \hat{L}(\hat{a}_2; \tilde{\rho}),
\]

where the rotating-wave approximation has been employed for the Hamiltonian \( \hat{H}'_{12}/\hbar = \delta_{12}\hat{a}_2\hat{a}_2^\dagger + g(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_1\hat{a}_2^\dagger) \) and \( \delta_{12} = \omega_2 - \omega_1 \). Assuming the resonators remain in coherent states, the dynamics of the expectation values \( \langle \hat{a}_1 \rangle \) and \( \langle \hat{a}_2 \rangle \) is described by a set of coupled differential equations

\[
\frac{d}{dt} \begin{pmatrix} \langle \hat{a}_1(t) \rangle \\ \langle \hat{a}_2(t) \rangle \end{pmatrix} = -i \begin{pmatrix} -i\kappa_1/2 & g \\ g & \delta_{12} - i\kappa_2/2 \end{pmatrix} \begin{pmatrix} \langle \hat{a}_1(t) \rangle \\ \langle \hat{a}_2(t) \rangle \end{pmatrix}.
\]

This system has two eigenmodes corresponding to the eigenvalues

\[
\lambda_{\pm} = \frac{1}{4} \left[ 2\delta_{12} - i\kappa_1 - i\kappa_2 \pm \sqrt{(2\delta_{12} - i\kappa_1 - i\kappa_2)^2 + 16g^2} \right]
\]

of the matrix in Eq. (5.2). The effective rates of decay of the eigenmodes are given by \( \kappa_{\text{eff},1} = -2\text{Im}(\lambda_+) \) and \( \kappa_{\text{eff},2} = -2\text{Im}(\lambda_-) \). In resonance, \( \kappa_{\text{eff},1} \) corresponds to the mode dominant in R1.
5.2 Experimental results

We use a measurement setup similar to the one described in Sec. 2.6 to measure the photon decay rates of R1.

Figure 5.2 shows the results of a time domain measurement where a constant drive tone is applied to R1 and subsequently turned off. During the natural ring-down of the resonator amplitude, we apply a rectangular pulse to the QCR, which temporarily increases the effective decay rate in both resonators. Importantly, the QCR allows us to tune the decay rate rapidly at a timescale of tens of nanoseconds to a value that is well predicted by our model. Note that for the time domain measurements, the resonators are detuned to decrease the damping such that the changes in the decay rate can be accurately quantified.

One of the main motivations of Publication III is to evacuate the photons in the primary resonator R1 as fast as possible. This is achieved by critical damping of R1, which is realized by operating the system at an exceptional point [115] of the parameter space spanned by δ_{12} and κ_2. At the exceptional point, the two eigenmodes coalesce, λ_+ = λ_−, and energy does not oscillate back and forth between the resonators. In our system the exceptional point is approximately at (δ_{12} = 0, κ_2 = 4g).

Figure 5.3 shows the extracted effective decay rates κ_{eff,1} and κ_{eff,2} of the two hybridized modes as functions of the bias voltage of the QCR. The effective rates are calculated with the help of Eq. (5.3) and values of κ_1 and κ_2(V_b) are obtained from separate measurements. We observe that both of the effective rates may be tuned by two orders of magnitude. Moreover, the
Controllable dissipation for resonator initialization

The mode that is predominant in R1 has the maximal decay rate \( \max(\kappa_{\text{eff},1}) = 2g \) at the exceptional point, limited by the coupling constant \( g \).

![Graph showing theoretical and experimental decay rates of hybridized modes](image)

**Figure 5.3.** Theoretical and experimentally extracted effective decay rates of the hybridized modes in the coupled system as a function of the bias voltage of the QCR, as indicated. The decay rate of the mode that is predominant in R1 is maximized at the exceptional point denoted by the red circle. The data is measured for two samples that nominally differ only in the maximum frequency of R2. Figure adapted from Publication III.
In this chapter, we summarize the protocol for initializing a resonator proposed in Publication IV. In contrast to Publication III, here the primary resonator is indirectly coupled to a tunable environment through a qubit instead of another resonator. The non-linearity of the qubit allows to exploit the state-refreshing interaction briefly discussed in Sec. 3.3 and the concept of the virtual origin discussed in Sec. 4.1.

In the proposed protocol, the operation point of the qubit–resonator system is periodically switched between a resonant interaction and a dispersive, strongly dissipative interaction. This allows us to stabilize the state of the resonator towards a desired target coherent state, including the vacuum state, without prior knowledge of the state.

The state is stabilized at a rate that is proportional to the maximum dissipation rate of the qubit. This is an improvement over driving the resonator directly, in the case of which the rate of convergence is limited by the decay rate of the resonator. Thus the protocol is most beneficial when operating weakly coupled resonators that are used, for example, as quantum memories [86]. Furthermore, with the help of the stabilization protocol, a single QCR device attached to a qubit is sufficient to reset the resonator–qubit system without additional devices.

### 6.1 Stabilization protocol

We consider once more a driven and damped Jaynes–Cummings system described by the Hamiltonian $\hat{H}_s = \hat{H}_{JC} + \hat{H}_{qd}$, where the interaction and qubit-driving terms are defined in Eqs. (2.5) and (2.7), respectively. The qubit is coherently driven at the frequency of the resonator with a constant amplitude $\Omega_q$ which displaces the origin of the resonator phase space to $\alpha_{vo} = -\Omega_q/g$, as explained in Sec. 4.1. Furthermore, the detuning $\Delta_q(t) = \omega_q(t) - \omega_r$ between the qubit and the resonator and the dissipation rate $\gamma_q(t)$ of the qubit are considered tunable control parameters.

In the following, we describe how the stabilization protocol steers the
Resonator state control using a coupler qubit

Figure 6.1. (a) Numerically simulated trajectories of the deviation $\delta(t)$ from the target coherent state $|\alpha_{\text{vo}} = 10\rangle$ during the interaction phase. The system is initially in state $|g\rangle|\alpha_{\text{vo}} + \delta(0)\rangle$ where different initial deviations $\delta(0)$ correspond to the crosses of different colors. (b) Control parameters of the simulated stabilization protocol which consists of alternating interaction and reset phases, shown as pink and white backgrounds, respectively. (c) Simulated amplitude deviation $\delta(t)$ from the target state $|\alpha_{\text{vo}} = \sqrt{10}\rangle$ (blue curve) and overlap error $P(t)$ with respect to the coherent state $|\alpha_{\text{vo}} + \delta(t)\rangle$ (green curve), during the stabilization sequence shown in (b). The dashed line shows an exponential fit corresponding to $\gamma_{\text{eff}} = 76$ MHz. The constant parameters used in the simulation are $\omega_r/(2\pi) = 6$ GHz, $g/(2\pi) = 50$ MHz, $t_{\text{reset}} = 9.8$ ns, $t_{\text{int}} = 3.6$ ns, and $\gamma_r = 30$ kHz. Figure adapted from Publication IV.

Resonator towards the coherent state $|\alpha_{\text{vo}}\rangle$. We aim to minimize the deviation from the target state, defined as $\delta(t) = \langle \hat{a}(t) \rangle - \alpha_{\text{vo}}$. In Publication IV, we analytically show that when interacting with a resonant qubit in the ground state, the resonator coherently oscillates towards the state where $\delta = 0$ regardless of the initial deviation, a phenomenon also exemplified in Fig. 6.1a. This is intuitively understood in the displaced frame, where the resonator swaps an excitation with the qubit and thus moves towards the origin.

In the stabilization protocol, we keep the resonant interaction in effect for a time $t_{\text{int}}$ such that the deviation is significantly decreased and subsequently turn the interaction off by detuning the qubit. The dissipation of the qubit is then increased to $\gamma_{\text{q,max}}$ for a duration of $t_{\text{reset}}$ in order to reset the qubit to its ground state. After the reset, the system is approximately in the initial state, with the exception that $|\delta|$ has decreased. Repetition of the two-step process thus serves to converge the state of the resonator to $|\alpha_{\text{vo}}\rangle$. 

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6.2 Results

We simulate the stabilization protocol by solving the master equation

$$\frac{d}{dt} \hat{\rho}(t) = -\frac{i}{\hbar} [\hat{H}_s, \hat{\rho}(t)] + \frac{\gamma_q(t)}{2} \hat{L} \left[ |g\rangle\langle e| ; \hat{\rho}(t) \right] + \frac{}{} \gamma_r \hat{L} \left[ \hat{a}; \hat{\rho}(t) \right],$$  \hspace{1cm} (6.1)

with a control scheme depicted in Fig. 6.1b. An example of the results is shown in Fig. 6.1c, where we observe that the deviation faithfully decreases as the process is repeated. Notably, the target state is reached at a rate three orders of magnitude higher than the natural decay rate $\gamma_r$ of the resonator. We also calculate the overlap error with a coherent state as $P(t) = 1 - \langle \alpha_{vo} + \delta(t) | \rho(t) | \alpha_{vo} + \delta(t) \rangle$ to confirm that the resulting state is a coherent state. After several iterations of the protocol, the deviation saturates to a level determined by $\gamma_r$ and $\alpha_{vo}$.

To assess the effectiveness of the protocol, we define the effective stabilization rate $\gamma_{\text{eff}}$ as the rate at which the deviation exponentially approaches zero before saturating, as indicated by the dashed line in Fig. 6.1c. We calculate the effective stabilization rate from a simplified model that neglects dissipation in the resonator and assumes that the density operator of the system is separable. In the limit of small initial deviations, $\delta(0) < 1$ the effective rate is given by

$$\gamma_{\text{eff}} = \frac{\gamma_{q,\text{max}}}{4} \frac{t_{\text{reset}}}{t_{\text{reset}} + t_{\text{int}}} \bar{S}(t_{\text{reset}}, t_{\text{int}}),$$  \hspace{1cm} (6.2)

where the factor $\bar{S} \in [0, 1]$ can be maximized by choosing experimentally feasible values for the durations $t_{\text{reset}}$ and $t_{\text{int}}$. Notably, the effective stabilization rate is proportional to the maximal decay rate of the qubit. The full expression for the effective stabilization rate is given in Publication IV, where it is also verified by comparing it to the simulated master equation Eq. (6.1).

Finally, we apply the protocol to cool down a thermal state of the resonator in a finite temperature heat bath. We simulate Eq. (6.1) with additional Lindblad terms that correspond to bath temperatures $T_q$ and $T_r$ for the qubit and the resonator, respectively, and set $\alpha_{vo} = 0$. We find that after several iterations of the stabilization protocol, the average number of photons in the resonator saturates to levels shown in Fig. 6.2. In cases where the environment of the qubit is colder than that of the resonator, for instance if the qubit is coupled to a QCR, the resulting thermal occupation is several orders of magnitude below the initial equilibrium value.
Figure 6.2. Residual average photon number $\bar{n} = \langle \hat{a}^\dagger \hat{a} \rangle$ of the resonator as a function of the resonator bath temperature $T_r$ after 15 iterations of the stabilization protocol. The dashed line indicates the number of photons in the initial thermal state at $T_r$ and the solid lines correspond to different qubit bath temperatures as indicated. The simulation parameters equal those in Fig. 6.1. Figure adapted from Publication IV.
7. Reconstruction approach to open quantum systems

In this chapter, we outline the key result of Publication V, a general method for solving the dynamics of an open bosonic quantum system.

Quantum master equations [87] have proven remarkably useful for describing open quantum systems where a small subsystem of interest is coupled to intractable degrees of freedom referred to as an environment or a bath. As discussed in Sec. 2.5, a master equation is commonly constructed by combining a phenomenologically or microscopically justified Lindblad operator with the unitary evolution of the system of interest, as is done in Publications II, III, and IV. However, the usual form of the Lindblad master equation calls for the secular approximation and the Born–Markov approximation, that is, the assumptions that the influence of the system on the environment is weak, that the system and the environment are initially uncorrelated, and that changes in the environment decay quickly [87]. Because of these requirements, the Lindblad equation is not necessarily justified, especially for strongly dissipative environments such as the QCR [116]. Furthermore, although many analytical methods for solving master equations exist [117–120], they may be challenging to apply to complex systems of several environments and interconnected components.

In Publication V, we present an alternative approach, referred to as the dynamical reconstruction method, to analytically calculate the temporal evolution of an open bosonic system. In general, our method does not rely on the above assumptions and is applicable to any system described by bosonic operators. Possible applications of the method include, for example, systems of harmonic and anharmonic oscillators, damped and driven oscillators, and components coupled to separate or common environments.

The dynamical reconstruction method is presented in Fig. 7.1. In summary, the system is transformed into the Heisenberg picture, where the problem is reduced to that of solving a set of coupled equations of motion for the annihilation operators of the system. The solved annihilation operators are then used to construct the density operator in the Schrödinger picture using a specific, straightforward reconstruction formula. Ultimately, it
Reconstruction approach to open quantum systems

Figure 7.1. Process chart of the dynamical reconstruction method. For simplicity, the example formulae are for a single bosonic mode. First, the Hamiltonian is transformed from the Schrödinger picture into the Heisenberg picture. Subsequently, the Heisenberg equations of motion for each discrete annihilation operator of the system are obtained and solved. In the last step, the density operator is reconstructed by inserting the solutions for the annihilation operators into the dynamical reconstruction formula, Eqs. (7.2) and (7.3). Figure adapted from Publication V.

depends on the problem whether the dynamics is more easily solvable in the Heisenberg picture.

In the following, we briefly outline our reconstruction method. Consider a general system of $N$ discrete modes $\hat{a}_j$ and $M$ environments, modeled as continua of oscillator modes $\hat{B}_j(\omega)$. Let the Hamiltonian of the system in the Schrödinger picture be a polynomial in the $N+M$ ladder operators \( \{ \hat{A}_j, \hat{A}^\dagger_j \} \), where $\hat{A}_j$ is used to denote both $\hat{a}_j$ and $\hat{B}_j(\omega)$. The Hamiltonian in the Heisenberg picture $\hat{H}_H(t)$ is obtained by the standard transformation between the two pictures of quantum mechanics [3, 87]. Each system operator evolves according to their associated Heisenberg equation of motion, 

$$ \frac{d}{dt} \hat{A}_j(t) = -i \frac{\hbar}{\hbar} \left[ \hat{A}_j(t), \hat{H}_H(t) \right] , \quad \text{for } j = 1, \ldots, N+M. \quad (7.1) $$

The equations are solved with the initial conditions $\hat{A}_j(0) = \hat{A}_j$, where the original Schrödinger picture operators $\hat{A}_j$ obey the usual bosonic commutation relations.

The density operator of the reduced system may be obtained by inserting the solved operators $\hat{a}_j(t)$ into the dynamical reconstruction formula

$$ \hat{\rho}(t) = \sum_{k_1,j_1,\ldots,k_N,j_N=0}^\infty \text{Tr} \left[ \hat{c}_{k_1,j_1,\ldots,k_N,j_N}(t) \hat{\rho}(0) \right] \prod_{j=1}^{N} \left[ \hat{a}_j^\dagger(0) \right]^k \hat{a}_j^l(0), \quad (7.2) $$

In the following, we briefly outline our reconstruction method. Consider a general system of $N$ discrete modes $\hat{a}_j$ and $M$ environments, modeled as continua of oscillator modes $\hat{B}_j(\omega)$. Let the Hamiltonian of the system in the Schrödinger picture be a polynomial in the $N+M$ ladder operators \( \{ \hat{A}_j, \hat{A}^\dagger_j \} \), where $\hat{A}_j$ is used to denote both $\hat{a}_j$ and $\hat{B}_j(\omega)$. The Hamiltonian in the Heisenberg picture $\hat{H}_H(t)$ is obtained by the standard transformation between the two pictures of quantum mechanics [3, 87]. Each system operator evolves according to their associated Heisenberg equation of motion,

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$$ \hat{\rho}(t) = \sum_{k_1,j_1,\ldots,k_N,j_N=0}^\infty \text{Tr} \left[ \hat{c}_{k_1,j_1,\ldots,k_N,j_N}(t) \hat{\rho}(0) \right] \prod_{j=1}^{N} \left[ \hat{a}_j^\dagger(0) \right]^k \hat{a}_j^l(0), \quad (7.2) $$

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Reconstruction approach to open quantum systems

\[ \hat{c}_{k_1,l_1} \cdots k_N,l_N(t) = \prod_{j=1}^{N} \sum_{p_j=-\min(k_j,l_j)}^{\infty} \frac{(-1)^{p_j}(k_j+l_j+p_j)!}{k_j!l_j!(k_j+p_j)!} \left[ \hat{a}_{k_j,l_j}^{\dagger} \right]^{k_j+p_j} \hat{a}_{k_j,l_j}^{k_j+p_j}. \]  

(7.3)

In Publication V, we demonstrate the utility of the dynamical reconstruction method by employing it to solve the system of two coupled damped resonators used in Publication III. Resonators R1 and R2 are coupled to a transmission line and a QCR, respectively, which constitute the environmental degrees of freedom \( \hat{B}_1(\omega) \) and \( \hat{B}_2(\omega) \). Following Ref. [112], the Hamiltonian of the total system is given by

\[ \hat{H}_2/\hbar = g \hat{a}_1^{\dagger} \hat{a}_2 + \text{H.c.} + \sum_{j=1}^{2} \omega_j \hat{a}_j^{\dagger} \hat{a}_j + \int d\omega \left\{ \omega \hat{B}_j^{\dagger}(\omega) \hat{B}_j(\omega) + \sqrt{\frac{k_j}{2\pi}} \left[ \hat{a}_j^{\dagger} \hat{B}_j(\omega) + \text{H.c.} \right] \right\}, \]

(7.4)

where \( k_j \) denotes the coupling strength of resonator \( j \) to its respective environment and the Markov and rotating-wave approximations have been employed.

Using the dynamical reconstruction method, we are able to solve the temporal evolution of the density operator of the system under \( \hat{H}_2 \) analytically. To our knowledge, such a result for this system has not been presented before. We illustrate the solution by showing the average photon numbers as functions of time in Fig. 7.2 in the case where R2 is strongly coupled to a heat bath. As expected, in resonance, the resonators swap excitations before converging to thermal states. The full solution further shows that the resonators remain in thermal states throughout the evolution. On the other hand, detuning the resonators greatly suppresses the rate of thermalization for R1.

Figure 7.2. Average photon numbers \( \bar{\pi}_j = \langle \hat{a}_j^{\dagger} \hat{a}_j \rangle \) for a system of two damped resonators scaled by the steady state values \( \bar{\pi}_j^\infty \) as functions of scaled time. Both resonators are initially in the vacuum state and resonator 2 is strongly coupled to a heat bath with strength \( k_2 = g/2 \) \((k_1 = 0)\). The resonators are either (a) in resonance or (b) detuned by \( 5g \).
8. Conclusions and outlook

In this dissertation, we have investigated three elementary operations necessary for the development of a quantum computer: single-qubit gates, qubit readout, and initialization. The experimental publications demonstrate devices and applied methods that may be implemented in a future prototype of a quantum processor, and the proposed theoretical protocols lay the foundation for future experiments. The fundamental research of the dissertation gives insight to the limitations of qubit gates and general treatment of bosonic systems.

The multichannel readout method presented in Publication II shows great potential in improving upon the dispersive readout with minimal overhead. Although the design parameters of the experimental sample were not specifically optimized for the multichannel readout, we were able to demonstrate that the multichannel scheme yields up to 50% smaller readout error compared with the dispersive scheme. Our results are further supported by the fully independent experiment by Touzard et al. [114], where they demonstrated the non-destructiveness of the multichannel readout. Another benefit of the new method is the possibility to unconditionally reset the resonator after the readout, potentially leading to a shorter total clock cycle of a quantum processor. In addition to optimizing the experimental parameters, future research should seek to improve the method to achieve state-of-the-art readout fidelity, and use a more sophisticated technique such as gate set tomography [121] to characterize different sources of error.

In Publication III, we implemented the most efficient evacuation of photons from a resonator by utilizing a coupler resonator and a QCR. The combination of the two components serves to quickly reset a resonator to the vacuum state on demand. In principle, the device may also be utilized to reset a qubit instead of a resonator. Future experiments may incorporate the resonator–QCR device into hardware running a quantum algorithm to investigate the possible speed gained by actively resetting of the resonator and improvement of coherence times due to reduced thermal excitations. Aside from the practical utility, the controllability of the
Conclusions and outlook

The stabilization protocol proposed in Publication IV is designed for a similar system where a qubit acts as the coupler between a resonator and an environment. We showed that by modulating the frequency of the strongly dissipative coupler qubit, the resonator may be initialized to a desired coherent state at a rate that is independent of the intrinsic decay rate of the resonator. The protocol was simulated with parameters that are suitable for a future experiment. Publications III and IV together illustrate the versatility of tunable environments for initialization tasks.

In addition to the above-mentioned applications, Publications I and V present novel fundamental research. In Publication I, we derived the greatest lower bound for single-qubit gate error implemented with a given amount of power within the Jaynes–Cummings model. In principle, the derived scaling law of the error is experimentally verifiable by implementing qubit gates with a variable amount of photons in a resonator. However, the results may differ if an anharmonic oscillator such as a transmon is used as the qubit.

Furthermore, we showed in Publication I that a single control pulse may be efficiently reused for multiple single-qubit gates, in contrast to previous claims [57, 61]. Such reuse of a control pulse may become a relevant option in future quantum processors, where the overhead from dedicated control lines would otherwise become impractical.

Finally, Publication V provides a general method to solve the dynamics of an open bosonic quantum system. As an alternative to frequently-used master equations, the dynamical reconstruction method requires fewer assumptions than Lindblad master equations and is especially suitable for solving systems with multiple interconnected components. The method is applicable to systems with strong coupling to the environment, such as those investigated in Publications III and IV. We demonstrated the power of our method by analytically solving the system of two damped quantum harmonic oscillators used in Publication III. In the future, the method may find use in the analytical treatment of even more complex systems consisting of driven, anharmonic, and damped quantum oscillators.
References

References


References


References


Errata

Publication II

In supplementary equation (S24), the second $\Omega_r$ should read $\Omega_q$. 
Superconducting circuits are one of the most promising and rapidly developing platforms for large-scale quantum computing. In quantum computing, an algorithm is run by executing a series of fast and precise operations on the quantum bits, or qubits. The most important operations are the quantum gates, along with the measurement and initialization of the states of the qubits and auxiliary components. In this dissertation, we theoretically and experimentally develop techniques to enhance the precision and speed of these three operations. The results and techniques may be used for improving the clock cycle and robustness of a future quantum computer.