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SPERNER CAPACITY OF SMALL DIGRAPHS

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Abstract. The classical concept of Shannon capacity of undirected graphs was extended by Gargano, Körner, and Vaccaro to digraphs in the early 1990s, and termed Sperner capacity. Shannon, in his seminal work, determined the capacities for all isomorphism classes of undirected graphs with up to five vertices, except for the 5-cycle, which was finally settled by Lovász in 1979. The work of Shannon is here paralleled for digraphs; the Sperner capacity is determined for all but 8 of the 9846 isomorphism classes of digraphs with at most 5 vertices.

1. Introduction

Shannon [16], in 1956, introduced the concept of zero-error capacity of a communication channel. Since the channels studied in [16] can be viewed via their so-called characteristic or confusion graphs, a related concept is the Shannon capacity of an undirected graph. The exact definitions of the Shannon capacity and other notions will be given in Section 2.1.

Shannon [16], among many other things in his remarkable paper, developed some basic techniques for determining the capacity of an undirected graph, and applied these techniques to all (isomorphism classes of) graphs with at most 5 vertices. It turned out that the only case he could not settle was the 5-cycle, which remained open until Lovász published his celebrated result [11] in 1979. In fact, Shannon [16] also considered the graphs with 6 vertices and determined the capacities of all but four of these and pointed out that the open cases could be settled through a solution for the 5-cycle. So currently the smallest open case has 7 vertices. Such a graph is the 7-cycle—or the complement of the 7-cycle, depending on the definition

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of Shannon capacity; see Section 2.1—bounds for which can be found in [17]. The general case of the Shannon capacity of (complements of) odd cycles is in fact a challenging open problem.

Gargano, Körner, and Vaccaro [7] generalized the concept of Shannon capacity to digraphs, and coined the term Sperner capacity. Undirected graphs can indeed be seen as a subclass of digraphs: replace each edge $\{u, v\}$ in an undirected graph by a pair of edges, (u, v) and (v, u), to get a corresponding digraph. Also Sperner capacity can be linked to capacities of certain communication channels [4, 13].

The seminal paper [7] triggered work on many problems related to Sperner capacity, one of which is that of determining the Sperner capacity of small digraphs. The two isomorphism classes of directed triangles is settled in [3]—see also [2]—but little has been known for particular graphs with more than three vertices. The work in this area has mainly been focused on the relationship between Shannon and Sperner capacity and restricted to certain types of graphs, such as (directed) self-complementary graphs, cycles, and complements of cycles [10, 15]. In the vein of Shannon, to identify challenging types of digraphs, the authors decided to carry out an exhaustive study despite the rapid growth in the number of isomorphism classes. We are able to determine the Sperner capacities of all but 8 (pictured in Fig. 4) of the 9846 isomorphism classes of digraphs with at most 5 vertices.

The paper is organized as follows. The mathematical definitions of the involved concepts are gathered in Section 2.1. Techniques that can be used to determine lower and upper bounds on the Sperner capacity are discussed in Sections 2.2 and 2.3. Whenever these bounds meet, the Sperner capacity of a digraph is established. The results for the digraphs with at most 5 vertices are presented in Section 3. The paper is concluded in Section 4.

2. Sperner capacity

2.1. Preliminaries.

To define the concepts of Shannon and Sperner capacity, we need the following two definitions.

Definition 1. For undirected graphs G and H, the co-normal product $G \cdot H$ has vertex set

$$V(G \cdot H) = V(G) \times V(H)$$

and edge set

$$E(G \cdot H) = \{\{\{x_1, y_1\}, \{x_2, y_2\}\} : \{x_1, x_2\} \in E(G) \text{ or } \{y_1, y_2\} \in E(H)\}.$$

Definition 2. For digraphs $\vec{G}$ and $\vec{H}$, the co-normal product $\vec{G} \cdot \vec{H}$ has vertex set

$$V(\vec{G} \cdot \vec{H}) = V(\vec{G}) \times V(\vec{H})$$

and edge set

$$E(\vec{G} \cdot \vec{H}) = \{((x_1, y_1), (x_2, y_2)) : (x_1, x_2) \in E(\vec{G}) \text{ or } (y_1, y_2) \in E(\vec{H})\}.$$

The nth co-normal power of a (directed or undirected) graph G is denoted by G^n.

A *clique* in a graph is a set of mutually adjacent vertices. A clique in a graph G with largest possible size is said to be a *maximum clique* and its size is denoted by $\omega(G)$. The Shannon capacity of a graph can now be stated formally.

Definition 3. The (nonlogarithmic) *Shannon capacity* of an undirected graph G is

$$\Theta(G) = \sup_{n \geq 1} \sqrt[n]{\omega(G^n)}.$$

Here we have to remark that we have to take the power of the complement of G, $\bar{G}$, to make the definition conform with the original one of Shannon [16]. Generally, it is a matter of convention whether one defines the Shannon capacity via the size of a maximum clique of a co-normal power of a characteristic graph or via the size of a maximum independent set of a strong power of a confusion graph (a confusion graph is the complement of a characteristic graph). However, for digraphs, where the edges also have directions, it is much more convenient to consider edges than nonedges, so in work on Sperner capacity—where we want to generalize the definition of Shannon capacity—Shannon capacity is generally defined as in Definition 3; cf. [6, 9, 10, 15].

In the definition of Sperner capacity we need the following concepts. A *transitive subtournament* in a digraph $\vec{G}$ is a set of vertices $V' \subseteq V(\vec{G})$ having at least one linear ordering $<^R$ such that $x <^R y$ implies $(x, y) \in E(\vec{G})$ for every vertex $x, y \in V'$. The size of a maximum transitive subtournament in a digraph $\vec{G}$ is denoted by $\omega_t(\vec{G})$.

Definition 4. The (nonlogarithmic) *Sperner capacity* of a digraph $\vec{G}$ is

$$\sigma(\vec{G}) = \sup_{n \geq 1} \sqrt[n]{\omega_t(\vec{G}^n)}.$$

By definition, the Sperner capacity of a digraph $\vec{G}$ with $(x, y) \in V(\vec{G})$ iff $(y, x) \in V(\vec{G})$ equals the Shannon capacity of the underlying undirected graph. Since removal of edges cannot increase the capacity, the Sperner capacity of any digraph is bounded from above by the Shannon capacity of the underlying undirected graph. Sperner capacity is a true generalization of Shannon capacity: there exist digraphs with Sperner capacity smaller than the Shannon capacity of the underlying undirected graph. The simplest of these is the cyclically oriented triangle [2, 3]; see also Section 3.

2.2. Lower bounds. A clique in any co-normal power of an undirected graph G gives a lower bound on the Shannon capacity of G, and analogously a transitive subtournament in any power of a digraph $\vec{G}$ gives a lower bound on the Sperner capacity of $\vec{G}$.

The digraphs considered in this work are comparably small, and finding a transitive subtournament leading to a lower bound for the Sperner capacity that meets an upper bound—obtained by the methods to be discussed in Section 2.3—is for most instances computationally a very easy task. In fact, in all cases in which we were able to determine the Sperner capacity it suffices to consider $\vec{G}$ or $\vec{G}^2$, that is, digraphs with at most 5 or 25 vertices. For determining the size of a maximum transitive subtournament, algorithms from [8] were used.

2.3. Upper bounds. A variety of methods for obtaining upper bounds on the Sperner capacity have been published in the literature. We shall here review the methods that are needed for our purpose of studying the Sperner capacity of small digraphs.

The local chromatic number was introduced by Erdős *et al.* [5] for undirected graphs and later by Körner, Pilotto, and Simonyi [10] for digraphs. The *closed* *out-neighborhood* of a digraph $\vec{G} = (V, E)$ is defined as

$$N^+_{\vec{G}}(v) := \{v\} \cup \{u : (v, u) \in E\}.$$

A coloring $c : V(\vec{G}) \to \mathbb{N}$ of a digraph $\vec{G} = (V, E)$ is a *proper coloring* if for every edge $(u, v) \in E$, we have $c(u) \neq c(v)$.

Definition 5. The *local chromatic number* of a digraph $\vec{G}$ is

$$\psi_d(\vec{G}) := \min_{c:V(\vec{G})\to\mathbb{N}} \max_{v\in V(\vec{G})} |\{c(w) : w \in N^+_{\vec{G}}(v)\}|,$$

where $c : V(\vec{G}) \to \mathbb{N}$ is a proper coloring of $\vec{G}$.

The local chromatic number gives directly an upper bound for the Sperner capacity [10, Theorem 1].

Theorem 1.

$$\sigma(\vec{G}) \leq \psi_d(\vec{G}).$$

A fractional version of the local chromatic number occasionally gives even better bounds than the local chromatic number. A set of vertices of a digraph is said to be an independent set if this set forms an independent set of the underlying undirected graph. Let $S(\vec{G})$ be the family of independent sets of a digraph $\vec{G} = (V, E)$ and $S_v = \{S' \in S(\vec{G}) : v \in S'\}$. A *fractional coloring* $w : S(\vec{G}) \to \mathbb{R}_+ \cup \{0\}$ is said to be *proper* if $\sum_{S'\in S_v} w(S') \geq 1$ holds for every vertex $v \in V(\vec{G})$.

Definition 6. The *fractional local chromatic number* of a digraph $\vec{G}$ is

$$\psi^*_d(\vec{G}) := \min_{w:S(\vec{G})\to\mathbb{R}_+\cup\{0\}} \max_{v\in V(\vec{G})} \sum_{\substack{S'\in S(\vec{G}) \\ N^+_{\vec{G}}(v)\cap S'\neq\varnothing}} w(S'),$$

where the minimization is over proper fractional colorings of $\vec{G}$.

The fractional local chromatic number of a digraph gives an upper bound on Sperner capacity [10, Theorem 4].

Theorem 2.

$$\sigma(\vec{G}) \leq \psi^*_d(\vec{G}).$$

A non-negative function $g : 2^{V(\vec{G})} \to \mathbb{R}_+ \cup \{0\}$ is called a *fractional cover* of $V(\vec{G})$ if $\sum_{U\in U_v} g(U) \geq 1$, where $U_v = \{U \subseteq V(\vec{G}) : v \in U\}$, holds for every vertex $v \in V(\vec{G})$. Denote the digraph induced by $U \subseteq V(\vec{G})$ by $\vec{G}[U]$. The following bound for Sperner capacity is obtained in [10, Theorem 6].

Theorem 3.

$$\sigma(\vec{G}) \leq \min_{g:2^{V(\vec{G})}\to\mathbb{R}_+\cup\{0\}} \sum_{U\subseteq V(\vec{G})} g(U)\psi^*_d(\vec{G}[U]),$$

where the minimization is over fractional covers g of $\vec{G}$.

Reversing the directions of all edges of a digraph does not affect the Sperner capacity of the digraph. Consequently, Theorems 1, 2, and 3 may be applied both to a given digraph and to the digraph obtained by reversing its edges, and the smaller value is taken as an upper bound.

The final result in this section is of a different type compared to the previous theorems. For a given digraph $\vec{G} = (V, E)$ with $V = \{1, 2, \dots, n\}$, let $\mathcal{M}$ be a collection of $n \times n$ matrices with 1s in the diagonal, $m_{ij} = 0$ whenever $(i, j) \in E$, and $m_{ij} \in \mathbb{R}$ whenever $(i, j) \notin E$. The following result is proved in [3].

Theorem 4.

$$\sigma(\vec{G}) \leq \min_{\mathbf{M} \in \mathcal{M}} rank(\mathbf{M}).$$

3. Results

To generate all nonisomorphic graphs, the program *nauty* [12] may be used. The numbers of undirected and directed graphs with up to $n = 7$ vertices are shown in Table 1; see also [14].

TABLE 1. Numbers of nonisomorphic graphs

n	Undirected	Directed
1	1	1
2	2	3
3	4	16
4	11	218
5	34	9608
6	156	1540944
7	1044	882033440

Any digraph $\vec{G}$ with at most three vertices can be settled by finding the local chromatic number of $\vec{G}$ and a transitive subtournament of the same size in $\vec{G}$. These results are summarized in Table 2. The two possible orientations of a triangle, with Sperner capacities 2 and 3, respectively, are shown in Fig. 1.

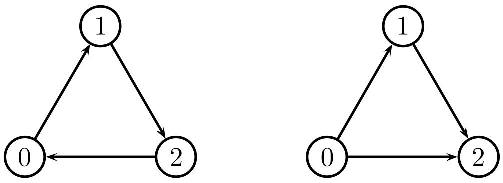

FIGURE 1. The two directed triangles

The upper bound for the cyclically oriented triangle can also be handled by Theorem 4, as the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

has rank 2. See [2, 3] for more details on the cases of directed triangles.

For digraphs with more than three vertices, the local chromatic number is still very useful for obtaining good upper bounds on the Sperner capacity, but for some instances we have to start considering other techniques as well.

For digraphs with four vertices, the local chromatic number attains the Sperner capacity for all but the four digraphs in Fig. 2. For all these digraphs, the local

Table 2. Sperner capacities of digraphs with up to 3 vertices

$\|V(\vec{G})\|$	$\sigma(\vec{G})$	#	$\|V(\vec{G})\|$	$\sigma(\vec{G})$	#
1	1	1	3	1	1
2	1	1	3	2	9
2	2	2	3	3	6

chromatic number is 4, because they all have a vertex 0 with an edge to all other vertices. In these cases we use Theorem 3 after partitioning the vertex set into $\{0\}$ and $\{1, 2, 3\}$, and obtain the upper bound 3 on the Sperner capacity. (For three of these digraphs we could also have used [1, Theorem 1.2].) The results for the digraphs with four vertices are summarized in Table 3.

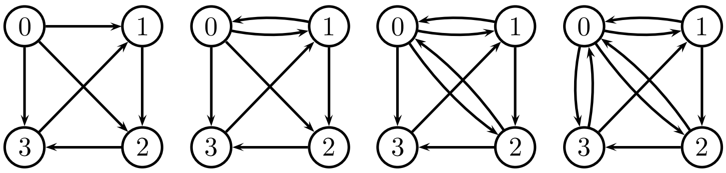

Figure 2. Four special 4-vertex digraphs

Table 3. Sperner capacities of 4-vertex digraphs

$\sigma(\vec{G})$	#
1	1
2	56
3	130
4	31

In the five-vertex case, we already have thousands of digraphs to be investigated. The local chromatic number can be used in 9144 cases to determine the capacity. Theorem 3 takes care of another 406 digraphs whose capacities were obtained by using the local chromatic number in Theorem 3; the optimization problem in the theorem was solved with linear programming.

Out of the remaining 57 digraphs, 20 have the 5-cycle as an underlying graph (ignoring multiple edges) so $\sqrt{5}$ is an upper bound on their Sperner capacity. For all these digraphs $\vec{G}$, we can find a transitive subtournament of size 5 in $\vec{G}^2$, so $\sqrt{5}$ is indeed their Sperner capacity.

For 29 of the 37 digraphs still remaining, we use Theorem 4. These 29 digraphs are drawn in Fig. 3 within 15 different digraphs so that in addition to the solid edges at most one dashed edge belongs to each depicted digraph. For example, the first digraph in the second row depicts three different digraphs, the one without the edges $(3, 2)$ and $(4, 2)$, the one without the edge $(3, 2)$, and the one without the edge $(4, 2)$. By Theorem 4, an upper bound of 3 is obtained for the Sperner capacity of any digraph in Fig. 3, and the lower bounds obtained by finding transitive subtournaments reach this upper bound.

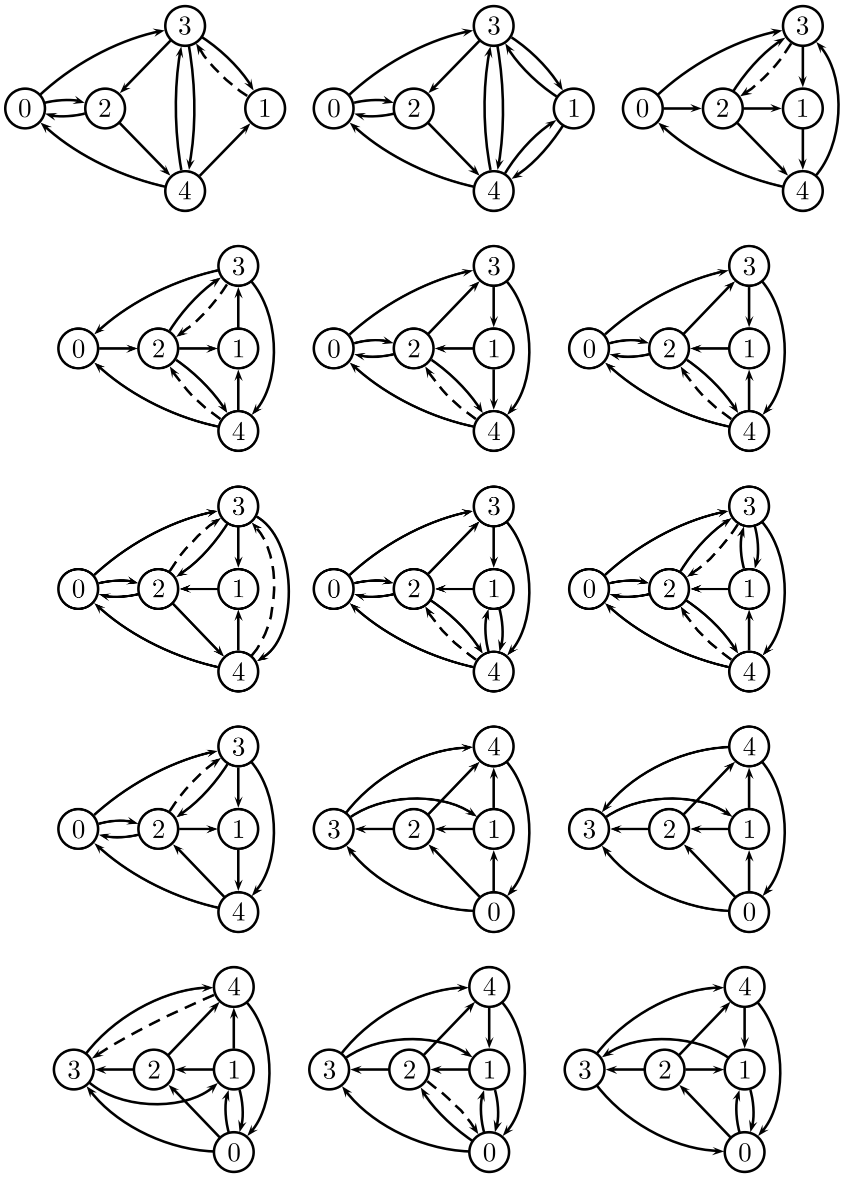

Figure 3. Digraphs handled by matrix rank method

For example, to obtain an upper bound on the Sperner capacity of the first digraph in Fig. 3 by Theorem 4, we observe that the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

has rank 3. In fact, for all the digraphs in Fig. 3 we obtain the upper bound 3 for their Sperner capacity in this manner.

Finally, we have 8 digraphs for which our upper and lower bounds do not meet. These digraphs are drawn in Fig. 4. For any digraph $\vec{G}$ among these, the digraph $\vec{G}^2$ has a transitive subtournament of size 5. (It actually suffices to consider only the first digraph, which is a subgraph of all other digraphs.) The best upper bound was obtained by Theorem 3, with $V_1 = \{2, 3\}$, $V_2 = \{0, 1, 2, 4\}$, and $V_3 = \{0, 1, 3, 4\}$, and the weight 1/2 for every subset. This leads to an upper bound of 5/2 for the Sperner capacity. Note that the first digraph in Fig. 4 is an induced subgraph of a seven vertex digraph discussed in [10] that has the same best known lower and upper bounds on its Sperner capacity. The results for digraphs with five vertices are summarized in Table 4.

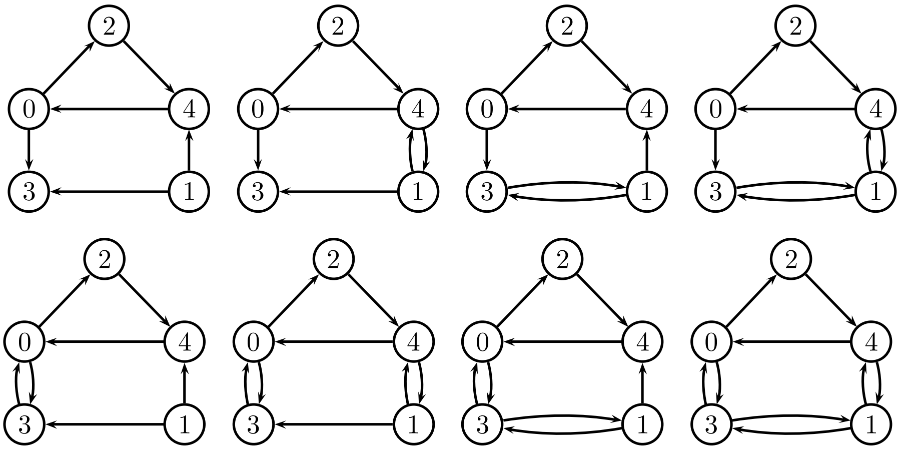

Figure 4. Eight unsolved 5-vertex digraphs

Table 4. Sperner capacities of 5-vertex digraphs

$\sigma(\vec{G})$	#
1	1
2	483
$\sqrt{5}$	20
$\sqrt{5} \leq \sigma(\vec{G}) \leq 5/2$	8
3	5299
4	3495
5	302

4. Conclusions

In the current work, the Sperner capacity of all but 8 of the digraphs with at most 5 vertices could be determined with the help of available theorems and techniques and by computer search. It is our hope that the open cases inspire further work in this area. Although there is a rapid growth of the number of digraphs with more than 5 vertices, it should be possible to push the limits a little bit further by an automatized, computer-aided approach. Such work should be carried out at the latest when the remaining open 5-vertex cases have been settled.

Related to Sperner capacity one of the most interesting questions is whether there exist undirected graphs, all orientations of which have smaller Sperner capacity than the Shannon capacity of the original graph [15]. None of the graphs with at most 5 vertices provide an example of such a graph.

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