Locally Verifiable Labelings and Nash Equilibria

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Abstract of the master's thesis

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Abstract

This thesis showcases a useful relation between game theory and distributed computing theory, and explores possible generalizations of this connection.

Game theory studies the interactions between rational agents. With origins in economics, it has applications on social sciences, computer science and logic. It is usual that agents, when interacting, want to change their strategy in order to improve their gain. When the rational agents interact in a stable way, meaning none of them wants to change their strategies, they are in a Nash equilibrium. This concept is widely studied. There are two types of Nash equilibria, pure and mixed, depending on which kinds of strategies players can have.

Distributed computing is a field of computer science that studies how multiple computers should interact in order to work as one distributed system. In distributed models, computational nodes, which could represent computers, produce outputs by communicating with its neighbors in a network. Locally verifiable labeling (LVL) is a type of problem where, to each node, it has be assigned a label from a set of possible labels, while satisfying specific conditions. These problems can be relaxed into more general formulations where the allowed set of labels is infinite, called a fractional relaxation.

Nash equilibria and locally verifiable labelings are related. For any game G there is always an LVL Π whose solutions are equivalent to Nash equilibria of G, and vice-versa.

In this thesis, a proof is provided showing that if the pure Nash equilibria of a game G are related to solutions of a locally verifiable labeling Π, then it is not necessarily true that there is a relation between mixed Nash equilibria of G and solutions of the fractional relaxation of Π.

Furthermore, it explores examples of the usefulness of linking the two areas of research described above: studying both a game and its equivalent LVL enables a broader understanding surrounding these two objects.

This work introduces a new tool and shows how it can be used. This tool can produce new results in both game theory and distributed computing and help understand existing results in a different way.

Keywords Game theory, Nash equilibria, distributed computing, locally verifiable labeling, fractional locally verifiable labeling, volunteer's dilemma, maximal independent set.

Preface

I want to thank professor Jukka Suomela for his incredible help and tutoring on writing this thesis and for providing recent problems to explore and work with. I wish this thesis will help future study both in game theory and distributed computing and show how helpful it is to think about one area when studying the other.

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João Afonso Batista

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1 Introduction

Science is vast and has many independent fields of knowledge and research. However, it is not uncommon to find bridges between such areas. These bridges enrich both connected subjects with new tools and results.

Fermat's Last Theorem was proven, 358 years after being formulated, by Andrew Wiles [1]. Even though Fermat's Last Theorem purely exists in the realm of number theory, its first proof used important results from harmonic analysis together with results that connected elliptical curves, a concept from number theory, with modular forms, a concept from harmonic analysis. This is one of the many examples where connections from seemingly independent areas can be fruitful.

A graphical game, being an interaction between n players played on a graph, is complex by nature, and generally depends on the underlying topology. Thus, a core concept in game theory, which allows a deeper understanding of games, are equilibria, the most common being Nash equilibria, introduced in 1950, by John Nash [2]. Throughout decades, this concept has been thoroughly studied and a lot is known about its properties and how to find them.

Distributed computing is a field of computer science where computational nodes communicate with each other in a network in order to output some result. Just like graphical games, the underlying topology is very important and can influence the complexity of problems and even the existence of solutions. As a theoretical computer science field, distributed computing results include the existence of algorithms that solve problems and the complexity study of such algorithms.

The authors of paper [3] show a strong connection between distributed computing and game theory, two fields of active research. This thesis grasps how far this connection goes and showcases the utility it provides to researchers in these areas.

Understanding a distributed problem, namely, its complexity and properties, can provide a lot of useful information to someone studying a game, from the level of difficulty of finding Nash equilibria, to algorithms that provide such equilibria as efficiently as possible, to the understanding of other properties about the game, such as mendability. On the other hand, when studying a distributed problem, understanding the related game might provide intuition about the problem and some practical examples where it might be applied, amongst other results.

Firstly, this thesis provides the definitions and propositions from both distributed computing theory and game theory necessary to understand the posed questions and the results provided. Then, there is an explanation of the main result from [3]: Nash equilibria of games are equivalent to solutions of a locally verifiable labeling (LVL). After all the ground work is done, there is a proof showing this relation cannot be generalized. Figure 1 provides a visual explanation of the generalization attempted on this thesis, which can be translated in the following question. Given a game G and an LVL Π and assuming the pure Nash equilibria of G represent solutions of Π and vice-versa, are mixed Nash equilibria (the natural generalization of pure Nash equilibria) somehow related to solutions of the fractional relaxation of Π (the natural generalization of an LVL)?

After showing the negative result about the previous posed question, this thesis

$$\begin{array}{ccc}
\text{Pure Nash Equilibria} & \Longleftrightarrow & \text{Solutions to LVL} \\
\Big\downarrow{\scriptstyle Generalization} & & \Big\downarrow{\scriptstyle Generalization} \\
\text{Mixed Nash Equilibria} & \overset{?}{\Longleftrightarrow} & \text{Solutions to Fractional LVL}
\end{array}$$

Figure 1: Question posed in this thesis represented on a diagram.

continues to show the usefulness of the bridge between the two areas with examples.

This thesis has two mains objectives: to spread the idea that thinking about game theory and distributed computing together provides richer conclusions and to prove a negative result regarding the generalization of the equivalence between these two areas.

2 Distributed computing and local verifiable labelings

2.1 Distributed computing

A key concept in the questions and results posed on this thesis is the locally verifiable labeling. In order to understand it, it is necessary to explain fundamental distributed computing concepts, such as models of distributed computing and algorithms. These have intricate mathematical definitions whose exhausting exploration falls outside the scope of this thesis. In the following subsections, a more intuitive and informal explanation of these ideas will be given. A more formal look into distributed computing can be found on [4].

2.1.1 LOCAL model

There are many models of distributed computing. In this thesis, the focus will be on the LOCAL model [5]. In this model, there is a graph $G = (V, E)$ where each node is a computational entity that can communicate with its neighbors. During synchronous rounds, every node changes its internal state according to some computations and sends messages to all of its neighbors. Note that there is no bound on the internal computation nor on the size of the messages sent. The LOCAL model also provides $O(\log(n))$-bit sized unique identifiers to the nodes, where $n = |V|$. After t rounds, when every node is in an output state, the computation ends and the local outputs of every node provide a global output of the graph. The combination of unique identifiers and no bound on the size of messages gives the LOCAL model the property that, in T rounds, any node $v \in G$ can learn everything about its T-neighborhood, denoted by $B(v, T)$. Furthermore, the LOCAL model allows that each node receives an initial input, although that will not be explored in this thesis.

2.1.2 Distributed algorithm

Intuitively, a distributed algorithm considers every node as a state machine. When designing an algorithm, one must define, for any state a node can have, which messages nodes send to its neighbors and to which state they change. It is also necessary to specify all output states. An algorithm is a set of instructions that each node, locally, must follow in order to arrive at a desired global state of the graph. As said before, there is a formal way to define the LOCAL model, what is an algorithm and its execution on a graph.

2.1.3 Example: 3-coloring cycle graphs

Let's consider an example: finding a 3-coloring in a cycle graph.

Finding a coloring in a graph is an old problem [6]. A graph is properly colored if every node v has a color c and none of v's neighbors has color c.

Consider a cycle graph G of length n with unique identifiers ranging from 0 to n^k, for some $k \in \mathbb{N}$, where $c(v)$ denotes node v's identifier. Also consider the set of colors as $\Sigma = \{-3, -2, -1\}$.

Algorithm 1 Finding a 3-coloring in cycles

while $c(v) \notin \{-3, -2, -1\}$ **do**
  send $c(v)$ to all neighbors. Let $N(v)$ denote the set of identifiers of the neighbors of v (which v knows since it received that information from its neighbors' messages).
  if $c(v) > \max(N(v))$ **then**
    $c(v) \leftarrow \max(\{-3, -2, -1\} \setminus N(v))$
  end if
end while

Let's check that if every node follows Algorithm 1, then the algorithm will output a 3-coloring when the execution finishes. Consider that, even after reaching an output state, the nodes continue sending their identifier to their neighbors each round.

Proposition 1. Algorithm 1 finishes its execution, in any cycle graph with length n, in $O(n)$ rounds.

Proof. For this algorithm, a node with an identifier in $\Sigma = \{-3, -2, -1\}$ is in an output state. So the execution stops if every node reaches an output state in finite time.

Note that every node has exactly 2 neighbors, since G is a cycle. Thus,

$$\forall_{n \in V} |N(v)| \leq 2 \text{ and therefore } \Sigma \setminus N(v) \neq \emptyset.$$

This means that, when a node tries to change into a new identifier in the set $\{-3, -2, -1\}$, it always has at least one number available to choose from.

Since all the nodes start with unique identifiers, there is a node v_1 whose identifier is the largest in the graph. In the first round, $c(v_1)$ will be larger than the identifiers of its neighbors, thus, it will change to an identifier in set Σ. After the first round, $c(v_1) \in \Sigma$, thus, following the same logic, there is a node v_2 whose unique identifier $c(v_2)$ is the largest in the graph. In the second round, v_2 will change its identifier to be in Σ. Following the previous logic, in round i, if the algorithm has not stopped yet, there will be a node v_i whose identifier is the largest in the graph. This proves that at least one node per round changes its identifier to be in Σ. Together with the fact that nodes in an output state never change states, it is true that, in the worst case, after n rounds, the algorithm will stop. □

Proposition 2. The outputs of the nodes, after executing Algorithm 1, form a solution to the 3-coloring problem.

Proof. When a node changes its identifier to be in Σ, it always does so in a way such that its new identifier is different from both its neighbors' identifiers. Furthermore, as every node has a different identifier than both its neighbors, no two adjacent nodes will ever change identifier in the same round. Thus, when every node has an identifier in Σ, the result is a 3-coloring of the graph. □

Propositions 1 and 2 show that Algorithm 1 is correct, meaning its execution stops in finite time and produces a solution to the desired problem.

2.2 Locally verifiable labeling

Definition 1 (Locally verifiable labeling). A **locally verifiable labeling** (LVL) is a pair $\Pi = (\Sigma, C)$, where Σ is an alphabet (possibly infinite) containing all the labels nodes can have and C is a set of configurations which defines what is a proper solution. A configuration $c \in C$ is a subgraph of G centered on some node v with radius at most k (in this thesis $k = 1$) where every node has a label in Σ. A **labeling** $f : V \to \Sigma$ is a **solution** to Π if and only if each f-labeled k-neighborhood of G is a configuration in C.

As an example, see how the previous problem, a 3-coloring in cycles, can be formalized as an LVL.

2.2.1 3-coloring in cycles as an LVL

Σ is the set of possible colors, it could, as formalized in the previous example, be set as $\Sigma = \{-3, -2, -1\}$. However, to simplify notation, let $\Sigma = \{1, 2, 3\}$. The set of configurations C defines the possible distribution of colors in a local 1-neighborhood. It contains every configuration where adjacent nodes have different colors. Since the problem is defined for cycles, it is only necessary to study 1-neighborhoods of nodes with degree 2. The set of configurations is expressed as follows $C = \{[x, y, z] \mid x, y, z \in \Sigma, x \neq y, x \neq z\}$. In a configuration $[x, y, z]$, x is the label of the center node and y and z are the labels of its two neighbors.

Considering some node v in a cycle of length 3 and its two neighbors u and w, a mapping f where $f(v) = 1$, $f(u) = 1$ and $f(w) = 3$ does not solve the problem since $[1, 1, 3] \notin C$, even if, when looking at node w, $[3, 1, 1] \in C$. However, if $f(v) = 1$, $f(u) = 3$ and $f(w) = 2$, then f does solve the problem, since, when looking at node v, $[1, 3, 2] \in C$, when looking at node u, $[3, 2, 1] \in C$ and when looking at node w, $[2, 1, 3] \in C$.

2.2.2 Maximal independent set

Maximal independent set [7] is a problem that will be studied in this thesis. An independent set I in a graph $G = (V, E)$ is a set of nodes, $I \subseteq V$, that does not contain adjacent nodes. It is maximal if there is no other independent set I' that contains I.

This problem can be formalized as an LVL for graphs with maximum degree Δ: $\Pi_1 = (\Sigma_1, C_1)$ where $\Sigma_1 = \{0, 1\}$ and

$$C_1 = \bigcup_{d=0}^{\Delta} \{[1, x_1, x_2, \ldots, x_d] | \sum_{i=1}^{d} x_i = 0\} \cup \bigcup_{d=1}^{\Delta} \{[0, x_1, x_2, \ldots, x_d] | \sum_{i=1}^{d} x_i \geq 1\}.$$

The previous notation used for configurations assumes the first label belongs to the center node and the other d labels belong to the center node's d neighbors. Intuitively, a node with label 1 belongs to the independent set and a node with label 0 does not. What the configurations in C mean is: either the center node is in the set, in which case none of its neighbors can be, or the center node is not in the set, in which case at least one of its neighbors must be in the set. The first condition ensures the resulting set is independent and the second that the resulting set is maximal. Two examples can be found on Figure 2.

As shown in [8], a maximal independent set in a graph G can be computed in $O(\log^*(n) + \Delta^2)$ rounds, where n is the number of nodes in G and Δ is the maximum degree of G. Asymptotically, for graphs with maximum degree Δ, the complexity of finding maximal independent sets is $O(\log^*(n))$ rounds.

2.3 Fractional problems

A pertinent question when faced with LVL where $\Sigma = \{0, 1\}$ is what happens if the alphabet becomes continuous, $\Sigma = [0, 1]$. If, instead of a node belonging to a set or not, what happens if a node can be labeled as belonging 0.3 to the set, or 0.6? This is called the fractional relaxation of a problem [9]. Before formalizing the relaxation, one must first try to find a simple, yet generalizable, characteristic that encompasses all the conditions a solution to the problem must possess. For the purposes of the current study, only fractional relaxation of the maximal independent set problem will be constructed.

2.3.1 Maximal fractional independent set

For this particular problem, the conditions a solution must possess can be characterized as follows. For the set to be independent, we have that, for any edge on the graph, the sum of the labels of the two adjacent nodes must be at most 1. For the set to be maximal, for every node, there must exist one adjacent edge for which the sum of the labels of its adjacent nodes is exactly 1. For the case of nodes with degree 0, with no adjacent edges, they must be labeled with 1. Having this in mind, a proper formalization of this problem is trivial: $\Pi_2 = (\Sigma_2, C_2)$ where $\Sigma_2 = [0, 1]$ and

$$C_2 = \{[1]\} \cup \bigcup_{d=1}^{\Delta} \{[x_0, x_1, x_2, \ldots, x_d] |\ \forall_{j>0}\ x_0 + x_j \leq 1,\ \exists_{j>0}\ x_0 + x_j = 1\}.$$

Note that any mapping f that is a solution to Π_1 is also a solution to Π_2. Examples can be found on Figure 2.

A solution to problem a Π is also a solution to its fractional relaxation. Thus, the fractional relaxation of a problem Π is never harder to solve than Π. It is not always the case that the fractional relaxation actually makes the problem easier, but for the maximal independent set it does. The constant time algorithm that outputs label 0.5 on every node whose degree is larger than 0 and label 1 on nodes with degree 0 solves the problem. In this case, for every edge, the sum of its two adjacent labels is exactly 1, satisfying both conditions of the problem.

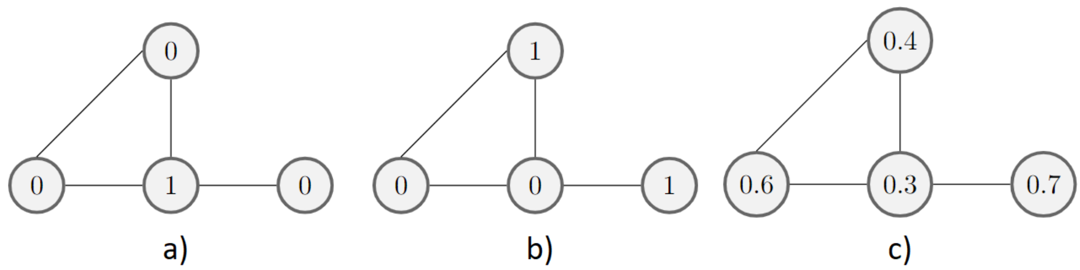

Figure 2: a),b) Maximal independent sets. c) Maximal fractional independent set.

2.4 Solvability and mendability

When studying problems in a distributed setting, a lot of research questions may be studied. Solvability and mendability are two such questions.

The authors in [10] properly define these concepts, however, for the purpose of this thesis, an informal definition suffices.

A problem Π is $O(T(n))$-solvable if there is an algorithm that finds a solution to Π in $O(T(n))$ rounds.

As seen before, problem Π_1, whose solutions are maximal independent sets, is $O(\log^*(n))$-solvable and problem Π_2, whose solutions are maximal fractional independent sets, is $O(1)$-solvable.

Mendability is more complex and intricate. Consider an LVL $\Pi = (\Sigma, C)$, a graph $G = (V, E)$ and a labeling $f : V \to \Sigma$. As mentioned in Section 2.2, f solves Π if, for every node $v \in V$, the labels of v and its neighbors $v_1, v_2, \dots, v_d$ form a configuration in C ($[f(v), f(v_1), f(v_2), \dots, f(v_d)] \in C$).

Now consider a partial labeling $f' : V \to \{\perp\} \cup \Sigma$, where a node associated to $\perp$ can be thought as not having a label. A node v_0 with neighbors $v_1, v_2, \dots, v_d$ is said to be *happy* if $[f'(v_0), f'(v_1), f'(v_2), \dots, f'(v_d)] \in C$ or if $\exists_j f'(v_j) = \perp$.

f' is said to be a partial solution for Π if every node v is *happy* and there is at least one node whose label is $\perp$. Notice how, given a partial labeling f', some nodes might not be labeled (meaning they are labeled with $\perp$). The problem of mendability revolves around, given a node v which is not labeled (meaning its partial label is $\perp$), trying to label v while maintaining the happiness of every node in G.

Definition 2 (Mendability). Consider a partial labeling f' where $f'(v) = \perp$. Problem Π is said to be $O(T(n))$-mendable if, in the worst case, in order to label v and keep the happiness of every node, it is necessary to change the labels of nodes up to distance $T(n)$ of v.

The following couple of examples will illustrate these definitions.

Consider the problem of finding a 3-coloring in cycles, defined as an LVL in Section 2.2.1. Algorithm 1 shows that this problem is $O(n)$-solvable. In fact, it is $O(\log^*(n))$-solvable, as shown in [4]. As for mendability, consider the following partial labelings of a cycle of length four with nodes $[v_1, v_2, v_3, v_4]$:

- $f_1 = [1, 2, 3, 2]$

- $f_2 = [1, \perp, 3, 2]$
- $f_3 = [\perp, 3, 3, 2]$

f_1 is a partial labeling where all the nodes are labeled. Note how every node v_i is *happy* since its 2 neighbors have different labels than v_i.

f_2 is a partial label where the second node does not have a label. Again, every node v_i is *happy*, since v_i is either 1) not labeled, or 2) a neighbor of a non-labeled node, or 3) in a regular situation, where the label of v_i is different than the labels of its two neighbors. It is possible to attribute label 2 to v_2, completing the labeling while keeping everyone *happy*. In this case, no other node needed to be changed other than v_2.

Lastly, f_3 is a partial labeling where every node is *happy*, for similar reasons of those for f_2. Note that v_2 is *happy* even though v_3 has the same label, since v_1 is not labeled. In this case, in order to label v_1 and make every node *happy*, it is necessary to change the label of v_2. For example, one could attribute label 1 to v_1 and change the label of v_2 to 2 and every node would be *happy*. The key aspect is that nodes up to distance 1 had to change their labels in order to label v_1 and keep every none *happy*.

This is an example of a problem that is mendable in 1 round, called a 1-mendable problem (as shown in [10]).

The previously defined problem of finding a maximal independent set is also 1-mendable and $O(\log^*(n))$-solvable.

As proven in [10], a problem which is efficiently mendable ($O(\log^*(n))$), is also efficiently solvable. As such, problems can be divided into 3 families: 1) problems which are efficiently solvable and efficiently mendable, 2) problems which are efficiently solvable but not efficiently mendable and 3) problems which are not efficiently solvable nor efficiently mendable.

3 Game theory

Now that all the necessary distributed computing theory concepts have been explained, let the focus turn to game theory, in particular to graphical games and Nash equilibria.

Definition 3 (Multiplayer game). A **multiplayer game** is a game with n players $i \in I$ where each player i has an action space A_i, potentially infinite. Furthermore, each player i is equipped with an utility function u_i that, given the actions of every player, returns the utility player i has. A joint action profile is denoted as $\vec{a} = (a_1, a_2, \ldots, a_n)$, where $a_i \in A_i$ and let $\vec{a}_{-i}$ denote the joint action profile of every player other than player i. To simplify notation, and when the joint action $\vec{a}$ is obvious by context, let $u_i(a_i, \vec{a}_{-i}) = u_i(a_i)$.

Definition 4 (Graphical game). A **graphical game** [11] is a multiplayer game over a graph $G = (V, E)$, where nodes are players, $I = V$ and the utility function u_i only depends on the 1-neighborhood of i.

Definition 5 (Pure and Mixed Strategies). A **mixed strategy** for player i is defined as $s_i \in \{x \mid x \in [0,1]^{|A_i|}$ and $\sum(x) = 1\}$. It is a probability distribution function (p.d.f) over the action space A_i. Let a mixed strategy $s_i = [c_1, c_2, \ldots, c_m]$ be a **full mixed strategy** if $|\{i|c_i = 0\}| = 0$.

A **pure strategy** is a deterministic probability distribution where s_i is composed of 0's and one 1. Let ps_i denote the pure strategy with a 1 on the i'th position, meaning that a player with a pure strategy ps_i is certain to choose action a_i. Again, let $\vec{s}$ denote a joint strategy profile and $\vec{s}_{-i}$ denote the joint strategy profile of every player other than player i.

The notion of full mixed strategy is not common or particularly useful. It is introduced here since it will simplify future results.

It is possible to generalize the utility function u_i so it has as input a joint strategy $\vec{s} = (s_1, s_2, \ldots, s_n)$ instead of a joint action $\vec{a}$. To keep the notation unambiguous, let the utility function that takes joint strategies as input be denoted as $w_i(\vec{s})$. It is the expected value of the utility over the strategies that define which probability each action has of being chosen.

Note that the utility function with pure strategies is the same as the utility function with actions:

$$u_i(a_{j_1}, a_{j_2}, \ldots, a_{j_n}) = w_i(ps_{j_1}, ps_{j_2}, \ldots, ps_{j_n}).$$

Furthermore, given that a mixed strategy is a p.d.f over the action space A_i, the following is true:

Proposition 3. Let the number of actions a player $i \in I$ can have be m, $A_i = \{a_1, a_2, \ldots, a_m\}$. The utility of a mixed strategy $s_i = [c_1, c_2, \ldots, c_m]$ is an affine combination of the utilities of pure strategies. The scalars are the probabilities of s_i. This affine combination can be described as:

$$w_i([c_1, c_2, \ldots, c_m], \overrightarrow{s_{-i}}) = \sum_{j=1}^{m} (c_j \cdot w_i(ps_j, \overrightarrow{s_{-i}})). \qquad (1)$$

Proof. A mixed strategy $s_i = [c_1, c_2, \ldots, c_m]$ is a probability distribution over the actions player i can have. Player i has a c_1 probability of doing action a_1, a c_2 probability of doing action a_2, etc. Given a joint strategy $\overrightarrow{s}$, player i has a c_1 probability of having utility $w_i(ps_1, \overrightarrow{s_{-i}})$, a c_2 probability of having utility $w_i(ps_2, \overrightarrow{s_{-i}})$, etc.

□

A Nash equilibrium is a very important concept both in game theory and for this thesis.

Definition 6 (Nash equilibrium). A **mixed Nash equilibrium** is a joint strategy profile $\overrightarrow{s^*}$ that, for every player i, satisfies the following:

$$\forall_{s_i} w_i(s_i^*, \overrightarrow{s_{-i}^*}) \geq w_i(s_i, \overrightarrow{s_{-i}^*}). \tag{2}$$

A **pure Nash equilibrium** is a mixed Nash equilibrium where the possible strategies that nodes can have are pure strategies.

Intuitively, a Nash equilibrium is a state where no player has any incentive to change strategy.

Proposition 3 gives us a tool to find mixed Nash equilibria.

Proposition 4. Assume a game is following a joint strategy profile $\overrightarrow{s^*}$. If, for every player, every pure strategy ps_i has the same utility, then $\overrightarrow{s^*}$ is a mixed Nash equilibrium.

Proof. Assuming, for a player $i \in I$ with $|A_i| = m$, that $w_i(ps_1) = w_i(ps_2) = \ldots = w_i(ps_m) = u$ then, for any strategy $s_i = [c_1, c_2, \ldots, c_m]$, the following is true

$$w_i(s_i, \overrightarrow{s_{-i}^*}) = \sum_{j=1}^{m}(c_j \cdot w_i(ps_j, \overrightarrow{s_{-i}^*})) = \sum_{j=1}^{m}(c_j \cdot u) = u \cdot \sum_{j=1}^{m} c_j = u.$$

In particular, $w_i(s_i^*, \overrightarrow{s_{-i}^*}) = u$. By Definition 6, it holds that

$$\forall s_i w_i(s_i^*, \overrightarrow{s_{-i}^*}) = u \geq u = w_i(s_i, \overrightarrow{s_{-i}^*}).$$

Thus, since this holds for any player $i \in I$, $\overrightarrow{s^*}$ is a mixed Nash equilibrium.

□

It is important to note that this proposition is not an equivalence: if every node has the same utility for every pure strategy, then it is an equilibrium, but if there is an equilibrium, it might be the case that the utilities of pure strategies are different for some nodes. Examples of this will be provided later. It is important to further build upon the previous proposition.

Proposition 5. Assume a game is following a joint strategy profile $\overrightarrow{s^*}$. If a player $j \in I$ has a full mixed strategy s_j^*, then $\overrightarrow{s^*}$ can only be a mixed Nash equilibrium if every pure strategy ps_i of player j has the same utility.

Proof. Consider a player $j \in I$ with $|A_j| = m$ whose full mixed strategy is $s_j^* = [c_1, c_2, \ldots, c_m]$ and whose pure strategies do not have all the same utility. Without loss of generality, consider that the actions are ordered by descending utility: $w_j(ps_1, \overrightarrow{s_{-j}^*}) \geq w_j(ps_2, \overrightarrow{s_{-j}^*}) \geq \ldots \geq w_j(ps_{i-1}, \overrightarrow{s_{-j}^*}) > w_j(ps_i, \overrightarrow{s_{-j}^*}) \geq \ldots \geq w_j(ps_m, \overrightarrow{s_{-j}^*})$. Then, it is the case that

$$w_j(ps_1, \overrightarrow{s_{-j}^*}) > w_j(s_j^*, \overrightarrow{s_{-j}^*})$$

since, from Proposition 3, $w_j(s_j^*, \overrightarrow{s_{-j}^*})$ is an affine combination of the utilities of pure strategies and ps_1 has a higher utility than ps_m. Note that this is only true since s_j^* is a full mixed strategy. If s_j^* had some zero probabilities, then the formulation of this proposition would be more convoluted, but this level of generality is enough for the results on this thesis.

Equation 3 contradicts Definition 6 and thus, $\overrightarrow{s^*}$ is not a mixed Nash equilibrium. □

Intuitively, this previous proposition means that, if a player i has a higher utility with a given action a, then raising the probability of doing action a will raise the utility of player i. That might not be possible if player i already has a pure strategy where it always does action a. Therefore, this proposition creates two cases for a player in a mixed Nash equilibrium:

- Player i has a full mixed strategy, in which case the utility of every pure strategy ps_i is the same.
- Player i has a mixed strategy which is not a full mixed strategy, in which case the utility of pure strategies might differ. (A mixed strategy which is not a full mixed strategy is a strategy which has at least one action whose probability is zero. This includes pure strategies.)

The necessary game theory concepts have been laid out. The next Section explores an algorithm to find pure Nash equilibria, which will be important for future analyses.

3.1 Best response dynamics

Best response dynamics [12] is an algorithm used to find pure Nash equilibria in graphical games. There are various variations of this algorithm. The one used by the authors in [3] and the one useful to this thesis can be found in Algorithm 2.

Algorithm 2 Best response dynamics

```
Initialize the game in a graph G. Each player has a starting pure strategy.
while Some player has changed strategy since previous iteration do
    Some adversary orders the players.
    In the the order given, and one at a time, each player changes its strategy to
    the optimal one.
end while
```

First, notice that when this algorithm stops, the output is a pure Nash equilibrium. The algorithm stops its execution when no node has a need to change strategy, meaning that, given their local neighborhood, their pure strategy is one of the pure strategies with the highest utility. This fits exactly in Definition 6. Notice how both line 3 and 4 are, respectively, linear on the number of players $|V|$ and linear on the number of possible pure strategies $|A|$.

A potential game is a game where the utility of a player can be defined by a global utility function. Graphical games are potential games.

As shown in [13]:

Theorem 1. In any finite potential game, best response dynamics always converge to a Nash equilibrium.

Since the algorithm is at least linear on the number of possible pure strategies, it cannot be applied directly to mixed strategies, where the number of possible strategies is uncountable.

In distributed computing, the complexity of an algorithm is measured by the number of rounds until it reaches an output state. In this case, the number of rounds the best response dynamics takes be the number of cycles it goes through.

3.2 Volunteer's dilemma

Now that the most important Definitions and results have been laid out, an example is helpful to build intuition. This example comes in the form of the volunteer's dilemma [14]. The flavor behind this game is that every player can either produce a good or not produce a good, like food or energy. Producing a good has an associated cost $c \in (0,1)$, which is a meta-parameter of the game, and provides access to the good to the player that produced it and to every neighbor. If a player has access to the good, then he receives a positive utility. The action space is the same for every player and is given by $A = \{$"do not produce good", "produce good"$\} = \{0,1\}$. The utility function is the same for every player v of degree d:

$$u_v(a_0, a_1, \dots, a_d) = \begin{cases} 1 & \text{if } a_0 = 0 \text{ and } \sum_{i=1}^{d} a_i \geq 1 \\ 1-c & \text{if } a_0 = 1 \\ 0 & \text{if } \sum_{i=0}^{d} a_i = 0. \end{cases}$$

3.2.1 Pure Nash equilibrium for the volunteer's dilemma

Studying the utilities of nodes in different situations leads to the understanding about what kinds of pure Nash equilibria there are for this game.

Considering a player v such that at least one of its neighbors produces the good, its utility is $1-c$ if he also produces the good and it is 1 if it does not. Thus, any pure Nash equilibrium cannot have two adjacent players with strategy ps_1.

On the other hand, if a player v has no neighbor that produces the good, its utility is $1-c$ if he produces the good and it is 0 if he does not. Therefore, any pure Nash equilibrium cannot have a player with strategy ps_0 if all of its neighbors also have strategy ps_0.

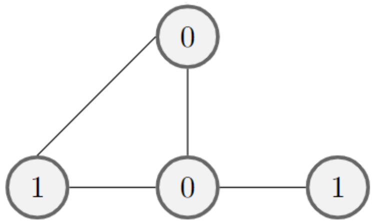

Figure 3: Pure Nash equilibrium

Figure 3 shows a pure Nash equilibrium since no player could raise its utility by changing strategies. Both players with strategy ps_1 have utility $1-c$ and would have utility 0 if they changed to strategy ps_0. Furthermore, both players with strategy ps_0 have utility 1 and would have utility $1-c$ were they to change to strategy ps_1.

Proposition 6. For the volunteer's dilemma, best response dynamics converges in 2 rounds.

Proof. This proof can also be found in paper [3]. Remember players change their strategy sequentially and every player has an opportunity to change every round. Also, pure strategy ps_0 represents always doing action "do not produce good" and pure strategy ps_1 represents always doing action "produce good".

Assume that two players v_1 and v_2 are neighbors. Notice that v_1 and v_2 have different places in the order so, without loss of generality, assume v_1 acts before v_2. In the case where v_1, after acting, has action ps_1, v_2 never chooses strategy ps_1, since the utility of ps_0 is 1 and the utility of ps_1 is $1-c$. Thus, after the first round of the best response dynamics algorithm, there are no instances of two neighbors whose strategies are ps_1.

Note that players with strategy ps_0 with at least one neighbor with strategy ps_1 do not have incentive to change strategy, since that would lower their utility from 1 to $1-c$.

Consequently, after the first round, players with strategy ps_1 will never change strategies, since all of their neighbors have strategy ps_0 and will never change it.

Lastly, consider a player v_3 whose strategy at the start of the second round is ps_0. When it is his turn to decide which strategy to choose, there are two cases. It might happen that he already has a neighbor with strategy ps_1, in which case v_3 keeps his strategy. If v_3 has no neighbor with strategy ps_1 then he will change its strategy since that raises its utility from 0 to $1-c$. In either case, after the second round, v_3 will never change strategies again.

Thus, after the second round, independently of the order of the players, each player has a strategy whose utility is higher than the other strategy. Thus, the algorithm stops.

□

3.2.2 Mixed Nash equilibrium

Note that, since the volunteer's dilemma is a two action game, any mixed strategy is of the form $s = [1-x, x]$, with $x \in [0,1]$. Thus, in order to simplify notation, it is unambiguous to denote s by x, that is, the probability of doing action "produce good".

The utility function 3 can be generalized to mixed strategies. The utility of a player is its probability of having access to the good minus the probability of having to pay the cost of producing the good times the cost:

$$w(s_0, s_1, \ldots, s_d) = 1 - \prod_{i=0}^{n}(1 - s_i) - s_0 c. \tag{3}$$

Using Proposition 4, one can try to find mixed Nash equilibria by forcing the utility of both pure strategies to be the same, for every player.

First, note that the utility of strategy ps_1 is always the same, independently of the strategies of the player's neighbors. If the player v always produces the good, then v always has access to that good and v always pays the cost of producing it so its utility, independently of the strategies of v's neighbors, is:

$$\forall_{\overrightarrow{s_{-i}}} w_i(ps_1, \overrightarrow{s_{-i}}) = 1 - c. \tag{4}$$

The utility of strategy ps_0 depends on the strategies of the player's neighbors. More concretely, it depends on the probability that at least one neighbor produces the good. Considering a player $i \in I$ with degree d, the utility of action "do not produce good" is

$$w_i(ps_0, \overrightarrow{s^*_{-i}}) = 1 - \prod_{n=1}^{d}(1 - s^*_n). \tag{5}$$

Proposition 7. Consider the following conditions

1. If the player has strategy $s = [1, 0]$ and the probability that at least one neighbor produces the good is greater than $1 - c$.
2. If the player has strategy $s = [0, 1]$ and the probability that at least one neighbor produces the good is smaller than $1 - c$.
3. If the probability that at least one neighbor produces the good is exactly equal to $1 - c$.

A joint strategy $\overrightarrow{s^*}$ is a mixed Nash equilibrium for the volunteer's dilemma if and only if every player $v \in V$ satisfies at least one of the conditions above.

Proof. Consider a player $v \in V$ whose 1-neighborhood has joint strategy $s_0, s_1, \ldots, s_d$.
($\Longrightarrow$)

This implication is done by contradiction. Assume a joint strategy $\overrightarrow{s^*}$ is a mixed Nash equilibrium but player v does not satisfy any of the 3 conditions.

There are 3 cases where this might happen:

Case 1: $s_0 = 0$ **and** $1 - \prod_{i=1}^{d}(1 - s_i) < 1 - c$**.**

In this case, it is true that

$$w_v(0, s_1, \dots, s_d) = 1 - \prod_{i=1}^{d}(1 - s_i) < 1 - c = w_v(1, s_1, \dots, s_d)$$

and thus, player v would have a higher utility if he changed strategy to 1, contradicting the fact that $\overrightarrow{s^*}$ is a Nash equilibrium.

Case 2: $s_0 = 1$ **and** $1 - \prod_{i=1}^{d}(1 - s_i) > 1 - c$**.**

Note that

$$w_v(0, s_1, \dots, s_d) = 1 - \prod_{i=1}^{d}(1 - s_i) > 1 - c = w_v(1, s_1, \dots, s_d).$$

Similarly, player v would have a higher utility if he changed strategy to 0, contradicting the fact that $\overrightarrow{s^*}$ is a Nash equilibrium.

Case 3: $s_0 \notin \{0, 1\}$ **and** $1 - \prod_{i=1}^{d}(1 - s_i) \neq 1 - c$**.**

Proposition 5 states that, since v has a full mixed strategy, $\overrightarrow{s^*}$ can only be a mixed Nash equilibrium if every pure strategy of v has the same utility, which is not true since

$$w_v(0, s_1, \dots, s_d) = 1 - \prod_{i=1}^{d}(1 - s_i) \neq 1 - c = w_v(1, s_1, \dots, s_d).$$

($\Longleftarrow$)

If a player v satisfies **condition 3**, meaning that $1 - \prod_{i=1}^{d}(1 - s_i) = 1 - c$, then it has no incentive to change strategies since, for v, the utility of ps_0 is the same as the utility of ps_1. From Proposition 3 it follows that v's current strategy s_0 is one of the strategies with highest utility.

If player v satisfies **condition 1**, then $s_0 = 0$ and $1 - \prod_{i=1}^{d}(1 - s_i) > 1 - c$. From the inequality, it follows that

$$w_v(0, s_1, \dots, s_d) = 1 - \prod_{i=1}^{d}(1 - s_i) > 1 - c = w_v(1, s_1, \dots, s_d).$$

Hence, strategy $s_0 = 0$ is the one with highest utility for player v, satisfying condition 6 for Nash equilibria. Intuitively, player v would prefer to diminish the probability of producing the good, but since that probability is already zero, he cannot lower it any further.

If player v satisfies **condition 2**, then $s_0 = 1$ and $1 - \prod_{i=1}^{d}(1 - s_i) < 1 - c$. Similarly as before, it is true that

$$w_v(1, s_1, \dots, s_d) = 1 - c > 1 - \prod_{i=1}^{d}(1 - s_i) = w_v(0, s_1, \dots, s_d).$$

That is, strategy $s_0 = 1$ is the one with highest utility for player v, satisfying condition 6 for Nash equilibria. Intuitively, player v would prefer to raise the probability he produces the good, but since that probability is already 1, he cannot do so.

□

Corollary 1. Let $i \in I$ be a player. Suppose that the probability that at least one of its neighbors produces the good is exactly 1-c. Then i has no incentive to change strategies.

Proposition 7 provides a tool to find mixed Nash equilibria for the volunteer's dilemma.

3.2.3 Mixed Nash equilibrium on paths

From Proposition 7, it follows that a player v with only one neighbor u only has 3 cases to be in an equilibrium.

1. $s_v = 0$ and $s_u \geq 1 - c$;
2. $s_v = 1$ and $s_u \leq 1 - c$;
3. $s_v \in (0, 1)$ and $s_u = 1 - c$.

Now consider a player v with degree 2 and a full mixed strategy. Let v's neighbors strategies be a and b, as seen in b) from Figure 5. From Corollary 1, one can find a relation between a and b, which can be graphically seen in Figure **??**. The utility of player v has to be equal to $1 - c$,

$$1 - [(1 - a).(1 - b)] = 1 - c,$$

therefore:

$$b = \frac{1 - c - a}{1 - a}. \tag{6}$$

These two previous facts about players with degree 1 and 2 allow a deep study of equilibrium on paths for this game. Figure 5 shows three examples of possible mixed Nash equilibria on paths.

These can be glued together to form numerous mixed Nash equilibria on paths. Look at Figure 6 and try to realize that no player can, alone, change its strategy without lowering its utility.

Note that **red players** are between one player with strategy $1 - c$ and another player with strategy 0 (or no other player in the case of the first player), thus the probability of a neighbor producing the good is $1 - c$. Furthermore, **blue players** are between two neighbors whose strategies satisfy relation 6, thus the probability of a neighbor producing the good is also $1 - c$. According to Corollary 1, these two types of players do not have incentive to change strategy. They satisfy condition 3 from Proposition 7.

On the other hand, **green players** are in a situation where the probability that a neighbor produces the good is equal to or higher than $1 - c$ (if $a_1 \leq a_2$), thus, they would have an incentive to lower their probability of producing the good. Since they

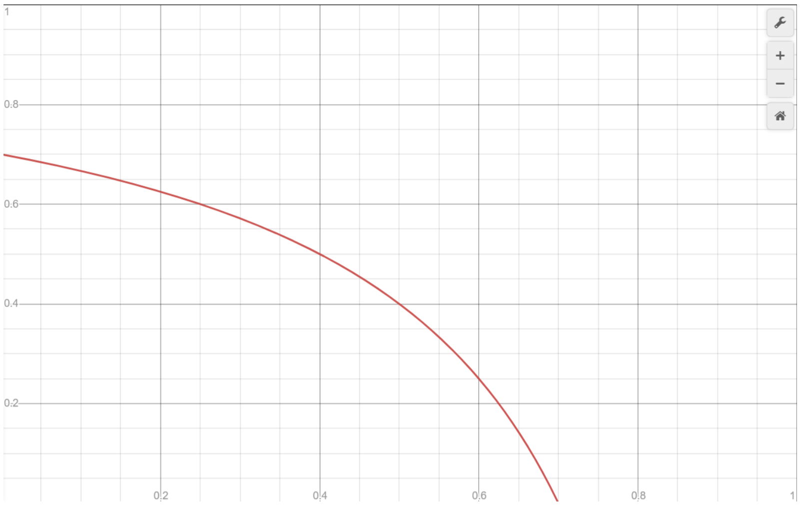

Figure 4: Relation between a and b with $c = 0.3$.

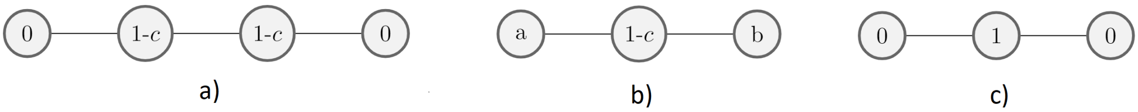

Figure 5: Different mixed Nash equilibria in path for the volunteer's dilemma.

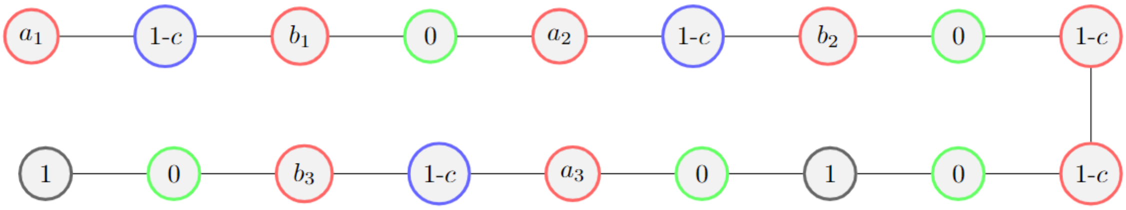

Figure 6: A mixed Nash equilibrium on a path which has all the previous artifacts. For this to be an equilibrium, $a_1 \leq a_2$ and a_i and b_i, for $i = 1, 2, 3$, must satisfy relation 6.

already have probability zero, they cannot lower it any further. These players satisfy condition 1 from Proposition 7.

Finally, **black players** are in the extreme case where the probability of a neighbor producing the good is zero. These players satisfy condition 2 from Proposition 7.

Every player in Figure 6 satisfies at least one condition from Proposition 7 and thus, by Proposition 7, the Figure represents a mixed Nash equilibrium for the volunteer's dilemma.

4 The equivalence

This Section introduces the main contributions from paper [3] and proposes the questions that will be answered on this thesis.

On their paper, the authors prove a powerful equivalence between game theory and distributed algorithms and they use it to provide complexity bounds on algorithms from both areas.

As shown by the authors, Nash equilibria of graphical games uniquely define solutions to LVL problems. As an example, one can prove that pure Nash equilibria from the volunteer's dilemma are solutions to the maximal independent set problem Π_1, and vice versa, as shown by the authors in [3].

Proposition 8. For the volunteer's dilemma, the set of players with strategy ps_1 ("produce good") in a pure Nash equilibrium is a maximal independent set.

Proof. This is a proof by contradiction. Assume that a joint strategy profile $\overrightarrow{s^*}$ is a pure Nash equilibrium and that there are two neighbors $i, j \in I$ whose strategies are ps_1, $s_i^* = ps_1$ and $s_j^* = ps_1$, meaning they are certain to do action "produce good". Then,

$$1 = w_i(ps_0, \overrightarrow{s_{-i}^*}) > w_i(s_i^*, \overrightarrow{s_{-i}^*}) = w_i(ps_1, \overrightarrow{s_{-i}^*}) = 1 - c,$$

which contradicts Definition 6. This proves that the set of players with strategy ps_1 ("produce good") in a pure Nash equilibrium is an independent set. To prove that it is maximal assume that, for a player $i \in I$, $s_i^* = ps_0$ and every neighbor of i also has the same strategy. Then,

$$0 = w_i(ps_0, \overrightarrow{s_{-i}^*}) = w_i(s_i^*, \overrightarrow{s_{-i}^*}) < w_i(ps_1, \overrightarrow{s_{-i}^*}) = 1 - c.$$

Again, this contradicts Definition 6, concluding the proof. □

Proposition 9. For a graph G, if every node in a maximal independent set I has strategy ps_1 and every node not in I has strategy ps_0, then $\overrightarrow{s^*}$ is a pure Nash equilibrium for the volunteer's dilemma.

Proof. Since the set of nodes with strategy ps_1 is independent, every node i with this strategy only has neighbors with strategy ps_0. Thus, i has no incentive to change strategy since

$$0 = w_i(ps_0, \overrightarrow{s_{-i}^*}) < w_i(s_i^*, \overrightarrow{s_{-i}^*}) = w_i(ps_1, \overrightarrow{s_{-i}^*}) = 1 - c.$$

Furthermore, since the set of nodes with strategy ps_1 is maximal, every node i with strategy ps_0 has at least one neighbor whose strategy is ps_1. Again, i has no incentive to change strategy since

$$1 = w_i(ps_0, \overrightarrow{s_{-i}^*}) = w_i(s_i^*, \overrightarrow{s_{-i}^*}) > w_i(ps_1, \overrightarrow{s_{-i}^*}) = 1 - c.$$

□

The fact that no player has a desire to change strategies ends this proof.

This equivalence allows a very powerful study of complexities.

Proposition 10. If an LVL requires in $T(n) = \Omega(\log^*(n))$ rounds, then the best response dynamics converges to the equivalent Nash equilibrium in $\Omega(T(n))$ rounds.

Proposition 11. If the best response dynamics converges in $T(n)$ rounds, then the corresponding LVL can be solved in $O(\log^*(n) + T(n))$ rounds.

In fact, these conclusions are not specific for best response dynamics, but rather for any algorithms that finds pure Nash equilibria.

The details concerning the proofs of these propositions fall outside the scope of this thesis and can be found on [3]. However, the utility of these statements can be easily understood: the extensive library of algorithms and their respective complexities studied in distributed computing theory allow a deeper understanding of the time complexity on the convergence of Nash equilibria for games. On the other hand, the complexity on the convergence of games can help understand distributed computing problems.

The authors use the volunteer's dilemma as an example, where, as seen before, the best response dynamics converges in two rounds to a pure Nash equilibrium and the corresponding LVL, maximal independent set, can be solved in $\Theta(\log^*(n))$.

The authors of [3] explore pure games, mentioning on the conclusion Section that mixed strategies could also be studied.

The main objective of this thesis can now be understood. That is to answer the question "Can this idea be generalized?"

The equivalence between LVL and Nash equilibria is not restrained for pure Nash equilibria, thus, mixed Nash equilibria also uniquely define solutions to an LVL. Since pure Nash equilibria and LVL are related and both have natural generalizations (mixed strategies and fractional problems), are these generalizations related in some way?

5 Fractional problems and mixed Nash equilibria

As mentioned in Section 4, the authors of paper [3] show a powerful equivalence between solutions to LVL problems, introduced in Section 2, and Nash equilibria, introduced in Section 3. However, their examples and focus span only pure Nash equilibria, even though the equivalence still holds for mixed strategies and infinite label alphabets.

This Section utilizes all the concepts introduced until now in order to try to prove or disprove the main question posed at the beginning, in Section 1: are mixed Nash equilibria and fractional problems related?

Figure 1 shows the question in visual form. Pure Nash equilibria and solutions of LVL problems are equivalent. Is it true that mixed Nash equilibria (the generalization of pure Nash equilibria) are related in the same way to solutions of fractional LVL problems (the generalization of LVL problems)?

Formulated in yet another way: for a game G, pure Nash equilibria are equivalent to solutions of an LVL Π_1. Furthermore, mixed Nash equilibria of G are also equivalent to solutions to an LVL Π_2. Is it true that Π_2 is the fractional generalization of Π_1?

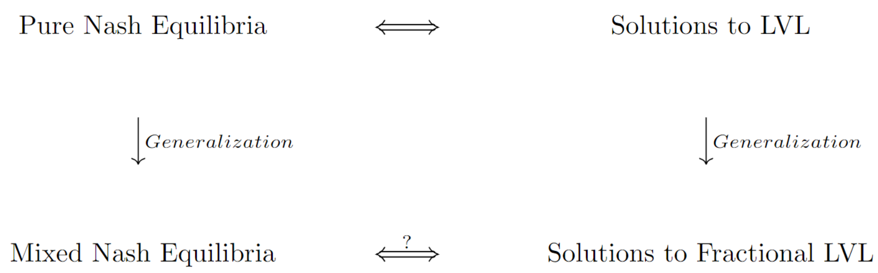

Figure 7: Question posed in this thesis represented on a diagram.

The answer to the question is negative.

Proposition 12. If pure Nash equilibria of a graphical game G represent solutions to a locally verifiable labeling problem $\Pi = (\Sigma, C)$, and vice-versa, then mixed Nash equilibria of G do not necessarily represent solutions to the fractional relaxation of Π, and vice-versa.

The following Section provides four different ways to show the truth behind this proposition, all of them using the volunteer's dilemma and the maximal independent set as a counterexample, each in their own perspective and providing interesting insights which will be discussed later.

5.1 Proving Proposition 12

5.1.1 Proof 1: a maximal fractional independent set which is not a mixed Nash equilibrium for the volunteer's dilemma

As shown in Section 4, pure Nash equilibria of the volunteer's dilemma are maximal independent sets, which represent solutions to the problem Π_1, introduced in Section 2.2.2. The fractional relaxation of Π_1 can be found on Section 2.3.1, and its solutions are maximal fractional independent sets.

Figure 2 c) represents a solution to the fractional problem Π_2. The numbers on the nodes represent their labels. It is a maximal fractional independent set since every f-labeled 1-neighborhood is a configuration in C_2. In other words, it satisfies the two necessary conditions:

1. For every edge, the sum of the labels of both adjacent nodes is at most 1.
2. For every node, there must exist one adjacent edge for which the sum of the labels of its adjacent nodes is exactly 1.

By changing the way one looks at the figure, one can imagine it represents 4 players playing the volunteer's dilemma and that the numbers represent the strategies of each player, that is, the probability of choosing action "produce good". Thinking this way, one can see that the joint strategy represented on the Figure is not a mixed Nash equilibrium.

To show this, it is only necessary to find a player that does not satisfy condition 2 found in the Definition of mixed Nash equilibria. It is enough to find a strategy for a node whose utility is higher than its current strategy.

Proof. Look at Figure 2 c), in particular, to the center node v whose label is 0.3. This means that its strategy is $s_v^* = [0.7, 0.3]$, that is, player v has a probability of 0.3 of doing action "produce good". With the current joint strategy $\vec{s^*} = [0.3, 0.6, 0.4, 0.7]$, and looking at the defined utility function for this game found in 3, the utility of player v is

$$w_v(0.3, 0.6, 0.4, 0.7) = 0.9496 - 0.3c.$$

Since player v has a full mixed strategy, from Proposition 5, it is the case that $\vec{s^*}$ can only be a mixed Nash equilibrium if the utilities of both pure strategies are the same for player v. Recall that the utility of any mixed strategy is an affine combination of the utilities of pure strategies (Proposition 3). Since the utilities of the pure strategies must be the same, they have to be $0.9496 - 0.3c$:

$$w_v(1, 0.6, 0.4, 0.7) = w_v(0, 0.6, 0.4, 0.7) = 0.9496 - 0.3c \tag{7}$$

From 4,

$$w_v(1, 0.6, 0.4, 0.7) = 1 - c \tag{8}$$

and, from 5,

$$w_v(0, 0.6, 0.4, 0.7) = 0.928. \tag{9}$$

Both sets of equations, 7, 8 and 7, 9 have solution $c = 0.072$.

This analysis leads to the statement: $\overrightarrow{s^*}$ can only be a mixed Nash equilibrium for the volunteer's dilemma if $c = 0.072$.

Let u be the player whose strategy is $s^*_u = [0.4, 0.6]$. Doing a similar analysis for player u will prove that c cannot be equal to 0.072.

Assume now that the cost of producing the good is set to be $c = 0.072$. Since player u also has a full mixed strategy, following a similar reasoning as before, $\overrightarrow{s^*}$ can only be a mixed Nash equilibrium if the utilities of both pure strategies are the same for player u. However,

$$w_u(1, 0.4, 0.3) = 1 - c = 0.928$$

and

$$w_u(0, 0.4, 0.3) = 0.58.$$

□

Since the utilities of both pure strategies for player u have different utilities, $\overrightarrow{s^*}$ is not a mixed Nash equilibrium for the volunteer's dilemma for any cost c.

5.1.2 Proof 2: a mixed Nash equilibrium for the volunteer's dilemma which is not a maximal fractional independent set

On the previous section, it was presented an example of a maximal fractional independent set which does not represent a mixed Nash equilibrium for any volunteer's dilemma. This Section focuses on the opposite: a mixed Nash equilibrium for any volunteer's dilemma which is not a maximal fractional independent set.

Figure 5 represents three different Nash equilibria for the volunteer's dilemma. The numbers on the nodes represent the player's strategies, that is, each number is the probability that a player chooses action "produce good". For instance, a) represents a mixed Nash equilibrium since every node satisfies Corollary 1. In fact, for any node, the probability that a neighbor produces the good is exactly $1 - c$, thus, the utility of both pure strategies is the same.

Again, by changing the way one interprets the meaning of the figure, one can think that the numbers on the nodes represent a mapping f that possibly solves problem Π_2, defined in Section 2.3.1. In this sense, the numbers represent the labels of the nodes. Thinking this way, f is a solution to Π_2 if and only if every neighborhood belong to C_2. In other words, the labels have to satisfy the two conditions:

1. For every edge, the sum of the labels of both adjacent nodes is at most 1.
2. For every node, there must exist one adjacent edge for which the sum of the labels of its adjacent nodes is exactly 1.

It will be see that, for any $c \in (0,1)$, that is never the case.

Proof. Looking at the end nodes whose labels are 0, condition 2. states that the sum of the labels of both adjacent nodes of its edge must be exactly 1. Since its label is 0 and its neighbor's label is $1-c$, this labeling could only be a solution to problem Π_2 (that is, a maximal fractional independent set) if $c = 0$, which is a value outside the possible space of this meta-parameter.

Even if c could be equal to zero, looking at the two center nodes, and assuming $c = 0$, a contradiction with condition 1. would arise. If both center nodes have label 1, then the sum of the labels of both adjacent nodes for the edge between them is 2, which is larger than 1, contradicting condition 1.

□

Therefore, the mixed Nash equilibrium for the volunteer's dilemma represented on Figure 5 a) does not represent a maximal fractional independent set.

5.1.3 Proof 3: turning a fractional LVL into a game

Solutions of the problem problem Π_2, defined in Section 2.3.1, are maximal fractional independent sets. Even though problem Π_2 has an infinite label alphabet Σ_2, it is still a locally verifiable labeling (LVL). As such, the results from paper [3] still hold. This means that some game G_2's Nash equilibria are equivalent to solutions of Π_2. This Section focuses on defining a G_2 that satisfies this condition and comparing it to the volunteer's dilemma.

Let G_2 be a graphical game with n players, $|I| = n$. Every player has the same action space $A = [0,1]$. This game, similarly to the volunteer's dilemma, has a meta-parameter $c \in (0,1)$.

The utility function of any player i with degree $d \geq 1$ depends only on its 1-neighborhood. With a joint action $\overrightarrow{a^*} = (a_0, a_1, \ldots, a_d)$, the utility of player i is as follows:

$$u_i(a_0, a_1, \ldots, a_d) = \min(1, \max_{1 \leq j \leq d}(a_0 + a_j)) - a_0 c. \tag{10}$$

Since $\max(\emptyset) = -\infty$, nodes with degree 0 would have $-\infty$ utility. Instead, let the utility of nodes with degree 0 be

$$u_i(a_0) = \min(1, a_0) - a_0 c = (1-c)a_0. \tag{11}$$

Game G_2 has an interpretation similar to the volunteer's dilemma. In the volunteer's dilemma, players choose to produce a good or not. For G_2, players choose to partially produce a good. For the volunteer's dilemma, there are only 2 pure strategies. As such, a player i with a mixed strategy $s = [0.8, 0.2]$ produces the good with a 0.2 chance. For G_2, there are infinitely many pure strategies, so if player j has the pure strategy equivalent to doing action $a_j = 0.2$, then player j, deterministically, produces 20% of the good. Furthermore, each player has access to the production of itself and of one other neighbor. Hence, for example, if every player had action $a = 0.3$, then a player, independently of its degree (as long as it has at least one neighbor), has only access to 60% of the good. One can think each player has only one carriage capable of transporting the good from one of its neighbors back to him. Like the volunteer's dilemma, having more than 100% of the good does not increase any further the utility of a player.

Proposition 13. Maximal fractional independent sets are pure Nash equilibria of game G_2.

Proof. Let $G = (V, E)$ be a graph and $f : V \to \Sigma_2$ be a solution to problem Π_2, defined in Section 2.3.1, meaning that each f-labeled 1-neighborhood of G belongs to C_2. In other words, the labels of the nodes form a maximal fractional independent set in G.

Let node $n_0 \in V$ have degree 0. The only configuration $c_0 \in C_2$ with a 1-neighborhood with only one label is $c_0 = [1]$. If one thinks of the label as the action of a player playing game G_2 in graph G then player n_0 has the following utility:

$$u_{n_0}(1) = (1 - c) \cdot 1 = 1 - c.$$

To prove that a maximal fractional independent set represents a pure Nash equilibrium for game G_2, any other action n_0 could choose has to have at most utility $1 - c$. This must hold because, if there was another action $a_0 \neq 1$ whose utility satisfied $u_{n_0}(a_0) > 1 - c = u_{n_0}(1)$ then, by Definition 6, the labels would not represent a Nash equilibrium.

Consider another action $a_0 = 1 - \varepsilon$ for node n_0, with $\varepsilon \in (0, 1]$. Its utility would be

$$\begin{aligned} u_{n_0}(1 - \varepsilon) &= (1 - c)(1 - \varepsilon) \\ =& 1 - c - (1 - c)\varepsilon < 1 - c \quad (\varepsilon > 0 \text{ and } c < 1). \end{aligned}$$

Thus, for players with degree 0, it holds that the action with highest utility is $a = 1$, which is exactly the label nodes with degree 0 have on maximal fractional independent sets.

Now let node $n \in V$ have degree larger than 0, $degree(n) = d \geq 1$. The set of configurations for 1-neighborhood with $d + 1$ nodes is $\{[x_0, x_1, x_2, \ldots, x_d] \mid \forall_{j>0}\ x_0 + x_j \leq 1,\ \exists_{j>0}\ x_0 + x_j = 1\}$. In other words, node n satisfies two conditions:

1. For every neighbor m of n, the sum of m's label and n's label is at most 1.

2. There is at least one neighbor w of n such that the sum of w's label and n's label is exactly 1.

Again, if one thinks of the labels as the actions of the nodes, then one can prove that this local joint action profile belongs to a pure Nash equilibrium.

Let $[x_0, x_1, \ldots, x_d]$ be the configuration of the 1-neighborhood centered on n, meaning the label of node n is x_0, $f(n) = x_0$, and the labels of n's neighbors are $x_1, \ldots, x_d$. Using game theory terms, the joint action profile for node n and its neighbors is $\vec{a} = (x_0, x_1, \ldots, x_d)$.

The utility of node n, with the current action profile, is the following:

$$\begin{aligned}
&u_i(x_0, x_1, \ldots, x_d) \\
=&\min(1, \max_{1\leq j\leq d}(x_0 + x_j)) - x_0 c \\
=&\min(1, 1) - x_0 c \quad \text{(From conditions 1. and 2.)} \\
=&1 - x_0 c.
\end{aligned}$$

To prove that node n satisfies the Definition of a pure Nash equilibrium, found in Definition 2, one needs to show that any other action has lower or equal utility. There are two cases to take into consideration.

Firstly, consider that n has action $x_0 + \varepsilon$ with $\varepsilon > 0$ and $x_0 + \varepsilon \leq 1$. Then,

$$\begin{aligned}
&u_i(x_0 + \varepsilon, x_1, \ldots, x_d) \\
=&\min(1, \max_{1\leq j\leq d}(x_0 + \varepsilon + x_j)) - (x_0 + \varepsilon)c \\
=&\min(1, 1 + \varepsilon) - (x_0 + \varepsilon)c \quad \text{(From condition 1. and 2.)} \\
=&1 - x_0 c - \varepsilon c \\
<&1 - x_0 c \quad (\varepsilon > 0 \text{ and } c > 0).
\end{aligned}$$

On the other hand, if node n has action $x_0 - \varepsilon$ with $\varepsilon > 0$ and $x_0 - \varepsilon \geq 0$, it holds that

$$\begin{aligned}
&u_i(x_0 - \varepsilon, x_1, \ldots, x_d) \\
=&\min(1, \max_{1\leq j\leq d}(x_0 - \varepsilon + x_j)) - (x_0 - \varepsilon)c \\
=&\min(1, 1 - \varepsilon) - (x_0 - \varepsilon)c \quad \text{(From condition 1. and 2.)} \\
=&1 - \varepsilon - x_0 c + \varepsilon c \\
=&1 - x_0 c - (1 - c)\varepsilon \\
<&1 - x_0 c \quad (\varepsilon > 0 \text{ and } c < 1).
\end{aligned} \tag{12}$$

Inequalities (**??**) and (12) show that choosing any other action would lower player n's utility. This holds for any player whose degree is larger than 0. Together with the previous proved fact for nodes with degree 0, by Definition 6, the labels of a maximal fractional independent set represent a pure Nash equilibrium of game G_2. □

Proposition 14. Pure Nash equilibria of game G_2 are maximal fractional independent sets.

Proof. Similarly to the previous proof, players with degree 0 and players with higher degree behave in different ways.

Look at the Definition of the utility for game G_2 concerning players with degree 0, found in 11. It is clear that the utility function is linearly increasing with the action's value. Thus, for a player with degree 0 to be in a Nash equilibrium joint action $\vec{a}$, its action must be 1, since

$$\forall_{a\in[0,1)} \quad u_i(a) = (1-c)a < (1-c) = u_i(1).$$

Think of the action of a player as the label of a node. Players with degree 0 in pure Nash equilibria, which have action 1, represent labelings of 1-neighborhoods with only one node, which belong to C_2, since $[1] \in C_2$.

Looking at players with at least one neighbor, a proof by contradiction allows to conclude that their actions on a pure Nash equilibrium also represent configurations in C_2, meaning their labels create a maximal fractional independent set.

Assume that the actions $\overrightarrow{a} = [a_0, a_1, \ldots, a_d]$ of a player i and its d neighbors belong to a joint action $\overrightarrow{a^*}$, which is a pure Nash equilibrium. Also assume that those actions do not represent a configuration in C_2, $[a_0, a_1, \ldots, a_d] \notin C_2$.

There are two ways of having $[a_0, a_1, \ldots, a_d] \notin C_2$:

Case 1: $\exists_{j>0}\ a_0 + a_j > 1$

Without loss of generality, let $a_0 + a_1 > 1$. With a joint action that satisfies this condition, the utility of player i is

$$\begin{aligned}
&u_i(a_0, a_1, \ldots, a_d) \\
=&\min(1, \max_{1\le j\le d}(a_0 + a_j)) - a_0c \\
=&1 - a_0c.
\end{aligned}$$

Notice how, if player i lowers its action in a way such that $a_0 - \varepsilon + a_1 > 1$, with $\varepsilon > 0$, then its utility raises:

$$\begin{aligned}
&u_i(a_0 - \varepsilon, a_1, \ldots, a_d) \\
=&\min(1, \max_{1\le j\le d}(a_0 - \varepsilon + a_j)) - (a_0 - \varepsilon)c \\
=&1 - (a_0 - \varepsilon)c \\
=&1 - a_0c + \varepsilon c \\
>&1 - a_0c \quad (\varepsilon > 0 \text{ and } c > 0)
\end{aligned}$$

From this inequality follows that a joint action where $\exists_{j>0}\ a_0 + a_j > 1$ cannot be a pure Nash equilibrium.

Case 2: $\forall_{j>0}\ a_0 + a_j < 1$

With a joint action that satisfies this condition, player i would have utility

$$\begin{aligned}
&u_i(a_0, a_1, \ldots, a_d) \\
=&\min(1, \max_{1\le j\le d}(a_0 + a_j)) - a_0c \\
=&\max_{1\le j\le d}(a_0 + a_j) - a_0c.
\end{aligned}$$

Note that in this case player i can raise its action while still satisfying the condition, increasing its utility. Consider $\varepsilon > 0$ such that $\forall_{j>0}\ a_0 + \varepsilon + a_j < 1$. Then,

$$
\begin{aligned}
&u_i(a_0+\varepsilon, a_1, \ldots, a_d) \\
=&\min(1, \max_{1\le j\le d}(a_0+\varepsilon+a_j)) - (a_0+\varepsilon)c \\
=&\max_{1\le j\le d}(a_0+\varepsilon+a_j) - (a_0+\varepsilon)c \\
=&\max_{1\le j\le d}(a_0+a_j) + \varepsilon - a_0c - \varepsilon c \\
=&\max_{1\le j\le d}(a_0+a_j) - a_0c + (1-c)\varepsilon \\
>&\max_{1\le j\le d}(a_0+a_j) - a_0c \quad (\varepsilon > 0 \text{ and } c < 1).
\end{aligned}
$$

From this inequality follows that a joint action where $\forall_{j>0}\ a_0 + a_j < 1$ cannot be a pure Nash equilibrium.

From these two cases it follows that a joint action $\overrightarrow{a^*} = [a_0, a_1, \ldots, a_n]$ which is a pure Nash equilibrium for game G_2 must satisfy, for each player i with d neighbors, that either $d = 0$ and $a_i = 1$ or $d > 0$ and $\forall_{j>0}\ a_0 + a_j \le 1$ and $\exists_{j>0}\ a_0 + a_j = 1$. This represents a maximal fractional independent set. □

Propositions 13 and 14 together show an equivalence between pure Nash equilibria of game G_2 and solutions of problem Π_2.

In other words, if the actions in a joint action $\overrightarrow{a^*}$ which represents a pure Nash equilibrium of game G_2 are seen as labels of nodes in graph G, then those labels are a maximal fractional independent set in G. The opposite is also true: if the labels of nodes in a maximal fractional independent set in G are seen as the actions of player playing game G_2, then their joint action $\overrightarrow{a^*}$ represents a pure Nash equilibrium for game G_2.

Lastly, note that game G_2 is different from the volunteer's dilemma. Both the action sets and the utility functions are different for players playing these two games. This shows the pretended result.

5.1.4 Proof 4: turning a game into a fractional LVL

There are three conditions, found in Proposition 7, that define the local neighborhoods that can be found in Nash equilibria for the volunteer's dilemma. These are enough to define a locally verifiable labeling (LVL) Π_3 whose solutions are equivalent to mixed Nash equilibria of the volunteer's dilemma. If Π_2 and Π_3 turn out to be different, then the result of this Section is proven.

Let $\Pi_3 = (\Sigma_3, C_3)$ be an LVL for graphs with maximum degree Δ, where $\Sigma_3 = [0, 1]$ and $C_3 = C_{31} \cup C_{32} \cup C_{33}$. Each of the configuration sets C_{3i}, $i = 1, 2, 3$, represents the configurations allowed by one of the conditions defined in Proposition 7. They are

- $C_{31} = \bigcup_{d=1}^{\Delta}\{[0, x_1, x_2, \ldots, x_d] | 1 - \prod_{i=1}^{d}(1 - x_i) > 1 - c\}$;
- $C_{32} = \bigcup_{d=0}^{\Delta}\{[1, x_1, x_2, \ldots, x_d] | 1 - \prod_{i=1}^{d}(1 - x_i) < 1 - c\}$;

- $C_{33} = \bigcup_{d=1}^{\Delta}\{[x_0, x_1, x_2, \dots, x_d] | 1 - \prod_{i=1}^{d}(1 - x_i) = 1 - c\}$.

Variable $c \in (0, 1)$ is the meta-parameter associated with the volunteer's dilemma.

Proposition 15. Solutions to problem Π_3 are mixed Nash equilibria of the volunteer's dilemma.

Proof. Assume mapping $f : V \to \Sigma_3$ is a solution to Π_3 for a given graph $G = (V, E)$. This means each f-labeled 1-neighborhood of G belongs to C_3. Now think about the labels of nodes as the probability of doing action "produce good", $\overrightarrow{s^*} = [f(v_1), f(v_2), \dots, f(v_n)]$.

Consider a player v_0 with d neighbors, $v_1, v_2 \dots, v_d$, whose strategies are, respectively, $f(v_0), f(v_1), \dots, f(v_d)$. Since the set of configurations C was built having the 3 conditions from Proposition 7 into account, then, by construction, the following is true. If $[f(v_0), f(v_1), \dots, f(v_d)] \in C_{3i}$, then v_0 satisfies **condition i** from Proposition 7, for $i \in \{1, 2, 3\}$.

Consequently, every player satisfies at least one of the conditions from Proposition 7, which, by Proposition 7, proves that $\overrightarrow{s^*}$ is a mixed Nash equilibrium for the volunteer's dilemma.

□

Proposition 16. Mixed Nash equilibria of the volunteer's dilemma are solutions to problem Π_3.

Proof. Assume a joint strategy $\overrightarrow{s^*} = [s_1^*, s_2^*, \dots, s_n^*]$ is a mixed Nash equilibrium for the volunteer's dilemma for a graph G.

Since $\overrightarrow{s^*}$ is a mixed Nash equilibrium, each player satisfies at least one condition from Proposition 7. Consider a player v_0 with d neighbors, $v_1, v_2 \dots, v_d$, whose strategies are, while in equilibrium, respectively, $s_0, s_1, s_2, \dots, s_d$. Now think about these strategies as labels of a mapping $f : V \to \Sigma_3$. Is f a solution to Π_3? There are 3 different cases to explore. Since problem Π_3 was designed to take these 3 conditions into consideration, this proof is almost completed by Definition.

On the first case, the probability player v_0 does action "produce good" is zero and the probability that at least one of its neighbors produces the good is larger than $1 - c$, meaning v_0 satisfies **condition 1** from Proposition 7. Thus, by construction, $[s_0, s_1, \dots, s_d] \in C_{31}$.

Similarly, if the probability player v_0 does action "produce good" is 1 and the probability that at least one of its neighbors produces the good is less than $1 - c$, meaning v_0 satisfies **condition 2** from Proposition 7, then $[s_0, s_1, \dots, s_d] \in C_{32}$.

Lastly, on every other case, the probability that at least one of v_0's neighbors produces the good is exactly $1 - c$, so $[s_0, s_1, \dots, s_d] \in C_{33}$.

Any other joint strategy where all 3 conditions are not met for some player, would not be a mixed Nash equilibrium and so the proof is done.

□

Propositions 15 and 16 show an equivalence between mixed Nash equilibria for the volunteer's dilemma and solutions for the problem Π_3.

Furthermore, note how problems Π_2 and Π_3 are different. Although the alphabet of labels is the same, the set of configurations is not.

Note how the configurations $c_1 = [0.5, 0.5] \in C_2$ and $c_2 = [0.5, 0.5, 0.5] \in C_2$. However, $c_1 = [0.5, 0.5] \in C_3$ implies that $c = 0.5$ and $c_2 = [0.5, 0.5, 0.5] \in C_3$ implies that $c = 0.25$. Thus, for any meta-parameter c, $C_2 \not\subset C_3$.

On the other hand, $c_3 = [0, 1 - c] \in C_3$ but, for any $c \in (0, 1)$, $c_3 = [0, 1 - c] \notin C_2$. Thus, for any meta-parameter c, $C_3 \not\subset C_2$.

5.2 Insights

Proofs 1 and 2 provide examples that show the desired conclusion. However, proofs 3 and 4 create, respectively, a new game G_2 and a new problem Π_3, different from the ones analysed previously.

From Proposition 12, it is clear that there is no implication in either direction, as clearly shown on proofs 1 and 2. However, further studying game G_2 and problem Π_3 might show some interesting properties related to the problem at hand. In particular, it should be asked if there is any relation between the mixed strategy generalization of G_2 and the fractional generalization of problem Π_3.

6 Usefulness of the equivalence

The theory of distributed computing is rich and growing. A lot is known about various problems and new results are discovered every year. As such, when studying a game G, identifying an LVL whose solutions are equivalent to Nash equilibria of G can lead to important insights, like efficient algorithms and complexity bounds.

On the other hand, game theory has decades of accumulated results. Every game has various real life applications. When studying an LVL, identifying which game is related to it can supply a lot of intuition, examples, and conclusions from game theory.

This Section provides three examples, showcasing how one can use results from distributed computing theory to enhance their knowledge about games and use game theory to build intuition about distributed computing. As explained in Section 2.4, there are three types of problems when it comes to complexity of solvability and mendability. This Section presents one example from each of those cases.

In each of the following subsections, a description of the game being considered will be given, followed by a description of the LVL whose solutions are equivalent to the considered Nash equilibria. Finally, some interesting results and insights are explained.

For instance, given a partial solution, meaning that almost every player has decided on its strategy and no player has incentive to change, then it might be easy or hard to find a Nash equilibrium according to the complexity of the mendability for the equivalent LVL. Furthermore, distributed computing theory might provide algorithms specialized in solving a particular LVL. The complexity of such algorithms show how fast Nash equilibria for the related game can be found. Using algorithms from distributed computing is a much more efficient way of finding Nash equilibria for that specific game rather than using general Nash equilibria finding algorithms.

6.1 Volunteer's dilemma and maximal independent set

The volunteer's dilemma has been used as an example during this thesis (its description and Definition can be found in Section 3.2). As explained in Section 4, pure Nash equilibria for the volunteer's dilemma represent maximal independent sets. As seen in Section 2.2.2, maximal independent sets can be computed efficiently. Furthermore, as proven in [3], maximal independent sets are also $O(1)$-mendable. This is an example of a problem from family 1) from Section 2.4.

As such, by analysing the distributed computing problem Π_1, whose solutions are maximal independent sets, it is possible to learn properties about the volunteer's dilemma. Namely, it is very efficient and easy to find pure Nash equilibria and that, given a partial solution where almost every player is *happy*, it is very easy to find a pure Nash equilibrium. When studying this game, it is probably more efficient to use algorithms to find maximal independent sets rather than generic algorithms to find Nash equilibria.

On the other hand, a distributed computing theorist can look into the volunteer's dilemma and get intuition. There are examples of communities of animals, like some birds and mammals, where this game is constantly being played. While some individuals eat food, others sacrifice their opportunity to look out for predators.

6.2 Minority game and locally optimal cuts

On the minority game [15], every player chooses between 2 possible actions, $A = \{0, 1\}$, with utility (for a player v with degree d)

$$u_v(a_0, a_1, \dots, a_d) = 1 + |\{a_i|\ i > 0, a_i \neq a_0\}| - |\{a_i|i > 0, a_i = a_0\}|.$$

This game is called an anti-coordination game, since players are encouraged to choose the action less common amongst its neighbors. It is usually associated with the situation where players have to choose between two possible restaurants to have dinner in. The restaurant which is less crowded is preferred.

Pure Nash equilibria of this game are locally optimal cuts, and vice-versa.

Consider a graph $G = (V, E)$ and two sets of nodes $I, J \subseteq V$ such that $I \cap J = \emptyset$ and $I \cup J = V$. Any edge $(v, u) = e \in E$ such that $v \in I$ and $u \in J$, or vice-versa, is considered a cut edge. The problem of finding a large cut, meaning, separating V into two sets in a way that produces a large amount of cut edges, is a hard one. For example, if a graph is 2-colored, then the size of the maximum cut is $|E|$, since a node with color 0 only has neighbors with color 1 and vice-versa. If I and J represent the nodes with color 0 and nodes with color 1, respectively, then every edge in E is a cut edge.

Locally optimal cuts are solutions to the following LVL defined for graphs with maximum degree Δ: $\Pi_4 = (\Sigma_4, C_4)$ where $\Sigma_4 = \{0, 1\}$ and

$$C_4 = \bigcup_{d=0}^{\Delta} \{[x_0, x_1, x_2, \dots, x_d]|\ |\{x_0 = x_i\}| \leq |\{x_0 \neq x_i\}|\}.$$

Similarly to the minority game, these configurations only allow nodes to be labeled with the label which is less common amongst its neighbors.

As mentioned in [3], the authors of [16] prove that finding a locally optimal cut is not efficient, its solvability is $\Omega(\log(n))$. Since Π_4 is not solved efficiently, then it is also not mendable efficiently. This is an example of a problem from family 3) from Section 2.4.

By only analysing Π_4, a lot can be said about the minority game. Namely, computing a pure Nash equilibrium for this game is hard, it will take $\Omega(\log(n))$ time. Furthermore, given a state where almost every player has chosen its strategy and no player has incentive to change, then it might be the case that, from that joint strategy profile, finding a pure Nash equilibrium is hard, and a lot of players might have to change strategies in order to achieve an equilibrium.

If one sets out to study this game and needs to find Nash equilibria, then the algorithms used to find locally optimal cuts can be used and will probably be faster than using more general game theory algorithms like best response dynamics or fictitious play.

Furthermore, when studying the optimal cut problem, the minority game might provide useful insights. There are a lot of social studies focused on it. The minority game, its equilibria and other game theory results around it will enable a lot of insights about the problem and might lead to the right questions and results.

6.3 Stag hunt and locally worst cut

The stag hunt game [17] is famously known and vastly studied. The stag hunt game is usually presented as a two player game, where players have two possible actions $A = \{''Hare'','' Stag''\} = \{0,1\}$. Figure 8 shows a possible return matrix for this game.

	Stag	Hare
Stag	4, 4	1, 3
Hare	3, 1	2, 2

Figure 8: Example of an utility matrix for the stag hunt game.

In its simplest form, it is a coordination game where both players try to do the same action/hunt the same animal. In fact, this return matrix has two pure Nash equilibria, {"Stag", "Stag"} and {"Hare", "Hare"}.

This concept can be generalized to a graphical game, where a player v plays this game individually with each neighbor. The utility of a player v is given by

$$u_v(a_0, a_1, \ldots, a_d) = \sum_{i=1}^{d} g(a_0, a_i),$$

where g represents the return matrix and is defined as

$$g(a_0, a_1) = \begin{cases} 2 & \text{if } a_0 = 0 \text{ and } a_1 = 0 \\ 3 & \text{if } a_0 = 0 \text{ and } a_1 = 1 \\ 1 & \text{if } a_0 = 1 \text{ and } a_1 = 0 \\ 4 & \text{if } a_0 = 1 \text{ and } a_1 = 1. \end{cases}$$

In a sense, the graphical game version of the stag hunt is the opposite of the minority game. As explained in the previous example, a player playing the minority game will try to choose the action less common amongst its neighborhood, whereas a player playing the stag hunt will try to choose the action most common amongst its neighborhood. While the minority game attempts to maximize the amount of cut edges on the graph, this game tries to minimize them. Naturally, this will lead to clusters or communities of nodes who choose the same action since inside a cluster there are no cut edges. The only cut edges are the ones between clusters.

In order to study the game using distributed computing theory, it is necessary to define the equivalent LVL: $\Pi_5 = (\Sigma_5, C_5)$ where $\Sigma_5 = \{0, 1\}$ and

$$C_5 = \bigcup_{d=0}^{\Delta} \{[1, x_1, x_2, \ldots, x_d] | \sum_{i=1}^{d} x_i \geq \lceil d/2 \rceil\} \cup \bigcup_{d=1}^{\Delta} \{[0, x_1, x_2, \ldots, x_d] | \sum_{i=0}^{d} x_i \leq \lfloor d/2 \rfloor\}.$$

This LVL has not been thoroughly studied. It will be referred to as the locally worst cut, since it is the opposite of the locally optimal cut.

Even in this case, where the considered game is related to a problem which is not yet studied, the relationship between the two areas is helpful. The tools of distributed computing and its way of solving problems can still be useful as one studies the stag hunt.

Notice how the algorithm that simply outputs label 1 in every node solves the locally worst cut problem. In this sense, the solvability of this LVL is $O(1)$.

If one adds the restriction that, for any given node v, the algorithm has a 50% chance of outputting 1 and a 50% chance of outputting 0, then this problem is definitely harder. There is a need for larger scale coordination. This type of scenario pops up in nature in various situations. One famous example is the orientation of magnetization on a magnetic material, like iron. Without an external source of orientation, each part of the material tries to coordinate with its neighbors where they should orient themselves. The result is the creation of clusters with different orientations. Metaphorically, some parts of the iron chose the action "Stag", while some other parts chose "Hare".

Figure 9 is a representation of possible magnetizations of a material [18]. The third image represents the result of the constant time algorithm described before, where every node outputs "1". This magnetization can only be achieved if there is an outside magnetic field that allows every part of the material to choose the same direction. Using game theory terms, this might be achieved through moral codes or strategizing before choosing the action. Without these mechanisms, Nash equilibria for the stag hunt and solutions for the locally worst cut look like the middle image, where different clusters of the graph choose different actions.

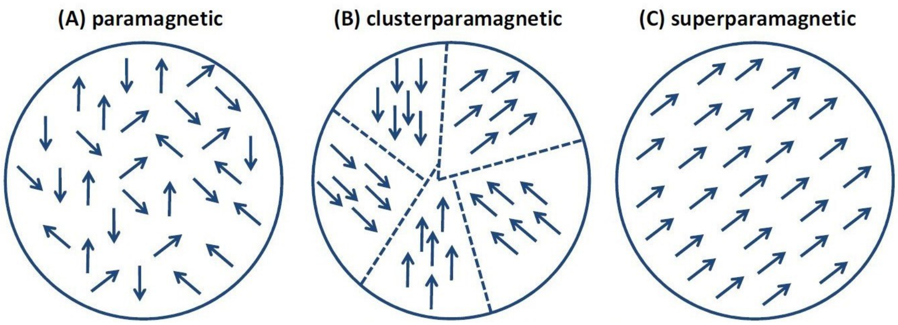

Figure 9: A representation of the magnetization of a magnetic material.

In terms of mendability, Figure 10 proves that Π_5 is $\Omega(\log(n))$-mendable.

Figure 10 shows a partial labeling on a 3-regular rooted tree, where every node has label 0, except for two brother leaves and their ascendants, which have label 1,

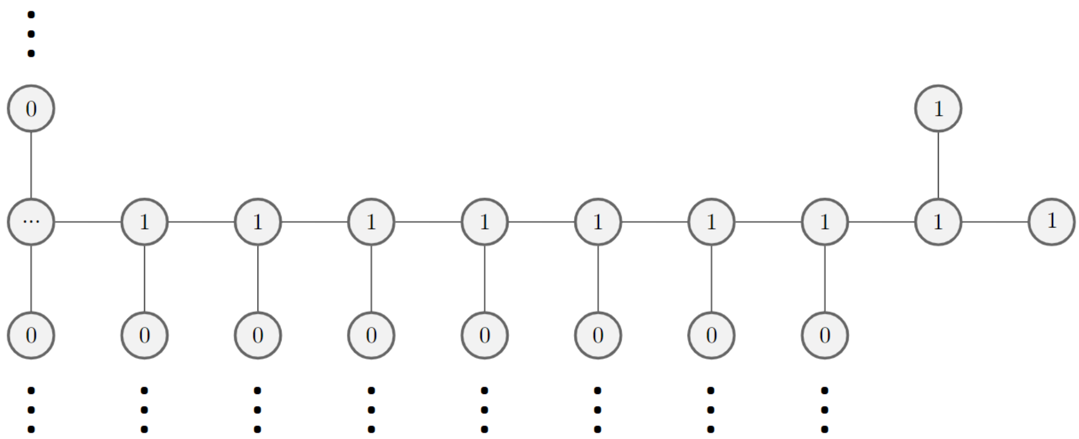

Figure 10: A partial labeling of a 3-regular rooted tree where $O(\log(n))$ nodes have to change label in order to find a solution.

and the root, which has label $\perp$. Notice how this partial labeling makes every node *happy*. The only way to label the root, while making it *happy*, is to label it with 0. Once one does so, its neighbor whose label is 1 becomes *unhappy*. Changing its label to 0 makes its neighbor whose label is 1 *unhappy*. This chain reaction continues until all the branch changes label to 0 (except for the two leaves and their parent). This is an example where, in order to turn a partial labeling to a solution, $O(\log(n))$ nodes need to change labels. It is possible to conclude that the locally worst cut is an example of a problem from family 2) from Section 2.4.

The solvability analyses for this game shows that, with a moral code or a previous agreement, finding a Nash equilibrium is trivial. If, before playing, everyone agrees to choose action 1 or 0, then doing so results in an equilibrium. However, without a previous consensus, with the game being symmetric, it becomes hard to find Nash equilibria and the resulting equilibrium is likely to produce clusters with different choices. Furthermore, the mendability analyses shows that even given a partial solution, where almost every node is *happy*, it might be the case that further almost-global coordination is necessary in order to make every node *happy*.

If one tries to study the locally worst cut, then looking into the stag hunt should be fruitful. A lot is known about what happens with different moral codes or different graph topologies and there is always a lot of intuition that derives from the way one thinks about games.

7 Conclusion and future work

This thesis has two main conclusions that the reader should keep: game theory and distributed computing are closely connected and results from both areas may be used together to produce powerful discoveries.

Either as a way of analysing games and their equilibria, or as a way of building intuition about distributed problems, thinking about one area of research when attempting to understand the other is bound to be productive.

Furthermore, this thesis also proves, using various tools, a limitation on the bridge that connects these two areas. Proposition 12 shows that, even if the pure Nash equilibria of a game G are equivalent to solutions of a problem Π, nothing can be said about the relation between mixed Nash equilibria of G and solutions to the fractional relaxation of Π.

This thesis also leaves the reader four possibly fruitful questions.

The first one it the question found in Section 5.2, formulated in a general sense: given a game G whose pure Nash equilibria are equivalent to solutions of problem Π, then 1) there is a game G' whose pure Nash equilibria are equivalent to solutions of the fractional relaxation of Π and 2) there is also a problem Π' whose solutions are equivalent to mixed Nash equilibria of G. Is there anything that can be said about the relation between mixed Nash equilibria of G' and solutions to the fractional relaxation of Π'? Image 11 provides a visual formalization of this question.

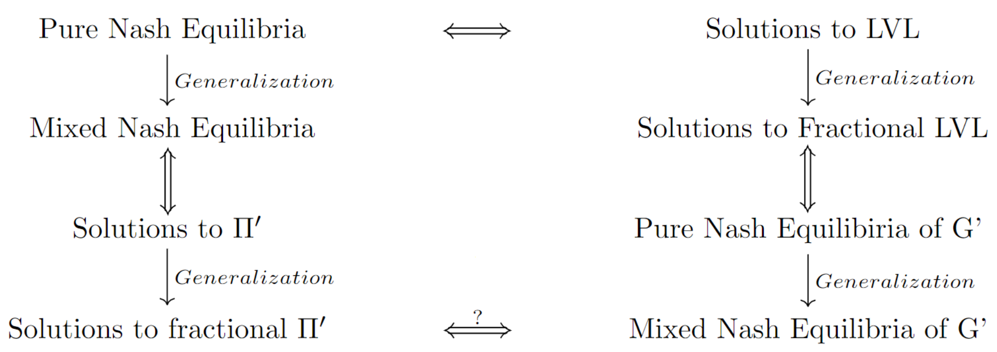

Figure 11: Question posed as future work represented on a diagram.

This thesis also introduces a new locally verifiable labeling: the locally worst cut. Studying this problem could lead to new insights in the area of distributed computing. From the stag hunt game, it is known that solutions for the locally worst cut are clusters of nodes with the same label. With some sort of global coordination, it is possible to achieve a labeling where every node has the same label. This thesis also contains the proof showing that the locally worst cut is $\Omega(\log(n))$-mendable.

Moreover, it might be interesting to apply complexity gaps into game theory. Different topologies and different problems have different complexity gaps. However, these results seem to show up everywhere in distributed computing. By using the equivalence between the two areas, these complexity gaps can be studied in the lens of game theory. This might have two consequences. A new understanding of complexity gaps, why they exist and how they can be found. And a new understanding of games, since complexity gaps have never been introduced into game theory.

The last suggestion revolves around the study of mendability. As mentioned in Section 2.4, there are three families of problems, when considering their mendability complexity. The locally worst cut belongs to the family of problems which can be easily solved but are hard to mend. However, there is a symmetrical condition which, when applied to the problem, makes it hard to solve as well. It might be the case that this characteristic is present in every problem from that family. This means that it is possible that every problem which is easy to solve and hard to mend has one additional condition which makes it hard to solve. That condition might very well be a symmetrical condition similar to the locally worst cut.

In science, it is imperative to prove useful results, as a way of expanding humanity's collective knowledge, and to ask the right questions that will lead others in the correct path. This thesis is a building block for future work. It showcases a new useful tool that can be used to understand new problems and view old ones in a different perspective.

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