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Object-dependent cloaking in the first-order Born approximation

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We consider the cloaking of a slab object in scalar wave theory within the first-order Born approximation. We show that in the forward direction cloaking is achieved for any transversally invariant, positively refracting, and absorbing object by using a lossy, negative-index metamaterial cloak. Cloaking is perfect and occurs for incident fields having any spatial structure and bandwidth. In the backward direction cloaking is found to be possible for self-imaging fields. In both cases the refractive-index distribution and dispersive properties of the cloak slab resemble those of the object slab. The method of object-dependent cloaking with weak scatterers may find useful applications.

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I. INTRODUCTION

Recent progress in metamaterials has enabled the control of light in extraordinary ways as evidenced, e.g., by negative refraction, reversed Doppler shift, and the phenomenon of perfect imaging [1,2]. In particular, the cloaking of objects, making them invisible or less detectable, has received much attention due to general interest and potential applications. Several cloaking methods have been put forward, such as coordinate transformations [3–8], reduction of the scattering cross section [9–13], use of embedded arrays of holes and dielectric particles in metal films [14,15], and utilization of anomalous resonances in certain systems [16,17]. These methods are either based on strong scatterers, involve embedded particle configurations, are not exact, are valid in a narrow spectral band, or utilize specific resonant structures. Several important results on invisibility have been discovered, albeit true cloaks operating at visible light are still awaiting.

In this work, we present a different type of cloaking based on classical scattering theory with weak slab scatterers. We demonstrate that any positively refractive and absorbing object with transversally uniform optical properties can be cloaked in the forward direction either by a positively refractive, amplifying (active) medium or by a negative-index, absorbing (passive) metamaterial slab. Cloaking is perfect, takes place for an incident field of any spatial and spectral distribution, and does not involve strong scatterers as do cloaks based on transformation optics. We further show that cloaking in the backward direction is not possible for arbitrary fields but is obtained for self-imaging fields, often with ordinary materials. In both directions the structure of the refractive-index distribution of the cloak and its dispersion properties resemble those of the object.

The paper is organized as follows. In Sec. II we present the geometry and derive the conditions for the scattering potential of the cloak to render an object invisible in the forward and backward directions. Sections III and IV concentrate on the

solutions for the forward and backward cloaking, respectively. Finally, Sec. V summarizes the work.

II. CLOAKING CONDITION

Consider the scattering of a monochromatic scalar field, $U^{(i)}(\mathbf{r}, \omega) \exp(-i\omega t)$ (ω is frequency, t is time, and \mathbf{r} is position), in a slab geometry illustrated in Fig. 1. The field is incident in vacuum along the z axis onto the object slab, located between $0 \leq z \leq a$ and having a z -dependent refractive-index distribution $n_o(z, \omega)$. Our aim is to find the refractive index $n_c(z, \omega)$ of a cloak slab, placed at $a_1 \leq z \leq a_2$ ($a_1 > a$), such that the field $U_c(\mathbf{r}, \omega)$ scattered from it exactly cancels the scattered field $U_o(\mathbf{r}, \omega)$ produced by the known object distribution $n_o(z, \omega)$. The refractive-index profiles are taken smooth and their contrasts with the surroundings low, so that a scalar treatment within the first-order Born approximation is adequate.

In the first-order Born approximation, omitting the temporal part, the total field, $U(\mathbf{r}, \omega)$, takes the form [18]

$$U(\mathbf{r}, \omega) = U^{(i)}(\mathbf{r}, \omega) + U_o(\mathbf{r}, \omega) + U_c(\mathbf{r}, \omega), \quad (1)$$

where the fields scattered from the object and cloak slabs, $U_o(\mathbf{r}, \omega)$ and $U_c(\mathbf{r}, \omega)$, respectively, are

$$U_j(\mathbf{r}, \omega) = \int_{V_j} G(\mathbf{r}, \mathbf{r}', \omega) F_j(\mathbf{r}', \omega) U^{(i)}(\mathbf{r}', \omega) d^3 \mathbf{r}', \quad (2)$$

with $j = (o, c)$. For $U_o(\mathbf{r}, \omega)$ the integration is performed over the volume V_o of the object slab, whereas for $U_c(\mathbf{r}, \omega)$ the integration is over the volume V_c of the cloak slab. The function $G(\mathbf{r}, \mathbf{r}', \omega) = \exp(ik_0|\mathbf{r} - \mathbf{r}'|)/|\mathbf{r} - \mathbf{r}'|$ is the outgoing free-space Green function, with the vacuum wave number $k_0 = 2\pi/\lambda$, where λ is the wavelength of light. The scattering potentials of the object and cloak slabs, $F_o(\mathbf{r}, \omega)$ and $F_c(\mathbf{r}, \omega)$, respectively, are $F_j(\mathbf{r}, \omega) = k_0^2 [n_j^2(\mathbf{r}, \omega) - 1]/4\pi$, for $j = (o, c)$. According to Eq. (1), cloaking is achieved when the fields scattered by the object and cloak cancel each other, i.e.,

$$U_o(\mathbf{r}, \omega) = -U_c(\mathbf{r}, \omega). \quad (3)$$

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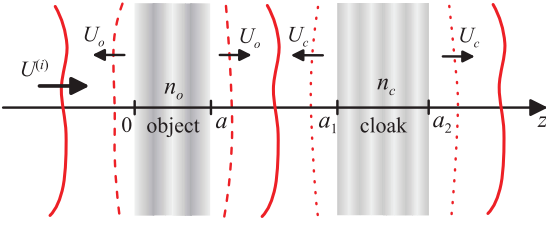


FIG. 1. (Color online) Illustration of slab geometry for cloaking. The field $U^{(i)}(\mathbf{r}, \omega)$ is incident along the z axis in vacuum on the object (at $0 \leq z \leq a$) and cloak (at $a_1 \leq z \leq a_2$) slabs having the refractive-index distributions of $n_o(z, \omega)$ and $n_c(z, \omega)$, respectively. The field scattered by the object is $U_o(\mathbf{r}, \omega)$ and that by the cloak is $U_c(\mathbf{r}, \omega)$.

By making use of the angular-spectrum representation the incident field, on neglecting the evanescent waves, is given by [19]

$$U^{(i)}(\mathbf{r}, \omega) = \iint_{p^2+q^2 \leq 1} a(p, q, \omega) e^{ik_0(px+qy+mz)} dp dq. \quad (4)$$

This formula expresses the field as a superposition of plane waves propagating with amplitudes $a(p, q, \omega)$ in directions specified by directional cosines (p, q, m) , with $m = (1 - p^2 - q^2)^{1/2}$ positive ($0 \leq m \leq 1$). The angular-spectrum representation of the free-space Green function is the Weyl representation [19], which by omitting the evanescent waves becomes

$$G(\mathbf{r}, \mathbf{r}', \omega) = \frac{ik_0}{2\pi} \iint_{p^2+q^2 \leq 1} \frac{1}{m} e^{ik_0[p(x-x') + q(y-y') + m|z-z'|]} dp dq. \quad (5)$$

By using Eqs. (4) and (5) in Eq. (2), the cloaking condition of Eq. (3) assumes the form

$$\int_0^a F_o(z', \omega) dz' = - \int_{a_1}^{a_2} F_c(z', \omega) dz' \quad (6)$$

for the half-space $z \geq a_2$, and

$$\int_0^a F_o(z', \omega) e^{2ik_0 m z'} dz' = - \int_{a_1}^{a_2} F_c(z', \omega) e^{2ik_0 m z'} dz' \quad (7)$$

for $z \leq 0$. Physically the exponent terms in Eq. (7) correspond to phase changes experienced by plane waves when traveling from $z = 0$ to position z' and back to plane $z = 0$. This phase change depends on the location z' at which the scattering takes place. In Eq. (6) such phase factors do not arise, since the phases induced on waves in transit through the system do not depend on the position z' of the forward scattering. These arguments suggest that cloaking in the backward direction is more involved than in the forward direction.

III. CLOAKING IN THE FORWARD DIRECTION

Consider first the condition for the half-space $z \geq a_2$, Eq. (6). Since it is independent of the incident field, cloaking

in the forward direction, if possible, occurs for any incident field. Changing the integration variable from z' to $z = a(z' - a_1)/(a_2 - a_1)$ on the right-hand side of Eq. (6), we obtain

$$\int_0^a F_o(z, \omega) dz = \int_0^a \left[\frac{a_1 - a_2}{a} F_c \left(\frac{a_2 - a_1}{a} z + a_1, \omega \right) \right] dz. \quad (8)$$

Assuming, for simplicity, that the object and cloak slabs are of the same thickness, $a_2 - a_1 = a$ (but with $a_1 > a$ arbitrary), and Eq. (8) implies cloaking when

$$F_c(z + a_1, \omega) = -F_o(z, \omega). \quad (9)$$

Since the object scatters weakly, its refractive index is close to unity (vacuum environment) and expressible as

$$n_o(z, \omega) = 1 + \epsilon(z, \omega) \exp[i\psi(z, \omega)], \quad (10)$$

where $0 < \epsilon(z, \omega) \ll 1$ and $\psi(z, \omega)$ is a real quantity. The functions $\epsilon(z, \omega)$ and $\psi(z, \omega)$ are continuous in both variables and slowly varying with respect to z . For this object, the condition in Eq. (9) then implies that the cloak is also a weak scatterer (as assumed) and its refractive-index distribution has two possible values,

$$n_c^\pm(z + a_1, \omega) = \pm 1 \mp \epsilon(z, \omega) \exp[i\psi(z, \omega)], \quad (11)$$

with either upper or lower signs taken. The refractive indices of the object and the cloak are illustrated in the complex plane in Fig. 2(a). For any positively refractive and absorbing object slab $n_o(z, \omega)$ lies in the vicinity of $+1$ in the upper-half plane. Furthermore, $n_c^+(z + a_1, \omega)$ of the cloak lies close to $+1$ in the lower half, whereas $n_c^-(z + a_1, \omega)$ is around -1 in the upper half of the complex plane. Thus any weakly scattering, positively refractive and absorbing slab can be cloaked either by a positively refractive and amplifying (active) medium, or in particular, by a medium which is negatively refractive and absorbing (passive). Cloaking is perfect, occurs for any incident field at all frequencies, and is independent of the cloak position a_1 . The dispersion properties of the cloak are explicitly specified by the parameters $\epsilon(z, \omega)$ and $\psi(z, \omega)$ of the object. Without external amplification all media, natural and artificial (metamaterials) alike, are dispersive and therefore lossy. Equations (10) and (11) indicate that the refractive indices of the object and cloak, although different in values, have similar spatial (and dispersive) structures.

At first sight the results for forward scattering might seem to contradict the well-known fact that, within the first-order Born approximation, no finite object can be nonscattering, even into a half-space, for all directions of the incident plane waves [20,21]. The resolution of the apparent paradox is that the slab structure considered here is infinite in extent. Furthermore, the results do not contradict the optical theorem which is also formulated for finite scatterers and is not restricted to the first-order Born approximation but holds for exact scattering [22].

IV. CLOAKING IN THE BACKWARD DIRECTION

Consider next the cloaking condition for backward scattering, Eq. (7). It is straightforward to show it can be satisfied only for discrete values of $0 \leq m \leq 1$. To prove this, we assume that Eq. (7) holds for a continuous range of m . Since both sides

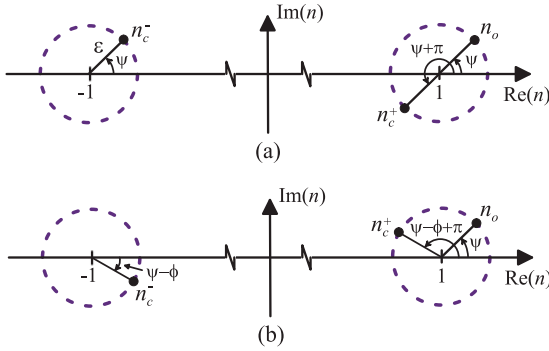


FIG. 2. (Color online) Illustration of the refractive indices of the object, $n_o(z, \omega)$, and the cloak, $n_c^\pm(z + a_1, \omega)$, in the complex plane: (a) forward scattering and (b) backward scattering. The parameters $\epsilon(z, \omega) \ll 1$ and $\psi(z, \omega)$ specify the refractive index of the object slab, whereas $\phi(\omega)$ is related to the incident field in the case of backscattering.

are analytic functions of m , it would then follow that Eq. (7) must hold for all m . However, integrating by parts both sides of Eq. (7), the left- and right-hand can, respectively, be written as

$$\int_0^a F_o(z', \omega) e^{2ik_0 m z'} dz' = \frac{F_o(a, \omega) e^{2ik_0 m a} - F_o(0, \omega)}{2ik_0 m} - O(m^{-2}), \quad (12)$$

$$- \int_{a_1}^{a_2} F_c(z', \omega) e^{2ik_0 m z'} dz' = \frac{F_c(a_1, \omega) e^{2ik_0 m a_1} - F_c(a_2, \omega) e^{2ik_0 m a_2}}{2ik_0 m} + O(m^{-2}). \quad (13)$$

Hence, for large m , the exponential term $\exp(2ik_0 m a)$ must be a linear combination of $\exp(2ik_0 m a_1)$, $\exp(2ik_0 m a_2)$, and a zero spatial-frequency wave. Owing to the linear independence of exponential functions this is impossible, and we conclude that Eq. (7) can be satisfied for discrete values of m only. Thus, unlike in the forward direction, cloaking in the backward direction is not generally possible.

With an analysis similar to that leading from Eq. (6) to Eq. (9) for forward scattering, Eq. (7) for backscattering implies the cloaking condition $F_c(z + a_1, \omega) = -F_o(z, \omega) \exp(-2ik_0 m a_1)$. This, in turn, indicates that cloaking is achieved, at least, if the right-hand side is independent of m . Therefore, choosing the scattering potential of the cloak slab as

$$F_c(z + a_1, \omega) = -F_o(z, \omega) e^{-i\phi(\omega)}, \quad (14)$$

with $\phi(\omega)$ a real constant, cloaking occurs for fields that are superpositions of plane waves with discrete values of m , given by

$$m_\alpha = \frac{\pi}{k_0 a_1} \left(\alpha + \frac{\phi(\omega)}{2\pi} \right), \quad (15)$$

where α is an integer such that $0 \leq m_\alpha \leq 1$. As a specific example we bring up a cloak slab for which $\phi(\omega) = \pi$. In

this case, cloaking is achieved for the incident fields with $m_\alpha = (\alpha + 1/2)\pi/k_0 a_1$, where $0 \leq \alpha \leq 2a_1/\lambda - 1/2$. The scattering potential of the cloak is $F_c(z + a_1, \omega) = F_o(z, \omega)$, showing that in reflection an exact replica of the object slab can also work as a cloak.

For an object characterized by Eq. (10), the condition in Eq. (14) implies that invisibility in the backward direction is achieved with two different cloaks specified as

$$n_c^\pm(z + a_1, \omega) = \pm 1 \mp \epsilon(z, \omega) \exp\{i[\psi(z, \omega) - \phi(\omega)]\}, \quad (16)$$

where either upper or lower signs are assumed. The refractive indices of the object and cloak in backscattering are illustrated in Fig. 2(b). Hence cloaking in reflection can be effected by positively and negatively refractive media, which are either lossy or amplifying (or both), depending on the incident field [via $\phi(\omega)$] and the object slab. For example, when $\phi(\omega) = 4\pi/5$ and $\psi(z, \omega) = \pi/4$ corresponding to a positively refractive lossy object, cloaking occurs, e.g., when $n_c(z + a_1, \omega) = 1 + \epsilon(z, \omega) \exp(i9\pi/20)$, which is a positively refractive and absorbing (ordinary) medium. Thus backward cloaking does not necessitate use of negative-index or amplifying materials. Furthermore, if for instance, $\phi(\omega) = 3\pi/4$ and $\psi(z, \omega) = 0$, cloaking takes place if $n_c(z + a_1, \omega) = 1 + \epsilon(z, \omega) \exp(i\pi/4)$. This result shows that in backward direction a dielectric slab object can be perfectly cloaked by an ordinary absorbing medium.

It is evident that a single value of m_α in Eq. (15) corresponds to an incident field that is a sum of plane waves with wave vectors on a conical surface of cone angle $\theta_c = \arccos(m_\alpha)$. Such a field is a (nondiffracting) Bessel beam, and thus Eq. (15) specifies a superposition of Bessel beams. Equation (15) shows that by increasing a_1 , i.e., moving the cloak slab farther from the object, the incident field can contain more values of m_α (Bessel beams). The adjacent m_α values are equally spaced with the separation $\Delta m_\alpha = \lambda/2a_1$, and therefore, the larger is a_1 the denser are m_α .

The plane waves constituting the incident field for which cloaking is achieved in half-space $z \leq 0$ form rings in the spatial-frequency (p, q) domain. The spacing of the rings, centered at origin, is not constant but depends on α . The angular spectrum of these fields in polar coordinates (ρ, θ) can generally be written as

$$a(\rho, \theta) = \sum_{\alpha=0}^{\alpha_{\max}} a_\alpha(\theta) \delta(\rho - \rho_\alpha), \quad (17)$$

where $\rho_\alpha = (1 - m_\alpha^2)^{1/2}$, $a_\alpha(\theta)$ is an arbitrary function, and α_{\max} is the largest integer satisfying Eq. (15). Since $a_\alpha(\theta)$ are arbitrary, an infinite number of incident fields with different intensity profiles exist for which backward cloaking is possible.

The functions $a_\alpha(\theta)$ are expressible as a (complex) Fourier series

$$a_\alpha(\theta) = \sum_{\beta=-\infty}^{\infty} a_{\alpha\beta} e^{i\beta\theta}, \quad a_{\alpha\beta} = \frac{1}{2\pi} \int_0^{2\pi} a_\alpha(\theta) e^{-i\beta\theta} d\theta. \quad (18)$$

Inserting Eqs. (17) and (18) into Eq. (4), transferring to polar coordinates, and making use of the representation

$$J_\beta(z) = \frac{i^{-\beta}}{2\pi} \int_0^{2\pi} e^{i(z \cos \theta + \beta \theta)} d\theta, \quad \beta = 0, \pm 1, \pm 2, \dots \quad (19)$$

where $J_\beta(z)$ is the β th-order Bessel function of the first kind, one finds that the incident field takes the form

$$U^{(i)}(\mathbf{r}, \omega) = \sum_{\alpha=0}^{\alpha_{\max}} \left(e^{ik_0 m_\alpha z} \sum_{\beta=-\infty}^{\infty} A_{\alpha\beta} J_\beta(k_0 \rho_\alpha r_\parallel) e^{i\beta\varphi} \right). \quad (20)$$

In this equation $A_{\alpha\beta} = 2\pi \rho_\alpha i^\beta a_{\alpha\beta}$, $r_\parallel = (x^2 + y^2)^{1/2}$, and the polar angle φ is determined by $\cos \varphi = x/r_\parallel$ and $\sin \varphi = y/r_\parallel$. Equation (20) expresses the incident field as a superposition of Bessel beams of various orders and widths [23]. Each term in the large parentheses (with fixed α) is a field whose transverse intensity distribution does not change on propagation. The sum of diffraction-free fields is not generally propagation invariant, but instead corresponds to a self-imaging field [23]. With the help of Eq. (15) one finds that the self-imaging (or Talbot) distance of the field is $z_T = 2a_1$. Thus, within the first-order Born approximation in a slab geometry, cloaking in half-space $z \leq 0$ is obtained for self-imaging fields by placing a cloak slab with the scattering potential given by Eq. (14) at $a_1 = Nz_T/2$, where N is an integer such that $a_1 \geq a$.

V. SUMMARY

In summary, we considered cloaking of a slab object in scalar wave theory within the first-order Born approximation.

We found that any absorbing object with transversally invariant refractive-index distribution can be perfectly cloaked in the forward direction (half-space $z \geq a_2$ in Fig. 1) by an absorbing, negative-index metamaterial slab. Cloaking takes place for incident fields with arbitrary spatial structure and spectral width, and is not based on use of subwavelength particles or strong scatterers. We also found that in backscattering (half-space $z \leq 0$) cloaking is achievable at least for incident self-imaging fields (exhibiting the Talbot effect). Forward cloaking is independent of the cloak position a_1 , whereas backward cloaking occurs only with certain values of a_1 . In general, a scattered field exists either in $z \leq 0$ or $z \geq a_2$ (or both); only in the special case when $\phi(\omega) = \pi$, corresponding to certain self-imaging fields, is cloaking perfect in both half-spaces.

We emphasize that our findings do not contradict the optical theorem and the known results on nonscattering scatterers. Interestingly, in both forward and backward cloaking, the refractive-index profile and dispersive properties of the cloak slab resemble those of the object (cf. “homeopathic principle”: the cure is similar to the illness). Weak scatterers frequently occur, for instance, in atmospheric optics and biophotonics.

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