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Higher integrability of the parabolic p-Laplace equation

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ABSTRACT
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We prove that the gradient of a weak solution to the parabolic p-Laplace equation is locally integrable to a higher exponent than assumed a priori. This result is based on reverse Hölder inequality which is shown using an energy estimate and a general Sobolev inequality. Higher integrability follows from a general self improvement property of reverse Hölder inequalities.

Because of the inhomogeneity of the parabolic p-Laplacian, we have to prove the reverse Hölder inequality in cylinders whose time scaling depends on the solution itself. This makes the arguments more delicate than in the linear case. The methods presented here continue to have relevance as they are robust, and can be applied for other types of parabolic PDEs.

Keywords:	Nonlinear parabolic PDEs, parabolic p-Laplacian, higher integrability, energy estimate, Sobolev inequality, reverse Hölder inequality
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Työssä todistetaan, että parabolisen p-Laplacen yhtälön heikko ratkaisu on lokaalisti integroituva korkeampaan eksponenttiin kuin alunperin oletettu. Tuloksen todistus pohjautuu käänteiseen Hölderin epäyhtälöön, joka osoitetaan energiaestimaatin ja yleisen Sobolevin epäyhtälön avulla. Korkeampi integroituvuus seuraa käänteisen Hölderin epäyhtälön ominaisuudesta parantaa itseään.

Parabolisen p-Laplacen yhtälön epähomogeenisuuden vuoksi käänteinen Hölderin epäyhtälö on todistettava sylintereissä, joiden aikaskaalaus riippuu ratkaisusta itsestään. Tämä tekee argumenteista mutkikkaampia kuin lineaarisessa tapauksessa. Työssä esitetyillä menetelmillä on edelleen merkitystä, sillä ne ovat robusteja ja sovellettavissa myös yleisemmille parabolisille osittaisdifferentiaal iyhtälöille.

Asiasanat: Epälineaariset paraboliset osittaisdifferentiaaliyhtälöt, parabolinen p-Laplacen yhtälö, korkeampi integroituvuus, energiaestimaatti, Sobolevin epäyhtälö, käänteinen Hölderin epäyhtälö

Kieli: Englanti

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Chapter 1

Introduction

This thesis concerns the regularity theory of the parabolic p-Laplace equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \tag{1.1}$$

which is a typical prototype for a nonlinear evolutionary partial differential equation. A solution to (1.1), referred to here as u, is a function of space and time, and ∇u denotes the spatial gradient of u. In case $p = 2$, (1.1) is the classical heat equation, but with other values of p the equation can model diffusive processes with different behaviour. For example, in the degenerate case $p > 2$, solutions to (1.1) have finite speed of propagation, in contrast to the heat equation. Equation (1.1) comes up typically in fluid dynamics when modelling non-Newtonian fluids or flow through an anisotropic medium.

In order to study existence of solutions to (1.1), an appropriate function space is needed. The minimization problem corresponding to (1.1) involves the integral of $|\nabla u|^p$, and therefore it is natural to consider weak solutions in the space $W^{1,p}$. In this thesis, we study whether (1.1) is strong enough to make a weak solution integrable to an exponent larger than p. This is called higher integrability of the gradient, or simply higher integrability. Our thesis is based on the article [11] by Kinnunen and Lewis, where higher integrability is shown for parabolic systems of p-Laplacian type.

Higher integrability is a fundamental step for obtaining other regularity results of PDEs. In terms of regularity, the most one can usually obtain, is partial regularity, i.e., Hölder regularity of the gradient outside a set of measure zero. For regularity theory of p-laplacian type equations, we refer to [4]. Showing partial regularity of the prototype equation (1.1) does not require higher integrability, but the methods used in [11] allow regularity results to be extended for more general PDEs. What is important is not only the result, but also the techniques used. In order to give an example of

other applications of higher integrability, we mention that showing stability of the p-Laplace equation with respect to change of p, requires higher integrability be able to compare solutions in the same function space, see [12]. Moreover, higher integrability lays the foundation for studying very weak solution, i.e., solutions for which less integrability is assumed a priori than for weak solutions.

The proof of higher integrability as presented in this thesis, is based on a reverse Hölder inequality for the gradient, which is shown with an energy estimate and a general Sobolev inequality. Then an argument using the self improvement property of reverse Hölder inequalities, known as Gehring's lemma, gives us higher integrability of the gradient. This method originates from the higher integrability of elliptic p-Laplace type systems, shown by Elcrat and Meyers in [14]. The method extends to parabolic systems with $p = 2$ straightforwardly enough, as shown in[8], but when $p \neq 2$, showing the reverse Hölder inequality proves to be problematic due to the inhomogeneity of (1.1). This problem was finally solved in [11] by proving the reverse Hölder inequality in cylinders with dimensions scaled to compensate for the inhomogeneity. This scaling has to depend on the solution itself and is therefore called intrinsic. The idea of intrinsic geometry goes back to the work of diBenedetto and Friedman in, e.g., [5].

Over the last two decades, the same strategy as by Kinnunen and Lewis in [11] has been used for showing higher integrability of other nonlinear PDEs, such as the porous medium equation, in [7] and [3], and doubly nonlinear equation, in [2]. With these equations a more complicated intrinsic geometry is needed, and this brings additional difficulties into the arguments. The results discussed so far are all local in nature, meaning that higher integrability is obtained in a region sufficiently far away from the parabolic boundary. However, higher integrability can also be extended to the whole domain, with some assumptions on the regularity of the boundary. For parabolic p-Laplace equation this was shown by Parviainen in [15].

This thesis is based mostly on the original article [11] by Kinnunen and Lewis. As an additional reference for the same arguments, we used the PhD thesis of Verena Bögelein [1]. Our aim is not to present the results in maximal generality, but to give a transparent argument with all the details included. To increase the clarity of presentation, we consider (1.1) in the most simple setting. More precisely, in [11] higher integrability is shown for systems of parabolic p-Laplacian type with $p > 2n/(n+2)$, but we consider only the scalar valued prototype equation (1.1) with $p \geq 2$. This way the core arguments become hopefully more visible.

As briefly mentioned, the proof of higher integrability is based on a reverse Hölder inequality, showing which has also independent importance. In order

to show the reverse Hölder inequality, we need two lemmas derived from the weak formulation of (1.1): an energy estimate and a so called gluing lemma. When proving the energy estimate, there is a technical difficulty when using u as a test function, as it has no smoothness in time. In [11] this is merely commented, but here we write in detail how to resolve the issue with mollification. Also for the gluing lemma, we give a complete proof, unlike in [11]. The energy estimate tells that $|\nabla u|^p$ can be estimated with an oscillation, and the oscillation can be further estimated by integral of $|\nabla u|$ to a lower exponent by the Sobolev embedding theorem. However, this can only be done at fixed times, and we need the gluing lemma in order to stitch together the reverse Hölder inequality in a space-time cylinder. The main difficulty in the proof of the reverse Hölder inequality, is that we have to compensate the inhomogeneity of (1.1) by conducting the proof of reverse Hölder inequality in intrinsic cylinders, whose time scaling depends on the integral average of $|\nabla u|^p$ over the cylinder itself. Heuristically, the idea is that we assume a constant λ to be comparable to $|\nabla u|$ in the intrinsic cylinder, when scaling time by λ^{p-2} would negate the diffusivity constant of (1.1).

Our proof for obtaining higher integrability is essentially self contained. It is based on a general self-improving property of uniform reverse Hölder inequality for upper level sets, which we show by elementary measure theoretic tools. In order to use our previously obtained reverse Hölder inequality, we construct intrinsic cylinders with a stopping time argument, and obtain a disjoint covering for the upper level set with a Vitali type lemma. Our presentation elaborates on the dense arguments of [11] and makes the proof more modular. In addition, we give a proof for the Vitali covering theorem for intrinsic cylinders, which was originally neglected in [11] and [1].

Chapter 2

Preliminaries

2.1 Weak solution

We will study solutions to (1.1) defined in $\Omega \times (0,T)$, where the spatial domain Ω is an open subset of $\mathbb{R}^n$, with $n \geq 2$. Throughout this thesis we assume that $p \geq 2$.

We say that $u \in L^p_{\mathrm{loc}}(0,T;W^{1,p}_{\mathrm{loc}}(\Omega))$, when

$$\int_{t_1}^{t_2} \|u\|^p_{W^{1,p}(K)}\, dt < \infty,$$

for any $0 < t_1 < t_2 < T$ and compact set $K \subset \Omega$. A function $u \in L^p_{\mathrm{loc}}(0,T;W^{1,p}_{\mathrm{loc}}(\Omega))$ is called a weak solution of (1.1) on $\Omega \times (0,T)$, when it satisfies

$$\int_0^T \int_\Omega -u\frac{\partial \phi}{\partial t} + |\nabla u|^{p-2}\nabla u \cdot \nabla \phi \, dx\, dt = 0 \tag{2.1}$$

for all test functions $\phi \in C_0^\infty(\Omega \times (0,T))$.

By restricting the test function in the weak formulation to a subinterval in time, and mollifying appropriately, we get that a weak solution u satisfies

$$\int_{t_1}^{t_2} \int_\Omega -u\frac{\partial \phi}{\partial t} + |\nabla u|^{p-2}\nabla u \cdot \nabla \phi \, dx\, dt + \left[\int_\Omega u\phi \, dx\right]_{t_1}^{t_2} = 0, \tag{2.2}$$

for all $\phi \in C_0^\infty(\Omega \times (0,T))$, and almost every $t_1, t_2 \in (0,T)$. We refer to these times where the above formula is satisfied as Lebesgue instants.

In order to consider u as a test function, even though it is not smooth, we formulate in the next lemma a mollified version of the weak formulation. We will denote a function f mollified in time with

$$[f]_\varepsilon(x,t) = \int_{\mathbb{R}} f(x,s)\psi_\varepsilon(t-s)\, dt,$$

where ψ_ε is a one-dimensional standard mollifier with support $(-\varepsilon, \varepsilon)$.

Lemma 2.1. *Let u be a weak solution to (1.1) and $0 < t_1 < t_2 < T$. We have*

$$\int_{t_1}^{t_2} \int_\Omega -[u]_\varepsilon \frac{\partial \phi}{\partial t} + [|\nabla u|^{p-2}\nabla u]_\varepsilon \cdot \nabla \phi \, dx \, dt + \left[\int_\Omega [u(x,t)]_\varepsilon \phi(x, t+s) \, dx \right]_{t_1}^{t_2} = 0,$$

for all $\phi \in C_0^\infty(\Omega \times (0,T))$ and $0 < 2\varepsilon < \min\{t_1, T - t_2\}$.

Proof. Let $\phi \in C_0^\infty(\Omega \times (0,T))$, $0 < 2\varepsilon < \min\{t_1, T - t_2\}$ and $s \in [-\varepsilon, \varepsilon]$. After an appropriate smooth cut-off near 0 or T, the shifted test function $\phi(x, t+s)$ is an admissible test function for (2.2) with integration bounds $t_1 - s$ and $t_2 - s$, and therefore we have

$$\int_{t_1 - s}^{t_2 - s} \int_\Omega -u(x,t) \frac{\partial \phi(x, t+s)}{\partial t} + |\nabla u(x,t)|^{p-2} \nabla u(x,t) \cdot \nabla \phi(x, t+s) \, dx \, dt + \left[\int_\Omega u(x,t) \phi(x, t+s) \, dx \right]_{t_1 - s}^{t_2 - s} = 0.$$

After a change of variables $t = t - s$, this becomes

$$\int_{t_1}^{t_2} \int_\Omega -u(x, t-s) \frac{\partial \phi(x,t)}{\partial t} + |\nabla u(x, t-s)|^{p-2} \nabla u(x, t-s) \cdot \nabla \phi(x,t) \, dx \, dt + \left[\int_\Omega u(x, t-s) \phi(x,t) \, dx \right]_{t_1}^{t_2} = 0.$$

To introduce the mollified solution inside the integrals, we multiply both sides with $\psi_\varepsilon(s)$ and integrate with respect to s over $[-\varepsilon, \varepsilon]$. This gives

$$\begin{aligned} &\int_{-\varepsilon}^{\varepsilon} \int_{t_1}^{t_2} \int_\Omega -u(x, t-s) \frac{\partial \phi(x,t)}{\partial t} \psi_\varepsilon(s) \, dx \, dt \, ds \\ &+ \int_{-\varepsilon}^{\varepsilon} \int_{t_1}^{t_2} \int_\Omega |\nabla u(x, t-s)|^{p-2} \nabla u(x, t-s) \cdot \nabla \phi(x,t) \psi_\varepsilon(s) \, dx \, dt \, ds \\ &+ \left[\int_{-\varepsilon}^{\varepsilon} \int_\Omega u(x, t-s) \phi(x,t) \psi_\varepsilon(s) \, dx \, ds \right]_{t_1}^{t_2} = 0, \end{aligned}$$

where we separated the integrals for more clarity. After changing the order of integration with Fubini's theorem, we have the desired formula

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega -\frac{\partial\phi(x,t)}{\partial t}\int_{-\varepsilon}^{\varepsilon} u(x,t-s)\psi_\varepsilon(s)\,ds\,dx\,dt \\
&+\int_{t_1}^{t_2}\int_\Omega \nabla\phi\cdot\int_{-\varepsilon}^{\varepsilon} |\nabla u(x,t-s)|^{p-2}\nabla u(x,t-s)\psi_\varepsilon(s)\,ds\,dx\,dt \\
&+\left[\int_\Omega\int_{-\varepsilon}^{\varepsilon} u(x,t-s)\psi_\varepsilon(s)\,ds\,dx\right]_{t_1}^{t_2} = 0.
\end{aligned}$$

□

2.2 Notation

Our notation is standard. We will write

$$B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$$

for open ball of radius $r > 0$ and center $x \in \mathbb{R}^n$, and

$$\Lambda_s(t_0) = (t_0 - s, t_0 + s).$$

for time intervals. Space-time cylinder centered at $(x,t) \in \mathbb{R}^{n+1}$ is denoted by

$$Q_{r,s}(x,t) = B_r(x)\times\Lambda_s(t).$$

The $n+1$ dimensional Lebesgue measure of a measurable $E \subset \mathbb{R}^{n+1}$ is denoted with $|E|$, and we use the notation

$$\fint_E f\,dx\,dt = \frac{1}{|E|}\int_E f\,dx\,dt$$

for the integral average of $f \in L^1(E)$ where $0 < |E| < \infty$. For brevity we will write

$$u_\rho(t) = m(B_\rho(x_0))^{-1}\int_{B_\rho(x_0)} u(x,t)\,dx$$

for the integral average of u over a ball at time t. Here m denotes the n-dimensional Lebesgue measure. The center of the cylinder, x_0, is omitted in this notation as it will generally be fixed within the forthcoming arguments.

Most of the time we will not keep track of constants, instead redefining the constant c from line to line when required.

2.3 Auxiliary lemmas

In this section we will state some general lemmas which will be used in the thesis. The first two are well known.

Lemma 2.2 (Young's inequality with ε)**.** *Let $a, b, \varepsilon > 0$ and $p, q > 1$ satisfying $1/p + 1/q = 1$. We have $c = c(\varepsilon, p)$ such that*

$$ab \leq ca^p + \varepsilon b^q.$$

Lemma 2.3 (Poincare inequality)**.** *Let $1 < p < \infty$ and $u \in W^{1,p}(\mathbb{R}^n)$. There exist $c(n, p)$, such that*

$$\fint_{B_r} |u - u_{B_r}|^p \, dx \leq cr^p \fint_{B_r} |\nabla u|^p \, dx,$$

for all balls $B_r \subset \mathbb{R}^n$.

The next lemma is a version of the Gagliado-Nirenberg-Sobolev inequality, for proof we refer to [10] Corollary 3.7.

Lemma 2.4. *Let $u \in W_0^{1,p}(\Omega)$ with $1 \leq p$ and $\Omega \in \mathbb{R}^n$ bounded. There exists $c = c(n, p)$, such that*

$$\left(\int_\Omega |u|^{\frac{pn}{n-p}} \, dx \right)^{\frac{n-p}{pn}} \leq c \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}}.$$

We need the following iteration lemma to conclude the proof of Lemma 3.2. For proof we refer to [13], Lemma 2.

Lemma 2.5. *Let $0 < \rho$, $0 < \theta < 1$, $a \geq 0$, $p > 0$ and $b \geq 0$, and let f be a nonnegative bounded function on $\mathbb{R}$, satisfying*

$$f(\rho_1) \leq \theta f(\rho_2) + a(\rho_2 - \rho_1)^{-p} + b, \tag{2.3}$$

for all $\rho_1, \rho_2 \in [\rho, 2\rho]$ with $\rho_1 < \rho_2$. Then there exists a constant $c = c(p, \theta)$, such that

$$f(\rho) \leq c(a\rho^{-p} + b).$$

Chapter 3

Fundamental estimates

In this chapter we prove three lemmas which allow us to prove a reverse Hölder inequality. Actually, showing Lemmas 3.1 and 3.2 is the only part in this thesis where we use the fact that u is a weak solution to (1.1). This makes the whole argument very flexible.

In the first lemma we use the behaviour of a solution, that is the weak formulation (2.1), to show that the difference of mean values of u at different times can be controlled by a spatial gradient. In order to get a reverse Hölder inequality, it is important that here the exponent of the gradient is less than p. This so called gluing lemma is needed to estimate an oscillatory integral over a cylinder with slice-wise oscillations. Here the fundamental difference with the elliptic case is that gradient in itself has no control over the oscillation in time direction, and thus Sobolev inequalities cannot be used directly in the whole cylinder.

Lemma 3.1. *Assume that u is a weak solution to (1.1) and $Q_{4\rho,s}(x_0,t_0) \subset \Omega \times (0,T)$. Then there exists $\widehat{\rho} \in (\rho, 2\rho)$, and a constant $c = c(n)$, such that*

$$|u_{\widehat{\rho}}(t_2) - u_{\widehat{\rho}}(t_1)| \leq cs\rho^{-1} \fint_{Q_{2\rho,s}(x_0,t_0)} |\nabla u|^{p-1}\, dx\, dt,$$

for almost all $t_1, t_2 \in \Lambda_s(t_0)$.

Proof. As everything is centered around (x_0,t_0), we omit notating the centers in the proof. Assume $t_1, t_2 \in \Lambda_s(t_0)$, with $t_1 \leq t_2$, are Lebesgue instants. Let $\widehat{\rho} \in (\rho, 2\rho)$ be a Lebesgue point for the function

$$g(r) = \int_{\partial B_r} |\nabla u|^{p-1}\, dS,$$

satisfying

$$\int_{\partial B_{\widehat{\rho}}} |\nabla u|^{p-1}\, dS \leq \fint_{\rho}^{2\rho} \int_{\partial B_r} |\nabla u|^{p-1}\, dS\, dr, \tag{3.1}$$

where S denotes the $n-1$ dimensional Hausdorff measure. It is clear that such $\widehat{\rho}$ exists as it is a point where g is less or equal than the average on $(\rho, 2\rho)$.

Let $0 < \delta < \rho$ be a parameter we will eventually let tend to zero. Let $\phi \in C_0^\infty(\Omega \times (0,T))$ be a cut-off function, satisfying

$$\begin{cases} \phi = 1 \text{ on } Q_{\widehat{\rho},s} \\ |\nabla \phi| \leq 2/\delta \\ \phi(\cdot, t) = 0 \text{ on } \Omega \setminus B_{\widehat{\rho}+\delta}. \end{cases}$$

Inserting ϕ into (2.2) with time interval (t_1, t_2), and taking absolute value of both sides, gives

$$\left| \left[\int_\Omega u\phi\, dx \right]_{t_1}^{t_2} \right| = \left| \int_{t_1}^{t_2} \int_\Omega -u\frac{\partial \phi}{\partial t} + |\nabla u|^{p-2}\nabla u \cdot \nabla \phi\, dx\, dt \right|,$$

and taking into account that $\frac{\partial \phi}{\partial t} = 0$, on $Q_{\widehat{\rho},s}$, and $\phi = 0$ outside $B_{\widehat{\rho}+\delta}$, we have

$$\left| \left[\int_{B_{\widehat{\rho}+\delta}} u\phi\, dx \right]_{t_1}^{t_2} \right| = \left| \int_{t_1}^{t_2} \int_{B_{\widehat{\rho}+\delta}} |\nabla u|^{p-2}\nabla u \cdot \nabla \phi\, dx\, dt \right|.$$

Estimating the right hand side with Cauchy-Schwarz inequality and using the properties of ϕ, gives

$$\begin{aligned} \left| \int_{t_1}^{t_2} \int_{B_{\widehat{\rho}+\delta}} |\nabla u|^{p-2}\nabla u \cdot \nabla \phi\, dx\, dt \right| &\leq \int_{t_1}^{t_2} \int_{B_{\widehat{\rho}+\delta}} |\nabla u|^{p-1} |\nabla \phi|\, dx\, dt \\ &\leq \int_{t_1}^{t_2} \frac{2}{\delta} \int_{B_{\widehat{\rho}+\delta} \setminus B_{\widehat{\rho}-\delta}} |\nabla u|^{p-1}\, dx\, dt \\ &= \int_{t_1}^{t_2} \fint_{\widehat{\rho}-\delta}^{\widehat{\rho}+\delta} \int_{\partial B_r} |\nabla u|^{p-1}\, dS\, dr\, dt, \end{aligned}$$

where we rewrote the integral over an annulus as an iterated integral over spheres.

As $\widehat{\rho}$ was defined to be a Lebesgue point for the appropriate function g, letting $\delta \to 0$ gives

$$\begin{aligned} |B_{\widehat{\rho}}|\,|u_{\widehat{\rho}}(t_2) - u_{\widehat{\rho}}(t_1)| &= \lim_{\delta\to 0}\left|\left[\int_{B_{\widehat{\rho}+\delta}} u\phi\,dx\right]_{t_1}^{t_2}\right| \\ &\leq \int_{t_1}^{t_2}\int_{\partial B_{\widehat{\rho}}} |\nabla u|^{p-1}\,dS\,dt. \end{aligned}$$

From the way $\widehat{\rho}$ was defined, that is (3.1), we get

$$\begin{aligned} \int_{t_1}^{t_2}\int_{\partial B_{\widehat{\rho}}} |\nabla u|^{p-1}\,dS\,dt &\leq \int_{t_1}^{t_2} \fint_{\rho}^{2\rho}\int_{\partial B_r} |\nabla u|^{p-1}\,dS\,dr\,dt \\ &\leq \rho^{-1}\int_{Q_{2\rho,s}} |\nabla u|^{p-1}\,dx\,dt, \end{aligned}$$

and consequently

$$\begin{aligned} |u_{\widehat{\rho}}(t_2) - u_{\widehat{\rho}}(t_1)| &\leq s\rho^{-1}\frac{|B_{2\rho}|}{|B_{\widehat{\rho}}|}\fint_{Q_{2\rho,s}} |\nabla u|^{p-1}\,dx\,dt \\ &\leq 2^n s\rho^{-1}\fint_{Q_{2\rho,s}} |\nabla u|^{p-1}\,dx\,dt, \end{aligned}$$

which finishes the proof. □

Next we prove a so called energy or Caccioppoli type estimate. As both terms on left hand side are positive, there are in fact two estimates included; one for the p-energy and another for the maximal slice-wise potential energy. These estimates are used separately when showing the reverse Hölder inequality.

As in the previous lemma, we proof the lemma by using a suitable test function in the weak formulation, in this case $(u-a)$ with an appropriate cut-off. Here it is important that $(u-a)$ is also a weak solution to (1.1). There is a technical problem as a weak solution has no smoothness in time direction, and thus cannot be used as a test function. This problem is circumvented by using the mollified formulation of Lemma 2.1. After dealing with the problematic time derivatives with integration by parts, we can take the limit and do rest of the calculations with regular u.

Lemma 3.2. *Let u be a weak solution to (1.1), $a \in \mathbb{R}$ a constant and $Q_{4\rho,4^2s}(x_0,t_0) \subset \Omega \times (0,T)$. There exists a constant $c = c(n,p)$, such that*

$$\begin{aligned}\int_{Q_{\rho,s}(x_0,t_0)} |\nabla u|^p \, dx\, dt + \operatorname*{ess\,sup}_{t\in\Lambda_s(t_0)} \int_{B_\rho(x_0)\times\{t\}} |u-a|^2 \, dx \\ \leq cs^{-1} \int_{Q_{2\rho,2^2s}(x_0,t_0)} |u-a|^2 \, dx\, dt \\ + c\rho^{-p} \int_{Q_{2\rho,2^2s}(x_0,t_0)} |u-a|^p \, dx\, dt\end{aligned}$$

Proof. As everything is centered around (x_0,t_0) or x_0 and t_0 respectively, we will omit notation of the centers.

In order to use the iteration argument of Lemma 2.5, we let $\rho \leq \rho_1 < \rho_2 \leq 2\rho$. We define a suitable test function for the mollified formulation with small $\varepsilon > 0$, by

$$\phi_\varepsilon = \eta\zeta^2([u]_\varepsilon - a),$$

where $\eta \in C_0^\infty(B_{\rho_2})$ and $\zeta \in C_0^\infty((0,T))$ are cut-off functions satisfying

$$\begin{cases}\eta = 1 \text{ in } B_{\rho_1} \\ 0 \leq \eta \leq 1 \\ |\nabla\eta| \leq c_\eta |\rho_2 - \rho_1|^{-1},\end{cases}$$

and

$$\begin{cases}\zeta = 0 \text{ on } (0, t_0 - 2^2 s] \\ \zeta = 1 \text{ on } [t_0 - s, t_0 + 2^2 s] \\ 0 \leq \zeta \leq 1 \\ 0 \leq \zeta' \leq s^{-1}.\end{cases}$$

Let $\tau \in \Lambda_{2^2s}(t_0)$ be a Lebesgue instant for $(u-a)$. As $(u-a)$ is also a weak solution to (1.1), we can use Lemma 2.1 with integration bounds $t_0 - 2^2 s$ and τ, and test function ϕ_ε. This gives

$$\begin{aligned}\int_{t_0-2^2s}^{\tau} \int_{B_{\rho_2}} -[u-a]_\varepsilon \frac{\partial \phi_\varepsilon}{\partial t} + [|\nabla u|^{p-2}\nabla u]_\varepsilon \cdot \nabla\phi_\varepsilon \, dx\, dt \\ + \int_{B_{\rho_2}\times\{\tau\}} [u-a]_\varepsilon \phi_\varepsilon \, dx = 0,\end{aligned} \tag{3.2}$$

as η vanishes outside B_{ρ_2} and ζ at $t_0 - 2^2 s$.

We begin by considering the first term with time derivative of ϕ_ε. Using elementary rules for derivatives and integration by parts in time variable, we get

$$\begin{aligned}
&\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} -[u-a]_\varepsilon\frac{\partial\phi_\varepsilon}{\partial t}\,dx\,dt\\
&\quad=\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} -[u-a]_\varepsilon\frac{\partial\eta\zeta^2([u]_\varepsilon-a)}{\partial t}\,dx\,dt\\
&\quad=\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} -2([u-a]_\varepsilon)^2\eta\zeta\zeta'\,dx\,dt\\
&\qquad-\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} \eta\zeta^2[u-a]_\varepsilon\frac{\partial[u-a]_\varepsilon}{\partial t}\,dx\,dt\\
&\quad=\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} -2([u-a]_\varepsilon)^2\eta\zeta\zeta'\,dx\,dt-\frac{1}{2}\int_{B_{\rho_2}\times\{\tau\}}\eta\zeta^2[u-a]_\varepsilon^2\,dx\\
&\qquad+\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} \eta\zeta\zeta'[u-a]_\varepsilon^2\,dx\,dt,
\end{aligned}$$

where the boundary term at t_0-2^2s was zero by the definition of ζ.

Now we no longer have a time derivative of u and can let $\varepsilon\to 0$. Because convolutions are dominated by maximal functions, for details see [10], Section 2.3., the mollifications converge and (3.2) becomes

$$\begin{aligned}
&\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} -|u-a|^2\,\eta\zeta\zeta'\,dx\,dt-\frac{1}{2}\int_{B_{\rho_2}\times\{\tau\}}\eta\zeta^2\,|u-a|^2\,dx\\
&\qquad+\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} |\nabla u|^{p-2}\nabla u\cdot\nabla(\eta\zeta^2(u-a))\,dx\,dt\\
&\qquad+\int_{B_{\rho_2}\times\{\tau\}}\eta\zeta^2\,|u-a|^2\,dx=0.
\end{aligned}$$

After expanding the gradient of the test function and reordering terms, we get

$$\begin{aligned}
&\frac{1}{2}\int_{B_{\rho_2}\times\{\tau\}}\eta\zeta^2\,|u-a|^2\,dx+\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}}\eta\zeta^2\,|\nabla u|^p\,dx\,dt\\
&\qquad=\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} |u-a|^2\,\eta\zeta\zeta'\,dx\,dt\\
&\qquad\quad-\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}}\zeta^2(u-a)|\nabla u|^{p-2}\nabla u\cdot\nabla\eta\,dx\,dt.
\end{aligned}\tag{3.3}$$

We estimate the signed term on the right hand side with absolute values and Cauchy-Schwarz inequality, and split the resulting product with Young's inequality. This, and taking into account the bounds for $\eta, |\nabla\eta|, \zeta$ and ζ', gives

$$\begin{aligned}
&\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} |u-a|^2\,\eta\zeta\zeta'\,dx\,dt - \int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} \zeta^2(u-a)|\nabla u|^{p-2}\nabla u\cdot\nabla\eta\,dx\,dt \\
&\qquad\le s^{-1}\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} |u-a|^2\,dx\,dt \\
&\qquad\quad + \int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} \zeta^2|u-a||\nabla u|^{p-1}|\nabla\eta|\,dx\,dt \\
&\qquad\le s^{-1}\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} |u-a|^2\,dx\,dt + \int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} c\zeta^2|\nabla\eta|^p|u-a|^p \\
&\qquad\quad + \frac{1}{2}\zeta^2\,|\nabla u|^p\,dx\,dt \\
&\qquad\le s^{-1}\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} |u-a|^2\,dx\,dt \\
&\qquad\quad + c|\rho_2-\rho_1|^{-p}\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} |u-a|^p\,dx\,dt \\
&\qquad\quad + \frac{1}{2}\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} \zeta^2\,|\nabla u|^p\,dx\,dt,
\end{aligned}$$

where Young's inequality was used with $\varepsilon = 1/2$ and exponents p and $p/(p-1)$. Inserting the above estimate into (3.3), and adding the first term on the left-hand side of (3.3) to both sides, we get

$$\begin{aligned}
&\int_{B_{\rho_1}\times\{\tau\}} \zeta^2\,|u-a|^2\,dx + \int_{t_0-2^2s}^{\tau}\int_{B_{\rho_1}} \zeta^2\,|\nabla u|^p\,dx\,dt \\
&\qquad\le \frac{1}{2}\int_{B_{\rho_2}\times\{\tau\}} \zeta^2\,|u-a|^2\,dx + \frac{1}{2}\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} \zeta^2\,|\nabla u|^p\,dx\,dt \\
&\qquad\quad + c|\rho_2-\rho_1|^{-p}\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} |u-a|^p\,dx\,dt \\
&\qquad\quad + s^{-1}\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} |u-a|^2\,dx\,dt,
\end{aligned}$$

where we also restricted the integration to B_{ρ_1} on the left-hand side. This is

exactly what we need to use Lemma 2.5 with

$$\begin{cases} f(r) = \int_{B_r\times\{\tau\}} \zeta^2 \left|u-a\right|^2 \, dx + \int_{t_0-2^2s}^{\tau}\int_{B_r} \zeta^2 \left|\nabla u\right|^p \, dx\, dt \\ a = c\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} \left|u-a\right|^p \, dx\, dt \\ b = s^{-1}\int_{t_0-2^2s}^{\tau}\int_{B_{\rho_2}} \left|u-a\right|^2 \, dx\, dt \\ \theta = \frac{1}{2}, \end{cases}$$

and therefore we have

$$\begin{aligned} &\int_{B_\rho\times\{\tau\}} \zeta^2 \left|u-a\right|^2 \, dx + \int_{t_0-2^2s}^{\tau}\int_{B_\rho} \zeta^2 \left|\nabla u\right|^p \, dx\, dt \\ &\qquad \leq c\rho^{-p}\int_{t_0-2^2s}^{\tau}\int_{B_{2\rho}} \left|u-a\right|^p \, dx\, dt \\ &\qquad\qquad + c\int_{t_0-2^2s}^{\tau}\int_{B_{2\rho}} \left|u-a\right|^2 \, dx\, dt. \end{aligned} \tag{3.4}$$

By the definition of supremum, there is a Lebesgue instant $\tilde{t} \in \Lambda_s(t_0)$, such that

$$\operatorname*{ess\,sup}_{t\in\Lambda_s(t_0)} \int_{B_\rho\times\{t\}} \left|u-a\right|^2 dx \leq 2\int_{B_\rho(x_0)\times\{\tilde{t}\}} \left|u-a\right|^2 dx.$$

By choosing $\tau = \tilde{t}$ in (3.4) and recalling that $\zeta = 1$ on $\Lambda_s(t_0)$, we conclude that

$$\begin{aligned} \operatorname*{ess\,sup}_{t\in\Lambda_s(t_0)} \int_{B_\rho\times\{t\}} \left|u-a\right|^2 dx &\leq 2\int_{B_\rho\times\{\tilde{t}\}} \left|u-a\right|^2 \, dx \\ &\leq cs^{-1}\int_{Q_{2\rho,2^2s}} \left|u-a\right|^2 \, dx\, dt \\ &\qquad + c\rho^{-p}\int_{Q_{2\rho,2^2s}} \left|u-a\right|^p \, dx\, dt. \end{aligned}$$

On the other hand, choosing $\tau = t_0 + 2^2 s$ in (3.4) gives

$$\begin{aligned} \int_{Q_{\rho,s}} \left|\nabla u\right|^p \, dx\, dt &\leq \int_{Q_{\rho,2^2s}} \zeta^2 \left|\nabla u\right|^p \, dx\, dt \\ &\leq cs^{-1}\int_{Q_{2\rho,2^2s}} \left|u-a\right|^2 \, dx\, dt \\ &\qquad + c\rho^{-p}\int_{Q_{2\rho,2^2s}} \left|u-a\right|^p \, dx\, dt, \end{aligned}$$

which finishes the proof. □

Finally, we show a general parabolic Sobolev inequality, which tells us that extra differentiability can be turned into extra integrability. In a way this is the origin for the reverse Hölder inequality, and thus the higher integrability as well. Naturally this general lemma alone is not enough to show the reverse Hölder inequality, and we need the previous energy to obtain the aforementioned extra differentiability, as well as the gluing lemma.

Lemma 3.3. *Let $1 \leq \nu < \infty$ and assume that $u \in L^\nu(\Lambda_{2s}(t_0); W^{1,\nu}(B_{2\rho}(x_0)))$. Then there is a constant c depending only on n and ν, such that*

$$\int_{Q_{\rho,s}(x_0,t_0)} |u(x,t) - u_\rho(t)|^{\nu(1+2/n)}\,dx\,dt \leq c \int_{Q_{2\rho,2s}(x_0,t_0)} |\nabla u(x,t)|^\nu\,dx\,dt \cdot \left(\operatorname*{ess\,sup}_{t\in(t_0-2s,t_0+2s)} \int_{B_{2\rho}(x_0)} |u(x,t) - u_{2\rho}(t)|^2\,dx \right)^{\nu/n}.$$

Proof. For brevity we omit notating the centers in the proof, as everything is centered around (x_0, t_0). Let $t \in \Lambda_{2s}$ be such that $x \mapsto u(x,t)$ belongs to $W^{1,\nu}(B_{2\rho})$. Let $\eta \in C_0^\infty(Q_{2\rho,2s})$ be a cutoff function satisfying

$$\begin{cases} \eta = 1 \text{ in } Q_{\rho,s} \\ 0 \leq \eta \leq 1 \\ |\nabla \eta| \leq 10/\rho. \end{cases}$$

We denote

$$\tilde{u}(x,t) = |u(x,t) - u_\rho(t)|\eta(x,t).$$

To ensure that $\tilde{u}$ is finite and differentiable, we do a standard mollification denoted by $[\tilde{u}]_\varepsilon = \tilde{u} * \psi_\varepsilon$.

We estimate the integral in the left hand side of the claim slice-wise. Hölder's inequality with exponents n and $n/(n-1)$ gives

$$\begin{aligned} J &= \int_{B_{2\rho}\times\{t\}} [\tilde{u}]_\varepsilon^{\nu(1+2/n)}\,dx \\ &= \int_{B_{2\rho}\times\{t\}} [\tilde{u}]_\varepsilon^{2/n} [\tilde{u}]_\varepsilon^{\nu+(2/n)(\nu-1)}\,dx \\ &\leq \left(\int_{B_{2\rho}\times\{t\}} [\tilde{u}]_\varepsilon^2\,dx \right)^{1/n} \left(\int_{B_{2\rho}\times\{t\}} [\tilde{u}]_\varepsilon^{(\nu+(2/n)(\nu-1))n/(n-1)}\,dx \right)^{(n-1)/n}. \end{aligned}$$

Assume first $\nu > 1$. By the Gagliardo-Nirenberg-Sobolev inequality in $W_0^{1,1}(B_{2\rho})$ see Lemma 2.4, and Hölder's inequality with exponents $\nu/(\nu-1)$

and ν, we have

$$\begin{aligned}
&\left(\int_{B_{2\rho}\times\{t\}} [\tilde{u}]_\varepsilon^{(\nu+(2/n)(\nu-1))n/(n-1)}\,dx\right)^{(n-1)/n} \\
&\quad\le c\int_{B_{2\rho}\times\{t\}} |\nabla\left([\tilde{u}]_\varepsilon^{\nu+(2/n)(\nu-1)}\right)|\,dx \\
&\quad= c\int_{B_{2\rho}\times\{t\}} (\nu+2/n(\nu-1))[\tilde{u}]_\varepsilon^{(\nu-1)(1+2/n)}|\nabla[\tilde{u}]_\varepsilon|\,dx \\
&\quad\le c\left(\int_{B_{2\rho}\times\{t\}} [\tilde{u}]_\varepsilon^{\nu(1+2/n)}\,dx\right)^{(\nu-1)/\nu}\left(\int_{B_{2\rho}\times\{t\}} |\nabla[\tilde{u}]_\varepsilon|^\nu\,dx\right)^{1/\nu},
\end{aligned}$$

where the equality in the third line comes from the chain rule for gradients. In case $\nu = 1$, already the Gagliardo-Nirenberg-Sobolev inequality implies the inequality above. Thus in both cases we have

$$J \le cJ^{(\nu-1)/\nu}\left(\int_{B_{2\rho}\times\{t\}} |\nabla[\tilde{u}]_\varepsilon|^\nu\,dx\right)^{1/\nu}\left(\int_{B_{2\rho}\times\{t\}} [\tilde{u}]_\varepsilon^2\,dx\right)^{1/n}.$$

Absorbing $J^{(\nu-1)/\nu}$ on the left hand side and raising both sides to power ν, gives

$$\int_{B_{2\rho}\times\{t\}} [\tilde{u}]_\varepsilon^{\nu(1+2/n)}\,dx \le c\int_{B_{2\rho}\times\{t\}} |\nabla[\tilde{u}]_\varepsilon|^\nu\,dx\left(\int_{B_{2\rho}\times\{t\}} [\tilde{u}]_\varepsilon^2\,dx\right)^{\nu/n}$$

Now we can let $\varepsilon \to 0$ and by the convergence of convolutions in Sobolev spaces, see [6], Section 5.3.1, we have

$$\int_{B_{2\rho}\times\{t\}} \tilde{u}^{\nu(1+2/n)}\,dx \le c\int_{B_{2\rho}\times\{t\}} |\nabla\tilde{u}|^\nu\,dx\left(\int_{B_{2\rho}\times\{t\}} \tilde{u}^2\,dx\right)^{\nu/n}. \tag{3.5}$$

We estimate the first integral on the right-hand side of above inequality.

Triangle inequality of $\|\cdot\|_{L^\nu}$ gives

$$\begin{aligned}
&\left(\int_{B_{2\rho}\times\{t\}} |\nabla\tilde{u}|^\nu \, dx\right)^{1/\nu} \\
&\quad = \left(\int_{B_{2\rho}\times\{t\}} \Big|\eta\nabla|u-u_\rho(t)| + |u-u_\rho(t)|\nabla\eta\Big|^\nu \, dx\right)^{1/\nu} \\
&\quad \le \left(\int_{B_{2\rho}\times\{t\}} |\eta|^\nu|\nabla u|^\nu \, dx\right)^{1/\nu} + \left(\int_{B_{2\rho}\times\{t\}} |\nabla\eta|^\nu|u-u_\rho|^\nu \, dx\right)^{1/\nu} \\
&\quad \le \left(\int_{B_{2\rho}\times\{t\}} |\nabla u|^\nu \, dx\right)^{1/\nu} + c\rho^{-1}\left(\int_{B_{2\rho}\times\{t\}} |u-u_\rho|^\nu \, dx\right)^{1/\nu},
\end{aligned}$$

where we used the product rule of gradient in the first equality and the properties of η in the last inequality. Furthermore, Poincare's inequality in $W^{1,\nu}(B_{2\rho})$, see Lemma 2.3, implies

$$\begin{aligned}
&\left(\int_{B_{2\rho}\times\{t\}} |u-u_\rho(t)|^\nu \, dx\right)^{1/\nu} \\
&\quad \le \left(\int_{B_{2\rho}\times\{t\}} |u-u_{2\rho}(t)|^\nu \, dx\right)^{1/\nu} + \left(\int_{B_{2\rho}\times\{t\}} |u_{2\rho}(t)-u_\rho(t)|^\nu \, dx\right)^{1/\nu} \\
&\quad \le (1+2^n)\left(\int_{B_{2\rho}\times\{t\}} |u-u_{2\rho}(t)|^\nu \, dx\right)^{1/\nu} \\
&\quad \le c\rho\left(\int_{B_{2\rho}\times\{t\}} |\nabla u|^\nu \, dx\right)^{1/\nu},
\end{aligned}$$

and hence

$$\int_{B_{2\rho}\times\{t\}} |\nabla\tilde{u}|^\nu \, dx \le c\int_{B_{2\rho}\times\{t\}} |\nabla u|^\nu \, dx. \tag{3.6}$$

We were able to merge the terms in the second line of the previous chain of

inequalities, as

$$\begin{aligned}
&\left(\int_{B_{2\rho}\times\{t\}} |u_{2\rho}(t)-u_\rho(t)|^\nu\,dx\right)^{1/\nu}\\
&\qquad = |u_{2\rho}(t)-u_\rho(t)||B_{2\rho}|^{1/\nu}\\
&\qquad = \left|\fint_{B_\rho} u_{2\rho}(t)-u\,dx\right| |B_{2\rho}|^{1/\nu}\\
&\qquad \le \fint_{B_\rho} |u_{2\rho}(t)-u|\,dx |B_{2\rho}|^{1/\nu}\\
&\qquad \le 2^n \fint_{B_{2\rho}\times\{t\}} |u_{2\rho}(t)-u|\,dx |B_{2\rho}|^{1/\nu}\\
&\qquad \le 2^n \left(\fint_{B_{2\rho}\times\{t\}} |u_{2\rho}(t)-u|^\nu\,dx\right)^{1/\nu} |B_{2\rho}|^{1/\nu}\\
&\qquad = 2^n \left(\int_{B_{2\rho}\times\{t\}} |u-u_{2\rho}(t)|^\nu\,dx\right)^{1/\nu}.
\end{aligned}$$

The same argument as above, but with ν replaced by 2, and the fact that $\eta \le 1$, gives

$$\begin{aligned}
\left(\int_{B_{2\rho}\times\{t\}} \tilde{u}^2\,dx\right)^{\nu/n} &\le \left(\int_{B_{2\rho}\times\{t\}} |u-u_\rho(t)|^2\,dx\right)^{\nu/n}\\
&\le c\left(\int_{B_{2\rho}\times\{t\}} |u-u_{2\rho}(t)|^2\,dx\right)^{\nu/n}.
\end{aligned} \tag{3.7}$$

Inserting inequalities (3.6) and (3.7) into (3.5), and integrating over Λ_s,

gives

$$\begin{aligned}
&\fint_{Q_{\rho,s}} |u-u_\rho(t)|^{\nu(1+2/n)}\,dx\,dt \\
&\qquad \le \fint_{\Lambda_s}\fint_{B_{2\rho}} \tilde{u}^{\nu(1+2/n)}\,dx\,dt \\
&\qquad \le c\fint_{\Lambda_s}\fint_{B_{2\rho}} |\nabla u|^\nu\,dx\left(\int_{B_{2\rho}} |u-u_{2\rho}(t)|^2\,dx\right)^{\nu/n} dt \\
&\qquad \le c\fint_{Q_{2\rho,2s}(x_0,t_0)} |\nabla u|^\nu\,dx\,dt \\
&\qquad\qquad \cdot\left(\operatorname*{ess\,sup}_{\tau\in(t_0-2s,t_0+2s)}\int_{B_{2\rho}} |u-u_{2\rho}(t)|^2\,dx\right)^{\nu/n},
\end{aligned}$$

which finishes the proof.

□

Chapter 4

Reverse Hölder inequality

Now we are ready to prove a reverse Hölder inequality, which is the basis for the higher integrability result of our thesis. The idea of the proof is to estimate the integral of $|\nabla u|^p$ by an oscillatory integral, with Lemma 3.2, thus allowing us to lower the exponent with a Sobolev embedding. However we cannot do this in a regular cylinder as problems emerge when trying to fit together the scale dependencies of Lemmas 3.1 and 3.2. The remedy is to show the reverse Hölder inequality in intrinsic cylinders with time scaling depending on the integral average of $|\nabla u|^p$ in same cylinder.

In the proof we assume λ^p to be comparable with the integral average of $|\nabla u|^p$, and scale time in the intrinsic cylinder by λ^{2-p}. Heuristically, this scaling be seen to compensate for the inhomogeneity $|\nabla u|^{p-2}$ of (1.1), thus making the equation behave more like heat equation, for which the reverse Hölder inequality was already shown in [8]. However, this is more of a motivation, and the real reasons for using exactly this intrinsic scaling are technical details in the proof.

Theorem 4.1. *Let u be a weak solution to (1.1) in $\Omega \times (0,T)$. Let $\lambda > 0$, denote $s = \lambda^{2-p}\rho^2$, and assume that $Q_{40\rho,40^2 s}(x_0,t_0) \subset \Omega \times (0,T)$. Denote $Q = Q_{\rho,s}(x_0,t_0)$, $Q' = Q_{4\rho,4^2 s}(x_0,t_0)$, $Q'' = Q_{20\rho,20^2 s}(x_0,t_0)$. Given $\delta \geq 1$ such that*

$$\begin{aligned}\delta^{-1}\lambda^p &\leq \fint_Q |\nabla u|^p \, dx\, dt \\ &\leq \delta \fint_{Q''} |\nabla u|^p \, dx\, dt \leq \delta^2 \lambda^p, +\end{aligned}$$

there is $c \geq 1$ depending on n, p and δ, such that

$$\fint_{Q''} |\nabla u|^p \, dx\, dt \leq c \left(\fint_{Q'} |\nabla u|^q \, dx\, dt \right)^{p/q}, \tag{4.1}$$

where $q = \max\{p-1, pn/(n+2)\}$.

Proof. As a preliminary step we use Lemma 3.1 to find $\widehat{\rho} \in (2\rho, 4\rho)$, such that

$$|u_{\widehat{\rho}}(t_2) - u_{\widehat{\rho}}(t_1)| \leq cs\rho^{-1} \fint_{Q'} |\nabla u|^p \, dx\, dt, \tag{4.2}$$

for almost all $t_1, t_2 \in (t_0 - 2s, t_0 + 2s)$. Note that it is not a problem to use the cylinder $Q' = Q_{4\rho,4^2s}(x_0, t_0)$ instead of the smaller $Q_{2\rho,s}(x_0, t_0)$, for which Lemma 3.1 was shown, as the integrand is positive and enlarging the cylinder over which the integral average is taken will only change the constant c. With $\widehat{\rho}$ fixed, let us denote $\widehat{Q} = Q_{\widehat{\rho},2s}(x_0, t_0)$.

We begin estimating the left-hand side of (4.1) using the estimates shown in Chapter 3. By Lemma 3.2 with $\rho_1 = \rho$, $\rho_2 = 2\rho$, $s_1 = s$, $s_2 = 2s$ and $a = u_{\widehat{Q}}$ we have

$$\begin{aligned}\fint_{Q_{\rho,s}(x_0,t_0)} |\nabla u|^p \, dx\, dt \leq & cs^{-1} \fint_{Q_{2\rho,2s}(x_0,t_0)} |u - u_{\widehat{Q}}|^2 \, dx\, dt \\ & + c\rho^{-p} \fint_{Q_{2\rho,2s}(x_0,t_0)} |u - u_{\widehat{Q}}|^p \, dx\, dt \\ = & T_1 + T_2,\end{aligned}$$

where the constants in Lemma 3.2 are multiplied by $2n^{n+1}$ as we moved to integral averages.

Next we show that it is enough to estimate T_1. As $p \geq 2$, we can use first Jensen's inequality and then Young's ε-inequality, with exponents $p/(p-2)$ and $p/2$, to get

$$\begin{aligned}T_1 &= c\lambda^{p-2}\rho^{-2} \fint_{Q_{2\rho,2s}(x_0,t_0)} |u - u_{\widehat{Q}}|^2 \, dx\, dt \\ &\leq c\lambda^{p-2}\rho^{-2} \left(\fint_{Q_{2\rho,2s}(x_0,t_0)} |u - u_{\widehat{Q}}|^p \, dx\, dt \right)^{2/p} \\ &\leq (2\delta)^{-1}\lambda^p + c\rho^{-p} \fint_{Q_{2\rho,2s}(x_0,t_0)} |u - u_{\widehat{Q}}|^p \, dx\, dt,\end{aligned}$$

where $\varepsilon = (2\delta)^{-1}\lambda^p$ was chosen so that the first term can be absorbed into the left hand side of the original inequality.

By assumption we have

$$\lambda^p \leq \delta \fint_{Q_{\rho,s}(x_0,t_0)} |\nabla u|^p \, dx\, dt,$$

and thus λ^p can be absorbed. Therefore we have

$$\fint_{Q_{\rho,s}(x_0,t_0)} |\nabla u|^p \, dx\, dt \le cT_2,$$

where c depends on n, p and δ.

In order to estimate T_2 with Lemma 3.3 we want to introduce the term $|u - u_{\widehat{\rho}}(t)|$ inside the integral. By convexity of $|\cdot|^p$ and estimating the space independent term with supremum over time, we get

$$\begin{aligned} T_2 &\le c\rho^{-p} \fint_{Q_{2\rho,2s}(x_0,t_0)} |u - u_{\widehat{\rho}}(t)|^p + |u_{\widehat{\rho}}(t) - u_{\widehat{Q}}|^p \, dx\, dt \\ &\le c_1 \rho^{-p} \fint_{\widehat{Q}} |u - u_{\widehat{\rho}}(t)|^p \, dx\, dt + c\rho^{-p} \operatorname*{ess\,sup}_{\tau \in (t_0-2s, t_0+2s)} |u_{\widehat{\rho}}(\tau) - u_{\widehat{Q}}|^p, \end{aligned} \tag{4.3}$$

where we also enlarged the cylinders to $\widehat{Q}$ in order to use Lemma 3.3 in the way it is stated.

In the rest of the proof we estimate both terms in (4.3) by an integral of $|\nabla u|$ to some power less than p. We start with the supremum on the right-hand side of (4.3).

By the gluing property (4.2) we have for almost all $\tau \in (t_0 - 2s, t_0 + 2s)$, that

$$\begin{aligned} |u_{\widehat{\rho}}(\tau) - u_{\widehat{Q}}| &= \left| u_{\widehat{\rho}}(\tau) - \frac{1}{|\widehat{Q}|} \int_{\widehat{Q}} u(x,\theta)\, dx\, d\theta \right| \\ &= \left| \fint_{t_0-2s}^{t_0+2s} \fint_{B_{\widehat{\rho}}(x_0)} u_{\widehat{\rho}}(\tau) - u(x,\theta)\, dx\, d\theta \right| \\ &\le \fint_{t_0-2s}^{t_0+2s} |u_{\widehat{\rho}}(\tau) - u_{\widehat{\rho}}(\theta)|\, d\theta \\ &\le cs\rho^{-1} \fint_{t_0-2s}^{t_0+2s} \fint_{Q'} |\nabla u|^{p-1} \, dx\, dt\, d\theta \\ &\le cs\rho^{-1} \fint_{Q'} |\nabla u|^{p-1} \, dx\, dt, \end{aligned}$$

as also the integration variable θ lies in the interval $(t_0 - 2s, t_0 + 2s)$, and the set of measure zero where (4.2) does not hold, can be discarded from the integration.

As the previous estimate is independent of τ, we have

$$\begin{aligned} c\rho^{-p} \operatorname*{ess\,sup}_{\tau\in(t_0-2s,t_0+2s)} & |u_{\widehat{\rho}}(\tau)-u_{\widehat{Q}}|^p \\ &\le cs^p\rho^{-2p}\left(\fint_{Q'}|\nabla u|^{p-1}\,dx\,dt\right)^p \\ &= c\lambda^{p(2-p)}\left(\fint_{Q'}|\nabla u|^{p-1}\,dx\,dt\right)^p \\ &\le c\delta^{2-p}\left(\fint_{Q'}|\nabla u|^{p-1}\,dx\,dt\right)^{p/(p-1)}, \end{aligned}$$

which is a suitable reverse Hölder type estimate for our theorem. Here the last inequality deserves more explanation: by assumption and Jensen's inequality, we have

$$\begin{aligned} \lambda^p &\ge \delta\fint_Q |\nabla u|^p\,dx\,dt \\ &\ge c\delta\fint_{Q'}|\nabla u|^p\,dx\,dt \\ &\ge c\delta\left(\fint_{Q'}|\nabla u|^{p-1}\right)^{p/(p-1)} \end{aligned}$$

and as $2-p\le 0$, it holds that

$$\begin{aligned} c\lambda^{p(2-p)}&\left(\fint_{Q'}|\nabla u|^{p-1}\,dx\,dt\right)^p \\ &\le c\delta^{2-p}\left(\fint_{Q'}|\nabla u|^{p-1}\right)^{\frac{p(2-p)}{p-1}+p} \\ &= c\delta^{2-p}\left(\fint_{Q'}|\nabla u|^{p-1}\right)^{p/(p-1)}. \end{aligned}$$

Now it is left to estimate the first term in (4.3). By the Parabolic Sobolev-Poincare inequality of Lemma 3.3 with $\nu = pn/(n+2)$, we have

$$\begin{aligned} c\rho^{-p}&\fint_{\widehat{Q}}|u-u_{\widehat{\rho}}(t)|^p\,dx\,dt \\ &\le c\rho^{-p}\fint_{2\widehat{Q}}|\nabla u|^{\frac{pn}{n+2}}\,dx\,dt\cdot\left(\operatorname*{ess\,sup}_{\tau\in\Lambda_{4s}}\int_{B_{2\widehat{\rho}}}|u(x,\tau)-u_{2\widehat{\rho}}(\tau)^2|\,dx\right)^{\frac{p}{n+2}}. \end{aligned} \tag{4.4}$$

The first integral above is already of the right type as $pn/(n+2) \leq p$, but it is not yet raised to the appropriate power. Therefore it is enough to estimate the essential supremum above by $\rho^{n+2}\lambda^2$, after which we can use Young's inequality to get the desired result. We begin by replacing the slice-wise mean $u_{\widehat{\rho}}(t)$ with a time-independent integral average over a large enough cylinder to use Lemma 3.2. In addition we want the cylinder to satisfy the gluing property of Lemma 3.1, therefore we choose $\tilde{\rho} \in (10\rho, 20\rho)$, such that

$$|u_{\tilde{\rho}}(t_2) - u_{\tilde{\rho}}(t_1)| \leq cs\rho^{-1} \fint_{20Q} |\nabla u|^{p-1} \, dx \, dt, \tag{4.5}$$

for almost all $t_1, t_2 \in \Lambda_{10s}$. Now we denote $\widetilde{Q} = Q_{\tilde{\rho},10s}$.

By the triangle inequality we have

$$\begin{aligned}
&\int_{B_{2\widehat{\rho}}} |u(x,\tau) - u_{2\widehat{\rho}}(\tau)^2| \, dx \\
&\qquad \leq c \int_{B_{2\widehat{\rho}}} |u(x,\tau) - u_{\widetilde{Q}}|^2 + |u - u_{2\widehat{\rho}}(\tau)|^2 \, dx \\
&\qquad \leq c \int_{B_{2\widehat{\rho}}} |u(x,\tau) - u_{\widetilde{Q}}|^2 \, dx,
\end{aligned}$$

and therefore the supremum estimate of Lemma 3.2 yields

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\tau \in \Lambda_{4s}} \int_{B_{2\widehat{\rho}}} |u(x,\tau) - u_{2\widehat{\rho}}(\tau)|^2 \, dx \\
&\qquad \leq c \operatorname*{ess\,sup}_{\tau \in \Lambda_{4s}} \int_{B_{2\widehat{\rho}}} |u - u_{\widetilde{Q}}|^2 \, dx \\
&\qquad \leq cs^{-1} \int_{\widetilde{Q}} |u - u_{\widetilde{Q}}|^2 \, dx \, dt + c\rho^{-p} \int_{\widetilde{Q}} |u - u_{\widetilde{Q}}|^p \, dx \, dt \\
&\qquad = T_1 + T_2.
\end{aligned}$$

Now we have essentially the same terms as in the beginning but instead of a reverse Hölder inequality we estimate these terms by $\rho^{n+2}\lambda^2$. However, the arguments used are similar to the ones already used. First we split the oscillatory integrals into slice-wise oscillations which yields four terms that are estimated one by one.

Once again using the convexity of $|\cdot|^p$ gives

$$\begin{aligned}
T_1 &\leq cs^{-1} \int_{\widetilde{Q}} |u - u_{\tilde{\rho}}(t)|^2 - |u_{\tilde{\rho}}(t) - u_{\widetilde{Q}}|^2 \, dx \, dt \\
&\leq c_1 s^{-1} \int_{\widetilde{Q}} |u - u_{\tilde{\rho}}(t)|^2 \, dx \, dt + c_2 s^{-1} |Q| \operatorname*{ess\,sup}_{t \in \Lambda_{10s}} |u_{\tilde{\rho}}(t) - u_{\widetilde{Q}}|^2 \\
&= T_{11} + T_{12}
\end{aligned}$$

and

$$\begin{aligned}
T_2 &\leq c\rho^{-p} \int_{\widetilde{Q}} |u - u_{\tilde{\rho}}(t)|^p - |u_{\tilde{\rho}}(t) - u_{\widetilde{Q}}|^p \, dx \, dt \\
&\leq c_1\rho^{-p} \int_{\widetilde{Q}} |u - u_{\tilde{\rho}}(t)|^p \, dx \, dt + c_2\rho^{-p}|Q| \operatorname*{ess\,sup}_{t\in\Lambda_{10s}} |u_{\tilde{\rho}}(t) - u_{\widetilde{Q}}|^p \\
&= T_{21} + T_{22}.
\end{aligned}$$

We can estimate the slice-wise oscillations T_{11} and T_{21} with the standard Poincare inequality of Lemma 2.3. As $p \geq 2$, we can use Jensen's inequality after applying Lemma 2.3, which gives

$$\begin{aligned}
T_{11} &= cs^{-1} \int_{\Lambda_{10s}} \int_{B_{\tilde{\rho}}} |u - u_{\tilde{\rho}}(t)|^2 \, dx \, dt \\
&\leq cs^{-1}\rho^2 \int_{\Lambda_{10s}} \int_{B_{\tilde{\rho}}} |\nabla u|^2 \, dx \, dt \\
&= c\rho^{n+2} \fint_{\widetilde{Q}} |\nabla u|^2 \, dx \, dt \\
&\leq c\rho^{n+2} \left(\fint_{\widetilde{Q}} |\nabla u|^p \, dx \, dt \right)^{2/p} \\
&\leq c\rho^{n+2}\lambda^2,
\end{aligned}$$

where we used the assumption that Q is intrinsic in the last inequality. This is an estimate of the right form. Recalling the definition of s we get similarly

$$\begin{aligned}
T_{21} &= c\rho^{-p} \int_{\Lambda_{10s}} \int_{B_{\tilde{\rho}}} |u - u_{\tilde{\rho}}(t)|^p \, dx \, dt \\
&\leq c \int_{\Lambda_{10s}} \int_{B_{\tilde{\rho}}} |\nabla u|^p \, dx \, dt \\
&= c\lambda^{2-p}\rho^{n+2} \fint_{\widetilde{Q}} |\nabla u|^p \, dx \, dt \\
&\leq c\rho^{n+2}\lambda^2.
\end{aligned}$$

We can estimate the supremum terms T_{12} and T_{22} in the same way as the supremum in 4.3. By (4.5), Jensen's inequality and the assumption that Q

is intrinsic, we have

$$\begin{aligned}
T_{12} &= c\rho^n \operatorname*{ess\,sup}_{\tau\in\Lambda_{10s}} \left| \fint_{\Lambda_{10s}} u_{\tilde{\rho}}(\tau) - u_{\tilde{\rho}}(\theta)\, d\theta \right|^2 \\
&\leq c\rho^n \left| s\rho^{-1} \fint_{20Q} |\nabla u|^{p-1}\, dx\, dt \right|^2 \\
&\leq cs^2\rho^{n-2} \left| \fint_{20Q} |\nabla u|^p\, dx\, dt \right|^{\frac{2(p-1)}{p}} \\
&\leq cs^2\rho^{n-2}\lambda^{2(p-1)} \\
&= c\lambda^{2(2-p)}\rho^{n+2}\lambda^{2(p-1)} = c\rho^{n+2}\lambda^2.
\end{aligned}$$

Similarly we get

$$\begin{aligned}
T_{22} &= c\rho^{-p+n} s \operatorname*{ess\,sup}_{\tau\in\Lambda_{10s}} \left| \fint_{\Lambda_{10s}} u_{\tilde{\rho}}(\tau) - u_{\tilde{\rho}}(\theta)\, d\theta \right|^p \\
&\leq c\rho^{-p+n} s \left| s\rho^{-1} \fint_{20Q} |\nabla u|^{p-1}\, dx\, dt \right|^p \\
&\leq c\rho^{-2p+n} s^{p+1} \left| \fint_{20Q} |\nabla u|^p\, dx\, dt \right|^{(p-1)} \\
&\leq c\rho^{-2p+n} s^{p+1}\lambda^{(p-1)p} \\
&= c\rho^{n+2}\lambda^{(2-p)(p+1)}\lambda^{(p-1)p} = c\rho^{n+2}\lambda^2.
\end{aligned}$$

From the previous estimates we conclude, that

$$\begin{aligned}
\operatorname*{ess\,sup}_{\tau\in\Lambda_{4s}} \int_{B_{2\widehat{\rho}}} |u(x,\tau) - u_{2\widehat{\rho}}(\tau)^2|\, dx &\leq T_{11} + T_{12} + T_{21} + T_{22} \\
&\leq c\rho^{n+2}\lambda^2.
\end{aligned}$$

Now we are ready finish our proof by absorbing the supremum in 4.4 to the left hand side of the claim. For brevity we write $q = pn/(n+2)$ for the exponent in the Sobolev embedding (4.4). Combining the embedding with

the inequality above we get

$$\begin{aligned}
&c\rho^{-p}\fint_{\widehat{Q}} |u-u_{\widehat{\rho}}(t)|^p\,dx\,dt \\
&\quad\le c\rho^{-p}\fint_{2\widehat{Q}} |\nabla u|^q\,dx\,dt\cdot\left(\operatorname*{ess\,sup}_{\tau\in\Lambda_{4s}}\int_{B_{2\widehat{\rho}}}|u(x,\tau)-u_{2\widehat{\rho}}(\tau)^2|\,dx\right)^{\frac{q}{n}} \\
&\quad\le c\lambda^{2q/n}\fint_{2\widehat{Q}} |\nabla u|^q\,dx\,dt \\
&\quad\le c\left(\fint_{2\widehat{Q}} |\nabla u|^q\,dx\,dt\right)^{p/q}+\lambda^p/(2\delta),
\end{aligned}$$

where in the last inequality we use Young's inequality with exponents p/q and $p/(p-q)$, and mass is distributed between the terms so that the term containing λ can be absorbed.

Now we have estimates for both terms in (4.3), that is

$$\begin{aligned}
\fint_Q |\nabla u|^p\,dx\,dt \le& c_1\left(\fint_{Q_{4\rho,4^2s}(x_0,t_0)}|\nabla u|^{p-1}\,dx\,dt\right)^{p/(p-1)} \\
&+c_2\left(\fint_{Q_{8\rho,4s}}|\nabla u|^{pn/(n+2)}\,dx\,dt\right)^{(n+2)/n}+\lambda^p/(2\delta).
\end{aligned}$$

As we have by assumption

$$\begin{aligned}
\fint_Q |\nabla u|^p\,dx\,dt-\lambda^p/(2\delta)&\ge\frac{1}{2}\fint_Q|\nabla u|^p\,dx\,dt \\
&\ge\frac{1}{2}\delta^2\fint_{Q''}|\nabla u|^p\,dx\,dt,
\end{aligned}$$

with Jensen and enlarging the cylinder, we can write our result in a clean form

$$\fint_{Q''}|\nabla u|^p\,dx\,dt\le c\left(\fint_{Q'}|\nabla u|^q\,dx\,dt\right)^{p/q},$$

where $q=\max\{p-1,np/(n+2)\}$. □

Chapter 5

Higher integrability

The fact that we can get higher integrability from a reverse Hölder inequality is based on a general self-improving result of Lemma 5.2. This lemma can be used to prove the well known Gehring's lemma, but we find it more direct to use Lemma 5.2, when showing the final higher integrability result. Its proof is elementary in nature, using nothing more than Cavalieri's principle and Fatou's lemma.

In order to make sure that a term in the proof is finite, we need the following truncated version of Cavalieri's principle. Time plays no special role in either result, but for coherence we formulate the following two lemmas in a cylinder $Q \subset \mathbb{R}^{n+1}$. However, for brevity we denote $z = (x,t)$ and omit Q in the notation of distribution sets, i.e., we write $\{f > \lambda\}$ instead of $\{(x,t) \in \mathbb{R}^{n+1} : f(z) > \lambda\}$.

Lemma 5.1. *Let $Q \subset \mathbb{R}^{n+1}$ be a cylinder and μ a measure on $\mathbb{R}^{n+1}$ such that $\mu(Q) < \infty$. For $0 < q < \infty$ and $0 \leq \lambda_0 \leq \lambda_1 < \infty$, we have*

$$\int_{\{\lambda_0 < f \leq \lambda_1\}} f^q \, d\mu = q \int_{\lambda_0}^{\lambda_1} \lambda^{q-1} \mu(\{f > \lambda\}) \, d\lambda + \lambda_0^q \mu(\{f > \lambda_0\}) - \lambda_1^q \mu(\{f > \lambda_1\}).$$

Lemma 5.2. *Let $1 \leq q < p < \infty$ and let $Q \subset \mathbb{R}^{n+1}$ be a cylinder. Assume that $f \in L^p(Q)$ is a non-negative function and that there exist $\lambda_0 \geq 1$ and $c_1 > 1$, such that*

$$\int_{\{f > \lambda\}} f^p \, dz \leq c_1 \lambda^{p-q} \int_{\{f > \lambda\}} f^q \, dz,$$

for all $\lambda \geq \lambda_0$. Then there exist $c = c(p, c_1)$, such that for $0 < \varepsilon < \frac{p-1}{c_1-1}$, we have

$$\int_Q f^{p+\varepsilon}\,dz \leq c\lambda_0^{\varepsilon} \int_Q f^p\,dz.$$

Proof. We denote $q = p + \varepsilon$. Clearly

$$\begin{aligned}\int_Q f^q\,dz &= \int_{\{f\leq\lambda_0\}} f^q\,dz + \int_{\{f>\lambda_0\}} f^q\,dz \\ &\leq \lambda_0^{q-p} \int_{\{f\leq\lambda_0\}} f^p\,dz + \int_{\{f>\lambda_0\}} f^q\,dz,\end{aligned}$$

and therefore it is enough to estimate f^q in the set where f is large. We need to ensure that the integral of f^q is finite to absorb terms later. Therefore let $\lambda_1 > \lambda_0$. We will do the estimation and then let t_1 go to infinity. By the truncated Cavalieri's principle of Lemma 5.1 with exponent $q - p$ and measure μ induced by f^p, that is $\mu(E) = \int_E f^p\,dz$, we have

$$\begin{aligned}\int_{\{\lambda_0<f\leq\lambda_1\}} f^q\,dz &= \int_{\{\lambda_0<f\leq\lambda_1\}} f^{q-p}\,d\mu \\ &= (q-p)\int_{\lambda_0}^{\lambda_1} \lambda^{q-p-1} \int_{\{f>t\}} f^p\,dz \\ &\quad + \lambda_0^{q-p} \int_{\{f>\lambda_0\}} f^p\,dz - \lambda_1^{q-p} \int_{\{f>\lambda_1\}} f^p\,dz.\end{aligned} \tag{5.1}$$

We use the reverse Hölder inequality of the assumption on the first term above, to get

$$\int_{\lambda_0}^{\lambda_1} \lambda^{q-p-1} \int_{\{f>t\}} f^p\,dz \leq c_1 \int_{\lambda_0}^{\lambda_1} \lambda^{q-2} \int_{\{f>t\}} f\,dz.$$

We note that the above is almost the first term on the right hand side of Lemma 5.1 with exponent $q-1$ and the measure μ induced by f, and therefore

$$\begin{aligned}&\int_{\lambda_0}^{\lambda_1} \lambda^{q-2} \int_{\{f>t\}} f\,dz \\ &\quad= \frac{1}{q-1}\left(\int_{\{\lambda_0<f\leq\lambda_1\}} f\,dz - \lambda_0^{q-1}\int_{\{f>\lambda_0\}} f\,dz + \lambda_1^{q-1}\int_{\{f>\lambda_1\}} f\,dz\right) \\ &\quad\leq \frac{1}{q-1}\left(\int_{\{\lambda_0<f\leq\lambda_1\}} f\,dz + \lambda_1^{q-p}\int_{\{f>\lambda_1\}} f^p\,dz\right),\end{aligned}$$

where we left out the term with negative sign and used the simple estimate

$$\begin{aligned}\int_{\{f>\lambda_1\}} f\,dz &\leq \int_{\{f>\lambda_1\}} f\left(\frac{f}{\lambda_1}\right)^{p-1} dz\\ &= \lambda_1^{1-p}\int_{\{f>\lambda_1\}} f^p\,dz.\end{aligned}$$

Inserting the obtained estimate to (5.1) gives

$$\begin{aligned}\int_{\{\lambda_0<f\leq\lambda_1\}} f^q\,dz \leq c_1\frac{q-p}{q-1}\int_{\{\lambda_0<f\leq\lambda_1\}} f^q\,dz + \lambda_0\int_{\{f>\lambda_1\}} f^p\,dz\\ + \left(c_1\frac{q-p}{q-1}-1\right)t_1^{q-p}\int_{\{f>\lambda_1\}} f^p\,dz.\end{aligned}$$

Because

$$\int_{\{\lambda_0<f\leq\lambda_1\}} f^q\,dz \leq t_1^q|Q| < \infty,$$

we can move this term to the left hand side, which is the reason why we did the truncation with t_1. Therefore

$$\begin{aligned}&\left(1-c_1\frac{q-p}{q-1}\right)\int_{\{\lambda_0<f\leq\lambda_1\}} f^q\,dz\\ &\quad\leq \lambda_0\int_{\{f>\lambda_1\}} f^p\,dz + \left(c_1\frac{q-p}{q-1}-1\right)t_1^{q-p}\int_{\{f>\lambda_1\}} f^p\,dz\end{aligned}$$

By our assumption on ε, we have $c_1\frac{q-p}{q-1} < 1$, and therefore dividing both sides of the previous estimate with $1-c_1\frac{q-p}{q-1}$, gives

$$\begin{aligned}\int_{\{\lambda_0<f\leq\lambda_1\}} f^q\,dz &\leq c\lambda_0^{q-p}\int_{\{f>\lambda_0\}} f^p\,dz - \lambda_1^{q-p}\int_{\{f>\lambda_1\}} f^p\,dz\\ &\leq c\lambda_0^{q-p}\int_{\{f>\lambda_0\}} f^p\,dz,\end{aligned}$$

where $c = \left(1-c_1\frac{q-p}{q-1}\right)^{-1}$. As this upper bound is independent of $\lambda-1$, we conclude with Fatou's lemma, that

$$\begin{aligned}\int_{\{f>\lambda_0\}} f^q\,dz &= \int_{\{f>\lambda_0\}} \lim_{\lambda_1\to\infty} f^q\chi_{\{f\leq\lambda_1\}}(z)\,dz\\ &\leq \lim_{\lambda_1\to\infty}\int_{\{\lambda_0<f\leq\lambda_1\}} f^q\,dz\\ &\leq c\lambda_0^{q-p}\int_{\{f>\lambda_0\}} f^p\,dz,\end{aligned}$$

where χ is the characteristic function. This finishes the proof. □

The previous lemma tells us that in order to show higher integrability, it is enough to show a uniform reverse Hölder inequality on upper level sets. We have already shown in Lemma 4.1 that on intrinsic cylinders a reverse Hölder inequality holds. Therefore our task remains to construct intrinsic cylinders with uniform tolerance constant δ, and to show that we can cover the upper level set with disjoint collection of such cylinders.

Normally Vitali covering theorem could be used to find the required disjoint subcollection, but here it turns out that our intrinsic cylinders have unbounded eccentricity, i.e., ratio of height and radius, and thus a general covering argument does not apply. However, we made sure in Theorem 5.1 that the scaling factors are bounded from above, and the scaling factors of neighbouring cylinders are comparable, and this is enough to construct Vitali covering. We show this in the next lemma, but first we define a suitable metric for parabolic geometry.

Definition 5.1. *We define the parabolic distance between two points $z_1, z_2 \in \mathbb{R}^{n+1}$, as*

$$\mathrm{d}_{\mathcal{P}}(z_1, z_2) = \max\left\{|x_1 - x_2|, \sqrt{|t_1 - t_2|}\right\}.$$

It is clear that such distance is a metric, with balls being cylinders of the form Q_{ρ,ρ^2}. We mention in passing that also the scaled distance

$$\mathrm{d}^{\gamma}_{\mathcal{P}}(z_1, z_2) = \max\left\{|x_1 - x_2|, \gamma^{-1}\sqrt{|t_1 - t_2|}\right\},$$

which generates balls of the form $Q_{\rho,\gamma\rho^2}$, is a metric. This will be useful later when constructing intrinsic cylinders.

Lemma 5.3. *Let $Q_0 = Q_{R,R^2}(z_0) \subset \mathbb{R}^{n+1}$ be a cylinder and $E \subset Q_0$. We denote the parabolic distance of $z \in Q_0$ from the boundary of Q_0, with*

$$d_z = \mathrm{d}_{\mathcal{P}}(z, \partial Q_0) = \inf_{(\tilde{x},\tilde{t})\in\partial Q_0} \min\left\{|x - \tilde{x}|, \sqrt{|t - \tilde{t}|}\right\}.$$

Assume we have for almost every $z \in E$ a cylinder

$$Q(z) = Q_{\rho_z,\gamma_z f\rho_z^2}(z),$$

with $\rho_z < d_z/10$ and $\gamma_z = k d_z^{\beta} \leq 1$, where $k, \beta > 0$ are constants. Then there exists a countable and pairwise disjoint subcollection $\{Q(z_j)\}_{j=1}^{\infty}$, such that

$$E \subset \bigcup_{j=1}^{\infty} \theta Q(z_j),$$

where $\theta = \max\{5, \sqrt{2^{\beta+3}+1}\}$.

Proof. We can follow the proof of the standard Vitali covering theorem but need to be more careful when showing covering property in time direction. For $i \in \mathbb{N}$ define subcollections of decreasing scale

$$\mathcal{F}_i = \left\{ Q(z) : \frac{R}{2^j} < \rho_z \leq \frac{R}{2^{j-1}} \right\}.$$

We construct a disjoint countable collection inductively. First, let $\mathcal{G}_1$ be a maximal disjoint collection of cylinders in $\mathcal{F}_1$. This collection is finite as γ is bounded from below for these cylinders. For $k \geq 2$, define G_k to be any maximal disjoint subcollection of

$$\left\{ Q \in \mathcal{F}_k : Q \cap Q' = \emptyset \text{ for any } Q' \in \bigcup_{j=1}^{k-1} \mathcal{G}_j \right\}.$$

This collection is again finite, and thus the collection

$$\mathcal{G} = \bigcup_{j=1}^{\infty} \mathcal{G}_j$$

is countable, as well as disjoint.

To conclude the proof, we show that any $Q(z)$ is included in some enlarged cylinder in the collection $\mathcal{G}$. From the way $\mathcal{G}$ is constructed, we know that there exists $Q(\tilde{z})$ satisfying $Q(\tilde{z}) \cap Q(z) \neq \emptyset$ and $\rho_z \leq 2\rho_{\tilde{z}}$. It clear that in space $Q(z)$ is contained in enlarged $Q(\tilde{z})$, that is, $B\rho_z(x) \subset 5B\rho_{\tilde{z}}(\tilde{x})$. The inclusion in time direction requires more careful inspection.

First we show that the scaling factors of the two cylinders are comparable. By triangle inequality, the intersection of the cylinders, and the assumptions on γ and $\rho_{\tilde{z}}$, we have

$$\begin{aligned} d_z &\leq d_{\tilde{z}} + 3\rho_{\tilde{z}} + \sqrt{\gamma_{\tilde{z}}\rho_{\tilde{z}}^2 + \gamma_z \rho_z^2} \\ &\leq d_{\tilde{z}} + \frac{3d_{\tilde{z}}}{10} + \sqrt{5\rho_{\tilde{z}}^2} \\ &\leq d_{\tilde{z}} + \frac{3d_{\tilde{z}}}{10} + \frac{3d_{\tilde{z}}}{10} \\ &\leq 2d_{\tilde{z}}. \end{aligned}$$

By the above inequality we have

$$\begin{aligned} \gamma_z \rho_z^2 &= k d_z^\beta \rho_z^2 \\ &\leq 2^{\beta+2} k d_{\tilde{z}}^\beta \rho_{\tilde{z}}^2 \\ &= 2^{\beta+2} \gamma_{\tilde{z}} \rho_{\tilde{z}}^2 \end{aligned}$$

It is clear that we have the required inclusion in time if we find θ, such that

$$\theta^2\gamma_{\tilde{z}}\rho_{\tilde{z}}^2 \geq 2\gamma_z\rho_z^2 + \gamma_{\tilde{z}}\rho_{\tilde{z}}^2.$$

Therefore the inclusion in time holds with constant $\sqrt{2^{\beta+3}+1}$. In order to have the inclusion in space as well, take $\theta = \max\{5, \sqrt{2^{\beta+3}+1}\}$. □

Now we have all the required tools for proving the higher integrability result. In the proof we construct intrinsic cylinders around points where $|\nabla u|^p$ is large in comparison to the integral average on an underlying cylinder, and then apply Theorem 4.1 to get a reverse Hölder inequality. The construction is based on a so called stopping time argument. The idea is that we fix the time scaling of a cylinder and enlarge it until the integral average matches with the scaling. As we have less room to enlarge cylinders near the boundary of the underlying cylinder, the time scaling has to depend on the distance to the boundary. In order to nevertheless get a uniform reverse Hölder inequality, we define an auxiliary function f.

Theorem 5.1. *Let u be weak solution to (1.1) and $Q_{4\theta r,(4\theta r)^2}(x_0,t_0) \subset \Omega \times (0,T)$, where $\theta > 10$ depends on n and p. Then there exists $\varepsilon > 0$ and $c = c(n,p,r)$, such that*

$$\int_{Q_{r,r^2}(x_0,t_0)} |\nabla u|^{p+\varepsilon}\, dx\, dt \leq c + c\left(\int_{Q_{2r,2^2r^2}(x_0,t_0)} |\nabla u|^p\, dx\, dt\right)^{\varepsilon/2+1}.$$

Proof. For brevity, denote $Q_r = Q_{r,r^2}(x_0,t_0)$ and $Q_{2r} = Q_{2r,(2r)^2}(x_0,t_0)$.

In order to use the stopping time argument successfully, we need to ensure that the gradient in the upper level set is large enough. Let

$$\kappa_0' = \left(\fint_{Q_{2r}} |\nabla u|^p\, dx\, dt\right)^{1/2},$$

and

$$\kappa > \kappa_0 = \max\{\kappa_0', (2r)^\alpha\},$$

be the level as in Lemma 5.2.

To get a uniform reverse Hölder inequality later, and to decide on which points we construct intrinsic cylinders, we define an auxiliary function

$$f = \hat{c}^{-1} d_{\tilde{z}}^\alpha\, |\nabla u|,$$

where $\hat{c}(n,p,r) > 1$ is a constant chosen to be large enough later, and $\alpha = (n+2)/2$. Let $\tilde{z} \in \Phi_\kappa = \{z \in Q_{2r} : f(z) > \kappa\}$ be a Lebesgue point for

$|\nabla u|^p$. Next we construct an intrinsic cylinder around this point to get the reverse Hölder inequality of Lemma 4.1. We choose the scaling factor for the intrinsic cylinder around $\tilde{z}$ to be

$$\gamma = \left(d_{\tilde{z}}^{-\alpha}\kappa\right)^{2-p}.$$

We note that as $d_{\tilde{z}}^{\alpha} \leq (2r)^{\alpha} < \kappa$ and $p \geq 2$, we have

$$\gamma = \left(d_{\tilde{z}}^{-\alpha}\kappa\right)^{2-p} \leq 1.$$

To find an intrinsic cylinder with this scaling factor we use the stopping time argument. Let $\theta = \max\left\{10, \sqrt{2^{\alpha+3}+1}\right\}$. From our definitions for κ, γ and α it follows that for $R \in (d_{\tilde{z}}/\theta, d_{\tilde{z}})$, we have

$$\begin{aligned}
\fint_{Q_{R,\gamma R^2}(\tilde{z})} |\nabla u|^p \, dx\, dt &\leq \frac{|Q_{2r}|}{|Q_{R,\gamma R^2}|} \fint_{Q_{2r}} |\nabla u|^p \, dx\, dt \\
&\leq (2r)^{n+p} R^{-(n+2)} \gamma^{-1} \kappa^2 \\
&\leq (2r)^{n+p} \cdot \theta^{n+2} d_{\tilde{z}}^{-(n+2)} \gamma^{-1} \kappa^2 \\
&= (2r)^{n+p} \cdot \theta^{n+2} d_{\tilde{z}}^{-2\alpha+\alpha(2-p)} \kappa^{p-2} \kappa^2 \\
&= \hat{c}^p \left(d_{\tilde{z}}^{-\alpha}\kappa\right)^p,
\end{aligned}$$

where we fixed $\hat{c} = (2r\theta)^{(n+2)/p}$.

On the other hand, we have by Lebesgue differentiation theorem on metric spaces, see [9], Theorem 1.8., that

$$\begin{aligned}
\lim_{r\to 0} \fint_{Q_{r,\gamma r^2}(\tilde{z})} |\nabla u|^p \, dx\, dt &= |\nabla u(\tilde{z})|^p \\
&= \hat{c}^p d_{\tilde{z}}^{-\alpha p} f(\tilde{z})^p > \hat{c}^p (d_{\tilde{z}}^{-\alpha}\kappa)^p,
\end{aligned}$$

where we used the definition of f and the fact that $f \in \Phi_\kappa$. As the integral average over a cylinder depends continuously on the radius, there exists $\rho = \rho_{\tilde{z}} \in (0, d_{\tilde{z}}/\theta]$, such that

$$\fint_{Q_{\rho,\gamma\rho^2}(\tilde{z})} |\nabla u|^p \, dx\, dt = \hat{c}^p (d_{\tilde{z}}^{-\alpha}\kappa)^p,$$

and

$$\fint_{Q_{R,\gamma R^2}(\tilde{z})} |\nabla u|^p \, dx\, dt \leq \hat{c}^p (d_{\tilde{z}}^{-\alpha}\kappa)^p,$$

for all $R \in (\rho, d_{\tilde{z}})$.

Now we are ready to get a reverse Hölder inequality from Theorem 4.1. Denote

$$\begin{cases} Q(\tilde{z}) = Q_{\rho,\gamma\rho^2}(\tilde{z}) \\ Q'(\tilde{z}) = Q_{2\rho,2^2\gamma\rho^2}(\tilde{z}) \\ Q''(\tilde{z}) = Q_{\theta\rho,10^2\gamma\rho^2}(\tilde{z}). \end{cases}$$

As $\hat{c} \geq 1$ and $\theta\rho \leq d_{\tilde{z}}$, by the inequalities above we have

$$(d_{\tilde{z}}^{-\alpha}\kappa)^p \leq \fint_{Q(\tilde{z})} |\nabla u|^p \, dx\, dt, \tag{5.2}$$

and

$$\fint_{Q''(\tilde{z})} |\nabla u|^p \, dx\, dt \leq \hat{c}^p (d_{\tilde{z}}^{-\alpha}\kappa)^p. \tag{5.3}$$

We assumed that we have enough room around Q_r, namely $Q_{4\theta\rho,(4\theta\rho)^2}(z_0) \subset \Omega \times (0,T)$. Hence as $\gamma \leq 1$, we have

$$\begin{aligned} 2Q''(\tilde{z}) &= Q_{2\theta\rho,\gamma(2\theta\rho)^2}(\tilde{z}) \\ &\subset Q_{4\theta\rho,(4\theta\rho)^2}(z_0) \subset \Omega \times (0,T). \end{aligned}$$

This means that the assumptions of Theorem 4.1 hold with $\lambda = d_{\tilde{z}}^{-\alpha}\kappa$ and $\delta = \hat{c}^p$, and we have the reverse Hölder inequality

$$\fint_{Q''(\tilde{z})} |\nabla u|^p \, dx\, dt \leq c \left(\fint_{Q'(\tilde{z})} |\nabla u|^q \, dx\, dt \right)^{p/q}, \tag{5.4}$$

where c depends only on n, p and r.

To get uniform bounds in terms of κ for the integrals above, we show next for f, instead of $|\nabla u|$, the reverse Hölder inequality

$$\begin{aligned} c^{-1}\kappa^p &\leq \fint_{Q''(\tilde{z})} f^p \, dx\, dt \\ &\leq \left(\fint_{Q'(\tilde{z})} f^q \, dx\, dt \right)^{p/q} \leq c^2\kappa^p. \end{aligned} \tag{5.5}$$

To show this, we first note that as $\gamma \leq 1$ and $\rho \leq d_{\tilde{z}}/\theta$, we have in Q'' the estimate

$$\begin{aligned} f &= \hat{c}^{-1} \, \mathrm{d}_{\mathcal{P}}^{\alpha} |\nabla u| \\ &\leq c d_{\tilde{z}}^{\alpha} |\nabla u| \end{aligned} \tag{5.6}$$

and similarly with geometric argument

$$\begin{aligned} |\nabla u| &= \hat{c}\, \mathrm{d}_{\mathcal{P}}^{-\alpha} f \\ &\leq c d_{\tilde{z}}^{-\alpha} f. \end{aligned} \tag{5.7}$$

Then the first inequality in (5.5) follows from (5.2) and (5.7), as

$$\begin{aligned} \kappa^p &\leq \fint_{Q(\tilde{z})} (d_{\tilde{z}}^{\alpha} |\nabla u|)^p \, dx \, dt \\ &\leq c \fint_{Q(\tilde{z})} f^p \, dx \, dt \\ &\leq c \fint_{Q''(\tilde{z})} f^p \, dx \, dt. \end{aligned}$$

The second inequality follows from the reverse Hölder inequality (5.4) and moving from f to $|\nabla u|$ by (5.6) and (5.7), that is

$$\begin{aligned} \fint_{Q''(\tilde{z})} f^p \, dx \, dt &\leq c r_{\tilde{z}}^{\alpha p} \fint_{Q''(\tilde{z})} |\nabla u|^p \, dx \, dt \\ &\leq c d_{\tilde{z}}^{\alpha p} \left(\fint_{Q'(\tilde{z})} |\nabla u|^q \, dx \, dt \right)^{p/q} \\ &\leq c \left(\fint_{Q'(\tilde{z})} f^q \, dx \, dt \right)^{p/q}. \end{aligned}$$

The third inequality follows from Jensen's inequality, (5.7) and (5.3), as

$$\begin{aligned} \left(\fint_{Q'(\tilde{z})} f^q \, dx \, dt \right)^{p/q} &\leq \fint_{Q'(\tilde{z})} f^p \, dx \, dt \\ &\leq c d_{\tilde{z}}^{\alpha p} \fint_{Q''(\tilde{z})} |\nabla u|^p \, dx \, dt \leq c \kappa^p. \end{aligned}$$

Finally, we get a reverse Hölder inequality for f in the upper level set Φ_κ in order to use Lemma 5.2. Let $1 > \eta(n,p) > 0$ be a small constant that is fixed later. Dividing $Q(\tilde{z})$ into parts according to whether f is larger or smaller than $\eta\kappa$, estimating f in the smaller part by $\eta\kappa$ and using the first

and last inequalities of (5.5), gives

$$\begin{aligned}
&\left(\fint_{Q(\tilde{z})} f^q \,dx\,dt\right)^{\frac{p}{q}} \\
&\quad= \left(\frac{1}{|Q(\tilde{z})|}\int_{Q(\tilde{z})\setminus\Phi_{\eta\kappa}} f^q \,dx\,dt + \frac{1}{|Q(\tilde{z})|}\int_{Q(\tilde{z})\cap\Phi_{\eta\kappa}} f^q \,dx\,dt\right)^{p/q} \\
&\quad\le c\eta^p\kappa^p + c\left(\fint_{Q(\tilde{z})} f^q \,dx\,dt\right)^{p/q-1}\frac{1}{|Q(\tilde{z})|}\int_{Q(\tilde{z})\cap\Phi_{\eta\kappa}} f^q \,dx\,dt \\
&\quad\le c_1\eta^p\left(\fint_{Q(\tilde{z})} f^q \,dx\,dt\right)^{\frac{p}{q}} + c_2\kappa^{p-q}\frac{1}{|Q(\tilde{z})|}\int_{Q(\tilde{z})\cap\Phi_{\eta\kappa}} f^q \,dx\,dt,
\end{aligned}$$

where we also used the convexity of $|\cdot|^{p/q}$ in the first inequality. Choosing $\eta = (2c)^{-1}$, we can absorb the first term of the right hand side of the above inequality into left hand side, on which using reverse Hölder inequality in (5.5) gives

$$\int_{Q''(\tilde{z})} f^p \,dx\,dt \le c\kappa^{p-q}\int_{\{Q(\tilde{z}):f>\eta\kappa\}} f^q \,dx\,dt. \tag{5.8}$$

Thus far we have shown that for almost every point $\tilde{z} \in \Phi_\kappa$ we have a cylinder $Q(\tilde{z})$ satisfying the above inequality. We formulated Lemma 5.3 so that it matches our setting, namely $\rho_{\tilde{z}} \le d_{\tilde{z}}/\theta \le d_{\tilde{z}}/10$ and $\gamma(\tilde{z}) \le 1$. Therefore we have by this Vitali type lemma a countable collection $\{Q(\tilde{z}_i)\}_{i=1}^{\infty}$ of disjoint cylinders, satisfying (5.8) and

$$\Phi_\kappa \subset \bigcup_{i=1}^{\infty} \theta Q(\tilde{z}_i) \subset Q_{2r}.$$

Consequently we have by (5.8) and disjointedness of the cylinders, that

$$\begin{aligned}
\int_{\Phi_\kappa} f^p \,dx\,dt &\le \sum_{i=1}^{\infty}\int_{\theta Q(\tilde{z}_i)} f^p \,dx\,dt \\
&\le c\kappa^{p-q}\sum_{i=1}^{\infty}\int_{Q_i\cap\Phi_{\eta\kappa}} f^q \,dx\,dt \\
&\le c\kappa^{p-q}\int_{\Phi_{\eta\kappa}} f^q \,dx\,dt.
\end{aligned}$$

In the part where f is small, that is $\Phi_\kappa \setminus \Phi_{\eta\kappa}$, we also have

$$\begin{aligned}\int_{\Phi_{\eta\kappa}\setminus\Phi_\kappa} f^q\,dx\,dt &\le \kappa^{p-q}\int_{\Phi_{\eta\kappa}\setminus\Phi_\kappa} f^q\,dx\,dt\\ &\le \kappa^{p-q}\int_{\Phi_{\eta\kappa}} f^q\,dx\,dt,\end{aligned}$$

as $f \le \kappa$ on $\Phi_\kappa \setminus \Phi_{\eta\kappa}$. Therefore we have

$$\begin{aligned}\int_{\Phi_{\eta\kappa}} f^p\,dx\,dt &\le \int_{\Phi_\kappa} f^q\,dx\,dt + \int_{\Phi_{\eta\kappa}\setminus\Phi_\kappa} f^q\,dx\,dt\\ &\le c\kappa^{p-q}\int_{\Phi_{\eta\kappa}} f^q\,dx\,dt.\end{aligned}$$

This means that for all $\lambda \ge \eta\kappa_0$ we have

$$\int_{\Phi_\lambda} f^p\,dx\,dt \le c\kappa^{p-q}\int_{\Phi_\lambda} f^q\,dx\,dt,$$

and therefore Lemma 5.2 gives

$$\int_{Q_{2r}} f^{p+\varepsilon}\,dx\,dt \le c\kappa_0^\varepsilon \int_{Q_{2r}} f^p\,dx\,dt.$$

Recalling the definition of κ_0, we have

$$\kappa_0 \le (2r)^\alpha + \left(\fint_{Q_{2r}} |\nabla u|^p\,dx\,dt\right)^{1/2},$$

and as $|\nabla u| \le \hat{c}f$ on Q_r and $f \le c\,|\nabla u|$ on Q_{2r}, we can write the higher integrability result in the form

$$\begin{aligned}\int_{Q_r} |\nabla u|^{p+\varepsilon}\,dx\,dt &\le c\left((2r)^\alpha + \left(\fint_{Q_{2r}} |\nabla u|^p\,dx\,dt\right)^{1/2}\right)^\varepsilon \int_{Q_{2r}} |\nabla u|^p\,dx\,dt\\ &\le c + c\left(\int_{Q_{2r}} |\nabla u|^p\,dx\,dt\right)^{\varepsilon/2+1},\end{aligned}$$

where c depends on n, p and r. □

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