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Efficient Bragg waveguide-grating analysis by quasi-rigorous approach based on Redheffer's star product

Jani Tervo [a], Markku Kuittinen [a,*], Pasi Vahimaa [a], Jari Turunen [a], Timo Aalto [b], Päivi Heimala [b], Matti Leppihalme [b]

[a] *Vaisala Laboratory, Department of Physics, University of Joensuu, P.O. Box 111, FIN-80101 Joensuu, Finland*

[b] *VTT Electronics, Microelectronics, P.O. Box 1101, FIN-02044 VTT, Finland*

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Abstract

We introduce a computationally efficient quasi-rigorous method for the analysis of corrugated planar waveguide structures. The method is based on rigorous diffraction theory of gratings. The computational efficiency is achieved by using Redheffer's star product and the so-called binary method for the involution of the transfer matrix. The developed method enables efficient rigorous analysis of corrugated waveguide structures without any limitations for the corrugation depth. Comparison with the thin-film stack method shows that the proposed method gives similar results for Bragg grating for the fundamental mode when the corrugations are shallow, but the results differ significantly when the corrugations are deep. Furthermore, the quasi-rigorous method also facilities the analysis of the coupling of light from the fundamental mode into the higher waveguide modes.

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1. Introduction

Bragg gratings are widely used in the rapidly growing field of optical telecommunications. In general, any perturbation formed as a periodic corrugation or refractive index modulation in an optical waveguide serves as a Bragg grating for some wavelength. Bragg gratings written in a photosensitive fibers as a periodic refractive index variation can be used to make a variety of devices such as filters, add/drop multiplexers, and dispersion compensators [1]. The advantage of the all-fiber devices is that they have low insertion losses. However, if cost efficiency, size reduction or separation of several wavelength channels is important, integrated optical Bragg gratings offer an attractive alternative. Silicon based waveguide technology eases mass production and enables monolithic integration of different gratings in almost arbitrary configurations. Especially in complex systems this eliminates many interconnections, thus reducing work load, costs, power losses, and size. Materials with high refractive index, such as silicon, allow to

* Corresponding author. Tel.: +358-13-251-2110; fax: +358-13-251-3290.

E-mail address: markku.kuittinen@joensuu.fi (M. Kuittinen).

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realize integrated optical grating components that are smaller than their fiber equivalents. In addition, integrated Bragg gratings offer a couple of completely new possibilities. First, gratings can be fabricated as corrugated structures in materials, which are not photorefractive. Second, other optical devices (couplers etc.), and even electrical components (modulators, control circuits etc.), can be monolithically integrated with the gratings.

The propagation of optical modes in periodic corrugated waveguides can be analyzed accurately by using the Floquet–Bloch theory [2,3]. If the depth of the corrugated structure is much lower than the waveguide thickness, coupled-mode theory [4,5] is widely used for the analysis of waveguide gratings. Wang [6] proposed a simple effective-index/impedance matching technique for solving the mode-coupling problem. Later Verly et al. [7,8] derived the effective-index method for corrugated gratings directly from Maxwell's equations. The effective-index method is known to give results equivalent with coupled-mode theory for shallow surface corrugations [9]. Effective-index method is very similar to the methods used to study reflection and transmission of light from thin-film stack. This thin-film stack approach can be implemented in a computationally efficient form for waveguide gratings by using Rouard's method [10], which is a recursive method used in thin-film coating design. The basic idea of the Rouard's method is the replacement of a thin-film layer characterized by an effective complex reflectivity by a single interface having the same properties. In fact, thin-film stack method is a numerical method for solving the coupled-mode equations and it has shown to be in excellent agreement with the coupled-mode method [10,11]. Coppola et al. have presented analytic approach [12] for analysis of the effects of errors in grating period and shape as well as other fabrication errors, which is an extension of the coupled-mode theory. The idea of this analytic extension is to separate the response of the ideal grating and the response of the errors.

All the methods mentioned above are limited either by the assumption of infinite grating structure or by the small-perturbation (shallow structure) hypothesis. Recently, a rigorous method of bidirectional mode expansion and propagation (BEP) [13,14] has been extended for efficient modeling of periodic structures by implementation of the Floquet theorem [15,16]. The BEP method can be cast into a consistent implementation of the mode-matching method for waveguide structures with strongly corrugated Bragg gratings. Also, some finite-difference beam propagation methods have been applied to the analysis of waveguide gratings [17], but these methods are still computationally rather inefficient, especially if the waveguide grating contains a large number of periods.

In this paper we introduce a computationally efficient quasi-rigorous method for analyzing corrugated Bragg gratings. It has been developed for the simulation of silicon-on-insulator (SOI) waveguide gratings [18–20], but it is applicable to other grating structures as well. The method is based on rigorous diffraction theory of gratings [21] and it is described in detail in Section 2. The comparison between the introduced method and thin-film stack method is given in Section 3, where we also demonstrate that the thin-film stack method cannot predict the reflectance of Bragg gratings with strong corrugations. The new method can be applied also to the error analysis of Bragg gratings. Some results for regular, stitching-type errors are given in Section 3.

2. Computational model

Vahimaa and Turunen [22] introduced an idea on how to apply rigorous grating diffraction theory to the analysis of waveguide gratings (see also Ref. [23]). A similar method has been used by Lalanne and Silberstein [24]. This method is highly accurate but computationally inefficient because the transfer of fields through the structure must be calculated sequentially, period by period. In this paper we describe how the S-matrix transfer algorithm with the star product [25,26] can be applied to reduce the computational effort from $\sim N$ to $\sim \log_2 N$, where N is the number of grating periods.

Let us consider the three-dimensional structure illustrated in Fig. 1. We assume that the ridge waveguide is etched by an amount h so that $J/2$ identical periods, where J is an even integer, are

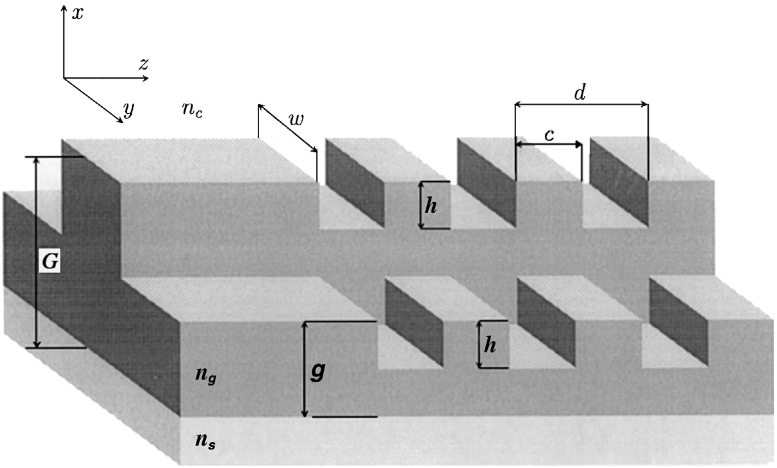

Fig. 1. Three-dimensional structure of an etched ridge-waveguide grating.

formed. If we assume that the constant width w of the guiding part is, at least, a few wavelengths, we may approximate this structure by a two-dimensional structure illustrated in Fig. 2, where n_{eff} is the effective refractive index of the ridge part calculated in the y-direction as described in Ref. [28]. In Fig. 2 we have already assumed that the structure is periodic, with period D, in the x-direction to make the use of the grating theory possible. To minimize the interaction between the fields in adjacent periods the artificial absorbers have to be added between the periods. The optimal selection for an absorber would be an optimized Bérenger layer [29], but in numerical simulations we noticed that Gaussian absorber, which does not require optimization, is sufficient. Moreover, the period D is assumed to be large enough to prevent the evanescent tails of the guided modes from interacting with the absorbers.

As explained in Ref. [21], the field expression in the jth layer in the case of TE-polarization is of the form

$$E_j(x,z) = \sum_{m=1}^{\infty} \left\{ a_m^j \exp\left[\mathrm{i}\gamma_m^j (z - z_j) \right] + b_m^j \exp\left[-\mathrm{i}\gamma_m^j \left(z - z_{j+1}\right) \right] \right\} X_m^j(x), \tag{1}$$

where a_m^j and b_m^j are the unknown amplitudes of the mth mode propagating in the positive and the negative z-directions, respectively, and γ_m^j are the propagation constants of the modes obtained by solving the eigenvalue equations as explained in Ref. [21]. Here the function $X_m^j(x)$ represents the transversal distribution of the mth mode. Applying the periodicity, $X_m^j(x)$ may be expanded as a Fourier series

$$X_m^j(x) = \sum_{q=-\infty}^{\infty} P_{mq}^j \exp(\mathrm{i}2\pi q x / D), \tag{2}$$

with the qth Fourier coefficient given by

$$P_{mq}^j = \frac{1}{D} \int_0^D X_m^j(x) \exp(-\mathrm{i}2\pi q x / D)\, \mathrm{d}x. \tag{3}$$

These coefficients P_{mq}^j are actually solved eigenvector coefficients from the eigenvalue problems, see Ref. [21]. In rigorous grating diffraction theory, the input and output regions are assumed to be homogeneous materials. However, in our model for waveguides both the input and output regions are periodically modulated, and the field expressions in these regions are similar to Eq. (1):

$$E_0(x,z) = \sum_{m=1}^{\infty} \left\{ a_m^0 \exp\left(\mathrm{i}\gamma_m^0 z \right) + b_m^0 \exp\left[-\mathrm{i}\gamma_m^0 (z - z_1) \right] \right\} X_m^0(x) \tag{4}$$

in the input region and

$$E_{J+1}(x,z) = \sum_{m=1}^{\infty} a_m^{J+1} \exp\left[\mathrm{i}\gamma_m^{J+1} (z - z_{J+1}) \right] X_m^{J+1}(x) \tag{5}$$

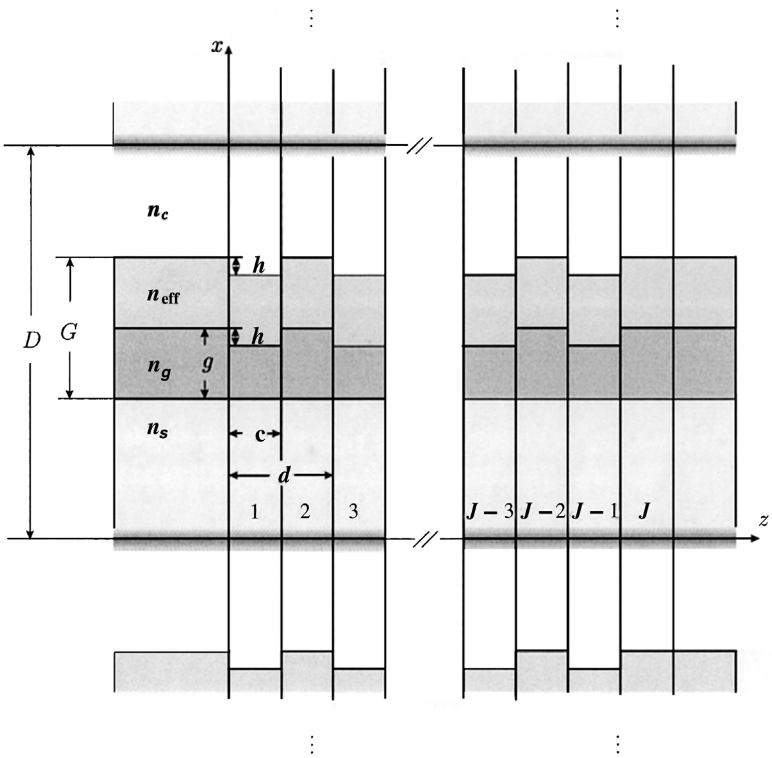

Fig. 2. Two-dimensional approximation of the geometry in Fig. 1.

in the output region. We note here that in Eq. (4) the first term in summation represents incoming field and the second term represents reflected field.

To solve the mode amplitudes a_m^j and b_m^j we use the requirement of the continuity of the electric field as well as its z-derivative at the boundary between the layers j and $j+1$, which yields

$$\begin{bmatrix} \boldsymbol{a}_{j+1} \\ \boldsymbol{b}_j \end{bmatrix} = \mathbf{S}_j \begin{bmatrix} \boldsymbol{a}_j \\ \boldsymbol{b}_{j+1} \end{bmatrix}, \tag{6}$$

where

$$\boldsymbol{S}_j = \begin{bmatrix} \mathbf{P}_{j+1} & -\mathbf{P}_j \\ \mathbf{P}_{j+1}\boldsymbol{\Gamma}_{j+1} & \mathbf{P}_j\boldsymbol{\Gamma}_j \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{P}_j\mathbf{E}_j & -\mathbf{P}_{j+1}\mathbf{E}_{j+1} \\ \mathbf{P}_j\boldsymbol{\Gamma}_j\mathbf{E}_j & \mathbf{P}_{j+1}\boldsymbol{\Gamma}_{j+1}\mathbf{E}_{j+1} \end{bmatrix}. \tag{7}$$

The elements of the column vectors $\boldsymbol{a}_j$ and $\boldsymbol{b}_j$ are a_m^j and b_m^j, respectively. Here $\boldsymbol{\Gamma}_j$ and $\mathbf{E}_j$ are diagonal matrices with elements γ_m^j and $\exp[\mathrm{i}\gamma_m^j(z_{j+1} - z_j)]$, respectively, while the elements of the matrix $\mathbf{P}_j$ are defined in Eq. (3). Therefore, all the matrices $\mathbf{S}_j$ can be formed once the eigenvalue problem of each layer is solved. One should notice that Eq. (7) is valid also at $z = z_1$ and $z = z_{J+1}$. In the latter case, however, $\mathbf{b}_{J+1} = 0$.

If only the amplitudes $\boldsymbol{b}_0$ and $\boldsymbol{a}_{J+1}$ are of interest, one may combine the matrices $\mathbf{S}_j$ by using Redheffer's star product [30] as described by Li [26]:

$$\begin{bmatrix} \boldsymbol{a}_{J+1} \\ \boldsymbol{b}_0 \end{bmatrix} = \mathbf{S}_0 * \mathbf{S}_1 * \cdots * \mathbf{S}_J \begin{bmatrix} \boldsymbol{a}_0 \\ 0 \end{bmatrix}. \tag{8}$$

Here the star product for $2N \times 2N$ matrices $\mathbf{C}$ and $\mathbf{D}$ is defined as

$$\begin{aligned} \mathbf{C} * \mathbf{D} &= \begin{bmatrix} \mathbf{c}_{11} & \mathbf{c}_{12} \\ \mathbf{c}_{21} & \mathbf{c}_{22} \end{bmatrix} * \begin{bmatrix} \mathbf{d}_{11} & \mathbf{d}_{12} \\ \mathbf{d}_{21} & \mathbf{d}_{22} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{d}_{11}(\mathbf{I} - \mathbf{c}_{12}\mathbf{d}_{21})^{-1}\mathbf{c}_{11} & \mathbf{d}_{12} + \mathbf{d}_{11}\mathbf{c}_{12}(\mathbf{I} - \mathbf{d}_{21}\mathbf{c}_{12})^{-1}\mathbf{d}_{22} \\ \mathbf{c}_{21} + \mathbf{c}_{22}\mathbf{d}_{21}(\mathbf{I} - \mathbf{c}_{12}\mathbf{d}_{21})^{-1}\mathbf{c}_{11} & \mathbf{c}_{22}(\mathbf{I} - \mathbf{d}_{21}\mathbf{c}_{12})^{-1}\mathbf{d}_{22} \end{bmatrix}, \end{aligned} \tag{9}$$

where $\mathbf{c}_{mq}$ and $\mathbf{d}_{mq}$ are $N \times N$ submatrices and $\mathbf{I}$ is an unit matrix.

If the structure is periodic so that layers j and $j+2$ are identical, we have $\mathbf{S}_j = \mathbf{S}_{j+2}$ for $j = 1, 2, \ldots, J-3$. Using the associativity of the star product, we may express Eq. (8) as

$$\begin{bmatrix} \boldsymbol{a}_{J+1} \\ \boldsymbol{b}_0 \end{bmatrix} = \mathbf{S}_0 * [(\mathbf{S}_1 * \mathbf{S}_2) * \cdots * (\mathbf{S}_{J-3} * \mathbf{S}_{J-2})] * \mathbf{S}_{J-1} * \mathbf{S}_J \begin{bmatrix} \boldsymbol{a}_0 \\ 0 \end{bmatrix} \tag{10}$$

$$= \mathbf{S}_0 * \mathbf{Z}^{(J-2)/2} * \mathbf{S}_{J-1} * \mathbf{S}_J \begin{bmatrix} \boldsymbol{a}_0 \\ 0 \end{bmatrix}, \tag{11}$$

where $\mathbf{Z} = \mathbf{S}_1 * \mathbf{S}_2$ and $\mathbf{Z}^n$ is an involution in terms of Eq. (9).

Due to the involution the computational effort may be reduced significantly. However, many different approaches to the evaluation of powers exist. Probably the most straightforward method (excluding the direct computation) is known as the binary method [31]. Because this method is fully explained in elementary textbooks of computational mathematics, we discuss it here only briefly.

The loop for calculating $\mathbf{C} = \mathbf{Z}^n$ consists of the following steps:

1. Initialize: Set $\mathbf{C} \leftarrow \mathbf{I}$. If $n = 0$, output $\mathbf{C}$ is the answer and one terminates the algorithm. Otherwise, set $p \leftarrow n$ and $\mathbf{D} \leftarrow \mathbf{Z}$.
2. Multiply: If p is odd, set $\mathbf{C} \leftarrow \mathbf{D} * \mathbf{C}$ and $p \leftarrow p - 1$.
3. Halve p: Set $p \leftarrow p/2$. If $p = 0$, the output is $\mathbf{C}$ and the algorithm is terminated. Otherwise, set $\mathbf{D} \leftarrow \mathbf{D} * \mathbf{D}$ and return to step 2.

This method requires the calculation of no more than $2 \log_2 n + 1$ star products and is thus superior to direct computation, especially with large n. For example, with $n = 29\,999$, as in the examples given later in this paper, evaluation of only 24 star products is needed. Of course, three additional products must be calculated when combining the. remaining matrices and one more is needed to form $\mathbf{Z}$. Hence, in this case, a total of 28 star products is needed. One should notice that, by using only values $n = 2^m$, where m is an integer, the number of matrix multiplications can be further reduced remarkably.

The binary method represented here does not always give the smallest possible number of multiplications. However, usually the difference in the number of calculations between different power-raising methods is relatively small and the determination of the best available method is not a simple task. For extensive discussion of the evaluation of powers with different methods, see Ref. [32].

3. Analysis and design results

As the first example we consider the back-reflection efficiency of a SOI waveguide Bragg grating [20] as a function of the wavelength. We assume that the waveguide is silicon on silica, with refractive indices $n_g = 3.48$, $n_c = 1$ (air) and $n_s = 1.46$, respectively. The dimensions of the waveguide and the grating are as follows: width $w = 7$ μm, height $G = 10$ μm, $g = 5$ μm, period $d = 220$ nm, and filling factor $c/d = 0.5$; see Figs. 1 and 2 for the notation. In the two-dimensional approximation used here, the refractive index of the ridge part can be replaced by the effective index calculated in the transverse (y) direction. For the given dimensions $n_{\text{eff}} \approx 3.4782$, i.e., very close to the refractive index of the core. The analysis is carried out for the TE-polarization of the ridge waveguide. However, one should notice that in the selected geometry the effective index of the ridge part must be calculated for TM-polarization.

In the following we will compare results of our quasi-rigorous method with the film-stack method. The latter is based on the calculation of multiple reflections and transmissions from a stack of uniform homogeneous layers, where the refractive index of each layer is the effective index of the fundamental mode in the corresponding waveguide cross-section. In the film-stack method the refractive indices of the both input and output regions are replaced by the effective index of the unmodulated waveguide. We have implemented our film-stack method as described in Ref. [33]. The results of these two methods are shown in Fig. 3 for the grating of 30 000 periods. The reflection peak of

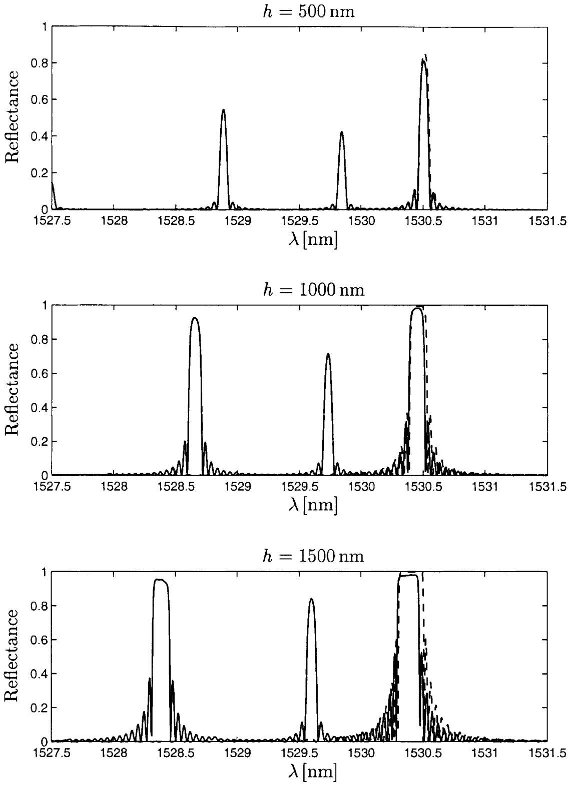

Fig. 3. Reflectances of waveguide Bragg gratings with three different etch depths h, calculated by the method presented here (solid lines) and by the thin-film stack method (dashed lines).

the fundamental guided mode is predicted very accurately by the thin-film stack method, but the efficiency of the modal method is somewhat lower because of the correctly handled losses. Moreover, the modal method predicts additional reflection peaks at shorter wavelengths. These peaks arise because of wavelength-dependent coupling of energy from the fundamental mode to higher-order modes. The wavelength of the reflected higher-order modes can be solved approximately from $\beta = (\beta_0 + |\beta_m|)/2$, where m is the number of the order and $\beta = 2\pi n_{\mathrm{eff}}/\lambda$ [27].

In addition to the above-considered relatively shallow structures we calculated reflection spectra also for strongly corrugated waveguides. As shown in Fig. 4, the difference between the modal method and the thin-film stack method is remarkable for such structures. By comparing Figs. 3 and 4 we

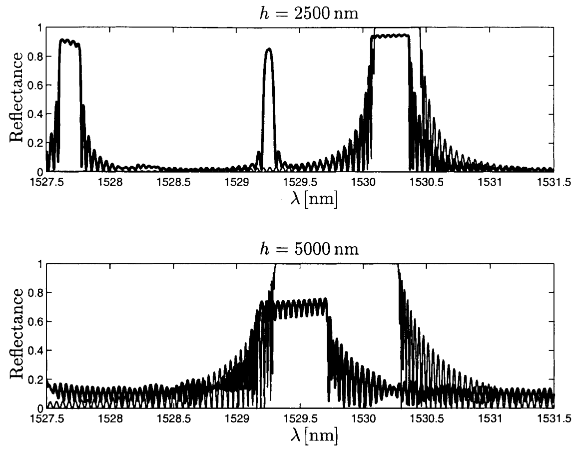

Fig. 4. Same as Fig. 3, but with deeper etch depths. Note, here thick line: presented method, thin line: thin-film stack method.

notice that the reflectance peak is shifted towards lower wavelengths when the etch depth is increased. This phenomenon originates from the fact that etching alters the average effective index of the waveguide.

In addition to the calculations mentioned above, we determined the effect of stitching errors, which typically occur in fabrication of waveguide Bragg gratings by electron beam lithography. We assumed that these errors take place with every 454th period and that they are identical to each other. In computations we can first construct one structure with the error and then multiply these basic blocks together by applying the matrix involution rule. Thus, these regular errors increase the computational effort only by few matrix multiplications. The etch depth used in these calculations was 1.5 μm. The results for two different errors, $\Delta = -10$ nm and $\Delta = +20$ nm (see Fig. 5 for notation), are illustrated in Fig. 6. By comparing Figs. 3 and 6 we immediately see that the reflectance peak of the fundamental guided mode

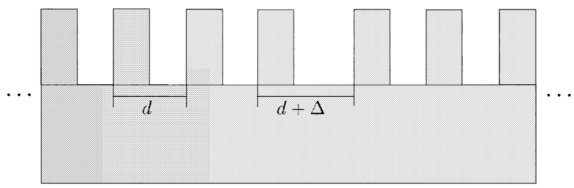

Fig. 5. Definition of the stitching error Δ used in this article.

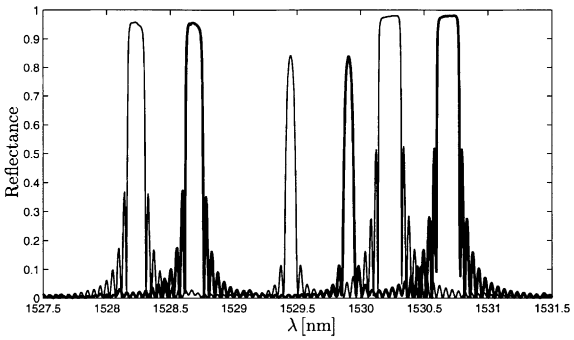

Fig. 6. Same as Fig. 3, but with stitching errors −10 nm (thin line) and +20 nm (thick line).

is shifted towards either shorter or longer wavelengths, depending on the sign of Δ.

4. Conclusions

We have presented a computationally efficient method for the analysis of corrugated waveguide structures, like Bragg gratings. The efficiency of the method is based on the possibility to apply the Redheffer's star product. By using the star product it is possible to construct the electromagnetic field transfer problem in a form of a matrix involution. This matrix involution is then solved using a so-called binary method which enables the solution of matrix involution in a basis of the powers of two. The approach reduces the computational effort of the rigorous analysis method to $\sim\log_2 N$, where N is the number of periods in a Bragg grating. The extension of the rigorous method for the analysis of the three-dimensional waveguide structures is straightforward, but mainly due to the computer memory limitations, it is not reasonable at this moment.

In this paper we have shown that the thin-film stack method gives rather accurate results when the corrugation of the waveguide is shallow. However, when the groove depth of the corrugation exceeds two tenths of the waveguide thickness, the results given by the thin-film stack method differ remarkably from the results given by the quasi-rigorous approach. One of the main benefits of the new numerical method is that it enables the analysis of coupling of light from the fundamental mode into higher-order modes. We have shown that this coupling, which occurs at shorter wavelengths than the fundamental-mode Bragg reflection, can be very strong. Thus, in wavelength-division-multiplexing systems this coupling will determine the free spectral range of the system. Furthermore, we have demonstrated that the presented quasi-rigorous method is a suitable tool for the analysis of regular, stitching type errors. Method can also be used for the analysis of other fabrication errors, like undercutting or tilted sidewalls. In these cases one has to divide a single period up into thin layers and then proceed as with the multilayer structure in rigorous diffraction problems. When the problem is solved for the single period, the star product and matrix involution can be applied as described in this paper.

Experimental work on the demonstration of the results is underway and we expect to publish characterization results of fabricated Bragg gratings on silicon waveguides elsewhere.

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