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Setting "Condition of Order Preservation" Requirements for the Priority Vector Estimate in AHP is not Justified

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Abstract

In this paper, we demonstrate that the "Condition of Order Preservation" proposed by Bana e Costa and Vansnick (2008) for the priority vector in AHP is based on the very restrictive assumption that only the direct (explicit) pairwise judgments matter in estimating the priority vector. Bana e Costa and Vansnick (2008) criticize the use of the eigenvalue method as an estimation technique because the derived priority vector at times may not satisfy the "Condition of Order Preservation" requirement. Their criticism, however, is not justified because the consistency measure of the AHP provides additional implicit information about the preference ratios that has to be taken into account and Saaty's (1977) eigenvalue method does this. In all the examples in this paper the priority vectors are estimated using both the eigenvalue method and the regression technique. The results are quite similar.

Keywords: Regression, Eigenvalue Method, Analytic Hierarchy Process, Estimation of Priority Vector, Decision Analysis.

1 Introduction

Since Saaty developed the Analytic Hierarchy Process (AHP) (see, e.g. Saaty, 1980), it has become a very popular decision support system, which is also widely applied. From the ISI Web of Knowledge database (February 6, 2008) we found 3191 articles (since 1986) using key words: *AHP or Analytic* Hierarch* Process*, and the total number of citations to those articles was 37 071 ("*" refers to a wild character). Correspondingly, the key words: *(AHP or Analytic* Hierarch* Process) and Application** produced 684 articles which were cited 8278 times indicating that there also exist quite many published and cited applications The popularity of AHP has developed very rapidly. In year 2007, there were 5125 citations to the AHP articles mentioned in the ISI Web of Knowledge, database whereas before 1992 there were less than 100 citations/year.

Even though the AHP is widely used, it is also widely criticized. Some of the key features that are critically discussed in literature are rank reversal (see, e.g. Belton and Gear, 1983), the normalization and aggregation rules (see, e.g. Barzilai and Golany, 1994), the scale (see, e.g. Salo and Hämäläinen, 1997; Leskinen, 2001), the method of estimation of priority vectors using the eigenvalue method (see, e.g. Alho, 1996), etc. Quite recently Bana e Costa and Vansnick (2008) presented a new criticism about the eigenvalue method by arguing that its estimates of the priority vector may violate a "Condition of Order Preservation (COP)" requirement, although such estimate exists. Without committing ourselves in the debate about which estimation method is best, we will show that the eigenvalue method works pretty well and provides logical and rational results. The requirement of satisfying the Condition of Order Preservation is based on the restrictive assumption that only the direct pairwise judgments matter in estimating the priority vector. We refer to the direct pairwise judgments as explicit judgments, but there are also implicit judgments for each position in the matrix that are a consequence of the inconsistency of the matrix. When the matrix is consistent the explicit and implicit judgments are the same for each position.

We will show that a very natural reason to cause a violation of the Condition of Order Preservation is a conflict between explicit and implicit preference information. The eigenvalue method takes into account this conflict information and incorporates both explicit and implicit information in the priority vector. In our considerations, we will use examples published in Bana e Costa and Vansnick (2008) and estimate the priority vectors using both the eigenvalue method and regression technique. Because the comparison matrices in the examples are quite consistent, the estimated priority vectors using these two methods are very close to each other. This is an additional indication that the "biased" results referred to by Bana e Costa and Vansnick are not caused by the method used to derive them being in error. Seemingly the "biased results" are simply caused by the inconsistent evaluations of the decision maker (DM). The consistency condition provides implicit information about preference ratios, and we will show that it has to be taken into account in the estimation as well.

In this paper, we go through three examples presented in the paper by Bana e Costa and Vansnick (2008) and show that their criticism is not justified. In Section 2, we briefly review the main principles of estimating the priority vector using the eigenvalue method and regression technique. In Section 3, we discuss the Condition of Order Preservation using an example of our own, and in Section 4, we consider the examples by Bana e Costa and Vansnick (2008) and explain why their "seemingly biased results" appear. Some discussion and concluding remarks are given in Section 5.

2 Some Theory

2.1 The Problem of Estimating the Priority Vector

Let us consider the elements $x_1, x_2, \ldots, x_n$. For convenience, we will refer to element x_i simply by its index *i.* The purpose in AHP (at some hierarchy level) is to find the weights of influence w_1, $w_2, \ldots, w_n$ ($w_i > 0$) for those elements. The vector $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)^T$ is called a *priority vector* and its elements w_i, $i = 1, 2, \ldots, n,$ are referred to as *priorities* or simply *weights*. Let the ordered pair (i, j) denote the comparison of *i* to *j,* and let the relative score of *i* to *j* be $a_{ij} > 0$. In the AHP procedure one forms an $n \times n$ comparison matrix A = (a_{ij}) by first asking the decision maker (DM) to elicit the values a_{ij} for $i < j$, and then defining $a_{ii} = 1$ and $a_{ji} = 1/a_{ij}$ for $i > j$. The matrix A is reciprocal. If $a_{ij}a_{jk} = a_{ik}$, for all i, j, k, then the elicited views are *consistent*, and the matrix is called *consistent* as well. When A is consistent, then it is easy to find weights for which

$$a_{ij} = w_i/w_j \, , \; i, j = 1, 2, \ldots, n. \tag{1}$$

To guarantee uniqueness, we scale $\boldsymbol{w}$ such that $w_1 + w_2 + \ldots + w_n = 1$. Thus

$$A = \boldsymbol{w}\boldsymbol{u}^T, \tag{2}$$

where $\boldsymbol{u} = (1/w_1, 1/w_2, \ldots, 1/w_n)^T$.

In practice, matrix A is not usually consistent, because it is based on subjective judgements on a pre-defined scale. This means that there exists no vector $\boldsymbol{w}$ fully satisfying equation (2). The estimation problem in the AHP is to find the "best" vector $\boldsymbol{w}$ for which

$$A \approx \boldsymbol{w}\boldsymbol{u}^T. \tag{3}$$

As a solution to the estimation problem, Saaty (1977) proposed the use of an eigenvalue method (see, Section 2.2). Another estimation technique is based on the use of the regression technique (see, e.g. Alho et al. (2001) (see Section 2.3). Without evaluating their superiority over each other, the methods would be expected to produce quite similar results if the matrix A is fully or nearly consistent. As a matter of face, in the fully consistent case both estimation methods give exactly the same results, of course.

When the matrix A is not consistent, the consistency condition

$$a_{ij}a_{jk} = a_{ik}, \text{ for all } i, j, k, \tag{4}$$

is not satisfied. When a DM states the preference ratio between elements i and k, (s)he provides explicit information a_{ik}. On the other hand, the expression $a_{ij}a_{jk}$, for all j, $j \neq i$ and $j \neq k$, also contains implicit or indirect information about what that preference ratio should be if the decision maker were to be consistent. That's why it is not correct to set requirements for the preference ratio of the elements of the priority vector only on the basis of the explicit preference ratio. There is a single explicit judgment given for a_{ik}, and it seems it should not be considered more representative of the underlying reality of the DM than the multiplicity of implied judgments.

To refer to an implicit preference ratio through j, we write $\wp_j(x_i, x_k)$. The notation $\wp(x_i, x_k)$ is used when we refer to a preference ratio without specifying whether it is explicit or implicit.

2.2 Estimating Priorities by Using an Eigenvalue Method

When a matrix A is consistent, then the solution vector $\boldsymbol{w}$ in (1) satisfies the equation

$$\mathrm{A}\boldsymbol{w} = n\boldsymbol{w} \tag{5}$$

This formula means that $\boldsymbol{w}$ is an eigenvector of A corresponding to eigenvalue n. The vector $\boldsymbol{w}$ is scaled such that $w_1 + w_2 + \ldots + w_n = 1$. When equation (5) does not hold, that is, when the matrix A is inconsistent, Saaty (1977) proposed the use of the eigenvalue method to estimate the vector $\boldsymbol{w}$. The principal eigenvector $\boldsymbol{z}$ that corresponds to the principal eigenvalue of A, λ_{max}, satisfies the equation

$$\mathrm{A}\boldsymbol{z} = \lambda_{max}\boldsymbol{z} \tag{6}$$

and can be used as an estimate for $\boldsymbol{w}$ ($w_i > 0$ and $w_1 + w_2 + \ldots + w_n = 1$). The value w_i can then be interpreted as the relative value of element i, because for a consistent set of elicited views we know that $\Sigma_j a_{ij}w_j = nw_i$. In fact, one can prove that in general $\lambda \geq n$ with the equality condition being satisfied if and only if the comparisons are exactly consistent (Saaty 1977). In addition, $\lambda_1 + \lambda_2 + \ldots + \lambda_n = n$. Moreover, it is known that small changes in A cause small changes in the eigenvalues and furthermore small changes in the eigenvectors as well. Saaty's (1977) proposal to use the eigenvector corresponding to the largest eigenvalue as the priority vector estimate is based on these facts.

The deviation of λ_{max} from n can be used to evaluate the closeness of the derived priority vector $\boldsymbol{w}$ to the "real", though unknown, underlying priority vector. Saaty (1977) proposed the use of the formula

$$(\lambda_{max} - n)/(n - 1) \tag{7}$$

as the indicator of "closeness to consistency", and called it the *consistency index* (C.I.). If one takes the ratio of the consistency index for the given matrix A to the average consistency index of a large number of randomly generated matrices using the same scale to make the judgments, a *consistency* *ratio* (C.R.) is obtained. The result is considered acceptable when the C.R.≤ 0.10, (Saaty, 1980).

2.3 Estimating Priorities by Using Least Squares Technique

In Saaty and Vargas (1984), the authors discussed other methods for estimating the priority vector including least squares and logarithmic least squares. The regression technique was introduced by De Jong (1984), and further studied by Crawford and Williams (1985), Carriere and Finster (1992), and Alho et al. (1996), among others. Alho et al. (1996) has further analyzed its statistical properties. We provide a brief introduction to the use of the logarithmic least squares technique as one of the estimation techniques. Our discussion primarily follows Alho et al. (1996, 2001), where many additional details and extensions can be found.

Suppose the weight of the true influence of element i is $v_i = \exp(\mu + \beta_i)$, $i = 1, 2, \ldots$, n. Thus the relative value of i to j is $\exp(\beta_i - \beta_j)$. Suppose the actual elicited relative preference a_{ij} is of the form $a_{ij} = \exp(\beta_i - \beta_j + \varepsilon_{ij})$, where the expectations of the random errors $E(\varepsilon_{ij}) = 0$. Note that in this formulation we think of error in relative terms. Defining $y_{ij} = \log(a_{ij})$, we get the regression model:

$$y_{ij} = \beta_i - \beta_j + \varepsilon_{ij} \quad \text{for } i < j,\ j = 1, 2, \ldots, n \tag{8}$$

For identifiability, one of β_i has to be set 0. We assume that the errors are uncorrelated and homoscedastic with $\sigma^2(\varepsilon_{ij}) = \sigma^2$. Thus the least squares solution for the estimates b_i for the regression coefficients β_i. is optimal. The parameter μ is defined as given below such that

$$\exp(\mu)(\exp(b_1 + b_2 + \ldots + b_n) = 1.$$

When the regression technique is used to estimate the priority vector, statistical theory provides tools for evaluating “closeness to consistency”. A multiple determination of R^2 and various tests are available for this purpose.

2.4 Comparing the Methods

When the matrix A is almost consistent, both methods obviously should produce quite similar estimates for the priority vector. When the matrix A is very inconsistent, the estimated vectors may be much different. The properties of the estimator vector using the eigenvalue method are not known as well as they are using the regression method provided that the underlying assumptions are fulfilled: $E(\varepsilon_{ij}) = 0$ and the errors are uncorrelated and homoscedastic.

The eigenvector method also requires that all $n(n - 1)/2$ comparisons be complete.[1] Completeness is not required using the regression technique.

Because in this paper all matrices in the examples below are almost consistent, we may expect that the results of the two estimation methods should be quite similar. No further discussion about the mutual advantages or disadvantages of the estimation techniques themselves is required.

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[1] If not all comparisons are available, one way is to replace missing values by 1, or to use the consistency condition in the matrix to replace missing values. Assume we have compared elements i to j, and j to k, but the evaluation of i to k is missing. Then we could use the value $a_{ij}a_{jk}$ as an approximation for a_{ik}.

3 Questioning the Condition of Order Preservation for a Priority Vector

In their paper, Bana e Costa and Vansnick (2008) argue that a necessary requirement for the estimated priority vector is that it has to satisfy a so-called "Condition of Order Preservation (COP)" when it is possible. Moreover, they question the use of the eigenvalue method, because priority vectors obtained by using it do not necessarily satisfy the condition.

The condition of order preservation may be defined as follows:

Definition 1. Let $x_1, x_2,\ldots, x_n$ be the set of elements being evaluated, and let *i, j, k,* and *h* be indices referring to those elements. The priority vector $\boldsymbol{w}$ preserves the condition of order preference if the order intensity of preference $w_i/w_j > w_k/w_h$ is true for all *i, j, k,* and *h* for which $a_{ij} > a_{kh}$, where the a_{ij} for all $i,j = 1,\ldots n$ are the judgments given by the decision maker in A, the pairwise comparison matrix.

The condition of order preservation assumes that only explicit pairwise information matters in the estimation of a priority vector. However, this is not a well justified assumption. Let us consider the following example (Table 1), where the information would be fully consistent, if 9 is replaced by 7.5 and 8 by 3.75 in the last column. For the fully consistent table we find the priority vector is $\boldsymbol{w}^g = (0.508, 0.254, 0.169, 0.068)^T$. According to Bana e Costa and Vansnick (2008) it is also an acceptable estimate for the priority vector of the original information in Table 1, because it satisfies the COP.

Table 1: A Pairwise Comparison Table

	x_1	x_2	x_3	x_4
x_1	1.00	2.00	3.00	9.00
x_2	0.50	1.00	1.50	8.00
x_3	0.33	0.67	1.00	2.50
x_4	0.11	0.13	0.40	1.00

When we estimate the priority vector using the eigenvalue method, we get the vector $\boldsymbol{w}^e$ = (0.497, 0.292, 0.159, 0.051), which does not satisfy the COP as seen from Table 2 which is constructed by forming ratios from the vector $\boldsymbol{w}^e$.

Table 2: Computed Pairwise Comparisons Using Vector $\boldsymbol{w}^e$

Priority Vector $\boldsymbol{w}^e$		x_1	x_2	x_3	x_4
0.497	x_1	1	1.70	3.13	9.75
0.292	x_2	0.59	1	1.84	5.73
0.159	x_3	0.32	0.54	1	3.12
0.051	x_4	0.10	0.17	0.32	1

Bana e Costa and Vansnick (2008) do not regard $\boldsymbol{w}^e$ as a feasible estimate, because $w^e_1/w^e_2 < w^e_2/w^e_3$, but originally $a_{12} > a_{23}$.[2] Vector $\boldsymbol{w}^e$ does not satisfy the condition of order preservation.

[2] To refer to numerical information presented in the tables, we use matrix notation.

Actually, satisfying the COP is not necessary if one considers the implicit information in the Table 1 as we shall show below.

Let us consider the kinds of preference information Table 1 contains for elements x_1 and x_2, and further for elements x_2 and x_3. The element a_{12} = 2 provides explicit information, the decision maker's judgment, about how strongly element x_1 is preferred to x_2. But there are other evaluations, such as those involving x_4, that give implicit preference information about x_1 and x_2. When we multiply the element a_{14} = 9 by a_{42} = 0.13, we get an approximation for the preference ratio: $\wp_4(x_1, x_2) = a_{14}a_{42} = 1.13$. Thus there is implicit pairwise information from the consistency condition obtained from considering judgments involving x_4 that tells us the preference ratio between x_1 and x_2 is actually smaller than the DM's explicit information 2. The decision maker is inconsistent. But which judgment is farthest off from the "real" situation? Is it a_{12} or a_{14} or a_{42}? In the same way, we can see that the implicit information: $\wp_4(x_2, x_3) = a_{24}a_{43} = 3.2$ for x_2 and x_3 is greater than the explicit information a_{23} = 1.5. In the light of implicit information it is quite understandable that $w^e_1/w^e_2 < a_{12}$ and $w^e_2/w^e_3 > a_{23}$. We believe that whether the condition of order preservation is satisfied or not should depend on all the preference information available in the comparison matrix. We cannot consider only explicit information.

4 Examples in Bana e Costa (2008)

In their paper, Bana e Costa and Vansnick (2008) present four examples supporting their argument that the priority vector does not satisfy the condition of order preservation, even if such vector exists in three cases. We revisit these three examples and explain why such a requirement is not reasonable. In addition, we demonstrate that the estimates produced by the regression technique also violate the COP.

4.1 Example 1

Table 3: Original Evaluations for Example 1

	x_1	x_2	x_3	x_4	x_5
x_1	1	2	3	5	9
x_2	1/2	1	2	4	9
x_3	1/3	1/2	1	2	8
x_4	1/5	1/4	1/2	1	7
x_5	1/9	1/9	1/8	1/7	1

First we estimate the priority vector from the data given in Table 3 using the Eigenvalue Method (EM) and Ordinary Least Squares (OLS) method. The results are given in Table 4.

Table 4: Estimates of the Priority Vector obtained using different Methods

Method	w_1	w_2	w_3	w_4	w_5
EM	0.426	0.281	0.165	0.101	0.027
OLS	0.424	0.284	0.169	0.098	0.026

Using the priority vector estimated using the Eigenvalue Method, we compute the estimated preference ratios for all pairs of the elements (Table 5). Bana e Costa and Vansnick (2008) considered the result unacceptable, because $w_1/w_4 = 4.22 > w_4/w_5 = 3.74$, but $a_{14} = 5 < a_{45} = 7$ in Table 3.

Table 5: Ratios Computed by Using the Priority Vector Estimated by the Eigenvalue Method

Priority Vector $\boldsymbol{w}$		x_1	x_2	x_3	x_4	x_5
0.426	x_1	1	1.52	2.58	4.22	15.78
0.281	x_2	0.66	1	1.70	2.78	10.41
0.165	x_3	0.39	0.59	1	1.63	6.11
0.101	x_4	0.24	0.36	0.61	1	3.74
0.027	x_5	0.06	0.10	0.16	0.27	1

In Table 6, we have analyzed the implicit preference information in addition to the explicit information for x_1 versus x_4, and x_4 versus x_5 in the original data in Table 3. For instance, from the consistency relation passing through x_2 we get $\wp_2(x_1, x_4) = a_{12}a_{24} = 2*4 = 8$. This value is clearly higher than the explicitly stated $a_{14} = 5$. Correspondingly, we can see from Table 6, that the explicit preference $a_{45} = 7$ is considerably different from all the implicit values. If we consider all the implicit information from Table 3, the consistency condition produces various estimated values for $\wp(x_1, x_4)$ and $\wp(x_4, x_5)$ shown in Table 6. The geometric means of these values are $\wp_g(x_1, x_4) = 4.19$ and $\wp_g(x_4, x_5) = 3.26$. These values are pretty close to the estimates the eigenvalue method provides. As a conclusion, we may say that the explicit value $a_{45} = 7$ does not agree very well with the implicit information for a_{45} obtained from the other explicit values in Table 3.

The results can be verified by estimating the priority vector by means of the Ordinary Least Squares (OLS) method (Table 4) by using model (8). The results are on the second row in. As we can see, the results obtained with the EM and OLS methods are quite similar, and the condition of order preservation is not satisfied for the OLS method either as we can see from Table 7.

Table 6: Information in Table 3 about the Evaluated Ratios $\wp(x_1, x_4)$ and $\wp(x_4, x_5)$

	$a_{12}a_{24}$	$a_{13}a_{34}$	a_{14}	$a_{15}a_{54}$	Geometric Mean
$\wp(x_1, x_4)$	8.00	6.00	5.00	1.29	4.19
	$a_{41}a_{15}$	$a_{42}a_{25}$	$a_{43}a_{35}$	a_{45}	
$\wp(x_4, x_5)$	1.80	2.25	4.00	7.00	3.26

Table 7: Ratios Computed by Using the Priority Vector Estimated by the Ordinary Least Squares Method

OLS Priority Vector $\boldsymbol{w}$		x_1	x_2	x_3	x_4	x_5
0.424	x_1	1	1.50	2.52	4.34	16.50
0.284	x_2	0.67	1	1.68	2.90	11.03
0.169	x_3	0.40	0.59	1	1.72	6.55
0.098	x_4	0.23	0.34	0.58	1	3.80
0.026	x_5	0.06	0.09	0.15	0.26	1

Bana e Costa and Vansnick (2008) give an estimate of the priority vector $\boldsymbol{w}$ = (0.385, 0.275, 0.195, 0.125, 0.020), which satisfies COP (Table 8), but only w_4/w_5 = 6.74 is closer to the given value than the corresponding preference ratio in Table 5 and Table 7. (Note that Bana e Costa and Vansnick (2008) do not argue that the priority vector is the best one, but only feasible.)

Table 8: Pairwise Ratios by Using the Priority Vector Proposed by Bana e Costa and Vansnick (2008)

Priority Vector $\boldsymbol{w}$		x_1	x_2	x_3	x_4	x_5
0.385	x_1	1	1.40	1.97	3.08	19.25
0.275	x_2	0.71	1	1.41	2.20	13.75
0.195	x_3	0.51	0.71	1	1.56	9.75
0.125	x_4	0.32	0.45	0.64	1	6.25
0.020	x_5	0.05	0.07	0.10	0.16	1

To demonstrate the meaning of the consistency condition, we assume that the explicit ratio evaluation a_{45} = 7 from Table 7 is correct and other evaluations concerning x_5 are not. According to this assumption, we modify the evaluations concerning x_5 in such a way that information is consistent with a_{45} = 7. Thus we have to make the following modifications: $a_{15} = 9 \rightarrow 35$, $a_{25} = 9 \rightarrow 28$, and $a_{35} = 8 \rightarrow 14$. Using the modified data, we find the OLS priority vector (Table 9) and compute the corresponding ratios from it as shown in Table 9.

Table 9: Priority Vector and Pairwise Ratios with the

OLS Priority Vector $\boldsymbol{w}$		x_1	x_2	x_3	x_4	x_5
0.433	x_1	1	1.47	2.60	4.71	30.91
0.295	x_2	0.68	1	1.77	3.21	21.05
0.166	x_3	0.38	0.56	1	1.81	11.88
0.092	x_4	0.21	0.31	0.55	1	6.57
0.014	x_5	0.03	0.05	0.08	0.15	1

The estimates for all pairwise ratios are good, and COP is satisfied. It is worth mentioning that this priority vector does not deviate very much from the original OLS priority vector, even though we have changed some ratios quite radically. The values in the last column differ very much from those given in the original matrix in Table 3.

4.2 Example 2

Table 10: Original Evaluations of Example 2

	x_1	x_2	x_3	x_4
x_1	1.00	2.50	4.00	9.50
x_2	0.40	1.00	3.00	6.50
x_3	0.25	0.33	1.00	5.00
x_4	0.11	0.15	0.20	1.00

First, we estimate the priority vector by using the eigenvalue method and the ordinary least squares method. As we can see from Table 11, the results are very similar. In both cases, the $w_1/w_3 > w_3/w_4$, even if $a_{13} < a_{34}$. The result is again very understandable, because the explicit evaluation of the ratio $\wp(x_1, x_3)$ deviates very much from the other information in the original evaluation Table 12. If we "correct" the explicit evaluation a_{34} with the value 3, which is closer to the geometric mean and estimate the priority vector, we see that $w_1/w_3 > w_3/w_4$ and $a_{13} > a_{34}$. This modification mainly affects the values of w_1 and w_4.

Table 11: Alternative Priority Vector Estimates

Method	Data	w_1	w_2	w_3	w_4
EM	Original	0.533	0.287	0.139	0.041
OLS	Original	0.535	0.286	0.138	0.041
OLS	Modified ($a_{34} = 5 \rightarrow 3$)	0.459	0.305	0.160	0.076

Table 12: Information in Table 10 about the Evaluated Ratios $\wp(x_1, x_3)$ and $\wp(x_3, x_4)$

	$a_{12}a_{23}$	a_{13}	$a_{14}a_{43}$	Geometric Mean
$\wp(x_1, x_3)$	7.50	4.00	1.90	3.85
	$a_{31}a_{14}$	$a_{32}a_{24}$	a_{34}	
$\wp(x_3, x_4)$	2.38	2.17	5.00	2.95

In summary, as in the previous example the explicit evaluation is inconsistent with the implicit information about the preference ratio between x_3 and x_4.

4.3 Example 3

As their third example, Bana e Costa and Vansnick (2008) consider Saaty's (1980, pp. 40-41) own example, where seven countries are evaluated by using the criterion "Wealth":

Table 13: Evaluation of Seven Countries with Criterion Wealth

	US	USSR	China	France	UK	Japan	W. Germany
US	1.00	4.00	9.00	6.00	6.00	5.00	5.00
USSR	0.25	1.00	7.00	5.00	5.00	3.00	4.00
China	0.11	0.14	1.00	0.20	0.20	0.14	0.20
France	0.17	0.20	5.00	1.00	1.00	0.33	0.33
UK	0.17	0.20	5.00	1.00	1.00	0.33	0.33
Japan	0.20	0.33	7.00	3.00	3.00	1.00	2.00
W. Germany	0.20	0.25	5.00	3.00	3.00	0.50	1.00

When we include from Table 13 both explicit and implicit preference ratios US (x_1) to USSR (x_2) and Japan (x_4) to France (x_6), we obtain the following:

Table 14: Preference Information in Table 13 about the Evaluated Ratios $\wp(x_1, x_2)$ and $\wp(x_6, x_4)$

	a_{12}	$a_{13}a_{32}$	$a_{14}a_{42}$	$a_{15}a_{52}$	$a_{16}a_{62}$	$a_{17}a_{72}$	Geometric Mean
$\wp(x_1, x_2)$	4.00	1.29	1.20	1.20	1.67	1.25	1.58
	$a_{61}a_{14}$	$a_{62}a_{24}$	$a_{63}a_{34}$	$a_{64}a_{54}$	a_{64}	$a_{67}a_{74}$	
$\wp(x_6, x_4)$	1.20	1.67	1.40	3.00	3.00	6.00	2.31

From Table 14 we see that there is "more evidence" that the US is dominant over the USSR by 1.58 and Japan is dominant over France by 2.31, that is the US is relatively less preferred to USSR than Japan to France, even though the explicit information was vice versa (the US is 4.00 over the USSR, while Japan is 3.00 over France, from Table 13). Thus it would seem quite natural that the priority vector should correspond to the first result no matter how it is derived. The pairwise information is computed from the priority vector derived using the Eigenvalue method in Table 15.

Table 15: Ratios Computed by Using the Priority Vector Estimated by the Eigenvalue Method

w		US	USSR	China	France	UK	Japan	W. Germany
0.417	US	1	1.80	20.97	7.80	7.80	3.25	4.34
0.231	USSR	0.55	1	11.64	4.33	4.33	1.81	2.41
0.020	China	0.05	0.09	1	0.37	0.37	0.16	0.21
0.054	France	0.13	0.23	2.69	1	1.00	0.42	0.56
0.054	UK	0.13	0.23	2.69	1.00	1	0.42	0.56
0.128	Japan	0.31	0.55	6.45	2.40	2.40	1	1.33
0.096	W. Germany	0.23	1.80	1.80	0.75	1.00	0.75	1

The priority vector estimates were done in two ways as shown in Table 16 and were quite similar. Note that the ratios for the US versus USSR and Japan versus France, composed from the OLS estimate of the priority vector were quite close to the Geometric Mean results in Table 14.

Table 16: Alternative Priority Vector Estimates

Method	Data	w_1	w_2	w_3	w_4	w_5	w_6	w_{7Q}
EM	Original	0.427	0.230	0.021	0.052	0.052	0.123	0.094
OLS	Original	0.417	0.231	0.020	0.054	0.054	0.128	0.096

5 Discussion and Concluding Remarks

In this paper, we have shown that the requirement of the condition of order preservation (Bana e Costa and Vansnick, 2008) is based on the assumption that only explicit information is relevant in the pairwise comparison matrix. It completely ignores the requirement of the consistency condition $a_{ij}a_{jk} = a_{ik}$, for all *i, j, k,* which plays a key role in determining the true preference ratios. When explicit evaluation a_{ik}, for some pair of elements *i* and *k,* clearly differs from implicit information calculated from $a_{ij}a_{jk}$, for all $j \neq i$ and $j \neq k$, it is clear that the implicit information "weighs" more in estimating the elements w_i and w_k of the priority vector. It means that we cannot set requirements for the elements of vector $\boldsymbol{w}$ on the basis of explicit information only.

As an additional verification of the independence of the estimation method, we used the alternate method of a regression technique along with the eigenvalue method. The results turned out to be quite similar.

The Condition of Order Preservation sounds like a nice property, but it is likely satisfied only when the implicit and explicit information is quite consistent. Or, to put it another way, it means that the whole comparison matrix is quite consistent. When explicit information is very close to the geometric mean, it is very plausible that the Condition of Order Preservation is satisfied.

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