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IMPORTANCE SAMPLING FOR SIMULATING RISK CONTRIBUTIONS OF CORPORATE LOAN PORTFOLIOS

Submitted in partial fulfilment of the requirements for the degree of Master of Science in Technology in the Degree Programme in Engineering Physics and Mathematics.

Espoo, 1 February 2016

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Author: Janne Kunnas	
Title: Importance Sampling for Simulating Risk Contributions of Corporate Loan Portfolios	
Degree Programme: Engineering Physics and Marhematics	
Major Subject: Systems and Operations Reseach	**Minor Subject:** Industrial Management
Chair (code): Mat-2	
Supervisor: Prof Ahti Salo	
Instructor: Ph.D. Matti Tienari	

The Basel II regulatory framework offers a simplified approach for quantifying credit risk of corporate loan portfolios. It is insufficient for banks and financial institutions as it does not take specific portfolio characteristics into consideration. Therefore, internal models are needed. Risk management practitioners are particularly interested in significant, but rare, losses caused by a large number of simultaneous defaults. Monte Carlo simulation models are widely utilized in finance to quantify risk of credit portfolios. But for a rare-event simulation the plain Monte Carlo method is inefficient.

The purpose of this thesis is to determine if the plain method can be improved using importance sampling to produce statistically significant estimates for a real life credit portfolio. We use R programming language and a conventional home office laptop to compute simulations for the portfolio and its individual loans as well. Additionally, we use stock market data to infer the correlation structure of our credit portfolio model. This thesis focuses on a simulation application but a detailed presentation of the theoretical background is provided.

Date: 1 of February 2016	**Language:** English	**Number of pages:** 75

Keywords: Credit risk, importance sampling, Monte Carlo, default correlation, risk contribution, stochastic loss given default, value-at-risk, expected shortfall, normal copula

Aalto-yliopisto
Perustieteiden korkeakoulu

AALTO-YLIOPISTO PERUSTIETEIDEN KORKEAKOULU PL 1100, FI-00076 AALTO http://www.aalto.fi	DIPLOMITYÖN TIIVISTELMÄ	
Tekijä: Janne Kunnas		
Työn nimi: Yritysluottojen riskipääoman simulointi painoarvo-otannalla		
Tutkinto-ohjelma: Teknillinen fysiikka ja matematiikka		
Pääaine: Systeemi- ja operaatiotutkimus	**Sivuaine:** Teollisuustalous	
Opetusyksikön koodi: Mat-2		
Valvoja: Prof. Ahti Salo		
Ohjaaja: FT Matti Tienari		

Basel II vakavaraisuuskehikko tarjoaa yksinkertaisen lähestymistavan yritysluottosalkun luottoriskin mittaamiseen. Se on kuitenkin riittämätön pankeille ja finanssi-instituutioille, koska se ei huomioi luottosalkun erityispiirteitä. Siksi tarvitaan sisäisiä malleja. Riskienhallinnan asiantuntijat ovat erityisen kiinnostuneita harvinaisen suurista tappioista, jotka johtuvat suuresta yhdenaikaisten maksukyvyttömyyksien määrästä. Monte Carlo -simulaatiomallit ovat laajalti rahoitusalalla käytössä luottosalkkujen riskien mittaamisessa. Mutta harvinaisten tapahtumien simuloinnissa tavanomainen Monte Carlo -menetelmä on tehoton.

Tämän työn tarkoitus on selvittää voidaanko tavanomaista Monte Carlo -menetelmää parantaa painoarvo-otannalla tuottamaan tilastollisesti merkitseviä estimaatteja reaalimaailman luottosalkulle. Käytämme R-ohjelmointikieltä ja tavanomaista kannettavaa kotitietokonetta simulaatioiden suorittamiseen koko salkulle sekä yksittäisille luotoille. Lisäksi käytämme osakekurssiaineistoa luottosalkkumallimme korrelaatiorakenteen määritykseen. Työn keskipiste on simulaatiosovelluksessa, mutta myös teoreettiset taustat esitellään yksityiskohtaisesti.

Päivämäärä: 1.2.2016	**Kieli:** Englanti	**Sivumäärä:** 75

Avainsanat: Luottoriski, painoarvo-otanta, Monte Carlo -simulointi, riskipääoma, maksukyvyttömyyskorrelaatio, stokastinen tappio-osuus, value-at-risk, odotettu vaje

Acknowledgements

This thesis was conducted while working in OP Financial Group's Risk Management Department. I would like to express my sincere gratitude to my instructor and colleague Matti Tienari for his pedantic comments and advice.

I also want to thank my supervisor professor Ahti Salo for his time and support towards this thesis and his high quality lectures during the years.

Finally, I would like to thank my family and friends for their support during my studies and writing this thesis.

Helsinki, 2016

Janne Kunnas

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Abbreviations and symbols

Abbreviations

EAD	Exposure at default
EL	Expected loss
ES	Expected shortfall
EC	Economic capital
FIRB	Foundation internal rating based approach
AIRB	Advanced internal rating-based approach
IS	Importance sampling
LGD	Loss given default
PD	Probability of default
UL	Unexpected loss
VaR	Value at risk

Symbols, operators and other notations

A	Lower triangular matrix of Cholesky factorization
a_{kd}	Factor loading of kth obligor for a systematic risk factor d
$B(\alpha, \beta, x)$	Beta-distribution function
c_k	Loss resulting from kth obligor
$Cov[X_i, X_j]$	Covariance of variable X_i and X_j
$E[S]$	Expectation of a random variable S
$E[S\|T]$	Expectation of a random variable S on conditional on T
g	Importance sampling density
$P[S]$	Probability of an event S
p_k	Marginal default probability of kth obligor
$\tilde{p}_k(\theta)$	Exponentially twisted marginal default probability kth obligor
I	Identity matrix
$\tilde{L}$	Loss variable of default event
L	Total loss of a portfolio

m	Number of obligors in a portfolio
$N(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2
q_α	Alpha-quantile of a distribution
R_k	Correlation between obligor and kth systematic factor
R	Pearson correlation coefficient
r	Conditional expectation of portfolio loss
S_i^A	Monthly logarithmic return of company A
T_i^A	Monthly logarithmic return of industry A
$Var[S]$	Variance of a random variable S
w	Likelihood ratio
x_k	Default threshold of kth obligor
Y	Default indicator (1 if default, 0 otherwise)
Y_k	Default indicator for kth obligor (1 if default, 0 otherwise)
Z	Systematic risk factor
Z	Systematic risk factor

$\Gamma(\alpha + \beta)$	Gamma-function
ε_k	Idiosyncratic risk factor
$\hat{\eta}$	Monte Carlo estimator
$\hat{\eta}_g$	Importance sampling Monte Carlo estimator
μ_i	Mean of factor i
μ^*	Optimal importance sampling drift
Σ	Covariance matrix
σ_i^2	Variance of factor i
ρ_{ij}	Correlation between factors i and j
$\Phi(Z)$	Standard normal cumulative distribution function

θ	Twisting parameter of exponential twisting
θ_x	Optimal twist of exponential twisting
$\psi(\theta)$	Cumulant generating function

1 Introduction

Credit risk is risk of loss due to default of a loan by an obligor or decrease of credit worthiness caused by a migration to a lower credit rating. An obligor defaults its loan when it fails to pay its debt or portion of it. Number of defaults in credit portfolio is dependent on the state of economy. Financial institutions suffer greater losses due to defaults in a recession than in other states of economy. Capital requirements, capital held against large future credit losses, for credit risk brought by the Basel Committee do not sufficiently take portfolio characteristics into consideration. Therefore, banks and financial institutions utilize internal models to accurately measure variation of credit portfolio losses and therefore capital required to sustain large losses.

Credit risk models are divided into two categories: reduced-form models and structural models. The CreditMetrics model of J.P Morgan is the most used reduced-form model alongside the CreditRisk+ developed by Credit Suisse Financial Products (Fatemi and Fooladi 2006). The KMV has developed structural model called the Portfolio Manager that was the most used model in 2002 (Smithson *et al* 2002). According to survey of European Central Bank 2007 most central banks use models based on the CreditMetrics framework. Differences and similarities between these models are widely studied. (Crouhy *et al* 2000) (Gordy 2000)

This thesis concentrates on the CreditMetrics framework that can be seen as an extension of Merton's option pricing approach based on a firm's asset value process (Crouhy *et al* 2000). The fundamental idea behind the model is that a change in an obligor's credit quality will affect the risk of the credit portfolio. When the change in downside is substantial the obligor is considered to default it's loan and the lender suffers a portion of the loan value as credit loss. We concentrate on a two-state model of the CreditMetrics meaning that the obligor is either in default state or not. The task comes to simulate the defaults of obligors of the portfolio. As the default events in corporate loan portfolios are rare the plain Monte Carlo method is inefficient for portfolio loss distribution estimation. Not only the rare events but the dependence structure of default events introduces great challenges. Large losses are greatly determined by dependencies between obligors. The tendency of obligors defaulting simultaneously is modelled via common systematic risk factors in the CreditMetrics framework. The model is called the normal copula model. For corporate loan portfolios it is computationally inefficient to generate large portfolio losses with the plain Monte Carlo method.

The purpose of this thesis is to determine whether plain Monte Carlo simulation can be improved with importance sampling to produce statistically significant estimates for portfolio tail probabilities and conditional expectations. Also, it is of great interest to see if conditional expectations can be estimated for individual exposures. To make simulations relevant dependence structure used with the normal model is inferred from real stock market data and portfolio under examination is constructed to imitate a real life corporate loan portfolio.

Thesis is organized as follows. Chapter 2 presents basic measures in defining risk of a corporate loan portfolio and how credit loss of a portfolio and an individual obligor is measured. Chapter 3 presents notations for the normal copula model and how correlated random variables are generated using cholesky factorization.

In Chapter 4 we calibrate our correlation structure using 311 publicly traded corporations listed in stock markets in Helsinki, NASDAQ OMX Helsinki, and Stockholm, NASDAQ OMX Stockholm. Common systematic factors, industry factors, of the normal copula model are defined as seven different GICS industries. In this thesis we use equity correlations inferred from stock market returns to describe the dependencies between obligors. A real life credit portfolio is constructed in chapter 4 assigning three different credit rating to every corporation that was used in defining correlation structure. The total sum of obligors in the real life portfolio is then 933.

Large corporate loan portfolios that consist of hundreds or thousands of transactions compose challenges to simulation. It is practically impossible to estimate tail probabilities or individual risk contributions without incorporating variance reduction methods into plain Monte Carlo simulation. Applying importance sampling is described in detail in chapter 5. Two different methods are presented. Exponential twisting increases default probabilities to generate default events more frequently. Factor shifting shifts expectation of portfolio loss distribution to a desired α-quantile to make simulation more efficient.

Risk measures value-at-risk and expected shortfall are presented in chapter 6. Addition to plain Monte Carlo method, chapter 7 presents three different importance sampling Monte Carlo algorithms to estimate portfolio tail probabilities and conditional expectations. For internal risk management purposes measuring credit risk of portfolio is just the first the step. Banks and financial institutions are also interested in allocating capital to single transactions. Therefore, decomposition of total portfolio credit risk, portfolio loss, to individual transactions is required. Definition of conditional expectation for a marginal risk contribution is provided in chapter 7.

Simulation results for the real life credit portfolio constructed in the chapter 4 are presented in chapter 8. First, we estimate tail probabilities and conditional expectations followed by marginal risk contribution estimation for individual exposures. Lastly, we will find out if our model calibrated for deterministic cost of default will be useful with stochastic cost of default.

Contrary to the Basel II regulatory capital which assumes infinite granularity of the portfolio, the credit risk model presented in this thesis takes concentration risk into consideration. For example, relatively large exposures exhibit greater capital charges relative to their exposure. All simulations are done in R programming language using home office laptop. Therefore, no excess computing power is employed.

2 Measuring credit risk

Quantifying credit risk of a portfolio begins from the individual obligor level. Each obligor and its loan has following risk characteristics: credit rating, collateral and its seniority, the amount of exposure and dependence on common economic factors. Credit rating describes the *creditworthiness* of a firm, an obligor, and in technical terms it is translated into firm's probability of not paying back its loan, *probability of default*. Collateral describes the risk profile of the individual loan and it secures portion of it in the event of default. The seniority of the collateral refers to the order of repayment when obligor has defaulted its loan. It transfers into *loss given default*, the portion of loan that borrower suffers when obligor defaults. The size of the exposure is the amount of capital that the obligor owes to the borrower at default, *exposure at default*.

The dependency on common economic factors measures the impact of a state of economy to the firm's ability to repay its loans. Firms are more likely to default their loans during economic downturn. Firms' simultaneous tendency of defaulting loans called *default correlation*.

2.1 Credit rating and probability of default

A credit rating describes creditworthiness of a firm. The rating is based on qualitative and quantitative assessment of credit quality of the firm. Altman (1968) introduced the Z-score approach to measure credit quality and applied it to manufacturing corporations. The Z-score is based on the assumption that past accounting information can be used to estimate the default probability of a firm. In the United States most issuers of public debt are rated by rating agencies Moody's, S&P and Fitch. In Europe this is not usually the case. Therefore banks need to have their own internal rating systems to measure creditworthiness of their customers. Commonly used financial information in credit scoring models is (Bluhm *et al* 2003) future earnings and cash flows, short- and long-term liabilities and financial obligations, a debt to equity ratio, liquidity of firm's assets. Political and social situation of the firm's home country and conditions of the market of the firm's main activities are usually taken into consideration. Often the quality of the firm's management and the general company structure is review for scoring.

Quantitative credit scorings models provide statistical analysis of the credit quality of a firm. The most import explanatory variables are found to be financial ratios measuring profitability, leverage and liquidity (Allen *et al* 2004). The best practice in banking is that an automatic credit scoring produced by statistical models are re-evaluated by a rating specialist before granting the final credit grade. The most commonly used credit scoring models are multivariate scoring models the linear regression probability model, the logit model, the probit model and the multiple discriminant analysis model. These are parametric models aiming to score explanatory variables to describe creditworthiness. The credit grade of a firm is related to probability of default PD using historical observed default rates. This is called default probability calibration among practitioners. (de Servigny and Renault 2004)

2.2 Loss given default

The portion of a loan that a bank suffers as a credit loss in the event of default is called *loss given default* LGD. Often banks require collateral or other guarantees from borrowers to secure repayment of loans. Collateral is a borrower's pledge of specific property to a lender. A firm investing a new production plant can pledge the plant to the lender. In the event of default, the lender liquidates the plant and receives the capital. The notional amount of a loan minus capital reclaimed as a results of liquidating the collateral bank considers as a credit loss. Key determinants of loss given default are the seniority of the instrument, available collateral or guarantees, the industry of the obligor, the current business cycle and the bargaining power of debt holders. The most important drivers of these are the quality and the quantity of the collateral and the seniority. The seniority refers to the order of repayment in the event of default. Collateral consists of assets that servers as a guarantee in the event of default.

Usually in commercial applications for measuring credit risk the probability of default and the loss given default are assumed to be independent of each other. However, empirical evidence suggests that there could be some co-movement. Altman *et al* (2001) showed that high default rates are linked to high loss given default rates because macroeconomic factors have similar influence on default rates and loss given default.

Another link between default rates and realized LGD is found by examining the value of collateral in recession. Frey (2000) showed that in an economic downturn default rates are relatively high and that collateral values seem to decline as it is difficult to liquidate assets such as real estate due to imbalance of supply and demand on the market. Some types of collateral may bear a substantial portion of market risk. Therefore one could expect the collateral's market value to decline in downturn.

2.3 Exposure at default

Exposure at default EAD is the quantity of exposure a bank has to its borrower. Generally it consists of liabilities already activated into bank's balance sheet such as loans and off-balance sheet items consisting of undrawn credit lines. In reality, obligors tend to draw off committed lines of credit in times of financial distress, usually prior to default. Undrawn credit lines are taken into consideration when assessing the amount of exposure at default.

Banks can model EAD as a stochastic process given the uncertainty of quantity drawn from off-balance sheet commitments at default. Usually the expected portion of the outstanding credit lines depends on creditworthiness of the obligor and the type of credit facility involved.

Banks can require covenants in a form of excess collateral to provide additional security in the time of financial stress. Covenants could also be applied in a way that allows banks to close committed credit lines due to triggers of early indicators of default.

2.4 Expected loss

To measure loss potential, of a loan, that can be expected due to default it is quite straight forward to calculate *expected loss*

$$\mathrm{EL} = \mathrm{E}[\tilde{\mathrm{L}}] = \mathrm{P(D)} * \mathrm{LGD} * \mathrm{EAD},$$

where $\mathrm{P(D)} = \mathrm{PD}$ is the probability of the default event, LGD and EAD are deterministic and independent of each other. The EL can been as a cost arising from lending activities amongst other expenditures. In risk management terms the sum of ELs of loans over bank's entire portfolio is the amount of *expected loss reserve* that banks hold as a capital buffer against expected future losses over a particular time horizon, usually one calendar year. In loan pricing the EL is sometimes referred to as *risk premium*.

The expected loss refers to expected value or mean value over long period of time or business cycle. The actual losses banks experience over the chosen time horizon deviates. To illustrate the variation of actual default rates, realized PDs, of companies figure 2.1 presents observed yearly default rates of Finnish companies between 1993 and 2012. The observed average default rate was 1.1% from 1993 to 2012 and 0.9% starting from 1999 when Finland adopted the Euro.

In the peak year 1993 the observed default rate was almost three times the average.

In this case, if a bank expected its yearly default rate of its corporate loan portfolio to be the long term average 1.1% the losses in the year 1993 would have been almost three times the expected loss EL, holding LGD and EAD unchanged. It is clear that banks need to reserve additional cushion on top of EL to sustain much larger losses than the portfolio EL.

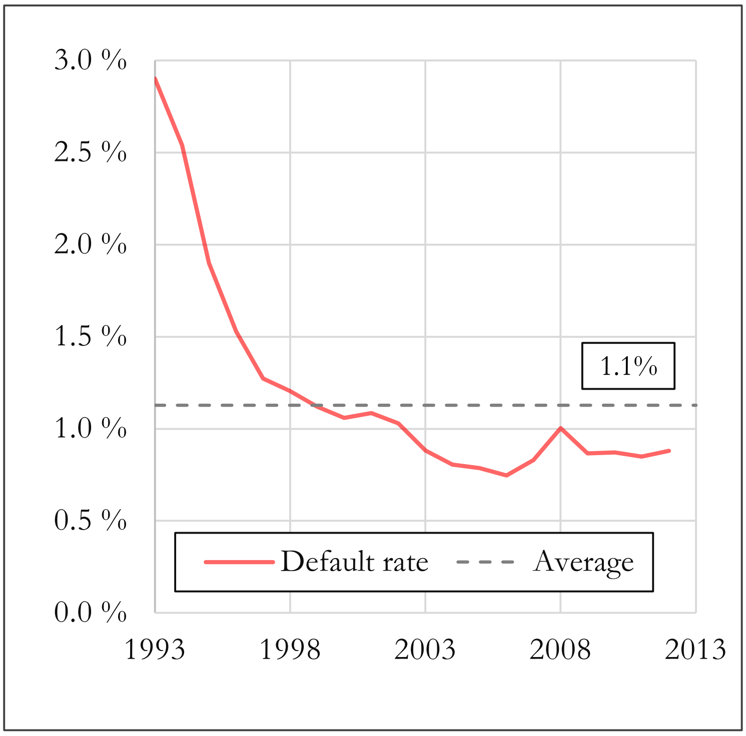

Figure 2.1: Yearly default rates of Finnish companies between 1993-2012. *Source: Statistics Finland*

2.5 Unexpected loss

In order to survive over periods of financial distress banks need to reserve capital to sustain losses exceeding the average experienced losses from past history. A convenient choice to measure losses greater than EL or *unexpected loss* is the standard deviation of EL

$$\mathrm{UL} = \sqrt{\mathrm{Var}\big(\mathrm{E}[\tilde{\mathrm{L}}]\big)}.$$

In practice, the excess *risk capital* saved for the cases of severe financial distress is called *economic capital* EC. Unexpected loss UL or it's multiples can be used to quantify EC. Although it is common for practitioners to define economic capital using α-quantile

$$\mathrm{EC}_{\alpha} = \mathrm{q}_{\alpha} - \mathrm{EL}_{\mathrm{portfolio}},$$

where q_{α} is the α-quantile of portfolio loss corresponding to confidence level α. The portfolio EL is deducted because economic capital is defined as a risk capital reserved to cover losses exceeding expected loss of a portfolio, see figure 2.2 for illustration of the EC. Decomposing total risk capital into expected loss and economic capital is essential as the EL is a portfolio independent measure, recall that $\mathrm{EL} = \mathrm{PD} * \mathrm{LGD} * \mathrm{EAD}$, and the EC strongly depends on the composition of the bank's portfolio[1]. Even one single loan can significantly change tail characteristics of portfolio loss distribution.

[1] Basel II Regulatory Capital Requirement assumes infinite granularity of bank's portfolio. Therefore FIRB and AIRB -capital changes are portfolio dependent.

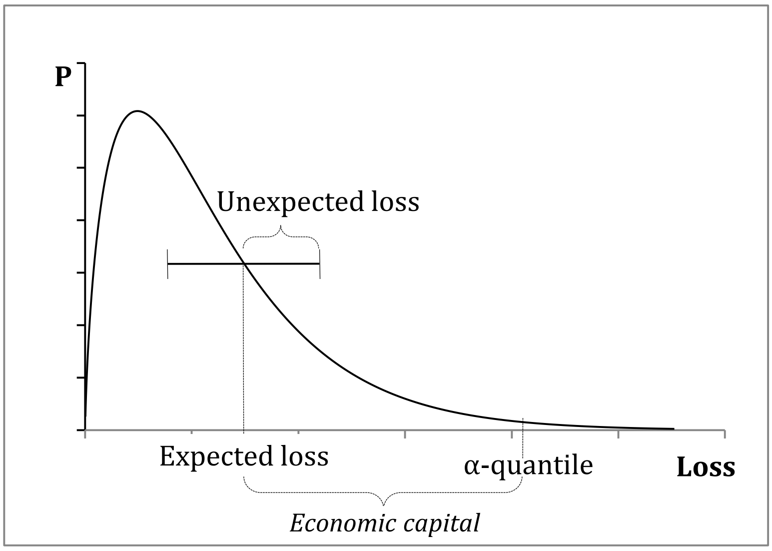

Figure 2.2: Loan portfolio loss distribution and economic capital measured as an α-quantile minus expected loss.

3 Credit portfolio model

The purpose of the normal copula model is to capture dependencies across defaults. In corporate loan portfolios the dependencies between obligors are usually the key determinant of portfolio tail loss behaviour. Dependencies between obligors is captured with a multivariate normal vector of latent variables describing creditworthiness. Changes in obligors credit quality is modelled as a multifactor model of systematic risk factors, which can be interpreted as industry factors having similar effect on companies operating in the same region.

3.1 Generating multivariate normal

Instead of sampling correlated multivariate random variables with covariance matrix it is more convenient to introduce correlation structure through factor loadings and uncorrelated standard normal variables. This requires linear transformation for a multivariate normal vector. Covariance between systematic factors are defined as follows

$$\mathrm{Cov}[X_i, X_j] = E[(X_i - \mu_i)(X_j - \mu_j)] = \Sigma_{ij}, \tag{3.1}$$

where X_i is an observation of factor i, μ_x is the mean and Σ_{ij} refers to a specific element on the covariance matrix Σ. The covariance matrix is implicitly defined through its diagonal elements σ_i^2 and correlations ρ_{ij}

$$\Sigma_{ij} = \sigma_i \sigma_j \rho_{ij} \tag{3.2}$$

and in matrix form

$$\Sigma = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} & & \rho_{11} \\ \rho_{12} & \rho_{22} & & \rho_{2d} \\ & & \ddots & \\ \rho_{1d} & \rho_{2d} & & \sigma_{dd} \end{pmatrix} \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{pmatrix}. \quad (3.3)$$

The Linear Transformation Property: Any linear transformation of normal vector is normal

$$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Rightarrow \mathbf{AX} \sim \mathrm{N}(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^{\mathbf{T}}), \quad (3.4)$$

for any d-vector $\boldsymbol{\mu}$, and any d x d matrix $\boldsymbol{\Sigma}$, and any k x d matrix $\mathbf{A}$.

The covariance matrix Σ and mean vector μ specifies a multivariate normal distribution $\mathbf{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Using the property (3.4) we have $\mathbf{Z} \sim \mathrm{N}(\overline{\mathbf{0}}, \mathbf{I})$ and $\mathbf{X} = \boldsymbol{\mu} + \mathbf{AZ}$ implying that $\mathbf{X} \sim \boldsymbol{\mu} + \mathrm{N}(\overline{\mathbf{0}}, \mathbf{AA}^{\mathrm{T}})$. Thus, we need to choose A that satisfies $\mathbf{AA}^{\mathrm{T}} = \boldsymbol{\Sigma}$ to sample the multivariate normal $\mathrm{N}(\overline{\mathbf{0}}, \boldsymbol{\Sigma})$. (Glasserman 2003)

3.1.1 Cholesky factorization

A lower triangular matrix A is an attractive choice because it reduces the calculation of the multivariate normal to following

$$\begin{aligned} X_1 &= \mu_1 + A_{11}Z_1 \\ X_2 &= \mu_2 + A_{21}Z_1 + A_{22}Z_2 \\ &\vdots \\ X_d &= \mu_d + A_{d1}Z_1 + A_{d2}Z_2 + \cdots + A_{dd}Z_d \end{aligned} \quad (3.5)$$

as a result the number of multiplications and additions are halved. The representation of $\boldsymbol{\Sigma}$ with a lower triangular matrix $\mathbf{A}$ is called *Cholesky factorization* and $\boldsymbol{\Sigma}$ has to be positive definite. A lower triangular matrix $\mathbf{A}$ is found by solving equations

$$
\begin{aligned}
A_{11}^2 &= \Sigma_{11} \\
A_{11}A_{21} &= \Sigma_{21} \\
&\vdots \\
A_{d1}A_{11} &= \Sigma_{d1} \\
A_{21}^2 + A_{22}^2 &= \Sigma_{22} \\
&\vdots \\
A_{d1}^2 + \cdots + A_{dd}^2 &= \Sigma_{dd}
\end{aligned} \tag{3.6}
$$

If we have $\mathbf{X} \sim N(\mathbf{0}, \mathbf{\Sigma})$ and $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{I})$ then (3.5) simplifies to

$$
\mathbf{X} = a_1 Z_1 + a_2 Z_2 + \cdots + a_d Z_d \tag{3.7}
$$

where a_j is the jth row of the matrix $\mathbf{A}$ and a_j is called factor loadings. (Golub and Van Loan 1996)

3.2 Normal copula model

The normal copula model for credit risk portfolios associated with the CreditMetrics (Gupton *et al* 1997) introduces correlations between obligors' default events via standard normal distributed variables $(X_1, X_2, \ldots, X_m)$. We use the same notations presented in Glasserman and Li (2005)

m = number of obligors in a portfolio
Y_k = default indicator of kth obligor, set to 1 if obligor defaults and 0 otherwise
p_k = marginal default probability of kth obligor
c_k = loss resulted from default of kth obligor
$L = c_1 Y_1 + c_2 Y_2 + \cdots + c_m Y_m$ = total loss of portfolio

Let X_k be standard normal distributed and $x_k = \Phi^{-1}(1 - p_k)$ with Φ^{-1} inverse of standard normal cumulative distribution function. The default threshold x_k can be interpreted as a default boundary value in association of the Merton's asset value process (Merton 1974). The marginal default probabilities p_k are expected to be known. For each obligor we have a default indicator

$$Y_k = \mathbf{1}\{X_k > x_k\}, \qquad k = 1 \ldots m. \tag{3.8}$$

Thus we have

$$\begin{aligned} P(Y_k = \mathbf{1}) &= P\big(X_k > \Phi^{-1}(1 - p_k)\big) \\ &= 1 - \Phi\big(\Phi^{-1}(1 - p_k)\big) = p_k. \end{aligned} \tag{3.9}$$

In our normal copula model dependencies across default indicators Y_k are captured with a multivariate normal vector $(X_1, X_2, \ldots, X_m)$ defined as a linear combination

$$\begin{gathered} X_k = a_{k1}Z_1 + \cdots + a_{kd}Z_d + b_k\varepsilon_k, \\ k = 1, \ldots, m, \end{gathered} \tag{3.10}$$

where $\mathbf{Z} = (Z_1, \ldots, Z_d)^T$ are systematic independent standard normal distributed risk factors, $\varepsilon_k \sim N(0,1)$ is an idiosyncratic risk factor of kth obligor, $a_{k1}, \ldots, a_{kd}$ are factor loadings for kth obligor $\sum_{i=1}^{d} a_{ki}^2 \leq 1$ and $b_k = \sqrt{1 - \left(a_{k1}^2 + \cdots + a_{kd}^2\right)}$.

Systematic risk factors $Z_1, \ldots, Z_d$ are interpreted as industry factors to witch obligors are exposed to. The correlation structure is defined by factor loadings $a_{k1}, \ldots, a_{kd}$ and correlation between latent variables X_k and X_j is $a_k\, a_j^T$.

From (3.9) and (3.10) we get conditional marginal default probability for the obligor k

$$\begin{aligned} p_k(Z) &= P(Y_k = 1|Z) = P(X_k > x_k|Z) \\ &= P\big(a_k Z + b_k \varepsilon_k > \Phi^{-1}(1 - p_k)\big) \\ &= P\big(b_k \varepsilon_k > -a_k Z + \Phi^{-1}(1 - p_k)\big) \\ &= \Phi\left(\frac{a_k Z + \Phi^{-1}(p_k)}{b_k}\right) \end{aligned} \quad (3.11)$$

where $a_k = (a_{k1}, .., a_{kd})$ are the factor loadings of kth obligor of systematic factors $Z_1, \dots, Z_d$. Factor loadings are defined in the next chapter.

3.3 Stochastic loss given default

The CreditMetrics framework uses the beta distribution for stochastic loss given default LGD. The beta distribution is fully specified with its mean and standard deviation, and it provides high degree of flexibility for modelling stochastic LGD (Gupton *et al* 1997). The general density of the Beta-distribution is

$$B(\alpha, \beta, x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}(1 - x)^{\beta - 1} x^{\alpha - 1},$$

where Γ denotes the Gamma-function. Parameters α and β can be solved if expectation and standard variation of LGD is known. In Basel II regulatory capital framework corporate bonds under foundation internal rating based approach FIRB are assigned with constant loss given default rate of 45%. We adopt this and set $E[\mathrm{LGD}] = 0.45$.

Tache (2004) examines modelling of loss given default and default events with single loss variable. He defines variance of LGD as a fixed percentage of maximally possible variance. It is common in credit portfolio models to use following representations for α and β

$$\alpha = \mathrm{LGD}\frac{1-v}{v}$$
$$\beta = (1-\mathrm{LGD})\frac{1-v}{v}, \tag{3.12}$$

where parameter v has fixed value of 0.25. Substituting $\mathrm{LGD} = 0.45$ into (3.12) we get $\alpha = 1.35, \beta = 1.65$. The density function $B(1.35, 1.65, x)$ is presented in figure 3.1.

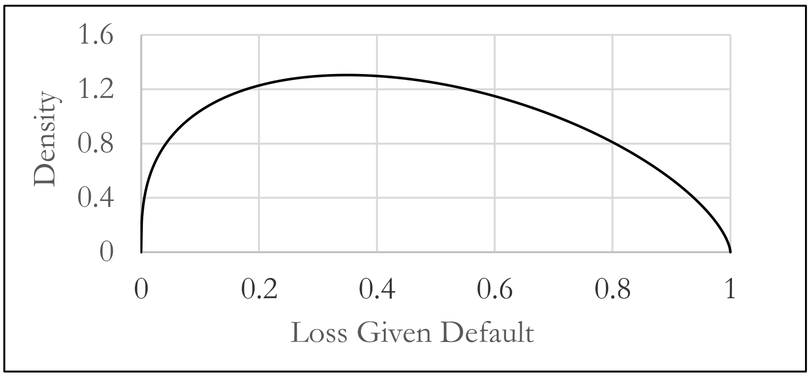

Figure 3.1: Density function of loss given default 45%, $\alpha = 1.35, \beta = 1.65$.

Figure 3.2 presents recovery rate distributions of corporate bonds for different seniorities. Recovery rate is $1 - \mathrm{LGD}$. The distribution in figure 3.1 is similar to senior unsecured corporate bonds in figure 3.2. So, we can use the density $B(1.35, 1.65, x)$ in our simulations.

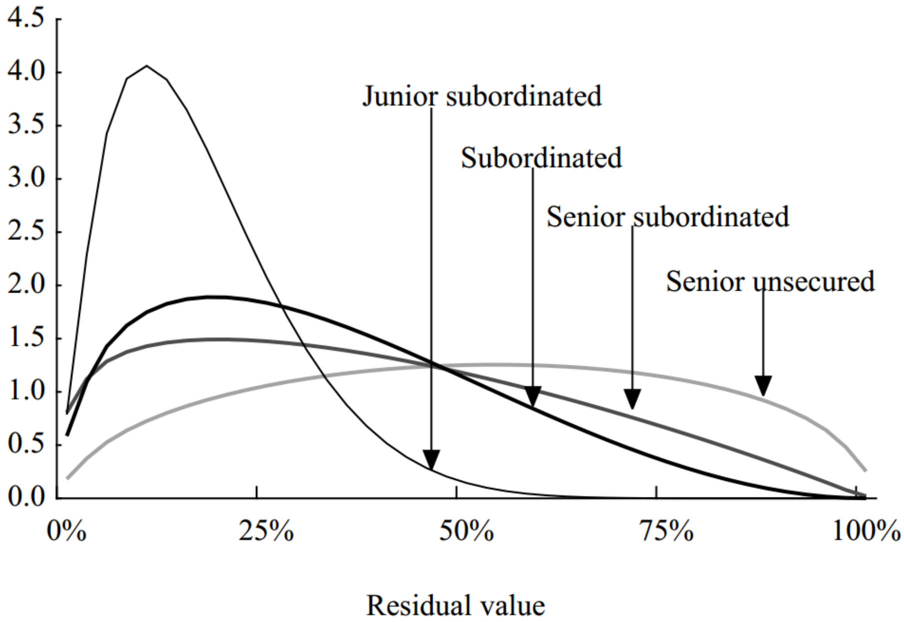

Figure 3.2: Recovery rate, *Residual value,* distribution for different seniority classes (Gupton *et al* 1997)

4 Model calibration

Default correlation refers to tendency for two, or more, companies to default at the period of time. In our model this trigger is one common risk factor. Companies operating in the same geographical region or in the same industry are exposed to same fluctuations in economic conditions. Default correlation is important in the determination of probability distribution for portfolio losses and vitally important in the determination of the α-quantile estimates for portfolio loss distribution.

This chapter describes a method to define correlation structure for our credit portfolio model. For a large portfolio consisting of hundreds or thousands of exposures it is not computationally convenient to define correlations between obligors individually because it would result very large covariance matrices. Portfolio consisting $m = 1\,000$ different obligors would require calculation of $m(m-1)/2 = 499\,500$ different correlation estimates. Thus, it is computationally efficient to introduce correlation through systematic risk factors. Separating idiosyncratic firm specific risk from common systematic risk also has its benefits. Firm specific risk is the part of risk that can be diversified away through portfolio diversification, while the risk contribution of systematic factors is non diversifiable. Correlation between two obligors is illustrated in figure 4.1. Parameters R_i represents the correlation between corporate and its industry and $\rho_{Ind_A Ind_B}$ represents the correlation between industries of corporation A and B

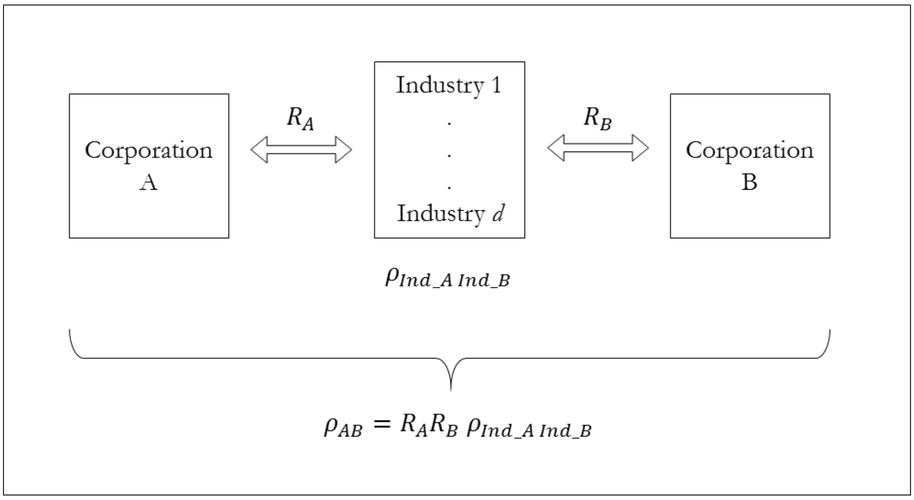

Figure 4.1: The default correlation between two obligors.

4.1 Correlation structure

Global Industry Classification Standard categorizes corporations by ten different sectors, see table 4.1. In this thesis, publicly traded corporations listed in stock markets in Helsinki, NASDAQ OMX Helsinki, and Stockholm, NASDAQ OMX Stockholm, are included constructing a correlation structure. The stock market data is from Bloomberg Data Services starting from the beginning of year 2001 and ending in October 2014. Monthly logarithmic returns S_i individual companies are calculated using every month's last trading day quotes.

Corporations having less than 24 months of data available are excluded. Sectors Energy and Utilities are also excluded due to small number of observations and Telecommunication Services and Information Technology are grouped to form one industry. There is possibility of over estimating correlation between obligor and the industry if small number of observations is used.

Return series T_i of industry factor, or index related to the industry, is calculated as monthly logarithmic returns of sum of market capitalization of companies assigned to each industry. For example, the return series of Materials industry is logarithmic returns of sum of market capitalizations of the 22 companies that are included in Materials industry. Number of corporations by industries are presented in table 4.2 totaling 311 different corporation.

Table 4.1: GICS-sectors.

Code	Sector	Subcode	Industry Groups
10	Energy	1010	Energy
15	Materials	1510	Materials
20	Industrials	2010	Capital Goods
		2020	Commercial & Professional Services
		2030	Transportation
25	Consumer Discretionary	2510	Automobiles & Components
		2520	Consumer Durables & Apparel
		2530	Hotels Restaurants & Leisure
		2540	Media
		2550	Retailing
30	Consumer Staples	3010	Food & Staples Retailing
		3020	Food, Beverage & Tobacco
		3030	Household & Personal Products
35	Health Care	3510	Health Care Equipment & Services
		3520	Pharmaceuticals & Biotechnology
40	Financials	4010	Banks
		4020	Diversified Financials
		4030	Insurance
		4040	Real Estate
45	Information Technology	4510	Software & Services
		4520	Technology Hardware & Equipment
		4530	Semiconductors & Semiconductor Equipment
50	Telecommunication Services	5010	Telecommunication Services
55	Utilities	5510	Utilities

Table 4.2: GICS-industries used to construct correlation structure and number of listed companies.

GICS Code	GICS	Observations
15	Materials	22
20	Industrials	93
25	Consmer Discretionary	46
30	Consumer Staples	13
35	Health Care	32
40	Financials	41
70	Inf Tech + Telecom	64

4.2 Inferring correlations from equity returns

We now estimate the correlation between companies and their industry return series. Pairwise correlation are calculated as a Pearson-correlation coefficient

$$R = \frac{\sum_{i=1}^{n}\left(S_i^A - \bar{S}^A\right)\left(T_i - \bar{T}\right)}{\sqrt{\sum_{i=1}^{n}\left(S_i^A - \bar{S}^A\right)^2}\sqrt{\sum_{i=1}^{n}\left(T_i - \bar{T}\right)^2}}, \tag{4.1}$$

where S_i is the monthly logarithmic return of a corporation A, T_i is the monthly logarithmic return of sum of market capitalizations of companies assigned to the same industry as corporation A, and $\bar{S}$ and $\bar{T}$ are the means of S_i and T_i. Summary of correlation coefficients R_{Pe} by industry is presented in table 4.3. Examining the obligor to industry correlations one will find that the distribution of correlations between obligor and industry relative to other industries is not significantly different. Industrial and Inf Tech + Telecom industries have one obligor each with very low correlation, NURMINEN LOGISTICS OYJ and SCANFIL OYJ. This could indicate misclassification of their industry. But from simulation point of view it is interesting to determine how their low correlation affect marginal risk contribution estimates. Correlations between individual companies and their industries are presented in appendix A together with number quotes used in estimating correlations.

Table 4.3: Summary of obligor to industry Pearson-correlation coefficients by industries.

	Min	Q1	Median	Mean	Q3	Max
MA	0,218	0,386	0,487	0,464	0,533	0,659
IN	0,005	0,431	0,512	0,513	0,633	0,817
CD	0,193	0,408	0,520	0,498	0,599	0,708
CS	0,275	0,429	0,457	0,455	0,506	0,643
HC	0,197	0,335	0,411	0,401	0,484	0,574
FI	0,205	0,409	0,518	0,496	0,588	0,796
IT	0,068	0,405	0,489	0,481	0,563	0,774

Industry to industry correlations are also defined as Pearson-correlation

$$\Sigma_{A,B} = \frac{\sum_{i=1}^{n}\left(T_i^A - \overline{T}^A\right)\left(T_i^B - \overline{T}^B\right)}{\sqrt{\sum_{i=1}^{n}\left(T_i^A - \overline{T}^A\right)^2}\sqrt{\sum_{i=1}^{n}(T_i^B - \overline{T}^B)^2}}, \tag{4.2}$$

where A and B refers to different industries. Industry correlation matrix $\mathbf{\Sigma}$ is presented in in table 4.4.

Table 4.4: Industry factor correlation matrix. AVG is the industry average correlation between industries.

	MA	IN	CD	CS	HC	FI	IT
MA	1,000	0,752	0,746	0,619	0,502	0,648	0,573
IN	0,752	1,000	0,890	0,633	0,702	0,817	0,793
CD	0,746	0,890	1,000	0,667	0,635	0,778	0,751
CS	0,619	0,633	0,667	1,000	0,447	0,584	0,461
HC	0,502	0,702	0,635	0,447	1,000	0,610	0,742
FI	0,648	0,817	0,778	0,584	0,610	1,000	0,708
IT	0,573	0,793	0,751	0,461	0,742	0,708	1,000
AVG	0,640	0,765	0,745	0,568	0,606	0,691	0,671

Materials	= MA
Industrials	= IN
Consmer Discretionary	= CD
Consumer Staples	= CS
Health Care	= HC
Financials	= FI
Inf Tech + Telecom	= IT

The correlation matrix in the table 4.4 seems valid. The least correlated industry is Consumer Staples which contains food industries and other non-cyclical retail industries. Industrials and Consumer Discretionary has the greatest average correlations between other industries. This is a logical result as Industrials and Consumer Discretionary consists of industry groups that are relatively highly cyclical and exposed to global economic factors and Consumer Staples represents less cyclical industry groups.

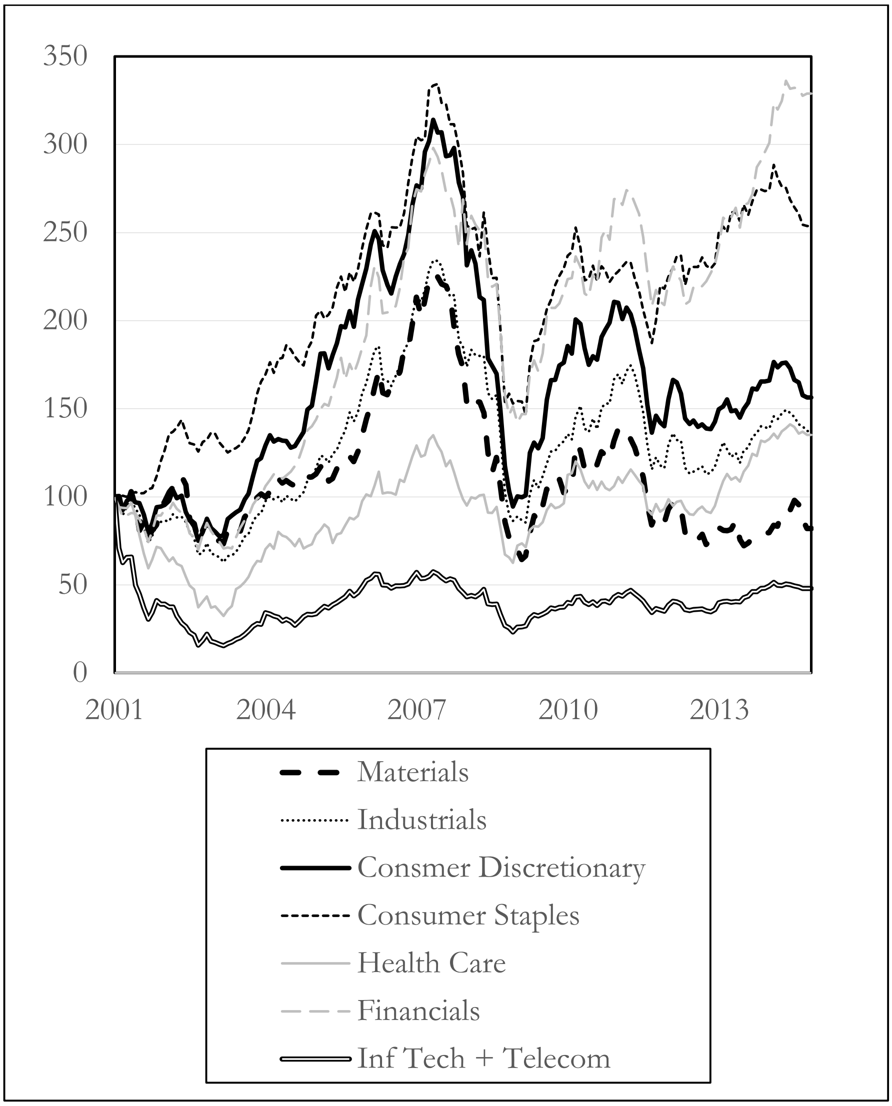

Figure 4.2: Return series on industry factors used to construct correlation structure.

Our model assumes that the default dependence between two obligors is entirely defined by the correlation between their respective industries and correlation between obligor and its industry, figure 4.1. This could lead to misclassification of correlation structure. But for our simulation purposes it is still adequate.

We now define the factor loadings $a_{k1}, \dots, a_{kd}$ of our normal copula model in such a way that they comprise the correlation structure presented in this chapter 3. In the CreditMetrics framework the level variation of company's assets explained by industry factor is the correlation R. Let $\mathbf{R}$ $m \times d$ matrix containing correlation coefficients of industry to obligor correlations with m referring to number of obligors and d referring to number of industry factors, systematic risk factor. We also have correlated systematic risk factors $\hat{\mathbf{Z}} = (\hat{Z}_1, \dots, \hat{Z}_d)^T$, $\hat{\mathbf{Z}} \sim N(\bar{0}, \Sigma_d)$. Therefore the latent variable describing asset movement is

$$X_k = R_{k1}\hat{Z}_1 + \cdots + R_{kd}\hat{Z}_d + b_k\varepsilon_k, \quad k = 1, \dots, m. \tag{4.3}$$

The systematic part in (4.3) is $\mathbf{X} = \mathbf{R}\hat{\mathbf{Z}}$ in matrix form. After Cholesky factorization we have $\mathbf{R}\hat{\mathbf{Z}} = \mathbf{R}\mathbf{A}^T\mathbf{Z}$, $\mathbf{Z} \sim N(\bar{0}, \mathbf{I}_d)$ and A is the lower triangular matrix. For simplicity let $\mathbf{C} = \mathbf{R}\mathbf{A}^T$. To ensure that $\mathbf{R}\hat{\mathbf{Z}}$ and $\mathbf{R}\mathbf{A}^T\mathbf{Z}$ have the variance after Cholesky factorization factor loadings of (3.10) are defined as

$$a_{kj} = C_{kj}\sqrt{\frac{\sum_{i=1}^{d} R_{ki}^2}{\sum_{i=1}^{d} C_{ki}^2}}, \quad j = 1, \dots, d, \tag{4.4}$$

and with $b_k = \sqrt{1 - (a_{k1}^2 + \cdots + a_{kd}^2)}$ we have $X_k \sim (0,1)$.

Inferring correlations from equity returns neglects capital structure of companies. Other companies are more leveraged than others, they have more debt respect to equity. Still, equity returns are widely used because they capture dependencies sufficiently (Hull 2005). Estimating correlations from asset returns would require information about capital structure and default probabilities to infer asset value movements from equity returns. Frey and McNeil (2001) argue that equity correlations do not sufficiently measure dependencies and that using them introduces excess model risk because the normal copula model fails to produce losses extreme enough. This thesis focuses on reducing computation times so it is sufficient to use equity returns to infer correlations.

Löffler (2004) states that asset returns are heavily tailed and asymmetric unlike the normal distribution. He uses *t*-distribution for asset returns to illustrate the correlation between obligors and explains the choice of degree of freedom parameter. Additionally he demonstrates that the tail behavior significantly differs when different degrees of freedom is used.

4.3 A real life credit portfolio

To measure performance differences between different Monte Carlo algorithms we need a real life corporate loan portfolio. Our credit portfolio is constructed using the same corporations that we used to estimate correlations structure. Each obligor is assigned with three credit ratings sampled randomly and their corresponding probability of defaults resulting 933 obligors in total. The credit rating distribution of obligors is presented in figure 4.3. The ratings correspond to Moody's rating system for corporate bonds and average observed default probabilities, see table 4.5. Ratings are sampled using a method to produce large number of obligors with low probabilities of default in every industry. The rating distribution represents a typical corporate loan portfolio credit rating distribution of a Nordic bank except having large portion of obligors with the highest credit rating.

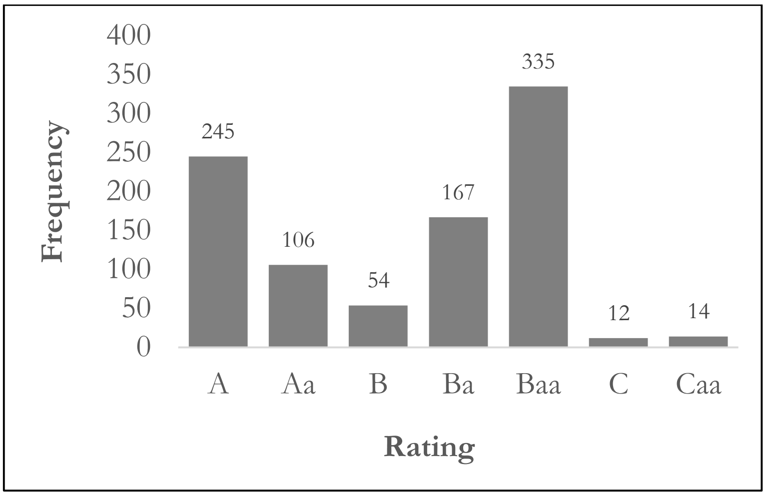

Figure 4.3: The sampled rating distribution of the real life portfolio.

Table 4.5: Moody's credit ratings and corresponding one-year default probabilities rates of corporate bonds between 1920-2007.

Rating	A	Aa	B	Ba	Baa	C	Caa
p	0,00009	0,0002	0,00192	0,01166	0,04663	0,15371	0,32905

5 Importance sampling

The plain Monte Carlo simulation becomes inefficient when estimating tail probabilities in rare-event simulation. Therefore, accurate measurement of credit risk in corporate loan portfolios often requires variance reduction when using Monte Carlo simulation. One commonly used method is the importance sampling (IS). An expectation under one probability measure is expressed as an expectation under a different measure to generate "important" outcomes thereby increasing sampling efficiency. Defining the change of probability distribution of risk factors is a critical step in developing IS-method.

There are two commonly used ways to apply importance sampling to the normal copula model. One way is to increase default probabilities by twisting the conditional default probabilities and second way is to shift systematic factors to generate more scenarios with large losses. Applying importance sampling to systematic factors of single-factor model has been suggested by (Avranitis and Gregory 2001) and (Kalkbrener *et al* 2003).

Mathematically importance sampling means changing the probability measure. For a more detailed mathematical treatment of importance sampling see (Glasserman 2003) and (McNeil *et al* 2005). Let us consider the problem of estimating

$$\eta = \mathrm{E}[h(X)] = \int h(x)f(x)\,dx \qquad (5.1)$$

where X is a random element of $\mathbb{R}^d$ that has a probability density function f and $h: \mathbb{R}^d \rightarrow \mathbb{R}$. Therefore, the ordinary Monte Carlo estimator is

$$\hat{\eta} = \hat{\alpha}(n) = \frac{1}{n}\sum_{i=1}^{n} h(X_i) \tag{5.2}$$

where X_i and X_j are independent and $X_i \sim f$. If we have g that is a probability density in $\mathbb{R}^d$ satisfying $f(x) > 0 \Rightarrow g(x) > 0 \; \forall \; x \in \mathbb{R}^d$ we can write alternatively

$$\eta = \int h(x)\frac{f(x)}{g(x)} g(x)dx. \tag{5.3}$$

This can be interpreted as an expectation respect to density g and therefore we can write

$$\eta = E_g\left[h(X)\frac{f(X)}{g(X)}\right], \tag{5.4}$$

where E_g represents the expectation of $X \sim g$. For independent $X_i, \dots, X_n$ we have the importance sampling estimator

$$\hat{\eta}_g = \hat{\eta}_g(n) = \frac{1}{n}\sum_{i=1}^{n} h(X_i)\frac{f(X_i)}{g(X_i)}, \tag{5.5}$$

where the importance sampling weight $f(X_i)/\,g(X_i)$ is the likelihood ratio or Radon-Nikodym derivative evaluated at X_i. From (5.4) we get that $E_g[\hat{\eta}_g] = \eta$ and that $\hat{\eta}_g$ is an unbiased estimator of η. With importance sampling we have second moment of $\hat{\eta}_g$

$$E_g\left[\left(h(X)\frac{f(X)}{g(X)}\right)^2\right] = E\left[h(X)^2\frac{f(X)}{g(X)}\right], \tag{5.6}$$

where E_g is the expectation of $X \sim g$. Without importance sampling the second moment is $E[h(X)^2]$. The problem comes to find an optimal importance sampling density g to make the second moment smaller. Two commonly used alternatives in credit portfolio simulations for importance sampling are presented in following sections.

5.1 Exponential twisting

Convenient choice of transforming probability measure is twisting it exponentially. In case of exponential change of measure we have cumulative distribution function F on $\mathbb{R}$ and cumulant generating function is defined by the logarithm of the moment generating function of F

$$\psi(\theta) = \log \int_{-\infty}^{\infty} e^{\theta x}\, dF(x). \tag{5.7}$$

If $\psi(\theta) < \infty$, we set

$$F_\theta(x) = \int_{-\infty}^{x} e^{\theta u - \psi(\theta)}\, dF(u) \tag{5.8}$$

so that each F_θ is a probability function. Now F transforms F_θ exponentially and if F has a density f, then F_θ has density

$$f_\theta(x) = e^{\theta x - \psi(\theta)} f(x) \tag{5.9}$$

In (5.8) the Radon-Nikodym derivative is $dF_\theta/dF = e^{\theta x - \psi(\theta)}$ which is the importance sampling weight. In (5.7) we have $\psi(\theta) = \log E[\, e^{\theta x})]$. The first derivative of cumulant generating function $\psi(\theta)$ equals to mean of F_θ

$$\psi'(\theta) = \frac{E[Xe^{\theta X}]}{E[e^{\theta X}]} = E[Xe^{\theta X - \psi(\theta)}] = E_\theta[X], \tag{5.10}$$

where $X \sim f$ and E_θ is expectation under twisted probability measure. Portfolio loss L is a sum of independent random variables Y_i with moment generating function

$$E[e^{\theta L}] = \prod_{k=1}^{m} E[e^{\theta Y_k c_k}] = \prod_{k=1}^{m} [p_k e^{\theta c_k} + (1 - p_k)]. \tag{5.11}$$

where c_k is cost of default and Y_i is the default indicator. The culumant generating function of distribution L can also be written in the form

$$\begin{aligned} \psi_{L|Z}(\theta) &= \log E[(e^{\theta L}|Z)] \\ &= \sum_{k=1}^{m} \log\left(p_k(Z)e^{\theta c_k} + (1 - p_k(Z))\right). \end{aligned} \tag{5.12}$$

Glasserman and Li (2005) propose increasing default probabilities p_k depending on cost of default c_k and default probability. Taking first derivative of (5.12) respect to θ for individual obligor results the exponentially twisted default probability

$$\tilde{p}_k(\theta(Z)) = \frac{p_k(Z)e^{\theta(Z)c_k}}{p_k(Z)e^{\theta(Z)c_k} + (1 - p_k(Z))}, \tag{5.13}$$

where $\theta(Z)$ is the twisting parameter conditional on systematic factor Z, $p_k(Z)$ is the conditional marginal default probability of obligor k and c_k is the cost of default. For $\theta(Z) > 0$ the default probabilities are increased and choosing $\theta(Z) = 0$ we get $\tilde{p}_k(0) = p_k(Z)$. The larger the cost of default, the larger the twisted probability. We want to generate large portfolio losses L. Therefore, we concentrate on increasing conditional default probabilities and having $\theta(Z) > 0$. Although negative values of $\theta(Z)$ are useful in estimating conditional expectations given $L = x$. (Glasserman 2003)

Now we can write the IS-estimator for exponential twisting

$$1\{L > x\} \underbrace{e^{-\theta(Z)L+\psi_L(\theta(Z))}}_{\text{IS-weight}}, \tag{5.14}$$

where $\mathbf{1}\{L > x\}$ is an indicator for portfolio loss exceeding loss level x and $L = \sum_i^k c_i Y_i$. To define optimal θ, the second moment of the IS-estimator can be minimized (Glasseman and Li 2005)

$$\begin{aligned} M_2(x, \theta) &= E_\theta\left[e^{-2\theta L+2\psi_L(\theta)}\, \mathbf{1}\{L > x\}\right] \\ &\leq e^{-2\theta x+2\psi_L(\theta)}, \end{aligned} \tag{5.15}$$

where E_θ stands for expectation under exponential twisting distribution with parameter θ. The upper bound to holds if $\theta \geq 0$. It is suggested to focus on minimizing the upper bound as it is far more convenient. So we need to maximize $\theta x - \psi(\theta)$. The cumulant generating function ψ_L is strictly convex, second derivative is positive, and $\psi_L(0) = 0$. For further theoretical background see Barndorff-Nielson (1978). The maximum is attained at

$$\theta_x(Z) = \begin{cases} \text{unique solution to } \psi'(\theta) = x, & x > \psi'(0) \\ 0 \qquad\qquad\qquad\qquad\qquad\quad , & x \leq \psi'(0), \end{cases} \tag{5.16}$$

where $\theta_x(Z)$ is the optimal twist conditional on systematic factor Z. A unique solution indeed exists because, for all Z, the derivative increases from $-\infty$ to ∞

as $\theta(Z)$ increases from $-\infty$ to ∞ (Glasserman and Li 2005). We solve numerically equation

$$\frac{\partial}{\partial\theta}\psi\big(\theta_x(Z)\big) = \sum_{k=1}^{m}\left[\frac{p_k(Z)\, e^{\theta_x(Z)c_k}}{p_k(Z)e^{\theta(Z)c_k} + \big(1 - p_k(Z)\big)}\right] = x, \qquad x > \psi'(0). \tag{5.17}$$

Setting $\theta(Z) = \theta_x(Z)$ we twist the expected conditional portfolio loss $E[L|Z]$ to x and the exponentially twisted conditional default probability is

$$\tilde{p}_{k,\theta(Z)}(Z) = \frac{p_k(Z)e^{\theta_x(Z)c_k}}{p_k(Z)e^{\theta_x(Z)c_k} + \big(1 - p_k(Z)\big)}. \tag{5.18}$$

Since we are interested in large losses we have to take into consideration what is the key driver of large portfolio losses. If we had independent obligors the exponential twisting would result meaningful reduction of variance. This is not necessarily the case when obligors are dependent because large losses occur when obligors defaults simultaneously especially when the dependence structure is strong (McNeil *et al* 2005). Meaning that the key driver of large portfolio losses is the systematic factor Z. In this case the exponential twisting of conditional default probabilities does not guarantee sufficient variance reduction or reduction at all. Therefore, it is suggested to apply importance sampling on systematic factors (Glasserman and Li 2005).

5.2 Shifting factor means

If obligors are highly dependent it is useful to apply importance sampling on systematic factors by shifting $Z = (Z_1, \ldots, Z_d)$. An attractive IS distribution for the factor Z would be the probability density proportional to the function

$$z \to P(L > x|Z = z)\, e^{-\frac{1}{2}z^T z}, \qquad (5.19)$$

where $P(L > x|Z = z)$ is the conditional expectation of portfolio loss exceeding x. Sampling with the density (5.19) is not feasible because it is required to be normalized with the value $P(L > x|)$ to make it a density. So it is proposed to use the normal density with the same mode (Glasserman and Li 2005). A similar problem arises in an option-pricing context in which Glasserman *et al* (1999) suggest to use the normal density with the same mode as the optimal density. This mode occurs at a solution to an optimization problem

$$\max_z P(L > x|Z = z)\, e^{-\frac{1}{2}z^T z}. \qquad (5.20)$$

The solution, the optimal shift μ^* of the systematic factor Z , is then also the mean of the approximating normal density distribution. After shifting we have $Z \sim N(\mu^*, I)$. The likelihood ratio that relates $N(0, I)$ and $N(\mu^*, I)$ is

$$w_\mu = e^{-\mu^T Z + \frac{1}{2}\mu^T \mu}. \qquad (5.21)$$

Because $P(L > x|Z = z)$ does not have representation in closed form it is difficult to find exact solution for μ^*. Therefore, it is suggested to simplify the problem through approximation. Glasserman and Li (2005) propose few different approaches for this and each approach produces different values for the optimal shift μ^*. They justify *tail bound approximation* method stating that it produces an asymptotically optimal solution when the number of obligors m approaches infinity. Conversely, Egloff *et al* (2005) argue that since credit portfolios are finite of their size and the interest lies on a particular α-quantile, or expected shortfall above a specific α-quantile, instead of the asymptotic tail of the loss distribution, the methods based on asymptotic arguments may not be the most effective way of reducing variance. It is shown by Egloff *et al* (2004) that importance sampling techniques in this thesis do not guarantee improvements for the plain Monte Carlo simulation and they suggest the adaptive stochastic approximation method of Robbins-Monro (Robbins and Monro 1951).

Kalkbrener *et al* (2004) have proposed *the homogenous portfolio approximation* to find the optimal importance sampling shift for systematic factors. In a homogenous portfolio, all loans are specified by identical default probability, cost of default and correlation between obligors. Authors found that their method significantly reduces the variance of the Monte Carlo estimates and computing time necessary to calculate high α-quantiles of the portfolio loss distribution.

Since the evidence does not clearly suggest one particular method over the others and all of the alternatives seem to produce roughly the same variance reduction, it is still unclear which of these approximation methods is the most robust one. Therefore, we use *the normal approximation* because it is computationally efficient and simple to incorporate into our normal copula model. For (5.19) we have normal approximation

$$P(L > x|Z = z) \approx 1 - \Phi\left(\frac{x - E[L \mid Z = z]}{\sqrt{Var[L \mid Z = z]}}\right), \qquad (5.22)$$

Recall that $E[L \mid Z = z] = \sum_{k=1}^{m} c_k p_k(z)$ and $Var[L \mid Z = z] = \sum_{k=1}^{m} c_k^2 [p_k(z) - p_k(z)^2]$. Substituting these into (5.19) and (5.22) we get a multi-dimensional optimization problem

$$\max_{z}\left[1-\Phi\left(\frac{x-\sum_{k=1}^{m} c_k p_k(z)}{\sqrt{\sum_{k=1}^{m} c_k^2 [p_k(z)-p_k(z)^2]}}\right)\right] e^{-\frac{1}{2}z^T z}. \qquad (5.23)$$

The R programming language has *nloptm* package for solving non-linear optimization problems using derivative-free algorithm *cobyla* with nonlinear inequality and equality constraints.

6 Risk measures

Value at Risk (VaR) is widely used measure of risk in financial risk management. It is simply a α-quantile of a portfolio loss distribution. VaR corresponds to a confidence level of $100(1-\alpha)$ percent and is defined as

$$\mathrm{VaR}_{\alpha}(\mathrm{X}) = \sup\{\mathrm{x}|\mathrm{P}[\mathrm{X} \geq \mathrm{x}] > \alpha\}, \tag{6.1}$$

where $\sup\{\mathrm{x}|\mathrm{A}\}$ is the upper limit of portfolio loss x given event an A. As a risk measure VaR is easy to understand, but it is not a coherent risk measure since it is not sub-additive. This means that risk of a portfolio could exceed the sum of the stand-alone risk of its components. VaR -measure also ignores losses beyond the chosen limit. Alternatively α-quantile is referred to as a tail probability.

Expected Shortfall (ES) is more convenient choice of a risk measure as it is both coherent and it accounts losses beyond chosen confidence level. ES is simply expected loss in a condition that portfolio loss exceeds a given limit

$$\mathrm{ES}_{\alpha}(\mathrm{X}) = \mathrm{E}[\mathrm{X}|\mathrm{X} \geq \mathrm{VaR}_{\alpha}(\mathrm{X})] \tag{6.2}$$

for a confidence level of $100(1-\alpha)$ percent. (Arzner *et al* 1999)

The expectation for portfolio loss conditioned on portfolio loss exceeding a threshold x is

$$\mathrm{r} = \mathrm{E}[\mathrm{L}|\mathrm{L} > \mathrm{x}] = \frac{\mathrm{E}[\mathrm{L}\,\mathbf{1}\{\mathrm{L} > \mathrm{x}\}]}{\mathrm{P}(\mathrm{L} > \mathrm{x})}, \tag{6.3}$$

where the loss level x corresponds to $\mathrm{VaR}_{\alpha}(\mathrm{X})$. The IS-estimator with respect to probability density g is defined as

$$\hat{r}^{IS} = \frac{\sum_{k=1}^{n} L_k\, w_k\, \mathbf{1}\{L_k > x\}}{\sum_{k=1}^{n}\, w_k\, \mathbf{1}\{L_k > x\}}, \tag{6.4}$$

where w_k and L_k are the likelihood ratio and portfolio loss of kth replication. The ratio is zero whenever the denominator is zero. Choosing $w_k = 1$ in (6.4) we get the plain Monte Carlo estimator. To measure the accuracy of (6.4) confidence intervals are used. With the L_k sampled under g probability density function, the distribution

$$\frac{\hat{r}^{IS} - r}{\hat{\sigma}^{IS}/\sqrt{n}} \tag{6.5}$$

converges to the standard normal and

$$\hat{r}^{IS} \pm z_{\delta/2}\frac{\hat{\sigma}^{IS}}{\sqrt{n}} \tag{6.6}$$

is an asymptotically valid $1-\delta$ confidence interval of r and $z_{\delta/2} = -\Phi^{-1}(\delta/2)$. Using the general result for ratio estimates (Glasserman 2003), we have the variance estimator

$$\hat{\sigma}^{IS} = \left(\frac{n\sum_{k=1}^{n}(L_k w_k - \hat{r} w_k)^2\, \mathbf{1}\{L_k > x\}}{\left(\sum_{k=1}^{n} w_k \mathbf{1}\{L_k > x\}\right)^2}\right)^{½}, \tag{6.7}$$

in which the ratio is zero whenever the denominator is zero.

7 Monte Carlo algorithms

This chapter presents algorithms to estimate portfolio tail loss probabilities and tail loss expectations with plain Monte Carlo and IS Monte Carlo simulation. There are three different alternatives for IS. The exponential twisting algorithm increases default probabilities and the factor shifting algorithm shifts systematic factor to generate higher conditional default probabilities. Two step IS Monte Carlo algorithm applies both exponential twisting and factor shifting. Simulated default indicators are weighted using IS-weights presented in chapter 5.

Algorithms are divided into steps a-d. Numerated steps describe one simulation round consisting of M number on repetitions. Final two steps describe computation of tail loss probability and conditional expectations. Factor loadings are solved using the same equation regardless of the algorithm. Portfolio loss $\hat{L}$ is sum of all exposures c_k of defaulted obligors.

7.1 Plain Monte Carlo algorithm

Plain Monte Carlo algorithm for multi-factor model is described in figure 7.1. In step a) we generate systematic factors Z and define whether obligor defaulted and then we sum defaulted exposures over portfolio. In step b) we calculate the portion of repetitions having portfolio loss greater than x, tail probability, and in step c) we calculate average loss when portfolio loss exceeds x, expected shortfall. In step a) p_k is computed using (3.11).

Multi-factor algorithm in figure 7.1 can be used as a single-factor model replacing a) 1-2 steps with steps presented in figure 7.2. In this case the systematic factor Z is a scalar.

a)	Repeat M times
1.	Solve factor loadings $\hat{a}_{kj} = C_{kj}\sqrt{\sum_{i=1}^{d} R_{ki}^2 / \sum_{i}^{d} C_{ki}^2}$
2.	Compute $\hat{b}_k = \sqrt{1 - (\hat{a}_{k1}^2 + \cdots + \hat{a}_{kd}^2)}$
3.	Generate standard normal distributed systematic factor $\hat{Z}$ and compute conditional marginal default probabilities $p_k(\hat{Z})$
4.	Generate standard normal distributed variables $\hat{X}_k$
5.	Define default indicators $\hat{Y}_k = \mathbf{1}\left\{\hat{X}_k > \Phi^{-1}\left(1 - p_k(\hat{Z})\right)\right\}$
6.	Sum portfolio loss $\hat{L} = \sum_{k=1}^{m} c_k \hat{Y}_k$
b)	Compute $\hat{P}(L > x) = \frac{1}{M}\sum_{j=i}^{M} \mathbf{1}\{\hat{L}_j > x\}$
c)	Compute $\hat{r}^{IS} = \sum_{j=i}^{M} \hat{L}_j \mathbf{1}\{\hat{L}_j > x\} / \sum_{j=i}^{M} \mathbf{1}\{\hat{L}_j > x\}$

Figure 7.1: The algorithm of the plain Monte Carlo simulation of a multi-factor model.

1.	Solve factor loadings $\hat{a}_{k1} = \hat{R}_k$, where $\hat{R}_k$ is a correlation between obligor and systematic factor
2.	Compute $\hat{b}_k = \sqrt{1 - \hat{a}_{k1}^2}$

Figure 7.2: Replacing steps of the plain Monte Carlo simulation of a single-factor model.

7.2 Importance sampling Monte Carlo algorithms

The exponential twisting algorithm begins by computing factor loadings in same way that we did with the plain algorithm. The optimal twist is solved using conditional marginal default probabilities (3.11) and from (5.19) we get twisted probabilities. Tail loss probabilities c) and conditional expected loss d) are computed using IS-weight w_θ. Note that the optimal twist θ_x is dependent on systematic factor $\hat{Z}$ and therefore it is required to be solved in every simulation round. This makes the algorithm significantly slower than the shifting algorithm. The optimal shift depends only on the specification of portfolio and therefore it is required to be solved only once. Exponential twisting Monte Carlo algorithm is described in figure 7.3.

a) Repeat M times

1. Solve factor loadings $\hat{a}_{kj} = C_{kj}\sqrt{\sum_{i=1}^{d} R_{ki}^2 / \sum_i^d C_{ki}^2}$
2. Compute $\hat{b}_k = \sqrt{1 - \left(\hat{a}_{k1}^2 + \cdots + \hat{a}_{kd}^2\right)}$
3. Generate standard normal distributed systematic factor $\hat{Z}$ and compute conditional marginal default probabilities $p_k(\hat{Z})$
4. Solve optimal twist $\theta_x(\hat{Z})$
5. Set $\theta_x = 0$ if $E[L] = \sum_{j=1}^{m} p_k(\hat{Z})c_k < x$
6. Compute exponentially shifted marginal default probabilities $\tilde{p}_k(\theta_x(\hat{Z}))$, equation (5.19)
7. Generate standard normal distributed variables $\hat{X}_k$
8. Define indicators $\hat{Y}_k = \mathbf{1}\left\{\hat{X}_k > \Phi^{-1}\left(1 - \tilde{p}_k\left(\theta_x(\hat{Z})\right)\right)\right\}$
9. Sum portfolio loss $\hat{L} = \sum_{k=1}^{m} c_k \hat{Y}_k$
10. Compute likelihood ratio $w_{\theta_x(\hat{Z})} = e^{-\theta_x(\hat{Z})L + \psi_L(\theta_x(\hat{Z}))}$

b) Compute $\hat{P}(L > x) = \frac{1}{M}\sum_{j=i}^{M} \mathbf{1}\{\hat{L}_j > x\}\, w_{\theta_x(\hat{Z})}^j$

c) Compute $\hat{r}^{IS} = \sum_{j=i}^{M} \hat{L}_j w_{\theta_x(\hat{Z})}^j \mathbf{1}\{\hat{L}_j > x\} / \sum_{j=i}^{M} w_{\theta_x(\hat{Z})}^j \mathbf{1}\{\hat{L}_j > x\}$

Figure 7.3: The algorithm of the exponentially twisted Monte Carlo simulation of a multi-factor model.

The shifting algorithm is presented in figure 7.4. First step solves optimal shift. Conditional marginal default probabilities are computed using (3.11) like in the plain algorithm but now with shifted systematic factor. Tail loss probabilities c) and expectations d) are computed using IS-weight w_μ.

Two-step algorithm combines exponential twisting and factor shifting. Twisting is applied into conditional marginal default probabilities computed with shifted systematic factor. Tail loss probabilities and expectations are scaled with both IS-weights. Algorithm is described in figure 7.5.

a) Solve optimal shift μ^* using (5.19)

b) Repeat M times

1. Solve factor loadings $\hat{a}_{kj} = C_{kj}\sqrt{\sum_{i=1}^{d} R_{ki}^2 / \sum_i^d C_{ki}^2}$

2. Compute $\hat{b}_k = \sqrt{1 - \left(\hat{a}_{k1}^2 + \cdots + \hat{a}_{kd}^2\right)}$

3. Generate systematic factor $\tilde{Z} \sim N(\mu^*, I)$ and compute shifted conditional marginal default probabilities $p_k(\tilde{Z})$, equation (3.11)

4. Generate standard normal distributed variables $\hat{X}_k$

5. Define indicators $\hat{Y}_k = \mathbf{1}\left\{\hat{X}_k > \Phi^{-1}\left(1 - p_k(\tilde{Z})\right)\right\}$

6. Sum portfolio loss $\hat{L} = \sum_{k=1}^{m} c_k \hat{Y}_k$

7. Compute likelihood ratio $w_{\mu^*} = e^{-\mu^{*T}\tilde{Z} + \frac{1}{2}\mu^{*T}\mu^*}$

c) Compute $\hat{P}(L > x) = \frac{1}{M}\sum_{j=i}^{M} \mathbf{1}\{\hat{L}_j > x\}\, w_{\mu^*}^j$

d) Compute $\hat{r}^{IS} = \sum_{j=i}^{M} \hat{L}_j w_{\mu^*}^j \mathbf{1}\{\hat{L}_j > x\} / \sum_{j=i}^{M} w_{\mu^*}^j \mathbf{1}\{\hat{L}_j > x\}$

Figure 7.4: The algorithm of the factor shifting Monte Carlo simulation of a multi-factor model.

a) Solve optimal shift μ^* using (5.19)

b) Repeat M times

1. Solve factor loadings $\hat{a}_{kj} = C_{kj}\sqrt{\sum_{i=1}^{d} R_{ki}^2 / \sum_{i}^{d} C_{ki}^2}$

1. Compute $\hat{b}_k = \sqrt{1 - \left(\hat{a}_{k1}^2 + \cdots + \hat{a}_{kd}^2\right)}$

2. Generate systematic factor $\tilde{Z} \sim N(\mu^*, I)$ and compute shifted conditional marginal default probabilities $p_k(\tilde{Z})$, equation (3.11)

3. Solve optimal shift $\theta_x(\tilde{Z})$

4. Set $\theta_x = 0$ if $E[L] = \sum_{j=1}^{m} p_k(\hat{Z}) c_k < x$

5. Compute exponentially shifted marginal default probabilities $\tilde{p}_k(\theta_x(\tilde{Z}))$, equation (5.19)

6. Generate standard normal distributed variables $\hat{X}_k$

7. Define indicators $\hat{Y}_k = \mathbf{1}\left\{\hat{X}_k > \Phi^{-1}\left(1 - \tilde{p}_k\left(\theta_x(\tilde{Z})\right)\right)\right\}$

8. Sum portfolio loss $\hat{L} = \sum_{k=1}^{m} c_k \hat{Y}_k$

9. Compute likelihood ratios $w_{\theta_x(\tilde{Z})} = e^{-\theta_x(\tilde{Z})L + \psi_L(\theta_x(\tilde{Z}))}$ and $w_{\mu^*} = e^{-\mu^{*T}\tilde{Z} + \frac{1}{2}\mu^{*T}\mu^*}$

c) Compute $\hat{P}(L > x) = \frac{1}{M}\sum_{j=i}^{M} \mathbf{1}\{\hat{L}_j > x\} w_{\theta_x(\tilde{Z})}^j w_{\mu^*}^j$

d) Compute $\hat{r}^{IS} = \frac{\sum_{j=i}^{M} \hat{L}_j w_{\theta_x(\tilde{Z})}^j w_{\mu^*}^j \mathbf{1}\{\hat{L}_j > x\}}{\sum_{j=i}^{M} w_{\theta_x(\tilde{Z})}^j w_{\mu^*}^j \mathbf{1}\{\hat{L}_j > x\}}$

Figure 7.5: The algorithm of the two-step IS Monte Carlo simulation of a multi-factor model.

A conditional expectation for individual exposures is estimated similarly to the portfolio conditional expectation taking weighted average loss in a condition that portfolio loss exceeds x

$$\hat{r}_k^{obl} = \frac{\sum_{j=i}^{M} c_k \widehat{Y}_k \, w_j^{IS} \mathbf{1}\{\hat{L}_j > x\}}{\sum_{j=i}^{M} \, w_j^{IS} \mathbf{1}\{\hat{L}_j > x\}}, \tag{7.1}$$

where k refers to kth obligor in the portfolio and j refers to jth simulation round.

7.2.1 Stochastic cost default

Our importance sampling algorithms are optimized for deterministic cost of default c_k. However, in real life there is uncertainty in recovery rates of defaulted exposures. We incorporate stochastic loss given default by replacing constant c_k with

$$c_k = e_k q_k(\alpha_k, \beta_k), \qquad q_k \sim B(\alpha_k, \beta_k), \tag{7.2}$$

where e_k is exposure at default, q_k is loss given default LGD, α_k and β_k are solved from (3.12) and (3.13) when $E[LGD]$ is known. Stochastic cost of default c_k is generated in every simulation round.

7.3 Simulation with homogeneous portfolio

We begin examining properties of our calibrated model using homogeneous portfolio consisting of identical exposures having same marginal default probability $p_k = p$, cost of default $c_k = c$, correlation $R_k = R$ and industry. When using only one industry the model is referred to as single factor model.

Simulation results of estimating VaR of homogeneous portfolio using two-step IS Monte Carlo with different number of obligors m are presented in figure 7.6. VaR is calculated using (6.1) with loss level x that corresponds to $VaR_{99.9\%}$ obtained from (7.3). Estimated portfolio VaR approaches to asymptotic $VaR_{99.9\%}$

of (7.3) when the number of obligors m increases. In other words, the smaller the number of obligors, the larger is the tail probability when loss level x is held constant. Asymptotic formula for portfolio α-quantile is

$$q_\alpha(L) = \Phi\left(\frac{\Phi^{-1}(p_k) + \sqrt{R^2}\,\Phi^{-1}(\alpha)}{\sqrt{1 - R^2}}\right), \tag{7.3}$$

when the number of obligors m in the portfolio approaches to infinity, $p_k = p$ is the marginal default probability of obligors and R is the correlation parameter (McNeil *et al* 2005).

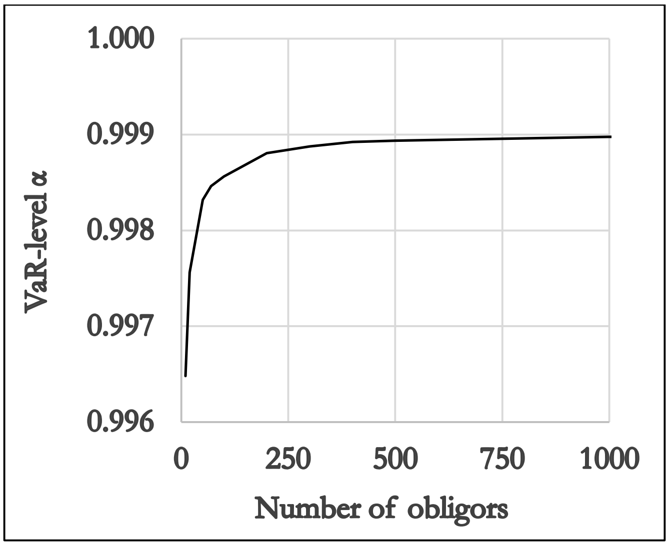

Figure 7.6: VaR -estimate of two-step IS Monte Carlo simulation using homogeneous portfolio with different number of obligors.

7.3.1 Relaxed homogeneous portfolio, single factor model

This subsection illustrates the differences between the algorithms tail probability -estimates and their confidence limits. Note that estimated confidence limits are not unbiased as they are estimated using sample variance. Homogeneous and relaxed homogeneous portfolios are used.

First, let us look at the performance of our four different algorithms with homogeneous portfolio described in table 7.1. Probability of default p and correlation R corresponds to an average values of our example portfolio described section 4.3.

Table 7.1: Specification of portfolio #A1.

	m	p	c	R
Obligor group #1	933	0,0121	1	0,485

Simulation results are illustrated in figure 7.7. The exponential twisting fails to reduce variance of tail probability estimate because the dominant driver of default events is the dependence across obligors. Note that because homogeneous portfolio is used the cost of default do not have an effect on the optimality of the twisting parameter and therefore the performance of the algorithm. Optimal exponential twisting is different in every simulation round but optimal factor shift is independent on simulation round. For loss levels x 100, 150 and 250 optimal shifts are 1.500669, 2.139131 and 3.497177 respectively. The greater the portfolio loss level x is the greater is the shift needed to produce large portfolio losses.

Factor shifting algorithm performs better than the twisting algorithm because the key driver of large losses is the relatively strong dependence between defaults. Performance of the shifting algorithm is independent on loss level unlike with the twisting algorithm. Confidence limits grow wider when loss level is increased. Two-step algorithm outperforms other algorithms as it combines both benefits of the two importance sampling methods.

Confidence limits are calculated using sample variance of 100 replications with normal distribution assumption. Simulations are run with $M = 1\ 000$.

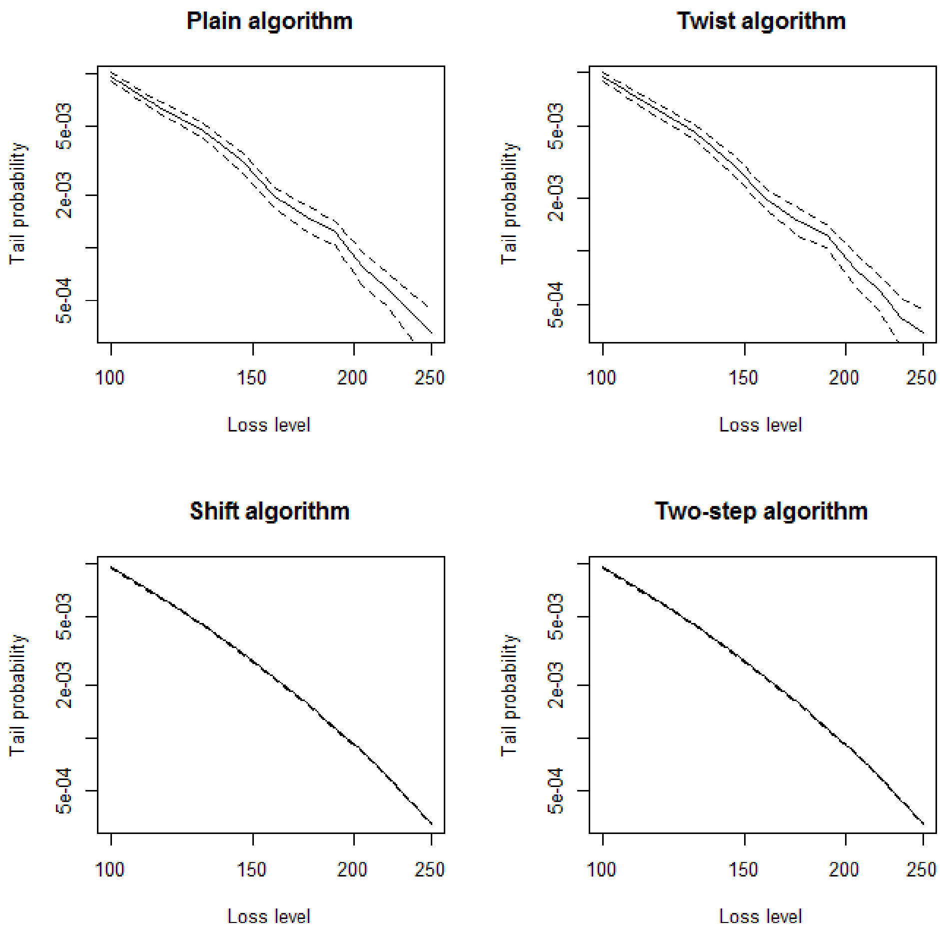

Figure 7.7: Estimated tail probabilities and 95% confidence limits for the portfolio #A1 with all algorithms.

We now relax the constraint of constant p among obligors and estimate tail probabilities for portfolio #A2 described in table 7.2. Obligors are divided into three different groups having same parameters except p. Simulation results are presented in figure 7.9. Twisting algorithm outperforms shifting algorithm because the key driver of default events is probability of default instead of dependence across obligors. Again two-step algorithm produces smallest confidence limits for tail probability estimates.

Table 7.2: Specification of portfolio #A2.

	m	p	c	R
Obligor group #1	311	0,00121	1	0,173
Obligor group #2	311	0,000121	1	0,173
Obligor group #3	311	0,0000121	1	0,173

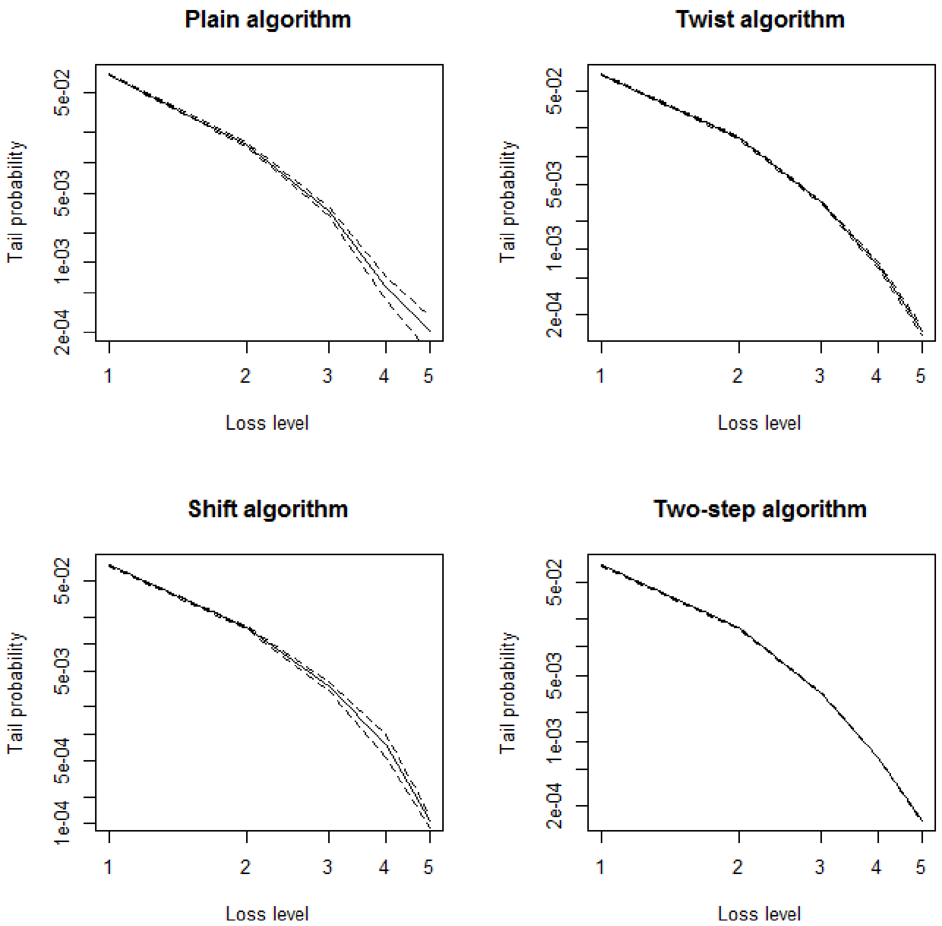

Figure 7.9: Estimated tail probabilities and 95% confidence limits for the portfolio #2 with all algorithms.

7.3.2 Simulation times

Simulation times are roughly the same with the factor shifting algorithm and the plain algorithm. With 933 obligors, solving the optimal shift takes less than one second. Computing 1000 rounds and 100 repetitions takes approximately 45 seconds with a conventional home office laptop. The twisting algorithm and the two-step algorithm are computationally more expensive. Because the optimal twist is dependent on systematic factor Z, it is solved in every single simulation round and it greatly increases computation time. The twisting algorithm and the two-step algorithm take four to five times the computation time of the plain and the shifting algorithms.

8 Risk measures for a real life credit portfolio

This chapter presents tail probability and conditional expectation estimates for our real life credit portfolio. Section 8.1 presents marginal risk contribution estimates for relaxed homogeneous portfolio with seven different obligor groups categorized by seven different industries. In section 8.2 we estimate tail probabilities and conditional expectations for our real life portfolio with varying deterministic cost of default using all four different algorithms. Section 8.3 presents marginal risk contributions for all obligors included in the real life portfolio simulated using the factor shifting algorithm. In section 8.4 we will determine whether our shifting algorithm optimized for a deterministic cost of default is able to produce statistically significant tail probability estimates for obligors having a stochastic cost of default.

8.1 A homogeneous multi-factor model

First, we examine our seven factor model and an example portfolio #B1 that consists of 100 obligors in each seven industry with the same parameters $m = 100$, $p = 0.0121$, $c = 1$ and $R = 0.485$. Each obligor has exposure of 1 and therefore the total exposure of portfolio is 700. Loss level $x = 115$ gives a tail loss probability $\widehat{P}(L > 115) \approx 0.0005$. Mean, min and max values of conditional expectation (7.1) $\hat{r}^{obl}(x = 115)$ estimates of obligor groups are presented in table 8.1. Mean, min and max values are calculated by industry over 100 obligors. The average industry to industry correlations are taken from table 4.4. The simulation was run with 1 000 000 rounds and $R = 0.485$ is the average over all 933 obligors. Note that the larger the average correlation between industries the larger the conditional expectation contribution because m, p, c, R are constant for all obligors. Deviations from mean value is solely caused by simulation error. For obligors in the Consumer Staples industry credit economic capital is then calculated as $EC = ES_{99.95\%} - EL = 0.11908 - 0.0121 = 0.11787$. In other words capital consumption is 11.79% of obligor's exposure using expected shortfall and confidence level of 99.95%.

Table 8.1: Mean, minimum and maximum of $\hat{r}^{obl}(x = 115)$ of the portfolio #B1 and the average correlation between industries.

Industry	Max	Mean	Min	Avg corr
Materials	0,14769	0,14400	0,14035	0,640
Industrial	0,19558	0,19092	0,18741	0,765
Consumer Discretionary	0,18866	0,18303	0,18017	0,745
Consumer Staples	0,12248	0,11908	0,11557	0,568
Heath Care	0,13631	0,13310	0,12886	0,606
Financials	0,16812	0,16320	0,15980	0,691
Inf Tech & Telecom	0,15876	0,15627	0,15309	0,671

8.2 Portfolio risk measures with deterministic cost of default

Simulation results for the real life portfolio are presented in table 8.2. All obligors have deterministic and homogeneous cost of default 1. The exponential twisting algorithm marginally decreases the variance of tail probability estimates. The shifting algorithm produces a substantial reduction compared to the plain algorithm. Standard deviations computed from 100 repetitions of tail probability estimates simulated with shift- and two-step algorithms do not differ significantly from each other. However, the two-step simulation takes almost five times the computation time of the shifting algorithm. Conditional expectation estimates and standard deviations in table 8.3 show that the two-step and shifting algorithms outperform the plain and the twisting algorithms. The twisting algorithm does not reduce the variance of tail probability estimates notably compared to the plain algorithm.

Table 8.2: Tail probability estimates and standard deviations for the real life portfolio with deterministic cost of default 1, simulation rounds 1000 with 100 repetitions.

	Plain		Twist		Shift		Two-step	
x	**Tail Pr**	**Std**	**Tail Pr**	**Std**	**Tail Pr**	**Std**	**Tail Pr**	**Std**
20	1,33E-01	1,00E-02	1,33E-01	9,70E-03	1,33E-01	5,64E-03	1,34E-01	4,65E-03
40	2,18E-02	4,52E-03	2,18E-02	4,49E-03	2,28E-02	1,15E-03	2,27E-02	1,01E-03
60	5,37E-03	2,50E-03	5,34E-03	2,34E-03	5,47E-03	3,31E-04	5,46E-03	2,84E-04
80	1,43E-03	1,08E-03	1,50E-03	1,11E-03	1,60E-03	1,00E-04	1,60E-03	8,62E-05
100	5,20E-04	6,59E-04	5,24E-04	6,47E-04	5,25E-04	3,41E-05	5,26E-04	3,12E-05
120	2,20E-04	4,62E-04	2,02E-04	4,26E-04	1,89E-04	1,28E-05	1,88E-04	1,11E-05
140	8,00E-05	2,73E-04	8,90E-05	2,75E-04	7,16E-05	4,99E-06	7,16E-05	4,06E-06

Table 8.3: Estimated conditional expectations and standard deviations for the real life portfolio with deterministic cost of default 1, simulation rounds 100 000.

	Plain		Twist		Shift		Two-step	
x	$\hat{r}$	**Std**	$\hat{r}$	**Std**	$\hat{r}$	**Std**	$\hat{r}$	**Std**
20	31,8	35,8	31,8	35,2	31,9	14,8	32,0	13,5
40	54,8	105,3	54,8	103,6	54,9	21,1	55,0	18,6
60	76,5	222,9	76,5	218,7	77,3	26,4	77,3	23,0
80	98,1	491,0	97,9	476,0	99,0	30,4	99,0	26,6
100	125,4	1457,6	127,2	1541,6	120,4	33,9	120,5	29,4
120	139,1	1558,7	138,5	1528,7	141,3	37,2	141,5	32,1
140	149,5	780,8	150,2	684,3	162,2	39,3	162,2	33,6

Next we take our real life portfolio and assign following cost of defaults $\{1, 2, 4, 8, 16, 32, 64, 128, 256, 1, 2, 4 \ldots\}$ to 933 obligors alphabetically. For example ASPOCOMP GROUP OYJ is assigned with $\{1, 2, 4\}$ and ADDTECH is assigned with $\{8, 16, 32\}$ and so on. The total exposure of the portfolio is 52 696. Tail probability estimates are presented in table 8.4 and conditional expectations in table 8.5. The plain and the twisting algorithms produce tail probability estimates with much greater variation than the shifting and the two-step algorithms. Simulating with the shifting algorithm 1 000 rounds with 1 000 repetitions we get a tail probability estimate $3.00084 * 10^{-4}$ and a 95 % confidence interval $[3.015797 * 10^{-4}, 2.985882 * 10^{-4}]$ for loss level $x = 9\ 410$.

Table 8.4: Estimated tail probabilities and standard deviations for real life portfolio with varying deterministic cost of default. Simulation rounds 1000 with 100 repetitions.

	Plain		**Twist**		**Shift**		**Two-step**	
x	**Tail Pr**	**Std**	**Tail Pr**	**Std**	**Tail Pr**	**Std**	**Tail Pr**	**Std**
2 000	1,60E-01	1,26E-02	1,60E-01	1,14E-02	1,59E-01	6,83E-03	1,59E-01	5,21E-03
3 500	3,43E-02	5,16E-03	3,51E-02	4,75E-03	3,43E-02	1,96E-03	3,47E-02	1,45E-03
5 000	8,95E-03	3,05E-03	9,13E-03	2,73E-03	8,85E-03	6,16E-04	8,93E-03	4,32E-04
6 500	2,74E-03	1,57E-03	2,84E-03	1,51E-03	2,63E-03	1,91E-04	2,63E-03	1,31E-04
8 000	6,80E-04	8,03E-04	7,19E-04	7,68E-04	8,30E-04	6,32E-05	8,34E-04	4,57E-05
9 500	2,60E-04	5,05E-04	2,46E-04	4,24E-04	2,83E-04	2,14E-05	2,82E-04	1,78E-05
11 000	8,00E-05	2,73E-04	8,52E-05	2,48E-04	9,85E-05	7,75E-06	9,83E-05	4,69E-06

Table 8.5 presents estimates for conditional expectations and standard deviations. Conditional expectation estimates of the plain and the twisting algorithms have is large enough of deviation to make estimates meaningless. The shifting and the two-step algorithms produce relatively accurate estimates. Loss level $x = 11\ 000$ corresponds to $\mathrm{VaR}_{99.9992\%}$ and estimated 95% confidence

intervals for the conditional expectation are $\hat{r} \approx 12\,446 \pm 19.2$ and $\hat{r} \approx 12\,449 \pm 14.0$ using the shifting and the two-step algorithms. Intervals are computed using (6.6) and (6.7).

Table 8.5: Estimated conditional expectations and standard deviations for real life portfolio with varying deterministic cost of default, simulation rounds 100 000.

	Plain		**Twist**		**Shift**		**Two-step**	
x	$\hat{r}$	**Std**	$\hat{r}$	**Std**	$\hat{r}$	**Std**	$\hat{r}$	**Std**
2 000	3004	2733	3010	2670	3014	1336	3016	1144
3 500	4606	6376	4611	6158	4618	1786	4626	1408
5 000	6201	12962	6192	12466	6241	2202	6252	1663
6 500	7908	27922	7922	27981	7844	2469	7833	1870
8 000	9434	49557	9333	45226	9397	2697	9397	2018
9 500	10846	55542	10778	52588	10927	2863	10926	2142
11 000	12071	88168	12065	93206	12446	3099	12449	2252

8.3 Marginal risk contributions

Now we examine the robustness of marginal risk contribution estimates. The marginal risk contribution is measured as a conditional expectation computed using (7.1) with loss level $x = 9\,410$ corresponding to $\mathrm{VaR}_{99.97\%}$. Table 8.6 lists the greatest difference between 95% upper confidence limit and mean value proportionate to the conditional expectation estimate among 933 obligors. The shifting algorithm is run with 1 000 000 simulation rounds and the two-step is run with 200 000 rounds so that the computation time is roughly the same. Both algorithms are run with the same number of repetitions $\{10, 20, 30, 40, 50\}$. The two-step performs better when the same number of simulations rounds is used. Measuring performance from computation time point of view the shifting algorithm is more efficient. Simulation time for 50 000 000 rounds with shifting algorithm takes just under eight hours.

Table 8.6: Greatest difference between 95% upper confidence limit and mean value proportionate to conditional expectation estimate of 933 obligors.

	Shift	**Two-step**
Repetitions	$\left(\hat{r}_{obl}^{upper} - \hat{r}_{obl}\right)/\hat{r}_{obl}$	$\left(\hat{r}_{obl}^{upper} - \hat{r}_{obl}\right)/\hat{r}_{obl}$
10	22,2%	60,5%
20	13,3%	30,2%
30	11,1%	22,1%
40	9,9%	19,1%
50	8,5%	19,0%

Table 8.7 presents marginal risk contribution estimates, upper and lower limits together with one-sided 95% confidence interval widths proportionate to the estimated conditional expectation of the top 10 widest confidence intervals. All obligors have very low probability of default p and low correlation R with their respective industry. This is a logical result as shifting systematic factor Z increases default probability with respect to correlation R. One could argue that NURMINEN LOGISTICS and SCANFIL have exceptionally low correlation with their respective industries, and that it should be revised whether their dependence on other obligors could be captured better by another industry factor. However, the simulation result is vitally important as it illustrates that the marginal contributions can be estimated even if we have obligors with very low correlation coupled with very low probability of default.

Table 8.7: Top 10 widest 95% confidence intervals of marginal risk contribution estimates simulated using shifting algorithm using 50 000 000 rounds.

OBLIGOR	R	p	c	Upper Limit	$\hat{r}^{obl}$	Lower Limit	Width	EL
NURMINEN LOGISTICS OYJ	0,00543	9,00E-05	64	0,00715	0,00659	0,00603	8,5%	0,00576
SCANFIL OYJ	0,06831	9,00E-05	1	0,00021	0,00020	0,00019	5,2%	0,00009
NURMINEN LOGISTICS OYJ	0,00543	2,00E-04	128	0,03119	0,02976	0,02833	4,8%	0,0256
SCANFIL OYJ	0,06831	2,00E-04	2	0,00091	0,00088	0,00086	3,1%	0,0004
TAKOMA OYJ	0,21242	9,00E-05	128	0,15537	0,15158	0,14779	2,5%	0,01152
ORION OYJ	0,25216	9,00E-05	1	0,00097	0,00094	0,00092	2,5%	0,00009
RAPALA VMC OYJ	0,20614	9,00E-05	64	0,06418	0,06268	0,06117	2,4%	0,00576
UNIFLEX AB	0,21849	9,00E-05	8	0,00957	0,00936	0,00914	2,3%	0,00072
TURVATIIMI OYJ	0,18963	9,00E-05	1	0,00089	0,00087	0,00085	2,3%	0,00009
TAKOMA OYJ	0,21242	9,00E-05	64	0,07605	0,07435	0,07265	2,3%	0,00576

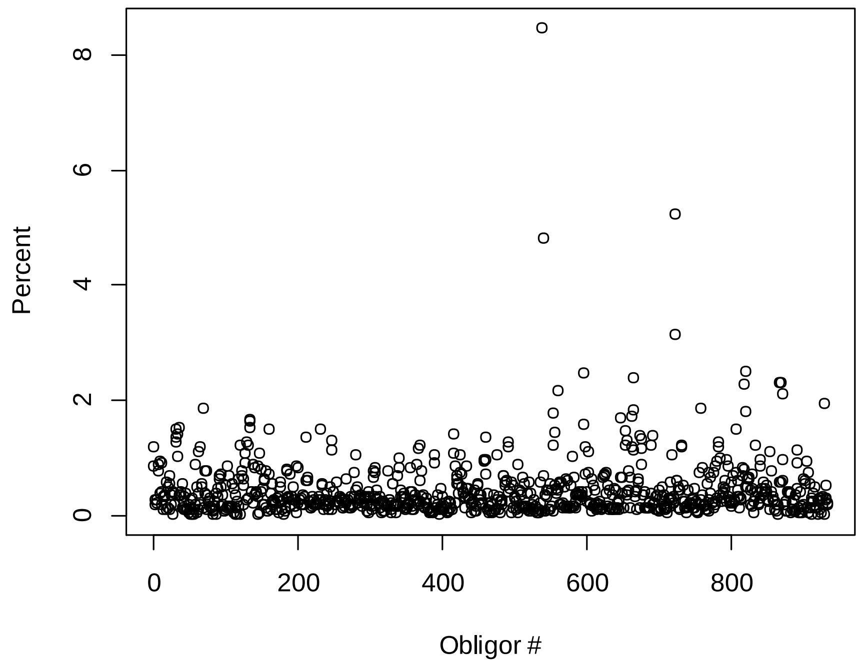

Figure 8.1: One-sided 95% confidence intervals proportionate to estimated conditional expectations by obligor.

Figure 8.1 shows one-sided 95% confidence intervals of marginal risk contributions estimated using the shifting algorithm. Out of 933 obligors 853 have smaller than 1% difference between the upper limit and the mean value. The portfolio conditional expectation with $x = 9\,410$ is $\hat{r} = ES_{99.97\%} = 10\,840.95 \pm 0.80$.

8.4 Tail probabilities with stochastic cost of default

Although the factor shifting algorithm is optimized for a deterministic cost of default it is still interesting to test if it can be used with a stochastic cost of default. Every obligor is now assigned with expectation of LGD of 45% and the LGD has density function $B(1.35, 1.65, x)$. Additionally, cost of defaults assigned to each obligor in section 8.2 are scaled by dividing with 0.45 to keep the expected loss EL of obligors unchanged.

Table 8.9: Tail probability estimates, standard deviations and 95% confidence limits for the real life portfolio with a stochastic LGD using the shifting algorithm with 1000 rounds and 500 repetitions.

x	Tail Pr	Std	Upper	Lower
2 000	1,663E-01	8,048E-03	1,656E-01	1,670E-01
3 500	3,789E-02	2,598E-03	3,766E-02	3,812E-02
5 000	9,963E-03	7,523E-04	9,897E-03	1,003E-02
6 500	2,949E-03	2,545E-04	2,927E-03	2,972E-03
8 000	9,436E-04	8,334E-05	9,509E-04	9,363E-04
9 500	3,196E-04	3,143E-05	3,224E-04	3,168E-04
11 000	1,137E-04	1,228E-05	1,148E-04	1,126E-04

Table 8.9 presents tail probability estimates, standard deviations and 95% confidence limits with the same loss levels we used in previous section with the deterministic cost of default. Increasing simulation repetition to 500 we get fairly small confidence limits. Small enough to conclude that the factor shifting algorithm produces statistically significant tail probability estimates. Simulating

1 000 rounds with 1 000 repetitions we get a tail probability estimate $2.992905 * 10^{-4}$ and confidence interval $[3.011056 * 10^{-4}, 2.974755 * 10^{-4}]$ for loss level $x = 9\,600$. It is roughly the same tail probability, $2.985882 * 10^{-4}$, that corresponds to the loss level $x = 9\,410$ with the deterministic cost of default. Therefore, stochastic LGD increases $\mathrm{VaR}_{99.97\%}$ 2.0% in our setting. If we had infinite number of obligors we would get the same VaR for the deterministic and the stochastic LGD.

9 Conclusions

The exponential twisting reduces variance of tail probability estimates when default events of obligors are independent or have relatively low correlation. Simulation results for the relaxed homogeneous portfolio suggest that the exponential twisting produces a greater variance reduction than the factors shifting with very low probabilities of default and very low correlations. In real world such low levels of correlation are rarely observed in corporate loan portfolios. It would be interesting to make the same comparison with the exponential twisting and the factor shifting algorithms for a retail credit portfolio. Because retail exposures tend to have weaker correlation structure than corporate exposures. Using our real life corporate loan portfolio we can conclude that the exponential twisting alone does not result meaningful a variance reduction of tail probability estimates.

The real life portfolio consisted of obligors having correlation structure inferred directly from logarithmic market returns of 311 companies listed in the NASDAQ OMX Helsinki and the NASDAQ OMX Stockholm stock exchanges. Credit ratings and corresponding PDs assigned to obligors represent a rating distribution of a typical corporate loan portfolio except having high number of obligors with the highest credit rating to ensure relevance of simulation results.

The normal approximation method was used in solving the optimal shift parameter for the factor shifting. Although the normal approximation method is optimal for portfolios having infinite number of obligors it still produces a substantial variance reduction for the estimates in our real life setting. The normal approximation gives optimal shift that can be used with stochastic cost of default and we still get a significant variance reduction.

The optimal shift aims to minimize variance of a specific tail probability estimate but we managed to estimate individual marginal risk contributions with relatively high degree of statistical significance. For obligors with very low correlation between their industries we were able to simulate marginal risk contribution estimates with one-sided 95% confidence intervals less than 8.5%.

The absolute quantity of exposures does not directly have an effect on simulation performance when estimating marginal risk contributions. However, the exposure distribution of a portfolio has an effect on the variance of risk contribution estimates, especially when the number of obligors is small. Meaning that if the total portfolio exposure is concentrated on small number of obligors the conditional expectation estimates for obligors with a small exposure, and small correlation and probability of default, could have a much greater variance. Our real life portfolio consisted of exposures ranging from 1 to 256. Examining obligors having the same correlation and the probability of default with differing exposures the confidence intervals of marginal risk contribution estimates were not statistically different. Thus, we can conclude that in our model the distribution of exposures does not have an impact on simulation performance. We would need a greater concentration of portfolio exposure for a smaller number of obligors to observe this effect.

We concluded that our model increases $\mathrm{VaR}_{99.97\%}$ by 2.0% when the stochastic cost of default is implemented with the real life portfolio. This increase could be seen as an additional concentration risk. Increasing the number of obligors would result smaller difference between the deterministic and the stochastic LGD and with infinite number of obligors the difference would go to zero.

Simulations need to be computed in batches because R programming runs everything in RAM and therefore a conventional home office laptop is only capable of storing limited amount of data at the time. The R is still very efficient because it enables to make computation in matrix form without for-loops. One must notice that the model developed in this thesis and its performance is highly dependent on the calibration and the constituents of the portfolio. However, we have proven that with very low default probabilities and very low correlations it is possible to estimate marginal risk contributions without excessive computing power relatively accurately and it would not take longer than one day at the office!

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Appendix A

TICKER	CORPORATION	GICS CODE	RETURNS	CORR
ACG1V FH	ASPOCOMP GROUP OYJ	70	166	0,3663
ADDTB SS	ADDTECH AB-B SHARES	20	158	0,6409
AEROB SS	AEROCRINE AB - B	35	89	0,4161
AFB SS	AF AB-B SHS	20	166	0,5345
AFE1V FH	AFFECTO OYJ	70	114	0,3754
AHL1V FH	AHLSTROM OYJ	15	104	0,4911
AKTAV FH	AKTIA BANK OYJ	40	62	0,2766
ALBBV FH	ALANDSBANKEN-B	40	166	0,2204
ALFA SS	ALFA LAVAL AB	20	150	0,5724
ALIV SS	AUTOLIV INC-SWED DEP RECEIPT	25	143	0,5828
ALN1V FH	ALMA MEDIA CORP	25	115	0,2423
ALNX SS	ALLENEX AB	35	95	0,3174
AMEAS FH	AMER SPORTS OYJ	25	166	0,5732
ANOT SS	ANOTO GROUP AB	70	166	0,4496
APETI FH	APETIT OYJ	30	163	0,4567
ASP SS	ASPIRO AB	70	166	0,3753
ASSAB SS	ASSA ABLOY AB-B	20	166	0,5420
ASU1V FH	ASPO OYJ	20	166	0,4071
ATCOB SS	ATLAS COPCO AB-B SHS	20	166	0,6812
ATRAV FH	ATRIA OYJ	30	166	0,5309
AVEGB SS	AVEGA GROUP AB-B SHS	70	83	0,5593
AXFO SS	AXFOOD AB	30	166	0,4283
AXIS SS	AXIS COMMUNICATIONS AB	70	166	0,4856
AZN SS	ASTRAZENECA PLC	35	166	0,2006
BALDB SS	FASTIGHETS AB BALDER-B SHRS	40	166	0,4094
BAS1V FH	BASWARE OYJ	70	166	0,4857
BBTOB SS	B&B TOOLS AB-B SHS	20	166	0,6334
BEGR SS	BE GROUP AB	20	96	0,7579
BEIAB SS	BEIJER ALMA AB	20	166	0,6365

BEIJB SS	G & L BEIJER AB-B SHS	20	166	0,5264
BELE SS	BEIJER ELECTRONICS AB	70	166	0,5382
BETSB SS	BETSSON AB	25	166	0,3883
BILIA SS	BILIA AB-A SHS	25	166	0,6459
BILL SS	BILLERUD AKTIEBOLAG	15	156	0,6037
BINV SS	BIOINVENT INTERNATIONAL AB	35	161	0,4739
BIOBV FH	BIOHIT OYJ-B	35	166	0,4908
BIOGB SS	BIOGAIA AB-B SHS	35	166	0,5712
BIOT SS	BIOTAGE AB	35	166	0,5737
BMAX SS	BYGGMAX GROUP AB	25	53	0,6643
BOL SS	BOLIDEN AB	15	166	0,4863
BONG SS	BONG LJUNGDAHL AB	20	160	0,3606
BORG SS	BJOERN BORG AB	25	119	0,4082
BOUL SS	BOULE DIAGNOSTICS INTERNATIO	35	41	0,3827
BRGB SS	BERGS TIMBER AB-B SHARES	15	166	0,3182
BTH1V FH	BIOTIE THERAPIES OYJ	35	166	0,2622
BTSB SS	BTS GROUP AB-B SHARES	20	161	0,5095
BURE SS	BURE EQUITY AB	40	166	0,3719
CAST SS	CASTELLUM AB	40	166	0,6015
CATE SS	CATENA AB	40	103	0,4007
CCC SS	CAVOTEC SA	20	37	0,3443
CDON SS	CDON GROUP	25	47	0,5447
CEVI SS	CELLAVISION AB	35	85	0,4969
CGCBV FH	CARGOTEC OYJ-B SHARE	20	113	0,8167
CLAB SS	CLOETTA AB-B SHS	30	71	0,2747
CLASB SS	CLAS OHLSON AB-B SHS	25	166	0,5217
CNC1V FH	CENCORP OYJ	70	166	0,4900
COIC SS	CONCENTRIC AB	20	41	0,7139
CONSB SS	CONSILIUM AB- B SHS	20	166	0,3862
CORE SS	COREM PROPERTY GROUP AB-B	40	166	0,2605
CPMBV FH	CAPMAN OYJ-B SHS	40	163	0,5090
CRA1V FH	CRAMO OYJ	20	166	0,6177
CTH1V FH	COMPONENTA OYJ	20	152	0,3730
CTL1V FH	COMPTEL OYJ	70	166	0,6277

CTT SS	CTT SYSTEMS AB	20	166	0,3684
CTY1S FH	CITYCON OYJ	40	163	0,5187
CYBE SS	CYBERCOM GROUP AB	70	166	0,6161
DEDI SS	DEDICARE AB-B	35	42	0,1965
DGC SS	DGC ONE AB	70	76	0,4654
DIG1V FH	DIGIA OYJ	70	163	0,5449
DIOS SS	DIOS FASTIGHETER AB	40	102	0,3594
DORO SS	DORO AB	70	102	0,2519
DOV1V FH	DOVRE GROUP OYJ	20	166	0,4320
DUNI SS	DUNI AB	25	84	0,6731
DURCB SS	DUROC AB-B SHS	20	166	0,5566
EBC1V FH	ELEKTROBIT OYJ	70	166	0,4878
ECEX SS	EAST CAPITAL EXPLORER AB	40	84	0,5729
EFO1V FH	EFORE OYJ	20	166	0,3944
EKTAB SS	ELEKTA AB-B SHS	35	166	0,3047
ELANB SS	ELANDERS AB-B SHS	25	166	0,5502
ELEAV FH	ELECSTER OYJ-A SHS	20	166	0,4310
ELEC SS	ELECTRA GRUPPEN AB	25	100	0,5358
ELI1V FH	ELISA OYJ	70	166	0,4923
ELOSB SS	ELOS AB	35	166	0,3409
ELUXB SS	ELECTROLUX AB-SER B	25	166	0,5366
ENEA SS	ENEA AB	70	166	0,4908
ENRO SS	ENIRO AB	25	166	0,4516
EQV1V FH	EQ OYJ	40	157	0,4481
ERICB SS	ERICSSON LM-B SHS	70	166	0,6487
ETT1V FH	ETTEPLAN OYJ	20	159	0,4423
EWRK SS	EWORK SCANDINAVIA AB	70	78	0,4665
EXL1V FH	EXEL COMPOSITES OYJ	20	163	0,5195
FABG SS	FABEGE AB	40	166	0,6261
FAG SS	FAGERHULT AB	20	159	0,3851
FEEL SS	FEELGOOD SVENSKA AB	35	163	0,3637
FIA1S FH	FINNAIR OYJ	20	166	0,4570
FINGB SS	FINGERPRINT CARDS AB-B	70	166	0,4103
FIS1V FH	FISKARS OYJ ABP	25	166	0,5185

FLG1S FH	FINNLINES OYJ	20	163	0,3814
FPAR SS	FASTPARTNER AB	40	166	0,6291
FPIP SS	FORMPIPE SOFTWARE AB	70	116	0,3787
FSC1V FH	F-SECURE OYJ	70	166	0,6311
GETIB SS	GETINGE AB-B SHS	35	166	0,5205
GHP SS	GLOBAL HEALTH PARTNER AB	35	73	0,4437
GLA1V FH	GLASTON OYJ ABP	20	163	0,4899
GUNN SS	GUNNEBO AB	20	166	0,6151
GVKOB SS	GEVEKO AB-B SHS	20	166	0,4573
HEBAB SS	HEBA FASTIGHETS AB-B	40	166	0,4943
HEMX SS	HEMTEX AB	25	109	0,5991
HEXAB SS	HEXAGON AB-B SHS	70	166	0,4484
HIQ SS	HIQ INTERNATIONAL AB	70	166	0,7740
HKSAV FH	HKSCAN OYJ-A SHS	30	166	0,6428
HLDX SS	HALDEX AB	20	166	0,7196
HMB SS	HENNES & MAURITZ AB-B SHS	25	166	0,4076
HMS SS	HMS NETWORKS AB	70	85	0,3249
HOLMB SS	HOLMEN AB-B SHARES	15	166	0,5131
HONBS FH	HONKARAKENNE OYJ-B SHS	25	166	0,3918
HPOLB SS	HEXPOL AB	20	77	0,6707
HUFVC SS	HUFVUDSTADEN AB-C SHS	40	153	0,5802
HUH1V FH	HUHTAMAKI OYJ	15	166	0,4205
HUSQB SS	HUSQVARNA AB-B SHS	25	101	0,5482
IARB SS	IAR SYSTEMS GROUP AB	70	166	0,6230
ICP1V FH	INCAP OYJ	20	149	0,3334
ICTAB SS	INTELLECTA AB-B SHARES	20	160	0,2666
IFA1V FH	INNOFACTOR OYJ	70	166	0,4277
IFSB SS	INDUST & FINANCIAL SYSTEM-B	70	166	0,6230
ILK2S FH	ILKKA-YHTYMA OYJ-II	25	156	0,4561
INDT SS	INDUTRADE AB	20	109	0,6602
INDUC SS	INDUSTRIVARDEN AB-C SHS	40	166	0,7961
INVEB SS	INVESTOR AB-B SHS	40	166	0,7368
IS SS	IMAGE SYSTEMS AB	70	162	0,3016
ITABB SS	ITAB SHOP CONCEPT AB	20	123	0,4455

JM SS	JM AB	25	166	0,5994
KABEB SS	KABE HUSVAGNAR AB-B SHS	25	166	0,4909
KAHL SS	KAPPAHL AB	25	105	0,5663
KARO SS	KARO BIO AB	35	166	0,4099
KCR1V FH	KONECRANES OYJ	20	166	0,7036
KDEV SS	KAROLINSKA DEVELOPMENT-B	35	43	0,3534
KELAS FH	KESLA OYJ-A	20	166	0,5126
KESBV FH	KESKO OYJ-B SHS	30	166	0,4363
KINVB SS	INVESTMENT AB KINNEVIK-B SHS	40	180	0,4742
KLED SS	KUNGSLEDEN AB	40	166	0,5880
KLOV SS	KLOVERN AB	40	147	0,4109
KNEBV FH	KONE OYJ-B	20	113	0,5248
KNOW SS	KNOW IT AB	70	162	0,6384
KRA1V FH	KEMIRA OYJ	15	166	0,3857
KSLAV FH	KESKISUOMALAINEN OYJ-A SHS	25	142	0,4014
LAGRB SS	LAGERCRANTZ GROUP AB-B SHS	70	158	0,5742
LAMMB SS	LAMMHULTS DESIGN GROUP AB	20	166	0,4750
LAT1V FH	LASSILA & TIKANOJA OYJ	20	166	0,5297
LATOB SS	INVESTMENT AB LATOUR-B SHS	40	166	0,5780
LEM1S FH	LEMMINKAINEN OYJ	20	166	0,6241
LIAB SS	LINDAB INTERNATIONAL AB	20	96	0,7869
LJGRB SS	ATRIUM LJUNGBERG AB-B SHS	40	166	0,4259
LUMI SS	LUNDIN MINING CORP-SDR	15	166	0,3953
LUNDB SS	LUNDBERGS AB-B SHS	40	166	0,6589
MARAS FH	MARTELA OYJ	20	166	0,4844
MEABB SS	MALMBERGS ELEKTRISKA AB-B	20	155	0,4915
MEDAA SS	MEDA AB-A SHS	35	166	0,4457
MEKO SS	MEKONOMEN AB	25	166	0,3836
MELK SS	MELKER SCHORLING AB	40	98	0,7056
MEO1V FH	METSO OYJ	20	166	0,7305
METSA FH	METSA BOARD OYJ	15	166	0,6064
MIC SS	MILLICOM INTL CELLULAR-SDR	70	125	0,4551
MIDWB SS	MIDWAY HOLDING AB-B SHS	20	166	0,2746
MMO1V FH	MARIMEKKO OYJ	25	166	0,5156

MOB SS	MOBERG DERMA AB	35	41	0,2987
MQ SS	MQ HOLDING AB	25	53	0,4066
MSCB SS	MSC KONSULT AB-B SHS	70	166	0,5184
MSONB SS	MIDSONA AB-B SHS	30	166	0,4497
MTGB SS	MODERN TIMES GROUP-B SHS	25	166	0,6184
MULQ SS	MULTIQ INTERNATIONAL AB	70	163	0,4608
MVIRB SS	MEDIVIR AB-B SHS	35	166	0,5587
NCCB SS	NCC AB-B SHS	20	166	0,5802
NDA1V FH	NORDEA BANK AB - FDR	40	166	0,4626
NEO1V FH	NEO INDUSTRIAL OYJ	20	166	0,3442
NETB SS	NET ENTERTAINMENT NE AB	70	90	0,4439
NETIB SS	NET INSIGHT AB-B	70	166	0,5923
NEWAB SS	NEW WAVE GROUP AB -B SHS	25	166	0,7077
NIBEB SS	NIBE INDUSTRIER AB-B SHS	20	166	0,4580
NLG1V FH	NURMINEN LOGISTICS OYJ-A	20	96	0,0054
NMAN SS	NEDERMAN HOLDING AB	20	90	0,5861
NOBI SS	NOBIA AB	25	149	0,6362
NOK1V FH	NOKIA OYJ	70	166	0,5270
NOLAB SS	NOLATO AB-B SHS	70	166	0,5933
NOMI SS	NORDIC MINES AB	15	95	0,3348
NOTE SS	NOTE AB	70	125	0,3130
NOVE SS	NOVESTRA AB	40	163	0,2712
NRE1V FH	NOKIAN RENKAAT OYJ	25	166	0,5603
NSPB SS	NORDIC SERVICE PARTNERS HLDG	25	102	0,5160
NTEKB SS	NOVOTEK AB-B SHS	70	166	0,5315
OASM SS	OASMIA PHARMACEUTICAL AB	35	106	0,4792
ODD SS	ODD MOLLY INTERNATIONAL AB	25	89	0,4503
OEMB SS	OEM INTERNATIONAL AB-B SHS	20	166	0,4706
OKDBV FH	ORIOLA-KD OYJ B SHARES	35	100	0,5050
OKM1V FH	OKMETIC OYJ	70	163	0,4207
OLVAS FH	OLVI OYJ-A SHARES	30	163	0,5098
ORES SS	ORESUND INVESTMENT AB	40	166	0,4158
ORI SS	ORIFLAME COSMETICS SA-SDR	30	128	0,4618
ORNBV FH	ORION OYJ-CLASS B	35	100	0,2522

ORTIB SS	ORTIVUS AB-B SHS	35	166	0,4108
ORX SS	OREXO AB	35	108	0,4328
OTE1V FH	OUTOTEC OYJ	20	97	0,7041
OUT1V FH	OUTOKUMPU OYJ	15	166	0,5228
PACT SS	PROACT IT GROUP AB	70	166	0,5748
PART SS	PARTNERTECH AB	70	166	0,7323
PEABB SS	PEAB AB	20	166	0,5011
PKC1V FH	PKC GROUP OYJ	20	166	0,7084
PKK1V FH	POHJOIS-KARJALAN KIRJAPAINO	25	102	0,1925
PNA1V FH	PANOSTAJA OYJ	40	166	0,2050
PON1V FH	PONSSE OYJ	20	160	0,5718
POOLB SS	POOLIA AB-B SH	20	166	0,4946
POY1V FH	POYRY OYJ	20	166	0,5564
PREC SS	PRECISE BIOMETRICS AB	70	166	0,5387
PREVB SS	PREVAS AB-B SHS	70	166	0,5279
PRICB SS	PRICER AB-B SHS	70	166	0,4856
PROB SS	PROBI AB	35	119	0,2242
PROEB SS	PROFFICE AB-B SHS	20	166	0,6307
PROFB SS	PROFILGRUPPEN AB-B SHS	15	166	0,4288
QPR1V FH	QPR SOFTWARE OYJ	70	146	0,3534
RABTB SS	REDERI AB TRANSATLANTIC	20	166	0,4087
RAIVV FH	RAISIO OYJ-V SHS	30	166	0,4294
RAP1V FH	RAPALA VMC OYJ	25	163	0,2061
RATOB SS	RATOS AB-B SHS	40	166	0,5726
RAYB SS	RAYSEARCH LABORATORIES AB	35	166	0,5100
REG1V FH	REVENIO GROUP OYJ	35	166	0,3466
REJLB SS	REJLERKONCERNEN AB-B SHARES	20	138	0,3560
REZT SS	REZIDOR HOTEL GROUP AB	25	96	0,6406
RMR1V FH	RAMIRENT OYJ	20	159	0,6592
RNBS SS	RNB RETAIL AND BRANDS AB	25	161	0,5250
RROS SS	ROTTNEROS AB	15	166	0,4425
RTIMB SS	RORVIK TIMBER AB-B SHS	15	166	0,3750
RUTAV FH	RAUTE OYJ-A SHS	20	166	0,5510
SAA1V FH	SANOMA OYJ	25	166	0,5995

SAABB SS	SAAB AB-B	20	166	0,5085
SAGCV FH	SAGA FURS OYJ	20	149	0,2061
SAMAS FH	SAMPO OYJ-A SHS	40	180	0,5180
SAND SS	SANDVIK AB	20	166	0,6893
SAS SS	SAS AB	20	166	0,3972
SCAB SS	SVENSKA CELLULOSA AB-B SHS	15	166	0,5076
SCI1V FH	SIEVI CAPITAL OYJ	40	166	0,3565
SCL1V FH	SCANFIL OYJ	70	34	0,0683
SDA1V FH	SPONDA OYJ	40	166	0,5206
SEBC SS	SKANDINAVISKA ENSKILDA BAN-C	40	166	0,5563
SECTB SS	SECTRA AB-B SHS	35	166	0,3684
SECUB SS	SECURITAS AB-B SHS	20	166	0,4623
SEMC SS	SEMCON AB	20	166	0,6390
SENS SS	SENSYS TRAFFIC AB	70	166	0,3266
SHBB SS	SVENSKA HANDELSBANKEN-B SHS	40	166	0,5689
SINT SS	SINTERCAST AB	20	166	0,4514
SKAB SS	SKANSKA AB-B SHS	20	166	0,5892
SKFB SS	SKF AB-B SHARES	20	166	0,6708
SKISB SS	SKISTAR AB	25	166	0,3864
SMF SS	SEMAFO INC	15	37	0,2354
SOBI SS	SWEDISH ORPHAN BIOVITRUM AB	35	98	0,4116
SOFB SS	SOFTRONIC AB-B SHS	70	166	0,6020
SOPRA FH	SOPRANO OYJ	70	87	0,2845
SOSI1 FH	SOTKAMO SILVER AB	15	28	0,2178
SRV1V FH	SRV GROUP OYJ	20	89	0,4714
SSABB SS	SSAB AB - B SHARES	15	166	0,5942
SSH1V FH	SSH COMMUNICATIONS SECURITY	70	166	0,4516
SSK1S FH	SUOMEN SAASTAJIEN KIINTEISTO	40	72	0,3431
STCBV FH	STOCKMANN OYJ ABP-B SHARE	25	166	0,6714
STERV FH	STORA ENSO OYJ-R SHS	15	166	0,6591
STQ1V FH	SOLTEQ OYJ	70	166	0,5374
SUY1V FH	SUOMINEN OYJ	30	157	0,4894
SVEDB SS	SVEDBERGS I DALSTORP AB-B SH	20	166	0,5121
SVIK SS	STUDSVIK AB	20	162	0,4683

SWECB SS	SWECO AB-B SHS	20	166	0,4514
SWMA SS	SWEDISH MATCH AB	30	166	0,2991
SWOLB SS	SWEDOL AB-B	25	101	0,6292
SYSR SS	SYSTEMAIR AB	20	85	0,6980
TAGR SS	TRIGON AGRI A/S	30	90	0,5064
TAM1V FH	TAKOMA OYJ	20	123	0,2124
TEL2B SS	TELE2 AB-B SHS	70	166	0,5529
TEM1V FH	TECNOTREE OYJ	70	163	0,5090
TIE1V FH	TIETO OYJ	70	166	0,6287
TIK1V FH	TIKKURILA OYJ	15	56	0,5334
TLS1V FH	TELIASONERA AB	70	143	0,3726
TLT1V FH	TELESTE OYJ	70	166	0,5839
TLV1V FH	TALVIVAARA MINING CO PLC-DI	15	61	0,4884
TPS1V FH	TECHNOPOLIS OYJ	40	166	0,5966
TRACB SS	AB TRACTION -B SHS	40	166	0,5220
TRAD SS	TRADEDOUBLER	70	108	0,5474
TRELB SS	TRELLEBORG AB-B SHS	20	166	0,6597
TRH1V FH	TRAINERS' HOUSE OYJ	20	166	0,3582
TRMO SS	TRANSMODE HOLDING AB	70	42	0,4222
TTM1V FH	TALENTUM OYJ	25	166	0,4577
TULAV FH	TULIKIVI OYJ-A SHS	20	166	0,5066
TUT1V FH	TURVATIIMI OYJ	20	160	0,1896
UFLXB SS	UNIFLEX AB	20	120	0,2185
UNIB SS	UNIBET GROUP PLC-SDR	25	123	0,3793
UNR1V FH	UPONOR OYJ	20	166	0,6285
UPM1V FH	UPM-KYMMENE OYJ	15	166	0,6426
VAC1V FH	VACON OYJ	20	166	0,5358
VAIAS FH	VAISALA OYJ- A SHS	70	166	0,5436
VBGB SS	VBG GROUP AB-B SHS	20	166	0,6326
VIK1V FH	VIKING LINE ABP	25	157	0,1996
VITB SS	VITEC SOFTWARE GROUP AB-B SH	70	159	0,2406
VITR SS	VITROLIFE AB	35	161	0,4813
VNIL SS	VOSTOK NAFTA INVESTMENT-SDR	40	88	0,4930
VOLVB SS	VOLVO AB-B SHS	20	166	0,6833

VRGB SS	VENUE RETAIL GROUP AB	25	166	0,5015
WALLB SS	WALLENSTAM AB-B SHS	40	166	0,6650
WAT1V FH	VAAHTO GROUP OYJ	20	166	0,2913
WIHL SS	WIHLBORGS FASTIGHETER AB	40	114	0,6124
WRT1V FH	WARTSILA OYJ ABP	20	166	0,6648
WUF1V FH	WULFF-GROUP OYJ	25	149	0,4306
XANOB SS	XANO INDUSTRI AB	20	166	0,4347
XNS1V FH	IXONOS OYJ	70	166	0,3878
YLEPS FH	YLEISELEKTRONIIKKA OYJ	70	166	0,2784
YTY1V FH	YIT OYJ	20	166	0,7090