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TITLE

Simplified of time averaged Navier–Stokes equations and Reynold's stress equations for an implicit sweep.

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ABSTRACT

This paper presents an improved method of upwind the Euler part of the fluxes with the Reynolds-stress closure model. This new method takes account the anisotropic nature of the Reynolds-stresses. Resulting eigenvalues show effect of the anisotropy. Because the characteristic variables and the right eigenvector matrix are relatively complex, a simplified diagonalization is introduced.

MAIN RESULT

Possible equations for upwinding

PAGES
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KEY WORDS
Diagonalization, Reynolds-stress model, upwinding, implicit sweep

APPROVED BY
Timo Siikonen May 15, 1995

When the closure model is applied the primitive, the convervative and the flux vector can be written as

$$V = \begin{pmatrix} \rho \\ u \\ v \\ w \\ e \\ \widetilde{u_1^" u_1^"} \\ \widetilde{u_1^" u_2^"} \\ \widetilde{u_1^" u_3^"} \\ \widetilde{u_2^" u_2^"} \\ \widetilde{u_2^" u_3^"} \\ \widetilde{u_3^" u_3^"} \end{pmatrix}, U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \\ \rho u_1^" u_1^" \\ \rho u_1^" u_2^" \\ \rho u_1^" u_3^" \\ \rho u_2^" u_2^" \\ \rho u_2^" u_3^" \\ \rho u_3^" u_3^" \end{pmatrix}, F = \begin{pmatrix} \rho u \\ \rho u^2 + p + \overline{\rho u_1^" u_1^"} \\ \rho v u + \overline{\rho u_1^" u_2^"} \\ \rho w u + \overline{\rho u_1^" u_3^"} \\ (E + p + \overline{\rho u_1^" u_1^"})u + \overline{\rho u_1^" u_2^"} v + \overline{\rho u_1^" u_3^"} w \\ \rho u \widetilde{u_i^" u_j^"} \end{pmatrix} \tag{0.1}$$

For simplicity, diagonalization is made for primitive equations. Next write inviscid forms of Reynold's equations. Continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j} = 0 \tag{0.2}$$

take derivatives out and take derivatives only in direction $j = 1$ and obtain

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \tag{0.3}$$

Navier–Stokes equations in tensor form

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j + \overline{\rho u_i^" u_j^"}) + \frac{\partial p}{\partial x_i} = 0 \tag{0.4}$$

Make same manipulations as for the continuity equation and also use the continuity equation and the following

$$dp = \frac{\partial p}{\partial \rho} d\rho + \frac{\partial p}{\partial e} de = p_\rho d\rho + p_e de$$

to obtain

$$\frac{\partial u}{\partial t} + (\frac{p_\rho}{\rho} + \frac{u^" u^"}{\rho})\frac{\partial \rho}{\partial x} + u\frac{\partial u}{\partial x} + \frac{p_e}{\rho}\frac{\partial e}{\partial x} + \frac{\partial u^" u^"}{\partial x} = 0 \tag{0.5}$$

$$\frac{\partial v}{\partial t} + \frac{u^" v^"}{\rho}\frac{\partial \rho}{\partial x} + u\frac{\partial v}{\partial x} + \frac{\partial u^" v^"}{\partial x} = 0 \tag{0.6}$$

$$\frac{\partial w}{\partial t} + \frac{u^" w^"}{\rho}\frac{\partial \rho}{\partial x} + u\frac{\partial w}{\partial x} + \frac{\partial u^" w^"}{\partial x} = 0 \tag{0.7}$$

To be able to diagonalized the Jacobian matrix we need to take account production term, which is not conservative, in RSM. Nonviscous part of Reynold's stresses with the production term

$$\frac{\partial u_i^" u_j^"}{\partial t} + u_1^" u_i^" \frac{\partial u_j}{\partial x} + u_1^" u_j^" \frac{\partial u_i}{\partial x} + u_1 \frac{\partial u_i^" u_j^"}{\partial x} = 0 \tag{0.8}$$

Production term was enclosed to the system of equations to make matrix diagonalized.

Energy equation is

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}\left[u_j E + u_j p + \overline{E^" u_j^"}\right] = 0 \tag{0.9}$$

where

$$E = \rho e + \frac{1}{2}\rho u_j^2 + \frac{1}{2}\rho \widetilde{u_j^" u_j^"} \tag{0.10}$$

and

$$\overline{E'' u_j''} = c_v \bar{\rho} \widetilde{T'' u_j} + \overline{\rho u_i'' u_j''} u_i + \rho \frac{\widetilde{u_j'' u_i'' u_i''}}{2} \tag{0.11}$$

The first and the last one in a right hand side are treated as a viscous terms and are neglected in this case. Substituting this in the energy equation we can get following conservative equation

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}\left[u_j E + u_j p + \overline{\rho u_i'' u_j''} u_i\right] = 0 \tag{0.12}$$

By using continuity, momentum, equation of total energy and some part RSM following is obtained

$$\frac{\partial e}{\partial t} + \frac{p}{\rho}\frac{\partial u}{\partial x} + u\frac{\partial e}{\partial x} = 0 \tag{0.13}$$

From these equation construct the A' matrix $A' dV = 0$

$$A' = \begin{pmatrix}
u & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{p_\rho}{\rho} + \frac{\widetilde{u''u''}}{\rho} & u & 0 & 0 & \frac{p_e}{\rho} & 1 & 0 & 0 & 0 & 0 & 0 \\
\frac{\widetilde{u''v''}}{\rho} & 0 & u & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{\widetilde{u''w''}}{\rho} & 0 & 0 & u & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & \frac{p}{\rho} & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\widetilde{u''u''} & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 \\
0 & \widetilde{u''v''} & \widetilde{u''u''} & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 \\
0 & \widetilde{u''w''} & 0 & \widetilde{u''u''} & 0 & 0 & 0 & u & 0 & 0 & 0 \\
0 & 0 & 2\widetilde{u''v''} & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 \\
0 & 0 & \widetilde{u''w''} & \widetilde{u''v''} & 0 & 0 & 0 & 0 & 0 & u & 0 \\
0 & 0 & 0 & 2\widetilde{u''w''} & 0 & 0 & 0 & 0 & 0 & 0 & u
\end{pmatrix} \tag{0.14}$$

Diagonalization is possible for a matrix A' but the eigenvectors are very complicated. Matrix with same characters with original one but the eigenvectors would be simplier was searched. Some effects of Reynold's stresses were ignored. Following simplified is matrix A_s' is obtained by trial and error.

$$A_s' = \begin{pmatrix}
u & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{p_\rho}{\rho} + \frac{\widetilde{u_1''u_1''}}{\rho} & u & 0 & 0 & \frac{p_e}{\rho} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & u & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & u & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & \frac{p}{\rho} & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\widetilde{u_1''u_1''} & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \widetilde{u_1''u_1''} & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \widetilde{u_1''u_1''} & 0 & 0 & 0 & u & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u
\end{pmatrix} \tag{0.15}$$

To produce eigenvalues use

$$Det(A' - \lambda I) = 0$$

eigenvalues are

$$\lambda = (\{u, c+u, u-\sqrt{\widetilde{u_1''u_1''}}, u-\sqrt{\widetilde{u_1''u_1''}}, -c+u, u, u+\sqrt{\widetilde{u_1''u_1''}}, u+\sqrt{\widetilde{u_1''u_1''}}, u, u, u\}) \tag{0.16}$$

where

$$c^2 = p_e p/\rho^2 + p_\rho + 3\widetilde{u''u''}$$

The system is:

$$A = MA^{'}M^{-1} = ML\Lambda L^{-1}M^{-1} = R\Lambda R^{-1}$$

First write matrixes without c and after that with it. With c's you can find in file /math/reiska/imps1 (tai ainakin jokin versio).

in this paper	at mathematica	
A	akoko	conservative Jacobi'an
$A^{'}$	asim	simplified
R	r	eigenvector
R^{-1}	rm	right eigenvector
M	m1	primitive to conservative
M^{-1}	mm1	conservative to primitive
L	l	eigenvector (primitive)
L^{-1}	lm	right eigenvectors (primitive)
L^{-1}	lms	simplified right eigenvectors (primitive) present here
λ	lambda3c	eigenvalue vector with c

$$M = \frac{\partial U}{\partial V} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w & 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ e + \frac{\widetilde{u_i''u_i''}}{2} + \frac{u_i^2}{2} & \rho u & \rho v & \rho w & \rho & \rho/2 & 0 & 0 & \rho/2 & 0 & \rho/2 \\ \widetilde{u_1''u_1''} & 0 & 0 & 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ \widetilde{u_1''u_2''} & 0 & 0 & 0 & 0 & 0 & \rho & 0 & 0 & 0 & 0 \\ \widetilde{u_1''u_3''} & 0 & 0 & 0 & 0 & 0 & 0 & \rho & 0 & 0 & 0 \\ \widetilde{u_2''u_2''} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho & 0 & 0 \\ \widetilde{u_2''u_3''} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho & 0 \\ \widetilde{u_3''u_3''} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{w}{\rho} & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-2\,e+u^2+v^2+w^2}{2\,\rho} & -\frac{u}{\rho} & -\frac{v}{\rho} & -\frac{w}{\rho} & \frac{1}{\rho} & \frac{-1}{2\,\rho} & 0 & 0 & \frac{-1}{2\,\rho} & 0 & \frac{-1}{2\,\rho} \\ -\frac{\widetilde{u_1''u_1''}}{\rho} & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 \\ -\frac{\widetilde{u_1''u_2''}}{\rho} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 \\ -\frac{\widetilde{u_1''u_3''}}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 \\ -\frac{\widetilde{u_2''u_2''}}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 \\ -\frac{\widetilde{u_2''u_3''}}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 \\ -\frac{\widetilde{u_3''u_3''}}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho} \end{pmatrix}$$

$$
A = M\,A^{'}\,M^{-1} = \begin{pmatrix}
0 \\
p_\rho - u^2 + \frac{p_e\left(-2\,e + u^2 + v^2 + w^2\right)}{2\,\rho} \\
-\widetilde{u''_1 u''_2} - u\,v \\
-\widetilde{u''_1 u''_3} - u\,w \\
u\,\left[p_\rho - H - \frac{p_e\,e}{\rho}\left(e - \frac{u^2}{2} - \frac{v^2}{2} - \frac{w^2}{2}\right) - \frac{\widetilde{u''_1 u''_2}\,v}{u} - \frac{\widetilde{u''_1 u''_3}\,w}{u}\right] \\
-3\,u\,\widetilde{u''_1 u''_1} \\
-u\,\widetilde{u''_1 u''_2} - \widetilde{u''_1 u''_1}\,v \\
-u\,\widetilde{u''_1 u''_3} - \widetilde{u''_1 u''_1}\,w \\
-u\,\widetilde{u''_2 u''_2} \\
-u\,\widetilde{u''_2 u''_3} \\
-u\,\widetilde{u''_3 u''_3}
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\,u - \frac{p_e\,u}{\rho} & -\frac{p_e\,v}{\rho} & -\frac{p_e\,w}{\rho} & \frac{p_e}{\rho} & 1 - \frac{p_e}{2\,\rho} & 0 & 0 & \frac{-p_e}{2\,\rho} & 0 & \frac{-p_e}{2\,\rho} \\
v & u & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
w & 0 & u & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
H - \frac{p_e\,u^2}{\rho} & -\frac{p_e\,u\,v}{\rho} & -\frac{p_e\,u\,w}{\rho} & u + \frac{p_e\,u}{\rho} & u - \frac{p_e\,u}{2\,\rho} & v & w & \frac{-p_e\,u}{2\,\rho} & 0 & \frac{-p_e\,u}{2\,\rho} \\
3\,\widetilde{u''_1 u''_1} & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 \\
\widetilde{u''_1 u''_2} & \widetilde{u''_1 u''_1} & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 \\
\widetilde{u''_1 u''_3} & 0 & \widetilde{u''_1 u''_1} & 0 & 0 & 0 & u & 0 & 0 & 0 \\
\widetilde{u''_2 u''_2} & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 \\
\widetilde{u''_2 u''_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 \\
\widetilde{u''_3 u''_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u
\end{pmatrix}
$$

where

$$
H = E/\rho + p/\rho + \widetilde{u''_1 u''_1}
$$

$$
L = \begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{c}{\rho} & 0 & 0 & -\frac{c}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\rho} & 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\rho} & 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 \\
-\frac{p_\rho + \widetilde{u''_1 u''_1}}{p_e} & \frac{p}{\rho^2} & 0 & 0 & \frac{p}{\rho^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2\,\widetilde{u''_1 u''_1}}{\rho} & 0 & 0 & \frac{2\,\widetilde{u''_1 u''_1}}{\rho} & -\frac{p_\rho + \widetilde{u''_1 u''_1}}{\rho} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\sqrt{\widetilde{u''_1 u''_1}}}{\rho} & 0 & 0 & 0 & \frac{\sqrt{\widetilde{u''_1 u''_1}}}{\rho} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\sqrt{\widetilde{u''_1 u''_1}}}{\rho} & 0 & 0 & 0 & \frac{\sqrt{\widetilde{u''_1 u''_1}}}{\rho} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho}
\end{pmatrix}
$$

$$
L^{-1} =
$$

$$
\begin{pmatrix}
\frac{p_e\, p}{\rho^2 c^2} & 0 & 0 & 0 & -\frac{p_e \left(p_\rho + 3\,\widetilde{u''_1 u''_1}\right)}{c^2 \left(p_\rho + \widetilde{u''_1 u''_1}\right)} & \frac{p_e\, p}{\rho c^2 \left(p_\rho + \widetilde{u''_1 u''_1}\right)} & 0 & 0 & 0 & 0 & 0 \\
\frac{p_\rho + \widetilde{u''_1 u''_1}}{2\, c^2} & \frac{\rho}{2\, c} & 0 & 0 & \frac{p_e}{2\, c^2} & \frac{\rho}{2\, c^2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{\rho}{2} & 0 & 0 & 0 & \frac{\rho}{2\sqrt{\widetilde{u''_1 u''_1}}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\rho}{2} & 0 & 0 & 0 & \frac{\rho}{2\sqrt{\widetilde{u''_1 u''_1}}} & 0 & 0 & 0 \\
\frac{p_\rho + \widetilde{u''_1 u''_1}}{2\, c^2} & -\frac{\rho}{2\, c} & 0 & 0 & \frac{p_e}{2\, c^2} & \frac{\rho}{2\, c^2} & 0 & 0 & 0 & 0 & 0 \\
\frac{2\, \widetilde{u''_1 u''_1}}{c^2} & 0 & 0 & 0 & \frac{2\, p_e\, \widetilde{u''_1 u''_1}}{c^2 \left(p_\rho + \widetilde{u''_1 u''_1}\right)} & -\frac{\rho c^2 - 2\rho\, \widetilde{u''_1 u''_1}}{c^2 \left(p_\rho + \widetilde{u''_1 u''_1}\right)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\rho}{2} & 0 & 0 & 0 & \frac{\rho}{2\sqrt{\widetilde{u''_1 u''_1}}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\rho}{2} & 0 & 0 & 0 & \frac{\rho}{2\sqrt{\widetilde{u''_1 u''_1}}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho
\end{pmatrix}
$$

$$
R = \begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
u & c + u & 0 & 0 & -c + u & u & 0 & 0 & 0 & 0 & 0 \\
v & v & -1 & 0 & v & v & 1 & 0 & 0 & 0 & 0 \\
w & w & 0 & -1 & w & w & 0 & 1 & 0 & 0 & 0 \\
-\frac{\rho \left(p_\rho + \widetilde{u''_1 u''_1}\right)}{p_e} + \frac{E}{\rho} & H + c\, u & -v & -w & H - c\, u & \frac{E}{\rho} - \frac{p_\rho}{2} & v & w & \frac{1}{2} & 0 & \frac{1}{2} \\
\widetilde{u''_1 u''_1} & 3\, \widetilde{u''_1 u''_1} & 0 & 0 & 3\, \widetilde{u''_1 u''_1} & -p_\rho & 0 & 0 & 0 & 0 & 0 \\
\widetilde{u''_1 u''_2} & \widetilde{u''_1 u''_2} & \sqrt{\widetilde{u''_1 u''_1}} & 0 & \widetilde{u''_1 u''_2} & \widetilde{u''_1 u''_2} & \sqrt{\widetilde{u''_1 u''_1}} & 0 & 0 & 0 & 0 \\
\widetilde{u''_1 u''_3} & \widetilde{u''_1 u''_3} & 0 & \sqrt{\widetilde{u''_1 u''_1}} & \widetilde{u''_1 u''_3} & \widetilde{u''_1 u''_3} & 0 & \sqrt{\widetilde{u''_1 u''_1}} & 0 & 0 & 0 \\
\widetilde{u''_2 u''_2} & \widetilde{u''_2 u''_2} & 0 & 0 & \widetilde{u''_2 u''_2} & \widetilde{u''_2 u''_2} & 0 & 0 & 1 & 0 & 0 \\
\widetilde{u''_2 u''_3} & \widetilde{u''_2 u''_3} & 0 & 0 & \widetilde{u''_2 u''_3} & \widetilde{u''_2 u''_3} & 0 & 0 & 0 & 1 & 0 \\
\widetilde{u''_3 u''_3} & \widetilde{u''_3 u''_3} & 0 & 0 & \widetilde{u''_3 u''_3} & \widetilde{u''_3 u''_3} & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$$R^{-1} =$$

$$\begin{pmatrix}
p_e \frac{p_\rho\, p + (p_\rho + 3\, \widetilde{u''_1 u''_1})\left(e - \frac{u_i^2}{2}\right)}{c^2\, \rho \left(p_\rho + \widetilde{u''_1 u''_1}\right)} & \frac{p_e\, u \left(p_\rho + 3\, \widetilde{u''_1 u''_1}\right)}{c^2\, \rho \left(p_\rho + \widetilde{u''_1 u''_1}\right)} & \frac{p_e \left(p_\rho + 3\, \widetilde{u''_1 u''_1}\right) v}{c^2\, \rho \left(p_\rho + \widetilde{u''_1 u''_1}\right)} & \frac{p_e \left(p_\rho + 3\, \widetilde{u''_1 u''_1}\right) w}{c^2\, \rho \left(p_\rho + \widetilde{u''_1 u''_1}\right)} & -\frac{p_e \left(p_\rho + 3\, \widetilde{u''_1 u''_1}\right)}{c^2\, \rho \left(p_\rho + \widetilde{u''_1 u''_1}\right)} \\
\frac{p_\rho - c\, u}{2\, c^2} - \frac{p_e}{\rho} \frac{e - \frac{u_i^2}{2}}{2\, c^2} & \frac{1}{2\, c} - \frac{p_e\, u}{2\, c^2\, \rho} & \frac{-(p_e\, v)}{2\, c^2\, \rho} & \frac{-(p_e\, w)}{2\, c^2\, \rho} & \frac{p_e}{2\, c^2\, \rho} \\
\frac{-\widetilde{u''_1 u''_2}}{2\sqrt{\widetilde{u''_1 u''_1}}} + \frac{v}{2} & 0 & -\frac{1}{2} & 0 & 0 \\
\frac{-\widetilde{u''_1 u''_3}}{2\sqrt{\widetilde{u''_1 u''_1}}} + \frac{w}{2} & 0 & 0 & -\frac{1}{2} & 0 \\
\frac{p_\rho + c\, u}{2\, c^2} - \frac{p_e}{\rho} \frac{e - \frac{u_i}{2}}{2\, c^2} & \frac{-(c\, \rho + p_e\, u)}{2\, c^2\, \rho} & \frac{-(p_e\, v)}{2\, c^2\, \rho} & \frac{-(p_e\, w)}{2\, c^2\, \rho} & \frac{p_e}{2\, c^2\, \rho} \\
\frac{\widetilde{u''_1 u''_1}\left[c^2 + 2\, p_\rho - 2\, \frac{p_e}{\rho}\, (e - \frac{u_i^2}{2})\right]}{c^2 \left(p_\rho + \widetilde{u''_1 u''_1}\right)} & \frac{-2\, p_e\, u\, \widetilde{u''_1 u''_1}}{c^2\, \rho \left(p_\rho + \widetilde{u''_1 u''_1}\right)} & \frac{-2\, p_e\, \widetilde{u''_1 u''_1}\, v}{c^2\, \rho \left(p_\rho + \widetilde{u''_1 u''_1}\right)} & \frac{-2\, p_e\, \widetilde{u''_1 u''_1}\, w}{c^2\, \rho \left(p_\rho + \widetilde{u''_1 u''_1}\right)} & \frac{2\, p_e\, \widetilde{u''_1 u''_1}}{c^2\, \rho \left(p_\rho + \widetilde{u''_1 u''_1}\right)} \\
\frac{-\widetilde{u''_1 u''_2}}{2\sqrt{\widetilde{u''_1 u''_1}}} - \frac{v}{2} & 0 & \frac{1}{2} & 0 & 0 \\
\frac{-\widetilde{u''_1 u''_3}}{2\sqrt{\widetilde{u''_1 u''_1}}} - \frac{w}{2} & 0 & 0 & \frac{1}{2} & 0 \\
-\widetilde{u''_2 u''_2} & 0 & 0 & 0 & 0 \\
-\widetilde{u''_2 u''_3} & 0 & 0 & 0 & 0 \\
-\widetilde{u''_3 u''_3} & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\begin{pmatrix}
\frac{p_e \left(2\, p + p_\rho\, \rho + 3\, \rho\, \widetilde{u''_1 u''_1}\right)}{2\, c^2\, \rho^2 \left(p_\rho + \widetilde{u''_1 u''_1}\right)} & 0 & 0 & \frac{p_e \left(p_\rho + 3\, \widetilde{u''_1 u''_1}\right)}{2\, c^2\, \rho \left(p_\rho + \widetilde{u''_1 u''_1}\right)} & 0 & \frac{p_e \left(p_\rho + 3\, \widetilde{u''_1 u''_1}\right)}{2\, c^2\, \rho \left(p_\rho + \widetilde{u''_1 u''_1}\right)} \\
\frac{1}{2\, c^2} - \frac{p_e}{4\, c^2\, \rho} & 0 & 0 & \frac{-p_e}{4\, c^2\, \rho} & 0 & \frac{-p_e}{4\, c^2\, \rho} \\
0 & \frac{1}{2\sqrt{\widetilde{u''_1 u''_1}}} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2\sqrt{\widetilde{u''_1 u''_1}}} & 0 & 0 & 0 \\
\frac{1}{2\, c^2} - \frac{p_e}{4\, c^2\, \rho} & 0 & 0 & \frac{-p_e}{4\, c^2\, \rho} & 0 & \frac{-p_e}{4\, c^2\, \rho} \\
-\frac{p_e\, p + p_\rho\, \rho^2 + p_e\, \rho\, \widetilde{u''_1 u''_1} + \rho^2\, \widetilde{u''_1 u''_1}}{c^2\, \rho^2 \left(p_\rho + \widetilde{u''_1 u''_1}\right)} & 0 & 0 & -\frac{p_e\, \widetilde{u''_1 u''_1}}{c^2\, \rho \left(p_\rho + \widetilde{u''_1 u''_1}\right)} & 0 & -\frac{p_e\, \widetilde{u''_1 u''_1}}{c^2\, \rho \left(p_\rho + \widetilde{u''_1 u''_1}\right)} \\
0 & \frac{1}{2\sqrt{\widetilde{u''_1 u''_1}}} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2\sqrt{\widetilde{u''_1 u''_1}}} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

the end