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Publication IV

V. Pietilä, M. Möttönen, T. Isoshima, J. A. M. Huhtamäki, and S. M. M. Virtanen, *Stability and dynamics of vortex clusters in nonrotated Bose-Einstein condensates*, Phys. Rev. A **74**, 023603 (2006).

http://link.aps.org/abstract/pra/v74/p023603

Stability and dynamics of vortex clusters in nonrotated Bose-Einstein condensates

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(Received 16 May 2006; published 4 August 2006)

We study stationary clusters of vortices and antivortices in dilute pancake-shaped Bose-Einstein condensates confined in nonrotating harmonic traps. Previous theoretical results on the stability properties of these topologically nontrivial excited states are seemingly contradicting. We clarify this situation by a systematic stability analysis. The energetic and dynamic stability of the clusters is determined from the corresponding elementary excitation spectra obtained by solving the Bogoliubov equations. Furthermore, we study the temporal evolution of the dynamically unstable clusters. The stability of the clusters and the characteristics of their destabilizing modes only depend on the effective strength of the interactions between particles and the trap anisotropy. For certain values of these parameters, there exist several dynamical instabilities, but we show that there are also regions in which some of the clusters are dynamically stable. Moreover, we observe that the dynamical instability of the clusters does not always imply their structural instability, and that for some dynamically unstable states annihilation of the vortices is followed by their regeneration, and revival of the cluster.

DOI: 10.1103/PhysRevA.74.023603 PACS number(s): 03.75.Lm, 03.75.Kk, 03.65.Ge

I. INTRODUCTION

Quantized vortices are topological defects in systems with a long-range quantum phase coherence. They have been extensively investigated in different systems and branches of physics such as helium superfluids [1,2], superconductors [3], cosmology [4,5], and optics [6]. Atomic Bose-Einstein condensates (BECs) are extremely convenient systems to investigate the characteristics of quantized vortices due to their experimental versatility; many properties of these systems can be manipulated with lasers and external magnetic fields. Furthermore, real-space imaging of BECs can be carried out by optical *in situ* or time-of-flight measurements.

In addition to extensive theoretical analysis, vortices and vortex clusters composed of several vortices with the same topological charge have been realized and investigated experimentally in gaseous BECs [7–10]. These vortex clusters are local minima of energy for rotated condensates and hence stable states. However, recently there have been theoretical suggestions that other kinds of vortex clusters can be stationary and stable states for nonrotated condensates [11,12]. These clusters typically consist of vortices and antivortices in specific configurations such that the various forces acting on the vortices exactly balance each other. Based on the seeming robustness under external perturbations of some of these topological excited collective states, they could also be considered as solitonic states—indeed, they seem to be stabilized only by strong enough nonlinearity of the condensate. The development of techniques for vortex creation in dilute BECs, and especially phase-imprinting methods [13], may enable direct construction of such vortex clusters in the future. Both from the experimental and theoretical points of view, their stability properties are a central issue.

These vortex structures have been investigated in the noninteracting limit [11,14] as well as in interacting systems [11,12,15,16]. In previous studies, three different stationary vortex cluster configurations have been found in interacting pancake-shaped condensates. The so-called vortex dipole and quadrupole states, shown in Figs. 1(a) and 1(c), were originally introduced by Crasovan *et al.* [11,12]. The stationary vortex quadrupole state was found to exist in both interacting and noninteracting condensates, whereas the stationary vortex dipole state exists only in condensates interacting strongly enough. Furthermore, the authors of Refs. [11,12] studied the stability against small external perturbations of these configurations by integrating the Gross-Pitaevskii equation in real time, and concluded them to be stable and robust in the nonlinear regime.

Subsequently, based on the Bogoliubov quasiparticle spectra of the cluster states, the stability properties of the stationary clusters were studied by Möttönen *et al.* [16]. It was found that at least for the parameter values studied, both the vortex dipole and vortex quadrupole states are energetically and dynamically unstable. The effects of the energetic instability can be neglected at low enough temperatures due to vanishingly small dissipation, but the dynamical instabilities can prevent these states from being long-lived. In addition, another stationary cluster, a vortex tripole, was introduced—see Fig. 1(b).

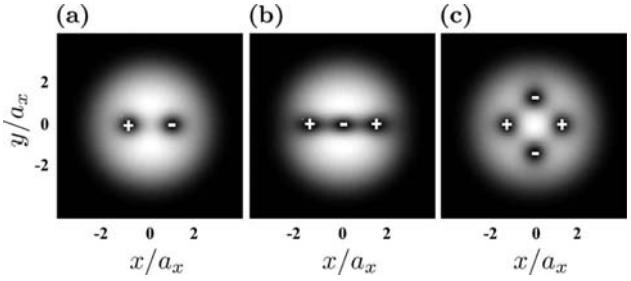

FIG. 1. Density profiles of the condensate for the stationary vortex dipole (a), tripole (b), and quadrupole (c) states. Vortices and antivortices are denoted by plus and minus signs at the vortex cores, respectively. The interaction strength parameter is $\tilde{g}=170$. The unit of length is the harmonic-oscillator length in the x direction.

In order to clarify the seeming contradiction between the results of Refs. [11,12] and Ref. [16], we present in this paper a systematic study of the stability properties of the above-mentioned stationary vortex clusters as functions of total particle number and trap anisotropy. We find the stationary vortex cluster states by directly minimizing an error functional for the Gross-Pitaevskii equation, and compute the corresponding instability modes, i.e., the quasiparticle excitations responsible for the dynamical instability, from the Bogoliubov equations. Finally, we investigate the nature of the instability modes by computing the time development of slightly perturbed cluster states. It is found that the characteristics of the instability modes only depend on the effective strength of the particle interactions and the trap anisotropy. Some dynamical instability modes do not destroy the cluster structure and the condensate is therefore referred to as structurally stable, whereas some lead to annihilation and subsequent revival of the vortices in the cluster.

II. MEAN-FIELD THEORY

The dynamics of weakly interacting gaseous Bose-Einstein condensates in the zero-temperature limit is accurately described by the Gross-Pitaevskii (GP) equation

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t)=\mathcal{H}\psi(\mathbf{r},t), \tag{1}$$

where the nonlinear operator $\mathcal{H}=\mathcal{H}[\psi]$ acting on the condensate order parameter ψ is given by

$$\mathcal{H}[\psi]=-\frac{\hbar^2}{2m}\nabla^2+V_{\mathrm{tr}}+g|\psi|^2. \tag{2}$$

The interaction parameter g is determined by the s-wave scattering length a as $g=4\pi\hbar^2a/m$. Above, m is the atomic mass, and the external trapping potential is denoted by $V_{\mathrm{tr}}(\mathbf{r})$. The order parameter is normalized according to $\int|\psi|^2d\mathbf{r}=N$, where N is the total number of atoms in the condensate. The total energy of the condensate can be calculated as

$$E[\psi]=\int d\mathbf{r}\psi^*(\mathbf{r})\left[-\frac{\hbar^2}{2m}\nabla^2+V_{\mathrm{tr}}(\mathbf{r})+\frac{g}{2}|\psi(\mathbf{r})|^2\right]\psi(\mathbf{r}). \tag{3}$$

The stationary states of the condensate with an eigenvalue μ are solutions to the GP equation of the form $\psi(\mathbf{r},t)=e^{-i\mu t/\hbar}\psi(\mathbf{r})$. Energetic and dynamic stability of a given stationary state can be inferred from the corresponding excitation spectrum given by the Bogoliubov equations

$$\begin{pmatrix}\mathcal{L}(\mathbf{r}) & g[\psi(\mathbf{r})]^2\\ -g[\psi^*(\mathbf{r})]^2 & -\mathcal{L}(\mathbf{r})\end{pmatrix}\begin{pmatrix}u_q(\mathbf{r})\\ v_q(\mathbf{r})\end{pmatrix}=\hbar\omega_q\begin{pmatrix}u_q(\mathbf{r})\\ v_q(\mathbf{r})\end{pmatrix}, \tag{4}$$

where $\mathcal{L}=\mathcal{H}+g|\psi|^2-\mu$, the quasiparticle amplitudes are denoted by $u_q(\mathbf{r})$ and $v_q(\mathbf{r})$, and ω_q is the eigenfrequency corresponding to the quasiparticle state with the index q.

If the quasiparticle spectrum contains excitations with positive norm $\int d\mathbf{r}(|u_q|^2-|v_q|^2)>0$ but negative eigenfrequency ω_q, the corresponding stationary state is energetically unstable, whereas eigenfrequencies with nonzero imaginary part indicate dynamical instability. The occupation of dynamical instability modes increases exponentially in time, and hence small perturbations of a dynamically unstable stationary state typically result in large modifications in its structure.

In pancake-shaped traps, the system in question has an SO(2) rotation symmetry. However, the vortex cluster states shown in Fig. 1 are not rotationally invariant. Thus, according to the Goldstone theorem [17,18], there should exist collective low-frequency modes that tend to restore the broken symmetry by rigidly rotating the cluster states. These modes are intrinsic to the vortex cluster states, since they are absent, e.g, for an axially symmetric single vortex state.

The existence of a symmetry-breaking induced long-lived collective mode is manifested as a $1/k^2$ divergence in the response function of a symmetry-restoring variable F [17]. Let us consider operators of the form [19]

$$F=\sum_q\int d\mathbf{r}f(\mathbf{r})\{[\psi^*(\mathbf{r})u_q(\mathbf{r})+\psi(\mathbf{r})v_q(\mathbf{r})]b_qe^{-i\omega_qt}+[\psi(\mathbf{r})u_q^*(\mathbf{r})+\psi^*(\mathbf{r})v_q^*(\mathbf{r})]b_q^\dagger e^{i\omega_qt}\}, \tag{5}$$

where $b_q^\dagger$ and b_q are the creation and annihilation operators for a quasiparticle with index q and $f(\mathbf{r})$ is a complex function. In the presence of dynamical instability modes, we may generalize the result presented Ref. [19] to obtain the response function as

$$\chi_F(\omega,t)=-\frac{1}{\hbar}\sum_q\left[\frac{A_q}{\omega-\omega_{q,0}+i\eta}e^{-2\ \mathrm{Im}\{\omega_q\}t}-\frac{B_q}{\omega+\omega_{q,0}+i\eta}e^{2\ \mathrm{Im}\{\omega_q\}t}\right], \tag{6}$$

with $A_q=|\int d\mathbf{r}f^*(\mathbf{r})[\psi^*(\mathbf{r})u_q(\mathbf{r})+\psi(\mathbf{r})v_q(\mathbf{r})]|^2$ and $B_q=|\int d\mathbf{r}f(\mathbf{r})[\psi^*(\mathbf{r})u_q(\mathbf{r})+\psi(\mathbf{r})v_q(\mathbf{r})]|^2$. The Bohr frequency $\omega_{q,0}$ is given by $\omega_{q,0}=(E_q-E_0)/\hbar$, where $E_q=\mathrm{Re}\{\hbar\omega_q\}$ is the energy of the excitation q and E_0 is the ground-state energy. From Eq. (6), we observe that a necessary condition for the response function χ_F to diverge as $1/\omega$, i.e., as $1/k^2$, is the existence of excitations with $\mathrm{Re}\{\omega_q\}=0$. In Sec. IV, we show that there indeed exists a zero-energy instability mode for all stationary vortex clusters and we therefore identify them as the Goldstone modes corresponding to broken rotational symmetry.

III. COMPUTATIONAL METHODS

In Ref. [16], the stationary vortex clusters considered were found to be energetically unstable. Thus, they cannot be found with the usual methods based on energy minimization. We employ the method introduced in Ref. [16], in which stationary solutions of the GP equation (1) are found by minimizing the error functional $F[\psi,\mu]=\int d\mathbf{r}|(\mathcal{H}[\psi]-\mu)\psi(\mathbf{r})|^2$. The chemical potential that minimizes the error functional is given by the familiar expression $\mu=\int d\mathbf{r}\psi^*(\mathbf{r})\mathcal{H}[\psi]\psi(\mathbf{r})/N$. Using the functional gradient

$$\frac{\delta F[\psi,\mu]}{\delta \psi^*} = [(\mathcal{H}[\psi]-\mu)^2 + 2g\,\mathrm{Re}\{\psi^*(\mathbf{r})(\mathcal{H}[\psi]-\mu)\psi(\mathbf{r})\}] \times \psi(\mathbf{r}), \tag{7}$$

one obtains the stationary solutions, e.g, by the method of steepest descent or by directly solving the equation $\delta F/\delta\psi^*=0$.

For computational simplicity, we consider a pancake-shaped system in the harmonic potential

$$V_{\mathrm{tr}}(\mathbf{r}) = \frac{1}{2}m(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2), \tag{8}$$

with $\omega_x,\omega_y \ll \omega_z$. Especially, we assume the z confinement to be tight enough such that the low-energy solutions of the GP equation and the Bogoliubov equations can be taken to be of the form $\sigma(z)=1/\sqrt{2\pi a_z}e^{-z^2/(2a_z^2)}$ in the z direction. If the harmonic-oscillator length $a_z=\sqrt{\hbar/(m\omega_z)}$ in the axial direction is much larger than the scattering length a, the condensate can be described by the usual GP equation with $g=4\pi\hbar^2a/m$ [20]. Substituting the ansatz $\psi(\mathbf{r},t)=\psi_{2\mathrm{D}}(x,y,t)\sigma(z)$ into Eq. (1), one obtains the effectively two-dimensional (2D) Gross-Pitaevskii equation in the dimensionless form

$$i\frac{\partial}{\partial\tilde{t}}\tilde{\psi} = \frac{1}{2}\left[-\tilde{\nabla}^2+(\tilde{x}^2+\lambda^2\tilde{y}^2)+\tilde{g}|\tilde{\psi}|^2+\frac{\omega_z}{\omega_x}\right]\tilde{\psi}, \tag{9}$$

where $\tilde{\psi}(\tilde{x},\tilde{y},\tilde{t})=a_x\psi_{2\mathrm{D}}(x,y,t)$ and $a_x=\sqrt{\hbar/(m\omega_x)}$. The dimensionless quantities denoted by the tilde are obtained from the original dimensional ones by scaling the length by a_x, time by $1/\omega_x$, and energy by $\hbar\omega_x$. The dimensionless constant $\lambda=\omega_y/\omega_x$ characterizes trap anisotropy in the xy plane. The normalization condition $\int d\tilde{x}\,d\tilde{y}|\tilde{\psi}|^2=1$ implies the effective dimensionless 2D coupling constant $\tilde{g}=4\sqrt{\pi}Na/a_z$. Equation (9) implies that the only relevant parameters in the problem are the coupling constant $\tilde{g}$ and the trap anisotropy λ. Thus it suffices to investigate the cluster states as functions of these parameters.

Based on the minimization of the error functional, we have computed stationary cluster states for different values of the interaction strength $\tilde{g}$ and trap asymmetry λ. To obtain a proper starting point for the minimization of the error functional $F[\psi,\mu]$, we first minimize the energy of ansatz wave functions for which the condensate phase is fixed. The found stationary states were computed to the relative final accuracy $F[\psi_s,\mu]/(N\mu^2)<10^{-15}$, which confirms that these states were, indeed, very accurate stationary states. The Bogoliubov equations were then solved for these states, identifying possible excitation modes with nonreal eigenfrequencies. Furthermore, we studied the nature of the found instability modes by solving the condensate temporal evolution for initial, slightly perturbed states of the form

$$\psi(\mathbf{r},0) = \psi_s(\mathbf{r}) + \kappa_q[u_q(\mathbf{r})+v_q^*(\mathbf{r})], \tag{10}$$

where $\psi_s(\mathbf{r})$ is the condensate order parameter for a stationary state, and $u_q(\mathbf{r}),v_q(\mathbf{r})$ are the quasiparticle amplitudes of an instability mode. The constant κ_q describes the initial population of the instability mode—by using the normalization $\int d\mathbf{r}(|u_q|^2+|v_q|^2)=1$ for the instability modes, we typically chose $\kappa_q\sim 10^{-2}-10^{-1}$ corresponding to less than 1% of the atoms in the unstable mode. We also point out that contrary to the real-frequency modes, the energy of the perturbed stationary state $\psi(\mathbf{r},0)$ in Eq. (10) is up to second order in κ_q equal to the energy of the stationary state $\psi_s(\mathbf{r})$ if the perturbation corresponds to a dynamical instability mode. Furthermore, our numerical computations have shown that in the presence of several instability modes in the spectrum of the state, it is typically sufficient to consider only the excitation with the largest $|\mathrm{Im}\{\omega_q\}|$. This is due to the fact that quasiparticle interactions necessarily excite also the fastest growing imaginary mode, which seems to always dominate the condensate dynamics. The temporal evolution of the perturbed state was solved numerically from the time-dependent GP equation using a combination of a split-operator method and an implicit Crank-Nicolson scheme.

IV. RESULTS

The vortex cluster states were analyzed by varying the interaction parameter in the range $0\leqslant\tilde{g}\leqslant 300$ for the rotationally symmetric trap with $\lambda=1$, and by separately varying the anisotropy parameter λ for fixed interaction parameter values. Specifically, we studied the existence of the stationary vortex clusters for $\tilde{g}=170$ (which was used in the previous studies in Ref. [16]) and the existence of certain instability modes for a wide range of the interaction parameter. Excitation spectra for all the stationary clusters considered contained negative energy excitations with positive norm, implying these configurations to be energetically unstable. On the other hand, dynamical instability of the clusters turned out to be a more subtle issue. The imaginary parts of the dynamical instability modes of the clusters in rotationally symmetric traps are presented in Fig. 2 as functions of the interaction strength $\tilde{g}$. Inspecting the temporal evolution of the cluster states in the presence of excited instability modes, we observe that there are two qualitatively different dynamical instability modes for each cluster, one that turns out to lead to decay of the cluster and one that tends to rotate the cluster rigidly. We identify the latter modes as the Goldstone modes discussed in Sec. II, since they have zero energy and they vanish if the trap becomes sufficiently anisotropic.

A. Vortex dipole

For a rotationally symmetric trap with $\lambda=1$, the stationary vortex dipole state was found to exist only for interaction strengths $\tilde{g}\gtrsim 42$, i.e., a certain amount of nonlinearity is required for the existence of the stationary dipole configuration. This agrees with the observations presented in Ref. [12]. For the vortex dipole, there exist two dynamical instability modes: one for all interaction strength values $\tilde{g}\simeq 42-300$ considered (marked with $\triangle$ in Fig. 2), and one that occurs only in the range $\tilde{g}\simeq 50-80$ (marked with ∇ in Fig. 2). The latter mode dominates in the region $\tilde{g}\simeq 50-80$ due to its larger $|\mathrm{Im}\{\omega_q\}|$. Time development of slightly perturbed vortex dipole states shows that the mode existing out-

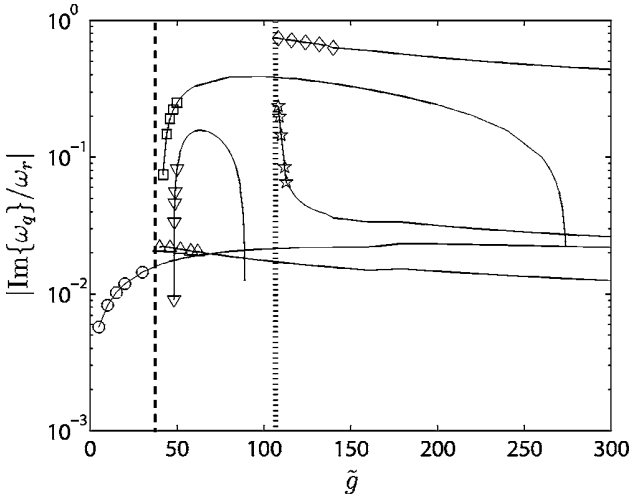

FIG. 2. Absolute values of the imaginary parts of the eigenfrequencies corresponding to instability modes as functions of the interaction strength $\tilde{g}$ for stationary dipole, tripole, and quadrupole clusters confined in a rotationally symmetric trap, with $\lambda=1$. Curves corresponding to the vortex dipole are marked with ∇ (decay mode) and $\triangle$ (rotational mode), the vortex tripole with $\diamond$ (decay mode) and $\star$ (rotational mode), and the vortex quadrupole with $\square$ (decay mode) and $\bigcirc$ (rotational mode). The dashed and the dotted line indicate the lower limits of the interaction strength for which a stationary vortex dipole and tripole were found to exist, respectively.

side the region $\tilde{g}\simeq 50-80$ corresponds very accurately to rigid rotation of the dipole configuration, as shown in Fig. 3. The structure of the dipole remains intact in this rotation, and hence the stationary dipole state can be considered structurally stable, although it is dynamically unstable in this region. Thus dynamical instability does not necessarily imply the cluster structure to be unstable.

The nature of the instability mode dominating vortex dipole dynamics in the region $\tilde{g}\simeq 50-80$ is very different from the rotational mode. Temporal evolution of a slightly perturbed dipole state in this regime is shown in Fig. 4. The dominant mode renders the vortices of the dipole first to

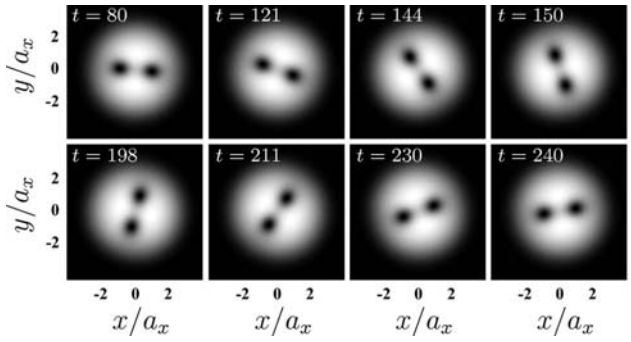

FIG. 3. Temporal evolution of the slightly perturbed stationary vortex dipole state for $\tilde{g}=160$ and $\lambda=1$. The interaction strength $\tilde{g}=160$ is in the region where only the rotational mode exists. Time is denoted by t and it is given in units of $1/\omega_x$. The structure of the cluster remains very close to the original stationary configuration.

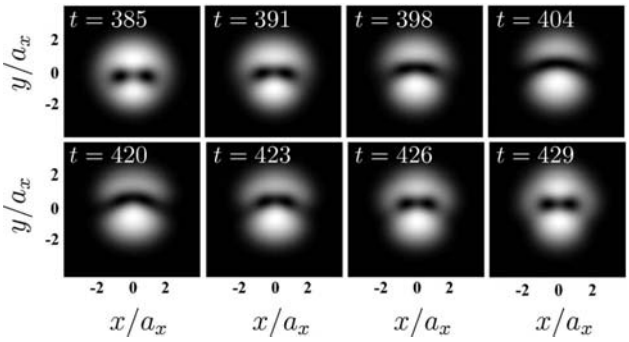

FIG. 4. Collapse and revival of the slightly perturbed stationary vortex dipole state for $\tilde{g}=60$ and $\lambda=1$. It should be noted that apart from oscillations of the vortex locations, the vortex dipole state remains essentially intact for a long time before it starts to decay. The vortices have opposite topological charges after the revival of the dipole state.

annihilate each other, but eventually the dipole configuration reappears from the vortex-free state. For $\tilde{g}\simeq 50$, the density distribution of this revived dipole configuration is almost the same as in the initial state, but the vorticity is changed such that the vortex becomes an antivortex and vice versa. Similar behavior has been observed in the numerical simulations of light propagation in a graded-index medium, where the vortex dipole nested in a light beam undergoes periodic collapse and revival along the direction of the light propagation [21]. The collapse and revival of the dipole configuration continues periodically for $\tilde{g}\simeq 50$, whereas for increasing interaction strength, nonlinearity gradually prevents the topological excitation from recombining and eventually the whole excitation disappears for $\tilde{g}\gtrsim 80$.

In asymmetric traps, the stationary vortex dipole state corresponding to interaction parameter $\tilde{g}=170$ was found to exist in the range $\lambda\simeq 0.9-1.5$. Interestingly, we found that even the tiny amount of asymmetry corresponding to $\lambda=1.005$ is sufficient to eliminate the rotational instability mode for all values of $\tilde{g}$. This is consistent with the fact that in an asymmetric trap, the system no longer has the SO(2) symmetry and the Goldstone mode should vanish. Thus by using a slightly asymmetric trap, the stationary vortex dipole state can be made fully dynamically stable in ranges $\tilde{g}\simeq 42-50$ and $\tilde{g}\gtrsim 80$. Experimentally, the transfer between the different stability and instability regimes can be accomplished by adjusting the total particle number and the trapping frequencies.

B. Vortex tripole

The vortex tripole state exists only in rather strongly interacting condensates: in symmetric traps with $\lambda=1$, the stationary vortex tripole state was found to exist only for $\tilde{g}\gtrsim 108$. The stationary vortex tripole has always two dynamical instability modes, as shown in Fig. 2. The real part of the eigenfrequency vanishes for both of these modes, but the imaginary part of the dominating mode is roughly an order of magnitude larger than that of the other one. The decay of the tripole configuration corresponding to the dominating in-

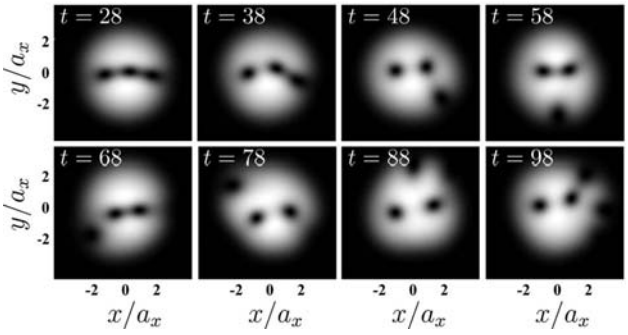

FIG. 5. Temporal evolution of the perturbed stationary vortex tripole state for $\tilde{g}=160$ and $\lambda=1$. The vortex tripole starts to decay almost immediately to the vortex dipole state due to the strong dynamical instability.

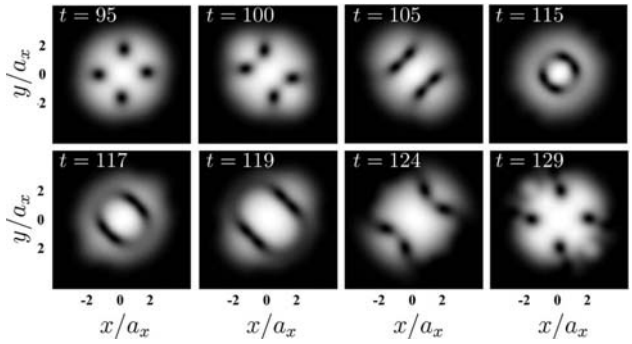

FIG. 6. Collapse and revival of the vortex quadrupole for $\tilde{g}=260$ and $\lambda=1$. In contrast to the vortex dipole state, the vortices have the same topological charge before and after the collapse and revival of the cluster.

stability mode is shown in Fig. 5. Under small perturbations, one of the outermost vortices starts to drift out of the condensate and eventually it reaches the surface of the cloud and excites surface modes, which can be observed in the last panel in Fig. 5. The remaining two vortices form a topologically neutral vortex dipole. Furthermore, the stationary vortex tripole exists as a stationary state only for $\tilde{g}\gtrsim 108$, which lies in the structurally stable region of the stationary vortex dipole. In the numerical simulations, the vortex tripole takes a time of order $1/\omega_x$ for the population $\kappa_q \sim 10^{-2}-10^{-1}$ of the excitation, before it starts to notably decay. Thus the decay of the vortex tripole to the vortex dipole is slow enough to be experimentally observable. The nature of the other instability mode, to which we refer as the slow mode, is more difficult to find out since this mode never seemed to dominate the temporal evolution. Numerical simulations showed that the slow mode tends to rotate the condensate, but the evolution of this mode excites also the dominant instability mode leading to the decay of tripole cluster to dipole cluster.

In anisotropic traps, the vortex tripole state turned out to be stationary in the same range $\lambda\simeq 0.9-1.5$ for $g\approx 170$ as the vortex dipole. Anisotropy of the trap corresponding to $\lambda\geqslant 1.01$ removes the slow instability mode from the spectrum, but the dominating mode always persists. This observation together with the characteristics of the initial decay dynamics suggest that the slow mode is analogous to the rotational mode of the vortex dipole and quadrupole. Since the dominant mode persists in an asymmetric trap, the vortex tripole state cannot be made dynamically stable by tuning the interaction strength or the trap anisotropy.

C. Vortex quadrupole

The vortex quadrupole was found to be stationary for all interaction strengths for $\lambda=1$. These results are in agreement with the previously reported calculations of Crasovan *et al.* [11], in which stationary vortex quadrupoles were found to exist both in interacting and noninteracting condensates. In this respect, the quadrupole configuration differs essentially from the dipole and tripole clusters. The dynamical instability modes of the quadrupole cluster resemble those of the vortex dipole: The stationary vortex quadrupole state in a rotationally symmetric trap has two instability modes, one that exists for all values of the interaction parameter, and one existing only in the range $\tilde{g}\simeq 50-280$, where it dominates the condensate dynamics, see Fig. 2. The mode that exists for all $\tilde{g}$ has $\mathrm{Re}\{\omega_q\}=0$, whereas the other mode has energy $\mathrm{Re}\{\hbar\omega_q\}\sim 10^{-1}\hbar\omega_x$.

In the region $\tilde{g}\simeq 50-280$, the dominant mode of the vortex quadrupole drives the vortices to merge together and annihilate each other, but eventually the quadrupole configuration reappears, as shown in Fig. 6. This mode also generates oscillations with increasing amplitude such that the condensate stretches and shrinks as the annihilation-revival cycles proceed. The amplitude of these oscillations increases gradually, and finally the vortices are driven out of the condensate. In the regions $\tilde{g}\lesssim 60$ and $\tilde{g}\simeq 280-300$, the existing instability mode is analogous to the dipole cluster mode that corresponds to rigid rotation of the cluster, i.e., the Goldstone mode. Thus the quadrupole cluster is structurally stable in this regime. Furthermore, one observes from Fig. 2 that the vortex quadrupole tends to become dynamically stable in the limit of noninteracting condensate. It has, however, been shown that in the noninteracting case the persistent current such as the vortex quadrupole is always structurally unstable against perturbations in the external trap parameters [22,23]. Thus dynamical stability does not necessarily imply stability against small perturbations in the trap parameters and, vice versa, dynamically unstable states can be structurally robust with respect to small perturbations.

As a function of the trap anisotropy parameter, the quadrupole cluster with $\tilde{g}=170$ was found to be stationary in the range $\lambda\simeq 0.9-1.2$. For increasing trap anisotropy parameter, the two vortices of the vortex quadrupole in the direction of the tight confinement move away from the center of the cloud, and eventually the state cannot be considered as a vortex quadrupole. This behavior is due to the increase of the buoyancy force in the direction of tight confinement and repulsive interaction between vortices with the same topological charge.

Analogously to the vortex dipole case, computations with different interaction strength values $\tilde{g}$ showed that for the stationary vortex quadrupole state, even the rather small

amount of asymmetry $\lambda=1.1$ is sufficient to remove the rotational instability mode from the quasiparticle spectrum. Hence, also the vortex quadrupole state can be made dynamically stable in the regimes $\tilde{g}\lesssim 60$ and $\tilde{g}\gtrsim 280$.

V. CONCLUSIONS

We have systematically studied the existence, stability, and dynamics of stationary vortex clusters in dilute pancake-shaped BECs confined in nonrotating harmonic traps. In contrast to the previous investigations [11,12], our approach utilizes the Bogoliubov equations to determine the quasiparticle spectrum, which reveals unambiguously not only the energetic and dynamic stability of a given state, but also yields explicitly the quasiparticle amplitudes of the instability modes. The nature of these modes was found out by computing the temporal evolution of slightly perturbed stationary cluster states.

It was observed that the dipole, tripole, and quadrupole clusters have various regimes of dynamical instability, but in some of these regimes the cluster configurations are structurally stable as they only tend to rotate rigidly. On the other hand, the vortex annihilation modes of the vortex dipole and quadrupole break the cluster structures altogether, but if energy dissipation is negligible, the clusters can reappear after a certain time. Furthermore, it was observed that even very small trap anisotropies suffice to remove the rotational instability modes, thus dynamically stabilizing the dipole and quadrupole clusters for suitable values of the interaction parameter.

ACKNOWLEDGMENTS

CSC-Scientific Computing Ltd. (Espoo, Finland) is acknowledged for computational resources and the Academy of Finland for financial support through the Center of Excellence in Computational Nanoscience. M.M. and J.H. thank the Finnish Cultural Foundation, and M.M. and T.I. the Vilho, Yrjö, and Kalle Väisälä Foundation for financial support.

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[23] It should be noted that the authors of Ref. [22] define structural instability as instability against small perturbations of the physical parameters describing the system.