Department of Radio Science and Engineering

Anisotropy in electrostatics

Solutions for inclusions with canonical shapes

Tommi Rimpiläinen

Aalto University

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Anisotropy in electrostatics

Solutions for inclusions with canonical shapes

Tommi Rimpiläinen

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Abstract

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Abstract

Inclusions that take a canonical shape have a special place in electrostatics. Their analysis involves methods with great mathematical subtlety and rigor. Where an arbitrarily shaped inclusion can only be solved with a method that involves approximation, some of the canonical shapes permit an approach that leads to an exact solution in a closed form.

What makes canonical shapes special, is that they have a particularly simple presentation in an important class of coordinate systems. There are 11 orthogonal coordinate systems where the coordinate curves are defined by equations of second degree. In these 11 systems, the Laplace equation is separable.

The particular canonical shapes that the thesis studies are spheres, spheroids, and ellipses. In their corresponding coordinate systems, they correspond to surfaces with a fixed coordinate. Because these three systems allow the Laplace equation to separate, much work can be done analytically before involving approximation or computation. Some of the presented solution in the thesis are exact and others are semi-analytical.

Although solutions for inclusions with the three canonical shapes exist in the literature, the thesis goes beyond the existing work by generalizing the materials of the inclusions. In all of the enclosed articles, the material of the inclusion is assumed to be anisotropic and inhomogeneous. More specifically, it is assumed that the axes of anisotropy correspond to the coordinate unit vectors of the relevant coordinate system. The term *radial anisotropy* has been used in the literature to refer to anisotropy in the spherical coordinates when the tangential components are equal but possibly differ from the normal component. The thesis employs a similar concept in spheroidal and elliptic systems. In the spherical coordinate system, the thesis relaxes the assumption that the two tangential components must be equal. The resulting inclusions is referred to as the *systropic sphere*.

Keywords anisotropy, Laplace equation, potential

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Tiivistelmä

Muodoltaan kanonisilla inkluusioilla on sähköstatiikassa keskeinen merkitys. Tämänkaltaisten inkluusioitten analysoimisessa sovelletaan matemaattisia menetelmiä, jotka ovat luonteeltaan perusteellisia ja täsmällisiä. Siinä, missä mielivaltaisen muotoisia inkluusioita pystyy tutkimaan vain likimääräisin menetelmin, joitakin kanonisia muotoja pystyy tutkimaan menetelmin, jotka antavat täsmällisen suljetun muodon kaavan.

Kanonisista muodoista tekee erityisiä se, että näillä muodoilla on poikkeuksellisen yksinkertainen esitystapa tärkeässä joukossa koordinaatistoja. Matematiikassa tunnetaan yksitoista ortogonaalista koordinaatistoa, jonka koordinaattikäyrät määrittelee toisen asteen yhtälö. Kaikilla näillä yhdellätoista koordinaatistolla on se ominaisuus, että niitten puitteissa Laplace-yhtälö separoituu.

Tämä väitöskirja tarkastelee kanonisista muodoista erityisesti palloja, sferoideja ja ellipsejä. Omissa koordinaatistoissaan nämä muodot vastaavat pintoja, joilla jokin koordinaatin arvo on vakio. Koska Laplace yhtälö separoituu valittujen muotojen kannalta olennaisissa koordinaatistoissa, suuri osa sähköstaattisen ongelman ratkaisemisesta saadaan aikaan analyyttisin menetelmin, ilman likiarvoistamista. Väitöskirjassa esitetään sähköstaattisten ongelmien ratkaisuja, joista jotkin ovat puolianalyyttisia ja loput kauttaaltaan matemaattisen täsmällisiä.

Vaikka kirjallisuus tunteekin valmiita ratkaisuja kaikille kolmelle väitöskirjan kannalta olennaiselle kanoniselle muodolle, olemassa oleva tutkimus jättää monta kysymystä avoimeksi. Väitöskirja poikkeaa aikaisemmasta tutkimuksesta siten, että se käsittelee materiaaliparametreiltaan aikaisempaa yleistetympiä inkluusioita. Väitöskirjan tutkimus pohjaa aikaisempaan *radiaalisesti anisotrooppisia* palloja koskevaan tutkimukseen, mutta luopuu siitä olettamuksesta, että pallon tangentiaalisten permittivisyyskomponenttien on oltava samat. Tämän yleistyksen aikaansaamaa uutta inkluusiota kutsutaan väitöskirjassa *systrooppiseksi palloksi*.

Avainsanat anisotropia, Laplace-yhtälö, potentiaali

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Preface

The efforts of many have flown together to make this thesis possible. Without my research group the thesis would certainly not have taken its present shape. I therefore feel obliged to reserve this place for an acknowledgement of my indebtedness. First, I would like to thank my supervisor Prof. Ari Sihvola whose comments have been invaluable for my research and my writing. Prof. Sihvola's experience in metamaterial research helped me to overcome many obstacles. His foregoing metamaterial research was also the inspiration for all the enclosed articles. Also, I would like to thank my instructor Henrik Wallén who despite his great many commitments in Aalto University found the time provide me assistance when needed. I have much benefited from his vast knowledge that easily covers the whole substance matter of the thesis but also extends to less theoretical issues like writing, presenting, and the maintenance of a computer system. I extend my warm gratitude to Dr. Henrik Kettunen who coauthored Publications I and II. Dr. Kettunen contributed valuable insights for the coauthored articles. His help in maintaining the efficacious but easygoing work mentality of our group was equally important.

Much help has come from within the research group but some important contributions have come from without. Special thanks go to Dr. Mikko Pitkonen who volunteered his help for Publication II. Even though assisting the research group in the study of systropic spheres was outside Dr. Pitkonen's professional liabilities, the help was promptly available when needed.

I also wish to thank Aalto ELEC Doctoral School, which funded the research from August 2013 to July 2016.

The thesis was written in the Department of Radio Science and Engineering, in the School of Electrical Engineering, in Aalto University. During my work term, the department's good reputation for sophisticated

environment has perpetually proved to be richly deserved. I wish to thank all the department's good people whose inspiring enthusiasm for academic research has spurred me onward.

Espoo, November 22, 2016,

Tommi Rimpiläinen

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List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I Tommi Rimpiläinen, Henrik Wallén, Henrik Kettunen, Ari Sihvola. Electrical Response of Systropic Sphere. *IEEE Transactions on Antennas and Propagation*, Vol. 60, No. 11, pp. 5348–5355, November 2012.

II Tommi Rimpiläinen, Mikko Pitkonen, Henrik Wallén, Henrik Kettunen, Ari Sihvola. General Systropy in Spherical Scatterers. *IEEE Transactions on Antennas and Propagation*, Vol. 62, No. 1, pp. 327–333, January 2014.

III Tommi Rimpiläinen, Henrik Wallén, Ari Sihvola. Radial Anisotropy in Spheroidal Scatterers. *IEEE Transactions on Antennas and Propagation*, Vol. 63, No. 7, pp. 3127–3135, July 2015.

IV Tommi Rimpiläinen, Henrik Wallén, Ari Sihvola. Polarizability of Radially Anisotropic Elliptic Inclusion. *Journal of Applied Physics*, Vol. 119, pp. 014107:1–8, January 2016.

V Tommi Rimpiläinen, Henrik Wallén, Ari Sihvola. Porcupic Concentrators and Bulbic Cloaks in Planar Configuration. *2016 URSI International Symposium on Electromagnetic Theory (EMTS)*, pp. 80–83, August 2016.

Author's Contribution

Publication I: "Electrical Response of Systropic Sphere"

The idea for the research came from Prof. Ari Sihvola. Prof. Sihvola suggested that the foregoing research concerning radial anisotropy could be generalized to treat spherical inclusions with three independent material parameters, the systropic spheres. The author coined the term "systropy" with help from Prof. Sihvola. The author derived the analytical formulas independently. Dr. Henrik Wallén and Dr. Henrik Kettunen helped the author to create the computer model for the numerical validation of the results. The article was written by the author. Prof. Sihvola, Dr. Wallén, and Dr. Kettunen helped the author to revise the manuscript.

Publication II: "General Systropy in Spherical Scatterers"

The research idea is a continuation of the study in the first article on systropy. The development of the method that the article outlines was long hindered by the difficulty of finding sufficiently regular special functions to express the solution. Dr. Pitkonen introduced suitable functions. His method canceled out the irregularity in two functions by taking a linear combination of them. Dr. Wallén improved on this method by showing how the result can be expressed with Gegenbauer polynomials, which are regular at all points and are also fast to compute. Dr. Wallén also showed how integrals between Gegenbauer polynomials can be computed efficiently with great accuracy. Apart from the issues concerning the special functions, the theory was developed by the author. The author calculated the results shown in the article. The article was written by the author. Prof. Sihvola, Dr. Wallén, and Dr. Kettunen helped the author to

revise the manuscript.

Publication III: "Radial Anisotropy in Spheroidal Scatterers"

The research idea is a continuation from existing research on radially anisotropic spheres. The idea was suggested by Prof. Sihvola. Prof. Ismo Lindell's astute advice was to restrict the research to spheroids instead of extending it to general ellipsoids. Dr. Wallén helped the author to formulate the Ansatz to the problem. Apart from this, the theory was developed by the author. The author calculated the results shown in the article. The article was written by the author. Prof. Sihvola, Dr. Wallén, and Dr. Kettunen helped the author to revise the manuscript.

Publication IV: "Polarizability of Radially Anisotropic Elliptic Inclusion"

The research idea came from Prof. Sihvola. The theory and the calculations were produced by the author. The article was written by the author. Prof. Sihvola and Dr. Wallén helped the author to revise the manuscript.

Publication V: "Porcupic Concentrators and Bulbic Cloaks in Planar Configuration"

Prof. Sihvola suggested the idea of studying the radially anisotropic ellipses that separate into two domains. The author took the initiative to extend the research to multidomain inclusions. The theory and the calculations were produced by the author. The article was written by the author. Prof. Sihvola and Dr. Wallén helped the author to revise the manuscript.

Symbols

a	radius of sphere
A	area
$A(\mathcal{C})$	area enclosed by planar curve
C	capacitance
$\mathcal{C}$	path or closed curve
d	distance
$\mathbf{D}$	electric flux density [As/m^2]
$\mathcal{D}$	domain
$\mathbf{E}$	electric field [V/m]
$\mathbf{E}^{\mathrm{p}}$	excitation field [V/m]
$\mathbf{E}^{\mathrm{s}}$	perturbation field [V/m]
f_{b}	volume fraction of background medium
f_{s}	volume fraction of spherical inclusions
$\mathbf{F}$	force [N]
$\mathcal{F}$	facet of pillbox
$\bar{\bar{I}}$	unit dyadic
L_ξ	depolarization factor
$\mathbf{n}$	unit normal vector of a surface
n_{brick}	block density
$\mathbf{p}$	dipole moment [Asm]
$\mathbf{P}$	dipole moment density [As/m^2]
P_n	Legendre polynomial of degree n
q	electric source charge [C]
q_{t}	electric test charge [C]
Q	electric charge [C]
$\mathbf{r}'$	location of source
$\mathbf{r}$	location of observation point
r	radial coordinate of spherical coordinate system

R	distance from source to observation point $\lvert\mathbf{r}-\mathbf{r}'\rvert$
$R(r)$	radial component of a function in spherical coordinate system
$\mathbf{R}$	separation between source and observation point $\mathbf{r}-\mathbf{r}'$
$\hat{\mathbf{R}}$	unit vector from source toward observation point
$\mathcal{S}$	surface-like domain
$\mathbf{u}_{\text{axis}}$	axis unit vector of uniaxial medium
$\mathbf{u}_{\xi}$	unit vector corresponding to variable ξ
V	volume [m^3]
$\mathcal{V}$	volume-like domain
α	normalized polarizability
$\bar{\bar{\alpha}}_{\text{abs}}$	polarizability (not normalized)
$\delta^3(\mathbf{r})$	Dirac delta function in three dimensions
$\delta_{\mathcal{S}}$	surface delta function
$\delta\xi$	displacement in ξ direction
Δf	change in function f
ϵ_0	vacuum permittivity $\approx 8{,}85 \cdot 10^{-12}$ As/Vm
$\epsilon_{\parallel}$	relative permittivity parallel to anisotropy axis
$\epsilon_{\perp}$	relative permittivity perpendicular to anisotropy axis
$\bar{\bar{\epsilon}}$	relative permittivity dyadic
ϵ_{eff}	relative effective permittivity
θ	polar coordinate of spherical coordinate system
λ	exponent in expression $r^{\lambda+1}$ of $R(r)$
λ	charge-per-unit-length [Q/m]
λ_{h}	density of blocks in horizontal direction
λ_{v}	density of blocks in vertical direction
ρ	charge-per-unit-volume [As/m^3]
ρ_{b}	density of bound charge [As/m^3]
ρ_{f}	density of free charge [As/m^3]
σ	charge-per-unit-area [As/m^2]
σ_{b}	density of bound surface charge [As/m^2]
ϕ	electrostatic potential [V]
ϕ^{p}	excitation potential [V]
ϕ^{s}	perturbation potential [V]
φ	azimuthal coordinate of spherical coordinate system
χ	electric susceptibility

Operators

$\lim_{\xi \to \Xi}$	limit
$\frac{\partial}{\partial \xi}$	partial derivative with respect to ξ
$\nabla f(\mathbf{r})$	gradient of function
$\nabla \times \mathbf{F}(\mathbf{r})$	curl of function
$\nabla \cdot \mathbf{F}(\mathbf{r})$	divergence of function
$\nabla^2 f$	Laplace operator
$\langle \mathbf{F} \rangle$	volume average of field

1. Field equations in electrostatics

All things obey fixed laws.

Titus Lucretius Carus

1.1 Origins of field equations

The equations that govern electrostatics were first developed within a discipline seemingly far removed from electricity, in the discipline of orbital mechanics. The formulation of electrostatics that has now become standard goes back to Pierre-Simon Laplace (1749–1827) and his researches into the workings of our solar system.

Although planetary science counts among one of the oldest sciences, some impactful open questions in this direction had persisted all the way to the late 18th century [1]. One of the prominent questions that remained regarded the stability of the solar system over long periods of time. Since Johannes Kepler's time (1571–1630) it had been known that each planet follows a near-elliptic orbit around the sun, so that the sun is located in one of focal points of the ellipse. However, Newton's theory of gravity predicts that the planets in the solar system are not only attracted by the sun but also by each other. During a single revolution around the sun, an interaction between two planets deviates their orbits from the elliptic shape only slightly, provided that the planets retain their customary celestial distance. However, the cosmological time periods involved in celestial mechanics provide ample time for small perturbations to accumulate so that the planetary orbits can change even drastically. While these new orbits retain approximately elliptic shapes, they may intersect in a way that makes collisions possible.

Among other things, collisions can drastically and permanently change the construct of the solar system. However, the astronomical data from

several millennia suggest that the solar system is stable. This raises the question, if the solar system is stable only for a period of time that is relatively small in the astronomical time scale or if the present state of the solar system is permanent, at least in the idealized model that excludes all catastrophes that are not caused by gravitation.

The question is deep enough to have eluded the great mind of Isaac Newton himself (1642–1727). Newton was, in fact, in a unique position to propose scientific determinism after having postulated his three fundamental laws of mechanics and his theory of gravitation in the seminal book *Philosophiæ Naturalis Principia Mathematica*, published in 1687. Claiming that these laws alone determine the fate of the solar system might have seemed like a plausible statement to make. However, the mathematical intricacies involved in solving the stability problem were beyond Newton's approach, which used rather elaborate geometric explanations to augment differential calculus. So puzzling was the dilemma, that Newton found himself at the end of his scientific insight. Having reached the limits of his scientific knowledge, he rather notoriously summoned a *deus ex machina* argument, explaining the stability of the solar system by a divine intervention. Newton discusses stability in his book *Opticks*. This book ends with "queries" where Newton answers questions on a diverse set of topics—many of the topics far outside the discipline after which the book is entitled. In these queries Newton wrote [2, p. 378]:

> For while Comets move in very excentrick Orbs in all manner of Positions, blind Fate could never make all the Planets move one and the same way in Orbs concentrick, some inconsiderable Irregularities excepted, which may have risen from the mutual Actions of Comets and Planets upon one another, and which will be apt to increase, till this System wants a Reformation.

Pierre–Simon Laplace sought to pursue the question of stability courageously, seeking a mathematical explanation to the conundrum [1]. His work, along most of the known planetary science of his time, is summarized in his five volume book *Méchanique céleste*, the first two volumes of which were published in 1799. Some of his findings are at the heart of the physical sciences of our day and thereby relevant to the present thesis. In particular, the third volume of *Méchanique céleste* introduces an equation that is now referred to as the *Laplace equation*. The Laplace equation is ubiquitous in all branches of theoretical physics. It will be introduced in

context of electrostatics to make it salient within the present work.

1.2 Electrostatic fields

The electrostatic force acting between two electrically charged particles in a vacuum is given by the Coulomb law. Let q be a *source charge* which is assumed to be at rest at a location indicated by a vector $\mathbf{r}'$ and let q_t be a *test charge* at the location $\mathbf{r}$. The force that the source charge produces on the test charge is

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qq_\mathrm{t}}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}') \tag{1.1}$$

The Coulomb law is an experimental result which can be regarded as the starting point for describing electrostatics. Likewise, it is found experimentally that the *vacuum permittivity* ϵ_0 takes the value

$$\epsilon_0 \approx 8.85 \times 10^{12}\,\frac{\mathrm{A\,s}}{\mathrm{V\,m}} \tag{1.2}$$

For brevity, we adopt some conventions to denote recurring vector quantities. The conventions apply throughout the thesis and are fairly standard. We denote vector quantities with bold-face symbols, like $\mathbf{G}$, and scalar quantities in normal mathematical typeface, like g. The norm $|\mathbf{G}| = (\mathbf{G}\cdot\mathbf{G})^{1/2}$ is denoted by G, in normal mathematical typeface. The unit vector $\mathbf{G}/|\mathbf{G}|$ to the direction of $\mathbf{G}$ is denoted by $\hat{\mathbf{G}}$. We always denote the place of a source charge with $\mathbf{r}'$ and the point of observation with $\mathbf{r}$. The separation $\mathbf{r}-\mathbf{r}'$ between these is denoted by $\mathbf{R}$. With these conventions in place, we can write (1.1) more concisely as

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qq_\mathrm{t}}{R^2}\hat{\mathbf{R}} \tag{1.3}$$

To extend the discussion to concern an ensemble of particles in a vacuum, it is necessary to introduce one more experimental result which gives the total force on a test charge produced by multiple source charges. The required law is the *principle of superposition* and it states that the combined force of many charges acting on a test charge is the sum of the forces of each source charge acting in isolation. To put this in symbols, let q_1, q_2, q_3,... be the source charges, and $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, ... be the forces they produce on a test charge q_t. The total force acting on the test charge q_t is then given by $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots$. The result is not entirely trivial because there are many important vector quantities in electromagnetic field theory that are not similarly additive [3, Sec. 3.A],[4, Sec. 2.1.1].

However, experiments agree with the principle of superposition. In conjunction with the Coulomb law, the principle is powerful enough to allow us to calculate electrostatic forces.

The Coulomb law exhibits some notable properties. The first consequence follows if we revert the arbitrary convention of labeling q_t the "test charge" and q the "source charge", so that instead q_t is the source charge and q the test charge. Then we find that the law retains the value of the force vector $\mathbf{F}$ apart from a change of sign. The Coulomb law thus stands in accordance to Newton's third law.

Another consequence of the Coulomb law follows when the law is employed in conjunction with the principle of superposition. Let us consider a distribution of source charges q_1, q_2, q_3,... in locations $\mathbf{r}'_1$, $\mathbf{r}'_2$, $\mathbf{r}'_3$,.... The force acting on a test charge q_t located at $\mathbf{r}$ is then given by

$$\mathbf{F} = \sum_n \mathbf{F}_n = q_\mathrm{t} \left(\frac{1}{4\pi\epsilon_0} \sum_n \frac{q'_n}{|\mathbf{r} - \mathbf{r}'_n|^3} (\mathbf{r} - \mathbf{r}'_n) \right) \tag{1.4}$$

The equation shows that the force that acts on a charge is proportional to the magnitude of the charge. If we introduce an auxiliary field quantity

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_n \frac{q_n}{|\mathbf{r} - \mathbf{r}'_n|^3} (\mathbf{r} - \mathbf{r}'_n) \tag{1.5}$$

the force that acts on the test charge is simply

$$\mathbf{F} = q_\mathrm{t} \mathbf{E} \tag{1.6}$$

If we vary the location of the test charge q_t so that the forces can be obtained for many different placements, we find that both the force and the auxiliary vector $\mathbf{E}$ depend on the position. Therefore we call them *vector fields*. The vector field $\mathbf{E}(\mathrm{r})$ is referred to as the *electric field intensity*.

If the source charge is not located at a discrete set of points but, instead, spreads smoothly over some region, the equations for the force (1.4) or the field (1.5) are not directly applicable. In that case, the sums in (1.4) and (1.5) become integrals, so that

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\mathbf{R}}}{R^2} \, \mathrm{d}q \tag{1.7}$$

where $\mathrm{d}q$ is an infinitesimal element of the source charge and the integral is taken over all the source charge in the relevant domain. The integral (1.7) can take several interesting forms, depending on the way the source charge is distributed. Starting with the one-dimensional case, if the source charge is distributed over a curve so that the *charge-per-unit-length* is λ in the element $\mathrm{d}l'$ of the curve, the element of charge is

$\mathrm{d}q = \lambda\,\mathrm{d}l'$ and the integral (1.7) becomes

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{R^2}\hat{\mathbf{R}}\,\mathrm{d}l' \tag{1.8}$$

The line charge is here introduced for completeness but the concept is not explicitly used within the context of the thesis. However, proceeding to the next, two-dimensional case, the concept of a continuous surface charge distribution turns out to be useful. If the charge is spread over a surface, the infinitesimal charge element takes the form $\mathrm{d}q = \sigma\,\mathrm{d}A'$, where $\mathrm{d}A'$ is an element of the relevant surface and σ is the *charge-per-unit-area*. In this case the integral (1.7) becomes

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{R^2}\hat{\mathbf{R}}\,\mathrm{d}A' \tag{1.9}$$

Lastly, the charge can be distributed continuously over a volume, in which case the element of source charge is $\mathrm{d}q = \rho\,\mathrm{d}V'$ inside the volume element $\mathrm{d}V'$ holding a *charge-per-unit-volume* ρ. The integral (1.7) for the electric field then becomes

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{R^2}\hat{\mathbf{R}}\,\mathrm{d}V' \tag{1.10}$$

In electrostatics, it is often the case that charge distributions over curves, surfaces, or volumes are not continuous in the whole domain but still retain continuity inside distinct subdomains that only share points at their boundaries. This lack of continuity often manifests itself, for example, at the interfaces between two adjacent domains of different materials. However, all of the integrals (1.8), (1.9), (1.10) still hold in the *piecewise continuous* case. That the integrals are also valid in this more general case, follows from an elementary property of integrals: piecewise continuous functions can always be integrated.

Although the concept of electric field was introduced as an auxiliary for calculating forces that act between electric charges, there are good reasons to hold electric fields as genuine physical entities. The conception that the electric fields physically permeate the space goes back, at least, to Michael Faraday (1791–1867), who argued that there cannot be force between two objects if there is nothing in between to carry the force. Faraday conjectured electric and magnetic interactions are mediated by what he called *lines of force*, which are able to permeate most materials and even vacuum [5]. Faraday's view contrasts sharply with the conception of *action at a distance* which is implicit in Newton's theory of gravitation, although Newton himself rejected the concept as artificial [6, p. xxiii]. James Clerk Maxwell (1831–1879) echoed Faraday's conception and proceeded to postulate, in his 1861 article *On Physical Lines*

of Force, that the electric and magnetic fields are stresses and strains in an invisible, universal medium, *ether*, which he called *luminiferous medium* [7]. Maxwell's mechanical interpretation of fields and the notion of ether have become largely repudiated after special relativity [8]. However, the concepts of electric and magnetic fields persist in the electromagnetic field theory. The prevailing paradigm does not seek to explain electric or magnetic fields in terms of anything mechanical, but rather reverses the thought process by choosing a different explicandum. Rather than taking mechanical forces for granted and using the concept to explain electric and magnetic forces, the electromagnetic theory takes electric and magnetic fields as given, fundamental physical quantities, the existence of which can be assumed when explaining the action of mechanical forces [9].

When charged particles can have no direct effect on other charges without mediation from the intervening field quantities, a change in a source charge's position might have instantaneous effect on the force acting on a test particle if the field would somehow manage to adjust itself to the change without delays. However, experiments demonstrate that the fields do take their time to adjust and that the schedule of the adjustment is constrained by the universal speed limit that dictates the constant propagation speed of light [10]. The finite adjustment time of the fields suggests that (1.7) is true only when enough time has elapsed to permit the fields to readjust after the source charges q_1, q_2, q_3,... have been suspended stationary after movement. The fields are not, in general, given by (1.7) if the charges q_1, q_2, q_3,... have only become stationary within the time it takes the fields to adjust or if they continue in motion. The process that makes the fields change after a change in the source, i.e the *transient response*, exemplifies a degree of independence between the fields and their sources. It follows that (1.7), which we used to introduce electric fields, allows us to compute electric fields for long-time stationary charges but that (1.7) does not define the concept or suffice in the treatment of changing electric fields.

1.3 Vector operators

It is possible to formulate the relationship between the fields and their sources without the problematic concept of action at a distance. The view that the fields and the sources can only affect their immediate surround-

ings, but not remote objects, is referred to as the *principle of locality*. In the theory of dynamic fields, the principle of locality is associated with Christiaan Huygens who formulated the famous Huygens–Fresnel principle in his book *Traité de la Lumière*, drafted in 1678 [11, p. 19]. The local view of electrostatics requires a formulation quite different from but compatible to (1.7). The fields need to be presented in terms that apply locally, rather than in the entirety of the infinite space.

The required mathematical tools are enshrined in a vector calculus that Josiah Willard Gibbs (1839–1903) developed in 1881, in his book *Elements of Vector Analysis* [12]. The concepts of vector calculus, that most concern electrostatic field theory, are the concepts of gradient, divergence, and curl. Each of these three can be viewed as operators that act in a local region of space, regardless of the size of the chosen region, even if the extent of the region concerned approaches infinitesimal dimension. The three operators act upon a field quantity defined in this region and turn it into a new field of the same region.

To begin with the gradient, the operation is intuitively understood as a procedure that tells the direction of steepest ascent in the value of a scalar field and also the slope of ascent in the given direction. The relevance of such knowledge is exemplified in a geological analogy that involves a steady stream of water running from a glacier on the top of a suitably smooth mountain. For the analogy to work, it is presumed that the stream runs slowly enough for the waters inertial effects to be predominated by friction and the force of gravity, so that the contours of the mountain alone will determine the direction, and intensity of the flow. If the height of the mountain is taken to be the scalar field for the gradient operation to act upon, the direction of the gradient vector in a location taken from the center of the stream points to the direction opposite from the flow of the stream from that location and the magnitude of the gradient field tells the velocity of the flow.

The gradient and the curl are easiest to formulate in mathematical terms that involve the Cartesian coordinate system. Although the choice of a coordinate system is nowhere present in qualitative descriptions of the concept, such as the above analogy, the Cartesian system turns out to be the one that permits the definitions in the simplest terms. In this system, we can define the "*del*" symbol ∇ so that

$$\nabla = \mathbf{u}_x \frac{\partial}{\partial x} + \mathbf{u}_y \frac{\partial}{\partial y} + \mathbf{u}_z \frac{\partial}{\partial z} \tag{1.11}$$

The gradient of the scalar field $\phi(\mathbf{r})$ is then defined by a "multiplication"

of sorts between the symbol ∇ and the field ϕ, so that

$$\nabla\phi(\mathbf{r}) = \mathbf{u}_x\frac{\partial\phi}{\partial x} + \mathbf{u}_y\frac{\partial\phi}{\partial y} + \mathbf{u}_z\frac{\partial\phi}{\partial z} \tag{1.12}$$

That the symbol ∇ should act on the scalar field $\phi(\mathbf{r})$ by way of multiplication is to be taken as a useful short-hand rather than an accurate description.

Whereas the gradient operates on a scalar field to give a vector field, the divergence does the reverse, so that it operates on a vector field and gives a scalar field. However, the reverse relationship holds only between the domains of the operators and not, in general, on any particular values taken from those domains, so that the divergence is not the inverse of the gradient. The combined operation where a divergence is taken after a gradient, transforms a scalar field into another specimen, so that the relationship between the two is non-trivial and useful.

The intuition behind the concept of divergence is that a positive divergence indicates those parts of a region where field-lines are created, i.e. the "*sources*", and a negative divergence indicates those parts where field lines end, i.e. the "*sinks*". The notion of sinks and sources can be formalized by considering a connected volume of infinitesimal size. The surface integral over this miniscule volume tells the difference between the number of field lines exuded from the volume and the number of field lines bound towards the volume. When this difference vanishes, there are as many in-bound field-lines as there are out-bound ones, indicating that no field-lines are created or destroyed. However, a positive surface integral indicates emergent field-lines and a negative surface integral indicates vanishing field-lines. Thus the divergence operation takes on the following formal expression

$$\nabla\cdot\mathbf{F} = \lim_{V(\mathcal{S})\to 0}\frac{\int_{\mathcal{S}}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}A}{V(\mathcal{S})} \tag{1.13}$$

where $\mathbf{n}$ is the unit normal vector of the infinitesimal area element $\mathrm{d}A$ pointing outwards from the volume $\mathcal{V}$ and $V(\mathcal{S})$ is the volume enclosed by the surface $\mathcal{S}$ of $\mathcal{V}$. The divergence can also be expressed by using partial derivatives. In the Cartesian coordinate system the alternative form for divergence is

$$\nabla\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{1.14}$$

where F_x, F_x, F_x are the Cartesian components of the vector field $\mathbf{F}$. In light of (1.11), (1.14) justifies the analogy between the dot product and the divergence. However, the analogy no longer holds in general curvilinear coordinates.

The remaining operator, that is referred to as *curl*, is aptly named because it indicates the propensity of the vector field to curl around a given point. The operator acts on a vector field and gives back another vector field, so that at each point, the vector that results from the operation is the axis of rotation for the original vector field and the magnitude of the resulting vector gives the degree of curledness. A formal meaning to this curledness can be obtained by measuring the degree of circulation of the vector field around a given axis, denoted by unit vector $\mathbf{u}$. Let $\mathcal{C}$ be a closed path around the axis, lying completely on a plane perpendicular to $\mathbf{u}$. Then for each choice of $\mathbf{u}$, a component of the curl in the given direction is the circulation integral

$$\mathbf{u} \cdot (\nabla \times \mathbf{F}) = \lim_{A(\mathcal{C}) \to 0} \frac{1}{A(\mathcal{C})} \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{l} \tag{1.15}$$

where $A(\mathcal{C})$ is the area enclosed by the planar path $\mathcal{C}$ and $\mathrm{d}\mathbf{l}$ is the directed element of the path [13, Sec. 3.6]. The expression for the curl is again given most succinctly in the Cartesian system of coordinates, so that the curl is

$$\nabla \times \mathbf{F} = \mathbf{u}_x \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \mathbf{u}_y \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \mathbf{u}_z \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \tag{1.16}$$

The curl operator coincides with the cross product between the vector field and the operator in (1.11) when the coordinate system is Cartesian. Similarly to the case of divergence, the analogy between cross product and the vector operator, curl, holds in the Cartesian coordinate system but not in a general curvilinear coordinate system.

The concepts of divergence and curl can be defined either through partial derivatives in a specified coordinate system or through limits of integrals, (1.13) and (1.15), without a specified system of coordinates. The last approach makes the interrelation between integrals and the ∇-operator more apparent. In fact, the definitions suggest the possibility of a more general relationship where the sizes of the integrals are not limited to infinitesimal dimension. For the proof of this hypothesis, we refer to literature [14, Sections 10.7, 10.9].

Let us choose an arbitrary domain $\mathcal{V}$ with finite volume and denote the surface of this volume with $\partial\mathcal{V}$. Furthermore, let us denote the unit normal vector that points away from the volume $\mathcal{V}$ with $\mathbf{n}$. Then, when it comes to the divergence operator, the definition (1.13) generalizes into

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{F}) \, \mathrm{d}V = \int_{\partial\mathcal{V}} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}A \tag{1.17}$$

for a sufficiently smooth vector field $\mathbf{F}$. This important result is known as the *Gauss theorem* but we refer to it as *divergence theorem* to avoid a confusion with the forthcoming Gauss law (1.37) of Sec. 1.5.

To generalize the definition (1.15) of the curl, we consider a surface $\mathcal{S}$ which can be open or closed, so that it does not necessarily enclose a volume. As before, $\mathbf{n}$ denotes the normal vector of the surface but, because the surface can be open and therefore fail to distinguish between inside and outside domains, the normal vector can take two different directions. We take $\mathcal{C}$ to be a path that circulates the edge of the surface $\partial\mathcal{S}$ counterclockwise with respect to the vector $\mathbf{n}$. Then, the definition (1.15) generalizes into

$$\int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}A = \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{l} \tag{1.18}$$

where $\mathrm{d}\mathbf{l}$ is a directed element of the path $\mathcal{C}$. This is known as the *Stokes theorem*. The theorem becomes useful when we, in Sec. 1.4, establish the path independence of potential in a conservative field.

A third theorem that becomes useful concerns the gradient. When a gradient is taken from a suitably smooth scalar field f, the path integral over a path $\mathcal{C}$ from $\mathbf{a}$ to $\mathbf{b}$

$$\int_{\mathcal{C}} \nabla f \cdot \mathrm{d}\mathbf{l} = f(\mathbf{a}) - f(\mathbf{b}) \tag{1.19}$$

gives the difference between the scalar field at the end points. This is known as the *fundamental theorem of gradients*.

1.4 Electrostatic potentials

If we substitute the identity

$$\frac{\hat{\mathbf{R}}}{R^2} = -\nabla \left(\frac{1}{R} \right) \tag{1.20}$$

to the Coulomb law (1.7) involving a general charge distribution and operate with the curl on both sides of the Coulomb law, we note that the curl of a gradient vanishes identically. The resulting equation

$$\nabla \times \mathbf{E} = 0 \tag{1.21}$$

gives a vanishing curl for all electric fields, regardless of the charge distribution, provided only that it be static. One of the consequences of the vanishing curl is that the vector field $\mathbf{E}(\mathbf{r})$ can be replaced with a scalar field $\phi(\mathbf{r})$ with no loss of information. This is possible because the three

scalar components of the vector field $\mathbf{E}(\mathbf{r})$ are not independent but they are bound by the constraint (1.21). The constraint (1.21) is automatically respected if the scalar field and the vector field are related by

$$\mathbf{E} = -\nabla\phi \tag{1.22}$$

because a curl of a gradient always vanishes identically.

Let us derive a scalar field that satisfies (1.22). We first choose a point $\mathbf{r}_0$ in the space. The choice is almost arbitrary, but it must not coincide with any singularities in the electric field. It follows from (1.21) and the Stokes theorem that an integral from $\mathbf{r}_0$ to another point $\mathbf{r}$ is independent of the integration path between the points. Therefore, if $\mathcal{C}(\mathbf{r})$ is a path from the point $\mathbf{r}_0$ to the point $\mathbf{r}$, a scalar field defined by a path integral

$$\phi(\mathbf{r}) = -\int_{\mathcal{C}(\mathbf{r})} \mathbf{E}\cdot\,\mathrm{d}\mathbf{l} \tag{1.23}$$

is unique up to the choice of $\mathbf{r}_0$ and it also satisfies (1.22). That the potential is independent of the choice of path, follows if we consider two paths $\mathcal{C}$ and $\mathcal{C}'$ from $\mathbf{r}_0$ to $\mathbf{r}$ so that the potential given by the path $\mathcal{C}$ is denoted by ϕ and the potential given by the path $\mathcal{C}'$ is denoted by ϕ'. The difference between these is

$$\phi(\mathbf{r}) - \phi'(\mathbf{r}) = \int_{\mathcal{C}} \mathbf{E}\cdot\,\mathrm{d}\mathbf{l} - \int_{\mathcal{C}'} \mathbf{E}\cdot\,\mathrm{d}\mathbf{l} = \int_{\tilde{\mathcal{C}}} \mathbf{E}\cdot\,\mathrm{d}\mathbf{l} \tag{1.24}$$

where $\tilde{\mathcal{C}}$ is the compound path that goes from $\mathbf{r}_0$ to $\mathbf{r}$ through $\mathcal{C}$ and follows the path $\mathcal{C}'$ in the reverse direction from $\mathbf{r}$ to $\mathbf{r}_0$. This compound path constitutes a closed loop, which vanishes because of the Stokes theorem (1.18) and the fact (1.21) that the electric field of static charges has no curl. Therefore the potentials ϕ and ϕ' are the same regardless of the choice of the path.

That the required identity (1.22) is indeed satisfied by the potential (1.23), follows from the fundamental theorem of gradients (1.19). By the theorem, the potential difference between two points a and b is

$$\phi(\mathbf{a}) - \phi(\mathbf{b}) = \int_{\mathcal{C}} \nabla\phi\cdot\,\mathrm{d}\mathbf{l} \tag{1.25}$$

The same difference can be expressed in terms of electric fields by (1.23), so that

$$\phi(\mathbf{a}) - \phi(\mathbf{b}) = -\int\limits_{\mathcal{C}(\mathbf{a},\mathbf{r}_0)} \mathbf{E}\cdot\,\mathrm{d}\mathbf{l} + \int\limits_{\mathcal{C}(\mathbf{b},\mathbf{r}_0)} \mathbf{E}\cdot\,\mathrm{d}\mathbf{l} = -\int\limits_{\mathcal{C}(\mathbf{a},\mathbf{b})} \mathbf{E}\cdot\,\mathrm{d}\mathbf{l} \tag{1.26}$$

Because the integrals are the same

$$\int_{\mathcal{C}} \nabla\phi\cdot\,\mathrm{d}\mathbf{l} = -\int_{\mathcal{C}} \mathbf{E}\cdot\,\mathrm{d}\mathbf{l} \tag{1.27}$$

for every choice of the points $\mathbf{a}$ and $\mathbf{b}$ and for every choice of the path $\mathcal{C}$ between the points, we conclude that the integrands must be the same, which gives $\mathbf{E} = -\nabla\phi$, confirming (1.22).

Also, the choice of the reference point $\mathbf{r}_0$ only fixes the mostly irrelevant constant level of the potential without affecting potential differences $\phi(\mathbf{a}) - \phi(\mathbf{b})$ between points. If we choose a different reference point $\mathbf{r}_0'$, the potential ϕ' with this new reference point is related to the integral (1.23) by

$$\phi'(\mathbf{r}) = -\int_{\mathbf{r}_0'}^{\mathbf{r}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\int_{\mathbf{r}_0'}^{\mathbf{r}_0} \mathbf{E} \cdot \mathrm{d}\mathbf{l} - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot \mathrm{d}\mathbf{l}$$

The difference between the two potentials ϕ and ϕ' thus takes a constant value, whereby both potentials give the same field according to (1.22).

The electric potential created by a point charge q at $\mathbf{r}'$ is found as follows. Because the electric field was found to be independent of the choice of $\mathbf{r}_0$, we can place this point at infinity. By (1.5), the electric field of a point charge is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{\mathbf{R}}$$

We evaluate the integral (1.23) so that the path is taken along the infinite straight line that connects $\mathbf{r}$ and $\mathbf{r}'$, so that the segment is from infinity to $\mathbf{r}$. The integral then becomes

$$\phi(\mathbf{r}) = -\int_{\infty}^{\mathbf{r}} \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{\mathbf{R}} \cdot \mathrm{d}\mathbf{l} = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{R} \frac{q}{R^2}\,\mathrm{d}R = \frac{1}{4\pi\epsilon_0}\frac{q}{R} \tag{1.28}$$

The electric fields created by a charge distribution is known to observe the superposition principle $\mathbf{E} = \int \mathbf{E}(\mathrm{d}q)$ where $\mathbf{E}(\mathrm{d}q)$ is the field created by the infinitesimal but otherwise arbitrary element of charge $\mathrm{d}q$. If we choose a point of reference $\mathbf{r}_0$ and take the integral (1.23) we find that the electric potential observes a similar principle

$$\phi = \int_Q \phi(\mathrm{d}q) \tag{1.29}$$

Let us consider a charge distribution that is confined within a definite finite region of the space so that the total charge in the region is finite. This *localized charge distribution* creates an electric field that vanishes infinitely far away from the sources, which is not generally true about non-localized charge distributions [4, Sec. 2.3.2]. We can then use the potential of a point charge (1.28) and the principle of superposition (1.29) to obtain the electric potential of a piecewise continuous charge distribution

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_Q \frac{\mathrm{d}q}{R} \tag{1.30}$$

In particular, the potential created by a localized piecewise continuous surface charge is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{S}} \frac{\sigma}{R}\,\mathrm{d}A' \tag{1.31}$$

the potential created by a localized piecewise continuous volume charge is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho}{R}\,\mathrm{d}V' \tag{1.32}$$

We omit the domain symbol $\mathcal{V}$ if the integral is over the whole space.

1.5 Laplace equation

One of the rewards from using the apparatus of vector calculus is that it allows us to formulate electrostatics in local terms without a reference to the problematic concept of action at a distance. In particular, the information in the Coulomb law can be recast into an equation that only employs local properties of the vector field. The equation in question, is the Laplace equation. The historical context of the Laplace equation was discussed in Sec. 1.1. This section discusses the mathematics behind the equation.

To derive the Laplace equation, we consider the two properties that characterize a vector field locally: the curl and the divergence. Starting with the curl, the vanishing curl (1.21) makes it possible to represent the electric field in terms of a potential, as discussed in Sec. 1.4. The electric potential only has one component instead of the three in the original electric field. When we choose to calculate in terms of the potential instead of the vector field, the curl equation (1.21) is tacitly satisfied. It follows, that we only need to consider the divergence equation (1.37) to characterize the electric field locally when we use the electric potential.

Moving on to divergence, the electric field from a volume charge distribution is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\mathbf{R}}}{R^2}\rho(\mathbf{r}')\,\mathrm{d}V' \tag{1.33}$$

When we apply the divergence operator on both sides, we have

$$\nabla\cdot\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \nabla\cdot\left(\frac{\hat{\mathbf{R}}}{R^2}\right)\rho(\mathbf{r}')\,\mathrm{d}V' \tag{1.34}$$

Some levity was involved in commuting the divergence operator with an improper integral. The divergence in (1.34) is known to equal a constant

multiplied by the three-dimensional Dirac delta function [4, Sec. 1.5.3]:

$$\nabla \cdot \left(\frac{\hat{\mathbf{R}}}{R^2} \right) = 4\pi \delta^3(\mathbf{R}) \tag{1.35}$$

The characterizing property of the three-dimensional Dirac delta function is that the integral

$$\int f(\mathbf{r}')\delta^3(\mathbf{r}' - \mathbf{r}) \, \mathrm{d}V' = f(\mathbf{r}) \tag{1.36}$$

holds for any sufficiently smooth function $f(\mathbf{r})$. When we substitute (1.35) to (1.34) and use the property (1.36) of the Dirac delta function, we get

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho(\mathbf{r}) \tag{1.37}$$

which is an important result, referred to as the *Gauss law*.

The above derivation is appealing because it yields the correct result straightforwardly, with a direct calculation. However, the derivation precariously commutes the divergence operator with an improper integral. We can take a less direct approach to circumvent the problem. The divergence theorem (1.17) gives us an alternative derivation of the Gauss law. The electric field from a volume charge distribution is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\hat{\mathbf{R}}}{R^2} \rho(\mathbf{r}') \, \mathrm{d}V'$$

A surface integral of the field over a closed surface $\mathcal{S}$ is

$$\int_{\mathcal{S}} \mathbf{E}(\mathbf{r}) \cdot \mathbf{n} \, \mathrm{d}A = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \rho(\mathbf{r}') \int_{\mathcal{S}} \frac{\hat{\mathbf{R}}}{R^2} \cdot \mathbf{n} \, \mathrm{d}A \, \mathrm{d}V'$$

The value of the surface integral is known from literature (see [15, Sec. 1.3] or [16, Eq. 1.2.9]) and it is

$$\int_{\mathcal{S}} \frac{\hat{\mathbf{R}}}{R^2} \cdot \mathbf{n} \, \mathrm{d}A = \int_{\mathcal{S}} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \cdot \mathbf{n} \, \mathrm{d}A = \begin{cases} 4\pi, & \mathbf{r}' \in \mathcal{V} \\ 0, & \mathbf{r}' \notin \mathcal{V}, \mathcal{S} \end{cases}$$

With this result, the preceding equation becomes

$$\int_{\mathcal{S}} \mathbf{E}(\mathbf{r}) \cdot \mathbf{n} \, \mathrm{d}A = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho(\mathbf{r}') \, \mathrm{d}V'$$

Now, the divergence theorem (1.17) can be used on the left hand side of the equation to give

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E}(\mathbf{r}') \, \mathrm{d}V' = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho(\mathbf{r}') \, \mathrm{d}V'$$

If the divergence $\nabla \cdot \mathbf{E}$ and the charge density ρ are sufficiently regular, the integrands must coincide for the equation to hold for an arbitrary volume $\mathcal{V}$. Therefore,

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho(\mathbf{r})$$

To write the divergence equation for the electric potential, we substitute (1.22) to the original equation (1.37), which gives the Poisson equation

$$\nabla^2 \phi = -\frac{\rho(\mathbf{r})}{\epsilon_0} \tag{1.38}$$

A special case of the Poisson equation is the famous Laplace equation, which applies in those regions of the space where the source charge ρ vanishes. In this case, the equation for the potential is concisely

$$\nabla^2 \phi = 0 \tag{1.39}$$

2. Electric fields in medium

> In your letter, you apply the word “imponderable” to a molecule. Don’t do that again.
>
> James Clerk Maxwell

2.1 Microscopic charge distributions

To study how a material interacts with a static electric field, we need a theory of electrostatics that involves microscopic configurations of source charges. This is necessary because the relevant sources of electrostatic fields inside a medium are the small building blocks of the material, like atoms or molecules. The field from any one of the building blocks would necessarily be small, barring infinitesimal distances from the source. But when several such sources act in unison the effect is far from negligible, so it enters into the analysis of macroscopic electrostatic fields.

We can analyze the fields originating from this microscopic domain so that we start from one of the basic building blocks. The charge distribution of this building block may be presumed localized to a small region but otherwise arbitrary. We may then continue to characterize the electric field of this microscopic charge distribution the way it is seen from afar.

Let us consider a source charge q displaced by a vector $\mathbf{r}'$ from the origin and let us observe the electric potential at a position $\mathbf{r}$, which is farther from the origin than the source, $|\mathbf{r}| > |\mathbf{r}'|$. Let us also denote the angle between these two vectors by θ. The electric potential at $\mathbf{r}$ is

$$\phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{2.1}$$

The trigonometric *cosine rule*

$$|\mathbf{r}-\mathbf{r}'| = r\sqrt{1+\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\theta} \tag{2.2}$$

may be applied to the denominator of the last factor, to give an expression that only involves the scalar coordinates r, r' and θ. If we denote $\cos\theta$ by x and r'/r by t and expand the reciprocal of the square root as a power series in t

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_0^\infty P_n(x)t^n, \quad |t|<1 \tag{2.3}$$

the coefficients of this series are polynomials $P_n(x)$ in the remaining variable x [17, Sec. 12.1]. These *Legendre polynomials* will play a prominent role in Publications I, II, and III where they are used to solve a generalization of the Laplace equation (1.39) in the spherical and the spheroidal coordinate systems. See (2.20) in Sec. 2.5 and (3.40) in Sec. 3.4. To use the Legendre polynomials in the expression (2.1), we revert back to the vector notation and obtain

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\sum_0^\infty \left(\frac{r'}{r}\right)^n P_n(\cos\theta) \tag{2.4}$$

A common type of microscopic charge distribution in electrostatics involves a positive charge q and a negative charge $-q$ of equal magnitude, separated by a distance d which is the *dipole moment axis*. If we put the origin at the middle of the dipole moment axis, the positions of the charges are $\mathbf{r}'$ and $-\mathbf{r}'$ respectively. The potential from the charge at $\mathbf{r}'$ is given by (2.4) when we multiply with the factor $(4\pi\epsilon_0)^{-1}q$. Furthermore, (2.4) also gives

$$\frac{1}{|\mathbf{r}+\mathbf{r}'|} = \frac{1}{r}\sum_{n=0}^\infty \left(\frac{r'}{r}\right)^n (-1)^n P_n(\cos\theta) \tag{2.5}$$

when we note that the polar angles of two opposite vectors are the opposites and apply the parity

$$P_n(-x) = (-1)^n P_n(x) \tag{2.6}$$

of the Legendre polynomials [18, Sec. 8.2.3]. We can use (2.4) and (2.5) to find the potential, which is the superposition

$$\begin{aligned}\phi(r) &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}+\mathbf{r}'|}\right) \\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r}\sum_{n=0}^\infty \left(\frac{r'}{r}\right)^n P_n(\cos\theta) - \frac{1}{r}\sum_{n=0}^\infty \left(\frac{r'}{r}\right)^n (-1)^n P_n(\cos\theta)\right)\end{aligned} \tag{2.7}$$

of the two relevant potentials. The opposite terms in the expansion cancel out, leaving behind a polynomial consisting in even exponents of r'/r. The

term of the remaining polynomial involves the exponent $(r'/r)^2$, while the other terms have higher values of the exponent and consequently taper of more quickly as a function of the distance r. Supposing that the charge displacement $r' = d/2$ is small compared to the observation distance r, the first term

$$\phi_{\mathrm{d}} = \frac{qd}{4\pi\epsilon_0}\frac{P_1(\cos\theta)}{r^2} \tag{2.8}$$

of the series predominates.

For interpretation, let us imagine that the axis of polarization d diminishes by several orders of magnitude while the charge q increases in proportion, so that throughout the process the *dipole moment* vector $\mathbf{p} = qd\hat{\mathbf{r}}$ is preserved. The first term in the series would then remain unchanged. It corresponds to an *ideal dipole* that has a strictly positive dipole moment with an infinitesimal axis of polarization. For an illustration of the potential and the electric field of an ideal dipole, see Smythe's book Static and Dynamic Electricity [19, Sec. 1.08].

An ideal dipole serves as an approximation of a single polarized molecule or an atom when observed from a distance. The contributions from the great multitude of other molecules in the material can similarly be modeled as ideal dipoles. When the effect of the material as a whole is considered, the resulting electric field is approximately a sum of the dipole fields created at the molecular level. The specific placements of these dipoles are unknown and, as it turns out, largely irrelevant. If we consider a volume $\mathrm{d}V'$ of the medium centered at $\mathbf{r}'$, so that the volume $\mathrm{d}V'$ is large compared to a single material building block but small compared to the distance of observation, the field emanating from this patch of material is approximately the field produced by a single dipole $\mathbf{p}_V$ at $\mathbf{r}'$ where the strength of the dipole combines

$$\mathbf{p}_V = \sum_i \mathbf{p}_i \tag{2.9}$$

the strengths of the constituents in the same patch $\mathrm{d}V'$. The more constituent elements are included in $\mathrm{d}V'$, the more terms there are in the sum (2.9), whereby the magnitude of $\mathbf{p}_V$ is increased in proportion to the increase of $\mathrm{d}V'$. Therefore the total dipole moment $\mathbf{p}_V$ depends on the choice of $\mathrm{d}V'$ and an average *dipole moment density* $\mathbf{P}$, so that instead of the sum (2.9) we can write the product $\mathbf{p}_V = \mathbf{P}\,\mathrm{d}V'$. The dipole moment density is a vector field, which usually varies from place to place, depending on the properties of the material. When such distributed dipole moment density is given for a macroscopic inclusion, occupying a volume V, the superposi-

tion principle (1.29) implies that the potential observed at $\mathbf{r}$ is an integral

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\mathbf{R}}}{R^2}\,\mathrm{d}V' \tag{2.10}$$

over this volume. If all of the dipole moment is confined to a volume $\mathcal{V}$ so that the obervation point $\mathbf{r}$ is far away from this volume, we can approximate the integral in (2.10) by disregarding the negligible variation in the quantity $\hat{\mathbf{R}}/R^2$. The integral then assumes the form

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{R}}}{R^2}\cdot\int_{\mathcal{V}} \mathbf{P}(\mathbf{r}')\,\mathrm{d}V' = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{R}}}{R^2} \tag{2.11}$$

where we identified the volume integral

$$\mathbf{p} = \int_{\mathcal{V}} \mathbf{P}\,\mathrm{d}V' \tag{2.12}$$

as the total dipole moment of the distribution.

2.2 Polarization of medium

The equations that were developed in Ch. 1 govern the whole extent of electrostatics, provided that everything is known about the sources of the electric fields. In this case, the known sources can be used in (1.4) and (1.7) to directly derive the corresponding fields. However, the source charges in unvarnished nature vastly surpass elementary textbook examples in complexity. For example, in the glass of a window pane the molecules arrange in a rigid but somewhat random configuration, conflated together by strong electrostatic bonds. Near a nucleus of an atom the electric field ascends to great magnitudes while its direction changes rapidly as a function of the location. Calculating the electric field of the great multitude of nuclei in the pane would already be challenging, but such calculation would ignore the effect from the electron cloud surrounding the nuclei. Electrostatics alone does not yield the shape of the electron cloud. Instead, the shape is explained by quantum mechanics [20].

Although calculating the electric fields inside a complicated material might be theoretically possible, there remains another question that relates to the end results of such calculation. Most of this detailed knowledge would be largely irrelevant barring very specialised applications of electrostatics. Extracting relevant information from the results would be extremely tedious because the obtained field would be vastly complex. In most cases, it suffices to confine the discussion to the macroscopic properties of the medium and to ignore the microscopic fluctuations in the

electric fields. When it becomes impractical to deal with the exact electric fields emanating from specific charges, the focus may shift to averaged source and field quantities.

To consider the electrostatics of such averaged quantities, requires us to revisit (1.4) and (1.7), so that the material properties can be accounted for. The Laplace equation (1.39), $\nabla^2\phi = 0$, also assumes a slightly different form when a background material is present. The materials in electrostatics are usually divided roughly into two categories, *conductors* and *insulators* [21, Sec. 1.7].

2.3 Polarization in conductors

The *conductors* permit essentially free movement of charges. When a conducting material inclusion is brought from a neutral state to a state where it is impinged by a static electric field, the newly introduced electric field compels the charge into movement. The movement continues until the electric field from the displaced electrons and the ionized molecules that they have left behind starts to counteract the effect of the impinging electric field and a new equilibrium is reached where the charge distribution of the inclusion differs from the neutral state.

Conducting materials vary in their composition and properties, but these details can normally be omitted from the discussion. For example, the resistance in superconductors is vanishingly small, facilitating rapid movement of electric charge even in very weak electric fields. In contrast, weak conductors, like sea water, require stronger electric fields to produce similar currents. If the relevant time frame is sufficiently long, the electrostatic field drives the charge into equilibrium. The process takes place even in a weakly conducting material, albeit more slowly. In this stasis all movement ceases and the electric currents vanish. Therefore, the conductivity difference between the different media is irrelevant in electrostatics [21, Sections 1.7, 3.2]. Electrostatic field theory makes use of the assumption that the conductors have infinite conductivity. Another useful idealization is to assume that the material has an infinite store of electric charge, so that when an inclusion is polarized by an electric field the displacement charges can indefinitely keep pace with an increasing magnitude of the impinging field because the inclusion will never run out of charge. When the material is assumed to be infinitely conductive and to have infinite amount of charge for polarization, the material is referred

to as *ideal conductor*.

2.4 Polarization in insulators

Conducting objects can straighforwardly accumulate new charge distributions when electrons in one part of the object dislocate. When it comes to *insulators*, also referred to as *dielectrics* the situation is more intricate, but it turns out that dielectric objects too exhibit accumulation of charge distribution. Although the electrons of a dielectric material are bound to their corresponding atoms or molecules by strong restoring forces that hinder their free movement, the microscopic dislocations in the electron clouds can still contribute a significant surplus of positive or negative charges for specific domains in the material.

Given the restrains in the movements of the electrons, the mechanisms behind the polarization of a dielectric material act at the level of the material's constituent building blocks, atoms and molecules. These building blocks may consist in molecules with permanent dipole moments, like the conjunction of two positive hydrogen ions and a negative oxygen ions that makes up a water molecule. Because the water molecule has a neutral total charge, the net force acting on it vanishes, given that the electric field that impinges the molecule is sufficiently homogeneous. However, the space between the center of the negative charge distribution and that of the positive charge distribution causes the exerted force to act differently on the different parts of the molecule. Therefore the molecule changes its orientation compelled by a torque from the electric field [4, Sec. 4.1.3]. The change continues until the torque vanishes as a result of the molecule's new orientation, which eventually aligns the *axis of polarization* with the electric field. In the material, where the molecule interacts with the other water molecules, a perfect alignment is prevented by the disturbances from thermal fluctuations. However, the thermal fluctuation cannot prevent the number of aligned molecules from becoming statistically significant if the field is sufficiently strong. Thus the average dipole moment over all the molecules tends to point the way of the electric field. The field acting on the material begets an equidirectional polarization. This *orientational polarization* thus results from the rotation and partial alignment or permanent dipoles [22, Sec. 2.1].

A second polarization mechanism acts on a given building block of a material by stretching it rather than changing its orientation. In a neutral

situation the center of the negative charge distribution coincides with the positive nucleus, so that the polarization axis conjoining these two vanishes. But an introduced electric field pulls the two charge distributions to opposite direction, elongating the polarization axis. While the field pulls the electron cloud away from the corresponding nucleus, the change diminishes the shadowing effect that the cloud exerts on the nucleus [4, Sec. 4.1.2]. When the shadowing effect declines, the electron cloud feels the pull from the nucleus with increased intensity. Thus the electric attraction between the nucleus and the electron cloud acts as a string of a kind, in the sense that it acts as a restoring force, counteracting perturbations. A stasis is reached when the restoring force exactly counterbalances the externally introduced electric field. The stasis has a strictly positive length of the polarization axis between the centers of the positive and the negative charge. Therefore, a dipole moment is introduced to a block of material even if it exhibited none before.

Other mechanisms too may contribute to material's polarization, but the above two mechanisms suffice in the present context. The thesis is mostly concerned with treating the polarization with an abstract mathematical formalism which overlooks the qualitative differences between the mechanisms and concentrates on the treatment of averaged quantitative properties. Also, the mechanisms of polarization are not mutually exclusive. The orientational polarization usually coexists with the stretching polarization.

A qualitative description of the material's polarization involves the concept of dipole moment density, introduced in Sec. 2.1. When a material is impinged by an electric field the induced microscopic dipoles may be represented by the dipole moment density $\mathbf{P}(\mathbf{r}')$. So long as the material can be assumed *linear*, the induced dipole moment density $\mathbf{P}(\mathbf{r}')$ at $\mathbf{r}'$ is proportional to the total averaged electric field $\mathbf{E}(\mathbf{r}')$ at the same point, so that

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E} \tag{2.13}$$

where χ is the *electric susceptibility*. Apart from linearity, (2.13) assumes that the dipole moment density $\mathbf{P}$ coextends with the field $\mathbf{E}$, but this is not true in general. The case where $\mathbf{P}$ and $\mathbf{E}$ may point to different directions requires a generalization of (2.13), which is the topic of Sec. 3.2. When the vectors $\mathbf{P}$ and $\mathbf{E}$ coextend for all $\mathbf{E}$, the material is *isotropic*.

2.5 Bound charges

When the electric field acts on a material, it polarizes the material by effecting rotations and elongations of the material building blocks. The microscopic deformations are one side of the interaction between the electric field and the material. The other side of the interaction comes from the material's newly accumulated charges, which change the total electric field by superimposing its own field to the externally induced *excitation field*. The source charges of this excitation field are extraneous to the material and they are referred to as *free charges*. Although the term, "free charge" might evoke the idea of free movement of electrons in conductors, these two concepts are distinct. In contrast to the excitation field, the sources of the superimposing *perturbation field* are in the material itself. They are referred to as *bound charges* to emphasize that the charges do not cover macroscopic distances when they dislocate but rather, the regional charge densities result from an accumulation of small dislocations.

The quantity of bound charge is obtained indirectly, by observing the electric field produced by a polarized material. If we temporaneously ignore the cause of the material polarization and only regard the effect, the field produced by the material is the aggretate of dipole fields, originating from a known distribution of dipole moment density $\mathbf{P}(\mathbf{r})$. In Sec. 2.1, the potential created by a dipole moment density $\mathbf{P}(\mathbf{r}')$ inside a given volume $\mathcal{V}$ was found to be (2.10)

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\mathbf{R}}}{R^2}\,\mathrm{d}V'$$

With the identity

$$\frac{\hat{\mathbf{R}}}{R^2} = \nabla'\left(\frac{1}{R}\right)$$

the equation changes into

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \nabla'\left(\frac{1}{R}\right)\cdot\mathbf{P}(\mathbf{r}')\,\mathrm{d}V'$$

If we integrate in parts, we obtain

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\int_{\mathcal{V}} \nabla'\cdot\left(\frac{\mathbf{P}(\mathbf{r}')}{R}\right)\,\mathrm{d}V' - \int_{\mathcal{V}}\frac{1}{R}\left(\nabla'\cdot\mathbf{P}(\mathbf{r}')\right)\,\mathrm{d}V'\right]$$

and the Gauss divergence theorem, when applied to the first integral, transforms the equation into

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{S}}\frac{1}{R}\mathbf{P}(\mathbf{r}')\cdot\mathbf{n}\,\mathrm{d}S' - \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{1}{R}\left(\nabla'\cdot\mathbf{P}(\mathbf{r}')\right)\,\mathrm{d}V'$$

The two terms that give the potential can now be identified as the potential that results from the bound surface charge

$$\sigma_{\mathrm{b}} = \mathbf{n} \cdot \mathbf{P} \tag{2.14}$$

and the potential that results from the bound volume charge

$$\rho_{\mathrm{b}} = -\nabla \cdot \mathbf{P} \tag{2.15}$$

so that the potential created by the dipole moment inside V is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{S}} \frac{\sigma_{\mathrm{b}}}{R} \, \mathrm{d}S' + \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho_{\mathrm{b}}}{R} \, \mathrm{d}V' \tag{2.16}$$

We can now make some further analysis on (1.37)

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho(\mathbf{r})$$

introduced in Sec. 1.5. Let us separate the total charge density ρ into its free charge part ρ_{f} and the bound charge part ρ_{b}, so that

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} (\rho_{\mathrm{b}} + \rho_{\mathrm{f}})$$

By (2.15) this becomes

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_{\mathrm{f}}$$

The field quantity $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ on the left hand side is referred to as the *electric flux density*. Using the definition of the electric flux density, the Gauss law in a medium is concisely

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{f}} \tag{2.17}$$

The equation becomes useful because it tacitly accounts for the bound charge, so that only the free charge is explicit.

So far, we have assumed nothing about the origin of the dipole moment density $\mathbf{P}$. Let us, however, now assume that the dipole moment density observes a linear relation $\mathbf{P}(\mathbf{r}') = \epsilon_0 \chi(\mathbf{r}') \mathbf{E}(\mathbf{r}')$ where χ can vary from point to point. The relation between the electric flux density and the electric field

$$\mathbf{D} = \epsilon \epsilon_0 \mathbf{E} \tag{2.18}$$

is then also linear, and the constant of proportionality $\epsilon = 1 + \chi$ is referred to as the *relative permittivity* of the material. All the permittivities in this thesis, apart from the vacuum permittivity ϵ_0, are relative permittivities, so that they are dimensionless.

The foregoing definition of permittivity helps us to derive Laplace equation for a medium. If we apply the definition of permittivity to the Gauss law in medium (2.17), the resulting equation

$$\nabla \cdot (\epsilon \mathbf{E}) = \frac{\rho_\mathrm{f}}{\epsilon_0}$$

turns into the Poisson equation for the potential

$$\nabla \cdot (\epsilon \nabla \phi) = -\frac{\rho_\mathrm{f}}{\epsilon_0} \tag{2.19}$$

from which the corresponding Laplace equation

$$\nabla \cdot (\epsilon \nabla \phi) = 0 \tag{2.20}$$

follows in an absence of free charges.

2.6 Potential at interface

The goal of the present section is to establish the conditions that determine when a solution to a given electrostatic problem can be known without ambiguity. For this, we summarize some of the earlier results. Sections 1.4 and 2.5 introduced the two equations

$$\nabla \times \mathbf{E} = 0 \tag{2.21}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{2.22}$$

that lie at the heart of electrostatics. Note that we reserve the symbol ρ for the free charge when the bound charge is only implicitly present in the electric flux density $\mathbf{D}$. A connection between the two field quantities $\mathbf{E}$ and $\mathbf{D}$ was furnished in Sec. 2.5 by means of the constitutive relation

$$\mathbf{D} = \epsilon \epsilon_0 \mathbf{E} \tag{2.23}$$

where the relative permittivity ϵ can vary over the volume. These three equations need to be supplemented with additional conditions to obtain unique solutions to electrostatic problems.

To convince ourselves that the equations (2.21), (2.22), and (2.23) are not sufficient, we can consider a pair of fields $\mathbf{E}_1, \mathbf{D}_1$ that is a solution to (2.21) and (2.22) and another pair $\mathbf{E}_2, \mathbf{D}_2$ that satisfies the correspoding homogeneous equation pair

$$\nabla \times \mathbf{E}_2 = 0 \tag{2.24}$$

$$\nabla \cdot \mathbf{D}_2 = 0 \tag{2.25}$$

The combination $\mathbf{E}_1+\mathbf{E}_2, \mathbf{D}_1+\mathbf{D}_2$ now satisfies (2.21) and (2.22). If both of the solutions also satisfy the constitutive relation (2.23) the combination of these also has the same property.

Similarly, a given solution ϕ_1 of the Poisson equation

$$\nabla \cdot (\epsilon \nabla \phi_1) = -\frac{\rho_\mathrm{f}}{\epsilon_0} \tag{2.26}$$

can be combined with any solution ϕ_2 of

$$\nabla \cdot (\epsilon \nabla \phi_2) = 0 \tag{2.27}$$

so that the sum $\phi_1 + \phi_2$ still satisfies the Poisson equation. Two different pairs of field quantities $\mathbf{E}_1, \mathbf{D}_1$ and $\mathbf{E}_2, \mathbf{D}_2$ can then be calculated from ϕ_1 and $\phi_1 + \phi_2$ so that both pairs satisfy (2.21) and (2.22).

There are several ways to define suitable boundary conditions. If the material inside the domain $\mathcal{D}$ has a positive permittivity ϵ, the solution to the generalized Laplace equation (2.20) becomes unique when we require that the potential at the boundary of the domain $\mathcal{D}$ takes a specified value which may vary over the surface. The condition is known as the *Dirichlet boundary condition*. The Dirichlet boundary condition applies, for example, at the surface of a conductor which is held at a predetermined potential [15, Sec. 1.9].

Another possible condition is to require that the normal derivative with respect to the surface $\partial\mathcal{D}$ of the domain must take a specified value. This second condition is known as the *Neumann boundary condition*. The Neumann boundary condition states that the normal component of the electric field at the surface of the domain is predetermined. The condition makes the electric field inside the domain $\mathcal{D}$ unique, provided that the material inside the domain has a positive permittivity ϵ. However, the level of the potential can be chosen arbitrarily so that the potential remains ambiguous.

A more general condition is

$$A(\mathbf{r})\phi(\mathbf{r}) + B(\mathbf{r})[\mathbf{n} \cdot (\nabla\phi)(\mathbf{r})] = C(\mathbf{r})|_{\mathbf{r}\in\partial\mathcal{D}}$$

This condition reduces into the Neumann condition if $A(\mathbf{r})$ vanishes and into the Dirichlet condition if $B(\mathbf{r})$ vanishes.

Once a suitable boundary condition is found, the solution to equations (2.22), (2.21), and (2.23) becomes unique [15, Sec 1.9]. However, by taking the derivatives in (2.22) and (2.21) we assume that the fields are continuous which might be unwarranted if the material parameter ϵ changes

abruptly. Let us imagine, for example, that the domain $\mathcal{D}$ is divided into two parts $\mathcal{D}_1$ and $\mathcal{D}_2$ so that the material parameter ϵ is piecewise continuous with a discontinuity at the interface $\mathcal{S}$ between the two parts of the domain. It then follows from the constitutive relation (2.23) that the continuity of $\mathbf{E}$ implies the discontinuity of $\mathbf{D}$ and vice versa. Therefore, one or both of the fields $\mathbf{E}$ and $\mathbf{D}$ need have discontinuity at the interface. Even so, the curl can be taken from $\mathbf{E}$ and the divergence from $\mathbf{D}$ if a suitably abstract definition for these operators is adopted at the discontinuity.

Let us assume that a continuous interface $\mathcal{S}$ divides the domain $\mathcal{D}$ into two parts $\mathcal{D}_1$ and $\mathcal{D}_2$ and that the interface $\mathcal{S}$ has no sharp corners, so that its surface normal vector is uniquely defined apart from sign. We denote the surface normal vector that points to the domain $\mathcal{D}_1$ with $\mathbf{n}_1$ and the one that points to the domain $\mathcal{D}_2$ with $\mathbf{n}_2$. The interface can coincide with a material interface, like in the example above, but this is not necessary. We can associate two step functions P_1 and P_2 with the two domains so that

$$P_1(\mathbf{r}) = \begin{cases} 1, & \mathbf{r} \in \mathcal{D}_1 \\ 1/2, & \mathbf{r} \in \mathcal{S} \\ 0, & \mathbf{r} \in \mathcal{D}_2 \end{cases} \tag{2.28}$$

$$P_2(\mathbf{r}) = \begin{cases} 1, & \mathbf{r} \in \mathcal{D}_2 \\ 1/2, & \mathbf{r} \in \mathcal{S} \\ 0, & \mathbf{r} \in \mathcal{D}_1 \end{cases} \tag{2.29}$$

Let the fields in either domain be $\mathbf{E}_n, \mathbf{D}_n$ and their sources ρ_n. The fields in the combined domain $\mathcal{D}$ can be written

$$\begin{aligned} \mathbf{E} &= P_1\mathbf{E}_1 + P_2\mathbf{E}_2 \\ \mathbf{D} &= P_1\mathbf{D}_1 + P_2\mathbf{D}_2 \end{aligned} \tag{2.30}$$

Also, the combined charge density is

$$\rho = P_1\rho_1 + P_2\rho_2 + \rho_\mathrm{s} \tag{2.31}$$

where ρ_n is the charge density in either domain and ρ_s is the charge density at the surface $\mathcal{S}$. All of the relevant charge densities, including ρ_s, have the dimensions of a volume charge density. The connection between ρ_s and the surface charge density σ will be outlined on p. 42.

The product rules of divergence and curl

$$\begin{aligned} \nabla \cdot (f\mathbf{G}) &= f(\nabla \cdot \mathbf{G}) + (\nabla f) \cdot \mathbf{G} \\ \nabla \times (f\mathbf{G}) &= f(\nabla \times \mathbf{G}) + (\nabla f) \times \mathbf{G} \end{aligned}$$

can now be applied to the combined field in (2.30). This yields

$$\begin{aligned}\nabla \times \mathbf{E} &= P_1(\nabla \times \mathbf{E}_1) + P_2(\nabla \times \mathbf{E}_2) + (\nabla P_1) \times \mathbf{E}_1 + (\nabla P_2) \times \mathbf{E}_2 \\ &= (\nabla P_1) \times \mathbf{E}_1 + (\nabla P_2) \times \mathbf{E}_2 = 0\end{aligned} \tag{2.32}$$

$$\begin{aligned}\nabla \cdot \mathbf{D} &= P_1(\nabla \cdot \mathbf{E}_1) + P_2(\nabla \cdot \mathbf{E}_2) + (\nabla P_1) \cdot \mathbf{D}_1 + (\nabla P_2) \cdot \mathbf{D}_2 \\ &= P_1\rho_1 + P_2\rho_2 + (\nabla P_1) \cdot \mathbf{D}_1 + (\nabla P_2) \cdot \mathbf{D}_2 = \rho\end{aligned} \tag{2.33}$$

where the last equation further reduces into

$$(\nabla P_1) \cdot \mathbf{D}_1 + (\nabla P_2) \cdot \mathbf{D}_2 = \rho_\mathrm{s} \tag{2.34}$$

The remaining question concerns the interpretation of the gradients ∇P_1 and ∇P_2. The functions P_1 and P_2 are not continuous, so that the gradients do not correspond to any ordinary functions. However, we may imagine that P_1 and P_2 are smoothened slighly to remove the abrupt change in the value, still retaining the identity $P_1 + P_2 = 1$ and the property that the function plateau to 0 and 1 in the respective domains. For simplicity, we concentrate on P_1. We can picture a pillbox that surrounds the surface $\mathcal{S}$ from both sides. From the mathematical view point, the pillbox is a straight cylinder with arbitrarily shaped facets—the facet $\mathcal{F}_1$ placed in the domain $\mathcal{D}_1$, the facet $\mathcal{F}_2$ of equal area and shape placed in the domain $\mathcal{D}_2$, and a diminishingly small distance Δl separating the two. Even though the facets are separated only by a diminishingly small distance, we assume that the function P_1 is sharp enough to plateau to 0 and 1 between the facets.

The volume integral of ∇P_1 is calculated as follows. We take a point from a surface element $\mathrm{d}A$ of the facet $\mathcal{F}_2$ in the domain $\mathcal{D}_2$ and draw a straight path $\mathcal{C}$ to the corresponding point on the opposing facet $\mathcal{F}_1$. Because the direction of the path and the direction of the gradient ∇P_1 coincide thoughout the path, the path integral is

$$\int_\mathcal{C} \nabla P_1 \,\mathrm{d}l = \mathbf{n}_1 \int_\mathcal{C} \nabla P_1 \cdot \,\mathrm{d}\mathbf{l} = \mathbf{n}_1 \tag{2.35}$$

where the last equality follows from the general property (1.19) of path integrals of gradients. The volume integral of ∇P_1 can now be subdivided into a surface integral over the facet $\mathcal{F}_2$ and a path integral over the path $\mathcal{C}$ at each surface element, so that

$$\int_\mathcal{V} \nabla P_1 \,\mathrm{d}V = \int_{\mathcal{F}_2} \int_\mathcal{C} \nabla P_1 \,\mathrm{d}l \,\mathrm{d}A = \int_{\mathcal{F}_2} \mathbf{n}_1 \,\mathrm{d}A \tag{2.36}$$

Furthermore, if the volume $\mathcal{V}$ does not conform to the designated shape of the pillbox, (2.36) is still valid. The gradient ∇P_1 vanishes outside the

neighborhood of the interface contributing nothing to the integral (2.36). The integral can therefore always be reduced to an integral over a pillbox with diminishing thickness.

In Sec. 1.5, we empoyed a mathematical tool referred to as Dirac delta function to derive the Gauss law (1.37). Here we take a recourse to an analogous tool, to interpret the result (2.36). In analogy to (1.36), we define for an arbitrary surface $\mathcal{S}$ a function $\delta_{\mathcal{S}}$ with the characterizing property

$$\int_{\mathcal{V}} f(\mathbf{r})\delta_{\mathcal{S}}\,\mathrm{d}V = \int_{\mathcal{V}\cap\mathcal{S}} f(\mathbf{r})\,\mathrm{d}A \tag{2.37}$$

which holds for an arbitrary volume $\mathcal{V}$ as long as the function $f(\mathbf{r})$ is sufficiently smooth. The surface integral in (2.37) is over the intersection $\mathcal{V}\cap\mathcal{S}$ which is the surface enclosed in the volume $\mathcal{V}$. By extension,

$$\int_{\mathcal{V}} \mathbf{F}(\mathbf{r})\delta_{\mathcal{S}}\,\mathrm{d}V = \int_{\mathcal{V}\cap\mathcal{S}} \mathbf{F}(\mathbf{r})\,\mathrm{d}A \tag{2.38}$$

holds for any sufficiently smooth vector function $\mathbf{F}(\mathbf{r})$ because (2.37) can be applied componentwise.

We finally arrive at the interpretation of the gradient ∇P_1. If, in (2.38), we set $\mathbf{F} = \mathbf{n}_1$ and compare the result with (2.36), we obtain

$$\int_{\mathcal{V}} \nabla P_1\,\mathrm{d}V = \int_{\mathcal{V}} \mathbf{n}_1\delta_{\mathcal{S}}\,\mathrm{d}V$$

For this to hold for an arbitrary volume $\mathcal{V}$, the integrands must coincide, whereby

$$\nabla P_1 = \mathbf{n}_1\delta_{\mathcal{S}} \tag{2.39}$$

When we apply (2.39) to (2.32), we obtain the interface condition

$$\mathbf{n}_1 \times \mathbf{E}_1 + \mathbf{n}_2 \times \mathbf{E}_2 = 0 \tag{2.40}$$

for the component of the electric field that is tangential to the surface $\mathcal{S}$. Similarly, we obtain

$$\mathbf{n}_1 \cdot \mathbf{D}_1 + \mathbf{n}_2 \cdot \mathbf{D}_2 = \sigma \tag{2.41}$$

for the normal component of the electric flux density. The charge ρ_{s} at the surface was expressed as $\rho_{\mathrm{s}} = \sigma\delta_{\mathcal{S}}$ in terms of the surface charge density σ and the relevant delta function. As before, the source is $\rho = P_1\rho_1 + P_2\rho_2 + \delta_{\mathcal{S}}\sigma$.

To see, how the interface conditions (2.40) and (2.41) translate into interface conditions for the electric potential ϕ, we consider two potentials ϕ_1 and ϕ_2 that are defined in their respective domains $\mathcal{D}_1$ and $\mathcal{D}_2$ so that

$\phi = P_1\phi_1 + P_2\phi_2$. When $\mathbf{D}_1$ and $\mathbf{D}_2$ are expressed in terms of these potentials, the interface condition (2.41) transforms into

$$\mathbf{n}_1 \cdot (\epsilon_1 \nabla\phi_1) + \mathbf{n}_2 \cdot (\epsilon_2 \nabla\phi_2) = 0 \tag{2.42}$$

Moreover, (2.40) transforms into

$$\mathbf{n}_1 \times (\nabla\phi_1) + \mathbf{n}_2 \times (\nabla\phi_2) = 0 \tag{2.43}$$

But $\mathbf{n}_1 \times (\nabla\phi_1)$ is the tangential component $(\nabla\phi_1)_\mathrm{t}$ of $\nabla\phi_1$ rotated counter clockwise by the angle $\pi/2$ with respect to the axis $\mathbf{n}_1$. The same goes for the tangential component $(\nabla\phi_2)_\mathrm{t}$, which equals $\mathbf{n}_2 \times (\nabla\phi_2)$ when it is rotated by the angle $\pi/2$ with respect to the axis $\mathbf{n}_2$ or equivalently $-\pi/2$ with respect to the axis $\mathbf{n}_1$. By (2.43), the sum of these rotated quantities must vanish, which shows that

$$(\nabla\phi_1)_\mathrm{t} = (\nabla\phi_2)_\mathrm{t} \tag{2.44}$$

Then, if we choose a point $\mathbf{r}_0$ on the surface $\mathcal{S}$ and take a path integral along a path $\mathcal{C}$ to an arbitrary point $\mathbf{r}$ on the same surface, we find

$$\begin{aligned} \phi_1(\mathbf{r}) - \phi_1(\mathbf{r}_0) &= \int_\mathcal{C} \nabla\phi_1 \cdot \mathrm{d}\mathbf{l} \\ &= \int_\mathcal{C} (\nabla\phi_1)_\mathrm{t} \cdot \mathrm{d}\mathbf{l} = \int_\mathcal{C} (\nabla\phi_2)_\mathrm{t} \cdot \mathrm{d}\mathbf{l} \\ &= \int_\mathcal{C} \nabla\phi_2 \cdot \mathrm{d}\mathbf{l} = \phi_2(\mathbf{r}) - \phi_2(\mathbf{r}_0) \end{aligned} \tag{2.45}$$

which shows that if the potentials agree at any given point $\mathbf{r}_0$ they must agree throughout the surface $\mathcal{S}$. However, the constant level of a potential is arbitrary, so that by adding an arbitrary constant potential $\phi_\mathrm{c} = C$ to one of the potentials ϕ_1 and ϕ_2 the potentials can always be brough to agree at one point. This gives the continuity of potentials across the whole surface $\mathcal{S}$. In summary,

$$\phi_1(\mathbf{r}) = \phi_2(\mathbf{r}), \qquad \mathbf{r} \in \mathcal{S} \tag{2.46}$$

$$\mathbf{n}_1 \cdot [\epsilon_1 \nabla\phi_1(\mathbf{r})] = \mathbf{n}_1 \cdot [\epsilon_2 \nabla\phi_2(\mathbf{r})], \qquad \mathbf{r} \in \mathcal{S} \tag{2.47}$$

hold for the potentials on an interface.

2.7 Polarization of inclusion

The preceding three sections, Sec. 2.1, Sec. 2.2, and Sec. 2.5, discussed the mechanism that causes microscopic charge distributions in a medium to

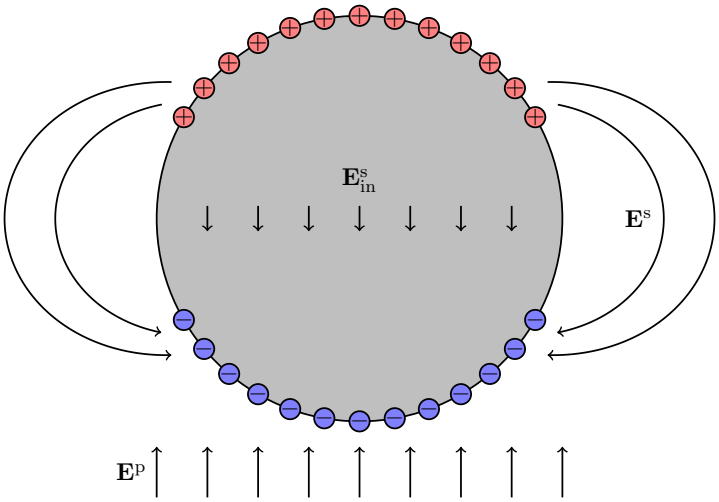

Figure 2.1. Spherical inclusion in an excitation potential. The excitation field causes charge to accumulate to opposite poles of the sphere. The accumulated surface charge creates a perturbation field $\mathbf{E}^{\mathrm{s}}$ which is a dipole field outside the sphere and a uniform field inside the inclusion.

have impact on a macroscopic domain. This section shows how the mechanism works in an *inclusion* that is surrounded by vacuum and that is impinged by a static electric field $\mathbf{E}^{\mathrm{p}}$, the *excitation field*. We make the further assumption that the sources of the excitation field are far away from the inclusion compared to the inclusion's dimensions, so that the excitation field varies very little in the neighborhood of the inclusions and can therefore be regarded effectively *uniform*. With the excitation field and the inclusion given, the problem is to find the secondary field $\mathbf{E}^{\mathrm{s}}$, the *perturbation field*, created by the bound charges of the inclusion's material, as described in Sec. 2.5. The problem occupies a central status in the present thesis. It is the problem that all of the enclosed articles—Publications I, II, III, IV, and V— confront.

The polarization of an inclusion will here be introduced by a canonical example. Suppose that the inclusion in question is a homogeneous sphere that has a constant relative permittivity ϵ throughout its volume. Let the z axis coextend with the excitation field $\mathbf{E}^{\mathrm{p}}$, so that $\mathbf{E}^{\mathrm{p}}$ and $\mathbf{u}_z$ point in the same direction. Also, let θ be the angle that $\mathbf{u}_z$ makes with a given vector $\mathbf{r}$, the *polar angle* coordinate of $\mathbf{r}$ in a spherical coordinate system based on $\mathbf{u}_z$. The problem is now solved in two parts, corresponding to the two-way interaction between the field and the material. The first part involves finding the dipole moment density and equivalently the bound charge when the electric field acting on the material is given. The second

part involves the reverse, finding the electric field produced by a given distribution of dipole moments.

As shown in Sec. 2.5, the total electric field is a superposition of the fields produced by free charges ρ_f and the fields produced by the bound charges ρ_b. The statement holds also in case of the inclusion and it is equally true both inside and outside the inclusion. The field produced by the implicit free charges ρ_f, which are not directly given, is the excitation field $\mathbf{E}^\mathrm{p}$ whereas the field produced by the bound charge ρ_b or bound surface charge σ_b is the perturbation field $\mathbf{E}^\mathrm{s}$. It may be assumed that inside the sphere $\mathbf{E}^\mathrm{s}$ has an equal or opposite direction to $\mathbf{E}^\mathrm{p}$ because the assumption turns out to be self-consistent. By (2.13), the uniform dipole moment density of the sphere is then

$$\mathbf{P} = \epsilon_0 \chi (\mathbf{E}^\mathrm{p} + \mathbf{E}^\mathrm{s}) \tag{2.48}$$

where χ is $\epsilon - 1$ as a direct result of the definition (2.18) of relative permittivity. Because the dipole moment density $\mathbf{P}$ is uniform the volume bound charge density ρ_b, by (2.15), vanishes throughout the inclusion. However, (2.14) implies that the inclusion has a surface bound charge

$$\sigma_\mathrm{b} = \mathbf{P} \cdot \mathbf{n} = P \cos\theta' \tag{2.49}$$

which is distributed as a function of the polar angle θ', that is, symmetrically with respect to the polarization axis $\mathbf{u}_z$.

When the surface bound charge is given by (2.49), the perturbation potential inside and outside the sphere can be calculated with (1.9) by integration or by indirect methods (see [4, Ex. 4.2]). The presently taken direction involves a direct integration of the potential produced by a charge at the surface of the inclusion. By (1.9), the perturbation potential ϕ^p is given by the integral

$$\phi^\mathrm{p} = \frac{1}{4\pi\epsilon_0} \int_\mathcal{S} \frac{\sigma_\mathrm{b}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \,\mathrm{d}A' \tag{2.50}$$

To evaluate the integral, we employ a generalization of (2.4) given in [23, Sec. 3.7]. Let the spherical coordinates of vector $\mathbf{r}$ be (r, θ, φ) and correspondingly the coordinates of $\mathbf{r}'$ be (r', θ', φ'). Let ℓ_m be the Neumann factor, which equals 1 for $m = 0$ and 2 for $m = 1, 2, 3, \ldots$, and let $R_<$ be the smaller of the values r and r' and $R_>$ correspondingly the greater of these. The reciprocal of the distance between two vectors in polar coordinates is (2.4)

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \ell_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta') P_n^m(\cos\theta) \cos m(\varphi - \varphi') \frac{R_<^n}{R_>^{n+1}} \tag{2.51}$$

Like (2.4), the above equation is rooted in the theory of multipoles. Note that here θ refers to the angle that the vector $\mathbf{r}$ makes with the z axis, unlike in (2.4) where θ indicated the angle between the vectors $\mathbf{r}$ and $\mathbf{r}'$.

When we substitute the multipole expression (2.51) and the surface charge distribution (2.49) to (2.50)—writing $P_1(\cos\theta')$ for $\cos\theta'$ because the first Legendre function is a simple identity function—and express the surface integral as a double integral over the coordinates θ' and φ', the potential of the perturbation field becomes

$$\phi^{\mathrm{p}} = \frac{P}{4\pi\epsilon_0} \int_{\theta=0}^{\pi} P_1(\cos\theta') \sum_{n=0}^{\infty} \sum_{m=0}^{n} \ell_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta') P_n^m(\cos\theta) \frac{R_<^n}{R_>^{n+1}} \int_{\varphi=0}^{2\pi} \cos m(\varphi - \varphi') a^2 \sin\theta' \,\mathrm{d}\varphi' \,\mathrm{d}\theta' \tag{2.52}$$

The last integral over φ' takes the value 2π when m is zero and it vanishes for positive values of m. Therefore the equation reduces into

$$\phi^{\mathrm{p}} = \frac{Pa^2}{2\epsilon_0} \int_{\theta=0}^{\pi} P_1(\cos\theta') \sum_{n=0}^{\infty} P_n(\cos\theta') P_n(\cos\theta) \frac{R_<^n}{R_>^{n+1}} \sin\theta' \,\mathrm{d}\theta' \tag{2.53}$$

The equation simplifies even further when the integration variable is changed into $\xi = \cos\theta'$ and the orthogonality property

$$\int_{-1}^{1} P_k^m(\xi) P_n^m(\xi) \,\mathrm{d}\xi = \frac{(n-m)!}{(n+m)!} \frac{2}{2n+1} \delta_{nk} \tag{2.54}$$

of Legendre functions is applied [18, Sections 8.14.11, 8.14.13]. Only one term of the original double sum in (2.52) remains after these operations. The perturbation potential simplifies into

$$\phi^{\mathrm{p}} = \frac{Pa^2}{3\epsilon_0} \frac{R_<}{R_>^2} \cos\theta \tag{2.55}$$

Therefore, the perturbation inside the sphere is

$$\phi_{\mathrm{in}}^{\mathrm{s}} = \frac{P}{3\epsilon_0} r \cos\theta \tag{2.56}$$

and the perturbation potential outside the sphere is

$$\phi_{\mathrm{out}}^{\mathrm{s}} = \frac{Pa^3}{3\epsilon_0} \frac{1}{r^2} \cos\theta \tag{2.57}$$

The perturbation potential equals the potential of an ideal dipole at the origin

$$\phi_{\mathrm{out}}^{\mathrm{s}} = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} \tag{2.58}$$

where the total dipole moment $\mathbf{p}$ is the dipole moment density integrated over the volume of the sphere

$$\mathbf{p} = \frac{4}{3}\pi a^3 \mathbf{P} \tag{2.59}$$

Because the perturbation potential inside the sphere changes in proportion to the z coordinate $z = r\cos\theta$, the perturbation field

$$\mathbf{E}^{\mathrm{s}}_{\mathrm{in}} = -\nabla\phi^{\mathrm{s}}_{\mathrm{in}} = -\frac{P}{3\epsilon_0}\mathbf{u}_z \tag{2.60}$$

is found to be uniform inside the sphere. Because the field inside the sphere is a superposition of a uniform excitation field and the perturbation field, which was found to be uniform, the starting assumption that the field inside the sphere is uniform turns out to be self-consistent.

We can now put together the two parts of the polarization problem. Polarization P depends on the fields $\mathbf{E}^{\mathrm{p}}$ and $\mathbf{E}^{\mathrm{s}}$ according to (2.48) whereas the perturbation field $\mathbf{E}^{\mathrm{s}}$ depends on the polarization P according to (2.60). If we take the scalar product with $\mathbf{u}_z$ on both sides of (2.48) to express the equation with scalar quantities and substitute (2.60), the resulting equation

$$P = \epsilon_0\chi(E^{\mathrm{p}} - \frac{P}{3\epsilon_0})$$

gives the dipole moment density

$$P = \frac{3\chi}{\chi+3}\epsilon_0 E^{\mathrm{p}} \tag{2.61}$$

which substituted back to (2.60) gives the internal perturbation field

$$\mathbf{E}^{\mathrm{s}}_{\mathrm{in}} = -\frac{\chi}{\chi+3}\mathbf{E}^{\mathrm{p}} \tag{2.62}$$

and the total internal field

$$\mathbf{E}_{\mathrm{in}} = \mathbf{E}^{\mathrm{p}} + \mathbf{E}^{\mathrm{s}}_{\mathrm{in}} = \frac{3}{\chi+3}\mathbf{E}^{\mathrm{p}} \tag{2.63}$$

An important question in electrostatics concerns finding the easiest way to quantify the inclusion's propensity to polarize in any given excitation field $\mathbf{E}^{\mathrm{p}}$. Because a material, seen at a microscopic level, mostly polarizes by creating new dipole moment and because the dipole field is the last multipole field to vanish as a function of a distance, it is common to quantify the inclusion's propensity to polarize by looking at the dipole field.

Continuing the example involving a homogeneous sphere in a uniform excitation field, the required dipole field is given by (2.58). This solution is always approximatively true at large distances but for a homogeneous sphere it is an exact solution everywhere outside the sphere, including the proximity of the sphere surface. It follows from (2.58) and (2.59) that the excitation field $\mathbf{E}^{\mathrm{p}}$ and the total dipole moment

$$\mathbf{p} = V\frac{3\chi}{\chi+3}\epsilon_0\mathbf{E}^{\mathrm{p}} \tag{2.64}$$

are proportional in a way that depends on the volume V of the sphere and the susceptibility χ. The constant of proportionality

$$\alpha_{\mathrm{abs}} = \epsilon_0 V \frac{3\chi}{\chi + 3} \tag{2.65}$$

is the *polarizability* of the sphere. To get a plain number as a result and to factor out the influence of the inclusion's size, the result is often expressed as

$$\alpha = \frac{\alpha_{\mathrm{abs}}}{\epsilon_0 V} = \frac{3\chi}{\chi + 3} \tag{2.66}$$

or equivalently with permittivity

$$\alpha = 3\frac{\epsilon - 1}{\epsilon + 2} \tag{2.67}$$

which is the inclusions *normalized polarizability*.

Although the concept of polarizability was introduced in connection to spheres, it applies generally to all inclusions in electrostatics. The normalized polarizability α of a general inclusion is defined by

$$\mathbf{p} = \epsilon_0 V \alpha \mathbf{E}^{\mathrm{p}} \tag{2.68}$$

Also, the normalized polarizability α need not be a scalar quantity if the directions of the excitation field $\mathbf{E}^{\mathrm{p}}$ and the total dipole moment $\mathbf{p}$ do not match. The question will be addressed in Ch. 3.

3. Anisotropic medium

When Nyambe lived on earth, people said that he had fallen from the sky. When he returned thither, he climbed up by means of a spider's thread.

African folktale

Section 2.4 discussed the polarization of an insulating medium at a microscopic level. The discussion postulated a linear dependence between the local, electric field $\mathbf{E}$ inside the material and the dipole moment density $\mathbf{P}$. It was assumed that the electric field $\mathbf{E}$ creates a dipole moment density $\mathbf{P}$ that points to the same direction as $\mathbf{E}$. If we relax this assumption while retaining a linear dependence between $\mathbf{E}$ and $\mathbf{P}$, the material is called *anisotropic*. Anisotropic materials will be treated in Sec. 3.2.

Section 2.7 made a similar assumption of isotropy for an electrostatic inclusion. The distinction was that the relevant quantities were the impinging electric field $\mathbf{E}^{\mathrm{p}}$ and the total dipole moment of an inclusion $\mathbf{p}$. As with the material, it may happen that the electric field $\mathbf{E}^{\mathrm{p}}$ and the dipole moment $\mathbf{p}$ of an inclusion fail to coextend but are still linearly related. An inclusion that polarizes in this more general way is called an *anisotropic* inclusion.

The concepts of anisotropy for material and for an inclusion are related in the sense that both have the same mathematical model. The modeling of anisotropy is treated in Sec. 3.1. However, the two concepts pertain to different domains of electrostatic analysis and are consequently distinct. Section 3.3 shows, how anisotropic inclusions can be created from an isotropic material and how isotropic inclusions can be created from an anisotropic material.

3.1 Concept of anisotropy

A spider climbing on a brick wall without the aid of its web is effectively confined to two-dimensional movement in the plane defined by the approximatively planar surface of the bricks. It can move to two linearly independent directions, vertically or horizontally. If the spider moves horizontally, it will deduce that the bricks are $250\,\mathrm{mm}$ apart or equivalently that the linear "brick density" is $1/250\,\mathrm{mm}^{-1}$. On the other hand, if it decides to move vertically, it find a different density $1/62\,\mathrm{mm}^{-1}$. For a typical European house brick, both of these numbers are correct. The spider must deduce that the brick density is *anisotropic*. The brick density depends on the direction the spider takes.

Let us assume for simplicity, that vertically adjacent bricks are stacked directly on top of each other in straight columns instead of the customary staggered configuration. Then, the two linear brick densities can be employed in calculating the number of bricks that the spider visits. Let λ_h be the density for a horizontal path and let λ_v be the density for a vertical path. If the spider moves in a straight line, so that its eventual horizontal displacement is positive δx and the vertical displacement positive δy, the number of bricks visited n_brick is approximatively

$$n_\mathrm{brick} = (\lambda_\mathrm{h}\mathbf{u}_x + \lambda_\mathrm{v}\mathbf{u}_y) \cdot (\delta x\mathbf{u}_x + \delta y\mathbf{u}_y) \tag{3.1}$$

If we want to calculate the number of mortar joints crossed, instead of the number of bricks visited, the procedure becomes slightly more complicated. Instead of one quantity, the number of bricks, we have two, the number of joints crossed horizontally n_h (the vertical joints) and the number of joints crossed vertically n_v (the horizontal joints). Two separate equations similar to (3.1) could be used, but a more eloquent way is to extend the customary operations of vector algebra—the scalar product $\cdot$ and the vector product $\times$—with a novel one, the *dyadic product*, which simply conflates two vectors together. A small introduction to the dyadic product will suffice to finish the brick calculation.

Dyadic algebra helps to represent linear transformations of vectors in a notation that is consistent with the Gibbsian vector algebra. If fact, the dyadic algebra and the vector algebra were both introduced by Josiah Willard Gibbs [24, 25]. The dyadic product of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is denoted by juxtaposing the vectors without an intervening multiplication sign. The vectors involved in the dyadic product can be real or complex. The order of the vectors in the product makes a difference, so that $\mathbf{v}_2\mathbf{v}_1$ is,

in general, different from $\mathbf{v}_1\mathbf{v}_2$. A dyadic product of two vectors is referred to as *dyad*. The sums of dyadic products are referred to as dyadic polynomials or simply *dyadics* [26]. The characterizing feature of the dyadic product is that it is bilinear

$$\begin{aligned}(\alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2)\mathbf{b} &= \alpha_1(\mathbf{a}_1\mathbf{b}) + \alpha_2(\mathbf{a}_2\mathbf{b}) \\ \mathbf{a}(\beta_1\mathbf{b}_1 + \beta_2\mathbf{b}_2) &= \beta_1(\mathbf{a}\mathbf{b}_1) + \beta_2(\mathbf{a}\mathbf{b}_2)\end{aligned} \tag{3.2}$$

Several types of products between vectors and dyadics were introduced in Gibbs' paper [25]. The only product that pertains to the thesis is the dot product, defined by

$$\begin{aligned}(\mathbf{a}\mathbf{b}) \cdot \mathbf{c} &= (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \\ \mathbf{c} \cdot (\mathbf{a}\mathbf{b}) &= (\mathbf{c} \cdot \mathbf{a})\mathbf{b}\end{aligned} \tag{3.3}$$

The virtue of this dyadic dot product with a vector is that represents a linear transformation from the vector space into itself. The concept is similar to the matrix product between a square matrix and a column vector. But we wish to exploit the elevated level of abstraction and the concision of notation that the Gibbsian dyadic algebra offers.

The spider problem is now solved by turning the two linear brick densities λ_h and λ_v into a brick density dyadic $\bar{\bar{\lambda}} = \lambda_\mathrm{h}\mathbf{u}_x\mathbf{u}_x + \lambda_\mathrm{v}\mathbf{u}_y\mathbf{u}_y$. If we denote the displacement vector by $\Delta\mathbf{r} = \delta x\mathbf{u}_x + \delta y\mathbf{u}_y$ and then combine the numbers of horizontal and vertical crossings of joints into a vector $\mathbf{n}_\mathrm{brick} = n_\mathrm{h}\mathbf{u}_x + n_\mathrm{v}\mathbf{u}_y$, the anisotropic relationship between the crossings $\mathbf{n}_\mathrm{brick}$ and the displacement $\Delta\mathbf{r}$ is

$$\mathbf{n}_\mathrm{brick} = \bar{\bar{\lambda}} \cdot \Delta\mathbf{r} \tag{3.4}$$

A special case of the spider problem involves a brick wall where the dimensions of each brick are equal and thus $\lambda_\mathrm{h} = \lambda_\mathrm{v} = \lambda$. In this case, we have

$$\bar{\bar{\lambda}} = \lambda(\mathbf{u}_x\mathbf{u}_x + \mathbf{u}_y\mathbf{u}_y)$$

The dyadic

$$\bar{\bar{I}}_{xy} = \mathbf{u}_x\mathbf{u}_x + \mathbf{u}_y\mathbf{u}_y \tag{3.5}$$

is the two-dimensional *unit dyadic*. Because

$$\bar{\bar{I}}_{xy} \cdot \mathbf{c} = \mathbf{c} \tag{3.6}$$

for any vector $\mathbf{c}$, the number of crossings reduces into

$$\mathbf{n}_\mathrm{brick} = \lambda\Delta\mathbf{r}$$

which translates into the intuitive result that, when the wall is isotropic instead of strictly anisotropic, the numbers of crossings in both directions are the same if the numbers of steps taken to these directions are the same.

The concept of a unit dyadic is equally applicable to three-dimensional space. If $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are three vectors of unit length and if these vectors point to mutually perpendicular directions, the vectors form an orthonormal basis. Then the three-dimensional unit dyadic is

$$\bar{\bar{I}} = \mathbf{u}_1\mathbf{u}_1 + \mathbf{u}_2\mathbf{u}_2 + \mathbf{u}_3\mathbf{u}_3 \tag{3.7}$$

This dyadic is independent of the particular choice of the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ as long as these vectors are orthonormal. The characterizing feature of the unit dyadic is that it corresponds to the identity mapping

$$\bar{\bar{I}} \cdot \mathbf{a} = \mathbf{a} \tag{3.8}$$

for any vector $\mathbf{a}$.

3.2 Anisotropy in medium

The bricks that build up a medium in electrostatics vary in shape and size. What counts as medium or inclusion, depends on the level of detail the analysis is expected to provide. The details that surrendered to averaging in Sec. 2.4 were the electric fields inside and in the neighborhood of the basic building blocks of material, atoms and molecules. But even less fine details can undergo averaging when a broader view is expedient.

Consider, for example, a medium where spherical inclusions of varying sizes and with uniform permittivity ϵ_s occupy random positions in a background medium with uniform permittivity ϵ_b. Let the respective volume fractions of the spheres and the background be f_s and f_b. An electric field that permeates the mixture varies between the materials, so that it is $\mathbf{E}_\mathrm{s}$ inside the spherical inclusions and $\mathbf{E}_\mathrm{b}$ elsewhere. We assume that the fields $\mathbf{E}_\mathrm{s}$ and $\mathbf{E}_\mathrm{b}$ are uniform [22, Sec. 3.1]. The constitutive relation (2.18) gives the electric flux densities

$$\begin{aligned} \mathbf{D}_\mathrm{s} &= \epsilon_0\epsilon_\mathrm{s}\mathbf{E}_\mathrm{s} \\ \mathbf{D}_\mathrm{b} &= \epsilon_0\epsilon_\mathrm{b}\mathbf{E}_\mathrm{b} \end{aligned} \tag{3.9}$$

The volume averages of the fields and the flux densities are

$$\begin{aligned} \langle\mathbf{E}\rangle &= f_\mathrm{s}\mathbf{E}_\mathrm{s} + f_\mathrm{b}\mathbf{E}_\mathrm{b} \\ \langle\mathbf{D}\rangle &= f_\mathrm{s}\epsilon_0\epsilon_\mathrm{s}\mathbf{E}_\mathrm{s} + f_\mathrm{b}\epsilon_0\epsilon_\mathrm{b}\mathbf{E}_\mathrm{b} \end{aligned} \tag{3.10}$$

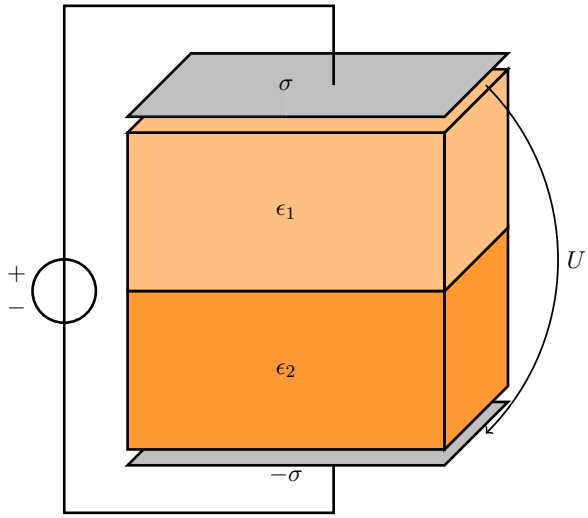

Figure 3.1. Two layers perpendicular to the field in a parallel plate capacitor. The effective permittivity that determines the capacitance of the capacitor is $2\epsilon_1\epsilon_2/(\epsilon_1+\epsilon_2)$.

These averages too have their own constitutive relation

$$\langle \mathbf{D} \rangle = \epsilon_0 \epsilon_{\text{eff}} \langle \mathbf{E} \rangle \tag{3.11}$$

where the *effective permittivity* ϵ_{eff} is the fraction

$$\epsilon_{\text{eff}} = \frac{f_{\text{s}}\epsilon_{\text{s}}A + f_{\text{b}}\epsilon_{\text{b}}}{f_{\text{s}}A + f_{\text{b}}} \tag{3.12}$$

with the field ratio A defined by $\mathbf{E}_{\text{s}} = A\mathbf{E}_{\text{b}}$. Such an average allows us to treat as medium even configurations of different materials where the constituent elements of the mixture occupy significant volumes rather than confining themselves to microscopic space.

While (3.12) gives the effective permittivity for a medium, it presumes that the electric fields are known, which is not true in general. The fields depend on the way the materials are combined in the mixture. Methods of determining the effective permittivity ϵ_{eff} for a given material are at the heart of metamaterial research in electromagnetics. These *electromagnetic mixing rules* fall into different categories depending on their use and scope of application. Some famous mixing rules are Maxwell-Garnett, Bruggeman, and Lorenz–Lorentz [22]. These mixing rules are more powerful than the thesis requires. Therefore, the discussion will be confined to a more rudimentary model of the effective permittivity ϵ_{eff}.

The capacitor in Fig. 3.1 comprises two dielectric layers with relative permittivities ϵ_1 and ϵ_2 stacked on top of each other in a planar configuration. Moreover, the two layers are assumed to occupy equal volume

fractions, so that each one has the thickness $d/2$. These dielectric layers are poised between two metal plates so that one of the plates is charged positively with a surface charge σ and the other negatively with a surface charge $-\sigma$, and the distance d between the plates accommodates the two dielectric layers. They produce an electric field that permeates the two dielectric layers. Let us assume that the capacitor plates and the dielectric layers extend to an area A which is large compared to the distance d between the plates. In this case, symmetry implies that the permeating electric field is uniform apart from the discontinuity at the interface between the layers.

Whereas the electric field suffers a discontinuity, it follows from the Gauss law and the symmetry of the plates that the electric flux density $\mathbf{D}$ is uniform throughout the space between the plates (see. [4, Sec. 2.5.4] for details). The flux density $\mathbf{D}$ points from the positive plate to the negative and its absolute value D equals the surface charge σ.

Let us find the capacitance of this capacitor in two different ways, first with averaged electric field $\mathbf{E}$ and then with specific electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$. By definition, the capacitance is the fraction between the total charge Q of the plates and the voltage U between the plates, so that the capacitance of a parallel plate capacitor is

$$C = \frac{Q}{U} = \frac{A\sigma}{Ed} \tag{3.13}$$

The averaged constitutive relation (3.11) allows us to replace σ in (3.13) by $\sigma = D = \epsilon_{\text{eff}} E$, which leads to a familiar equation

$$C = \epsilon_0 \epsilon_{\text{eff}} \frac{A}{d} \tag{3.14}$$

for the plate capacitor.

In the second place, we can substitute the specific electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ to the definition (3.13) and write

$$C = \frac{AD}{\frac{1}{2}dE_1 + \frac{1}{2}dE_2} \tag{3.15}$$

for capacitance. If we use the constitutive relations $D = \epsilon_1 E_1$ and $D = \epsilon_2 E_2$, the equation simplifies into

$$C = \epsilon_0 \frac{2\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \frac{A}{d} \tag{3.16}$$

Because (3.14) and (3.16) must give the same value for the capacitance C, the effective permittivity

$$\epsilon_{\text{eff}} = \frac{2\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \tag{3.17}$$

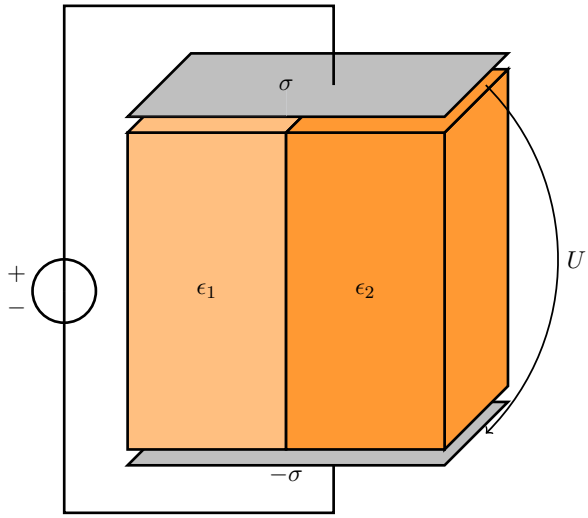

Figure 3.2. Two layers parallel to the field in a parallel plate capacitor. The effective permittivity that determines the capacitance of the capacitor is $(\epsilon_1 + \epsilon_2)/2$.

can be identified.

A different effective permittivity ϵ_{eff} can be bestowed on the capacitor while retaining the former values of the material parameters ϵ_1 and ϵ_2, the volume fractions $1/2$ and $1/2$, and the distance d between the plates. Instead of the former configuration where the dielectric layers were separated horizontally, consider the configuration in Fig. 3.2 where the layers are separated vertically. The thickness d of each layer is double the former value whereas the face area is reduced by half, to $A/2$. As before, the capacitance that employs averaging is given by (3.14) because nothing was assumed about the configuration of the materials in deriving this formula. However, (3.16) undergoes a revision when the configuration changes.

Each of the capacitor plates constitutes an equipotential which means that the voltage gap $U = Ed$ must be constant throughout the capacitor. For the constant voltage gap to be consistent with the implication of the Gauss law that D_n equals σ_n in each domain and with the constitutive relations $D_n = \epsilon_n E$, the charges must be distributed unevenly

$$\frac{\sigma_1}{\epsilon_1} = \frac{\sigma_2}{\epsilon_2} \tag{3.18}$$

on each plate. If ϵ_1 and ϵ_2 are positive real numbers, more charge will accumulate to the domain where the permittivity is higher. The capacitance

$$C = \frac{\frac{1}{2}A\sigma_1 + \frac{1}{2}A\sigma_2}{Ed} \tag{3.19}$$

combines the charges $\sigma_1 = \epsilon_0\epsilon_1 E$ and $\sigma_2 = \epsilon_0\epsilon_2 E$ from these two domains

so that the value with specific fields and flux densities is

$$C = \epsilon_0 \frac{\epsilon_1 + \epsilon_2}{2} \frac{A}{d} \tag{3.20}$$

and a comparison to (3.14) gives the effective permittivity

$$\epsilon_{\text{eff}} = \frac{\epsilon_1 + \epsilon_2}{2} \tag{3.21}$$

The striking similarity of (3.16) and (3.20) to the equations that circuit theory gives for capacitors in parallel and in series is not coincidental. In the first example the dielectric layers were stacked vertically, in a pile. In that case, the interface separating the two layers constitutes an equipotential surface, so that it can be filled with an infinitesimally thin metallic layer without changing the electric field or flux densities in either layer. The metallic layer separates the capacitor into two capacitors in series. The capacitances are found directly from the usual formula

$$C_i = \epsilon_0 \epsilon_i \frac{A}{\frac{1}{2}d} \tag{3.22}$$

for a plate capacitor. Plainly these two capacitors are in series, whereby the capacitance is

$$C = \frac{C_1 C_2}{C_1 + C_2} = \epsilon_0 \frac{2\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \frac{A}{d} \tag{3.23}$$

which coincides with (3.16).

When the dielectric layers are situated laterally, the two domains with respective permittivities ϵ_1 and ϵ_2 can be moved a part without changing the electric flux densities or electric fields anywhere apart from the junction between the domains, which was assumed to be small in comparison to the capacitor as a whole. To allow the charge to move freely between the upper capacitor plates, the upper plates must be connected with a metal wire. The same goes for the bottom plates. Thus, the original setup of the capacitor turns out to be equivalent to two capacitors in parallel. The capacitances of the two separate capacitors are

$$C_i = \epsilon_0 \epsilon_i \frac{\frac{1}{2}A}{d} \tag{3.24}$$

and in parallel they make up a capacitance of

$$C = C_1 + C_2 = \epsilon_0 \frac{\epsilon_1 + \epsilon_2}{2} \frac{A}{d} \tag{3.25}$$

which coincides with (3.20), as expected.

The importance of the discussion that led to effective permittivities in (3.17) and (3.21), in connection to the present thesis, is that it shows the

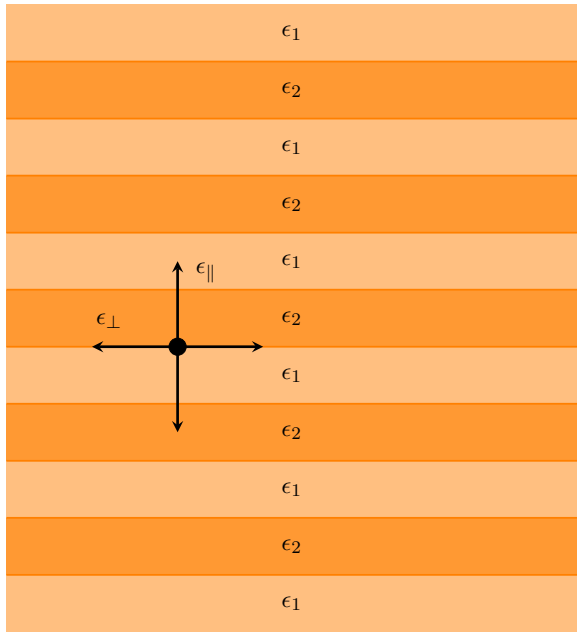

Figure 3.3. A stack of layers with alternating values for permittivity. The effective permittivity of the structure is anisotropic. Note that, in the adopted convention, "parallel" refers to that which is parallel to the axis of anisotropy, not to that which is parallel to the layers. The indexing in $\epsilon_\parallel$ and $\epsilon_\perp$ reflects this.

way that anisotropy arises in electrostatics. To distinguish the effective permittivity of the capacitor with the dielectric layers in series and the effective permittivity of the capacitor with the dielectric layers in parallel, we denote the first of these values by $\epsilon_\parallel$ and the second by $\epsilon_\perp$. The two effective permittivities

$$\epsilon_\parallel = \frac{2\epsilon_1\epsilon_2}{\epsilon_1+\epsilon_2}, \quad \epsilon_\perp = \frac{\epsilon_1+\epsilon_2}{2} \tag{3.26}$$

are in general different. In fact, if we assume that ϵ_1 and ϵ_2 are positive real numbers with separate values, $\epsilon_1 \neq \epsilon_2$, the permittivity $\epsilon_\perp$ of layers in parallel is strictly greater than the permittivity $\epsilon_\parallel$ of layers in series.

Moving on to a more complicated structure, consider a collection of dielectric layers in a column, so that each layer has the same thickness d and the material alternates between permittivity values ϵ_1 and ϵ_2, like in Fig. 3.3. It seems reasonable, that the homogenization technique that led to (3.17) and (3.21) gives the correct effective permittivities $\epsilon_\parallel$ and $\epsilon_\perp$ for the layered material, even when there are no capacitor plates present. After all, if the field is perpendicular to the layers, the interfaces between the layers are equipotentials where metal layers can be freely substituted without distraction to the fields or fluxes and one may calculate the ca-

pacitance as before. Also, if the field is parallel to the layers, any plane that is perpendicular to the field is an equipotential and thereby serves as a possible plate for an imagined capacitor.

If the capacitor plates are not explicit in the layered medium, what counts as dielectric layers in parallel or dielectric layers in series, depends on the orientation of the permeating electric field $\mathbf{E}$. If the field $\mathbf{E}$ is oriented perpendicularly to the layers, the layers are effectively in series and have the effective permittivity $\epsilon_{\parallel}$. If, in contrast, the field is coplanar with the layers, the layers are effectively parallel and have the effective permittivity $\epsilon_{\perp}$.

A general field vector $\mathbf{E}$ decomposes to a component $\mathbf{E}_{\parallel}$ perpendicular to the layers and another component $\mathbf{E}_{\perp}$ coplanar with the layers. Similarly, the flux density vector $\mathbf{D}$ decomposes into a perpendicular component $\mathbf{D}_{\parallel}$ and a coplanar component $\mathbf{D}_{\perp}$. The constitutive relations $\mathbf{D}_{\parallel} = \epsilon_0 \epsilon_{\parallel} \mathbf{E}_{\parallel}$ and $\mathbf{D}_{\perp} = \epsilon_0 \epsilon_{\perp} \mathbf{E}_{\perp}$ can be combined into

$$\mathbf{D} = \epsilon_0 \epsilon_{\parallel} \mathbf{E}_{\parallel} + \epsilon_0 \epsilon_{\perp} \mathbf{E}_{\perp} \tag{3.27}$$

and if we introduce a dyadic

$$\bar{\bar{\epsilon}} = (\bar{\bar{I}} - \mathbf{u}_{\text{axis}} \mathbf{u}_{\text{axis}}) \epsilon_{\perp} + \mathbf{u}_{\text{axis}} \mathbf{u}_{\text{axis}} \epsilon_{\parallel} \tag{3.28}$$

the general constitutive relation between $\mathbf{D}$ and $\mathbf{E}$ is succintly

$$\mathbf{D} = \epsilon_0 \bar{\bar{\epsilon}} \cdot \mathbf{E} \tag{3.29}$$

The type of anisotropy defined by (3.28) has two distinct components, so that one of the components acts along a specified axis—$\mathbf{u}_{\text{axis}}$ in the example—and the other one of which acts on a plane perpendicular to the given axis. If the permittivity of the material can be written in the form (3.28), the material is *uniaxial*. Uniaxial anisotropy is a central concept in Publications III, IV, and V, whereas the permittivity dyadic in Publications I and II is more general because it has three independent material parameters, one for each orthogonal direction.

Regardless of the implementation of the material, (3.28) can be taken as the definition of uniaxial material. However, if the perpendicular component $\epsilon_{\perp}$ of the dyadic in (3.28) is greater than the axial component $\epsilon_{\parallel}$, one of the possible implementations is to use the alternating dielectric layers. Because the alternating dielectric layers are vaguely reminiscent of the layers in an onion, the Ancient Greek word for onion, "βολβός", is used in the nomenclature. A material with $\epsilon_{\perp} > \epsilon_{\parallel}$ is referred to as *bulbic*.

If the greater permittivity component is, in contrast, the axial component $\epsilon_{\|}$, implementation with the alternating dielectric layers is more complicated. The required inequality $\epsilon_{\|} > \epsilon_{\perp}$ never holds when the permittivities of the layers ϵ_1 and ϵ_2 are real and positive. However, a different implementation is possible. Instead of dielectric layers, we may use a collection of codirectional needles, with permittivities ϵ_2 in a homogeneous background medium, with permittivity ϵ_1. The needles are known to polarize more strongly along their symmetry axis than in the perpendicular directions which means that their contribution to the effective permittivity is greater in the axial direction (see [27, Sec. 4.2] for details). Therefore, the mixture that consists of needles and the background material manages to implement the property $\epsilon_{\|} > \epsilon_{\perp}$ that the dielectric layers failed to implement.

Like the bulbic materials, the axially anisotropic materials that satisfy the inequality $\epsilon_{\|} > \epsilon_{\perp}$ call for a special nomenclature. The material with the required property can be implemented in many ways, but the implementation that guides the naming convention is the one that employs the needles. Because the needles are vaguely reminiscent of the needles of a porcupine, we refer to the material as *porcupic*.

3.3 Anisotropy in inclusion

To see how inclusions may exhibit anisotropy even when the material of the inclusion is homogeneous and isotropic, we build on the example of Sec. 2.7 that involved a sphere in a static, uniform excitation field. In the example, the sphere was assumed to consist of isotropic and homogeneous material. Also, it follows from the spherical symmetry of the inclusion, that the inclusion's normalized polarizability α does not depend on the orientation of the excitation field. The sphere, therefore, counts as an isotropic inclusion. However, let us choose an arbitrary direction, indicated by a unit vector $\mathbf{u}_z$, and elongate the sphere linearly with respect to this direction, so that the distance a_z from the former origin to the new farthest point at the surface is longer than the original radius a while the circular cross-section centered at the origin retains the radius a. The resulting inclusion is referred to as *prolate spheroid* whereas an inclusion that suffers a compression instead of an elongation is referred to as *oblate spheroid*. The distances a_z and $a_x = a_y = a$ are the semi-axes of the inclusion. The polarization of either prolate or oblate spheroid in

the direction of the symmetry axis is distinct from the polarization in the perpendicular directions. Therefore, the inclusion is anisotropic.

The process of making the inclusion anisotropic can be carried even further by choosing another direction $\mathbf{u}_x$, perpendicular to $\mathbf{u}_z$. When the inclusion is elongated or compressed linearly in this direction, the resulting inclusion has three distinct semi-axes a_x, a_y, and a_z. The inclusion takes the shape of an *ellipsoid*. It has a different polarization for each of the directions $\mathbf{u}_x$, $\mathbf{u}_y$, $\mathbf{u}_z$. This ellipsoidal geometry is the most general geometry that becomes relevant within the present thesis.

Let us consider a homogeneous, ellipsoidal inclusion in a vacuum background and impinged by a uniform static electric field $\mathbf{E}^{\mathrm{p}}$. The goal is to find the perturbation field both inside and outside the inclusion, similarly to Sec. 2.7. An ellipsoidal inclusion, with three distinct semi-axes a_x, a_y, and a_z, clearly lacks the spherical symmetry of the homogeneous sphere, discussed in Sec. 2.7. Therefore the formerly applied method is not directly applicable to a problem involving a homogeneous ellipsoid. However, the solution proceeds along the same lines.

Any uniform and static excitation field $\mathbf{E}^{\mathrm{p}}$ can be expressed as a linear combination of three fields, directed along the semi-axes $\mathbf{u}_x$, $\mathbf{u}_y$, $\mathbf{u}_z$. There is consequently no loss of generality in assuming that the excitation field $\mathbf{E}^{\mathrm{p}}$ points to one of these directions. Let us take this direction to be $\mathbf{u}_z$. As before, in Sec. 2.7, the dipole moment density $\mathbf{P}$ is dictated by the total electric field that penetrates the medium, which is the sum of the excitation field $\mathbf{E}^{\mathrm{p}}$ and the perturbation field $\mathbf{E}^{\mathrm{s}}$:

$$P = \epsilon_0 \chi (E^{\mathrm{p}} + E^{\mathrm{s}}_{\mathrm{in}}) \tag{3.30}$$

Also, the perturbation field $\mathbf{E}^{\mathrm{s}}$ is again related to the dipole moment density $\mathbf{P}$ through the surface charge density σ, assuming that the perturbation field is uniform inside the inclusion. In particular, the magnitude of $\mathbf{E}^{\mathrm{s}}_{\mathrm{in}}$ must be proportional to the magnitude of $\mathbf{P}$, so that

$$E^{\mathrm{s}}_{\mathrm{in}} = -\frac{L_z}{\epsilon_0} P \tag{3.31}$$

where L_z is one of the ellipsoids *depolarization factors* L_x, L_y, and L_z. To derive the depolarization factors would occupy some space, so we refer to Stratton's book *Electromagnetic Theory* for details and for the proof that the fields inside the inclusion are indeed uniform [21, Sec. 3.27]. The depolarization factor L_z of the ellipsoid is

$$L_z = \frac{a_x a_y a_z}{2} \int_0^\infty \frac{\mathrm{d}s}{(s + a_z^2)\sqrt{(s + a_x^2)(s + a_y^2)(s + a_z^2)}} \tag{3.32}$$

and the other two depolarization factors L_x and L_y are given by the same formula, when one interchanges a_z and a_x (a_z and a_y) to obtain L_x (L_y) [22, Sec. 4.2.1]. The two equations, (3.30) and (3.31), can then be solved for the dipole moment density

$$P_z = \frac{\chi}{1 + L_z\chi}\epsilon_0 E^{\mathrm{p}} \tag{3.33}$$

and the internal perturbation field $E^{\mathrm{s}}_{\mathrm{in}}$ which combined with the excitation field E^{p} gives the total internal field

$$E_{\mathrm{in}} = \frac{1}{1 + L_z\chi}E^{\mathrm{p}} \tag{3.34}$$

To express the external perturbation field $\mathbf{E}^{\mathrm{s}}$ with mathematical precision would require us to discuss ellipsoidal harmonics (see [21, Sec. 3.25]). But as noted in Sec. 2.1, the perturbation field of any inclusion resembles a dipole field when observed sufficiently far from the object. This allows us to characterize the external perturbation field in terms of the dipole moment $\mathbf{p}$ withheld by the inclusion. The far field is then given by (2.8). The dipole moment of the inclusion is the integral

$$\mathbf{p} = \int_{\mathcal{V}} \mathbf{P}\,\mathrm{d}V = PV\mathbf{u}_z \tag{3.35}$$

of the dipole moment density over the volume $\mathcal{V}$ of the inclusion, which reduces to a simple multiplication because of the uniformity of the dipole moment density $\mathbf{P}$. The required internal field has now been obtained, as well as an approximation for the external field in the far-region of the inclusion.

The depolarization factors given by (3.32) are distinct if the semi-axes a_x, a_y, and a_y are distinct. The result accords with the intuition that, when the symmetry of an inclusion is broken, the polarization in the inclusion becomes dependent on the orientation of the excitation field $\mathbf{E}^{\mathrm{p}}$. It follows from the fact that the depolarization factors are distinct and the way the internal field depends on these factors (3.34) that the internal field $\mathbf{E}_{\mathrm{in}}$ and the excitation field $\mathbf{E}^{\mathrm{p}}$ generally point to different directions and that they only coextend in the special case where $\mathbf{E}^{\mathrm{p}}$ is aligned with one of the principal axes of the ellipsoid. Because the material was assumed isotropic and homogeneous, the uniform dipole moment is oriented in the direction of the internal field. Therefore, the total dipole moment $\mathbf{p}$, which is the integral (2.12) of the dipole moment density $\mathbf{P}$, also points to the same direction with the internal field. This result shows the anisotropy of the ellipsoidal inclusion which is one of the important characteristics of the inclusion.

To see how anisotropy manifests itself in the mathematical view point, we generalize the definition (2.68) of the normalized polarizability into

$$\mathbf{p} = \epsilon_0 V \bar{\bar{\alpha}} \cdot \mathbf{E}^{\mathrm{p}} \tag{3.36}$$

This definition permits us to express the total dipole moment $\mathbf{p}$ in terms of the excitation field $\mathbf{E}^{\mathrm{p}}$ in one formula, instead of three formulas of the type (2.68). The components of the normalized polarizability dyadic $\bar{\bar{\alpha}}$ are found by substituting the dipole moment density (3.33) into the integral (3.35). The coordinate independent dyadic expression of the normalized polarizablity is

$$\bar{\bar{\alpha}} = \sum_{j=x,y,z} \frac{\chi}{1 + L_j \chi} \mathbf{u}_j \mathbf{u}_j \tag{3.37}$$

Only two of the depolarization factors of an ellipsoid are independent parameters because of the constraint

$$L_x + L_y + L_z = 1 \tag{3.38}$$

which follows from (3.32). If we take the limiting case where the inclusion is a sphere, the isotropy of the inclusion implies that all of the depolarization factors have the same value $L_{x/y/z} = 1/3$. As expected, we recover the polarizability of the sphere which was given by (2.65) when we substitute this value to (3.37). The depolarization factors for a sphere could also be found without the integral in (3.32) by substituting the definition of the depolarization factors (3.31) to the expression (2.60) of the internal perturbation field of a sphere.

While the symmetry argument directly provided the depolarization factors for a sphere, the case of a spheroid is slightly more intricate. Let z axis be the symmetry axis of the spheroidal inclusion, so that the two perpendicular semiaxes a_x and a_y are equally long. It follows from the rotation symmetry with respect to z axis that the depolarization factors L_x and L_y take the same value, denoted by $L_\perp$ while the depolarization factor $L_z = L_\parallel$ along the axis of symmetry retains, in general, a distinct value. The special case where this rotation symmetry can be assumed allows the integral in (3.32) to take a closed form expression [22, Sec. 4.2.1]. Two extreme cases that help to validate results in Publication III are the disk, with $a_z \ll a_x, a_y$, and the needle with, with $a_z \gg a_x, a_y$. The depolarization factors of a disk are $L_\parallel = 1$ and $L_\perp = 0$, and the depolarization factors of the needle are $L_\parallel = 0$ and $L_\perp = 1/2$.

Because the depolarization factors of a sphere are all equal, the polarizability dyadic (3.37) reduces into a multiple $\bar{\bar{\alpha}} = \alpha \bar{\bar{I}}$ of a unit dyadic for

a sphere. However, the polarizability of an ellipsoid with distinct semi-axes a_x, a_y, and a_z never assumes this simplified form $\alpha\bar{\bar{I}}$ if its material is homogeneous and isotropic. If we compare the result to the expression (3.28) of an isotropic material, a distinction can be noted between the discussion of an anisotropic material and an anisotropic inclusion. While the anisotropy of a material manifested itself in a non-unitary permittivity $\epsilon \neq \epsilon\bar{\bar{I}}$ or equivalently non-unitary susceptibility $\bar{\bar{\chi}} \neq \chi\bar{\bar{I}}$, the anisotropy of an inclusion manifests itself in a non-unitary polarizability $\bar{\bar{\alpha}} \neq \alpha\bar{\bar{I}}$.

3.4 Radially anisotropic sphere

The articles enclosed in the present thesis relate to the concept of the radially anisotropic sphere and the different variations of this concept. *Radial anisotropy* is a specific type of uniaxial anisotropy that was discussed in Sec. 3.2. While Sec. 3.2 considered uniaxial structures where the axis of anisotropy $\mathbf{u}_{\text{axis}}$ in (3.28) pointed to a constant direction throughout the medium, this assumption is relaxed in the definition of the radially anisotropic sphere. If the axis of anisotropy is permitted to vary over the volume, we call the medium *locally uniaxial* as opposed to the *rectilinearly uniaxial* medium where the direction of the axis of anisotropy is uniform throughout the volume. If, in a given locally uniaxial medium, an origin can be found so that the axis of anisotropy points away from the origin at all points of the medium, the permittivity dyadic of the medium assumes the form

$$\bar{\bar{\epsilon}} = (\bar{\bar{I}} - \mathbf{u}_r\mathbf{u}_r)\epsilon_\perp + \mathbf{u}_r\mathbf{u}_r\epsilon_\parallel \tag{3.39}$$

where the vector $\mathbf{u}_r$ indicates the direction of displacement between a given point and the origin. When the permittivity is so defined, the medium is referred to as radially anisotropic.

The spheres with radial anisotropy fall in the two categories that were discussed in Sec. 3.2, viz. bulbic and porcupic. In a spherical geometry, the allusions to the onion and to the porcupine, which motivated the choice of nomenclature, are more persuasive than in the rectilinear case. In the first place, a bulbic radially anisotropic sphere can be imagined as consisting of densely stratified spherical layers. If the layers are sufficiently thin, the separation between the different layers is small compared to the sphere's radius of curvature at a given point. Therefore, the arguments presented in Sec. 3.2 still apply locally, leading to the expression (3.39) where $\epsilon_\perp$ is greater than $\epsilon_\parallel$. Similarly, if the material consist of

needles that point away from a given origin, the porcupic variety of radial anisotropy manifests itself, with the radial component $\epsilon_{\parallel}$ exceeding the tangential component $\epsilon_{\perp}$.

The present thesis draws ideas from a great amount of former research on radially anisotropic spheres. The seminal paper on the subject was likely the one by Roth and Dignam, published in 1973 [28]. The paper treated an isotropic, homogeneous sphere with a radially anisotropic coating in a time-harmonic excitation field. Another approach to radial anisotropy was taken in a paper by Schulgasser, published in 1982 [29]. This paper considered the problem of homogenizing an *intact* radially anisotropic sphere, that is, a radially anisotropic sphere without embedded inclusions, such as a homogeneous sphere at the center. The paper also differed from the 1973 paper by discussing thermal conductivity instead of electrodynamics. This latter distinction, however, is not as critical as it might seem because Schulgasser emphasizes that the theory of thermal conductivity is mathematically analogous to the theory of electrostatics, which in turn may be regarded as a limiting case of electrodynamics where the frequency becomes diminishingly small and equivalently the wavelength becomes increasingly long. Schulgasser's influence in the early research on radially anisotropic sphere has left its mark on the naming conventions, so that the radially anisotropic sphere is sometimes referred to by its alternative name *Schulgasser sphere*. The term "Schulgasser sphere" appears in a 1991 article of Helsing and Helte where the concept is used as an example to illustrate aggregates of anisotropic particles [30]. A 1992 paper by Wong and Chen applied the theory of Mie scattering to the radially anisotropic sphere, so that one of the interesting parameters in electrodynamic scattering theory, the *radar cross section*, could be obtained from an analytical formula. The study assumed the radial anisotropy to be confined to the permittivity, whereas the analogous magnetic material parameter, *permeability*, was assumed to be homogeneous and isotropic. This restriction was relaxed in a 2007 paper by Qiu et al. [31]. In this study, both the permittivity and the permeability were allowed to exhibit radial anisotropy. An image theory for a radially anisotropic sphere was developed in a 1995 paper by Sten [32]. In this paper an electrostatic point charge is brought in the proximity of the radially anisotropic sphere, so that the excitation field $\mathbf{E}^{\mathrm{p}}$ is not uniform in the neighborhood of the sphere but varies with location. A certain type of generalization of radial anisotropy was introduced by Monzon [33].

Apart from advancement of theory, the radially anisotropic sphere has been studied from the view point of possible applications. The applications come in two varieties. First, some existing materials resemble radially anisotropic spheres closely enough to be modeled in terms of the idealized model. Second, the ideal radially anisotropic sphere in the theoretical model is found to have interesting properties when the material parameters $\epsilon_{\parallel}$ and $\epsilon_{\perp}$ are suitably chosen. Therefore, if a real life implementation of these interesting radially anisotropic spheres can be found, an avenue for new technologies opens. When it comes to the first category, that of preexisting materials, it can be found that some nematic microdroplets closely resemble radially anisotropic spheres. The nematic microdroplets are important because they are central in the theory of liquid crystals. The scattering of light from nematic microdroplets has been studied with approximative methods [34]. Turning to the second category, that of the possibly implementable new devices, the radially anisotropic sphere is known to concentrate electric field in its center when it is strongly porcupic and to repulse electric fields from the center when strongly bulbic. The applicability of radially anisotropic structures, not necessarily spheres, for cloaking and concentration of energy has been studied [35, 36, 37]. Publication III and Publication IV in this thesis study field concentration and cloaking with radially anisotropic structures.

Radial anisotropy comes in different varieties, some of which relate to geometries that are not spherical. The radially anisotropic structure can deviate from the symmetric, spherical shape by assuming, for example, the shape of a spheroid. If the principal axis of the uniaxial material aligns with the radial unit vector of the spheroidal coordinate system, the inclusion is referred to as *radially anisotropic spheroid*. If, furthermore, the radially anisotropic spheroid decomposes into many domains, so that the material parameters take constant values within each domain but differ between the neighboring domains, the so constituted layered radially anisotropic spheroid can be used as an approximative model of a human head [38]. The radially anisotropic spheroid is treated in Publication III.

The concept of radial anisotropy also naturally develops into a different direction. Instead of generalizing the spherical case into a spheroidal one, the study can be constrained to two dimensions instead of the usual three. In two-dimensional space, the relevant radially anisotropic structures are the radially anisotropic ellipse and its special case, the radially anisotropic disk. These concepts are treated in Publication IV.

Turning now to the quantitative analysis of the radially anisotropic sphere, let us assume that an intact radially anisotropic sphere is surrounded by vacuum and impinged by a static, uniform excitation field $\mathbf{E}^{\mathrm{p}}$. The goal is to find the field inside the inclusion and the perturbation field outside the inclusion. Similar problem was confronted in Sec. 2.7 but without the complications that come with radial anisotropy. In particular, Sec. 2.7 solved the problem without explicitly using the Laplace equation (2.20). To solve the problem involving a radially anisotropic sphere, we make use of the Laplace equation for the first time in this thesis.

The Laplace equation for a medium was derived in Sec. 2.5. An analogous calculation gives a generalization of the Laplace equation (2.20) for an anisotropic medium:

$$\nabla \cdot (\bar{\bar{\epsilon}} \cdot \nabla \phi) = 0 \tag{3.40}$$

Although (3.40) is more general than (2.20) for which the uniqueness conditions where postulated in Sec. 2.6, the Dirichlet condition and the Neumann condition also prove useful when anisotropy is involved. The Dirichlet condition guarantees the uniqueness of a solution to (3.40) if the permittivity dyadic is taken to be *positive definite*, that is,

$$\mathbf{x}^* \cdot \bar{\bar{\epsilon}} \cdot \mathbf{x} > 0$$

for all non-vanishing vectors $\mathbf{x}$ and the corresponding complex conjugates $\mathbf{x}^*$. However, the Neumann condition changes into

$$A(\mathbf{r})[\mathbf{n} \cdot \bar{\bar{\epsilon}} \cdot (\nabla \phi)(\mathbf{r})] = B(\mathbf{r})|_{\mathbf{r} \in \partial \mathcal{D}}$$

After this revision, the Neumann condition guarantees the uniqueness, with the same proviso that the permittivity $\bar{\bar{\epsilon}}$ be positive definite.

To apply the Laplace equation in the present electrostatic scattering problem, we consider two cases. Inside the spherical inclusion the permittivity $\bar{\bar{\epsilon}}$ takes the radially anisotropic form, given by (3.39). Outside the inclusion, the permittivity is just the vacuum permittivity, so that the relevant equation is the ordinary Laplace equation $\nabla^2 \phi = 0$ of a vacuum.

Because of the symmetry of the inclusion, the excitation field $\mathbf{E}^{\mathrm{p}}$ can be assumed to point to the direction of the positive z axis. The excitation potential in a spherical coordinate system, based on z axis is then

$$\phi^{\mathrm{p}}(r, \theta) = -U_0 \frac{r}{a} \cos\theta \tag{3.41}$$

where U_0 is half of the difference in the excitation potential ϕ^{p} at the

opposite sides of the sphere. We seek the solution in the form

$$\phi(r,\theta) = \begin{cases} \phi_{\mathrm{in}}(r,\theta), & r < a \\ \phi^{\mathrm{p}}(r,\theta) + \phi^{\mathrm{s}}(r,\theta), & r > a \end{cases} \tag{3.42}$$

The angular dependence of the potential ϕ is dictated by the angular dependence of the excitation potential ϕ^{p}. Therefore, the perturbation potential and the internal potential are

$$\begin{aligned} \phi_{\mathrm{in}}(r,\theta) &= \frac{R_{\mathrm{in}}(r)}{r}\cos\theta \\ \phi^{\mathrm{s}}(r,\theta) &= \frac{R^{\mathrm{s}}(r)}{r}\cos\theta \end{aligned} \tag{3.43}$$

where the radial parts R_{in} and R^{s} are heretofore unknown. The denominator r was introduced for later convenience.

The effect of the generalized Laplace operator on the unknown potentials ϕ_{in} and ϕ^{s} can be found by using the product rule of the gradient operator

$$\nabla(fg) = f\nabla g + g\nabla f$$

and the product rule of the divergence operator

$$\nabla\cdot(g\mathbf{F}) = (\nabla g)\cdot\mathbf{F} + g(\nabla\cdot\mathbf{F}) \tag{3.44}$$

The generalized Laplace operator for a product then becomes

$$\begin{aligned} \nabla\cdot(\bar{\bar{\epsilon}}\cdot\nabla(fg)) =& f\left[\nabla\cdot(\bar{\bar{\epsilon}}\cdot\nabla g)\right] + g\left[\nabla\cdot(\bar{\bar{\epsilon}}\cdot\nabla f)\right] + \\ & (\bar{\bar{\epsilon}}\cdot\nabla f)\cdot\nabla g + (\bar{\bar{\epsilon}}\cdot\nabla g)\cdot\nabla f \end{aligned} \tag{3.45}$$

Let the functions in the product fg be

$$\begin{aligned} f &= \frac{R(r)}{r} \\ g &= \cos\theta \end{aligned} \tag{3.46}$$

and let the relative permittivity $\bar{\bar{\epsilon}}$ be defined by (3.39). It follows that the gradients ∇f and ∇g point to the directions of the radial unit vector $\mathbf{u}_r$ and the polar unit vector $\mathbf{u}_\theta$ of the spherical coordinate system. They are consequently orthogonal. Furthermore, they are eigenvectors of the dyadic $\bar{\bar{\epsilon}}$, so that

$$\begin{aligned} \bar{\bar{\epsilon}}\cdot\nabla f &= \epsilon_{\mathrm{n}}\nabla f \\ \bar{\bar{\epsilon}}\cdot\nabla g &= \epsilon_{\mathrm{t}}\nabla g \end{aligned} \tag{3.47}$$

Consequently, the Laplace operator of the product fg simplifies into

$$\nabla\cdot(\bar{\bar{\epsilon}}\cdot\nabla(fg)) = \epsilon_{\mathrm{n}}(g\nabla^2 f) + \epsilon_{\mathrm{t}}(f\nabla^2 g) \tag{3.48}$$

Instead of the generalized Laplace operator, we can now apply the ordinary Laplace operator ∇^2 which in spherical coordinates is

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} \tag{3.49}$$

With (3.48) and (3.49), the generalized Laplace equation

$$\nabla\cdot\left(\bar{\bar{\epsilon}}\cdot\nabla\left(\frac{R(r)}{r}\cos\theta\right)\right) = 0 \tag{3.50}$$

reduces into

$$\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} - 2\frac{\epsilon_\mathrm{t}}{\epsilon_\mathrm{n}}\frac{R(r)}{r^2} = 0 \tag{3.51}$$

A solution to this ordinary second order differential equation is $r^{\lambda+1}$ where the exponent is so far unknown. Upon substitution to (3.51), the exponent can be found from

$$\lambda(\lambda+1) - 2\frac{\epsilon_\mathrm{t}}{\epsilon_\mathrm{n}} = 0 \tag{3.52}$$

The quadratic equation has two solutions, one of which is

$$\lambda = -\frac{1}{2} + \sqrt{2\frac{\epsilon_\mathrm{t}}{\epsilon_\mathrm{n}} + \frac{1}{4}} \tag{3.53}$$

The symbol λ is taken to refer to this solution where the positive sign for the square root is chosen. Because the two solutions add up to -1, the second solution must be $-\lambda-1$. The general solution for $R(r)$ must therefore be the linear combination

$$\frac{R(r)}{r} = A\left(\frac{r}{a}\right)^{\lambda} + B\left(\frac{r}{a}\right)^{-\lambda-1} \tag{3.54}$$

so that the solution to the generalized Laplace equation with the proper angular dependence is finally

$$\phi(r,\theta) = \left[A\left(\frac{r}{a}\right)^{\lambda} + B\left(\frac{r}{a}\right)^{-\lambda-1}\right]\cos\theta \tag{3.55}$$

We first apply solution (3.55) outside the sphere. This outside domain was assumed to consist in vacuum, which by (3.53) is a special case of radially anisotropic medium with $\lambda = 1$. To make solution (3.55) conform to (3.42), we must have $A = -U_0$ in this domain. Inside the sphere, λ is a positive real number when the permittivity components ϵ_n and ϵ_t are positive real numbers. Therefore, we must set B to zero to keep the potential finite at the origin. These considerations leave us with the potential

$$\phi = \begin{cases} \left[-U_0\left(\frac{r}{a}\right) + B\left(\frac{r}{a}\right)^{-2}\right]\cos\theta, & r \geq a \\ A\left(\frac{r}{a}\right)^{\lambda}\cos\theta, & r \leq a \end{cases} \tag{3.56}$$

with the coefficients A and B yet to be solved.

The interface conditions (2.46) and (2.47) that we developed in Sec. 2.6 deliver the unknown constants A and B. Applied to the potential in (3.56), these interface conditions give

$$\begin{cases} B = A + U_0 \\ -U_0 - 2B = \epsilon_{\mathrm{n}} \lambda A \end{cases}$$

which gives the coefficients

$$\begin{cases} A = \dfrac{-3}{\epsilon_{\mathrm{n}} \lambda + 2} U_0 \\ B = \dfrac{\epsilon_{\mathrm{n}} \lambda - 1}{\epsilon_{\mathrm{n}} \lambda + 2} U_0 \end{cases}$$

The significance of the coefficient B is that it determines the magnitude of the perturbation potential ϕ^{s}. Far enough from the inclusion, this perturbation potential should coincide with a potential

$$\phi_{\mathrm{d}} = \frac{1}{4\pi\epsilon_0} \frac{p \cos\theta}{r^2} \tag{3.57}$$

created by an ideal dipole $\mathbf{p} = p\mathbf{u}_z$ at the origin because Sec. 2.1 demonstrated (see Eq. 2.4) that the potential from an arbitrary distribution of charges can be represented as a sum of multipole potentials of which the dipole potential vanishes least rapidly as a function of distance. However, the perturbation term in the potential (3.56) perfectly coincides with an ideal dipole potential even at the proximity of the inclusion. When we compare the dipole potential (3.57) with the term that represents the perturbation potential in (3.56) we find more specifically that the potentials have the same magnitude when

$$\frac{p}{4\pi\epsilon_0} = Ba^2$$

or equivalently

$$p = 3 \frac{\epsilon_{\mathrm{n}} \lambda - 1}{\epsilon_{\mathrm{n}} \lambda + 2} \epsilon_0 V E^{\mathrm{p}}$$

where V is the volume of the inclusion. On the other hand, the definition of the normalized polarizability (2.68) gives the dipole moment

$$p = \epsilon_0 V \alpha E^{\mathrm{p}}$$

whereby consistency requires that the normalized polarizability of the sphere must be

$$\alpha = 3 \frac{\epsilon_{\mathrm{n}} \lambda - 1}{\epsilon_{\mathrm{n}} \lambda + 2} \tag{3.58}$$

The equation takes the form

$$\alpha = 3 \frac{\epsilon_{\mathrm{n}} \sqrt{1 + 8 \frac{\epsilon_{\mathrm{t}}}{\epsilon_{\mathrm{n}}}} - \epsilon_{\mathrm{n}} - 2}{\epsilon_{\mathrm{n}} \sqrt{1 + 8 \frac{\epsilon_{\mathrm{t}}}{\epsilon_{\mathrm{n}}}} - \epsilon_{\mathrm{n}} + 4} \tag{3.59}$$

when we substitute the value of λ from (3.53).

A notable property of (3.58) is that it closely resembles the equation (2.67) for the polarizability of a homogeneous and isotropic sphere, derived with a different method in Sec. 2.7. The polarizability

$$\alpha = 3\frac{\epsilon - 1}{\epsilon + 2}$$

of a homogeneous sphere matches that of an radially anisotropic sphere exactly when the homogeneous permittivity ϵ is set to $\lambda\epsilon_{\mathrm{n}}$. We can therefore refer to the quantity

$$\epsilon_{\mathrm{eff}} = \lambda\epsilon_{\mathrm{n}} \tag{3.60}$$

as the *effective permittivity* of the radially anisotropic sphere.

3.5 After radially anisotropic sphere

3.5.1 Systropy

The radially anisotropic sphere in Sec. 3.4 is the last technical concept that this introductory part of the thesis discusses in detail. The first article of the thesis, Publication I, departs from the place where the introduction ceases. Publication I discusses a generalization of the radially anisotropic sphere. Unlike the radially anisotropic sphere, this generalized inclusion has three independent material parameters—one for each coordinate of the spherical coordinate system. Then, the permittivity dyadic takes the form

$$\bar{\bar{\epsilon}} = \epsilon_r \mathbf{u}_r\mathbf{u}_r + \epsilon_\theta \mathbf{u}_\theta\mathbf{u}_\theta + \epsilon_\varphi \mathbf{u}_\varphi\mathbf{u}_\varphi \tag{3.61}$$

which is a generalization of (3.39). For illustration, see Fig. 1 in Publication I. The inclusion that is so constituted clearly differs from the radially anisotropic sphere because it lacks the radially anisotropic sphere's spherical symmetry. The new inclusion does, however, retain a rotational symmetry with respect to the z axis.

Because of the symmetry, the permittivity component ϵ_φ contributes nothing to the perturbation field when the excitation field $\mathbf{E}^{\mathrm{p}}$ is oriented in the z direction. In this case, the polarizability of the sphere $\alpha_\parallel$ must be exactly the same as it would be for a radially anisotropic sphere with the permittivity components $\epsilon_{\mathrm{n}} = \epsilon_r$ and $\epsilon_{\mathrm{t}} = \epsilon_\theta$. However, this line of reasoning fails when the direction of the excitation field $\mathbf{E}^{\mathrm{p}}$ is in the xy plane. In fact, the analysis for the perpendicular polarizability $\alpha_\perp$ is substantially

more intricate than the analysis for the radially anisotropic sphere. It turns out that the lack of spherical symmetry makes the polarizability of the inclusion anisotropic, so that in general $\alpha_{\parallel} \neq \alpha_{\perp}$.

The mathematical intricacy of finding the polarizability component $\alpha_{\perp}$ compels much analysis. Publications I and II are devoted to this topic. This novel inclusion that takes the material parameters (3.61) differs substantially enough from the radially anisotropic sphere to deserve its own independent place in the nomenclature. Publications I and II refer to it as the *systropic sphere*.

3.5.2 Canonical shapes

The thesis also develops the concept of the radially anisotropic sphere into another direction. The geometry of the radially anisotropic sphere can be generalized into other canonical shapes while retaining the two independent material parameters ϵ_{n} and ϵ_{t}. Apart from the sphere, the other canonical shapes that the thesis involves are the spheroid and the two-dimensional ellipse. The spheroid is studied in Publication III and the ellipse in Publications IV and V.

What distinguishes canonical shapes, is that they have a natural presentation in a group of coordinate systems that allow a special approach to the Laplace equation $\nabla^2\phi = 0$. There are 11 coordinate systems where the coordinate curves are defined by equations of second degree and where such curves meet orthogonally, at right angles at any point where they are well-defined [39, Sec. 1]. In each of these 11 systems, the Laplace equation separates into three ordinary differential equations for functions of different coordinate variables. When the Laplace equation separates like this, much can be accomplished analytically before involving approximation. In some cases, the solution for an inclusion with a canonical shape is exact.

The canonical shapes are defined in a suitable coordinate system as follows. We can choose one of the 11 special coordinate systems and set one of the system's coordinates to a constant value. The chosen value of the coordinate specifies a surface, referred to as an *isotimic surface* [16, Sec. 1.1]. This surface can be open or closed depending on the coordinate system and on the choice of the coordinate. If the surface is closed, it constitutes a surface of a hypothetical object. The term *canonical* refers to a shape that is defined by such closed isotimic surface.

Although canonical shapes have been discussed in literature, (see the

references in the beginning of Sec. 3.4), the radial anisotropy adds its own flavor to the analysis. Publications III, IV, and V discuss the canonical shapes for radially anisotropic inclusions. Publication III solves the electrostatic problem for the spheroid and Publication IV for the ellipse. Publication V solves the problem for ellipses with multiple layers.

4. Conclusions

If you do not change direction,
you may end up where you are
heading.

Laozi

4.1 Summary of introduction

This introduction outlined the basics of electrostatics. The purpose of the introduction was to precede the enclosed articles with the theory that the research is based upon. Therefore special emphasis was endowed on anisotropy, which features prominently in the enclosed articles. Anisotropy enters field theory in a plethora of different forms. So abundant are the variations of this theme, that to enumerate them would easily turn a thesis into a catalog—an outcome that the author has aspired to avoid. Because the thesis only treats electrostatics, the anisotropy has been constrained to the electrical material parameters of a medium, with magnetic material parameters ignored. In contrast, a dynamic theory of electromagnetic fields would lead a researcher to consider, not only magnetic but also magnetoelectric materials. Such complexities were tacitly avoided when the fields were presumed to remain static.

The introduction took its point of departure from a familiar place—the electrostatic force that acts between stationary, charged particles. This well-known interaction abides by the Coulomb law. Sec. 1.2 argued that the principle of locality requires us to state the Coulomb law in a form that dispenses the problematic notion of action at a distance. The argument motivated the notion of electric fields and invited the reader to consider it as an integral part of the realm of physics as opposed to a mathematical contrivance. The limits of the Coulomb law were also in-

dicated. Sec. 1.2 showed how the Coulomb law becomes untenable when charges move erratically at high velocities. To hold in this situation, the Coulomb law would require the particles to interact instantaneously over great distances, which defies experimental evidence and physical intuition. In Sec. 1.2, the Coulomb law was substituted by an equivalent set of differential equations for the electric field. This turn of thought was aided by the theory of vector operations, given in Sec. 1.3. The theory included discussion of the gradient, the divergence, and the curl. As a reward of this development, the equations for the electric field could be expressed in a form that respects the principle of locality. It was shown that the Coulomb law transforms into two equations—one for the curl of the electric field and one for the divergence. More specifically, Sec. 1.4 showed that the curl vanishes and used this finding to establish the concept of electric potential. Furthermore, Sec. 1.5 showed that the divergence of the electric field depends on the charge density. This result led to the Poisson equation and to the Laplace equation in free space.

The Coulomb force is closely analogous to the force of gravitation that governs the planetary orbits. Both forces cover great enough distances to be regarded as practically universal in their reach. Also, both forces decrease in proportion to the inverse-square of the distance between the interacting particles. The analogy between the two forces is historically significant, because it has permitted ideas in planetary science to migrate to electrostatics. A prominent example of such migration is the Laplace equation, discussed in Sec. 1.5.

The Laplace equation holds a special status in electrostatics because most it features in most non-trivial problems in the discipline. However, Sec. 2.7 showed how an ideally simple problem in electrostatics can be solved without a recourse to the Laplace equation. Part of the equation's usefulness results from an elevated level of abstraction which entails that some of the physical features, like the distribution of secondary sources of fields, are concealed. Sec. 2.7, instead of employing the Laplace equation, treated the classic problem of a dielectric sphere more directly. The section turned to the first principles of electrostatics and explained how the perturbation fields emanate from a concentration of secondary sources inside the inclusion.

The discussion concerning the dielectric sphere required preceding treatment of microscopic charge distributions. This topic was discussed in Sec. 2.1 and Sec. 2.5. It was shown how the perturbation potential can

be explained in two different ways, either as a potential that emanates from multiple microscopic dipole sources or as a potential that emanates from a conglomeration of charge, the bound charge, that accumulates unevenly in the material because of the variations in the material's dipole moment density. It turned out that the relevant charge in the case of the dielectric sphere was a charge on the surface of the sphere. None of the bound charge was found inside the sphere because the dipole moment density was uniform in this domain.

The central topic of the thesis—anisotropy—was introduced in Ch. 3. Sec. 3.1 discussed anisotropy in general whereas Sec. 3.2 discussed anisotropy in the context of electrostatics. Sec. 3.3 turned to the question of anisotropic inclusions consisting in isotropic material. The homogeneous ellipsoid furnished an example. Some important special cases were discussed, including needle and disk. Finally, Sec. 3.4 discussed a spherically symmetric inclusion consisting of radially anisotropic material. The Laplace equation was employed in its generalized form. It was found that the formula for the normalized polarizability of the radially anisotropic sphere closely resembles the corresponding formula for the homogeneous and isotropic sphere. The two formulas were found to concur in the special case of radial isotropy. Because the two results were derived by different methods, the study of homogeneous sphere counts as further corroboration to the results for the radially anisotropic sphere.

4.2 Summary of publications

Publication I: Electrical response of systropic sphere

This article discusses the electrostatic scattering from a spherical scatterer that is anisotropic in such way that the axes of anisotropy are determined by the spherical coordinates. A medium with this kind of material parameters is called *systropic* in this article. The polarizability dyadic of the systropic sphere has two independent components: the axial and the transverse polarizabilities. These two quantities are calculated by different methods. An analytical formula is derived for the axial polarizability. The transverse polarizability is calculated by a semi-analytical approach. The accuracy of the results is studied with two different methods. The first method is to compare different semi-analytical results, which are

calculated using a varying system matrix size. The second method is to compare the semi-analytical results with the numerical ones.

Publication II: General systropy in spherical scatterers

This article continues the discussion of the systropic sphere. Electrostatic scattering from systropic spherical objects is studied with greater generality than in the preceding article where simplifying assumptions were necessary. This article provides a generalized method for studying the systropic sphere. More specifically, the article focuses on the electrostatic scattering of the systropic sphere described by two quantities: the axial and the transverse polarizabilities. The axial polarizability does not depend on the azimuthal permittivity component, which simplifies the problem sufficiently to make it a subject to an analytical approach. In contrast, the transverse polarizability depends on the azimuthal permittivity component, which makes the problem complex enough to require a semi-analytic approach. This article describes a novel way to study the systropic sphere. An important parameter in the systropy research is the ratio between the two tangential permittivity components, called systropy ratio. The novel method drastically extends the scope of the systropy research, by allowing a wider range of systropy ratios, a range that covers the whole set of positive real numbers. While the range of systropy ratios has been formerly restricted to square numbers, the new method that uses Gegenbauer polynomials, can solve the more general case.

Publication III: Radial anisotropy in spheroidal scatterers

This article discusses a spheroidal scatterer in a uniform, static electric field. The scatterer is assumed to be anisotropic and inhomogeneous in a way that relates to the spheroidal coordinate systems, that come in two varieties. The study expands on former research conducted upon radially anisotropic spheres. The new class of objects concerned resembles the formerly studied radially anisotropic spheres, with the distinction that spheroidal instead of spherical coordinate system determines the axes of anisotropy. An analytical method to tackle the problem is developed. The results are validated analytically by comparison to well-known special cases of spheroids: spheres, disks, and needles.

Publication IV: Polarizability of radially anisotropic elliptic inclusion

This article discusses a two-dimensional electrostatic scattering problem where an elliptic inclusion is suspended in a homogeneous background and impinged by an electric field which is uniform and static. The novelty of the discussion stems from the inclusion's material parameters. The material of the inclusion is assumed to be axially anisotropic, so that the axis of anisotropy aligns itself with the radial unit vector of the elliptic coordinate system. Similar varieties of anisotropy have formerly been referred to as *radial anisotropy*, and the same term is employed herein. The radially anisotropic elliptic inclusions are studied with an analytic method. The validation is likewise analytic. The validation method compares the new results with the results for radially anisotropic circles and homogeneous two-dimensional needles. The elliptic inclusion is found to facilitate both cloaking and field concentration.

Publication V: Porcupic concentrators and bulbic cloaks in planar configuration

This article continues the discussion of the radially anisotropic elliptic inclusions. It extends the foregoing study on radially anisotropic ellipses by introducing a multilayer structure. The analysis assumes a planar configuration, so that only two dimensions are involved in the problem. A method is developed which gives an exact solution for a layered radially anisotropic inclusion with a finite but otherwise arbitrary number of layers. Two different implementations for radial anisotropy are represented, one that leads to *porcupic* inclusions, and another that leads to *bulbic* inclusions. These two kinds of radially anisotropic inclusions are shown to be qualitatively different.

4.3 Laplace demon

This concluding chapter of the introduction would fail to collect the threads of the subject matter, would it not return to the issue regarding the stability of the solar system. The issue has an amazingly long history. The Roman philosopher Titus Lucretius Carus (c. 99–c. 55 BC) considered the question in his philosophical writings on astronomy. In the book *De Rerum Natura* (On the nature of things) he expresses his view in verse [40,

Book 1, lines 1024–1032]:

For sure it was not by a set design
These primal germs then took their proper place.
Of their own knowledge, nor did they arrange
Their future motions, but because they were
In number many, changed in many ways.
Shifted about by blows through endless time.
Trying all motions and all unions too.
They reached at last to dispositions such
As now has formed this universe of ours.

Scholars of ancient literature hypothesize that this passage was written as a response to the doctrine of a Pre-Socratic Greek philosopher Anaxagoras (c. 510–c. 428 BC) who upon contemplating the order the solar system conjectured that the perfect order of the planets betokens design [41, Notes, p. 126]. Apparently, Isaac Newton's call for celestial reformism and the reaction against the idea have an ancient origin.

The quoted passage and many similar ones that Lucretius wrote, shows that the Epicurean philosopher possessed an intuitive understanding of chaos and its role in structuring the universe. Lucretius thought that chaos is needed before the universe can beget complicated but regular structures, such as the solar system or, indeed, even life. For Lucretius, this uncanny philosophical insight was obviously speculation rather than an established truth. And hypothetical the insight remained, until the matter was perused by Simon Pierre Laplace and his contemporaries.

Although Laplace contributed much to our knowledge when he studied the stability of the solar system, it was not through the scientific work of Laplace that chaos migrated from philosophy to mathematics. Laplace held that the solar system is essentially stable and employed perturbation theory to prove his point. The proof was almost complete but it relied on the convergence of an infinite series. Laplace decided to assume the convergence but the matter remained open to question [42, Ch. 11].

A particularly picturesque example of Laplace's faith in the predictability of the universe comes from his famous thought experiment where a hypothetical indefinitely knowledgeable entity examines the universe in detail and forms accurate predictions about the past and the future, expanding its observations over the infinitude of space, experiencing all

events—however remote in time and place—as vividly as it were a direct observation. Laplace wrote [43, p. 4]:

> We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.

The unnamed entity in Laplace's thought experiment is often referred to as the *Laplace demon*, although Laplace himself did not specify the character of the hypothetical entity.

What Laplace outlines by his thought experiment, is the doctrine of *causal determinism*. The doctrine holds that the force of necessity drives all events in past and future, making them knowable if the present is known in a sufficient detail. Having tentatively proved the predictability of the planetary orbits over indefinite intervals of time, Laplace must have found causal determinism a simple postulate to endorse. Causal determinism, alongside with the principle of locality that Sec. 1.3 outlined, remains an enduring principle in classical physics, but some prominent interpretations of quantum mechanics are incompatible with both principles.

Although Laplace's work ostensibly proved the universe to be predictable, the issue turned out to be more intricate. When Henri Poincaré (1853–1912) continued Laplace's research on celestial stability, he found that the series that Laplace had used did not converge in all cases. Moreover, he encountered complications when he attempted to solve the orbits of three mutually attractive, massive bodies. The problem is known as the *three-body problem*. The frustrated efforts in solving the problem made Poincaré understand that the problem was inherently unsolvable. We now know that the difficulties arise because the three-body system is chaotic. Poincaré himself did not apply the term "chaos" to the three-body system, but the concept started to penetrate into mathematics [42, Ch. 11].

We may, perhaps, find it paradoxical that chaos was first understood in celestial mechanics. After all, other disciplines, like biology or meteorol-

ogy, abound in chaos. An observer of planets spots little change in the solar system, even over the lapse of several millennia, whereas an observer of weather witnesses hailstorms that strike without forewarning and an observer of birds sees a murmuration of starlings erratically alter its course. An astronomer fixes his telescope to catch a glimpse of chaos where it constitutes a thing of curiosity; the chaos that reigns in his tea cup, when he pours the water, goes unnoticed.

But turning to the solar system for novel ideas is not irrational. Nor is it coincidental that the Laplace equation was first formulated for celestial mechanics, even when the equation is equally applicable in such grounded disciplines as hydrodynamics and hydrostatics. The solar system offers an ideal test bed for scientific ideas. When we experimentally validate a hypothesis, we seek to reduce the number of relevant parameters to minimum. The more factors contribute in the experiment, the more difficult it becomes to pinpoint any single factor as the cause of a perceived effect. In celestial mechanics, we need no labor to isolate extraneous factors. The solar system toils on our behalf [44, Ch. 1].

In view of the present understanding, the solar system once was significantly more crowded than it is on our days. It was occupied by the early building blocks of planets, referred to as *planetesimals*. These planetesimals were nothing but misshapen chunks of rock, only a few kilometers across. In this primordial state of the system, celestial collisions have been relatively frequent. But it was not only collisions that drastically altered the form of the solar system. Near encounters with the massive gas giants influenced the orbits of the planetoids—sometimes pushing them so near the sun as to vaporize them in the excess irradiance, sometimes throwing them into the interstellar abyss, and sometimes disintegrating them in the grips of tidal forces [45, Ch. 5]. As the encounters continued, the crowd of planetesimals dispersed, leaving behind those bodies of matter that happened to circulate relatively stable orbits. And thus, from eon to eon, from cataclysm to cataclysm, the blind hands of chaos wielded their hammer and chisel in sculpting the celestial forms toward such perfection that befits the firmament.

Henri Poincaré may have temporarily dispelled the Laplace demon by indicating the inherent long-term unpredictability of celestial events. What if we were to awaken the knowledgeable creature from its centurial slumber? The creature would probably look at the celestial landscape and self-complacently smile at its powers of prediction. Never mind that it would

awaken to a perplexing reality where knowability is the *de jure* of science and limits of knowledge the *de facto*. Even so, nothing more drastic than a quirky meteor shower would differ from its astronomical foretellings; the celestial sphere would be as it presumed. The success would likely make the demon endorse Newton's and Laplace's dreamy thought experiments, especially upon a contemplation on how these ushered in an era of discovery. The brightest of minds apparently deigned to indulge into a play with ideas; if only as an expedient to find results that pass the scrutiny of a most sober-minded scholar. That play on thought is the force that forges experience into iron-clad principles—principles that constitute a supreme apparatus for progress. It is this predilection that pulsates in the heart of the spirit of science.

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Errata

Publication I

- In the second sentence of Sec. III-D, the parenthetical expression that exempts the rightmost dashed curve of Fig. 2 from having equal permittivity components $\epsilon_{\theta\theta}$ and ϵ_{rr} is superfluous. The components are equal also for this curve.

- The items in the legend of Fig. 8 appear in a jumbled order. The permittivity ratio $\epsilon_{\varphi\varphi}/\epsilon_{\rho\rho}$ should be 5 for the leftmost curve, 1 for the curve at the center, and $1/5$ for the rightmost curve.

- In Sec. V, the reference to Fig. 7 at the beginning of the third paragraph should refer to Fig. 8.

- In the last paragraph of Sec. VI, the reference to Fig. 4 should refer to Fig. 2.

Publication III

- In the caption of Fig. 1, ψ instead of ϕ should represent the coordinate variable.

- The appropriate form of (4) is

$$L_x = \frac{a_x a_y a_z}{2} \int_0^\infty \frac{\mathrm{d}s}{(s+a_x^2)\sqrt{(s+a_x^2)(s+a_y^2)(s+a_z^2)}}$$

where the numerator consists of the differential $\mathrm{d}s$ instead of the integration variable s.

- In the second paragraph of Sec. VII, the reference to Sec. VII should refer to Sec. VI.

Publication IV

In the third paragraph of Sec. III-B, the intended component of the normalized polarizability is the parallel component, denoted by $\alpha_{\parallel}$, although the symbol $\alpha_{\perp}$ of the perpendicular component is mistakenly used.

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