Fair division of infinitely divisible goods

Tuukka Himanka

School of Business
Department of Economics

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Aalto University
School of Business

Aalto University
School of Business

Aalto University, P.O. BOX 11000, 00076 AALTO
www.aalto.fi
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Author Tuukka Himanka

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Abstract

In my thesis, I examine the theory of fair division of infinitely divisible heterogenous goods from measure theoretic context. I present the main existence theorems of divisions, give graphical representations of division and give an introduction to the algorithmic literature in the area.

To present the results, I first define the formal measure theoretic model used in the literature and discuss the concept of fairness. I present the existence theorems of Weller and Dubins and Spanier that give results for the existence of a number of different divisions. With the existence theorems I also discuss the limitations of the model, and look into the impact of some assumptions of the formal model concerning the existence theorems. I present two different geometrical interpretations that can be used to understand and find fair allocations, and in the final part, I consider three algorithms to achieve a fair division.

Keywords Fair division, economics, measure theory

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1 Introduction

An economist is hosting a birthday party and wants to divide a cake between their guest who all have equal rights to the cake. Each guest has their own preferences for different parts of the cake with different toppings spread out unevenly. It is of utmost importance for the economist that the division is fair. But what *fair* means? How can they know such a division exists?

The question of fair division is an age-old problem with first written records of it appearing already in the Bible (Brams & Taylor 1996). First time in academics, the Polish mathematicians studied the problem during the World War II and the first formal description of the problem was given by Hugo Steinhaus (1948). The question has been studied ever since by researchers across multiple disciplines including economics, mathematics, computer science, and operations research. The issue is relevant not only in academics with issues of income distribution, education policy, health care, and many others closely related to the notion of fairness.

Due to the broad nature of the subject, I limit the scope of my thesis to infinitely divisible goods. The model I use applies both to homogenous and heterogeneous goods. Similarly to the presentation in modern fair division literature, I approach the subject from measure theoretic context.

The structure of the thesis is as follows. Firstly some definitions and the formal model is needed. Then I will look into how *fair* division can be defined. Before continuing any further, it is reassuring to know whether the aimed division exists and thus an overview of theorems of the existence of fair divisions is needed. Also, I will present two kinds of graphical representations of division that hopefully help give better intuition of the subject. Finally, I briefly discuss the literature on getting to the fair divisions and present a few algorithms relating to it. In the end, I also discuss the possible applications of fair division literature. Along the way, I will present the most important theorems and results from the over 70-year history of the formal study of this subject.

Throughout the thesis, I will attempt to keep the notation the same that is introduced in the first section and therefore, it may differ from the original material. Despite the rather mathematical style of my thesis, I will omit proofs and reasonings for some of the theorems and result by citing to the original material aside from the most important theorems where the ideas leading to the main results are presented.

2 Model

The "cake" that is denoted by Ω is a set that needs to be divided between the set $I = \{1, 2, \dots n\}$ of players who each have equal rights to get a share. Typically Ω is the closed interval $[0, 1] \in \mathbb{R}$, but it can be any set. The objective is to find disjoint subsets of Ω for each player. The collection of the disjoint subsets, denoted by $\mathcal{B}$ is a σ-algebra on Ω. Mathematically Ω and the σ-algebra $\mathcal{B}$ form a measurable space $(\Omega, \mathcal{B})$. $\mathcal{B}$ has the following properties (Folland 1999):

1. $\Omega \in \mathcal{B}$.
2. If $E_i \in \mathcal{B} \ \ \forall i \in \mathbb{N}$, $\cup_1^\infty E_i \in \mathcal{B}$ (closed under countable unions).
3. If $E \in \mathcal{B}$, then $E^c \in \mathcal{B}$ (closed under complements).

Players use probability measure $\mu : \mathcal{B} \to [0, 1]$ to assess their utility on $\mathcal{B}$. μ equipped with the following two properties (Folland 1999):

1. $\mu(\emptyset) = 0$
2. $\mu(\cup_1^\infty E_j) = \sum_1^\infty \mu(E_j)$ given that $\{E_j\}_1^\infty$ is disjoint.

Often μ is also assumed to be non-atomic, i.e. for any $A \in \Omega$, if $\mu(A) > 0$, there exists $B \subset A$ such that $\mu(A) > \mu(B) > 0$. Additionally, if μ_i is called absolutely continuous with respect to μ_j, then for every set $A \subset \Omega$ if $\mu_i(A) = 0$, then also $\mu_j(A) = 0$.

Now that the formal framework is defined, the goal of fair division can be discussed. In this work, it is assumed that the players prefer a bigger part of Ω though in some literature also a smaller is better approach is considered. To characterise different divisions specifically and to discuss the notion of fairness, the following definition of some most used properties of division is needed.

Definition 1. *Let $P = (A_1, P_2, \dots A_n)$ be a partition of Ω. It is (Barbanel 2005):*

- Proportional *if and only if* $\mu_i(A_i) \geq \frac{1}{n} \quad \forall i \in I$.
- Envy-free *if and only if* $\mu_i(A_i) \geq \mu_i(A_j) \quad \forall i, j \in I$.
- Equitable *if and only if* $\mu_i(A_i) = \mu_j(A_j) \quad \forall i, j \in I$.
- Pareto efficient *if and only if* $\nexists P' = (B_1, \dots B_n) \neq P$ *such that* $\mu_i(B_i) \geq \mu_i(A_i) \quad \forall i \in I$ *with at least one inequality being strict.*

Note that in some earlier literature the term equitable was used to mean envy-free, but in modern literature, envy-free is the used term. Also, proportional division is called super-proportional if all inequalities are strict.

Verbally, a division is proportional if each player thinks that they get at least the average size of the cake. Envy-freeness means that each player values their own piece at least as much as everyone else's. Equitable division requires everyone's valuation of their own piece to be the same. If no other division exists such that someone's valuation increases, the division is called Pareto efficient.

3 Theory of fairness

The question of what is fair is a rather philosophical one. Given the cake analogy, the intuitive answer is to divide the cake into equal parts but is it fair? What if some of the players dislike some subset of the cake that they are given in equal division and would prefer some other piece? In other words, some player envies the piece of some other player. Can the division be called then fair? In the context of a division between two players, the classical solution to this is the "I cut, you choose" protocol to keep both happy which also has been generalised for any number of players discussed in section 6.1. This indeed guarantees proportionality and envy-freeness but is it the definitive answer? The purpose of this section is not to take a normative stance on what is fair but rather introduce some the different concepts of fairness proposed that are needed later in this thesis.

The term theory of fairness was first used in Varian (1974), and the notion of fairness has been extensively researched by economists since the 1960's when the envy-freeness was first proposed as an allocation rule (Foley 1967). Envy-freeness was later expanded with Pareto efficiency by Schmeidler and Yaari (Varian 1974) that is called here Varian-fair to avoid confusions. This definition of fairness has been extensively used by welfare economists ever since.

Definition 2 (Varian-fair (Varian 1974))**.** *A partition* $P = (A_1, A_2, \dots A_n)$ *of* Ω *is a (Varian) fair allocation if it is both envy-free and Pareto optimal.*

Varian (Varian 1976) does give some reasoning that does make this definition useful in economics. Firstly, as the envy-freeness condition only includes "internal" comparisons, no one needs to know the utilities of other players. Envy-freeness also implies that the allocation is stable (Varian 1974) since no player would trade their part with anyone else. Lastly, if there are two envy-free allocations but with the other, each player is better off, why not choose it and hence the Pareto efficiency. But as will be seen, in practice Pareto efficiency is hard to achieve without knowing every players' utilities.

In addition to the Varian's definition fairness, many other goals of fair division have been used. Most notably, perhaps a more common goal in the earlier mathematics and computer science literature proportional or equitable divisions have been considered. Another concept of optimal division needed in this work is the one Dubins and Spanier (1961) defined as optimal division, called here DS-optimality:

Definition 3 (DS-optimality (Dubins & Spanier 1961)). *Arrange utilities $\mu_j(A_j)$ of every partition $P = (A_1, \ldots A_n)$ in non-decreasing order and designate the resulting sequence as*

$$a_1(P) \leq \cdots \leq a_n(P)$$

P is a DS-optimal partition if for any P' either $a_i(P) = a_i(P')$ $\forall i$ or if j is the smallest i such that $a_i(P) \neq a_i(P')$, then $a_j(P') < a_j(P)$.

The idea in DS-optimality is to maximise the utility of the player whose utility is the smallest. From the remaining partitions find the one that maximises the utility of the second last person and continue towards the first person. The problem from an economics point of view is that to achieve this, one would need to know the utilities of each player which is difficult in practice. As noted in Weller (1985), DS-optimal divisions are also necessarily Varian-fair under countably additive measures and is much stricter requirement than the Varian-fairness.

The review of different definitions of fairness here is far from exhaustive (see Thomson (2011) for a comprehensive review) but is sufficient for the needs of this work and should be enough the understand what kind of questions are related to fairness. As no single definition is taken here, multiple notions of fairness are considered here and the constraints they impose.

However, one could argue that Pareto efficiency is a reasonable goal in all cases and it is commonly the aim in economics literature and more recent fair division works even though efficiency doesn't really characterise fairness in the same way than the other notions of fairness introduced. Nevertheless, as Thomson (2011) noted, the efficiency issues were largely ignored by mathematical literature until recently, and therefore, the older results do not consider efficiency issues.

4 Existence of fair allocation

The fundamental question before trying to divide anything fairly is that does such division exist in the first place. Two most notable papers on the existence of divisions in measure theoretic models are presented here, the arguably more mathematical Dubins & Spanier (1961) one and the one by Weller (1985) that was analogous to Varian's approach albeit from measure theoretic context. Additionally, a short discussion on the countable additivity assumption is given at the end.

4.1 Dubins & Spanier approach

The existence of DS-optimal division and a few others was shown by Dubins and Spanier (1961) in their paper the main results of which are presented here. Their proof relied much on the Lyapunov convexity theorem that is also generally important in fair division literature and hence also presented here.

Theorem 1 (Lyapunov's convexity theorem (Diestel & Uhl 1977))**.** *Let Σ be σ-algebra of subsets of Ω, X a finite dimensional Banach space and $\mu : \Sigma \to X$ a non-atomic, countably additive vector measure. The range of μ is a compact and convex subset of X.*

The idea of Dubins and Spanier in using the Lyapunov's theorem was as follows. Let $P = (A_1, \dots A_k)$ be an ordered partition of Ω. Then associate P with $M(P) = [\mu_i[A_j]_{1 \leq j \leq k}]_{j \in I}$, an $n \times k$ matrix that maps each element of partition P to the measure of each agent. The Lyapunov's theorem was then applied to show that the range $\mathcal{R}$ of function M is convex and that it is compact given that each μ is non-atomic and countably additive.

Now it only remains to show that this result implies that different fair divisions exist that is replicated here. First division considered was the exact one.

In other words, given non-negative real numbers $\alpha_1, \dots, \alpha_k$ such that $\sum \alpha_i = 1$, there exists a partition $P = (A_1, \dots, A_k)$ such that $\mu_i(A_j) = \alpha_j$ for all $i < n, j < k$. Now for all $j < k$ let $P_j = (A_1, \dots, A_k)$ be a partition in which element $A_j = 1$ and $A_i = \emptyset \ \forall i \neq j$. As a result, Convexity of $\mathcal{R}$ implies that $\sum \alpha_j M(P_j) \in \mathcal{R}$ and hence there exists a partition P such that the jth columns of $M(P)$ and $\sum \alpha_j M(P_j)$ are equal. Note that this result does not imply that everyone gets a single piece, the piece may be a union of multiple pieces from the cake. Note that if weights are chosen at $1/n$, this result implies that equitable division is achievable.

Next corollary of the compactness and convexity of $\mathcal{R}$ was that a super-proportional division exists if at least two measures are not identical. Or again given non-negative real numbers $\alpha_1, \ldots, \alpha_n$ such that $\sum \alpha_i = 1$, there exists a partition $P = (A_1, \ldots, A_n)$ such that $\mu(A_i) > \alpha_i \ \forall i$. Now let $\mu_1 \neq \mu_2$ and hence there exists A such that $\mu_1(A) > \mu_2(A)$ and if $B = \complement A$, $\mu_2(B) > \mu_1(B)$. By symmetry, it can now be seen that $\mu_1(A)/\alpha_1 \geq \mu_2(B)/\alpha_2$. Let P_0 be a partition where A is allocated for player 1 B for player 2 and nothing for the other players and let P_i for $i > 2$ be a partition where the entire Ω is given for player i. The Lyapunov's theorem now implies that for each non-negative numbers $x_1, \ldots, x_n$ with $\sum x_i = 1$ there is a partition $P = (A_1, \ldots, A_n)$ such that

$$M(P) = x_1 M(P_0) + \sum_{i \geq 2} x_i M(P_i). \tag{1}$$

The primary interest of this is the diagonal of $M(P)$, and each x_i needs to be found such that the entries in the diagonal are in the same ratio as α_i, i.e. the equations

$$x_1 \mu_1(A) = \lambda \alpha_1, \tag{2}$$

$$x_1 \mu_2(B) = \lambda \alpha_2, \tag{3}$$

$$x_i = \lambda \alpha_i \quad \text{for } i > 2. \tag{4}$$

By solving (2)–(4) for x_i, summing them and using $\sum \alpha_i = 1$,

$$\lambda = \left(1 + \frac{\alpha_1}{\mu_1(A)}(1 - \mu_1(A) - \mu_2(B))\right)^{-1}. \tag{5}$$

As $1 - \mu_1(A) - \mu_2(B) < 0$ and $\frac{\alpha_1}{\mu_1(A)} < 1$, it must be so that $\lambda > 1$ and therefore choosing $x_i = \alpha_i$, the ith entry of the diagonal of (1) is $\lambda \alpha_i > \alpha_i$.

Finally, given that $\mathcal{R}$ is compact, it follows immediately that a DS-optimal division exists. With these results, most of the notions of fairness in Definition 1 are known to exist in addition to the DS-optimality.

4.2 Weller approach

Varian (1974) used Knaster-Kuratowski-Mazurikiewicz lemma along with a number of assumptions on preferences to prove that Varian-fair division exists if either preferences are convex or no two weakly Pareto efficient allocations exist that all agents regard as indifferent and Svensson (1983) showed later than weaker assumptions also are sufficient.

The goal of Weller (1985) however was to show the same result in measure theoretic model with the main result being that a Varian-fair division of measurable space exists. Their proof relied on something they called combinatorial optimality concept and the construction of Pareto efficient allocation similarly as in Section 5.2 and then showing that they exist along with envy-freeness. In this subsection, the measures need not to be absolutely continuous.

Weller constructed the $n-1$ dimensional simplex with Radon-Nikodym derivatives of the measure functions that later was named as Radon-Nikodym Set introduced in Section 5.2 with a bit different notation. The idea was to map each point of the cake to a simplex depending on the relative utility of it for the players. The theorem that Weller then prove was that given a point in the interior of the simplex, if n closed areas can be formed for each player who gets the parts of the cake that belongs to their area every such division is Pareto efficient.

After establishing this, the efficiency had to be combined with envy-freeness. Weller used the Kakutani's fixed point theorem to show that the mapping from the interior of the simplex to the set of subsets of the simplex contains a fixed point $y = (y_1, \ldots, y_n)$. The result from a fixed point is that an efficient partition $P = (A_1, \ldots, A_n)$ exists such that

$$y_i = \frac{\mu_i(A_i)}{\sum_{j \in I} \mu_j(A_j)}. \tag{6}$$

This division now only needs to be envy-free. If a point $x = (x_1, \ldots, x_n)$ is taken from the simplex, Weller showed that for any two agents

$$\frac{m_i(A_i)}{x_i} \geq \frac{\mu_j(A_i)}{x_j}. \tag{7}$$

By taking a ratio for two players of (6) and combining with (7) one can get the definition of an envy-free division, and hence a Varian-fair division exists.

As a final note, Weller also related Pareto efficiency and envy-freeness to DS-optimality. The conclusion is that DS-optimality is much stricter than Pareto efficiency and if measures are assumed to be absolutely continuous, all DS-optimal divisions are fair. DS-optimality being stricter than efficiency is rather intuitive. Take Pareto efficient partition P is arranged to non-decreasing order and from the second-worst off player a sufficiently small measurable part is taken to the worst off player, the result is more optimal according to Dubins & Spanier.

4.3 Problem of countable additivity

For economists, a major issue with the existence theorems is that the measures need to be countably additive, and hence linear, which becomes problematic if for example, decreasing marginal utility is assumed. Furthermore, some players may get allocated a piece that is a union of multiple measurable subsets of Ω, which is potentially problematic in some applications, say land division. The economists' guests may also be somewhat disappointed if they are given a plateful of crumbles from the cake. With this motivation, some remarks are valuable concerning to the existence theorems if countable additivity of measures is not guaranteed.

Some of the results of Berliant et al. (1992) are presented here to give some reassurance for the economist's guests. The issue is that if countable additivity is not assumed, much can not be said of some general set functions and hence some assumptions are needed. To relate with decreasing marginal utility, the set functions can be intuitively assumed to be concave and and subadditive (Berliant et al. 1992). Subadditivity for set function $\mu_i : \mathcal{B} \to \mathbb{R}$ means that for all $A, B \in \mathcal{B}$ s.t. $A \cap B = \emptyset$, $\mu_i(A \cup B) \leq \mu_i(A) + \mu_i(B)$.

Berliant et al. (1992) impose a modified Hausdorff topology where the set of measurable partitions is compact. If all the set functions are assumed to be continuous, each μ_i is continuous and with this, it follows that Pareto efficient, DS-optimal, and utilitarian partitions exist.

In conclusion, the objectives of division need not to be abandoned if measures are not linear, but in this approach, the properties of measure functions imply that they can not be used the same way.

5 Graphical representations

To give a picture of what the division looks like and additionally to see the importance of the Lyapunov's theorem, two different graphical representations of fair division are introduced. The examples may only be intuitively drawn for two or three players, but the concepts are also usable in higher dimensions, albeit not graphically.

First, the Individual Pieces Set, or IPS is introduced that gives the feasible set of all possible partitions.

5.1 Individual Pieces Set

The Individual Pieces Set term has been introduced by Barbanel, whose examples (Barbanel 2000) are adopted here. Without getting any further into details, the definition of the Individual Pieces Set (IPS) is needed.

Definition 4. *Individual Pieces Set (Barbanel 2000) The Individual Pieces Set that associates* Ω *and probability measures* $(\mu_1, \mu_2, \ldots, \mu_n)$ *is the set*

$$\{\mu_1(A_1), \ldots, \mu_n(A_n) \mid P = (A_1, \ldots, A_n) \text{ is a partition of } \Omega\}$$

Intuitively, the IPS contains all the possible points that the measure functions can give with possible partitions of Ω.

Before going to the pictures, properties of the IPS (Barbanel 2000) should be discussed. First notable property by the Lyapunov's theorem is that the IPS is a closed and convex set. Also as the measure functions are here probability measures, the IPS is a subset of $[0,1]^n$ where n is the number of players. Finally, the IPS contains the points $(0, \ldots 0, 1, 0, \ldots 0)$ where 1 is at index i for all $i \in I$ as the entire Ω is given to a single player, and their measure of the entire cake is then 1.

With these properties established, the construction of a few examples when $n = 2$ can be begun. The easiest and the most obvious example is the case of $\mu_1 = \mu_2$.

Firstly based on the previous, discussion it is known that points $(0,1)$ and $(1,0)$ are in the IPS. The points $(0,1)$ and $(1,0)$ must be then connected such that the IPS is closed and convex by the Lyapunov's theorem. Hence a line connecting the two points is at least included in the IPS, but in this case, as the measures are identical, the straight line between the two points must contain all the possible points. This is because the utility of one player is always the utility of the other with the same part of the cake and one player gets the complement of the part given to the other player. Hence it is easy to see that the hyperplane $\mu_1 + \mu_2 = 1$ gives all the points for the IPS given that the entire cake is divided.

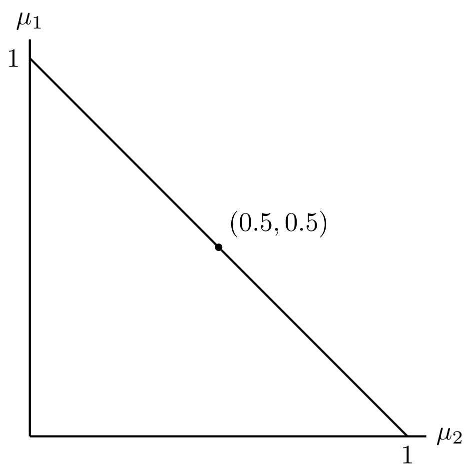

Figure 1: IPS when $\mu_1 = \mu_2$

In this case, all points on the line are Pareto efficient since there is not any way to make one better off without taking utility from the other. It may also be seen that the middle point $(\frac{1}{2}, \frac{1}{2})$ clearly fulfills each of the criteria in Definition 1.

To continue from the somewhat trivial nature of the first example, the next step is to see what the IPS can look like when the two measures are not identical. Again the points $(0, 1)$ and $(1, 0)$ are in the IPS and the hyperplane $\mu_1 + \mu_2 = 1$ has to be on it in order to construct a convex set between the two points. With these facts, it is now easy to draw such IPS with an example given in figure 2.

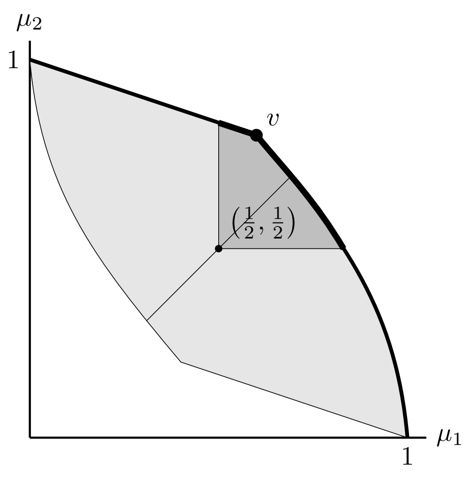

Figure 2: IPS when $\mu_1 \neq \mu_2$

There are a few interesting qualities of the IPS region that is coloured in light grey in figure 2. Firstly, the IPS is symmetric about the point $(\frac{1}{2}, \frac{1}{2})$, a property with all possible two-player Individual Pieces Sets. This not hard to see as the complement of any piece is given to the other player. A second interesting part of the figure is the bolded outer boundary of the IPS. As shown by Barbanel & Zwicker (1997) this outer boundary includes all the Pareto efficient points which is rather intuitive as no point exists on the IPS where both measures are higher than any point on the outer boundary.

What about the other notions of fairness? In two-player context envy-freeness is equal to proportionality. The dark grey area includes the points envy-free and equitable points as in that area $\mu_1, \mu_2 \geq 1/2$. Hence the boldest part of the outer bounder includes the Varian fair points. Now a diagonal line that connects all points where $\mu_1 = \mu_2$ in the light grey area gives all possible equitable divisions. The point where the equitable line meets the efficient boundary is then also clearly a DS-optimal allocation.

The interesting question now is what the IPS looks like if $n > 2$. As already noted, it is by Theorem 1 closed and convex n-dimensional object, and the points that limit it are known. The intuitive first idea by the symmetry of two-player case is that it would be symmetric by point $(\frac{1}{n}, \dots, \frac{1}{n})$. However, one can give a simple counterexample in a three-player context as the complement of one player's piece is divided between two other players. Then there is the question of proportionality and envy-freeness that were so clearly visible in figure 2. Envy-freeness and proportionality are obviously not the same with more than two players. As it turns out, in higher dimensions the IPS picture does not alone give a picture about envy-freeness (Barbanel 2005) and also the notion of outer border is somewhat vague on higher dimensions so of efficiency much cannot be said with IPS alone either. Proportionality, on the other, hand will always be found.

5.2 Radon-Nikodym Set

The Radon-Nikodym Set (RNS) was first introduced by Dubins & Spanier (1961) and has been extensively discussed later by Barbanel, who also introduced the term Radon-Nikodym Set. The RNS was also used by Weller in their proof of the existence of Varian-fair allocation (Weller 1985). The idea is essentially to replace the measures with another function and map each point of the cake to a simplex. As earlier, the definition is needed before getting any further.

Definition 5 (Radon-Nikodym Set (Barbanel 2000))**.** *Define measure* $\sigma = \mu_1 + \mu_2 + \cdots + \mu_n$ *and let* $f_i : \Omega \to \mathbb{R}$ *be the Radon-Nikodym derivative of* m_i*. For* *each* $p \in \Omega$*, let* $f(p) = (f_1(p), f_2(p), \dots f_n(p))$*. The Radon-Nikodym Set is the set*

$$\{f(a) \mid a \in \Omega\}.$$

The Radon-Nikodym theorem (Folland 1999) says that for any subset A of Ω, $\mu_i(A) = \int_A f_i \, \mathrm{d}\sigma$. If absolute continuity is assumed from the measure *sigma* with respect to μ_i, $f(a) = 1$ for nearly all $a \in \Omega$. The only possibility for $f(a)$ to be zero is by absolute continuity then that all measures are zero. It is assumed here that the functions are redefined at the possible point of $f(a) = 0$ to simplify the treatment of the examples. Based on the definition it can be seen that the RNS is an $n-1$-dimensional simplex in $\mathbb{R}^n$ denoted by Δ^{n-1} where each player is located at vertices of the simplex and the value of the Radon-Nikodym derivatives give a relative utility of a single part of the cake for each player. Intuitively, players prefer points that are closest to their vertex in the simplex.

With this definition, some examples can now be given that are adopted from Barbanel (2005). The easiest example again is the case of two players and identical measures. As the measures are identical, the Radon-Nikodym derivatives are equal. Therefore both Radon-Nikodym derivatives have a constant value $\frac{1}{2}$ and the RNS consists of just a single point, $\left(\frac{1}{2}, \frac{1}{2}\right)$.

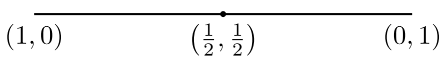

Figure 3: RNS when $\mu_1 \neq \mu_2$ and $n = 2$

With this, it is easy to see that with three players and identical measures, the RNS is at the centre of an equilateral triangle and with four players the centre of a regular tetrahedron and so on.

To get a better understanding, it would be helpful to draw this image to more players with nonequal measures. So what the image looks like in three-player context with $\mu_1 = \mu_2 \neq \mu_3$? Since $\mu_1 = \mu_2$, all points lie on the line $x = y$. In this example, the ratio of values that players 1 and 2 assign to each bit of Ω is assumed to be continuous distribution compared to the values of player 3.

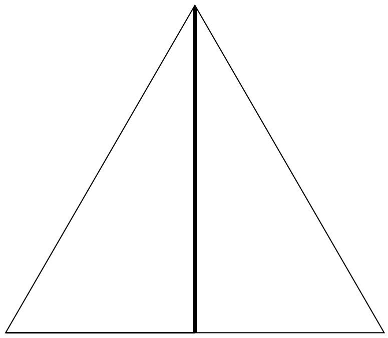

Figure 4: RNS when $\mu_1 = \mu_2 \neq \mu_3$ and $n = 3$

Now that few examples are presented the interesting properties of the RNS may be discussed. The next example is similar to the one presented by Weller (1985) in their proof with the language of Barbanel (2000).

Denote the interior points of the simplex with $\mathring{\Delta}^{n-1}$ and thus

$$\mathring{\Delta}^{n-1} = \left\{ x_1, x_2, \ldots, x_n \;\middle|\; x_i > 0 \,\forall i, \sum x_i = 1 \right\}.$$

By redefining f on possible zero points, the image of f is $\mathring{\Delta}$, i.e. $f : \Omega \to \mathring{\Delta}$. As seen in figure 4 and noted by Barbanel (2000), it is important to understand that f is not necessarily surjective. With this in mind, take partition $P = (A_1, \ldots, A_n)$ and a point $x = (x_1, \ldots, x_n) \in \mathring{\Delta}$. P is said to be *associated with* x if and only if

$$\frac{f_i(z)}{f_j(z)} \geq \frac{x_i}{x_j} \quad \forall i, j \in I, i \neq j$$

for all but measure 0 of $z \in P_i$. If P is associated with x, it is denoted by $P \in x^*$. From Weller's results, it is known that every partition in x^* is Pareto efficient and as proven by Barbanel (1999), the implication also holds conversely.

Given this, the construction of Pareto efficient partition in the RNS with three players can be drawn. Take arbitrary closed disjoint regions R_1, R_2, R_3 of the simplex. Given a point $x \in \mathring{\Delta}$ and partition $P = (A_1, A_2, A_3)$, x is in x^* if and only if each point of Ω mapped to R_i in the simplex with f goes to player i. If a point is on the boundary of R_i, it goes to any of the players whose region is on the boundary and there are multiple efficient allocations for the chosen disjoint regions. Figure 5 represents such point x and the three regions.

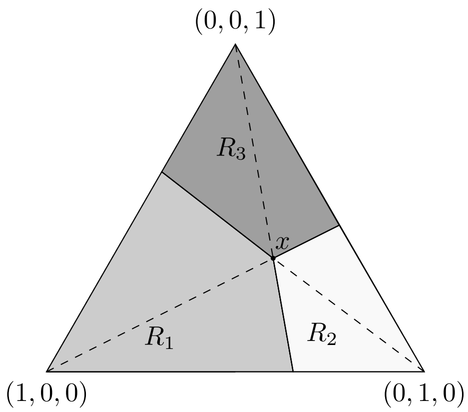

Figure 5: Pareto efficient point

The next interesting question is how the RNS relates to the IPS. As an example, the IPS in figure 2 is converted to RNS. The IPS represents a two-player game, and hence the RNS is on a line between $(1, 0)$ and $(0, 1)$. Remembering that the outer boundary of the IPS has all the Pareto efficient points, the outer boundary is therefore in x^*. Given an arbitrary x between the two players, if all points of Ω mapped to points between $(1, 0)$ and x, or R_1 similarly to the previous example are given to player 1 and all points between x and $(0, 1)$, or R_2 are given to player 2, the division is Pareto efficient. Denote the set of all partitions that are in x^* given by an arbitrary x by $m(x) = \{(m_1(A_1), m_2(A_2)) \mid (A_1, A_2) \in x^*\}$. As a result, $m(x)$ is in the outer boundary of IPS for all x.

The outer boundary of figure 2 has three interesting parts: a straight line, a non-differentiable point u and a curve. If the point x in the corresponding RNS of figure 6 is between points $(0, 1)$ and v, $m(x)$ consists of only a partition where the entire Ω is given to player 1 for every such x. At the point v, $m(v)$ consists of partitions where A_1 includes only points of Ω mapped to v. As shown by Barbanel, the ratio $-v_2/v_1$ equals the slope between the two partitions in $m(v)$ (Barbanel 2000), i.e. slope of the straight line in the IPS of figure 2. Continuing on the RNS, the points between v and the start of the line, w give a single partition $P = (A_1, A_2)$ where piece A_1 is at v and A_2 at v or w. So this part depicts the non-differentiable point u in figure 2. Finally, the curve of the IPS as one might guess is depicted on the RNS as the continuous line. That is since on the curve, there can't be single points of Ω that give a positive measure and there are no gaps where positive measure points could exist.

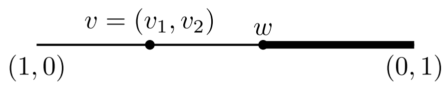

Figure 6: RNS picture corresponding to figure 2

The last question as with IPS is how this generalises to n players? The graphical pictures are not viable with more than four players, but the concept can be used to construct Pareto efficient partitions as done by Weller in their proof (Weller 1985) and the pictures shown here are just a few special cases with a limited number of players.

6 Algorithms

The last question considered in my thesis is how can a fair division be achieved. The theoretical results of existence are of very little actual use if the actual allocation is too impractical or complex to solve, and therefore, it is interesting to find out some of the algorithms proposed in this context. The aim is not to give a comprehensive review of the algorithmic study of the area but to introduce some of the more notable and interesting algorithms for different notions of fairness in games where players can't be fully aware of other players' utilities. Three such algorithms are discussed that reflect the evolving goals in the literature since the era of Steinhaus.

The intuitive, classic way of dividing a cake between two players is the "I cut you choose" algorithm where one person cuts the cake and the other chooses first. As already noted, it does guarantee proportionality and envy-freeness but the all so obvious issue arising is that what if there are more than two players. That's what Polish mathematician Steinhaus and their students Banach and Knaster were researching during World War II (Brams & Taylor 1996).

6.1 Banach–Knaster last-diminisher procedure

The result Banach and Knaster discovered in the 1940s for the n-player problem described by Steinhaus (1948, 1949) was to be called the last-diminisher procedure which is in a sense a generalisation of the classic two-player solution. The procedure does guarantee proportionality, but envy-freeness or efficiency was not yet achieved.

The process goes as follows. The players are ordered, and in the first round, the first player cuts an arbitrary part from the cake. The second player has an option to diminish the cut part. Then the third player has the option to diminish the part that is possibly already diminished, and this continues until the last player. The last diminisher of the part is then obliged to keep it. The second round is now the same without the last diminisher as an $n-1$ game and the last round with two players is the same as the "I cut you choose" protocol. The leftover pieces when a part is diminished are assumed to be combined back into the cake.

As explained by Brams & Taylor (1996) the strategy ensuring proportionality is where each player diminishes a part they value exactly $1/n$ or passes the piece forward untouched knowing that another piece must come that is at least $1/n$. But envy-freeness can be assumed true only for the last two players as they can take any of the pieces already diminished. And as with two players, efficiency is not guaranteed as everyone diminishes a piece that they value at $1/n$.

In addition to the last-diminisher procedure some other more efficient algorithms for proportional division have been proposed as the worst case for the last-diminisher procedure is simply $\mathcal{O}\left(\frac{n(n-1)}{2}\right) = \mathcal{O}(n^2)$. Notably Even and Paz (1984) have proposed an $\mathcal{O}(n \log n)$ algorithm which was later shown also to be the lower bound for proportional division (Woeginger & Sgall 2007).

6.2 Brams & Taylor procedure

The next major challenge in the algorithmic literature of fair division was to find an envy-free procedure. Some algorithms have been proposed for three and four players, but the first n player algorithm was discovered in Brams and Taylor (1995).

The algorithm is explained in 14 steps in Brams and Taylor (1995) for four players, and due to the length the reader is referred there for a detailed explanation, but it starts with one player proposing a division, and each player is asked if they objects to proposed allocation.

The issue of Brams & Taylor procedure from an algorithmic point of view is that it is not bounded for the number of queries or cuts. As a result, the number of steps can not be known in advance, and the next major step in the research was to find a bounded algorithm for envy-free division. Additionally, the procedure does not give connected pieces of the cake for the players.

6.3 $n^{n^{n^{n^{n^n}}}}$ queries

In 2008 Stromquist showed that there are no finite algorithms for envy-free cake division for three or more players (Stromquist 2008) and in 2009 Procaccia gave a lower bound of $\Omega(n^2)$[1] for the number steps to find an envy-free division (Procaccia 2009).

The question of what is the algorithm that provides unconnected envy-free division was open until 2016 when Aziz and Mackenzie described an algorithm that has an upper bound of $n^{n^{n^{n^{n^n}}}}$ queries that consist of five protocols (Aziz & Mackenzie 2016). Again due to the complexity of this algorithm, the reader is referred to the original publication for a detailed explanation. Even though this breakthrough result is considerably more complex than the lower bound established by Procaccia, it is significantly better than an unbounded solution and the best to date.

Now the economist might ask that what about Pareto efficiency. The problem with efficiency requirement is that it needs comparisons between the players' measures which in the theoretical results is not a problem, but in practice, the players should be able to give an explicit function representing their utility that could be compared with the functions of the other players. Hence in practice, achieving efficiency is rather difficult. If, however, the utilities of each player are known a Varian-fair allocation may be solved analytically without any effort from the players as done in Weller (1985). In the special case of a one-dimensional interval that needs to be divided into connected pieces envy-free allocation if efficient if preferences are strictly monotonic (Berliant et al. 1992) but as no finite procedures exist for connected pieces the only way is an approximation algorithm described by Su (1999).

7 Conclusion

In this thesis, I tried to open some of the essential aspects of fair division theory since the 1940s. The goals of fair division have evolved along the years, and different notions of fairness have been introduced, but the mathematical framework of measure theory gives a robust way to study the subject. Even though fairness is a reoccurring question in societies, fair division literature as such has seen surprisingly few applications outside of economists' birthday cake division problems.

[1] Note the use of big-omega notation which is not to be confused with the cake Ω.

Fair land division is, however, a subject that has seen these concepts used in a geometrical way with a little different model (Segal-Halevi 2017). Brams and Taylor also cover some possible applications such as auction theory where for example the last-diminisher procedure may be thought as a Dutch auction where the price is diminished until someone takes it (Brams & Taylor 1996). Brams also has a book where the fair division procedures are applied to democracy concepts such as coalition government formation (Brams 2008). One way to view fair division theory from an economics perspective is studying if fair allocations can be achieved in economies where formal models are different from measure theory (Piketty 1994, Pazner & Schmeidler 1974).

The algorithmic study of fair division is an area that has probably seen most attention in recent years where, for example, the discovery of a bounded envy-free algorithm was a significant breakthhrough. Computer scientists have been recently interested in fair division also outside of pure algorithmic study where the object of division can be for example CPU time or memory between different users or processes and the utilities are known (Dolev et al. 2011, Gutman & Nisan 2012).

Many of the results I introduced have been somewhat abstract and at times, challenging to give real-world analogues. The abstractness for their part also makes it difficult to find meaningful applications outside of research in other fields. On the other hand, the precise mathematical language makes it possible to discuss many of the results of fair division literature spanning over seven decades over multiple disciplines with the same formal model.

The multidisciplinarity of fair division literature has been evident in my thesis even though many interesting and notable papers have been left out as I tried to keep a certain economics perspective in my thesis with providing an overview that gives a grasp to pursue further exploration of the subject.

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