

Department of Mathematics and Systems Analysis

Regularity for nonlinear parabolic partial differential equations

Casimir Lindfors

Regularity for nonlinear parabolic partial differential equations

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Abstract

In this thesis we study the regularity of solutions for generalizations of the parabolic p -Laplace equation. The main focus is on equations with Orlicz type growth conditions for which we prove various regularity results, such as local boundedness of both weak solutions and their gradients. Moreover, we show the existence of a continuous solution up to the boundary for the Cauchy-Dirichlet problem. As a by-product we develop new approaches and techniques to handle the difficult nature of the equation that can be both degenerate and singular simultaneously. We also apply the obtained results to prove the existence of a unique solution to the related obstacle problem, and moreover, we show that in case the obstacle is continuous, the solution is as well.

The thesis also contains a section on phase transition problems. More precisely, we study the degenerate two-phase Stefan problem and show that there exists a solution to the Cauchy-Dirichlet problem that is continuous up to the boundary. Moreover, we derive an explicit modulus of continuity at the boundary. The main difficulty stems from the additional degeneracy caused by the jump at the transition point. This is overcome by considering the equation in three different intrinsic geometries instead of the usual one for the p -Laplacian.

The employed methods are mostly based on similar ideas to the ones typically used for the p -Laplace equation, for example De Giorgi's method is applied in many of the proofs. However, due to the generality of the equations, it has been necessary to also find some new tools and ideas.

Keywords partial differential equations, nonlinear analysis, regularity theory, parabolic equations, p -Laplace equation, Orlicz spaces, general growth conditions, regularity of solutions, obstacle problem, phase transition, Stefan problem, boundary regularity, method of intrinsic scaling

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Tekijä

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Väitöskirjan nimi

Epälineaaristen parabolisten osittaisdifferentiaaliyhtälöiden säännöllisyys

Julkaisija Perustieteiden korkeakoulu**Yksikkö** Matematiikan ja systeemianalyysin laitos**Sarja** Aalto University publication series DOCTORAL DISSERTATIONS 62/2018**Tutkimusala** Matematiikka**Käsikirjoituksen pvm** 22.03.2018**Väitöspäivä** 26.04.2018**Julkaisuluvan myöntämispäivä** 20.03.2018**Kieli** Englanti **Monografia** **Artikkeliväitöskirja** **Esseeväitöskirja****Tiivistelmä**

Tässä väitöskirjassa tutkitaan ratkaisujen säännöllisyyttä parabolisen p -Laplace-yhtälön yleistyksille. Keskeisessä osassa ovat yhtälöt, jotka toteuttavat Orlicz-tyyppiset kasvuehdot. Näille todistetaan useita säännöllisyytuloksia, kuten ratkaisujen sekä niiden gradienttien lokaali rajoittuneisuus. Lisäksi osoitetaan reunalle asti jatkuvan ratkaisun olemassaolo Cauchy-Dirichlet-ongelmalle. Sivutuotteena kehitetään uusia lähestymistapoja ja tekniikoita yhtälön, joka voi olla samanaikaisesti sekä degeneroitunut että singulaarinen, hankalan luonteen käsittelyyn. Saatuja tuloksia myös käytetään todistamaan yhtälöön liittyvälle esteongelmalle yksikäsitteisen ratkaisun olemassaolo, ja lisäksi osoitetaan, että jos este on jatkuva, niin ratkaisukin on.

Väitöskirja sisältää myös osion, joka käsittelee faasitransitio-ongelmia. Täsmällisemmin sanottuna siinä tutkitaan degeneroitunutta kaksifaasista Stefan-ongelmaa ja näytetään, että on olemassa Cauchy-Dirichlet-ongelman ratkaisu, joka on jatkuva reunalle asti. Lisäksi johdetaan eksplisiittinen jatkuvuusmoduli reunalla. Todistuksen hankaluus johtuu pääosin transitiopisteessä olevan hypyn aiheuttamasta ylimääräisestä degeneroituneisuudesta. Tämä ratkaistaan tarkastelemalla yhtälöä kolmessa eri geometriassa tavanomaisen p -Laplace-yhtälöön liittyvän geometrian sijaan.

Käytetyt menetelmät perustuvat enimmäkseen samankaltaisiin ideoihin kuin p -Laplace-yhtälölle on tyyppillisesti käytetty, esimerkiksi De Giorgin menetelmää sovelletaan useissa todistuksissa. Yhtälöiden yleisyydestä johtuen on kuitenkin ollut tarpeen löytää myös uusia työkaluja ja ideoita.

Avainsanat osittaisdifferentiaaliyhtälöt, epälineaarinen analyysi, säännöllisyysteoria, paraboliset yhtälöt, p -Laplace-yhtälö, Orlicz-avaruudet, yleiset kasvuehdot, ratkaisujen säännöllisyys, esteongelma, faasitransitio, Stefan ongelma, reunasäännöllisyys, intrinsic scaling -menetelmä

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Preface

First, I would like to express my deepest gratitude towards my advisor Professor Tuomo Kuusi for his patient guidance throughout the process of writing this thesis. I especially appreciate how understanding and encouraging he has been during particularly difficult times. I would also like to thank him for introducing me to the world of PDEs and for sharing his knowledge in an enthusiastic, yet understandable, way.

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Espoo, March 21, 2018,

Casimir Lindfors

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List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

- I** Paolo Baroni and Casimir Lindfors. The Cauchy-Dirichlet problem for a general class of parabolic equations. *Annales de l'Institut Henri Poincaré (C) Analyse Non Linéaire*, Volume 34, Issue 3, pages 593–624, DOI: 10.1016/j.anihpc.2016.03.003, May–June 2017.
- II** Casimir Lindfors. Obstacle problem for a class of parabolic equations of generalized p -Laplacian type. *Journal of Differential Equations*, Volume 261, Issue 10, pages 5499–5540, DOI: 10.1016/j.jde.2016.08.015, November 2016.
- III** Paolo Baroni, Tuomo Kuusi, Casimir Lindfors and José Miguel Urbano. Existence and boundary regularity for degenerate phase transitions. *SIAM Journal on Mathematical Analysis*, Volume 50, Issue 1, pages 456–490, DOI: 10.1137/17M1121585, January 2018.

Author's Contribution

Publication I: "The Cauchy-Dirichlet problem for a general class of parabolic equations"

The author has performed a substantial part of the research

Publication II: "Obstacle problem for a class of parabolic equations of generalized p -Laplacian type"

The author has performed all of the research

Publication III: "Existence and boundary regularity for degenerate phase transitions"

The author has performed a substantial part of the research

1. Introduction

In this thesis we study generalizations of the nonlinear parabolic partial differential equation

$$\partial_t u - \operatorname{div}(|Du|^{p-2}Du) = 0, \quad 1 < p < \infty, \quad (1.0.1)$$

also known as the evolutionary p -Laplace equation. The p -Laplace equation is used to model for example the heat distribution in a given region over time. Other applications include filtration of non-Newtonian fluids through a rigid porous medium and asset pricing in finance.

When $p = 2$ equation (1.0.1) is the heat equation. The heat equation is a linear PDE and it is well understood. Also the evolutionary p -Laplace equation has been widely studied, but it has been necessary to develop several new techniques to overcome its nonlinearity. Moreover, many of these techniques differ significantly depending on the size of p . In particular, the equation is degenerate when $p > 2$ and singular when $p < 2$, causing the equation to behave quite differently in these two cases.

Equation (1.0.1) can be further generalized by replacing the polynomial growth of the gradient in the elliptic part with more general growth. Our focus is on equations with Orlicz type growth, that is,

$$\partial_t u - \operatorname{div}\left(\frac{g(|Du|)}{|Du|}Du\right) = 0, \quad (1.0.2)$$

where g is an increasing C^1 function satisfying the so-called Orlicz condition

$$g_0 - 1 \leq \frac{sg'(s)}{g(s)} \leq g_1 - 1, \quad s > 0, \quad (1.0.3)$$

for some $1 < g_0 \leq g_1 < \infty$. The p -Laplace equation can be retrieved from (1.0.2) by taking $g(s) = s^{p-1}$, which corresponds to $g_0 = g_1 = p$. Equation (1.0.2) has applications for example in mechanics, fluid dynamics and magnetism, where a more complex model is needed to accurately describe certain phenomena.

Our aim is to study the regularity of weak solutions to (1.0.2). This obviously includes all the same issues as the p -Laplacian but also some new ones. The main difficulty caused by the general nature of the equation is the fact that, whereas the

p -Laplace equation is always either degenerate or singular, equation (1.0.2) can be both at the same time, and moreover so that it can oscillate between the two cases depending on the size of the gradient. Indeed, we strive to find methods that can handle both cases at the same time.

Another generalization of equation (1.0.1) that finds immediate applications in physics is the so-called Stefan problem, which describes for example the evolution of heat distribution in a system where phase transitions occur. Motivation for phase transition problems comes from the fact that, while in a system with a single phase the temperature is linearly proportional to its thermal energy, this is no longer the case in multi-phase systems due to part of the energy being spent on (or released from) the phase transitions without changing the temperature. This can be modeled by adding suitable jumps to the parabolic part of (1.0.1) accounting for the phase transitions and corresponding latent heats.

In this thesis we concentrate on the degenerate ($p \geq 2$) two-phase Stefan problem

$$\partial_t(u + \mathcal{L}_h H_a(u)) \ni \operatorname{div}(|Du|^{p-2} Du), \quad (1.0.4)$$

where

$$H_a(s) = \begin{cases} 0 & \text{if } s < a \\ [0, 1] & \text{if } s = a \\ 1 & \text{if } s > a \end{cases}$$

is the Heaviside graph centered at $a \in \mathbb{R}$, and $\mathcal{L}_h > 0$ is the latent heat. The regularity of solutions to (1.0.4) is well known when $p = 2$, but little is known for the degenerate case. The difficulty stems from the fact that not only do we have to handle the degeneracy of the p -Laplacian, but also the one caused by the jump.

The thesis is organized as follows. In Chapter 2 we introduce the evolutionary p -Laplace equation and give a brief overview of the regularity results related to this thesis known thus far. Chapter 3 discusses the equations with Orlicz type growth and the challenges brought by the generalization. In Chapter 4 we summarize the results and methods used in Publications I and II. Chapter 5 is devoted to the phase transition problems, and finally, in Chapter 6 we summarize the work done in Publication III. The last part of the thesis contains the original publications.

2. Evolutionary p -Laplace equation

The evolutionary p -Laplace equation is a parabolic PDE that can be used to model physical phenomena such as the evolution of heat distribution in a given domain. It is defined as

$$\partial_t u - \operatorname{div}(|Du|^{p-2} Du) = 0 \quad \text{in } \Omega_T, \quad (2.0.1)$$

where $\Omega_T := \Omega \times (0, T) \subset \mathbb{R}^{n+1}$, $u : \Omega_T \rightarrow \mathbb{R}$ is the solution to the equation, ∂_t denotes partial derivative with respect to the time variable t , div divergence with respect to the spatial variable x , and Du is the spatial gradient of u . The parameter p can be between one and infinity, although often a narrower range is considered.

In the case $p = 2$ equation (2.0.1) is the well-known heat equation

$$\partial_t u - \Delta u = 0,$$

where Δ is the usual Laplace operator. Due to the linear structure of the heat equation there is a wide range of techniques available for studying it. A characteristic property of the heat equation is that its solutions have infinite speed of propagation, meaning that any disturbances in the initial values ($t = 0$) will have spread everywhere for any $t > 0$. In reality nothing can be faster than the speed of light, which motivates to study more general equations such as the evolutionary p -Laplace equation that indeed has finite speed of propagation when $p \neq 2$. A comprehensive presentation on the heat equation can be found for example in [24], see also [25, 38] for linear equations with more general coefficients.

Whenever $p \neq 2$, equation (2.0.1) is nonlinear and thus several techniques used for the heat equation become unavailable. Adding to the difficulty, in the case $p > 2$ the modulus of ellipticity of the spatial part $|Du|^{p-2}$ vanishes at points where $|Du| = 0$ making the equation degenerate, and for $1 < p < 2$ the equation becomes singular, since the modulus of ellipticity is unbounded at such points. This causes the equation to behave very differently in these two cases, which leads to the proofs of many results to be quite different as well.

2.1 Weak solutions

Rigorously speaking writing equation (2.0.1) as such would require u to have at least one time derivative and two spatial derivatives. This is quite restrictive and proving existence of a solution with such regularity requirements could be very hard, or even impossible. Therefore one usually considers so-called *weak solutions* that *a priori* are not differentiable in time and only possess one weak derivative in space.

There are many ways to define weak solutions but they all share the same idea of multiplying the equation by a suitable test function, then integrating over a space-time cylinder, and finally integrating by parts to move one derivative from the solution to the test function. This method works not only for the p -Laplacian but for equations in divergence form in general. Weak sub- and supersolutions are defined similarly, but instead of equality we have inequality and the test functions are assumed nonnegative. See (3.2.1) for a definition of weak solutions in a more general setting.

2.2 Boundedness of solutions

Existence of weak solutions to (2.0.1) with given boundary values can be proved using for example Galerkin approximations, we refer to [38, 43]. After the existence of a solution is established the natural question to ask is how regular the solutions are. Local boundedness of solutions is a consequence of the following *a priori* estimates by Porzio [46], see also [17]. In the degenerate case ($p > 2$) we have

$$\operatorname{ess\,sup}_{\frac{1}{2}Q(r,r^p)} u \leq \max \left\{ c(n, p) \left(\int_{Q(r,r^p)} u^p \, dx \, dt \right)^{1/2}, 1 \right\}, \quad (2.2.1)$$

where $Q(r, r^p) = B_r(x_0) \times (t_0 - r^p, t_0) \Subset \Omega_T$, $r > 0$, is the natural space-time cylinder related to parabolic p -Laplace type equations and $\frac{1}{2}Q(r, r^p) = Q(r/2, r^p/2)$. In the singular case ($p < 2$) we need an additional lower bound for p ; indeed, the boundedness may fail unless $p > 2n/(n+2)$. This is often called the critical exponent. For the supercritical range $2n/(n+2) < p < 2$ the *a priori* estimate reads as

$$\operatorname{ess\,sup}_{\frac{1}{2}Q(r,r^p)} u \leq \max \left\{ c(n, p) \left(\int_{Q(r,r^p)} u^2 \, dx \, dt \right)^{p/(p(n+2)-2n)}, 1 \right\}. \quad (2.2.2)$$

Observe that estimates (2.2.1) and (2.2.2) are not homogeneous in u , meaning that the estimates do not stay the same, if u is multiplied by a constant. This is a characteristic property of evolutionary p -Laplace type equations.

2.3 Intrinsic scaling

Due to the inhomogeneity of the equation the question whether weak solutions are Hölder continuous stayed open for a long time. A positive answer was provided by DiBenedetto in [16] for the degenerate case and some time later also for the singular case [10]. The proofs for these two cases are significantly different, which inevitably stems from the different nature of the equation, but the main idea, so-called *intrinsic scaling*, is the same. The key point is to consider the equation in a geometry that somehow depends on the solution itself, such that, in this specific geometry, the equation looks like the heat equation.

The correct geometry can be seen by looking at the scaling of the equation. Consider equation (2.0.1) in the cylinder $B_R(0) \times (-T, 0)$ with $R, T > 0$. If we now scale the solution u by defining

$$v(x, t) := \frac{u(Rx, Tt)}{\omega}$$

for $\omega > 0$, then v is a solution to equation (2.0.1) in the unit cylinder $B_1(0) \times (-1, 0)$ if and only if

$$T = \omega^{2-p} R^p.$$

Thus, whenever $p \neq 2$, the time scale depends on the scaling factor ω , which is often chosen to depend on the oscillation of u , making the geometry indeed intrinsic. The geometry also provides more insight on the different nature of the degenerate and singular cases, since (ω usually being very small) when $p > 2$, the cylinders are long and thin, while in the case $p < 2$ they are flat, requiring rather different methods for handling them. See [17, 52] for more on the subject.

The intrinsic geometry makes it possible to balance the inhomogeneity of the equation, making it in some sense indeed behave like the heat equation in this geometry. This allows the use of De Giorgi's iteration method, which was first used to prove Hölder continuity for uniformly elliptic linear PDEs [13]. In fact, the linearity plays no role in the proof, and thus the method could be applied to more general equations, such as the elliptic p -Laplace equation as done in [39] by Ladyzenskaja and Ural'tzeva, and even the parabolic one as shown by DiBenedetto.

2.4 Higher regularity

Similar tools can also be used for obtaining higher regularity results. When proving results for the solution u , the only information that is taken from the equation is the so-called Caccioppoli inequality, which allows estimating the gradient of u with u itself. In order to do the same for the gradient, the key is to show that $v := |Du|^2$ is a

nonnegative subsolution to a parabolic equation of porous medium type, to be precise

$$\partial_t v - \operatorname{div} \left(v^{\frac{p-2}{2}} A(Du) Dv \right) = 0, \quad (2.4.1)$$

where

$$A(Du) := I + (p-2) \frac{Du \otimes Du}{|Du|^2}.$$

This is done by formally differentiating equation (2.0.1) with respect to x_k and then summing over k . Also (2.4.1) is in divergence form, so it is possible to derive another Caccioppoli inequality using similar methods to get the machinery started again.

Lieberman proved in [40] that also the gradient of a solution to (2.0.1) is bounded and, moreover, in [17] it is shown that it satisfies the *a priori* estimate

$$\operatorname{ess\,sup}_{\frac{1}{2}Q(r,r^2)} |Du| \leq \max \left\{ c(n,p) \left(\int_{Q(r,r^2)} |Du|^p dx dt \right)^{1/2}, 1 \right\}, \quad (2.4.2)$$

when $p > 2$, and

$$\operatorname{ess\,sup}_{\frac{1}{2}Q(r,r^2)} |Du| \leq \max \left\{ c(n,p) \left(\int_{Q(r,r^2)} |Du|^p dx dt \right)^{2/(p(n+2)-2n)}, 1 \right\}, \quad (2.4.3)$$

when $2n/(n+2) < p < 2$. Furthermore, the gradient is Hölder continuous, as shown by DiBenedetto and Friedman [19], see also [17].

2.5 Global regularity

Often equation (2.0.1) is studied together with some additional boundary condition, a typical example being the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u - \operatorname{div}(|Du|^{p-2} Du) = 0 & \text{in } \Omega_T, \\ u = \psi & \text{on } \partial_p \Omega_T, \end{cases} \quad (2.5.1)$$

where ψ is the boundary value function with suitable regularity assumptions and $\partial_p \Omega_T = (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times (0, T))$ is the parabolic boundary of Ω_T . Moreover, the domain Ω is usually assumed to be bounded and to satisfy certain regularity properties. The interpretation of $u = \psi$ depends on the context; in the case of continuous functions we may consider pointwise equality, but if ψ is for example only assumed to be a Sobolev function, $u = \psi$ should be understood in the sense of traces on the lateral boundary, and on the initial boundary one typically considers L^2 initial values, see [17].

Global boundedness of solutions to (2.5.1) can be proved using essentially the same methods as in the local case. Also the corresponding *a priori* estimates are very similar. Moreover, it is possible to show continuity up to the boundary with minor tweaks to the interior proof. We once again refer to [17].

2.6 Obstacle problems

Another closely related problem is the obstacle problem asking for the smallest weak supersolution to (2.0.1) lying above a given obstacle function. Besides being a useful tool in PDEs and potential theory, obstacle problems also have numerous applications in many other areas of science, including physics, chemistry, biology, and even finance. See [5, 32] for an overview of the classical theory and applications.

Existence of a solution to the obstacle problem related to equation (2.0.1) with a continuous obstacle was proved by Korte, Kuusi and Siljander in [34]. Moreover, they showed that also the solution is continuous. For more irregular but still bounded obstacles the existence result was obtained by Lindqvist and Parviainen [44] using potential theoretical arguments. An essential part of their proof was the fact that weak supersolutions to (2.0.1) have a lower semicontinuous representative, which was proved by Kuusi in [35]. Possibly unbounded obstacles are treated in [8] by Bögelein, Duzaar, and Mingione, in fact, they only assume that the time derivative of the obstacle is in $L^{p'}(\Omega_T)$. Even further improvements have been made by Scheven, see for example [48].

3. Equations of generalized p -Laplacian type

Sometimes it might be of interest to find more precise models for the complex phenomena of nature. This, in addition to purely mathematical curiosity, motivates to study more general PDEs. A natural way to generalize the elliptic p -Laplace equation was introduced by Lieberman in [41], and this idea can be extended to parabolic equations as well.

Indeed, consider the equation

$$\partial_t u - \operatorname{div} \left(\frac{g(|Du|)}{|Du|} Du \right) = 0 \quad \text{in } \Omega_T, \quad (3.0.1)$$

where $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a C^1 function satisfying the so-called Orlicz condition

$$g_0 - 1 \leq \frac{sg'(s)}{g(s)} \leq g_1 - 1, \quad s > 0, \quad (3.0.2)$$

with $1 < g_0 \leq g_1 < \infty$. When g_0 and g_1 are both equal to, say, p , (3.0.2) reduces to a simple ordinary differential equation whose solution is $g(s) = s^{p-1}$ up to a constant factor, giving back the p -Laplace equation. Analogously to the p -Laplacian the lower bound for g_0 is often taken to be the critical exponent $2n/(n+2)$.

3.1 Difficulties caused by the generalization

Equation (3.0.1) can be studied using many of the same techniques that are used for the evolutionary p -Laplace equation. Some properties of the power functions, such as the multiplicative property $a^p b^p = (ab)^p$, are no longer available for the function g , and thus some extra effort is required. The main difficulty compared to the p -Laplace case, however, is the fact that the general nature of the function g makes it possible for equation (3.0.1) to be both degenerate and singular. Indeed, this might happen when $g_0 < 2$ and $g_1 > 2$, and moreover, g could oscillate wildly between the two cases, see [41] and Publication I. Concretely this issue often comes up in the calculations when monotonicity of mappings like $s \mapsto g(s)/s$ would be needed but is not available, unlike in the p -Laplace case where $s \mapsto s^{p-2}$ is always monotone - either increasing or decreasing, depending on p .

3.2 Weak solutions and Orlicz spaces

Equation (3.0.1) should again be understood in a weak sense. We say that u is a *weak solution* to (3.0.1) in a space-time cylinder $\mathcal{K} = \mathcal{D} \times (t_1, t_2) \subset \Omega_T$, if u is in a suitable function space and satisfies the integral identity

$$-\int_{\mathcal{K}} u \partial_t \varphi \, dx \, dt + \int_{\mathcal{K}} \frac{g(|Du|)}{|Du|} Du \cdot D\varphi \, dx \, dt = 0 \quad (3.2.1)$$

for every compactly supported smooth test function φ . Weak supersolutions (sub-solutions) are defined by replacing $=$ with \geq (\leq) and considering nonnegative test functions only.

The correct space can be seen by formally testing equation (3.2.1) with $\varphi = u$ and requiring that the emerging terms be finite. The first term is finite, if u is in the space $L^\infty(t_1, t_2; L^2(\mathcal{D}))$ that consists of functions in $L^2(\mathcal{D})$ whose L^2 norm is finite for almost every $t \in (t_1, t_2)$. For the second term we need so-called Orlicz spaces which in a way generalize the L^p spaces. Let $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be the antiderivative of g , that is,

$$G(s) := \int_0^s g(\sigma) \, d\sigma.$$

We say that u belongs to the Orlicz space $L^G(A)$, if $\int_A G(|u|) \, dx < \infty$. Moreover, if u has a weak spatial gradient Du that also belongs to $L^G(A)$, then u is in the Orlicz-Sobolev space $W^{1,G}(A)$. For more on Orlicz spaces see for example [1].

Now, since also the function G satisfies the Orlicz condition (3.0.2) with the bounds g_0 and g_1 , the second term of (3.2.1) after testing with the solution u is finite, if $Du \in L^G(\mathcal{K})$. We call this space $V^G(\mathcal{K})$ and we denote by

$$V^{2,G}(\mathcal{K}) := L^\infty(t_1, t_2; L^2(\mathcal{D})) \cap V^G(\mathcal{K})$$

the space in which we assume the solution u to be. In the case of the p -Laplacian ($G(s) \approx s^p$) we have

$$V^{2,p}(\mathcal{K}) := L^\infty(t_1, t_2; L^2(\mathcal{D})) \cap L^p(t_1, t_2; W^{1,p}(\mathcal{D})). \quad (3.2.2)$$

3.3 Regularity of solutions

The regularity of weak solutions to the elliptic counterpart of (3.0.1) is well understood and has been studied by several authors along with Lieberman, see for example [3, 11, 21, 23]. In the parabolic case, however, only little is known due to the difficulties described above. In [42] Lieberman proved that if the gradient of a solution is bounded, then it is Hölder continuous. The missing link is provided in Publication I by showing the existence of a solution to the Cauchy-Dirichlet problem related

to equation (3.0.1) and that its gradient is indeed locally bounded. A similar result, also for systems, was obtained independently by Diening, Scharle and Schwarzacher in [22] using a different approach.

Results obtained for equation (3.0.1) can be quite easily extended to cover more general equations

$$\partial_t u - \operatorname{div} \mathcal{A}(Du) = 0 \quad \text{in } \Omega_T, \quad (3.3.1)$$

where \mathcal{A} is a vector field satisfying certain structural conditions that model same type of growth as the prototype $\xi \mapsto \frac{g(|\xi|)}{|\xi|} \xi$. When studying the gradient, one needs to differentiate the equation, and thus the vector field \mathcal{A} should be assumed C^1 regular.

In this case the structural conditions read as

$$\begin{cases} \langle D\mathcal{A}(\xi)\lambda, \lambda \rangle \geq \nu \frac{g(|\xi|)}{|\xi|} |\lambda|^2 \\ |D\mathcal{A}(\xi)| \leq L \frac{g(|\xi|)}{|\xi|} \end{cases} \quad (3.3.2)$$

for any $\xi \in \mathbb{R}^n \setminus \{0\}$, $\lambda \in \mathbb{R}^n$ and with constants $0 < \nu \leq 1 \leq L$. When dealing with the solution u , we may consider weaker assumptions of the form

$$\begin{cases} \langle \mathcal{A}(\xi), \xi \rangle \geq \tilde{\nu} G(|\xi|) \\ |\mathcal{A}(\xi)| \leq \tilde{L} \frac{G(|\xi|)}{|\xi|} \end{cases} \quad (3.3.3)$$

for $\xi \in \mathbb{R}^n \setminus \{0\}$ with constants $0 < \tilde{\nu} \leq 1 \leq \tilde{L}$; in this case \mathcal{A} is only assumed continuous.

In Publication II we prove local boundedness of solutions to equation (3.3.1) under the structural conditions (3.3.3) whenever $g_0 > 2n/(n+2)$. It still remains a difficult open problem to show that bounded solutions are also Hölder continuous for the whole range of exponents. However, Lieberman and Hwang proved this in the degenerate ($g_0 \geq 2$) and singular ($g_1 \leq 2$) cases separately, see [28, 29], leaving the intermediate case $g_0 < 2 < g_1$ still open.

4. Summaries of Publications I–II

4.1 Publication I

In Publication I we consider the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u - \operatorname{div} \mathcal{A}(Du) = 0 & \text{in } \Omega_T, \\ u = \psi & \text{on } \partial_p \Omega_T, \end{cases} \quad (4.1.1)$$

where \mathcal{A} is a C^1 vector field satisfying the Orlicz growth conditions (3.3.2) and the boundary value function ψ is continuous. Instead of $g_0 > 2n/(n+2)$ we assume the weaker condition

$$g(s) \geq c_\ell s^{\frac{n-2}{n+2} + \epsilon} \quad (4.1.2)$$

for some $c_\ell, \epsilon > 0$ and any $s \geq 1$. Moreover, we assume that the boundary of Ω is $C^{1,\alpha}$ regular. We define the solution to problem (4.1.1) as a function $u \in C(\overline{\Omega}_T)$ that is a weak solution to (4.1.1)₁ in Ω_T and satisfies (4.1.1)₂ pointwise on $\partial_p \Omega_T$.

We prove two main results. First is the existence of a unique solution to problem (4.1.1). This is achieved by regularizing the equation, proving continuity up to the boundary for the regularized equation independent of the regularization parameter, and finally applying the theorem of Ascoli-Arzelà to find a suitable candidate that also turns out to solve the original problem. As a by-product we obtain the second result, which states that the gradient of u is locally bounded.

4.2 Regularization

The regularization method we employ is a combination of mollification and adding a nondegenerate p -Laplace type viscosity term. To be precise, we define for $\varepsilon \in (0, 1)$

$$\mathcal{A}_\varepsilon(\xi) := (\phi_\varepsilon * \mathcal{A})(\xi) + \varepsilon(1 + |\xi|)^{\tilde{g}_1 - 2} \xi, \quad (4.2.1)$$

where $\phi_\varepsilon(\xi) = \phi(\xi/\varepsilon)/\varepsilon^n$ and ϕ is a standard mollifier with $\int_{\mathbb{R}^n} \phi \, dx = 1$. The number \tilde{g}_1 is chosen larger than both g_1 and 2 so that near infinity the latter term dominates. The vector field \mathcal{A}_ε satisfies the structural conditions (3.3.2) (with some other constants) when g is replaced with

$$g_\varepsilon(s) := \frac{g(s + \varepsilon)}{s + \varepsilon} s + \varepsilon(1 + s)^{\tilde{g}_1 - 2} s. \quad (4.2.2)$$

It is important to observe that g_ε satisfies the Orlicz condition (3.0.2) with constants independent of ε . This enables us to eventually prove uniform estimates in ε .

Consider now solutions u_ε to the Cauchy-Dirichlet problem related to the regularized equation

$$\begin{cases} \partial_t u_\varepsilon - \operatorname{div} \mathcal{A}_\varepsilon(Du_\varepsilon) = 0 & \text{in } \Omega_T, \\ u_\varepsilon = \psi & \text{on } \partial_p \Omega_T. \end{cases} \quad (4.2.3)$$

Existence and uniqueness of such solutions can be obtained from the literature, see for example [30]. Moreover, since the vector field \mathcal{A}_ε also satisfies nondegenerate p -Laplacian growth conditions with $p = \tilde{g}_1 > 2$, by standard theory [17] the solution u_ε enjoys further regularity; for our purposes it suffices that

$$u_\varepsilon, Du_\varepsilon \in C(\Omega_T), \quad u_\varepsilon \in L^2_{\text{loc}}(0, T; W^{2,2}_{\text{loc}}(\Omega)). \quad (4.2.4)$$

4.3 Boundedness of the gradient

4.3.1 Differentiating the equation

With the extra regularity at hand, we may differentiate the equation in order to get our hands on the gradient. To be more precise, we show that the function

$$v := |Du_\varepsilon|^2$$

is a weak subsolution to the similar equation

$$\partial_t v - \operatorname{div}(D\mathcal{A}_\varepsilon(Du_\varepsilon)Dv) = 0 \quad \text{in } \Omega_T. \quad (4.3.1)$$

The aim is to prove an L^∞ bound for v (and hence for the gradient of u_ε) not depending on ε using a standard De Giorgi iteration. We follow closely the proof of the corresponding result for the evolutionary p -Laplacian presented in [36] by Kuusi and Mingione.

As usual, the first step when studying a new PDE is deriving a Caccioppoli type energy estimate. This is done by testing the weak formulation of equation (4.3.1)

with essentially the solution v itself and using Young's inequality and the structural conditions. As a result we obtain the inequality

$$\begin{aligned} \sup_{\tau \in (t_1, t_2)} \int_{\mathcal{D}} [(v-k)_+^2 \varphi^2](\cdot, \tau) dx + \int_{\mathcal{K}} \frac{g_\varepsilon(|Du_\varepsilon|)}{|Du_\varepsilon|} |D(v-k)_+|^2 \varphi^2 dx dt \\ \leq c \int_{\mathcal{K}} (v-k)_+^2 \left(\frac{g_\varepsilon(|Du_\varepsilon|)}{|Du_\varepsilon|} |D\varphi|^2 + |\partial_t \varphi| \right) dx dt, \end{aligned} \quad (4.3.2)$$

which holds for every $\mathcal{K} = \mathcal{D} \times (t_1, t_2) \Subset \Omega_T$, $k \in \mathbb{R}$ and $\varphi \in C^\infty(\mathcal{K})$ vanishing in a neighborhood of $\partial_p \mathcal{K}$.

4.3.2 The intrinsic geometry and reverse Hölder's inequality

Inequality (4.3.2) looks a lot like the Caccioppoli inequality for the heat equation, the only difference being the weight $g_\varepsilon(|Du_\varepsilon|)/|Du_\varepsilon|$ appearing in two terms. Indeed, the key idea is to essentially consider the weight as a constant by somehow controlling it from below and from above, and thus treating the equation as though it were the heat equation.

In practise this is done as follows. Take a cylinder $Q_\rho(x_0, t_0) \subset \Omega_T$, let $\lambda \geq 1$ be such that

$$\lambda \geq \frac{1}{4} \sup_{Q_\rho(x_0, t_0)} |Du_\varepsilon|, \quad (4.3.3)$$

and set

$$\theta_\lambda := \frac{g_\varepsilon(\lambda)}{\lambda}.$$

We introduce the intrinsic cylinder

$$Q_\rho^\lambda \equiv Q_\rho^\lambda(x_0, t_0) := \begin{cases} B_\rho(x_0) \times (t_0 - \theta_\lambda^{-1} \rho^2, t_0), & \theta_\lambda \geq 1 \\ \theta_\lambda^{1/2} B_\rho(x_0) \times (t_0 - \rho^2, t_0), & 0 < \theta_\lambda < 1, \end{cases} \quad (4.3.4)$$

intrinsic meaning that the cylinder depends on the solution itself, in this case through (4.3.3).

Observe that when $g_\varepsilon(s) = s^{p-1}$, Q_ρ^λ reduces to the intrinsic cylinder used to handle the parabolic p -Laplacian.

Now in the level set $Q_\rho^\lambda \cap \{v \geq k\}$ for $k \geq \lambda^2$ we have

$$\frac{g_\varepsilon(|Du_\varepsilon|)}{|Du_\varepsilon|} \approx \theta_\lambda$$

due to (4.3.3), making the weights in the Caccioppoli inequality in a sense constant. By combining the Caccioppoli inequality with Sobolev inequality, we obtain an intrinsic reverse Hölder's inequality

$$\left(\int_{Q_{\rho/2}^\lambda} (v-k)_+^{2\gamma} dx dt \right)^{1/(2\gamma)} \leq c \left(\int_{Q_\rho^\lambda} (v-k)_+^q dx dt \right)^{1/q},$$

where $q > 0$ and

$$\gamma := 2 - \frac{2}{2^*} > 1, \quad 2^* := \begin{cases} \frac{2n}{n-2} & n > 2, \\ 4 & n = 2. \end{cases}$$

All the radii ρ and also θ_λ balance out thanks to the clever choice of geometry; indeed, the constant c depends only on the data and q . Finally, a De Giorgi iteration yields the pointwise estimate

$$|Du_\varepsilon(x_0, t_0)| \leq \lambda + c \left(\int_{Q_\rho^\lambda(x_0, t_0)} (|Du_\varepsilon|^2 - \lambda^2)_+^q dx dt \right)^{1/(2q)}. \quad (4.3.5)$$

4.3.3 Towards a uniform bound

In order to get a uniform bound for Du_ε , the right hand side of (4.3.5) still needs to be estimated from above independent of ε ; note that also λ and Q_ρ^λ have an ε dependence. This is done by considering two different cases depending on which term of g_ε dominates at λ and choosing the exponent q accordingly. Observing also that $Q_\rho^\lambda \subset Q_\rho := B_\rho(x_0) \times (t_0 - \rho^2, t_0)$ we obtain a nonintrinsic version of (4.3.5), and some real analytic tools eventually yield

$$\|Du_\varepsilon\|_{L^\infty(Q_R)} \leq c \left(\int_{Q_{2R}} [G_\varepsilon(|Du_\varepsilon|) + 1] dx dt \right)^{\tilde{\eta}}, \quad (4.3.6)$$

where $Q_{2R} \Subset \Omega_T$, G_ε is the antiderivative of g_ε , and $\tilde{\eta} := \max\{\frac{1}{2}, \frac{2}{\varepsilon(n+2)}\}$.

To conclude, we prove another Caccioppoli inequality, this time for equation (4.2.3)₁, and this allows us to essentially control the integral of $G_\varepsilon(|Du_\varepsilon|)$ by integrals of $G_\varepsilon(|u_\varepsilon|)$ and u_ε^2 (see (4.6.2)). These can be then uniformly bounded using the maximum principle, since $u_\varepsilon = \psi$ on the boundary of Ω_T for every ε .

4.4 The existence result

The local boundedness of Du_ε readily implies local Lipschitz continuity of u_ε , and since the bound is independent of ε , also the continuity is uniform. In addition to this, we prove uniform continuity of u_ε up to the boundary of Ω_T by building explicit barriers and utilizing the assumed $C^{1,\alpha}$ regularity of $\partial\Omega$. The achieved global equicontinuity together with equiboundedness (again by the maximum principle) allows us to apply the theorem of Ascoli-Arzelà (see for example [18]) to find a function $u \in C(\overline{\Omega}_T)$ such that the functions u_ε converge uniformly to u . The proof is finished by showing that $Du_\varepsilon \rightarrow Du$ almost everywhere implying that u is indeed a solution to the original problem (4.1.1). Now by taking $\varepsilon \rightarrow 0$ in the previous

results for u_ε - recalling that all the constants are independent of ε - we obtain the corresponding results for u . In particular, u is locally Lipschitz continuous and its gradient satisfies the estimate

$$\|Du\|_{L^\infty(Q_R)} \leq c \left(\int_{Q_{2R}} [G(|Du|) + 1] dx dt \right)^{\max\left\{\frac{1}{2}, \frac{2}{\varepsilon(n+2)}\right\}}.$$

4.5 Publication II

In Publication II we study the same equation as in Publication I, that is,

$$\partial_t u - \operatorname{div} \mathcal{A}(Du) = 0 \quad \text{in } \Omega_T \quad (4.5.1)$$

with \mathcal{A} as in (4.1.1), but instead of the gradient of the solution we focus on the solution itself. We prove several smaller results for weak (super)solutions and as one of the main results we show that the solutions are locally bounded. Moreover, we obtain an interesting *a priori* result that is also used to prove the lower semicontinuity of weak supersolutions. Everything is put together to prove the existence of a unique solution to the related obstacle problem, furthermore showing that if the obstacle is continuous, then so is the solution.

Most of the results hold also for vector fields \mathcal{A} satisfying the weaker assumptions (3.3.3). The main reason for assuming (3.3.2) is the fact that we use the results in Publication I in proving the existence result for the obstacle problem. On the other hand, in Publication II we are required to assume $g_0 > 2n/(n+2)$ instead of the weaker condition (4.1.2).

4.6 Boundedness of the solution

Being a generalization of the evolutionary p -Laplace equation, the proof of the boundedness of solutions to (4.5.1) uses - as one might expect - many of the same ideas as for example DiBenedetto in [17] for the p -Laplacian. The general guideline consists of a Caccioppoli inequality, Sobolev inequality and De Giorgi's method. However, the more general structure also requires some new ideas, and moreover, it leads to the result being only qualitative.

Again, a key element of the proof is finding the correct geometry. Looking at the cylinders $Q(\rho, \theta) := B_\rho(x_0) \times (t_0 - \theta, t_0)$ with $\theta = k^{2-p}\rho^p$ and $k > 0$ used for the parabolic p -Laplace equation, it would seem that taking $\theta = k^2/G(k)G(\rho)$ is a good candidate. However, it turns out this is not the case. By studying the scaling of the

equation we find that the correct time scale is

$$\theta = k^2 G\left(\frac{k}{\rho}\right)^{-1}. \quad (4.6.1)$$

4.6.1 De Giorgi's method

Let u be a nonnegative subsolution. When proving the boundedness result for the p -Laplacian, the method used differs slightly depending on the size of p . In the degenerate case ($p > 2$) the quantity we want to iterate is essentially $\int_{Q_j} (u - k_j)_+^p dx dt$, where Q_j and k_j are suitably chosen sequences, while in the singular case ($p < 2$) we have $\int_{Q_j} (u - k_j)_+^2 dx dt$. Since our equation covers both of these cases and more, we want to find a way to handle everything at the same time. This turns out to be possible using the quantity

$$Y_j := G\left(\frac{k}{\rho}\right)^{-1} \int_{Q_j} \left(G\left(\frac{(u - k_j)_+}{\rho}\right) + \frac{(u - k_j)_+^2}{\theta} \right) dx dt,$$

where

$$k_j = (1 - 2^{-j})k, \quad Q_j := Q(\rho_j, \theta_j),$$

and

$$\rho_j = (1 + 2^{-j})\frac{\rho}{2}, \quad \theta_j = (1 + 2^{-j})\frac{\theta}{2}.$$

In order to use De Giorgi's method we need the Caccioppoli inequality

$$\begin{aligned} & \frac{1}{\theta_j} \operatorname{ess\,sup}_{(t_0 - \theta_j, t_0)} \int_{B_{\rho_j}} (u - k_j)_+^2 \varphi_j^{g_1} dx + \int_{Q_j} G(|D(u - k_j)_+|) \varphi_j^{g_1} dx dt \\ & \leq c \int_{Q_j} G(|D\varphi_j|(u - k_j)_+) dx dt + c \int_{Q_j} (u - k_j)_+^2 |\partial_t \varphi_j| dx dt, \end{aligned} \quad (4.6.2)$$

where φ_j are suitably chosen cutoff functions. This is proved completely analogously to the corresponding inequality for the p -Laplacian. Moreover, we need a new parabolic Sobolev inequality

$$\begin{aligned} & \int_{B_R \times \Gamma} G\left(\frac{|u|}{R}\right)^{1/q} |u|^{2(1-1/q^*)} dx dt \\ & \leq c \operatorname{ess\,sup}_{\Gamma} \left(\int_{B_R} |u|^2 dx \right)^{1-1/q^*} \left(\int_{B_R \times \Gamma} G(|Du|) dx dt \right)^{1/q} \end{aligned} \quad (4.6.3)$$

that holds for every $u \in V_0^{2,G}(B_R \times \Gamma)$, $1 \leq q < \min\{n, g_0\}$, and $B_R \times \Gamma \subset \mathbb{R}^{n+1}$.

Here q^* denotes the Sobolev conjugate of q , that is,

$$q^* := \frac{nq}{n - q}.$$

The proof of (4.6.3) is a technical combination of Sobolev, Hölder's, and Young's inequalities together with some basic tools of Young functions. The parabolic Sobolev inequality is applied with $q = 1$ and $q = 2n/(n + 2)$, the latter being the reason for having to assume the stronger lower bound for g_0 .

4.6.2 *A priori* Lemma

After the iteration we obtain an important lemma for nonnegative weak subsolutions stating that there exists a constant $\sigma \in (0, 1)$ depending on the data such that whenever

$$\int_{Q(\rho, \theta)} \left(G\left(\frac{u}{\rho}\right) + \frac{u^2}{\theta} \right) dx dt \leq \sigma G\left(\frac{k}{\rho}\right) \quad (4.6.4)$$

holds, we have

$$\operatorname{ess\,sup}_{Q(\rho/2, \theta/2)} u \leq k.$$

Observe that θ depends on k , making the result more complex than it might appear at first.

4.6.3 Case by case analysis

Next the *a priori* Lemma is used to prove the local boundedness of weak solutions. At this point it is necessary to consider separately different cases depending on the growth properties of the function G . The first case corresponds to the degenerate case ($p > 2$) of the p -Laplacian and is defined as $\limsup_{s \rightarrow \infty} s^2/G(s) = 0$. In the second case we suppose $\liminf_{s \rightarrow \infty} s^2/G(s) = \infty$, which for the p -Laplacian translates to the singular case ($p < 2$). However, since it is possible for G to have more general than power-like growth, we also need a third case consisting of everything that is left over. In a way this case falls between the degenerate and singular ones (it occurs when $g_0 \leq 2 \leq g_1$), hence we call it the intermediate case.

The idea is to apply the lemma in each case by showing that there is a finite k and a neighborhood $Q(\rho, \theta)$ of $(x_0, t_0) \in \Omega_T$ such that (4.6.4) holds. The proof is quite technical but the idea is as follows.

Degenerate case. In the degenerate case we choose a really flat cylinder, that is, for a fixed $r > 0$ we take $\rho = r$ and $\theta = k^2 G(k/r)^{-1}$ according to (4.6.1). Since $\limsup_{s \rightarrow \infty} s^2/G(s) = 0$, for large s the term $G(s)$ dominates over s^2 , and thus we can estimate the cylinder $Q(\rho, \theta)$ from above by the nonintrinsic cylinder $Q(r, r^2)$.

We divide the integral in (4.6.4) into two parts. In the set $\{u < \varepsilon k\}$ we can easily estimate the integral by $\sigma/2 G(k/\rho)$ by taking ε small enough. On the other hand, when $u \geq \varepsilon k$, we may estimate u^2/θ by $G(u/r)$, but only for a large enough k . Thus we have

$$\begin{aligned} \int_{Q(\rho, \theta)} \left(G\left(\frac{u}{\rho}\right) + \frac{u^2}{\theta} \right) \chi_{\{u \geq \varepsilon k\}} dx dt \\ \leq c \frac{G(k/r)}{|B_r| k^2} \int_{Q(r, r^2)} G\left(\frac{u}{r}\right) dx dt \leq \frac{\sigma}{2} G\left(\frac{k}{\rho}\right) \end{aligned}$$

by taking

$$k \geq cr \left(\int_{Q(r,r^2)} G\left(\frac{u}{r}\right) dx dt \right)^{1/2}.$$

The size of k remains unknown due to the unpredictable behavior of $s^2/G(s)$; this will become apparent by studying the proof. However, we have found a finite k satisfying (4.6.4), and therefore we have the result in the degenerate case.

Singular case. The singular case is in a way dual to the degenerate case, making the proof very similar. This time we need the cylinder $Q(\rho, \theta)$ to be thin and long, while still satisfying (4.6.1); hence we take $\rho = k[G^{-1}(k^2/r^2)]^{-1}$ and $\theta = r^2$. We again divide the integral into two parts but now estimate $G(u/\rho)$ by u^2/r^2 in the set $\{u \geq \varepsilon k\}$ when k is large enough. Then by choosing

$$k \geq cr \left(\int_{Q(r,r^2)} \frac{u^2}{r^2} dx dt \right)^{1/(n+2-2n/g_0)}$$

we obtain the result also in the singular case. Again, the proof yields no quantitative information on the size of k .

Intermediate case. The intermediate case turns out to be the easiest, because now it is possible to find a sequence $s_m \rightarrow \infty$ as $m \rightarrow \infty$ and $M > 0$ such that

$$\frac{1}{M} \leq \frac{s_m^2}{G(s_m)} \leq M$$

for all $m \in \mathbb{N}$. The cylinder $Q(\rho, \theta)$ can be chosen to be for example the same as in the degenerate case. Now (4.6.4) holds by taking $k = r s_{m^*}$, where $0 < r < 1/M$ and m^* is the smallest m satisfying

$$s_m \geq \left(\frac{1}{\sigma r} \int_{Q(r,r)} \left(G\left(\frac{u}{r}\right) + M\left(\frac{u}{r}\right)^2 \right) dx dt \right)^{1/2}.$$

Also in this case no upper bound for k can be obtained, since we only have qualitative information about the sequence s_m .

4.6.4 Lower semicontinuity of supersolutions

Lower semicontinuity of weak supersolutions to the evolutionary p -Laplace equation up to a set of measure zero was proved by Kuusi in [35] using *a priori* estimates for the supremum of nonnegative subsolutions. Such estimates are not available for the equation with general growth as explained above, however, we manage to obtain the same result by employing the *a priori* Lemma in Section 4.6.2.

4.7 The obstacle problem

4.7.1 Existence

The last part of Publication II is devoted to the obstacle problem related to equation (4.5.1). We say that a function u is a solution to the obstacle problem with the obstacle ψ in Ω_T , if u is the smallest weak supersolution in Ω_T that lies above ψ almost everywhere. We prove that when $\psi \in L^\infty(\Omega_T)$, then there exists a unique solution $u \in L^\infty(\Omega_T)$ to the obstacle problem with the obstacle ψ . For this we need the lower semicontinuity result as well as the facts that both the minimum of weak supersolutions and the limit of a bounded sequence of weak supersolutions converging almost everywhere are also weak supersolutions. The latter are known facts for the p -Laplacian but - although quite obviously true - not for the more general equation, and hence we include proofs for the sake of completeness.

4.7.2 Continuity

Finally, we prove that if the obstacle ψ is moreover continuous, then the solution to the obstacle problem will be as well. This was proved for the evolutionary p -Laplace equation in [34], and our approach is very similar, in addition proving some smaller results that have been missing for the more general equation. The key idea is using a modification of the Schwarz alternating method to construct a sequence of functions converging to a continuous function. Then by showing that the limit is a solution to the obstacle problem, we obtain the result by uniqueness.

To be more precise, we define a countable and dense family of cylinders

$$\mathcal{F} = \left\{ Q^k \subset \Omega_T : Q^k = B_{r_k}(x_k) \times (\tau_k, T), r_k, \tau_k \in \mathbb{Q}, x_k \in \mathbb{Q}^n \right\}$$

and a sequence of functions

$$\varphi_0 = \psi, \quad \varphi_{k+1} = \max\{\varphi_k, v_k\},$$

where v_k is a weak solution in Q^k coinciding with φ_k in the complement of Q^k , in particular $v_k = \varphi_k$ on $\partial_p Q^k$. By the continuity result of Publication I the functions v_k are continuous and hence the functions φ_k are as well. Moreover, the sequence φ_k is uniformly bounded by the maximum principle. Since the sequence is also nondecreasing, it has a limit, which we call u^* .

We prove that the function u^* is a solution to the obstacle problem by showing that it is a weak supersolution; it is then easy to see that there can be no smaller weak supersolutions. To this end, we introduce so-called \mathcal{A} -superparabolic functions, which are defined via the comparison principle, analogously to the \mathcal{A} -superharmonic

functions used in the elliptic theory, see for example [27]. It turns out that also in the parabolic setting continuous \mathcal{A} -superparabolic functions are weak supersolutions.

It is fairly simple to see that the function u^* is \mathcal{A} -superparabolic. For the continuity we use a Poisson modification of the obstacle ψ and show that the solution to the modified obstacle problem is continuous by applying the gradient estimate (4.3.6) in Publication I. Then, since the obstacle itself is not too far from the modified obstacle, also the corresponding solutions differ only little, and thus the function u^* is continuous as well.

5. Phase transition problems

While the evolutionary p -Laplace equation describes the temperature and how it evolves in a system with only one phase, phase transition problems take into account systems that can possibly consist of homogeneous matter in two or even multiple different phases. A typical example of such a system is water with blocks of ice. Other applications of phase transition problems include financial models, Lotka-Volterra models in biology, and flows of solutes or gases in porous media. A characteristic feature of phase transition problems is that the boundary between the different phases is unknown and it evolves in time, making them free boundary problems.

The reason why systems with different phases cannot be treated with the same equations as one-phase systems comes directly from physics. When heat energy is brought into a system, within the same phase its temperature will rise linearly depending on the thermal capacity of the matter in that phase. However, with enough energy the system will reach a point where the matter will start transitioning between phases, for example ice will start melting into water once heated enough. Since some of the energy is needed for the phase transition, for a certain amount of heat the temperature will not rise. This is called *latent heat* and it depends on the matter and the two phases in question.

5.1 The mathematical model and concept of solutions

One way to take the latent heat into account in the mathematical model describing the system is by adding a jump to the parabolic part of the evolutionary p -Laplace equation. In practice this is done by considering the differential inclusion

$$\partial_t(u + \mathcal{L}_h H_a(u)) \ni \operatorname{div}(|Du|^{p-2} Du) \quad \text{in } \Omega_T, \quad (5.1.1)$$

where

$$H_a(s) = \begin{cases} 0 & \text{if } s < a \\ [0, 1] & \text{if } s = a \\ 1 & \text{if } s > a \end{cases} \quad (5.1.2)$$

is the Heaviside graph centered at $a \in \mathbb{R}$ (the “melting” point), and $\mathcal{L}_h > 0$ is the latent heat, often scaled to be one. Instead of equality we have set membership, since now the parabolic term is a graph. Problem (5.1.1) models systems with two phases. More phases can be considered by adding more jumps in the same spirit as above but in this work we shall only consider two-phase problems. For multi-phase problems see for instance [20].

The definition of weak solutions to (5.1.1) is similar to how we define weak solutions to the evolutionary p -Laplacian, but now we have to somehow deal with the ambiguity caused by the graph. This is done by choosing a function $v \in u + \mathcal{L}_h H_a(u)$ in the sense of graphs and saying that a weak solution to (5.1.1) in a space-time cylinder $\mathcal{K} = \mathcal{D} \times (t_1, t_2) \subset \Omega_T$ is a pair (u, v) such that u is in the correct function space and the integral identity

$$-\int_{\mathcal{K}} v \partial_t \varphi \, dx \, dt + \int_{\mathcal{K}} |Du|^{p-2} Du \cdot D\varphi \, dx \, dt = 0 \quad (5.1.3)$$

holds for every compactly supported smooth test function φ . The correct function space for the function u is the same as in the p -Laplace case, that is, $V^{2,p}(\mathcal{K})$, see (3.2). There are many alternative ways to define the weak solutions depending on what kind of properties the test functions φ are desired to have, but they are usually easily shown to be equivalent. See Publication III for an example of a slightly different definition.

5.2 The classical Stefan problem

Problem (5.1.1) when $p = 2$ is also known as the two-phase Stefan problem. The classical formulation of the problem, originally proposed by Stefan in 1890 [49], asks to solve the heat equation in the two different regions (say ice and water) separated by the free boundary with an additional condition on the free boundary known as the Stefan condition. The Stefan condition describes the local velocity of the free boundary in terms of quantities evaluated on both sides of the boundary, typically one takes

$$V_n = |Du^+| - |Du^-|,$$

where u^+ and u^- are the limits of the solution u when approaching the free boundary from $\{u > a\}$ and $\{u < a\}$, respectively, and V_n is the outward normal velocity of the free boundary with respect to the set $\{u > a\}$.

Classical solutions are weak solutions as shown by Friedman in [26], see also [38]. Instead of weak solutions one could also consider viscosity solutions as introduced by Crandall and Lions [12], and further developed by Caffarelli and others, see [2, 47]. Also viscosity solutions turn out to be weak solutions, and moreover the converse is true if $|\{u = a\}| = 0$, see [31].

Continuity of weak solutions to the two-phase Stefan problem was proved by Caffarelli and Evans in [9]. The same result for more general structures (although still linear with respect to the gradient) was obtained by DiBenedetto in [14]. Both of these proofs give a modulus of continuity of the type

$$\omega(r) = \left[\log \log \left(\frac{Ar_0}{r} \right) \right]^{-\sigma}, \quad 0 < r \leq r_0 \quad (5.2.1)$$

for some $A, \sigma > 0$. Continuity up to the boundary for the Cauchy-Dirichlet problem was proved by Ziemer in [53], and in [15] DiBenedetto quantifies the result by showing that the solutions have (5.2.1) as the modulus of continuity, if the boundary data is Hölder continuous.

5.3 The degenerate case

For the degenerate two-phase Stefan problem, that is, (5.1.1) with $p > 2$, only few results are available. Existence of solutions was obtained by Urbano in [50] using an approximation method. In [51] he was able to prove that at least one of the solutions is continuous, however, only an implicit modulus of continuity was obtained.

A significant improvement was obtained by Urbano together with Baroni and Kuusi in [4], where they apply the weak Harnack inequality for supersolutions to show that weak solutions to (5.1.1) with $p \geq 2$ are continuous with an explicit modulus of continuity

$$\omega(r) = \left[p + \log \left(\frac{r_0}{r} \right) \right]^{-\alpha(n,p)}, \quad 0 < r \leq r_0, \quad (5.3.1)$$

which they conjecture to be sharp. Thus, advancements were made not only in the range of the exponent p but also in the achieved modulus of continuity, which only has one logarithm instead of the two in (5.2.1). Moreover, the precise value of the exponent α is obtained.

The result in [4] assumes that weak solutions can be obtained as a uniform limit of Hölder continuous solutions to a regularized problem approximating (5.1.1). In Publication III this is shown to be possible by proving the existence of a solution to the Cauchy-Dirichlet problem with continuous boundary data using such a regularization method. Moreover, an explicit modulus of continuity up to the boundary is obtained, see (6.0.3).

6. Summary of Publication III

In Publication III we complete the work started in [4] by proving the existence of so-called *physical solutions* to the degenerate two-phase Stefan problem

$$\begin{cases} \partial_t [\beta(u) + H_a(\beta(u))] \ni \operatorname{div} \mathcal{A}(x, t, u, Du) & \text{in } \Omega_T, \\ u = g & \text{on } \partial_p \Omega_T, \end{cases} \quad (6.0.1)$$

where $\beta : \mathbb{R} \rightarrow \mathbb{R}$ is a sufficiently smooth function, H_a the Heaviside graph centered at $a \in \mathbb{R}$ (see (5.1.2)), and g a continuous boundary value function. The vector field \mathcal{A} is modeled after the degenerate p -Laplacian ($p \geq 2$) and is assumed to satisfy

$$\begin{aligned} |\mathcal{A}(x, t, u, \xi)| &\leq \Lambda |\xi|^{p-1}, & \langle \mathcal{A}(x, t, u, \xi), \xi \rangle &\geq \Lambda^{-1} |\xi|^p, \\ \langle \mathcal{A}(x, t, u, \xi) - \mathcal{A}(x, t, u, \zeta), \xi - \zeta \rangle &> 0, \end{aligned} \quad (6.0.2)$$

for almost every $(x, t) \in \Omega_T$, $(u, \xi, \zeta) \in \mathbb{R} \times \mathbb{R}^{2n}$ with $\zeta \neq \xi$, and for some $\Lambda \geq 1$. In addition to the existence result we obtain an explicit boundary modulus of continuity

$$\omega(r) = \frac{1}{\vartheta} \left[\log \log \left(\frac{\lambda_0 R_0}{r} \right) \right]^{-\alpha}, \quad 0 < r \leq R_0, \quad (6.0.3)$$

for some $\vartheta, \lambda_0, \alpha$ depending on the data.

6.1 Physical solutions

Similarly to (5.1.3) we define local weak solutions to (6.0.1)₁ as a pair (u, v) with

$$v \in \beta(u) + H_a(\beta(u))$$

in the sense of graphs such that $u \in V_{\text{loc}}^{2,p}(\Omega_T)$ (see (3.2.2)) and the integral identity

$$\int_{\mathcal{K}} [v \varphi](\cdot, \tau) dx \Big|_{\tau=t_1}^{t_2} + \int_{\mathcal{K} \times [t_1, t_2]} [-v \partial_t \varphi + \langle \mathcal{A}(\cdot, \cdot, u, Du), D\varphi \rangle] dx dt = 0 \quad (6.1.1)$$

holds for all $\mathcal{K} \Subset \Omega$ and almost every $t_1, t_2 \in \mathbb{R}$ such that $[t_1, t_2] \Subset (0, T]$, and for every test function $\varphi \in L_{\text{loc}}^p(0, T; W_0^{1,p}(\mathcal{K}))$ such that $\partial_t \varphi \in L^2(\mathcal{K} \times [t_1, t_2])$.

The function β is assumed to be an increasing C^1 diffeomorphism satisfying the bi-Lipschitz condition

$$\Lambda^{-1}|u - v| \leq |\beta(u) - \beta(v)| \leq \Lambda|u - v|, \quad (6.1.2)$$

and therefore also $v \in L_{\text{loc}}^\infty(0, T; L_{\text{loc}}^2(\Omega))$, making (6.1.1) well-defined. The function β is included in the model to account for the thermal properties of the medium, which can change slightly with respect to the temperature, see [14, 45].

A function u is called a physical solution to problem (6.0.1), if $u \in C(\overline{\Omega_T})$ is a local weak solution to (6.0.1)₁ and $u = g$ pointwise on $\partial_p \Omega_T$. The result in [4] was obtained under the assumption that physical solutions exist, leaving the proof to Publication III.

6.2 Regularization

What increases the difficulty of the Stefan problem significantly compared to the evolutionary p -Laplace equation is the jump in the parabolic part, which causes a further degeneracy in addition to the one from the elliptic part. The idea is to smooth out the graph by mollifying away the discontinuity in H_a . To this end, we define for $\varepsilon > 0$

$$H_{a,\varepsilon}(s) := (\rho_\varepsilon * H_a)(s), \quad s \in \mathbb{R},$$

where ρ_ε is a standard mollifier supported in $(-\varepsilon, \varepsilon)$. Observe that $H_{a,\varepsilon}$ is smooth and it satisfies

$$\text{supp } H'_{a,\varepsilon} \subset (a - \varepsilon, a + \varepsilon), \quad \int_{\mathbb{R}} H'_{a,\varepsilon}(v) dv = 1. \quad (6.2.1)$$

Let u_ε be a solution to problem (6.0.1) with H_a replaced by $H_{a,\varepsilon}$. Furthermore, we set

$$w_\varepsilon := \beta(u_\varepsilon), \quad w_0 := \beta(g),$$

and thus arrive at the regularized problem

$$\begin{cases} \partial_t w_\varepsilon - \text{div } \bar{\mathcal{A}}(x, t, w_\varepsilon, Dw_\varepsilon) = -\partial_t H_{a,\varepsilon}(w_\varepsilon) & \text{in } \Omega_T, \\ w_\varepsilon = w_0 & \text{on } \partial_p \Omega_T, \end{cases} \quad (6.2.2)$$

where $\bar{\mathcal{A}}$ is a structurally similar vector field to \mathcal{A} .

By standard regularity theory for degenerate parabolic equations (see [17, 37, 52]) solutions to (6.2.2)₁ are Hölder continuous, since $\beta(u_\varepsilon) + H_{a,\varepsilon}(\beta(u_\varepsilon))$ is a diffeomorphism. This kind of regularity depends, however, on the regularization and as such it will deteriorate as $\varepsilon \rightarrow 0$. Nonetheless, we may assume that the solution of the regularized equation is continuous having, in particular, pointwise values.

6.3 Reducing the oscillation

After rescaling the equation for convenience we end up studying the problem

$$\begin{cases} \partial_t v - \operatorname{div} \tilde{\mathcal{A}}(x, t, v, Dv) = -\partial_t H_{b,\varepsilon}(v) & \text{in } \Omega_T, \\ v = \tilde{g} & \text{on } \partial_p \Omega_T, \end{cases} \quad (6.3.1)$$

where $\tilde{\mathcal{A}}$ is another structurally similar vector field to \mathcal{A} , \tilde{g} is the boundary function rescaled, and b replaces a as the location of the jump after rescaling. The goal is to prove an oscillation reduction estimate at the boundary for the solution v independent of the regularization parameter ε .

As usual, the first step is to derive a Caccioppoli type inequality at the boundary. The proof is similar to the p -Laplace case, the treatment of the parabolic part can be found for example in [4]. The resulting inequality

$$\begin{aligned} & \sup_{\tau \in \Gamma \cap (0, T)} \frac{1}{|\Gamma \cap (0, T)|} \int_{B \cap \Omega} \left[\int_k^v H'_{b,\varepsilon}(\xi)(\xi - k)_+ d\xi \phi^p \right](\cdot, \tau) dx \\ & + \sup_{\tau \in \Gamma \cap (0, T)} \frac{1}{|\Gamma \cap (0, T)|} \int_{B \cap \Omega} [(v - k)_+^2 \phi^p](\cdot, \tau) dx + \int_{Q \cap \Omega_T} |D(v - k)_+ \phi|^p dx dt \\ & \leq c \int_{Q \cap \Omega_T} \left[(v - k)_+^p |D\phi|^p + (v - k)_+^2 (\partial_t \phi^p)_+ \right] dx dt \\ & \quad + c \int_{Q \cap \Omega_T} \int_k^v H'_{b,\varepsilon}(\xi)(\xi - k)_+ d\xi (\partial_t \phi^p)_+ dx dt \end{aligned} \quad (6.3.2)$$

holds for any $Q = B \times \Gamma$ such that $Q \cap \partial_p \Omega_T \neq \emptyset$, any $k > \sup_{\overline{Q} \cap \partial_p \Omega_T} \tilde{g}$ and any test function $\phi \in C^\infty(Q)$ vanishing on $\partial_p Q$. Observe that the two middle lines are precisely the Caccioppoli inequality for the evolutionary p -Laplace equation, however, due to the jump we also have two additional terms.

6.3.1 The intrinsic geometry

As shown by DiBenedetto for the evolutionary p -Laplacian (see for example [17]), the way to handle the inhomogeneity caused by the nonlinearity of the equation is by using the method of intrinsic scaling, see also Section 2.3. This is indeed the way to do it also for the degenerate Stefan problem but, as it turns out, on the lateral boundary we need three different geometries to also account for the degeneracy caused by the jump depending on its location. We consider the three different alternatives separately.

To this end, the following notation is needed. Let $(x_0, t_0) \in \partial_{\text{lat}} \Omega_T$, $\omega \in (0, 1]$, and define for $\varepsilon_1 \in (0, 1)$ to be chosen

$$\tilde{\omega} := \varepsilon_1 \omega \exp\left(-[\varepsilon_1 \omega]^{-p'q}\right),$$

where $q > 2$ is a number related to the Sobolev exponent. We set for $r > 0$ the three time scales

$$T^1 = [\varepsilon_1 \omega]^{2-p_r p}, \quad T^2 = [\varepsilon_2 \tilde{\omega}]^{2-p_r p}, \quad T^3 = \tilde{\omega}^{1-p_r p},$$

where $\varepsilon_2 \in (0, 1)$ is to be chosen such that $\varepsilon_1 \leq \varepsilon_2^{p-2}$. Moreover, we denote for $\sigma > 0$

$$\sigma Q^i := (B_{\sigma r}(x_0) \times (t_0 - \sigma T^i, t_0)) \cap \Omega_T, \quad i = 1, 2, 3.$$

Observe that $Q^1 \subset Q^2 \subset Q^3$. We shall also need the following restriction on the size of the regularization parameter:

$$\varepsilon \leq \frac{\tilde{\omega}}{2}.$$

Finally, let

$$\mu^+ := \sup_{Q^3} v, \quad \mu^- := \inf_{Q^3} v \quad (6.3.3)$$

and suppose $b \in [\mu^-, \mu^+]$, that is, the value of the jump is reached by the solution v . Otherwise the problem reduces to the evolutionary p -Laplace equation and for that the result is known.

Alternative 1. In the first alternative we assume that the jump is far away from the supremum of v , more precisely

$$b \leq \mu^+ - 2\tilde{\omega}. \quad (\text{Alt. 1})$$

Then it is possible to reduce the oscillation while remaining above the jump, and therefore the proof is essentially the same as for the p -Laplacian. Indeed, here we work in the cylinder Q^2 . After combining the Caccioppoli inequality (with the terms with the jump omitted due to the assumption) with Poincaré inequality and also using a Sobolev inequality, a De Giorgi type iteration yields

$$\sup_{\frac{1}{16}Q^2} v \leq \mu^+ - \varepsilon_2 \tilde{\omega} \quad (6.3.4)$$

fixing ε_2 depending on the data.

Alternative 2.1. In the second alternative we have the opposite case

$$b > \mu^+ - 2\tilde{\omega}. \quad (\text{Alt. 2})$$

Now the jump is close to the supremum of v , therefore having an actual effect on the behavior of the solution. We further divide this alternative into two cases, the first one being

$$\sup_{\max\{0, t_0 - \frac{1}{4}T^1\} < t < t_0} \int_{B_{r/4} \cap \Omega} \int_{\mu^+ - 3\tilde{\omega}}^{v(\cdot, t)} H'_{b, \varepsilon}(\xi) d\xi dx > \varepsilon_3^{-1} [\varepsilon_1 \omega]^q, \quad (\text{Alt. 2.1})$$

where $\varepsilon_3 \in (0, 1)$ is to be chosen. Essentially this means that the solution has a high peak of energy close to the jump. This time the terms with $H'_{b, \varepsilon}$ in the Caccioppoli

inequality cannot be ignored, and thus in order to rebalance the inhomogeneity we have to choose Q^3 as the geometry. Now De Giorgi's method gives

$$\sup_{\frac{1}{4}Q^3} v \leq \mu^+ - 4\tilde{\omega}, \tag{6.3.5}$$

fixing ε_3 in terms of the data.

Alternative 2.2. Finally, suppose that (Alt. 2) holds but instead of (Alt. 2.1) we have

$$\sup_{\max\{0, t_0 - \frac{1}{4}T^1\} < t < t_0} \int_{B_{r/4} \cap \Omega} \int_{\mu^+ - 3\tilde{\omega}}^{v(\cdot, t)} H'_{b, \varepsilon}(\xi) d\xi dx \leq \varepsilon_3^{-1} [\varepsilon_1 \omega]^q. \tag{Alt. 2.2}$$

Then the jump is again close to the supremum of v , but now the solution has low energy levels close to the jump for all times. This means that the presence of the jump does not influence the behavior of the solution quite as much, making the equation again similar to the p -Laplace equation. Indeed, we choose Q^1 as the geometry, and after iterating we obtain

$$\sup_{\frac{1}{16}Q^1} v \leq \mu^+ - \varepsilon_1 \omega, \tag{6.3.6}$$

which finally fixes ε_1 depending on the data and q .

6.3.2 Conclusion

Combining (6.3.4), (6.3.5) and (6.3.6), and subtracting $\inf_{\frac{1}{16}Q^1} v \geq \mu^-$ from both sides, we get that in every case the inequality

$$\operatorname{osc}_{\frac{1}{16}Q^1} v \leq \operatorname{osc}_{Q^3} v - \varepsilon_1^2 \omega \exp(-[\varepsilon_1 \omega]^{-p'q}) \tag{6.3.7}$$

holds. To be precise, the above results are obtained under the assumption that

$$\sup_{\overline{Q^3} \cap \partial_p \Omega_T} \tilde{g} \leq \mu^+ - \frac{\omega}{8} \tag{6.3.8}$$

holds. If we instead have

$$\inf_{\overline{Q^3} \cap \partial_p \Omega_T} \tilde{g} \geq \mu^- + \frac{\omega}{8}, \tag{6.3.9}$$

then (6.3.7) is still true, since $-v$ solves a problem similar to (6.3.1) with boundary datum $-\tilde{g}$. In the remaining case where both (6.3.8) and (6.3.9) fail, the oscillation of v is controlled by the oscillation of \tilde{g} .

6.3.3 Initial boundary

The proof for the reduction of oscillation at the initial boundary is almost identical to that for the evolutionary p -Laplacian. We prove a standard logarithmic lemma (see for example [17]) in cylinders of the type

$$Q = (B_r(x_0) \cap \Omega) \times (0, \min\{\omega^{2-p} r^p, T\})$$

for $x_0 \in \overline{\Omega}$ and $\omega, r > 0$. Then another De Giorgi iteration yields

$$\sup_{\frac{1}{4}Q} v \leq \sup_Q v - \varepsilon_4 \omega \quad (6.3.10)$$

for some $\varepsilon_4 \in (0, 1)$ depending on the data and q .

6.4 Uniform modulus of continuity

The lateral and initial boundary cases are put together with the interior estimate from [4] to obtain the following iterative estimate. Let $(x_0, t_0) \in \overline{\Omega}_T$, $\alpha := \frac{1}{p'q}$, and define for $j \in \mathbb{N}_0$

$$\begin{aligned} \tilde{\omega}_j &:= \tau \omega_j \exp\left(-[\tau \omega_j]^{-1/\alpha}\right) \\ R_{j+1} &:= \exp\left(-\frac{\vartheta}{\alpha} [\vartheta \omega_j]^{-1/\alpha}\right) R_j, \quad T_j := \tilde{\omega}_j^{1-p} R_j^p \\ Q^j &:= (B_{R_j}(x_0) \times (t_0 - T_j, t_0 + T_j)) \cap \overline{\Omega}_T, \end{aligned} \quad (6.4.1)$$

where $\{\omega_j\}_{j \in \mathbb{N}_0}$ is a decreasing sequence such that

$$\omega_0 := 1, \quad \omega_{j+1} \geq \omega_j \left(1 - \vartheta \exp\left(-[\vartheta \omega_j]^{-1/\alpha}\right)\right) \quad (6.4.2)$$

and the constants $\vartheta, \tau \in (0, 1/2)$ depend only on the data and q . Then whenever

$$\operatorname{osc}_{Q^j} v \leq \omega_j \quad (6.4.3)$$

for some $j \in \mathbb{N}_0$, we have

$$\operatorname{osc}_{Q^{j+1}} v \leq \max\left\{\omega_{j+1}, 2 \frac{\operatorname{osc}_{\tilde{Q}^j} \tilde{g}}{\tilde{Q}^j \cap \partial_p \Omega_T}\right\}. \quad (6.4.4)$$

The proof is fairly technical and the main difficulty is ensuring that all the different cylinders from the different cases can be boxed in a suitable way. Observe in particular the lateral boundary case, where we have geometry of the type Q^1 on the left hand side of inequality (6.3.7) and geometry of the type Q^3 on the right hand side. Thus, it is necessary to find a way to fit a smaller cylinder of the type Q^3 (smaller in the sense that j is increased by one) inside one 16th of the cylinder of the type Q^1 .

Using real analytical tools we show that (6.4.2) is satisfied by the sequence $\omega_j = \omega(R_j)$, where ω is defined as in (6.0.3). This means that by iterating the above result we find a modulus of continuity $\bar{\omega}$ depending on ω and the regularity of the boundary data \tilde{g} such that

$$\operatorname{osc}_{Q_r(x_0, t_0) \cap \overline{\Omega}_T} v \leq \bar{\omega}(r) + h(\varepsilon), \quad (6.4.5)$$

where $Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^p, t_0 + r^p)$ and $h(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. The term $h(\varepsilon)$ is due to the assumption (6.3.3). Observe, however, that the modulus of continuity $\bar{\omega}$ does not depend on ε .

6.5 Convergence

Scaling back to the problem

$$\begin{cases} \partial_t \beta(u_\varepsilon) - \operatorname{div} \mathcal{A}(x, t, u_\varepsilon, Du_\varepsilon) = -\partial_t H_{a,\varepsilon}(\beta(u_\varepsilon)) & \text{in } \Omega_T, \\ u_\varepsilon = g & \text{on } \partial_p \Omega_T, \end{cases}$$

and also using the regularity of the function β , we see that the solution u_ε is continuous up to the boundary with “almost uniform” modulus of continuity, in the spirit of (6.4.5). Moreover, by the maximum principle we have

$$\sup_{\Omega_T} |u_\varepsilon| \leq \sup_{\partial_p \Omega_T} |g|. \quad (6.5.1)$$

Thus a modification of the theorem of Ascoli-Arzelà yields a function $u \in C(\overline{\Omega_T})$ such that a subsequence of u_ε converges uniformly to u . Then by proving that the limit u is a physical solution to the original problem (6.0.1), we obtain the existence result. Moreover, by taking $\varepsilon \rightarrow 0$ in the previous estimates, we see that if the boundary data is regular enough, then the solution u has the modulus of continuity ω defined in (6.0.3).

In order to show that u is a weak solution to (6.0.1)₁ we study separately the cases where the convergence occurs near the jump and away from the jump. In the latter case the equation reduces to the evolutionary p -Laplace equation and the proof is nearly identical to that of Theorem 5.3 in [33], see also [6, 7]. Near the jump we test the regularized equation with a truncation (as in [33]) of the difference between the solution and the jump and show that the set where $|u - a|$ is small has negligible effect.

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